GEOMETRIC INVARIANT THEORY

1. Introduction

These notes present core results in geometric invariant theory. We mostly follow monography [Mumford et al., 1994]. We extensively use the language of schemes.
Throughout these notes we fix a field k and a group scheme $\mathbf{G}$ over k with the identity $e : \operatorname{Spec} k \to \mathbf{G}$ and the multiplication $\mu : \mathbf{G} \times_k \mathbf{G} \to \mathbf{G}$.

2. Basic properties of quotients

We start by discussing some properties of submersive morphisms.

Fact 2.1. *Submersive morphisms of schemes are local on target.*

Proof. Fix a morphism $q : X \to Y$ and suppose that there exists an open cover $\mathcal{V}$ of Y such that for every $V \in \mathcal{V}$ the restriction $q^{-1}(V) \to V$ of q is submersive. Clearly q is surjective. Fix a subset U of Y such that $q^{-1}(U)$ is open. A set $q^{-1}(U \cap V) = q^{-1}(U) \cap q^{-1}(V)$ is an open subset of X for every $V \in \mathcal{V}$. Since the restriction $q^{-1}(V) \to V$ is submersive for every $V \in \mathcal{V}$, we derive that $U \cap V$ is open for every $V \in \mathcal{V}$. Thus

$$U = \bigcup_{V \in \mathcal{V}} U \cap V$$

is open in X. Therefore, q is submersive.
On the other hand if $q : X \to Y$ is submersive, then for every open subscheme V the restriction $q^{-1}(V) \to V$ is submersive.

□

Fact 2.2. *Submersive and universally submersive morphisms descent along faithfuly flat and quasi-compact morphisms.*

Proof. It suffices to prove that submersive morphisms have descent property. This follows from the fact that faithfully flat and quasi-compact morphism are submersive. Details are left for the reader. □

In the remaining part of this section we fix a k-scheme X equipped with an action of $\mathbf{G}$ determined by morphism $a : \mathbf{G} \times_k X \to X$. The following result gives scheme-theoretic criterion for topological quotient in the case of group scheme actions.

Proposition 2.3. *Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Assume that q is submersive and the morphism $\mathbf{G} \times_k X \to X \times_Y X$ induced by a and pr_X is surjective. Then the diagram*

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel in the category topological spaces.

Proof. Let π_1 and π_2 be distinct projections $X \times_Y X \to X$. Pick points x_1 and x_2 in X such that $q(x_1) = q(x_2)$. Then there exists a field extension K over k such that $k(x_1) \subseteq K$ and $k(x_2) \subseteq K$.

These give rise to K-points $\overline{x_1}$ and $\overline{x_2}$ of X such that their images under q is the same K-point of Y. Since we have an identification

$$\big(X \times_Y X\big)(K) = X(K) \times_{Y(K)} X(K)$$

induced by π_1 and π_2, we derive that there exists a K-point $\overline{z}$ of $X \times_Y X$ such that $\pi_1(\overline{z}) = \overline{x_1}$ and $\pi_2(\overline{z}) = \overline{x_2}$. Let z be the point of $X \times_Y X$ corresponding to $\overline{z}$. Then $\pi_1(z) = x_1$ and $\pi_2(z) = x_2$. By assumption a and pr_X induce surjection $\mathbf{G} \times_k X \twoheadrightarrow X \times_Y X$. Thus there exists a point u of $\mathbf{G} \times_k X$ such that $a(u) = x_1$ and $\mathrm{pr}_X(u) = x_2$. Thus x_1 and x_2 are identified by an equivalence relation on the underlying set of X which is determined by the pair (a, pr_X). Therefore, fibers of q are equivalence classes with respect to this relation. Since q is submersive, this implies that the diagram

$$\begin{tikzcd} \mathbf{G} \times_k X \arrow[r, shift left, "a"] \arrow[r, shift right, "\mathrm{pr}_X"'] & X \arrow[r, "q"] & Y \end{tikzcd}$$

is a cokernel in the category topological spaces. $\square$

Definition 2.4. Let K be a field extension of k and suppose that $\overline{x}$ is a K-point of X. We consider $\overline{x}$ as a morphism $\operatorname{Spec} K \to X$. Then the morphism

$$\mathbf{G} \times_k \operatorname{Spec} K \xhookrightarrow{1_{\mathbf{G}} \times_k \overline{x}} \mathbf{G} \times_k X \xrightarrow{a} X$$

is called *the orbit morphism of* $\overline{x}$.

The following result is usefull.

Proposition 2.5. *Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Then the following assertions are equivalent.*

- **(i)** *Fix a point y in Y. Consider a geometric point $\overline{x} : \operatorname{Spec} K \to X$ such that $q(\overline{x}) = \overline{y}$ is the geometric point with y as the underlying point. For every K with sufficiently large transcendence degree over k the orbit morphism $o_{\overline{x}} : \mathbf{G} \times_k \operatorname{Spec} K \to X$ induces a surjection $\mathbf{G} \times_k \operatorname{Spec} K \twoheadrightarrow X_y$.*
- **(ii)** *The morphism $\mathbf{G} \times_k X \to X \times_Y X$ induced by $\langle a, \mathrm{pr}_x \rangle : \mathbf{G} \times_k X \to X \times_k X$ is surjective.*

Proof. We start by proving the implication **(i)** $\Rightarrow$ **(ii)**. Assume that **(i)** holds. Consider a point z in $X \times_Y X$. Let y be a point of Y such that $q\left(\mathrm{pr}_X(z)\right) = y = q\left(a(z)\right)$. Consider a geometric point $\overline{x} : \operatorname{Spec} K \to X$ such that $q(\overline{x}) = \overline{y}$ is the geometric point with y as the underlying point. We may assume according to **(i)** that the orbit morphism $o_{\overline{x}} : \mathbf{G} \times_k \operatorname{Spec} K \to X$ induces a surjection $\mathbf{G} \times_k \operatorname{Spec} K \twoheadrightarrow X_y$. Now suppose that L is an algebraically closed field containing K such that there exists an L-point $\overline{z}$ of $X \times_Y X$ with z as the underlying point and the map

$$\mathbf{G}(L) \longrightarrow X_y(L)$$

induced by $o_{\overline{x}}$ on L-points is surjective. Then there exists an L-point g of $\mathbf{G}$ such that $g \cdot \mathrm{pr}_X(\overline{z}) = a(\overline{z})$. Hence the map

$$\mathbf{G}(L) \times X(L) \longrightarrow X(L) \times_{Y(L)} X(L)$$

induced by $\langle a, \mathrm{pr}_x\rangle : \mathbf{G} \times_k X \to X \times_k X$ contains $\overline{z}$ in its image. Indeed, $\big(g, \mathrm{pr}_X(\overline{z})\big)$ is sent to $\overline{z}$ under this map. Thus the set-theoretic image of the morphism $\mathbf{G} \times_k X \to X \times_Y X$ contains z. This shows that **(ii)** holds.
Suppose now that **(ii)** holds. Pick a point y in Y. Let K be an algebraically closed field over k such that there is a surjective map

$$\mathbf{G}(K) \times X(K) \twoheadrightarrow X(K) \times_{Y(K)} X(K)$$

induced by $\langle a, \mathrm{pr}_x\rangle : \mathbf{G} \times_k X \to X \times_k X$. Assume also that K contains all residue fields of points in X_y. Pick a point x in X_y and let $\overline{x}$ be a K-point with x as the underlying point. Next fix any other point z in X_y and let $\overline{z}$ be a K-point with z as the underlying point. Since the map

$$\mathbf{G}(K) \times X(K) \twoheadrightarrow X(K) \times_{Y(K)} X(K)$$

is surjective, we derive that there exists $g \in \mathbf{G}(K)$ such that $g \cdot \overline{x} = \overline{z}$. This implies that the map $\mathbf{G}(K) \to X_y(K)$ induced by the orbit map $o_{\overline{x}}$ contains $\overline{z}$ in its image. Therefore, the morphism $\mathbf{G} \times_k \operatorname{Spec} K \to X_y$ induced by $o_{\overline{x}}$ contains z in its set-theoretic image. Hence it is surjective, since z is an arbitrary point of X_y □

Now we prove a series results concerning fpqc descent. For this we fix a k-scheme Y with the trivial action of $\mathbf{G}$ and a $\mathbf{G}$-equivariant morphism $q : X \to Y$. Let $g : Y' \to Y$ be a morphism of k-schemes and consider the cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

of k-schemes. Note that X' admits a unique action a' of $\mathbf{G}$ such that the square above consists of $\mathbf{G}$-equivariant morphism (we consider g as a $\mathbf{G}$-equivariant morphism between trivial $\mathbf{G}$-schemes).

Fact 2.6. *Suppose that g is faithfully flat and quasi-compact. Then the canonical morphism $X' \times_{Y'} X' \to X \times_Y X$ is faithfully flat and quasi-compact and there is the cartesian square*

$$\begin{tikzcd} \mathbf{G} \times_k X' \arrow[r] \arrow[d] & \mathbf{G} \times_k X \arrow[d] \\ X' \times_{Y'} X' \arrow[r] & X \times_Y X \end{tikzcd}$$

in which the left vertical arrow is induced by $\langle a', \mathrm{pr}_{X'}\rangle : \mathbf{G} \times_k X' \to X' \times_k X'$, the right vertical arrow is induced by $\langle a, \mathrm{pr}_X\rangle : \mathbf{G} \times_k X \to X \times_k X$ and the bottom horizontal morphism is the canonical morphism.

