

# INTRODUCTION TO MEASURE THEORY

## 1. FAMILIES OF SETS

In this section we study various families of sets that are important in the development of measure theory.

**Definition 1.1.** Let  $X$  be a set and  $\mathcal{F} \subseteq \mathcal{P}(X)$  be a family of subsets of  $X$ . We define the following types of families.

- (1)  $\mathcal{F}$  is an *algebra* if it contains  $X$  and is closed under finite unions, intersections and complements.
- (2)  $\mathcal{F}$  is a  $\sigma$ -*algebra* if it is an algebra and is closed under countable unions.
- (3)  $\mathcal{F}$  is a *monotone family* if it is closed under unions of countable non-decreasing sequences and under intersections of countable non-increasing sequences.
- (4)  $\mathcal{F}$  is a  $\pi$ -*system* if it is closed under finite intersections.
- (5)  $\mathcal{F}$  is a  $\lambda$ -*system* if it contains  $X$  and is closed under complements and countable disjoint unions.

**Fact 1.2.** Let  $X$  be a set and  $\{\mathcal{F}_i\}_{i \in I}$  be a class of families subsets of  $X$ . Suppose that  $\mathcal{F}_i$  is an algebra ( $\sigma$ -algebra, monotone family,  $\pi$ -system,  $\lambda$ -system) for every  $i \in I$ . Then the intersection  $\bigcap_{i \in I} \mathcal{F}_i$  is an algebra ( $\sigma$ -algebra, monotone family,  $\pi$ -system,  $\lambda$ -system).

*Proof.* Left as an exercise. □

**Definition 1.3.** Let  $\mathcal{F}$  be a family of subsets of  $X$ . We denote by  $\sigma(\mathcal{F})$ ,  $\lambda(\mathcal{F})$  and  $\mathcal{M}(\mathcal{F})$  intersections of all  $\sigma$ -algebras,  $\lambda$ -systems and monotone families containing  $\mathcal{F}$ , respectively. We call them  $\sigma$ -*algebra*,  $\lambda$ -*system* and *monotone family generated by*  $\mathcal{F}$ , respectively.

**Theorem 1.4** (Dynkin's  $\pi$ - $\lambda$  lemma). Let  $X$  be a set and  $\mathcal{P}$  be a  $\pi$ -system of its subsets. Then  $\lambda(\mathcal{P}) = \sigma(\mathcal{P})$ .

For the proof we need the following result.

**Lemma 1.4.1.** Let  $\mathcal{L}$  be a  $\lambda$ -system. Then for every  $A \in \mathcal{L}$  family

$$\mathcal{L}_A = \{B \in \mathcal{L} \mid A \cap B \in \mathcal{L}\}$$

is a  $\lambda$ -system.

*Proof of the lemma.* Since  $A \in \mathcal{L}$ , we have  $X \in \mathcal{L}_A$ . Suppose now that  $B \in \mathcal{L}_A$ . Then  $A \cap B \in \mathcal{L}$ . Since  $X \setminus A \in \mathcal{L}$ , we derive that also  $(A \cap B) \cup (X \setminus A) \in \mathcal{L}$  and hence

$$A \cap (X \setminus B) = X \setminus ((A \cap B) \cup (X \setminus A)) \in \mathcal{L}$$

Thus  $X \setminus B \in \mathcal{L}_A$ . Finally note that  $\mathcal{L}_A$  is closed under countable disjoint unions. □

*Proof of the theorem.* Fix  $A \in \mathcal{P}$ . Define  $\mathcal{L}_A$  as in Lemma 1.4.1 with  $\mathcal{L} = \lambda(\mathcal{P})$ . Then  $\mathcal{L}_A$  is a  $\lambda$ -system. Moreover,  $\mathcal{L}_A$  contains  $\mathcal{P}$ . Hence  $\mathcal{L}_A = \lambda(\mathcal{P})$ . This shows that  $\lambda(\mathcal{P})$  is closed under intersections with members of  $\mathcal{P}$ . Now fix  $A \in \lambda(\mathcal{P})$  and define  $\mathcal{L}_A$  as in Lemma 1.4.1 with  $\mathcal{L} = \lambda(\mathcal{P})$ . Then  $\mathcal{P} \subseteq \mathcal{L}_A$  and  $\mathcal{L}_A$  is a  $\lambda$ -system. Thus  $\mathcal{L}_A = \lambda(\mathcal{P})$ . This proves that  $\lambda(\mathcal{P})$  is a  $\pi$ -system. A  $\pi$ -system that is simultaneously a  $\lambda$ -system is a  $\sigma$ -algebra. Thus  $\sigma(\mathcal{P}) \subseteq \lambda(\mathcal{P})$ . Since it is clear that  $\lambda(\mathcal{P}) \subseteq \sigma(\mathcal{P})$ , we derive that  $\lambda(\mathcal{P}) = \sigma(\mathcal{P})$ . □

**Theorem 1.5** (Halmos's lemma on monotone classes). *Let  $X$  be a set and  $\mathcal{A}$  be an algebra of its subsets. Then  $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$ .*

For the proof we need the following easy results. Their proofs are left to the reader.

