BIAŁYNICKI-BIRULA FUNCTORS

1. Introduction

In this notes we study Białynicki-Birula functors. In the first section we prove some results concerning the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$, where $\mathbf{M}$ is an affine monoid k-scheme and $\mathbf{G}$ is its group of units (we assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$). These results will be used in the following sections.
We assume that k is a field. In these notes we use the following notational convention.

Remark 1.1. Since the Yoneda embedding $\mathbf{Sch}_k \hookrightarrow \widehat{\mathbf{Sch}_k}$ is full and faithful, we identify $\mathbf{Sch}_k$ with the subcategory of $\widehat{\mathbf{Sch}_k}$ consisting of representable presheaves on $\mathbf{Sch}_k$. In particular, if X is a k-scheme, then we denote by the same symbol the presheaf representable by X.

2. Tannakian formalism for quotient stacks

In this section we discuss an application of the main result of [Hall and Rydh, 2019]. For this we need to briefly discuss *algebraic stacks*, although for our purposes there is no need to use any technical details of this language. We refer the interested reader to the excellent exposition [Olsson, 2016] of this subject. We note the following facts.

(1) *An algebraic stack* is a category fibered over $\mathbf{Sch}_k$ satisfying certain extra conditions described in [Olsson, 2016, Definition 4.6.1] and [Olsson, 2016, Definition 8.1.4]. By [Olsson, 2016, Definition 8.2.1, Example 8.2.3] there are well defined notions of *locally noetherian, noetherian and excellent algebraic stacks*.

(2) *A morphism of algebraic stacks* is a morphism of fibered categories over $\mathbf{Sch}_k$. If $\mathcal{X}$ and $\mathcal{Y}$ are algebraic stack, then we denote by $\mathrm{Mor}(\mathcal{X}, \mathcal{Y})$ the corresponding category of morphisms.

(3) For every locally noetherian algebraic stack $\mathcal{X}$ there exists an abelian monoidal category $\mathfrak{Coh}(\mathcal{X})$ of coherent sheaves on $\mathcal{X}$ ([Olsson, 2016, Definition 9.1.14]). If $\mathcal{X}$ and $\mathcal{Y}$ are locally noetherian algebraic stacks, then we denote by $\mathrm{Hom}_{r,\otimes,\cong}(\mathfrak{Coh}(\mathcal{X}), \mathfrak{Coh}(\mathcal{Y}))$ the category of right exact, monoidal functors $\mathfrak{Coh}(\mathcal{X}) \to \mathfrak{Coh}(\mathcal{Y})$ with natural isomorphism as morphisms.

(4) If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of locally noetherian algebraic stacks, then f induces the functor $f^* : \mathfrak{Coh}(\mathcal{Y}) \to \mathfrak{Coh}(\mathcal{X})$ such that $f^* \in \mathrm{Hom}_{r,\otimes,\cong}(\mathfrak{Coh}(\mathcal{X}), \mathfrak{Coh}(\mathcal{Y}))$.

(5) Let $\mathbf{G}$ be a locally algebraic group k-scheme and let X be a k-scheme equipped with an action of $\mathbf{G}$. We consider $\mathbf{Sch}_k$ as a Grothendieck site with respect to *fppf topology* ([Olsson, 2016, Example 2.1.14]). Next the quotient fibered category $[X/\mathbf{G}]$ ([Monygham, 2020b, Definition 9.5]) with respect to this topology is an algebraic stack by [Olsson, 2016, Example 8.1.12].

(6) In **(5)** if k-scheme X is locally noetherian (noetherian, excellent), then $[X/\mathbf{G}]$ is a locally noetherian (noetherian, excellent) by [Olsson, 2016, Definition 8.2.1, Example 8.2.3] and [Olsson, 2016, Example 8.1.12].

(7) In **(5)** if k-scheme X is locally noetherian, then there exists an equivalence of monoidal categories $\mathfrak{Coh}([X/\mathbf{G}]) \cong \mathfrak{Coh}_{\mathbf{G}}(X)$ ([Olsson, 2016, Exercise 9.H]). Moreover, this equivalence is functorial with respect to $\mathbf{G}$-equivariant morphism. That is if Y is another locally noetherian k-scheme with action of $\mathbf{G}$ and $f : X \to Y$ is a $\mathbf{G}$-equivariant morphism, then f induces a morphism $[f/\mathbf{G}] : [X/\mathbf{G}] \to [Y\mathbf{G}]$ by [Monygham, 2020b, Theorem 9.7] and the square

$$\begin{tikzcd}
\mathfrak{Coh}([Y/\mathbf{G}]) \arrow[r, "{[f/\mathbf{G}]^*}"] \arrow[d, "\cong"] & \mathfrak{Coh}([X/\mathbf{G}]) \arrow[d, "\cong"] \\
\mathfrak{Coh}_{\mathbf{G}}(Y) \arrow[r, "f^*"'] & \mathfrak{Coh}_{\mathbf{G}}(X)
\end{tikzcd}$$

of categories and functors is commutative.

(8) If $\mathbf{G}$ is smooth and affine over k, then $[X/\mathbf{G}]$ has *affine stabilizers*.

Remark 2.1. Let $\mathrm{Spec}\, k$ be a point equipped with the trivial action of a smooth and affine group $\mathbf{G}$. Then **(7)** together with [Monygham, 2020a, Example 4.7] impy that $\mathfrak{Coh}([\mathrm{Spec}\, k/\mathbf{G}])$ can be identified with the category $\mathbf{Repf}_{\mathbf{G}}$ of finite dimensional representations of $\mathbf{G}$.

Let us state the main result of [Hall and Rydh, 2019].

Theorem 2.2 ([Hall and Rydh, 2019, Theorem 1.1])**.** *Let $\mathcal{X}$ be a noetherian algebraic stack with affine stabilizers. For every locally excellent algebraic stack $\mathcal{T}$ the functor*

$$\mathrm{Mor}\,(\mathcal{X},\mathcal{T}) \xrightarrow{f\mapsto f^*} \mathrm{Hom}_{r,\otimes,\cong}\,(\mathfrak{Coh}(\mathcal{T}),\mathfrak{Coh}(\mathcal{X}))$$

is an equivalence of categories.