Proof. Note that squares

$$
\begin{tikzcd}
X' \times_{Y'} X' \arrow[r] \arrow[d] & X' \times_Y X' \arrow[d] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \times_Y X' \arrow[r] \arrow[d] & X \times_Y X \arrow[d] \\
X' \times_k X' \arrow[r, "g' \times_k g'"'] & X \times_k X
\end{tikzcd}
$$

are cartesian. Since both g and $g' \times_k g'$ are faithfully flat and quasi-compact, we derive that both morphisms $X' \times_{Y'} X' \to X' \times_Y X'$ and $X' \times_Y X' \to X \times_Y X$ are faithfully flat and quasi-compact. Then their composition i.e. the canonical morphism $X' \times_{Y'} X' \to X \times_Y X$ is faithfully flat and quasi-compact. □

Fact 2.7. *Suppose that there exists an open cover $\mathcal{V}$ of Y such that for every V in $\mathcal{V}$ we have a surjection $\mathbf{G} \times_k V \twoheadrightarrow q^{-1}(V) \times_V q^{-1}(V)$ induced by pr_V and the restriction of the action to $q^{-1}(V)$. Then the morphism $\mathbf{G} \times_k X \to X \times_Y X$ induced by pr_X and a is surjective.*

Proof. It follows from the fact that

$$X \times_Y X = \bigcup_{V \in \mathcal{V}} q^{-1}(V) \times_V q^{-1}(V)$$

□

Finally we need the following notion

Definition 2.8. Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Consider a pair

$$q_*\mathcal{O}_X \overset{q_*a^\#}{\underset{q_*\mathrm{pr}_X^\#}{\rightrightarrows}} q_*\left(\mathrm{pr}_X\right)_* \mathcal{O}_{\mathbf{G}\times_k X} = q_* a_* \mathcal{O}_{\mathbf{G}\times_k X}$$

of morphisms of sheaves of rings on Y. Suppose that $q^\# : \mathcal{O}_Y \to q_*\mathcal{O}_X$ is a kernel of this pair. Then *$\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q.*

Proposition 2.9. *Suppose that g is faitfully flat and quasi-compact. Assume that q' is quasi-compact, semiseparated and $\mathcal{O}_{Y'}$ is the sheaf of $\mathbf{G}$-invariants for q'. Then $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q.*

Proof. We denote by a' the action of $\mathbf{G}$ on X'. First note that q is semiseparated and quasi-compact morphism as these classes of morphisms admit descent along quasi-compact and faithfully flat morphisms. Since q is quasi-compact, semiseparated and g is flat, we derive that for every quasi-coherent sheaf $\mathcal{F}$ on X the canonical morphism $q'_* g'^* \mathcal{F} \to g^* q_* \mathcal{F}$ is an isomorphism. Thus the diagram

$$\mathcal{O}_{Y'} \xrightarrow{q^\#} q'_*\mathcal{O}_{X'} \overset{q'_*a'^\#}{\underset{q'_*\mathrm{pr}_{X'}^\#}{\rightrightarrows}} q'_*\left(\mathrm{pr}_{X'}\right)_* \mathcal{O}_{\mathbf{G}\times_k X'} = q'_* a'_* \mathcal{O}_{\mathbf{G}\times_k X'}$$

is isomorphic to the diagram

$$g^*\mathcal{O}_Y \xrightarrow{g^*q^\#} g^*\left(q_*\mathcal{O}_X\right) \overset{g^*q_*a^\#}{\underset{g^*q_*\mathrm{pr}_X^\#}{\rightrightarrows}} g^*\left(q_*\left(\mathrm{pr}_X\right)_* \mathcal{O}_{\mathbf{G}\times_k X}\right) = g^*\left(q_* a_* \mathcal{O}_{\mathbf{G}\times_k X}\right)$$

Since $\mathcal{O}_{Y'}$ is the sheaf of $\mathbf{G}$-invariants for q', the first diagram is a kernel diagram. Hence the second is a kernel diagram. According to the fact that g is faithfully flat we deduce that the diagram

$$\mathcal{O}_Y \xrightarrow{q^{\#}} q_*\mathcal{O}_X \underset{q_*\mathrm{pr}_X^{\#}}{\overset{q_*a^{\#}}{\rightrightarrows}} q_*\left(\mathrm{pr}_X\right)_*\mathcal{O}_{\mathbf{G}\times_k X} = q_*a_*\mathcal{O}_{\mathbf{G}\times_k X}$$

is also a kernel diagram. Thus $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q. $\square$

Proposition 2.10. *Suppose that there exists an open cover $\mathcal{V}$ of Y such that $\mathcal{O}_V$ is the sheaf of $\mathbf{G}$-invariants for the restriction $q^{-1}(V) \to V$ of q for every V in $\mathcal{V}$. Then $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q.*

Proof of the lemma. The diagram

$$\mathcal{O}_V \xrightarrow{(q^{\#})_{|V}} (q_*\mathcal{O}_X)_{|V} \underset{(q_*\mathrm{pr}_X^{\#})_{|V}}{\overset{(q_*a^{\#})_{|V}}{\rightrightarrows}} \left(q_*\left(\mathrm{pr}_X\right)_*\mathcal{O}_{\mathbf{G}\times_k X}\right)_{|V} = \left(q_*a_*\mathcal{O}_{\mathbf{G}\times_k X}\right)_{|V}$$

is a kernel for every $V \in \mathcal{V}$. Since kernels are local, we derive that

$$\mathcal{O}_Y \xrightarrow{q^{\#}} q_*\mathcal{O}_X \underset{q_*\mathrm{pr}_X^{\#}}{\overset{q_*a^{\#}}{\rightrightarrows}} q_*\left(\mathrm{pr}_X\right)_*\mathcal{O}_{\mathbf{G}\times_k X} = q_*a_*\mathcal{O}_{\mathbf{G}\times_k X}$$

is a kernel of a pair $\left(q_*a^{\#}, q_*\mathrm{pr}_X^{\#}\right)$. Thus $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariant for q. $\square$

3. Categorical and geometric quotients

In this section we fix a k-scheme X equipped with an action of $\mathbf{G}$ determined by morphism $a : \mathbf{G} \times_k X \to X$.

Definition 3.1. Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Suppose that the following conditions hold.

(1) q is submersive.

(2) The morphism $\mathbf{G} \times_k X \to X \times_Y X$ induced by $\langle a, \mathrm{pr}_x\rangle : \mathbf{G} \times_k X \to X \times_k X$ is surjective.

(3) $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariant for q.

Then q is *a geometric quotient of X*.

Corollary 3.2. *Let q be a geometric quotient of X. Then the diagram*

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel in the category of ringed spaces.

Proof. Due to the fact that $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q it suffices to prove that

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

is the cokernel in the category of topological spaces. This follows from Proposition 2.3. $\square$

Corollary 3.3. *Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Then the following assertions are equivalent.*

- **(i)** *There exists an open affine cover $\mathcal{V}$ of Y such that for every V in $\mathcal{V}$ the restriction $q^{-1}(V) \to V$ of q is a geometric quotient.*
- **(ii)** *There exists an open cover $\mathcal{V}$ of Y such that for every V in $\mathcal{V}$ the restriction $q^{-1}(V) \to V$ of q is a geometric quotient.*
- **(iii)** *q is a geometric quotient.*

Proof. This is a consequence of Facts 2.1, 2.7 and Proposition 2.10. □

Definition 3.4. Let $q : X \to Y$ be a morphism of k-schemes such that the diagram

$$\mathbf{G} \times_k X \overset{a}{\underset{\mathrm{pr}_X}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel in the category of k-schemes. Then $q : X \to Y$ is *a categorical quotient of X*.

Fact 3.5. *Every geometric quotient is categorical.*

Proof. Categorical quotient is a cokernel in the category of k-schemes. On the other hand geometric quotient is a cokernel in the category of locally ringed spaces and hence it also satisfies cokernel property in its full subcategory of k-schemes. Thus every geometric quotient is categorical. □

Let $q : X \to Y$ be a morphism of k-schemes such that $q \cdot \mathrm{pr}_X = q \cdot a$. For a morphism $g : Y' \to Y$ of k-schemes consider the cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Then there exists a unique action $a' : \mathbf{G} \times_k X' \to X'$ of $\mathbf{G}$ on X' such that the square above consists of $\mathbf{G}$-equivariant morphism (we consider Y, Y' as $\mathbf{G}$-schemes equipped with trivial $\mathbf{G}$-actions). Keeping this in mind we have the following.

Corollary 3.6. *Let $g : Y' \to Y$ be a faithfully flat and quasi-compact morphism. Suppose that q' is a geometric quotient and a semiseparated morphism, then q is a geometric quotient.*

Proof. This follows from Facts 2.2, 2.6 and Proposition 2.9. □

Definition 3.7. A morphism $q : X \to Y$ is *a uniform categorical (geometric) quotient of X* if for every flat morphism $g : Y' \to Y$ of k-schemes a base change $q' : X' \to Y'$ of q along g is a categorical (geometric) quotient of X'.

Definition 3.8. A morphism $q : X \to Y$ is *a universal categorical (geometric) quotient of X* if for every morphism $g : Y' \to Y$ of k-schemes a base change $q' : X' \to Y'$ of q along g is a categorical (geometric) quotient of X'.

Now we show that uniform and universal categorical quotients are local on the target.

Theorem 3.9. *Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Then the following assertions are equivalent.*

(i) *There exists an open cover $\mathcal{V}$ of Y such that for every V in $\mathcal{V}$ morphism $q^{-1}(V) \to V$ is a universal (uniform) categorical quotient.*

(ii) *q is a universal (uniform) categorical quotient.*

(iii) *For every affine k-scheme Y' and a (flat) morphism $g : Y' \to Y$ of k-schemes a base change $q' : X' \to Y'$ of q along g is a categorical quotient.*

(iv) *There exists an open affine cover $\mathcal{V}$ of Y such that for every V in $\mathcal{V}$ morphism $q^{-1}(V) \to V$ is a universal (uniform) categorical quotient.*

For the proof we need the following.

Lemma 3.9.1. *Let Y be a k-scheme with the trivial action of* $\mathbf{G}$ *and let $q : X \to Y$ be a* $\mathbf{G}$*-equivariant morphism. If there exists an open cover $\mathcal{V}$ of Y such that for every V in $\mathcal{V}$ morphism $q^{-1}(V) \to V$ is a uniform categorical quotient, then q is a categorical quotient.*

Proof of the lemma. We first prove categorical case. For every open subscheme W of Y we denote by q_W the restriction $q^{-1}(W) \to W$. For this pick a $\mathbf{G}$-equivariant morphism $g : X \to Z$ into a scheme with the trivial $\mathbf{G}$-action. Since the restriction q_V is a categorical quotient for every $V \in \mathcal{V}$, there exists a unique morphism $f_V : V \to Z$ such that

$$g_{|q^{-1}(V)} = f_V \cdot q_V$$

Suppose that $V_1, V_2 \in \mathcal{V}$. Then

$$g_{|q^{-1}(V_1 \cap V_2)} = \left(f_{V_1}\right)_{|V_1 \cap V_2} \cdot q_{V_1 \cap V_2}$$

and

$$g_{|q^{-1}(V_1 \cap V_2)} = \left(f_{V_2}\right)_{|V_1 \cap V_2} \cdot q_{V_1 \cap V_2}$$

Since q_{V_1} and q_{V_2} are uniform categorical quotients, we derive that $q_{V_1 \cap V_2}$ is also categorical quotient. Thus equalities above show that $\left(f_{V_1}\right)_{|V_1 \cap V_2} = \left(f_{V_2}\right)_{|V_1 \cap V_2}$. Hence $\{f_V\}_{V \in \mathcal{V}}$ glue to a morphism $f : Y \to Z$ such that $g = f \cdot q$. The uniqueness of f follows from uniqueness of $\{f_V\}_{V \in \mathcal{V}}$. Thus q is a categorical quotient. □

Proof of the theorem. Implications **(ii)** ⇒ **(iii)** and **(iv)** ⇒ **(v)** are obvious.
We prove **(i)** ⇒ **(ii)**. Suppose that **(i)** holds. Pick a (flat) morphism $g : Y' \to Y$ and fix a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Then $\mathcal{V}' = \{g^{-1}(V) \,|\, V \in \mathcal{V}\}$ is an open cover of Y' such that for every $V \in \mathcal{V}'$ the morphism $q'^{-1}(V) \to V$ is a uniform categorical quotient. By Lemma 3.9.1 we derive that q' is a categorical quotient. This is **(ii)**.
Assume that **(iii)** holds. Pick an open affine subset V of Y. Consider a (flat) morphism $g : V' \to V$ and pick a cartesian square

$$\begin{tikzcd} U' \arrow[r, "g'"] \arrow[d, "q_{V'}"'] & q^{-1}(V) \arrow[d, "q_V"] \\ V' \arrow[r, "g"'] & V \end{tikzcd}$$

where $q_V : q^{-1}(V) \to V$ is the restriction of q. Then for every open affine subset W of V' the restriction $q_{V'}^{-1}(W) \to W$ of $q_{V'}$ is a universal (uniform) categorical quotient according to **(iii)** (and the fact that $W \hookrightarrow V'$ composed with g is flat). By Lemma 3.9.1 it follows that $q_{V'}$ is a categorical quotient. Thus q_V is a universal (uniform) categorical quotient. This holds for every open affine subset V of Y. This is **(iv)** and hence **(iii)** $\Rightarrow$ **(iv)** holds. □

Similar result holds for uniform and universal geometric quotients.