**Lemma 1.5.1.** *Let  $\mathcal{M}$  be a monotone family. Then for every  $A \in \mathcal{M}$  family*

$$\mathcal{M}_A = \{B \in \mathcal{M} \mid A \cap B \in \mathcal{M}\}$$

*is monotone.*

**Lemma 1.5.2.** *Let  $\mathcal{M}$  be a monotone family. Then a family*

$$\mathcal{M}^c = \{A \in \mathcal{M} \mid X \setminus A \in \mathcal{M}\}$$

*is monotone.*

*Proof of the theorem.* Fix  $A \in \mathcal{A}$ . Define  $\mathcal{M}_A$  as in Lemma 1.5.1 with  $\mathcal{M} = \mathcal{M}(\mathcal{A})$ . Then  $\mathcal{M}_A$  is a monotone family. Moreover,  $\mathcal{M}_A$  contains  $\mathcal{A}$ . Hence  $\mathcal{M}_A = \mathcal{M}(\mathcal{A})$ . This shows that  $\mathcal{M}(\mathcal{A})$  is closed under intersections with members of  $\mathcal{A}$ . Now fix  $A \in \mathcal{M}(\mathcal{A})$  and define  $\mathcal{M}_A$  as in Lemma 1.5.1 with  $\mathcal{M} = \mathcal{M}(\mathcal{A})$ . Then  $\mathcal{A} \subseteq \mathcal{M}_A$  and  $\mathcal{M}_A$  is a monotone family. Thus  $\mathcal{M}_A = \mathcal{M}(\mathcal{A})$ . This proves that  $\mathcal{M}(\mathcal{A})$  is closed under finite intersections. According to Lemma 1.5.2 we derive that  $\mathcal{M}(\mathcal{A})^c$  is a monotone family and contains  $\mathcal{A}$ . Hence  $\mathcal{M}(\mathcal{A})^c = \mathcal{M}(\mathcal{A})$  and  $\mathcal{M}(\mathcal{A})$  is closed under complements. Therefore,  $\mathcal{M}(\mathcal{A})$  is a  $\sigma$ -algebra. Thus  $\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$ . Since it is clear that  $\mathcal{M}(\mathcal{A}) \subseteq \sigma(\mathcal{A})$ , we derive that  $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$ .  $\square$

## 2. MEASURABLE SPACES AND MEASURES

**Definition 2.1.** A pair  $(X, \Sigma)$  consisting of a set  $X$  together with a  $\sigma$ -algebra  $\Sigma$  of its subsets is called a *measurable space*.

**Definition 2.2.** Let  $(X_1, \Sigma_1)$  and  $(X_2, \Sigma_2)$  be measurable spaces. A function  $f : X_1 \rightarrow X_2$  is called a *measurable map* if  $f^{-1}(A) \in \Sigma_1$  for every  $A \in \Sigma_2$ .

Measurable spaces and their morphisms form a category.

**Definition 2.3.** Let  $X$  be a set and  $\Sigma$  be an algebra of its subsets. A function  $\mu : \Sigma \rightarrow [0, +\infty]$  such that  $\mu(\emptyset) = 0$  and

$$\mu\left(\bigcup_{n=0}^m A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

for every family  $\{A_n\}_{0 \leq n \leq m}$  of pairwise disjoint subsets in  $\Sigma$  is called an *additive function*. If in addition  $\mu$  satisfies

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

for every family  $\{A_n\}_{n \in \mathbb{N}}$  of pairwise disjoint subsets in  $\Sigma$  such that  $\bigcup_{n \in \mathbb{N}} A_n \in \Sigma$ , then  $\mu$  is called a  $\sigma$ -*additive function*. Moreover, if  $\mu : \Sigma \rightarrow [0, +\infty]$  is a  $\sigma$ -additive function and  $\Sigma$  is a  $\sigma$ -algebra, then  $\mu$  is called a *measure*.