Keeping our previous remarks in mind we deduce the following result.

Corollary 2.3. *Let $\mathbf{G}$ be an smooth affine group k-scheme and let X, Z be k-schemes equipped with an action of $\mathbf{G}$. Suppose that Z is noetherian and X is locally of finite type over k. Then*

$$\mathrm{Mor}\,([Z/\mathbf{G}],[X/\mathbf{G}]) \xrightarrow{f\mapsto f^*} \mathrm{Hom}_{r,\otimes,\cong}\Big(\mathfrak{Coh}\big([X/\mathbf{G}]\big),\mathfrak{Coh}\big([Z/\mathbf{G}]\big)\Big)$$

is an equivalence of categories.

Proof. Note that $[Z/\mathbf{G}]$ is a noetherian algebraic stack according to **(5)** and **(6)**. It has affine stabilizers according to **(8)**. Similary by **(5)** $[X/\mathbf{G}]$ is an algebraic stack. Moreover, it is locally excellent according to the fact that X is locally excellent (it is locally of finite type over k and k is a field) and **(6)**. Then by Theorem 2.2 we derive that the functor in the statement is an equivalence of categories. $\square$

Corollary 2.4. *Let $\mathbf{G}$ be an smooth affine group k-scheme and let X, Z be k-schemes equipped with an action of $\mathbf{G}$. Suppose that Z is noetherian and X is locally of finite type over k. Then we have a bijection*

$$\big\{f : Z \to X \,\big|\, f \text{ is } \mathbf{G}\text{-equivariant}\big\} \xrightarrow{f\mapsto f^*} \big\{F \in \mathrm{Hom}_{r,\otimes,\cong}\,(\mathfrak{Coh}_{\mathbf{G}}(X),\mathfrak{Coh}_{\mathbf{G}}(Z)) \,\big|\, F\cdot p_X^* = p_Z^*\big\}$$

where $p_X^* : \mathbf{Repf}(\mathbf{G}) \to \mathfrak{Coh}_{\mathbf{G}}(X)$ and $p_Z^* : \mathbf{Repf}(\mathbf{G}) \to \mathfrak{Coh}_{\mathbf{G}}(Z)$ are functors induced by $\mathbf{G}$-equivariant morphisms $p_X : X \to \mathrm{Spec}\, k$ and $p_Z : Z \to \mathrm{Spec}\, k$, respectively.

Proof. Since fppf topology is subcanonical, [Monygham, 2020b, Theorem 9.7] shows that there exists a bijection

$$\big\{f : Z \to X \,\big|\, f \text{ is } \mathbf{G}\text{-equivariant}\big\} \xrightarrow{f\mapsto [f/\mathbf{G}]} \big\{h : [Z/\mathbf{G}] \to [X/\mathbf{G}] \,\big|\, [p_X/\mathbf{G}]\cdot h = [p_Y/\mathbf{G}]\big\}$$

Corollary 2.3 implies that there exists a bijection

$$\left\{h : [Z/\mathbf{G}] \to [X/\mathbf{G}] \,\middle|\, [p_X/\mathbf{G}] \cdot h = [p_Y/\mathbf{G}]\right\} \xrightarrow{h \mapsto h^*} \left\{F \in \mathrm{Hom}_{r,\otimes,\cong}(\mathfrak{Coh}([X/\mathbf{G}]), \mathfrak{Coh}([Z/\mathbf{G}])) \,\middle|\, F \cdot [p_X/\mathbf{G}]^* = [p_Z, \mathbf{G}]^*\right\}$$

Next **(7)** implies that there exists a bijection

$$\left\{F \in \mathrm{Hom}_{r,\otimes,\cong}(\mathfrak{Coh}([X/\mathbf{G}]), \mathfrak{Coh}([Z/\mathbf{G}])) \,\middle|\, F \cdot [p_X/\mathbf{G}]^* = [p_Z, \mathbf{G}]^*\right\} \longrightarrow \left\{F \in \mathrm{Hom}_{r,\otimes,\cong}(\mathfrak{Coh}_{\mathbf{G}}(X), \mathfrak{Coh}_{\mathbf{G}}(Z)) \,\middle|\, F \cdot p_X^* = p_Z^*\right\}$$

and for a $\mathbf{G}$-equivariant morphism $f : Z \to X$ the image of $[f/\mathbf{G}]^* : \mathfrak{Coh}([X/\mathbf{G}]) \to \mathfrak{Coh}([Z/\mathbf{G}])$ under this bijection is $f^* : \mathfrak{Coh}_{\mathbf{G}}(X) \to \mathfrak{Coh}_{\mathbf{G}}(Z)$. These imply that the map of classes

$$\left\{f : Z \to X \,\middle|\, f \text{ is } \mathbf{G}\text{-equivariant}\right\} \xrightarrow{f \mapsto f^*} \left\{F \in \mathrm{Hom}_{r,\otimes,\cong}(\mathfrak{Coh}_{\mathbf{G}}(X), \mathfrak{Coh}_{\mathbf{G}}(Z)) \,\middle|\, F \cdot p_X^* = p_Z^*\right\}$$

is a bijection. □

Note that Corollary 2.4 relies on some asumptions regarding $\mathbf{G}$, X and Z. It is worth noting that Joachim Jelisiejew and the author were able to obtain a slightly more general (yet unpublished) result.