Theorem 3.10. *Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Then the following assertions are equivalent.*

- **(i)** *There exists an open cover $\mathcal{V}$ of Y such that for every V in $\mathcal{V}$ morphism $q^{-1}(V) \to V$ is a universal (uniform) geometric quotient.*
- **(ii)** *q is a universal (uniform) geometric quotient.*
- **(iii)** *For every affine k-scheme Y' and a (flat) morphism $g : Y' \to Y$ of k-schemes a base change $q' : X' \to Y'$ of q along g is a geometric quotient.*
- **(iv)** *There exists an open affine cover $\mathcal{V}$ of Y such that for every V in $\mathcal{V}$ morphism $q^{-1}(V) \to V$ is a universal (uniform) geometric quotient.*

Proof. The implication **(i)** $\Rightarrow$ **(ii)** follows from Corollary 3.3. Implication **(ii)** $\Rightarrow$ **(iii)** and **(iv)** $\Rightarrow$ **(i)** are obvious. It suffices to prove that **(iii)** $\Rightarrow$ **(iv)**. Assume that **(iii)** holds. Pick an open affine subset V of Y. Consider a (flat) morphism $g : V' \to V$ and pick a cartesian square

$$\begin{tikzcd} U' \arrow[r, "g'"] \arrow[d, "q_{V'}"'] & q^{-1}(V) \arrow[d, "q_V"] \\ V' \arrow[r, "g"'] & V \end{tikzcd}$$

where $q_V : q^{-1}(V) \to V$ is the restriction of q. Then for every open affine subset W of V' the restriction $q_{V'}^{-1}(W) \to W$ of $q_{V'}$ is a universal (uniform) geometric quotient according to **(iii)** (and the fact that $W \hookrightarrow V'$ composed with g is flat). By Corollary 3.3 it follows that $q_{V'}$ is a geometric quotient. Thus q_V is a universal (uniform) geometric quotient. This holds for every open affine subset V of Y. This implies **(iv)** and hence **(iii)** $\Rightarrow$ **(iv)** holds. □

In the next result we give a simple example of a universal geometric quotient.

Proposition 3.11. *Suppose that $\mathbf{G}$ is a quasi-compact group scheme over k. Let Y be a k-scheme and consider $\mathbf{G} \times_k Y$ with the action of $\mathbf{G}$ induced by the regular action on the left factor. Then $\mathrm{pr}_Y : \mathbf{G} \times_k Y \to Y$ is a universal geometric quotient.*

Proof. Clearly pr_Y is univerally submersive (it is even universally open). Let $\mu : \mathbf{G} \times_k \mathbf{G} \to \mathbf{G}$ be the multiplication morphism and let $\pi_{23} : \mathbf{G} \times_k \mathbf{G} \times Y \to \mathbf{G} \times_k Y$ be the projection on the last two factors. Then the morphism

$$\mathbf{G} \times_k \mathbf{G} \times_k Y \to (\mathbf{G} \times_k Y) \times_Y (\mathbf{G} \times_k Y) = \mathbf{G} \times_k \mathbf{G} \times_k Y$$

induced by $\langle \mu \times_k 1_Y, \pi_{23} \rangle : \mathbf{G} \times_k \mathbf{G} \times_k Y \to (\mathbf{G} \times_k Y) \times_k (\mathbf{G} \times_k Y)$ is an isomorphism. We show that $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for pr_Y. For this pick an affine open subset V of Y. It suffices to check that the diagram

$$\Gamma(V,\mathcal{O}_Y) \xrightarrow{\mathrm{pr}_Y^{\#}} \Gamma\left(\mathbf{G}\times_k V,\mathcal{O}_{\mathbf{G}\times_k Y}\right) \underset{\pi_{23}^{\#}}{\overset{(\mu\times_k 1_Y)^{\#}}{\rightrightarrows}} \Gamma\left(\mathbf{G}\times_k \mathbf{G}\times_k V,\mathcal{O}_{\mathbf{G}\times_k\mathbf{G}\times_k Y}\right)$$

is a kernel. Since $\mathbf{G}$ is quasi-compact and separated (every group k-scheme is separated), we derive that the diagram above is isomorphic with

$$\Gamma(V,\mathcal{O}_Y) \xrightarrow{f\mapsto 1\otimes f} \Gamma(\mathbf{G},\mathcal{O}_{\mathbf{G}})\otimes_k\Gamma(V,\mathcal{O}_Y) \underset{\chi\otimes f\mapsto 1\otimes\chi\otimes f}{\overset{\chi\otimes f\mapsto\mu^{\#}(\chi)\otimes f}{\rightrightarrows}} \Gamma(\mathbf{G},\mathcal{O}_{\mathbf{G}})\otimes_k\Gamma(\mathbf{G},\mathcal{O}_{\mathbf{G}})\otimes_k\Gamma(V,\mathcal{O}_Y)$$

Thus the first diagram is the kernel diagram if $f\mapsto 1\otimes f$ induces an isomorphism of $\Gamma(V,\mathcal{O}_Y)$ with subspace of $\Gamma(\mathbf{G},\mathcal{O}_{\mathbf{G}})\otimes_k\Gamma(V,\mathcal{O}_Y)$ given by formula

$$\left\{\chi\in\Gamma(\mathbf{G},\mathcal{O}_{\mathbf{G}})\,\middle|\,\mu^{\#}(\chi)=1\otimes\chi\right\}\otimes_k\Gamma(V,\mathcal{O}_Y)$$

Hence it suffices to prove that

$$\left\{\chi\in\Gamma(\mathbf{G},\mathcal{O}_{\mathbf{G}})\,\middle|\,\mu^{\#}(\chi)=1\otimes\chi\right\}=\text{constant functions on }\mathbf{G}$$

For this pick a k-algebra A and let $g:\operatorname{Spec}A\to\mathbf{G}$ be an A-point. Next let $e:\operatorname{Spec}A\to\mathbf{G}$ be an A-point of $\mathbf{G}$ which corresponds to the identity element of $\mathbf{G}$. Suppose that a regular function χ in $\mathbf{G}$ satisfies $\mu^{\#}(\chi)=1\otimes\chi$. Then

$$g^{\#}(\chi)=\langle g,e\rangle^{\#}\mu^{\#}(\chi)=\langle g,e\rangle^{\#}(1\otimes\chi)=e^{\#}(\chi)$$

Recall that e is given by the composition of the structural morphism $\operatorname{Spec}A\to\operatorname{Spec}k$ and the k-point $\operatorname{Spec}k\to\mathbf{G}$ determined by the identity of $\mathbf{G}$. Thus $g^{\#}(\chi)$ is an element of k. Since this follows for every $g:\operatorname{Spec}A\to\mathbf{G}$, we derive that χ is a constant function. This completes the proof of our claim that

$$\Gamma(V,\mathcal{O}_Y) \xrightarrow{\mathrm{pr}_Y^{\#}} \Gamma\left(\mathbf{G}\times_k V,\mathcal{O}_{\mathbf{G}\times_k Y}\right) \underset{\pi_{23}^{\#}}{\overset{(\mu\times_k 1_Y)^{\#}}{\rightrightarrows}} \Gamma\left(\mathbf{G}\times_k \mathbf{G}\times_k V,\mathcal{O}_{\mathbf{G}\times_k\mathbf{G}\times_k Y}\right)$$

is the kernel diagram and hence $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for pr_Y. Therefore, we proved that pr_Y is a geometric quotient of $\mathbf{G}\times_k Y$. Consider any morphism $Y'\to Y$. Then base change of pr_Y along this morphism is $\mathrm{pr}_{Y'}$. We conclude that pr_Y is a universal geometric quotient for $\mathbf{G}\times_k Y$. □

4. Geometric quotients of separated actions

In this section we fix a k-scheme X equipped with an action of $\mathbf{G}$ determined by morphism $a:\mathbf{G}\times_k X\to X$.

Definition 4.1. The action of $\mathbf{G}$ on X is *separated* if the morphism $\langle a,\mathrm{pr}_X\rangle:\mathbf{G}\times_k X\to X\times_k X$ has closed set-theoretic image.

Theorem 4.2. *Let $q:X\to Y$ be a geometric quotient of X. Assume that q is universally submersive. Then the following assertions are equivalent.*

- **(i)** *The action of $\mathbf{G}$ on X is separated.*
- **(ii)** *Y is separated.*

Proof. We have a cartesian square

$$\begin{tikzcd} X \times_Y X \arrow[r, hook] \arrow[d] & X \times_k X \arrow[d, "q \times_k q"] \\ Y \arrow[r, hook, "\Delta_Y"'] & Y \times_k Y \end{tikzcd}$$

It follows that $X \times_Y X \hookrightarrow X \times_k X$ is a locally closed immersion. Since q is a geometric quotient, we derive that $\langle a, \mathrm{pr}_X \rangle$ factors as a surjective morphism $\mathbf{G} \times_k X \twoheadrightarrow X \times_Y X$ followed by the immersion $X \times_Y X \hookrightarrow X \times_k X$. Thus the action of $\mathbf{G}$ on X is separated if and only if $X \times_Y X$ is a closed subscheme of $X \times_k X$. Since q is universally submersive, we derive that $q \times_k q$ is submersive. As the square above is cartesian we derive that $\Delta_Y(Y) \subseteq Y \times_k Y$ is closed if and only if $X \times_Y X \subseteq X \times_k X$ is closed. Therefore, Y is separated if and only if the action of $\mathbf{G}$ on X is separated. □

5. Geometric quotients of free actions and principal bundles

In this section we fix a k-scheme X equipped with an action of $\mathbf{G}$ determined by morphism $a : \mathbf{G} \times_k X \to X$.

Definition 5.1. The action of $\mathbf{G}$ on X is *free* if the morphism $\langle a, \mathrm{pr}_X \rangle : \mathbf{G} \times_k X \to X \times_k X$ is a closed immersion.

Definition 5.2. Let x be a k-point of X. We consider x as a morphism $\mathrm{Spec}\, k \to X$. Suppose that *the orbit morphism* $\mathbf{G} \to X$ *of* x given by the composition

$$\mathbf{G} = \mathbf{G} \times_k \mathrm{Spec}\, k \xhookrightarrow{1_{\mathbf{G}} \times_k x} \mathbf{G} \times_k X \longrightarrow X$$

is a closed immersion. Then the action of $\mathbf{G}$ on X has *a closed free orbit at* x.