**Definition 2.4.** A tuple  $(X, \Sigma, \mu)$  consisting of a measurable space  $(X, \Sigma)$  and a measure  $\mu : \Sigma \rightarrow [0, +\infty]$  is called a *space with measure*.

**Definition 2.5.** Let  $(X, \Sigma, \mu)$  be a space with measure. We say that it is *finite* if  $\mu(X)$  is finite. We say that it is  $\sigma$ -*finite* if there exists a sequence  $\{X_n\}_{n \in \mathbb{N}}$  of subsets of  $\Sigma$  such that  $\mu(X_n)$  is finite for every  $n \in \mathbb{N}$  and  $X = \bigcup_{n \in \mathbb{N}} X_n$ .

**Theorem 2.6.** Let  $(X, \Sigma)$  be a measurable space and  $\mu_1, \mu_2 : \Sigma \rightarrow [0, +\infty]$  be measures such that  $\mu_1(X) = \mu_2(X)$  is finite. Suppose that  $\mathcal{P}$  is a  $\pi$ -system of subsets of  $X$  such that  $\Sigma = \sigma(\mathcal{P})$  and  $\mu_1(A) = \mu_2(A)$  for every  $A \in \mathcal{P}$ . Then  $\mu_1 = \mu_2$ .

*Proof.* Define  $\mathcal{F} = \{A \in \Sigma \mid \mu_1(A) = \mu_2(A)\}$ . Straightforward verification shows that  $\mathcal{F}$  is a  $\lambda$ -system. By assumption  $\mathcal{P} \subseteq \mathcal{F}$ . Therefore,  $\lambda(\mathcal{P}) \subseteq \mathcal{F}$ . By Theorem 1.5 we deduce that  $\Sigma = \sigma(\mathcal{P}) = \lambda(\mathcal{P}) \subseteq \mathcal{F} \subseteq \Sigma$ . Hence  $\mathcal{F} = \Sigma$ .  $\square$

**Definition 2.7.** Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be spaces with measures. A function  $f : X_1 \rightarrow X_2$  is called a *morphism of spaces with measures* if  $f$  is a morphism of measurable spaces and for every  $A \in \Sigma_2$  we have equality  $\mu_2(A) = \mu_1(f^{-1}(A))$ .

Spaces with measures and their morphisms form a category.

### 3. OUTER MEASURES AND CARATHÉODORY'S CONSTRUCTION

**Definition 3.1.** Let  $X$  be a set and  $\mu^* : \mathcal{P}(X) \rightarrow [0, +\infty]$  be a function. Suppose that  $\mu^*(\emptyset) = 0$ ,  $\mu^*(A) \leq \mu^*(B)$  for every subset  $A$  of a set  $B$  contained in  $X$  and

$$\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)$$

for every family  $\{A_n\}_{n \in \mathbb{N}}$  of subsets of  $X$ . Then we say that  $\mu^*$  is an *outer measure* on  $X$ .

**Theorem 3.2** (Carathéodory's construction). *Let  $X$  be a set and  $\mu^* : \mathcal{P}(X) \rightarrow [0, +\infty]$  be an outer measure on  $X$ . We define a family of sets  $\Sigma_{\mu^*} \subseteq \mathcal{P}(X)$  by condition*

$$A \in \Sigma_{\mu^*} \Leftrightarrow \forall E \subseteq X \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$$

*Then the following assertions hold.*

- (1)  $\Sigma_{\mu^*}$  is an  $\sigma$ -algebra of subsets of  $X$ .
- (2) For every family  $\{A_n\}_{n \in \mathbb{N}}$  of pairwise disjoint subsets of  $\Sigma_{\mu^*}$  and every subset  $E$  of  $X$  we have

$$\mu^*\left(E \cap \bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu^*(E \cap A_n)$$

*In particular,  $\mu^*_{|\Sigma_{\mu^*}}$  is a measure.*

- (3) Every subset  $A$  of  $X$  such that  $\mu^*(A) = 0$  is contained in  $\Sigma_{\mu^*}$ . In particular,  $\mu^*_{|\Sigma_{\mu^*}}$  is complete.

The proof is encapsulated in two lemmas.