Theorem 2.5 ([Jelisiejew and Sienkiewicz, 2020, Theorem A.2])**.** *Let $\mathbf{G}$ be an affine algebraic group over k. Let Z, X be k-schemes equipped with an action of $\mathbf{G}$ and assume that X is quasi-compact and quasi-separated. Suppose that $F : \mathfrak{Qcoh}_{\mathbf{G}}(X) \to \mathfrak{Qcoh}_{\mathbf{G}}(Z)$ is a cocontinuous, monoidal functor such that $F \cdot p_X^* = p_Z^*$. Then there exists a unique $\mathbf{G}$-equivariant morphism $f : Z \to X$ such that $f^* = F$.*

3. Relations between representations of a monoid and its group of units

In this section we study the relation between the category $\mathbf{Rep}(\mathbf{M})$ of representations of an affine monoid k-scheme $\mathbf{M}$ and the category $\mathbf{Rep}(\mathbf{G})$ of representations of its group of units $\mathbf{G}$. Let $i : k[\mathbf{M}] \to k[\mathbf{G}]$ be the morphism of k-bialgebras induced by $\mathbf{G} \hookrightarrow \mathbf{M}$. Let us first note the following elementary result.

Fact 3.1. *Assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. Then i is an injective morphism of k-algebras.*

Proof. This follows from [Görtz and Wedhorn, 2010, Proposition 9.19]. □

Fact 3.2. *The forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$ creates colimits and finite limits.*

Proof. This follows from [Monygham, 2020d, Theorem 14.3, Theorem 14.4] and the commutative triangle

$$\begin{tikzcd} \mathbf{Rep}(\mathbf{M}) \arrow[rr] \arrow[dr] & & \mathbf{Rep}(\mathbf{G}) \arrow[dl] \\ & \mathbf{Vect}_k & \end{tikzcd}$$

of functors. □

The theorem below characterizes representations of $\mathbf{G}$ which are contained in the image of the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$.

Theorem 3.3. *Assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. Let V be a $\mathbf{G}$-representation. Then the following are equivalent.*

- **(i)** *V is in the image of the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$.*
- **(ii)** *The coaction $d : V \to k[\mathbf{G}] \otimes_k V$ factors through $i \otimes_k 1_V : k[\mathbf{M}] \otimes_k V \hookrightarrow k[\mathbf{G}] \otimes_k V$.*

Proof. In the proof we denote by $\Delta_{\mathbf{M}}$ and $\Delta_{\mathbf{G}}$ comultiplications of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. We also denote by $\xi_{\mathbf{M}}$ and $\xi_{\mathbf{G}}$ counits of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. According to Fact 3.1 i is an injective morphism of k-algebras.
Clearly **(i)** $\Rightarrow$ **(ii)**. We prove the converse. Suppose that **(ii)** holds. Let $c : V \to k[\mathbf{M}] \otimes_k V$ be a unique morphism such that $d = (i \otimes_k 1_V) \cdot c$. It suffices to prove that c is the coaction of the bialgebra $k[\mathbf{M}]$ on V. Observe that

$$(i \otimes_k i \otimes_k 1_V) \cdot \left(1_{k[\mathbf{M}]} \otimes_k c\right) \cdot c = (i \otimes_k d) \cdot c = \left(1_{k[\mathbf{G}]} \otimes_k d\right) \cdot d = (\Delta_{\mathbf{G}} \otimes_k 1_V) \cdot d =$$

$$= (\Delta_{\mathbf{G}} \otimes_k 1_V) \cdot \big((i \otimes_k 1_V) \cdot c \big) = ((\Delta_{\mathbf{G}} \cdot i) \otimes_k 1_V) \cdot c = (i \otimes_k i \otimes_k 1_V) \cdot (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot c$$

Since $i \otimes_k i \otimes_k 1_V$ is a monomorphism, we deduce that $\left(1_{k[\mathbf{M}]} \otimes_k c\right) \cdot c = (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot c$. Moreover, we have

$$(\xi_{\mathbf{G}} \otimes_k 1_V) \cdot d = (\xi_{\mathbf{G}} \otimes_k 1_V) \cdot ((i \otimes_k 1_V) \cdot c) = (\xi_{\mathbf{M}} \otimes_k 1_V) \cdot c$$

and hence $(\xi_{\mathbf{M}} \otimes_k 1_V) \cdot c$ is the canonical isomorphism $V \cong k \otimes_k V$. Thus c is the coaction of $k[\mathbf{M}]$ and $d = (i \otimes_k 1_V) \cdot c$. Therefore, V is in the image of $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$. $\square$

Theorem 3.4. *Assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. Then $\mathbf{Rep}(\mathbf{M})$ is a full subcategory of $\mathbf{Rep}(\mathbf{G})$ closed under subobjects and quotients.*

Proof. In the proof we denote by $\Delta_{\mathbf{M}}$ and $\Delta_{\mathbf{G}}$ comultiplications of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. We also denote by $\xi_{\mathbf{M}}$ and $\xi_{\mathbf{G}}$ counits of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. According to Fact 3.1 i is an injective morphism of k-algebras.
We first prove that $\mathbf{Rep}(\mathbf{M})$ is a full subcategory of $\mathbf{Rep}(\mathbf{G})$. For this consider $\mathbf{M}$-representations V, W and a their morphism $f : V \to W$ as $\mathbf{G}$-representations. Let c_V and c_W be coactions of $k[\mathbf{M}]$ on V and W, respectively. Our goal is to prove that f is a morphism of $\mathbf{M}$-representations. Consider the diagram

$$\begin{tikzcd}
k[\mathbf{G}] \otimes_k V \arrow[r, "1_{k[\mathbf{G}]} \otimes_k f"] & k[\mathbf{G}] \otimes_k W \\
k[\mathbf{M}] \otimes_k V \arrow[u, "i \otimes_k 1_V"] \arrow[r, dotted, "1_{k[\mathbf{M}]} \otimes_k f"] & k[\mathbf{M}] \otimes_k W \arrow[u, "i \otimes_k 1_W"'] \\
V \arrow[u, "c_V"] \arrow[r, "f"'] & W \arrow[u, "c_W"']
\end{tikzcd}$$

in which the outer square is commutative. Our goal is to prove that the bottom square is commutative. We have

$$(i \otimes_k 1_W) \cdot c_W \cdot f = \left(1_{k[\mathbf{G}]} \otimes_k f\right) \cdot (i \otimes_k 1_V) \cdot c_V = (i \otimes_k 1_W) \cdot \left(1_{k[\mathbf{M}]} \otimes_k f\right) \cdot c_V$$

Since $i \otimes_k 1_W$ is a monomorphism, we deduce that $c_W \cdot f = \left(1_{k[\mathbf{M}]} \otimes_k f\right) \cdot c_V$. Hence f is a morphism of $\mathbf{M}$-representations.