Fact 5.3. *If the action of* $\mathbf{G}$ *on* X *is free, then every* k*-point of* X *has a closed free orbit.*

The following result states that over special type of local complete noetherian k-algebras geometric quotients of free actions correspond to trivial $\mathbf{G}$-bundles.

Theorem 5.4. *Suppose that* k *is an algebraically closed field and* $\mathbf{G}$ *is a smooth algebraic group over* k*. Let* $q : X \to Y$ *be a geometric quotient and a morphism locally of finite type and let* Y *be the spectrum of a complete local noetherian* k*-algebra such that the residue field of the closed point of* Y *is* k*. Then the following assertions hold.*

(1) *If* x *is a* k*-point of* X *which has a closed free orbit, then there exists a* $\mathbf{G}$*-equivariant, étale and surjective morphism* $f : \mathbf{G} \times_k Y \to X$ *such that the triangle*

$$\begin{tikzcd} \mathbf{G} \times_k Y \arrow[rr, "f"] \arrow[dr, "\mathrm{pr}_Y"'] & & X \arrow[dl, "q"] \\ & Y & \end{tikzcd}$$

is commutative and the morphism

$$Y = \mathrm{Spec}\, k \times_k Y \xhookrightarrow{e \times_k 1_Y} \mathbf{G} \times_k Y \xrightarrow{f} X$$

is a section of q*.*

(2) *If the action of* $\mathbf{G}$ *on* X *is free, then* f *is an isomorphism.*

The proof relies on two algebraic lemmas. The first describe free actions in context of complete rings (it is [Mumford et al., 1994, lemma on page 18]) and the second is a version of Hensel's lemma.

Lemma 5.4.1. *Let $(R, \mathfrak{m}, k)$ be a complete local noetherian k-algebra and let $\sigma : R \to R[[x_1, ..., x_n]]$ be a local morphism into a ring of formal power series over R. Assume that the composition*

$$R \xrightarrow{\sigma} R[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod (x_1, ..., x_n)} R$$

is the identity and the composition

$$R \xrightarrow{\sigma} R[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod \mathfrak{m}} (R/\mathfrak{m})\,[[x_1, ..., x_n]] = k[[x_1, ..., x_n]]$$

is surjective. Consider elements $y_1, ..., y_n$ of R such that $\sigma(y_i) \bmod \mathfrak{m} = x_i$ for $i = 1, ..., n$. Then the composition

$$R \xrightarrow{\sigma} R[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod (y_1, ..., y_n)} (R/(y_1, ..., y_n))\,[[x_1, ..., x_n]]$$

is an isomorphism.

Proof of the lemma. For convienience let ϕ denote the morphism given by the rule $r \mapsto \sigma(r) \bmod (y_1, ..., y_n)$. Also denote $R/(y_1, ..., y_n)$ by S. According to assumptions we have

$$\sigma(y_i) = x_i + y_i + \sum_{j=1}^{n} x_j \cdot \mathfrak{m}[[x_1, ..., x_n]]$$

for each i. Thus $\phi(y_i) = \sum_{j=1}^{n} f_{ij} \cdot x_j$ where $f_{ij} \in S$ are elements such that the matrix $\left[f_{ij}\right]_{1 \le i,j \le n}$ is invertible in S. Hence

$$S[[x_1, ..., x_n]] = S[[\phi(y_1), ..., \phi(y_n)]]$$

and ϕ composed with $S[[\phi(y_1), ..., \phi(y_n)]] \twoheadrightarrow S$ is the quotient morphism $R \twoheadrightarrow S$. From this observations we derive that ϕ is surjective. It remains to prove that it is injective. Consider z in R such that $\phi(z) = 0$. Suppose that $z \in (y_1, ..., y_n)^m$ for some $m \in \mathbb{N}$. Write

$$z = \sum_{\alpha \in \Lambda} c_\alpha \cdot y_1^{\alpha_1} ... y_n^{\alpha_n}$$

for some $c_\alpha \in R$ where $\Lambda = \{\alpha \in \mathbb{N}^n \,|\, \alpha_1 + ... + \alpha_n = m\}$. Hence

$$0 = \phi(z) = \sum_{\alpha \in \Lambda} \phi(c_\alpha) \cdot \phi(y_1)^{\alpha_1} ... \phi(y_n)^{\alpha_n}$$

Thus $\phi(c_\alpha) \in (\phi(y_1), ..., \phi(y_n))$ for every $\alpha \in \Lambda$. Since ϕ composed with $S[[\phi(y_1), ..., \phi(y_n)]] \twoheadrightarrow S$ is the quotient morphism $R \twoheadrightarrow S$, we derive that

$$c_\alpha \bmod (y_1, ..., y_n) = \phi(c_\alpha) \bmod (\phi(y_1), ..., \phi(y_n)) = 0$$

for every $\alpha \in \Lambda$. Thus $c_\alpha \in (y_1, ..., y_n)$ for every $\alpha \in \Lambda$, which implies that $z \in (y_1, ..., y_n)^{m+1}$. Thus we proved that

$$\phi(z) = 0 \text{ and } z \in (y_1, ..., y_n)^m \Rightarrow z \in (y_1, ..., y_n)^{m+1}$$

By $\mathfrak{m}$-adic completeness of R this implies that $\phi(z) = 0$ if and only if $z = 0$. Hence ϕ is also injective. □

Lemma 5.4.2. *Let $(R, \mathfrak{m})$ be a complete local noetherian ring and let $R \to S$ be a local morphism into a local noetherian ring. Suppose that there exists a finitely generated R-submodule N of S such that*

$$S = N + \mathfrak{m}S$$

Then $S = N$.

Proof of the lemma. Pick s in S. Since $S = N + \mathfrak{m}S$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in \mathfrak{m}^n N$ and

$$s - \sum_{i\leq n} x_i \in \mathfrak{m}^{n+1}S$$

According to the assumption that $(R, \mathfrak{m})$ is complete with respect to $\mathfrak{m}$-adic topology and N is finitely generated over R, we deduce that N is complete with respect to $\mathfrak{m}$-adic topology. Hence there exists a unique element x in N such that

$$x = \sum_{n\in\mathbb{N}} x_n$$

where above series is convergent with respect to $\mathfrak{m}$-adic topology. Note also that

$$x - \sum_{i\leq n} x_i \in \mathfrak{m}^{n+1}N$$

for every $n \in \mathbb{N}$. Thus we have

$$s - x = \left(s - \sum_{i\leq n} x_i\right) - \left(x - \sum_{i\leq n} x_i\right) \in \mathfrak{m}^{n+1}S + \mathfrak{m}^{n+1}N = \mathfrak{m}^{n+1}S$$

for every $n \in \mathbb{N}$. Hence

$$x - s \in \bigcap_{n\in\mathbb{N}} \mathfrak{m}^n S$$

Since $R \to S$ is local morphism and S is a local ring, we deduce that $\mathfrak{m}S$ is contained in the maximal ideal of S. By assumptions S is noetherian. Therefore, S is separated with respect to $\mathfrak{m}$-adic topology. This implies that

$$0 = \bigcap_{n\in\mathbb{N}} \mathfrak{m}^n S$$

Thus $s - x = 0$ and we infer that s is an element of N. This completes the proof that $S = N$. □

In what follows we shall denote by $\mathbf{G}x$ the closed subscheme determined by the orbit morphism $\mathbf{G} \to X$ of a k-point x of X which has a closed free orbit. For readers convienience we include the following lemmas, which have topological content.

Lemma 5.4.3. *Let $q : X \to Y$ be a geometric quotient and assume that Y is the spectrum of a local k-algebra such that the residue field of the closed point o of Y is k. Let x be a k-point of X with free closed orbit, then $q^{-1}(o) = \mathbf{G}x$ as closed subschemes of X.*

Proof of the lemma. Morphism q induces the morphism of residue fields $k(q(x)) \hookrightarrow k(x) = k$ over k. This implies that $k(q(x)) = k$ and hence $q(x)$ is a k-point of Y. Note that o is the unique k-point of Y. Thus $q(x) = o$. Clearly $q^{-1}(o)$ is a closed $\mathbf{G}$-stable subscheme of X (it is the preimage of o under $\mathbf{G}$-equivariant q), that contains x. Since $\mathbf{G}x$ is the smallest closed $\mathbf{G}$-stable subscheme of X containing x, we deduce that $\mathbf{G}x \subseteq q^{-1}(o)$ scheme-theoretically. Consider the pair of arrows

$$\mathbf{G} \times_k X \overset{a}{\underset{\mathrm{pr}_X}{\rightrightarrows}} X$$

Passing to functors of points we obtain that $a^{-1}(\mathbf{G}x) = \mathrm{pr}_X(\mathbf{G}.x)$. Since q is the cokernel of the pair (a, pr_X) in the category of topological spaces, we deduce that there exists a closed subset Z of Y such that $q^{-1}(Z) = \mathbf{G}x$. Clearly $o \in Z$ and hence $q^{-1}(o) \subseteq \mathbf{G}x$ set-theoretically. On the other hand above we proved that $\mathbf{G}x \subseteq q^{-1}(o)$ scheme-theoretically. This can only happen if $q^{-1}(o) = \mathbf{G}x$ as closed subschemes of X. □

Lemma 5.4.4. *Let $q : X \to Y$ be a geometric quotient and assume that Y is the spectrum of a local k-algebra such that the residue field of the closed point o of Y is k. Let U be an open $\mathbf{G}$-stable subset of X which contain a k-point. Then $U = X$.*

Proof of the lemma. Consider the pair of arrows

$$\begin{tikzcd} \mathbf{G} \times_k X \arrow[r, shift left, "a"] \arrow[r, shift right, "\mathrm{pr}_X"'] & X \end{tikzcd}$$

Since U is $\mathbf{G}$-stable open subset of X, we derive that $\mathrm{pr}_X^{-1}(U) = a^{-1}(U)$. Next by definition q is the cokernel of the above pair in the category of topological spaces. Hence there exists an open subset V of Y such that $U = q^{-1}(V)$. Since U contains a k-point of X, we deduce as in Lemma 5.4.3 that $o \in V$. Thus $V = Y$ and finally $U = q^{-1}(V) = X$. □