**Lemma 3.2.1.**  $\Sigma_{\mu^*}$  is an algebra of sets.

*Proof of the lemma.* Clearly  $\emptyset \in \Sigma_{\mu^*}$  and  $A \in \Sigma_{\mu^*} \Leftrightarrow X \setminus A \in \Sigma_{\mu^*}$ . It suffices to prove that  $\Sigma_{\mu^*}$  is closed under unions. For a subset  $B$  of  $X$  we denote  $X \setminus B$  by  $B^c$ . Now assume that  $A_1, A_2 \in \Sigma_{\mu^*}$  and pick a subset  $E$  of  $X$ . Then

$$\mu^*(E) = \mu^*(E \cap A_1) + \mu^*(E \cap A_1^c) = \mu^*(E \cap A_1) + \mu^*(E \cap A_1^c \cap A_2) + \mu^*(E \cap A_1^c \cap A_2^c)$$

Since we have equalities

$$E \cap A_1 = (E \cap (A_1 \cup A_2)) \cap A_1, E \cap A_1^c \cap A_2 = (E \cap (A_1 \cup A_2)) \cap A_1^c$$

we derive that  $\mu^*(E \cap (A_1 \cup A_2)) = \mu^*(E \cap A_1) + \mu^*(E \cap A_1^c \cap A_2)$ . Similarly we have equality

$$E \cap A_1^c \cap A_2^c = E \cap (A_1 \cup A_2)^c$$

and hence  $\mu^*(E \cap A_1^c \cap A_2^c) = \mu^*(E \cap (A_1 \cup A_2)^c)$ . Therefore, we have

$$\mu^*(E) = \mu^*(E \cap (A_1 \cup A_2)) + \mu^*(E \cap (A_1 \cup A_2)^c)$$

Thus we proved that  $A_1 \cup A_2 \in \Sigma_{\mu^*}$ . Therefore,  $\Sigma_{\mu^*}$  is a family of subsets of  $X$  closed under finite unions, complements and containing  $\emptyset$ . Thus  $\Sigma_{\mu^*}$  is an algebra of sets.  $\square$

**Lemma 3.2.2.** Let  $\{A_n\}_{n \in \mathbb{N}}$  be a family of pairwise disjoint subsets of  $\Sigma_{\mu^*}$ . Then  $\bigcup_{n \in \mathbb{N}} A_n \in \Sigma_{\mu^*}$  and for every subset  $E$  of  $X$  there is an equality

$$\mu^* \left( E \cap \bigcup_{n \in \mathbb{N}} A_n \right) = \sum_{n \in \mathbb{N}} \mu^*(E \cap A_n)$$

*Proof of the lemma.* We prove that  $\bigcup_{n \in \mathbb{N}} A_n \in \Sigma_{\mu^*}$ . For this observe that we have

$$\begin{aligned} \mu^*(E) &\leq \mu^* \left( E \cap \bigcup_{n \in \mathbb{N}} A_n \right) + \mu^* \left( E \setminus \bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n \in \mathbb{N}} \mu^*(E \cap A_n) + \mu^* \left( E \setminus \bigcup_{n \in \mathbb{N}} A_n \right) = \\ &= \lim_{m \rightarrow +\infty} \left( \mu^* \left( E \cap \bigcup_{n=0}^m A_n \right) + \mu^* \left( E \setminus \bigcup_{n=0}^m A_n \right) \right) \leq \lim_{m \rightarrow +\infty} \left( \mu^* \left( E \cap \bigcup_{n=0}^m A_n \right) + \mu^* \left( E \setminus \bigcup_{n=0}^m A_n \right) \right) = \mu^*(E) \end{aligned}$$

and the last equality holds, since  $\bigcup_{n=0}^m A_n \in \Sigma_{\mu^*}$  by Lemma 3.2.1. This implies that we have equalities everywhere above. Hence

$$\mu^* \left( E \cap \bigcup_{n \in \mathbb{N}} A_n \right) = \sum_{n \in \mathbb{N}} \mu^*(E \cap A_n)$$

and  $\bigcup_{n \in \mathbb{N}} A_n \in \Sigma_{\mu^*}$ . □

*Proof of the theorem.* Lemma 3.2.1 and Lemma 3.2.2 imply that  $\Sigma_{\mu^*}$  is a  $\sigma$ -algebra and statement (2) holds. It suffices to verify that statement (3) holds. For this pick a subset  $A$  of  $X$  such that  $\mu^*(A) = 0$ . Then for every subset  $E$  of  $X$  we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \setminus A) = \mu^*(E \setminus A) \leq \mu^*(E)$$

Hence  $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$  and thus  $A \in \Sigma_{\mu^*}$ . □

Next result is a general tool of constructing measures.