Next we prove that $\mathbf{Rep}(\mathbf{M})$ is a subcategory of $\mathbf{Rep}(\mathbf{G})$ that is closed under subquotients. Consider an $\mathbf{M}$-representation V and its quotient $\mathbf{G}$-representations $q : V \twoheadrightarrow W$. We show that W is a quotient $\mathbf{M}$-representation of V. Let c_V be the coaction of $\mathbf{M}$ on V and let d_W be the coaction of $\mathbf{G}$ on W. We have a commutative diagram

$$\begin{tikzcd} k[\mathbf{G}] \otimes_k V \arrow[r, two heads, "1_{k[\mathbf{G}]} \otimes_k q"] & k[\mathbf{G}] \otimes_k W \\ k[\mathbf{M}] \otimes_k V \arrow[u, "i \otimes_k 1_V"] & \\ V \arrow[u, "c_V"] \arrow[r, two heads, "q"'] & W \arrow[uu, "d_W"'] \end{tikzcd}$$

and hence $d_W(W) \subseteq k[\mathbf{M}] \otimes_k W$. Thus Theorem 3.3 implies that W is a representation of $\mathbf{M}$ and q is a morphism of $\mathbf{M}$-representations. This shows that $\mathbf{Rep}(\mathbf{M})$ is a subcategory of $\mathbf{Rep}(\mathbf{G})$ closed under quotients. Next let $j : U \hookrightarrow V$ be a $\mathbf{G}$-subrepresentation of a $\mathbf{M}$-representation V. By what we proved above the cokernel $q : V \twoheadrightarrow W$ of j in $\mathbf{Rep}(\mathbf{G})$ is contained in $\mathbf{Rep}(\mathbf{M})$. Since both $\mathbf{Rep}(\mathbf{M})$ and $\mathbf{Rep}(\mathbf{G})$ are abelian and the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$ is exact, we derive that the kernel of q in $\mathbf{Rep}(\mathbf{M})$ coincides with its kernel in $\mathbf{Rep}(\mathbf{G})$. Thus U is a $\mathbf{M}$-representation and $j : U \hookrightarrow V$ is a morphism of $\mathbf{M}$-representations. Hence $\mathbf{Rep}(\mathbf{M})$ is the category of $\mathbf{Rep}(\mathbf{G})$ closed under subobjects. □

Theorem 3.5. *Assume that* $\mathbf{G}$ *is open and schematically dense in* $\mathbf{M}$. *Let* V *be a* $\mathbf{G}$*-representation of* $\mathbf{G}$. *There exists an* $\mathbf{M}$*-representation* W *and a surjective morpism* $q : V \twoheadrightarrow W$ *of* $\mathbf{G}$*-representations such that for every* $\mathbf{M}$*-representation* U *and a morphism* $f : V \to U$ *of* $\mathbf{G}$*-representations there exists a unique morphism* $\tilde{f} : W \to U$ *of* $\mathbf{M}$*-representations making the triangle*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, two heads, "q"'] & U \\ W \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

commutative.

Proof. Assume first that V is finite dimensional. Let $\mathcal{K}$ be a set of $\mathbf{G}$-subrepresentations of V that consists of all $K \subseteq V$ such that V/K carries a structure of $\mathbf{M}$-representation. Clearly $\mathcal{K} = \emptyset$ because $\{0\} \in \mathcal{K}$. Since V is finite dimensional, there exists a finite subset $\{K_1, ..., K_n\} \subseteq \mathcal{K}$ such that

$$\bigcap_{i=1}^{n} K_i = \bigcap_{K \in \mathcal{K}} K$$

Then a morphism

$$V/\left(\bigcap_{K \in \mathcal{K}} K\right) \ni v \mapsto \left(v \bmod K_i\right)_{1 \leq i \leq n} \in \bigoplus_{i=1}^{n} V/K_i$$

is a monomorphism and hence by Theorem 3.4 the quotient $W = V/\left(\bigcap_{K \in \mathcal{K}} K\right)$ is an $\mathbf{M}$-representation. Let $q : V \twoheadrightarrow W$ be the canonical epimorphism. Consider now a morphism $f : V \to U$ of $\mathbf{G}$-representations, where U is an $\mathbf{M}$-representation. Then $\operatorname{im}(f)$ is a $\mathbf{G}$-subrepresentation of U and by Theorem 3.4 we derive that $\operatorname{im}(f)$ is an $\mathbf{M}$-representation. This implies that $\ker(f)$ is in $\mathcal{K}$. Hence f factors through q. Thus there exists a unique morphism $\tilde{f} : W \to U$ of $\mathbf{G}$-representations such that $\tilde{f} \cdot q = f$. This completes the proof in case when V is finite dimensional.

Now consider the general V. Let $\mathcal{F}$ be the set of all finite dimensional $\mathbf{G}$-representations of V. According to [Monygham, 2020d, Corollary 15.2] we deduce that $V = \mathrm{colim}_{F\in\mathcal{F}} F$. By the case considered above we deduce that for every F in $\mathcal{F}$ there exists a universal morphism $q_F : F \to W_F$ of $\mathbf{G}$-representations into an $\mathbf{M}$-representation W_F. Note that if $F_1 \subseteq F_2$ are two elements of $\mathcal{F}$, then

$$\begin{tikzcd} F_1 \arrow[r, two heads, "q_{F_1}"] \arrow[d, hook] & W_{F_1} \arrow[d] \\ F_2 \arrow[r, two heads, "q_{F_2}"'] & W_{F_2} \end{tikzcd}$$