Proof of the theorem. We first prove **(1)**. Denote by o the closed point of Y. Assume that x is a k-point of X which has a closed free orbit. Consider the surjective morphism $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{\mathbf{G},e}$ induced by the orbit morphism $\mathbf{G} \hookrightarrow X$ of x. Since $\mathbf{G}$ is smooth over k, the ring $\mathcal{O}_{\mathbf{G},e}$ is regular. Pick a system of parameters $x_1,...,x_n$ of $\mathcal{O}_{\mathbf{G},e}$ and let $y_1,...,y_n$ be elements of $\mathcal{O}_{X,x}$ such that y_i is send to x_i by the morphism $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{\mathbf{G},e}$ for $1 \le i \le n$. Define S to be the quotient ring $\mathcal{O}_{X,x}/(y_1,...,y_n)$. The morphism q induces the morphism $q^{\#} : \mathcal{O}_{Y,o} \to \mathcal{O}_{X,x}$ and hence the morphism $\mathcal{O}_{Y,o} \to S$. By Lemma 5.4.3 we have

$$S/\mathfrak{m}_o S = k$$

where $\mathfrak{m}_o$ is the maximal ideal of $\mathcal{O}_{Y,o}$. According to Lemma 5.4.2 we derive that $\mathcal{O}_{Y,o} \to S$ is surjective. Let $f : \mathbf{G} \times_k \operatorname{Spec} S \to X$ be the unique $\mathbf{G}$-equivariant morphism induced by the surjection $\mathcal{O}_{X,x} \twoheadrightarrow S$. We have a commutative square

$$\begin{tikzcd} \mathbf{G} \times_k \operatorname{Spec} S \arrow[r, "f"] \arrow[d, "\mathrm{pr}_{\operatorname{Spec} S}"'] & X \arrow[d, "q"] \\ \operatorname{Spec} S \arrow[r, hook, "j"'] & Y \end{tikzcd}$$

where j is a closed immersion induced by $\mathcal{O}_{Y,o} \twoheadrightarrow S$. According to assumptions q is locally of finite type. Moreover, $\mathbf{G}$ is an algebraic group over k and hence $\mathrm{pr}_{\operatorname{Spec} S}$ is locally of finite type. These two assertions together with the fact that $\operatorname{Spec} S \hookrightarrow Y$ is a closed immersion of noetherian schemes (and thus is of finite type) imply that f is locally of finite type. Then by Lemma 5.4.1 we deduce that f induces an isomorphism

$$\widehat{\mathcal{O}_{X,x}} \longrightarrow \hat{S}[[x_1,...,x_n]] = \hat{S}\hat{\otimes}_k \widehat{\mathcal{O}_{\mathbf{G},e}}$$

of complete local rings. Since f is locally of finite type, it follows that f is étale at a k-point of $\mathbf{G} \times_k \operatorname{Spec} S$ determined by the unique k-point of $\operatorname{Spec} S$ and $e \in \mathbf{G}$. Let U be an étale locus of f. It contains a k-point and hence it is nonempty. Moreover, U is open (it is étale locus) subset of X. Since f is $\mathbf{G}$-equivariant, we derive that U is $\mathbf{G}$-stable. Similarly $f(U)$ is open $\mathbf{G}$-stable subset of X and $x \in f(U)$. Thus by Lemma 5.4.4 we deduce that

$$U = \mathbf{G} \times_k \operatorname{Spec} S,\ f(U) = X$$

Therefore, f is étale and surjective. Now we pullback $q : X \to Y$ along the closed immersion $\operatorname{Spec} S \hookrightarrow Y$. We obtain a cartesian square

$$\begin{tikzcd} \tilde{X} \arrow[r, hook, "\tilde{j}"] \arrow[d, "\tilde{q}"'] & X \arrow[d, "q"] \\ \operatorname{Spec} S \arrow[r, hook, "j"'] & Y \end{tikzcd}$$

Then f factors as a morphism $\mathbf{G} \times_k \operatorname{Spec} S \to \tilde{X}$ followed by a closed immersion $\tilde{j}$. Since f is étale and surjective, we deduce that $\tilde{j}$ is étale and surjective. This implies that $\tilde{j}$ is an isomorphism of schemes. By definition of geometric quotient we derive that the diagram

$$\mathcal{O}_Y \xrightarrow{q^\#} q_*\mathcal{O}_X \underset{q_*\mathrm{pr}_X^\#}{\overset{q_*a^\#}{\rightrightarrows}} q_*\left(\mathrm{pr}_X\right)_*\mathcal{O}_{\mathbf{G}\times_k X} = q_*a_*\mathcal{O}_{\mathbf{G}\times_k X}$$

is the kernel in the category of sheaves on Y. Hence $q^\# : \mathcal{O}_Y \to q_*\mathcal{O}_X$ is a monomorphism of sheaves. On the other hand we have

$$q^\# = j_*q_*\left(\tilde{j}^{-1}\right)^\# \cdot j_*\tilde{q}^\# \cdot j^\#$$

and thus $j^\#$ is a monomorphism. Since j is a closed immersion, we infer that j is an isomorphism. Therefore, we can identify $\operatorname{Spec} S$ with Y. Then f is a morphism which makes the triangle

$$\begin{tikzcd} \mathbf{G}\times_k Y \arrow[rr, "f"] \arrow[dr, "\mathrm{pr}_Y"'] & & X \arrow[dl, "q"] \\ & Y & \end{tikzcd}$$

commutative. This completes the proof of **(1)**.
For the proof of **(2)** consider the section $s : Y \hookrightarrow X$ described in **(1)**. Then f fits into a cartesian square

$$\begin{tikzcd} \mathbf{G}\times_k Y \arrow[r, "f"] \arrow[d, "1_{\mathbf{G}}\times_Y s"'] & X\times_Y Y = X \arrow[d, "1_X\times_Y s"] \\ \mathbf{G}\times_k X \arrow[r, hook, "\phi"'] & X\times_Y X \end{tikzcd}$$

where ϕ is a closed immersion induced by the closed immersion $\langle a, \mathrm{pr}_X\rangle : \mathbf{G}\times_k X \hookrightarrow X\times_k X$ (the action of $\mathbf{G}$ on X is free). Thus f is a closed immersion. By **(1)** it is étale and surjective. Therefore, f is an isomorphism. □

Remark 5.5. We expect that Theorem 5.4 holds for prime spectra of strictly henselian rings.

Now we introduce sufficient condition for smoothness of geometric quotient in case of locally algebraic k-schemes.

Corollary 5.6. *Suppose that $\mathbf{G}$ is a smooth algebraic group over k. Let $q : X \to Y$ be a morphism of finite type between k-schemes locally of finite type. Assume that q is a uniform geometric quotient of X and x is a k-point of X with closed free orbit. Then q is smooth at x.*

Proof. Since smoothness descent along faithfully flat morphisms, we may assume that k is algebraically closed. Let $y = q(x)$. Then y is a k-point of Y. Now $1_{\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}}} \times_k q$ is a geometric quotient and $\widehat{\mathcal{O}_{Y,y}}$ is a complete local noetherian k-algebra with k as a residue field. Moreover, x is a k-point of $\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}} \times_k X$ with a closed free orbit. By Theorem 5.4 we deduce that $1_{\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}}} \times_k q$ is smooth. From descent for smoothness we infer that q is smooth at x. □

Definition 5.7. Let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism into a k-scheme Y equipped with the trivial $\mathbf{G}$-action. Suppose that q is faithfully flat, quasi-compact morphism and the square

$$\begin{tikzcd} \mathbf{G} \times_k X \arrow[r, "a"] \arrow[d, "\mathrm{pr}_X"'] & X \arrow[d, "q"] \\ X \arrow[r, "q"'] & Y \end{tikzcd}$$

is cartesian. Then q is *a principal* $\mathbf{G}$*-bundle*.

Now we use Theorem 5.4 to describe principal $\mathbf{G}$-bundles in the category of locally algebraic k-schemes.

Theorem 5.8. *Suppose that* $\mathbf{G}$ *is a smooth algebraic group over* k*. Let* $q : X \to Y$ *be a morphism of finite type between* k*-schemes locally of finite type. Then the following assertions are equivalent.*

- **(i)** *q is a universal geometric quotient and the action of* $\mathbf{G}$ *on* X *is free.*
- **(ii)** *q is a uniform geometric quotient and the action of* $\mathbf{G}$ *on* X *is free.*
- **(iii)** *q is a principal* $\mathbf{G}$*-bundle.*

Proof. Clearly **(i)** ⇒ **(ii)**. Suppose that **(ii)** holds. Let $\overline{k}$ be an algebraic closure of k. Then $1_{\mathrm{Spec}\,\overline{k}} \times_k q$ is a uniform quotient and the action of $\mathrm{Spec}\,\overline{k} \times_k \mathbf{G}$ on $\mathrm{Spec}\,\overline{k} \times_k X$ induced by the action of $\mathbf{G}$ on X is free. Moreover, if $1_{\mathrm{Spec}\,\overline{k}} \times_k q$ is a principal $\mathrm{Spec}\,\overline{k} \times_k \mathbf{G}$-bundle, then q is a $\mathbf{G}$-bundle. This follows from the observation that property of being a principal bundle descents along faithfuly flat and quasi-compact base extensions. Thus we may assume that k is algebraically closed. Next we pick a k-point y of Y and consider base change $1_{\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}}} \times_Y q$. This is a geometric quotient (because morphism $\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}} \to Y$ is flat) and a morphism of finite type. Moreover, the action of $\mathbf{G}$ on $\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}} \times_Y X$ is free. Since $\widehat{\mathcal{O}_{Y,y}}$ is a complete noetherian k-algebra with residue field k, we derive by Theorem 5.4 that $\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}} \times_Y q$ is isomorphic as a $\mathbf{G}$-equivariant morphism with $\mathrm{pr}_{\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}}}$. By faithfuly flat descent for flat morphism we deduce that q is flat at every point in the fiber $q^{-1}\left(\mathrm{Spec}\,\mathcal{O}_{Y,y}\right)$. Since y is an arbitrary k-point, it follows that q is flat at every point of X which specializes to a k-point. Every point of X is a generization of a k-point (X is locally of finite type and k is algebraically closed). Thus q is flat. It is also surjective (as it is a geometric quotient) and quasi-compact (it is of finite type). Therefore, it is faithfully flat and quasi-compact morphism. In order to obtain **(iii)** it remains to prove that the morphism $\Phi : \mathbf{G} \times_k X \to X \times_Y X$ induced by a and pr_X is an isomorphism. Note that it is a closed immersion (the action of $\mathbf{G}$ on X is closed). Moreover, $1_{\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}}} \times_Y \Phi$ is an isomorphism due to the fact that $1_{\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}}} \times_Y q$ is isomorphic as a $\mathbf{G}$-equivariant morphism with $\mathrm{pr}_{\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}}}$. By faithfully flat descent we infer that $1_{\mathrm{Spec}\,\mathcal{O}_{Y,y}} \times_Y \Phi$ is an isomorphism. This holds for every k-point y in Y. Thus Φ induces an isomorphism $\mathcal{O}_{X \times_Y X, \Phi(z)} \to \mathcal{O}_{\mathbf{G} \times_k X, z}$ for every k-point z of $X \times_Y X$. Hence a closed immersion Φ is an isomorphism. This completes the proof of **(ii)** ⇒ **(iii)**.
Assume now that **(iii)** holds. Then the square

$$\begin{tikzcd} \mathbf{G}\times_k X \arrow[r, "a"] \arrow[d, "\mathrm{pr}_X"'] & X \arrow[d, "q"] \\ X \arrow[r, "q"'] & Y \end{tikzcd}$$

is cartesian and q is faithfully flat and quasi-compact. By Proposition 3.11 morphism pr_X is a universal geometric quotient. According to Corollary 3.6 we derive that q is universal geometric quotient. Moreover, the cartesian square above shows that the morphism $\mathbf{G}\times_k X \to X\times_Y X$ induced by a and pr_X is an isomorphism. Thus the action of $\mathbf{G}$ on X is free. This is **(i)**. Hence **(iii)** $\Rightarrow$ **(i)** holds. $\square$

6. Good categorical quotients

In this section we fix a k-scheme X equipped with an action of $\mathbf{G}$ determined by morphism $a : \mathbf{G}\times_k X \to X$. We start by the following criterion for categorical quotients.

Theorem 6.1. *Let $q : X \to Y$ be a morphism into a k-scheme Y equipped with the trivial $\mathbf{G}$-action. Assume that the following assertions hold.*

- **(1)** *q is $\mathbf{G}$-equivariant.*
- **(2)** *$\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q.*
- **(3)** *If Z is a $\mathbf{G}$-stable closed subset of X, then $q(Z)$ is a closed subset of Y.*
- **(4)** *If $\{Z_i\}_{i\in I}$ is a family of closed $\mathbf{G}$-stable subsets with the empty intersection, then the intersection $\{q(Z_i)\}_{i\in I}$ is empty.*

Then q is submersive and it is a categorical quotient of X.

Proof. Clearly $q(X)$ is closed in Y. Hence $V = Y\smallsetminus q(X)$ is open. Moreover, $q^{\#} : \mathcal{O}_Y \to q_*\mathcal{O}_X$ is a monomorphism of sheaves of k-algebras. Thus we have a monomorphism $\mathcal{O}_V \hookrightarrow (q_*\mathcal{O}_X)_{q^{-1}(V)}$. We have $(q_*\mathcal{O}_X)_{q^{-1}(V)} = 0$ and hence $\mathcal{O}_V = 0$. This implies that $V = \varnothing$. Thus q is surjective. Suppose that Z is a subset of Y such that $q^{-1}(Z)$ is a closed subset of X. Then $q^{-1}(Z)$ is a $\mathbf{G}$-stable closed subset and hence $q\left(q^{-1}(Z)\right)$ is closed. Note that $q\left(q^{-1}(Z)\right) = Z$ because q is surjective. Thus Z is closed. This completes the proof that q is submersive.
Now we show that q is a categorical quotient of X. For this pick a $\mathbf{G}$-equivariant morphism $p : X \to Z$ where Z is a k-scheme with the trivial $\mathbf{G}$-action. Consider open affine cover $\{W_i\}_{i\in I}$ of Z. Then $X\smallsetminus p^{-1}(W_i)$ is a closed $\mathbf{G}$-stable closed for $i\in I$. Define $V_i = Y\smallsetminus q\left(X\smallsetminus p^{-1}(W_i)\right)$ for each i. Thus V_i is an open subset of Y for every $i\in I$. Moreover, we have

$$\bigcap_{i\in I} X\smallsetminus p^{-1}(W_i) = \varnothing$$

and hence $\{V_i\}_{i\in I}$ form an open cover of Y. Note that for every $i\in I$ we have $q^{-1}(V_i)\subseteq p^{-1}(V_i)$. Consider the composition

$$\Gamma\left(W_i,\mathcal{O}_Z\right) \xrightarrow{p^{\#}} \Gamma\left(p^{-1}(W_i),\mathcal{O}_X\right) \xrightarrow{f\mapsto f_{|q^{-1}(V_i)}} \Gamma\left(q^{-1}(V_i),\mathcal{O}_X\right)$$

for every i in I. Since the action of $\mathbf{G}$ on Z is trivial, we derive that the image of the morphism above consists of $\mathbf{G}$-invariant functions on $q^{-1}(V_i)$. This means that the morphism above factors uniquely through $q^{\#}_{V_i} : \Gamma\left(V_i,\mathcal{O}_Y\right) \to \Gamma\left(q^{-1}(V_i),\mathcal{O}_X\right)$. Since W_i is affine for every i in I, we obtain a unique morphism $f_i : V_i \to W_i$ such that $f_i\cdot q_{|q^{-1}(V_i)} = p_{|q^{-1}(V_i)}$ for each i. By construction the family $\{f_i\}_{i\in I}$ glue to a morphism $f : Y\to Z$ such that $f\cdot q = p$. This morphism is unique due to the fact that f_i are unique for every i. This finishes the proof of the fact that q is a categorical quotient of X. □

Definition 6.2. Let $q : X \to Y$ be a morphism into a k-scheme Y equipped with the trivial $\mathbf{G}$-action. Suppose that q satisfies conditions **(1)-(4)** of Theorem 6.1. Then q is *a good categorical quotient of* X.

Proposition 6.3. *Let $q : X \to Y$ be a morphism into a k-scheme Y equipped with the trivial $\mathbf{G}$-action. Assume that X is quasi-compact and the following assertions hold.*

(1) *q is $\mathbf{G}$-equivariant.*

(2) *$\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q.*

(3) *If Z is a $\mathbf{G}$-stable closed subset of X, then $q(Z)$ is a closed subset of Y.*

(4) *If Z_1 and Z_2 are closed $\mathbf{G}$-stable subsets with the empty intersection, then $q(Z_1) \cap q(Z_2) = \emptyset$.*

Then q is a good categorical quotient of X.

Proof. Suppose that $\{Z_i\}_{i \in I}$ is a family of closed $\mathbf{G}$-stable subsets with the empty intersection. Since X is quasi-compact, there exists a finite subset $\{i_1, ..., i_n\} \subseteq I$ such that the family $\{Z_{i_1}, ..., Z_{i_n}\}$ has empty intersection. Then

$$\bigcap_{i \in I} q(Z_i) \subseteq \bigcap_{j=1}^{n} q(Z_{i_j}) = \emptyset$$

according to **(4)**. This implies that q is a good categorical quotient. □

As in case of categorical and geometric quotients one can introduce the following notion.

Definition 6.4. A morphism $q : X \to Y$ is *a universal (uniform) good categorical quotient of* X if for every (flat) morphism $g : Y' \to Y$ of k-schemes a base change $q' : X' \to Y'$ of q along g is a good categorical quotient of X'.

Corollary 6.5. *If $q : X \to Y$ is a uniform good categorical quotient, then it is universally submersive.*

Proof. Let $g : Y' \to Y$ be a morphism of k-schemes. Then we can factor g as a closed immersion $Y' \hookrightarrow Z$ followed by a flat morphism $Z \to Y$. Since $q' = 1_Z \times_Y q$ is a good categorical quotient, we derive that it is submersive by Theorem 6.1. Hence the restriction $q'^{-1}(Y') \to Y'$ of q' to a closed subset is also submersive. Therefore, $1_{Y'} \times_Y q$ is submersive. □

Theorem 6.6. *Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Then the following assertions are equivalent.*

(i) *There exists an open cover $\mathcal{V}$ of Y such that for every V in $\mathcal{V}$ the restriction $q^{-1}(V) \to V$ of q is a universal (uniform) good categorical quotient.*

(ii) *q is a universal (uniform) good categorical quotient.*

(iii) *For every affine k-scheme Y' and a (flat) morphism $g : Y' \to Y$ of k-schemes a base change $q' : X' \to Y'$ of q along g is a good categorical quotient.*

(iv) *There exists an open affine cover $\mathcal{V}$ of Y such that for every V in $\mathcal{V}$ the restriction $q^{-1}(V) \to V$ of q is a universal (uniform) good categorical quotient.*

For the proof we need the following.

Lemma 6.6.1. *Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Suppose that there exists an open cover $\mathcal{V}$ of Y such that for every V in $\mathcal{V}$ the restriction $q^{-1}(V) \to V$ of q is a good categorical quotient. Then q is a good categorical quotient.*

Proof of the lemma. Pick a closed $\mathbf{G}$-stable subset Z of X. Then $q(Z)\cap V$ is closed in V for every $V\in\mathcal{V}$. Thus $q(Z)$ is closed in Y. Suppose that $\{Z_i\}_{i\in I}$ are closed $\mathbf{G}$-stable subsets of X with empty intersection. Then

$$V\cap\bigcap_{i\in I}q(Z_i)=\varnothing$$

and hence the intersection of $\{q(z_i)\}_{i\in I}$ is empty. According to Proposition 2.10 we derive that $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariant functions for q. Thus q is a good categorical quotient. $\square$

Proof of the theorem. Implications **(ii)** $\Rightarrow$ **(iii)** and **(iv)** $\Rightarrow$ **(v)** are obvious.
We prove **(i)** $\Rightarrow$ **(ii)**. Suppose that **(i)** holds. Pick a (flat) morphism $g:Y'\to Y$ and fix a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Then $\mathcal{V}'=\{g^{-1}(V)\,|\,V\in\mathcal{V}\}$ is an open cover of Y' such that for every $V\in\mathcal{V}'$ the morphism $q'^{-1}(V)\to V$ is a uniform categorical quotient. By Lemma 6.6.1 we derive that q' is a good categorical quotient. This is **(ii)**.
Assume that **(iii)** holds. Pick an open affine subset V of Y. Consider a (flat) morphism $g:V'\to V$ and pick a cartesian square

$$\begin{tikzcd} U' \arrow[r, "g'"] \arrow[d, "q_{V'}"'] & q^{-1}(V) \arrow[d, "q_V"] \\ V' \arrow[r, "g"'] & V \end{tikzcd}$$

where $q_V:q^{-1}(V)\to V$ is the restriction of q. Then for every open affine subset W of V' the restriction $q_{V'}^{-1}(W)\to W$ of $q_{V'}$ is a universal (uniform) good categorical quotient according to **(iii)** (and the fact that $W\hookrightarrow V'$ composed with g is flat). By Lemma 6.6.1 it follows that $q_{V'}$ is a good categorical quotient. Thus q_V is a universal (uniform) good categorical quotient. This holds for every open affine subset V of Y. This is **(iv)** and hence **(iii)** $\Rightarrow$ **(iv)** holds. $\square$

7. Affine case

In this section we fix a k-scheme X equipped with an action of $\mathbf{G}$ determined by morphism $a:\mathbf{G}\times_k X\to X$. We make the first important step towards existence of quotients by proving that good categorical quotients exists for affine k-schemes equipped with an action of geometrically reductive groups.