Thus $\{W_F\}_{F\in\mathcal{F}}$ together with morphisms $W_{F_1} \to W_{F_2}$ for $F_1 \subseteq F_2$ in $\mathcal{F}$ form a diagram parametrized by the poset $\mathcal{F}$. The category $\mathbf{Rep}(\mathbf{M})$ has small colimits ([Monygham, 2020d, Corollary 14.5]) and we define $W = \mathrm{colim}_{F\in\mathcal{F}} W_F$. This is also a colimit of this diagram in the category $\mathbf{Rep}(\mathbf{G})$ by Fact 3.2. We also define $q = \mathrm{colim}_{F\in\mathcal{F}} q_F : V = \mathrm{colim}_{F\in\mathcal{F}} F \to W$. Since a colimit of a family of epimorphisms is an epimorphism, we derive that q is an epimorphism of $\mathbf{G}$-representations. Suppose now that $f : V \to U$ is a morphism of $\mathbf{G}$-representations and U is an $\mathbf{M}$-representation. Then $f_{|F}$ uniquely factors through q_F for every F in $\mathcal{F}$. Hence by universal property of colimits we derive that f factors through q in a unique way. This completes the proof. □

4. Białynicki-Birula functors

In this section we fix an affine group k-scheme $\mathbf{G}$. Let $\mathbf{M}$ be an affine monoid k-scheme with zero $\mathbf{o}$ such that $\mathbf{G}$ is its group of units.

Definition 4.1. Let X be a k-scheme equipped with an action of $\mathbf{G}$. For every k-scheme Y we define

$$\mathcal{D}_X(Y) = \left\{\gamma : \mathbf{M} \times_k Y \to X \,\middle|\, \gamma \text{ is } \mathbf{G}\text{-equivariant}\right\}$$

This gives gives rise to a subfunctor $\mathcal{D}_X$ of $\mathrm{Mor}_k(\mathbf{M} \times_k (-), X) : \mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Set}$. We call it *the Białynicki-Birula functor of X*.

Fact 4.2. *Let X be a scheme equipped with an action of $\mathbf{G}$. Then $\mathcal{D}_X$ is a Zariski sheaf.*

Proof. This is a consequence of the fact that $\mathrm{Mor}_k(\mathbf{M} \times_k (-), X)$ is a Zariski sheaf and if we glue $\mathbf{G}$-equivariant morphisms, then the result is $\mathbf{G}$-equivariant. Indeed, this shows that $\mathcal{D}_X$ is a Zariski subsheaf of $\mathrm{Mor}_k(\mathbf{M} \times_k (-), X)$. □

Remark 4.3. Let X be a k-scheme equipped with an action of $\mathbf{G}$. Then there are canonical morphism of functors

$$\begin{tikzcd} \mathcal{D}_X \arrow[r, "i_X"] \arrow[d, "r_X"] & X \\ X^{\mathbf{G}} \arrow[u, bend left, "s_X"] & \end{tikzcd}$$

which we define now. First let us explain that in the diagram X stands for the presheaf representable by the k-scheme X (Remark 1.1) and $X^{\mathbf{G}}$ denotes the functor of fixed points of X ([Monygham, 2020c, Definition 7.1]). Now fix k-scheme Y and $\gamma \in \mathcal{D}_X(Y)$, then we define

$$i_X(\gamma) = \gamma_{|\{e\}\times_k X} = \gamma \cdot \langle e, 1_X\rangle,\ r_X(\gamma) = \gamma_{|\{o\}\times_k X} = \gamma \cdot \langle o, 1_X\rangle$$

where $e : \operatorname{Spec} k \to \mathbf{M}$ is the unit of $\mathbf{M}$ and $\mathbf{o} : \operatorname{Spec} k \to \mathbf{M}$ is the zero. Next if $f : Y \to X$ is a morphism in $X^{\mathbf{G}}(Y)$, then we define

$$s_X(f) = f \cdot pr_Y$$

where $pr_Y : \mathbf{M} \times_k Y \to Y$ is the projection. Finally note that $r_X \cdot s_X = 1_{X^{\mathbf{G}}}$.

Remark 4.4. Let X be a k-scheme equipped with an action of $\mathbf{G}$. Then $\mathbf{M}$ (actually the presheaf of monoids represented by $\mathbf{M}$) acts on $\mathcal{D}_X$. Indeed, fix k-scheme Y, $\gamma \in \mathcal{D}_X(Y)$ and $m : Y \to \mathbf{M}$. Then we define the product

$$m\gamma = \gamma \cdot \langle m, 1_Y \rangle$$

and this determines an action of $\mathbf{M}$ on $\mathcal{D}_X$. Moreover, with respect to this action i_X is $\mathbf{G}$-equivariant and r_X, s_X are $\mathbf{M}$-equivariant ($X^{\mathbf{G}}$ is equipped with trivial action of $\mathbf{M}$).

Remark 4.5. Let X, Y be k-schemes equipped with actions of $\mathbf{G}$ and let $f : X \to Y$ be a $\mathbf{G}$-equivariant morphism, then there exists a morphism of functors $\mathcal{D}_f : \mathcal{D}_X \to \mathcal{D}_Y$ given by

$$\mathcal{D}_f(\gamma) = f \cdot \gamma$$

for every element γ of the functor $\mathcal{D}_X$.

Let X be a k-scheme equipped with an action of $\mathbf{G}$. It is useful to discuss subfunctors of $\mathcal{D}_X$ defined by closed $\mathbf{G}$-stable subschemes of X.

Theorem 4.6. *Let X be a k-scheme equipped with an action of the group $\mathbf{G}$. Suppose that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. If $j : Z \hookrightarrow X$ is a closed $\mathbf{G}$-stable subscheme of X, then the square*

$$\begin{tikzcd} \mathcal{D}_Z \arrow[r, "\mathcal{D}_j"] \arrow[d, "i_Z"'] & \mathcal{D}_X \arrow[d, "i_X"] \\ Z \arrow[r, "j"'] & X \end{tikzcd}$$

is cartesian in the category of presheaves on $\mathbf{Sch}_k$.