Proposition 7.1. *Suppose that $\mathbf{G}$ is geometrically reductive group and X is an affine k-scheme. Let Z_1,Z_2 be nonempty closed $\mathbf{G}$-stable subsets of X such that $Z_1\cap Z_2=\varnothing$. Then there exists $\mathbf{G}$-invariant regular function f on X such that $f_{|Z_1}=1$ and $f_{|Z_2}=0$.*

Proof. By [Monygham, 2020, Corollary 5.4] we may consider Z_1 and Z_2 as a closed $\mathbf{G}$-stable subschemes of X. Since $Z_1\cap Z_2=\varnothing$, we have

$$\Gamma(Z_1\cup Z_2,\mathcal{O}_X)=\Gamma(Z_1,\mathcal{O}_X)\times\Gamma(Z_2,\mathcal{O}_X)$$

In particular, there exists a regular invariant function

$$g\in\Gamma(Z_1,\mathcal{O}_X)^{\mathbf{G}}\times\Gamma(Z_2,\mathcal{O}_X)^{\mathbf{G}}=\Gamma(Z_1\cup Z_2,\mathcal{O}_X)^{\mathbf{G}}$$

such that $g_{|Z_1} = 1$ and $g_{|Z_2} = 0$. Consider the canonical morphism

$$\Gamma(X, \mathcal{O}_X)^{\mathbf{G}} \longrightarrow \Gamma(Z_1 \cup Z_2, \mathcal{O}_X)^{\mathbf{G}} = \Gamma(Z_1, \mathcal{O}_X)^{\mathbf{G}} \times \Gamma(Z_2, \mathcal{O}_X)^{\mathbf{G}}$$

According to [Monygham, 2021, Theorem 2.4] there exists $f \in \Gamma(X, \mathcal{O}_X)$ and a positive integer r such that $f_{|Z_1 \cup Z_2} = g^r$. Then $f_{|Z_1} = 1$ and $f_{|Z_2} = 0$. □

Theorem 7.2. *Suppose that X is an affine k-scheme and $\mathbf{G}$ is a geometrically reductive group. Let $Y = \operatorname{Spec} \Gamma(X, \mathcal{O}_X)^{\mathbf{G}}$ and let $q : X \to Y$ be the canonical morphism. Then q is a uniform good categorical quotient of X. Moreover, the following assertions hold.*

(1) *If X is of finite type over k, then Y is of finite type over k.*

(2) *If $\mathbf{G}$ is linearly reductive, then q is a universal good categorical quotient of X.*

For the proof we need to following results.

Lemma 7.2.1. *Let $\mathbf{G}$ be an algebraic group which acts on $\operatorname{Spec} A$ for some k-algebra A. Fix a flat $A^{\mathbf{G}}$-algebra B. Then the canonical morphism $B \to (A \otimes_{A^{\mathbf{G}}} B)^{\mathbf{G}}$ is an isomorphism of k-algebras.*

Proof of the lemma. For every linear representation V of $\mathbf{G}$ we have a left exact sequence

$$0 \longrightarrow V^{\mathbf{G}} \longrightarrow V \xrightarrow{x \mapsto c(x) - 1 \otimes x} k[\mathbf{G}] \otimes_k V$$

where $c : V \to k[\mathbf{G}] \otimes_k V$ is the coaction. Now we denote by d the coaction on A. Thus we have left exact sequences

$$0 \longrightarrow A^{\mathbf{G}} \otimes_{A^{\mathbf{G}}} B \longrightarrow A \otimes_{A^{\mathbf{G}}} B \xrightarrow{x \otimes 1 \mapsto d(x) \otimes 1 - 1 \otimes x \otimes 1} k[\mathbf{G}] \otimes_k A \otimes_{A^{\mathbf{G}}} B$$

and

$$0 \longrightarrow (A \otimes_{A^{\mathbf{G}}} B)^{\mathbf{G}} \longrightarrow A \otimes_{A^{\mathbf{G}}} B \xrightarrow{x \otimes 1 \mapsto d(x) \otimes 1 - 1 \otimes x \otimes 1} k[\mathbf{G}] \otimes_k A \otimes_{A^{\mathbf{G}}} B$$

Note that $A \otimes_{A^{\mathbf{G}}} B \ni x \otimes 1 \mapsto d(x) \otimes 1 \in k[\mathbf{G}] \otimes_k A \otimes_{A^{\mathbf{G}}} B$ is the coaction induced by c on the base change $A \otimes_{A^{\mathbf{G}}} B$. This implies that there is canonical isomorphism

$$B = A^{\mathbf{G}} \otimes_{A^{\mathbf{G}}} B \cong (A \otimes_{A^{\mathbf{G}}} B)^{\mathbf{G}}$$

□

Lemma 7.2.2. *Let $\mathbf{G}$ be geometrically reductive group which acts on $\operatorname{Spec} A$ for some k-algebra A. If $f_1, ..., f_n \in A^{\mathbf{G}}$ and*

$$f \in \left(\sum_{i=1}^{n} A f_i \right) \cap A^{\mathbf{G}}$$

then there exists positive integer r such that

$$f^r \in \sum_{i=1}^{n} A^{\mathbf{G}} f_i$$

Moreover, if $\mathbf{G}$ is linearly reductive, then r can be chosen to be 1.

Proof of the lemma. Let $d : A \to k[\mathbf{G}] \otimes_k A$ be the coaction of $\mathbf{G}$ on A. The proof goes on induction on n. Write $f = a_1 f_1 + ... + a_n f_n$ for $a_1, ..., a_n \in A$. Consider $\mathfrak{a} = \mathrm{ann}(\mathrm{f}_1) + Af_2 + ... + Af_n$. This is a $\mathbf{G}$-stable ideal in A. We show now that a_1 is $\mathbf{G}$-invariant modulo $\mathfrak{a}$. Indeed, we have

$$(1 \otimes f_1)\left(d(a_1) - 1 \otimes a_1\right) = d(f_1)d(a_1) - 1 \otimes f_1 a_1 = d(f) - 1 \otimes f = 0$$

Hence

$$d(a_1) - 1 \otimes a_1 \in k[\mathbf{G}] \otimes_k \mathrm{ann}(f_1) \subseteq k[\mathbf{G}] \otimes_k \mathfrak{a}$$

and this shows that a_1 is $\mathbf{G}$-invariant modulo $\mathfrak{a}$. Therefore, according to [Monygham, 2021, Theorem 2.4] there exists positive integer r and $a_1' \in A^{\mathbf{G}}$ such that $a_1^r - a_1' \in \mathfrak{a}$. Thus

$$f^r \in f_1^r a_1^r + Af_2 + ... + Af_n = f_1^r a_1' + Af_2 + ... + Af_n$$

Now if $n = 1$, then we have $f^r = f_1^r a_1' \in A^{\mathbf{G}} f_1$ and the assertion holds. On the other hand if $n \geq 2$, then we can apply inductive hypothesis to

$$f^r - f_1^r a_1' \in (Af_2 + ... + Af_n) \cap A^{\mathbf{G}}$$

and obtain that

$$(f^r - f_1^r a_1')^d \in A^{\mathbf{G}} f_2 + ... + A^{\mathbf{G}} f_n$$

for some positive integer d. Then

$$f^{rd} \in A^{\mathbf{G}} f_1 + A^{\mathbf{G}} f_2 + ... + A^{\mathbf{G}} f_n$$

and the assertion holds. □

Lemma 7.2.3. *Let* $\mathbf{G}$ *be geometrically reductive group which acts on* Spec A *for some* k*-algebra* A*. Then the morphism* Spec $A \to$ Spec $A^{\mathbf{G}}$ *is surjective.*

Proof of the lemma. Pick a prime ideal $\mathfrak{p} \in \mathrm{Spec}\, A^{\mathbf{G}}$. Consider $f \in A\mathfrak{p} \cap A^{\mathbf{G}}$. Then there exist $f_1, ..., f_n \in \mathfrak{p}$ such that

$$f \in (Af_1 + ... + Af_n) \cap A^{\mathbf{G}}$$

By Lemma 7.2.2 we have

$$f^r \in A^{\mathbf{G}} f_1 + ... + A^{\mathbf{G}} f_n \subseteq \mathfrak{p}$$

for some positive integer r. Since $\mathfrak{p}$ is a prime ideal, we derive that $f \in \mathfrak{p}$. Thus $A\mathfrak{p} \cap A^{\mathbf{G}} = \mathfrak{p}$. Thus we have an injective morphisms $A^{\mathbf{G}}/\mathfrak{p} \hookrightarrow A/A\mathfrak{p}$ of k-algebras. This implies that the morphism $k(\mathfrak{p}) \to k(\mathfrak{p}) \otimes_{A^{\mathbf{G}}} A$ is also injective, where $k(\mathfrak{p})$ is a residue field of $\mathfrak{p}$ in $A^{\mathbf{G}}$. We infer that the fiber of Spec $A \to$ Spec $A^{\mathbf{G}}$ is nonempty. □

Lemma 7.2.4. *Let* $\mathbf{G}$ *be geometrically reductive group which acts on* Spec A *for some* k*-algebra* A*. Suppose that* $\mathfrak{a}$ *is an ideal in* $A^{\mathbf{G}}$*. Then* $(A/A\mathfrak{a})^{\mathbf{G}}$ *is canonically isomorphic with* $A^{\mathbf{G}}/\mathfrak{a}$*.*

Proof of the lemma. Lemma 7.2.2 shows that $A\mathfrak{a} \cap A^{\mathbf{G}} = \mathfrak{a}$. Since $\mathbf{G}$ is linearly reductive, we have a canonical identification

$$(A/A\mathfrak{a})^{\mathbf{G}} = A^{\mathbf{G}}/(A\mathfrak{a})^{\mathbf{G}} = A^{\mathbf{G}}/A\mathfrak{a} \cap A^{\mathbf{G}} = A^{\mathbf{G}}/\mathfrak{a}$$

□

Proof of the theorem. Since X is quasi-compact, we may verify conditions of Proposition 6.3. First let us denote by A the k-algebra of global regular functions $\Gamma(X, \mathcal{O}_X)$. Suppose that $V \subseteq \mathrm{Spec}\, A^{\mathbf{G}} = Y$ is an open affine subset. Then $B = \Gamma(V, \mathcal{O}_Y)$ is a flat $A^{\mathbf{G}}$-algebra and by Lemma 7.2.1 we have canonical isomorphism

$$B \cong (A \otimes_{A^{\mathbf{G}}} B)^{\mathbf{G}}$$

This implies that $\Gamma(V, \mathcal{O}_Y) \cong \Gamma(q^{-1}(V), \mathcal{O}_X)^{\mathbf{G}}$ and hence $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q. Fix now a closed $\mathbf{G}$-stable subset Z of X. By [Monygham, 2020, Corollary 5.4] there exists a $\mathbf{G}$-stable ideal $\mathfrak{a} \subseteq A$ such that its vanishing set is equal to Z. Consider a commutative square

$$\begin{tikzcd} \operatorname{Spec} A/\mathfrak{a} \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec}\,(A/\mathfrak{a})^{\mathbf{G}} \arrow[r] & \operatorname{Spec} A^{\mathbf{G}} \end{tikzcd}$$

with canonically defined arrows. Note that $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec}\,(A/\mathfrak{a})^{\mathbf{G}}$ is surjective (Lemma 7.2.3) and according to [Monygham, 2021, Theorem 2.4] morphism

$$\operatorname{Spec}\,(A/\mathfrak{a})^{\mathbf{G}} \to \operatorname{Spec} A^{\mathbf{G}}/\mathfrak{a} \cap A^{\mathbf{G}}$$

is surjective. Thus the set-theoretic image of $\operatorname{Spec} A/\mathfrak{a}$ under the map $\operatorname{Spec} A \to \operatorname{Spec} A^{\mathbf{G}}$ is a closed subset given by $\operatorname{Spec} A^{\mathbf{G}}/\mathfrak{a} \cap A^{\mathbf{G}}$. Hence $q(Z)$ is a closed subset of Y.
Fix now two closed $\mathbf{G}$-stable subsets Z_1, Z_2 and assume that $Z_1 \cap Z_2 = \emptyset$. We claim that $q(Z_1) \cap q(Z_2) = \emptyset$. For this we may assume that Z_1, Z_2 are both nonempty. Proposition 7.1 implies that there exists $f \in \Gamma(X, \mathcal{O}_X)^{\mathbf{G}}$ such that $f_{|Z_1} = 1$ and $f_{|Z_2} = 0$. Then f viewed as a function on Y satisfies $f_{|q(Z_1)} = 1$ and $f_{|q(Z_2)} = 0$. Thus $q(Z_1) \cap q(Z_2) = \emptyset$.
This completes the proof that q is a good categorical quotient. Lemma 7.2.1 and Theorem 6.6 imply that q is a uniform good categorical quotient.
If X is of finite type over k, then by [Monygham, 2021, Theorem 3.1] we deduce that Y which is the prime spectrum of $\Gamma\,(X, \mathcal{O}_X)^{\mathbf{G}}$ is of finite type over k.
If $\mathbf{G}$ is linearly reductive, then by Lemmas 7.2.1, 7.2.4 and Theorem 6.6 we deduce that q is a universal good categorical quotient. □

8. Quotients determined by linearization

We start by discussing some preliminary result concerning $\mathbf{G}$-linearizations of quasi-coherent sheaves. We assume in this section that $\mathbf{G}$ is an affine group scheme over k.