Proof. The fact that the square is commutative follows by examination of definitions in Remarks 4.3 and 4.5. Pick k-scheme Y, $f : Y \to Z$ and $\gamma \in \mathcal{D}_X(Y)$ such that $j \cdot f = i_X(\gamma)$. This is depicted in the diagram

$$\begin{tikzcd} & \gamma \arrow[d, mapsto, "i_X"] \\ f \arrow[r, mapsto, "j"'] & j \cdot f = \gamma_{|\{e\} \times_k X} \end{tikzcd}$$

Our goal is to show that there exists a unique $\mathbf{G}$-equivariant morphism $\eta : \mathbf{M} \times_k Y \to U$ such that $\mathcal{D}_j(\eta) = \gamma$ and $i_Z(\eta) = f$. This is depicted by the diagram

$$\begin{tikzcd} \eta \arrow[r, mapsto, "\mathcal{D}_j"] \arrow[d, mapsto, "r_U"'] & \gamma = j \cdot \eta \\ f = \eta_{|\{e\} \times_k X} & \end{tikzcd}$$

It suffices to prove that γ factors through j. First note that the assumption $\gamma_{|\{e\}\times_k Y} = j\cdot f$ implies that

$$\gamma_{|\mathbf{G}\times_k Y} = j\cdot f\cdot pr_Y$$

where $pr_Y : \mathbf{G}\times_k Y \to Y$ is the projection. This implies that $\gamma_{|\mathbf{G}\times_k}$ factors through j. Consider scheme-theoretic preimage $\gamma^{-1}(Z)$. Then $\gamma^{-1}(Z)$ is a closed $\mathbf{G}$-stable (as an inverse image of a $\mathbf{G}$-stable closed subscheme under the $\mathbf{G}$-equivariant morphism) subscheme of $\mathbf{M}\times_k Y$, which contains $\mathbf{G}\times_k Y$. Since $\mathbf{G}$ is open, schematically dense in $\mathbf{M}$ and k is a field, we derive that $\mathbf{G}\times_k Y$ is open and schematically dense in $\mathbf{M}\times_k Y$. Thus $\gamma^{-1}(Z) = \mathbf{M}\times_k Y$ and hence γ factors through j. $\square$

In order to prove interesting result in the spirit of Theorem 4.6 which concerns open $\mathbf{G}$-stable subschemes, we need to assume that $\mathbf{M}$ is a Kempf monoid.

Theorem 4.7. *Let X be a k-scheme equipped with an action of the group $\mathbf{G}$ of units of a Kempf monoid $\mathbf{M}$. If $j : U \hookrightarrow X$ is an open $\mathbf{G}$-stable subscheme of X, then the square*

$$\begin{tikzcd} \mathcal{D}_U \arrow[r, "\mathcal{D}_j"] \arrow[d, "r_U"'] & \mathcal{D}_X \arrow[d, "r_X"] \\ U^{\mathbf{G}} \arrow[r, "j^{\mathbf{G}}"'] & X^{\mathbf{G}} \end{tikzcd}$$

is cartesian in the category of presheaves on $\mathbf{Sch}_k$.

As we shall see this result follows from the following.

Lemma 4.7.1. *Let K be an algebraicaly closed field over k. Suppose that*

$$\mathbf{M}_K = \operatorname{Spec} K\times_k \mathbf{M},\ \mathbf{G}_K = \operatorname{Spec} K\times_k \mathbf{G}$$

and let $\mathbf{o}_K$ be the unique K-point of $\mathbf{M}_K$ lying over $\mathbf{o}$. Let V be an open $\mathbf{G}_K$-stable subscheme of $\mathbf{M}_K$ such that $\mathbf{o}_K \in V$. Then $V = \mathbf{M}_K$.

Proof of the lemma. Since $\mathbf{M}$ is a Kempf monoid, there exists a closed embedding of monoids $v : \mathbb{A}^1_K \hookrightarrow \mathbf{M}_K$ preserving zeros such that $v_{|\mathbb{G}_{m,K}} \subseteq \mathbf{G}_K$. Fix a point $p \in \mathbf{M}_K$ and let $u : \operatorname{Spec} k(p) \to \mathbf{M}_K$ be the associated morphism of K-schemes. Consider the composition

$$\begin{tikzcd} \mathbb{A}^1_{k(p)} = \mathbb{A}^1_K \times_K \operatorname{Spec} k(p) \arrow[r, hook, "v\times_K u"] \arrow[rr, bend left, "h"] & \mathbf{M}_K\times_K \mathbf{M}_K \arrow[r, hook] & \mathbf{M}_K \end{tikzcd}$$

where the second morphism is the multiplication. Clearly h is $\mathbf{G}_{m,k(p)}$-equivariant. Hence $h^{-1}(V)$ is an open $\mathbf{G}_{m,k(p)}$-stable subscheme of $\mathbb{A}^1_{k(p)}$ containing zero of this monoid $k(p)$-scheme (because $\mathbf{o}_K \in V$ by assumption). Since the only open $\mathbf{G}_{m,k(p)}$-stable subscheme of $\mathbb{A}^1_{k(p)}$ containing zero is $\mathbb{A}^1_{k(p)}$, we derive that $h^{-1}(V) = \mathbb{A}^1_{k(p)}$. Thus $p \in V$. Since p is arbitrary point of $\mathbf{M}_K$, we derive that $V = \mathbf{M}_K$. $\square$

Proof of the theorem. The fact that the square is commutative follows by examination of definitions in Remarks 4.3 and 4.5. Pick k-scheme Y, $f \in U^{\mathbf{G}}$ and $\gamma \in \mathcal{D}_X(Y)$ such that $j^{\mathbf{G}}(f) = r_X(\gamma)$. This is depicted in the diagram

$$\begin{tikzcd} & \gamma \arrow[d, mapsto, "r_X"] \\ f \arrow[r, mapsto, "j^{\mathbf{G}}"'] & j\cdot f = \gamma_{|\{\mathbf{o}\}\times_k X} \end{tikzcd}$$