Proposition 8.1. *Let $\mathcal{F}$ be a quasi-coherent sheaf on X and let $\tau : a^*\mathcal{F} \to \mathrm{pr}_X^*\mathcal{F}$ be a $\mathbf{G}$-linearization of $\mathcal{F}$. Suppose that X is quasi-compact and semiseparated. Then the morphism*

$$\Gamma(X, \mathcal{F}) \ni s \mapsto \tau(a^* s) \in \Gamma(\mathbf{G} \times_k X, \mathrm{pr}_X^*\mathcal{F}) = k[\mathbf{G}] \otimes_k \Gamma(X, \mathcal{F})$$

is a coaction of $\mathbf{G}$ on $\mathcal{F}$.

Proof. We denote the morphism in the statement by c. Fix $s \in \Gamma(X, \mathcal{F})$. Write

$$c(s) = \sum_{i=1}^{n} a_i \otimes s_i \in k[\mathbf{G}] \otimes_k \Gamma(X, \mathcal{F})$$

Then

$$\left(1_{k[\mathbf{G}]} \otimes_k c\right)(c(s)) = \sum_{i=1}^{n} a_i \otimes c(s_i) = \sum_{i=1}^{n} a_i \otimes \tau(a^* s_i) = \left(\mathrm{pr}_{23}^*\tau\right)\left(\sum_{i=1}^{n} a_i \otimes a^* s_i\right) =$$

$$= \left(\mathrm{pr}_{23}^*\tau\right)\left(\sum_{i=1}^{n} (1_{\mathbf{G}} \times_k a)^* (a_i \otimes s_i)\right) = \left(\mathrm{pr}_{23}^*\tau \cdot (1_{\mathbf{G}} \times_k a)^* \tau\right)\left((1_{\mathbf{G}} \times_k a)^* a^* s\right) =$$

$$= \left((\mu \times_k 1_X)^* \tau\right)\left((\mu \times_k 1_X)^* a^* s\right) = (\mu \times_k 1_X)^* (\tau(a^* s)) = \left(\Delta \otimes_k 1_{\Gamma(X,\mathcal{F})}\right)(c(s))$$

where $\Delta : k[\mathbf{G}] \to k[\mathbf{G}] \otimes_k k[\mathbf{G}]$ is the comultiplication. Moreover, we also have

$$\left(\xi \otimes_k 1_{\Gamma(X,\mathcal{F})}\right)(c(s)) = (e \times_k 1_X)^* (\tau(a^* s)) = \left((e \times_k 1_X)^* \tau\right)\left((e \times_k 1_X)^* a^* s\right) =$$

$$= \left((e \times_k 1_X)^* a^* s\right) = 1 \otimes s$$

where $\xi : k[\mathbf{G}] \to k$ is the counit. These imply that c is the coaction of $k[\mathbf{G}]$ on the space of global sections of $\mathcal{F}$. □

Definition 8.2. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and let $\tau : a^*\mathcal{F} \to \mathrm{pr}_X^*\mathcal{F}$ be a $\mathbf{G}$-linearization of $\mathcal{F}$. Suppose that X is quasi-compact and semiseparated. Then Proposition 8.1 shows that $\Gamma(X, \mathcal{F})$ is a linear representation of $\mathbf{G}$. We call it *the linear representation induced by* $\mathbf{G}$*-linearization* τ.

Now we study properties of the linear representation induced by $\mathbf{G}$-linearization in case of a line bundle.

Proposition 8.3. *Suppose that X is quasi-compact and semiseparated. Let $\mathcal{L}$ be a line bundle on X and let $\tau : a^*\mathcal{L} \to \mathrm{pr}_X^*\mathcal{L}$ be a $\mathbf{G}$-linearization of $\mathcal{L}$. Then the following assertions hold.*

(1) *If $s \in \Gamma(X, \mathcal{L})$ is $\mathbf{G}$-invariant with respect to the structure of linear representation of $\mathbf{G}$ induced by τ, then the open subscheme*
$$X_s = \left\{x \in X \,\middle|\, s(x) \neq 0\right\}$$
of X is $\mathbf{G}$-stable.

(2) *If $t, s \in \Gamma(X, \mathcal{L})$ are $\mathbf{G}$-invariant with respect to the structure of linear representation of $\mathbf{G}$ induced by τ, then the regular function $\frac{t}{s} \in \Gamma(X_s, \mathcal{O}_X)$ is $\mathbf{G}$-invariant.*

Proof. Suppose that $s \in \Gamma(X, \mathcal{L})$ is $\mathbf{G}$-invariant with respect to the structure of linear representation of $\mathbf{G}$ induced by τ. Then $\tau(a^*s) = \mathrm{pr}_X^*s$. Since τ is an isomorphism of line bundles on $\mathbf{G} \times_k X$, nonvanishing sets of $a^*s \in \Gamma(\mathbf{G} \times_k X, a^*\mathcal{L})$ and $\mathrm{pr}_X^*s \in \Gamma\left(\mathbf{G} \times_k X, \mathrm{pr}_X^*\mathcal{L}\right)$ coincide. Next the nonvanishing set of a^*s is $a^{-1}(X_s)$. On the other hand the nonvanishing set of pr_X^*s is $\mathrm{pr}_X^{-1}(X_s)$. Therefore, $a^{-1}(X_s) = \mathrm{pr}_X^{-1}(X_s)$ and hence X_s is open $\mathbf{G}$-stable subscheme of X. This completes the proof of **(1)**.
Suppose that $t, s \in \Gamma(X, \mathcal{L})$ are $\mathbf{G}$-invariant. Clearly $(\mathcal{O}_X)_{|X_s} \to \mathcal{L}_{|X_s}$ given by multiplication by s is an isomorphism. Recall that $\frac{t}{s}$ is a unique element $r \in \Gamma(X_s, \mathcal{O}_X)$ such that $r \cdot s_{|X_s} = t_{|X_s}$. Since X_s is $\mathbf{G}$-invariant, r is $\mathbf{G}$-invariant if
$$a^*r = \mathrm{pr}_X^*r$$
Since s is $\mathbf{G}$-invariant, we have a commutative triangle

$$\begin{tikzcd}
a^*\mathcal{L}_{|\mathbf{G}\times_k X_s} \arrow[rr, "\tau_{|\mathbf{G}\times_k X_s}"] & & \mathrm{pr}_X^*\mathcal{L}_{|\mathbf{G}\times_k X_s} \\
& \mathcal{O}_{\mathbf{G}\times_k X_s} \arrow[ul, "a^*s_{|\mathbf{G}\times_k X_s}\cdot(-)"] \arrow[ur, "\mathrm{pr}_X^*s_{|\mathbf{G}\times_k X_s}\cdot(-)"'] &
\end{tikzcd}$$

in which all morphisms are isomorphisms. By $\mathbf{G}$-invariance of s and t we have
$$\mathrm{pr}_X^*s_{|\mathbf{G}\times_k X_s} \cdot a^*r = \tau\left(a^*s_{|\mathbf{G}\times_k X_s}\right) \cdot a^*r = \tau\left(a^*s_{|\mathbf{G}\times_k X_s} \cdot a^*r\right) =$$
$$= \tau(a^*t_{|\mathbf{G}\times_k X_s}) = \mathrm{pr}_X^*t_{|\mathbf{G}\times_k X_s} = \mathrm{pr}_X^*s_{|\mathbf{G}\times_k X_s} \cdot \mathrm{pr}_X^*r$$
and hence $a^*r = \mathrm{pr}_X^*r$. This finishes the proof of **(2)**. □

The following notion introduced by Mumford in [Mumford et al., 1994] is fundamental.

Definition 8.4. Let $\mathcal{L}$ be a line bundle on X and let $\tau : a^*\mathcal{L} \to \mathrm{pr}_X^*\mathcal{L}$ be a $\mathbf{G}$-linearization of $\mathcal{L}$. Consider a point x in X. Then we say that:

(1) *x is semistable with respect to τ* if there exists a $\mathbf{G}$-invariant section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ with respect to $\tau^{\otimes n}$ for some n such that X_s is affine and contains x.

(2) *x is stable with respect to τ* if there exists a $\mathbf{G}$-invariant section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ with respect to $\tau^{\otimes n}$ for some n such that X_s is affine, contains x and the action of $\mathbf{G}$ on X_s has closed orbits for geometric points.

We also denote by

$$X^{ss}(\tau), X^{s}(\tau)$$

sets of semistable and stable points of X with respect to τ, respectively.

Theorem 8.5. *Suppose that* $\mathbf{G}$ *is geometrically reductive and* X *is of finite type over* k. *Let* $\mathcal{L}$ *be a line bundle on* X *which admits a* $\mathbf{G}$*-linearization* $\tau : a^*\mathcal{L} \to \mathrm{pr}_X^*\mathcal{L}$. *Then there exists a uniform good categorical quotient* $q : X^{ss}(\tau) \to Y$ *of* $X^{ss}(\tau)$ *by* $\mathbf{G}$. *Moreover, the following assertions hold.*

- **(1)** *q is affine and universally submersive.*
- **(2)** *There exists an ample line bundle $\mathcal{M}$ on Y such that $q^*\mathcal{M} = \mathcal{L}^{\otimes n}$ for some n.*
- **(3)** *There exists an open subscheme $\tilde{Y}$ of Y such that $q^{-1}(\tilde{Y}) = X^s(\tau)$ and the morphism $X^s(\tau) \to \tilde{Y}$ induced by q is a uniform geometric quotient of $X^s(\tau)$ by $\mathbf{G}$.*

References

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