Our goal is to show that there exists a unique $\mathbf{G}$-equivariant morphism $\eta : \mathbf{M} \times_k Y \to U$ such that $\mathcal{D}_j(\eta) = \gamma$ and $r_U(\eta) = f$. This is depicted by the diagram

$$\begin{tikzcd} \eta \arrow[r, mapsto, "\mathcal{D}_j"] \arrow[d, mapsto, "r_U"'] & \gamma = j\cdot \eta \\ f = \eta_{|\{\mathbf{o}\}\times_k X} & \end{tikzcd}$$

Fir this it suffices to prove that γ factors through j. Consider $W = \gamma^{-1}(U)$. Note that W is an open $\mathbf{G}$-stable (as an inverse image of a $\mathbf{G}$-stable open subscheme under the $\mathbf{G}$-equivariant morphism) subscheme of $\mathbf{M} \times_k Y$, which contains $\{\mathbf{o}\} \times_k Y$. Lemma 4.7.1 asserts that for every geometric point $\overline{y}$ of Y we have $W_{\overline{y}} = \mathbf{M}_{k(\overline{y})}$, where $W_{\overline{y}}$ is the fiber over $\overline{y}$ of the projection $\mathbf{M} \times_k Y \to Y$ restricted to W. Since W is open subscheme of $\mathbf{M} \times_k Y$, this implies that $W = \mathbf{M} \times_k Y$ and hence γ factors through j. □

As we shall see below both Theorems are extremely useful properties of Białynicki-Birula functors. Now we introduce a formal version of this functor.

Definition 4.8. Let $\mathbf{M}$ be an affine monoid k-scheme with zero $\mathbf{o}$ and let $\mathbf{G}$ be its group of units. For every $n \in \mathbb{N}$ let $\mathbf{M}_n \hookrightarrow \mathbf{M}$ be an n-th infinitesimal neighborhood of $\mathbf{o}$ in $\mathbf{M}$. Let X be a k-scheme equipped with an action of $\mathbf{G}$. For every k-scheme Y we define

$$\widehat{\mathcal{D}}_X(Y) = \left\{ \{\gamma_n : \mathbf{M}_n \times_k Y \to X\}_{n\in\mathbb{N}} \,\middle|\, \forall_{n\in\mathbb{N}}\, \gamma_n \text{ is } \mathbf{G}\text{-equivariant and } \gamma_{n+1|\mathbf{M}_n\times_k Y} = \gamma_n \right\}$$

This gives gives rise to a functor $\widehat{\mathcal{D}}_X$. We call it *the formal Białynicki-Birula functor of X*.

Remark 4.9. Let $\mathbf{M}$ be an affine monoid k-scheme with zero $\mathbf{o}$ and let $\mathbf{G}$ be its group of units. Let X be a k-scheme equipped with an action of $\mathbf{G}$. Then there exists a canonical morphism of functors $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ given by

$$\gamma \mapsto \{\gamma_{|\mathbf{M}_n\times_k Y}\}_{n\in\mathbb{N}}$$

for every $\gamma \in \mathcal{D}_X(Y)$ and every k-scheme Y.

5. Representability of Białynicki-Birula functor for Kempf monoids

In this section we prove various results concerning representability of Białynicki-Birula functors.

Theorem 5.1. *Let $\mathbf{M}$ be an affine monoid k-scheme with open and schematically dense group of units $\mathbf{G}$. Suppose that X is an affine k-scheme equipped with an ation of $\mathbf{G}$. Then $\mathcal{D}_X$ is representable and i_X is a closed immersion of k-schemes.*

Proof. Since X is an affine k-scheme, the action of $\mathbf{G}$ on X corresponds to the coaction of $k[\mathbf{G}]$ by $c : \Gamma(X, \mathcal{O}_X) \to k[\mathbf{G}] \otimes_k \Gamma(X, \mathcal{O}_X)$. Note that c is a morphism of k-algebras. By Theorem 3.5 there exists a universal morphism $q : \Gamma(X, \mathcal{O}_X) \twoheadrightarrow W$ of $\mathbf{G}$-representations into a $\mathbf{M}$-representation. Let $I \subseteq \Gamma(X, \mathcal{O}_X)$ be the ideal generated by $\ker(q)$. Fix f in I. Then

$$f = \sum_{i=1}^{n} g_i \cdot f_i$$

where $g_i \in k[\mathbf{G}]$ and $f_i \in \ker(q)$ for $1 \leq i \leq n$. Then

$$c(f) = c\left(\sum_{i=1}^{n} g_i \cdot f_i\right) = \sum_{i=1}^{n} c(g_i)\cdot c(f_i) \subseteq \left(k[\mathbf{G}] \otimes_k \Gamma(X, \mathcal{O}_X)\right)\cdot\left(k[\mathbf{G}] \otimes_k \ker(q)\right) \subseteq k[\mathbf{G}] \otimes_k I$$

Thus $c(I) \subseteq k[\mathbf{G}] \otimes_k I$ and hence I is a $\mathbf{G}$-representation. Consider

$$X^+ = V(I) = \operatorname{Spec}\Gamma(X, \mathcal{O}_X)/I \hookrightarrow X$$

Since $\Gamma(X, /cO_X)/I$ is the quotient $\mathbf{G}$-representation of W, we deduce by Theorem 3.5 that $\Gamma(X, \mathcal{O}_X)/I$ is a $\mathbf{M}$-representation. Hence X^+ is a k-scheme equipped with action of $\mathbf{M}$ and $X^+ \hookrightarrow X$ is $\mathbf{G}$-equivariant. Suppose now that Y is an affine k-scheme. Then $\mathbf{M} \times_k Y$ is a $\mathbf{M}$-scheme with respect to the left-hand side action of $\mathbf{M}$ and hence $\Gamma(\mathbf{M} \times_k Y, \mathcal{O}_{\mathbf{M}\times_k Y})$ is a $\mathbf{M}$-representation. Now Theorem 3.5 implies that if $\gamma : \mathbf{M} \times_k Y \to X$ is a $\mathbf{G}$-equivariant morphism, then a morphism $\gamma^\# : \Gamma(X, \mathcal{O}_X) \to \Gamma(\mathbf{M} \times_k Y, \mathcal{O}_{\mathbf{M}} \times_k Y)$ of k-algebras and $\mathbf{G}$-representations factors through $q : \Gamma(X, \mathcal{O}_X) \twoheadrightarrow W$ and thus by construction of I we have

$$\begin{tikzcd} \Gamma(X, \mathcal{O}_X) \arrow[r, two heads] \arrow[rr, bend left, "\gamma^\#"] & \Gamma(X, \mathcal{O}_X)/I \arrow[r, "f"] & \Gamma(\mathbf{M} \times_k Y, \mathcal{O}_{\mathbf{M}} \times_k Y) \end{tikzcd}$$

for some morphism f of k-algebras and $\mathbf{G}$-representations. Since both $\Gamma(X, \mathcal{O}_X)/I$ and $\Gamma(\mathbf{M} \times_k Y, \mathcal{O}_{\mathbf{M}} \times_k Y)$ are $\mathbf{M}$-representations and by Theorem 3.4 the subcategory $\mathbf{Rep}(\mathbf{M}) \subseteq \mathbf{Rep}(\mathbf{G})$ is full, we derive that f is a morphism of $\mathbf{M}$-representations. Thus f corresponds to a unique $\mathbf{M}$-equivariant morphism $\eta : \mathbf{M} \times_k Y \to X^+$ such that the diagram

$$\begin{tikzcd} & X^+ \arrow[d, hook] \\ \mathbf{M} \times_k Y \arrow[ur, bend left, "\eta"] \arrow[r, "\gamma"'] & X \end{tikzcd}$$

is commutative. Now this result can be extended to an arbitrary k-scheme Y, since $\mathrm{Mor}_k(\mathbf{M} \times_k (-), X^+)$ is a Zariski sheaf and a morphism that is $\mathbf{M}$-equivariant locally on the domain is $\mathbf{M}$-equivariant. Thus for every h-scheme Y we have a bijection

$$\mathcal{D}_X(Y) \ni \gamma \mapsto \eta \in \left\{\mathbf{M}\text{-equivariant morphisms } \mathbf{M} \times_k Y \to X^+\right\}$$

Since we also have a bijection

$$\left\{\mathbf{M}\text{-equivariant morphisms } \mathbf{M} \times_k Y \to X^+\right\} \ni \eta \mapsto \eta \cdot \langle e, 1_{X^+}\rangle \in \mathrm{Mor}_k(Y, X^+)$$

and both this bijections are natural, we derive that $\mathcal{D}_X$ is represented by X^+ and moreover, $i_X : \mathcal{D}_X \to X$ is a closed immersion $X^+ \hookrightarrow X$. □

Corollary 5.2. *Let $\mathbf{G}$ be a group k-scheme and $\mathbf{M}$ be a Kempf monoid having $\mathbf{G}$ as a group of units. Suppose that X is a k-scheme equipped with an action of $\mathbf{G}$ such that there exists a family $\mathcal{U}$ of open affine $\mathbf{G}$-stable open subschemes of X such that functors $\{U^{\mathbf{G}}\}_{U\in\mathcal{U}}$ form an open cover of $X^{\mathbf{G}}$. Then $\mathcal{D}_X$ is representable.*

Proof. Note that $\mathbf{G}$ is affine group k-scheme as a unit group of an affine monoid $\mathbf{M}$ ([Monygham, 2020d, Proposition 12.4]). Moreover, $\mathbf{M}$ is a Kempf monoid and hence $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. By Theorem 5.1 each $\mathcal{D}_U$ is representable by a k-scheme. On the other hand by Theorem 4.7 for each $U \in \mathcal{U}$ we have a cartesian square

$$\begin{tikzcd} \mathcal{D}_U \arrow[r] \arrow[d, "r_U"'] & \mathcal{D}_X \arrow[d, "r_X"] \\ U^{\mathbf{G}} \arrow[r] & X^{\mathbf{G}} \end{tikzcd}$$

of functors. This implies that $\{\mathcal{D}_U \hookrightarrow \mathcal{D}_X\}_{U \in \mathcal{U}}$ is an open cover of $\mathcal{D}_X$ as a pullback of an open cover $\{U^{\mathbf{G}} \hookrightarrow X^{\mathbf{G}}\}_{U \in \mathcal{U}}$. Hence Fact 4.2 and [Görtz and Wedhorn, 2010, Theorem 8.9] (or if you like [Monygham, 2019, Theorem 4.6]) imply that $\mathcal{D}_X$ is representable. □

Corollary 5.3. *Let $\mathbf{G}$ be group k-scheme and $\mathbf{M}$ be a Kempf monoid having $\mathbf{G}$ as a group of units. Suppose that X is a locally linear $\mathbf{G}$-scheme. Then $\mathcal{D}_X$ is representable.*

Proof. This is a consequence of Corollary 5.2. Indeed, X admits a cover $\mathcal{U}$ by open $\mathbf{G}$-stable affine subschemes. Then $\{U^{\mathbf{G}}\}_{U \in \mathcal{U}}$ is an open cover of $X^{\mathbf{G}}$. □

In order to prove more interesting result we need to recall some fact.

Remark 5.4. Let $\mathbf{G}$ be a geometrically connected and locally algebraic group over k. It follows by [Monygham, 2020c, Theorem 7.2] that for every k-scheme X equipped with an action of $\mathbf{G}$ there exists closed subscheme $X^{\mathbf{G}}$ of X representing the fixed point functor.

Theorem 5.5. *Let $\mathbf{G}$ be a geometrically connected and locally algebraic group k-scheme and $\mathbf{M}$ be a Kempf monoid having $\mathbf{G}$ as a group of units. Suppose that X is a k-scheme equipped with action of $\mathbf{G}$. Then $\widehat{\mathcal{D}}_X$ is representable.*

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