**Theorem 3.3** (Carathéodory extension). Let  $X$  be a set and  $\Sigma$  be some algebra of its subsets. Suppose that  $\mu : \Sigma \rightarrow [0, +\infty]$  is a  $\sigma$ -additive function. Now for every subset  $A$  in  $X$  we define

$$\mu^*(A) = \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_n) \mid A_n \in \Sigma \text{ for every } n \in \mathbb{N} \text{ and } A \subseteq \bigcup_{n \in \mathbb{N}} A_n \right\}$$

Then  $\mu^* : \mathcal{P}(X) \rightarrow [0, +\infty]$  is an outer measure,  $\Sigma \subseteq \Sigma_{\mu^*}$  and  $\mu^*_{|\Sigma} = \mu$ . Moreover, if  $\mu(X)$  is finite, then  $\mu^*_{|\sigma(\Sigma)}$  is a unique extension of  $\mu$  to a measure on  $\sigma(\Sigma)$ .

*Proof.* Standard verification shows that  $\mu^*$  is an outer measure. Note that

$$\mu^*(A) = \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_n) \mid \{A_n\}_{n \in \mathbb{N}} \text{ is a family of pairwise disjoint subsets of } \Sigma \text{ and } A \subseteq \bigcup_{n \in \mathbb{N}} A_n \right\}$$

for every subset  $A$  of  $X$ . Let  $A$  be element of  $\Sigma$  and let  $E$  be an arbitrary subset of  $X$ . Fix  $\epsilon > 0$ . By the remark above there exists a family  $\{A_n\}_{n \in \mathbb{N}}$  of pairwise disjoint elements of  $\Sigma$  such that

$$E \subseteq \bigcup_{n \in \mathbb{N}} A_n, \quad \sum_{n \in \mathbb{N}} \mu(A_n) \leq \mu^*(E) + \epsilon$$

By definition of  $\mu^*$  we have  $\mu^*(E \cap A) \leq \sum_{n \in \mathbb{N}} \mu(A_n \cap A)$ ,  $\mu^*(E \setminus A) \leq \sum_{n \in \mathbb{N}} \mu(A_n \setminus A)$  and hence

$$\begin{aligned} \mu^*(E) &\leq \mu^*(E \cap A) + \mu^*(E \setminus A) \leq \sum_{n \in \mathbb{N}} \mu(A_n \cap A) + \sum_{n \in \mathbb{N}} \mu(A_n \setminus A) = \\ &= \sum_{n \in \mathbb{N}} (\mu(A_n \cap A) + \mu(A_n \setminus A)) = \sum_{n \in \mathbb{N}} \mu(A_n) \leq \mu^*(E) + \epsilon \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary, we derive that  $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$  and hence  $A \in \Sigma_{\mu^*}$ . Thus  $\Sigma \subseteq \Sigma_{\mu^*}$ . Once again fix  $A \in \Sigma$ . Then for every family  $\{A_n\}_{n \in \mathbb{N}}$  of pairwise disjoint elements of  $\Sigma$  such that  $A \subseteq \bigcup_{n \in \mathbb{N}} A_n$  we have  $\mu(A) = \sum_{n \in \mathbb{N}} \mu(A_n \cap A) \leq \sum_{n \in \mathbb{N}} \mu(A_n)$  and thus  $\mu(A) \leq \mu^*(A)$ . Obviously  $\mu^*(A) \leq \mu(A)$ . Therefore, for every  $A \in \Sigma$  we have  $\mu(A) = \mu^*(A)$ . Together with

$\Sigma \subseteq \Sigma_{\mu^*}$  this implies that  $\mu^*_{|\sigma(\Sigma)}$  is a measure that extends  $\mu$ . Now we prove the uniqueness of extension under the assumption that  $\mu(X)$  is finite. This follows immediately from Theorem 2.6.  $\square$

#### 4. OUTER METRIC MEASURES

**Definition 4.1.** Let  $X$  be a topological space. The  $\sigma$ -algebra  $\mathcal{B}_X$  generated by all open sets of  $X$  is called the  $\sigma$ -algebra of Borel subsets of  $X$ .

**Definition 4.2.** Let  $(X, d)$  be a metric space and  $\mu^* : \mathcal{P}(X) \rightarrow [0, +\infty]$  be an outer measure. We say that  $\mu^*$  is a *metric outer measure* if

$$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$$

for any two subsets  $E_1, E_2$  of  $X$  with  $\text{dist}(E_1, E_2) = \inf_{x_1 \in E_1, x_2 \in E_2} d(x_1, x_2) > 0$ .

**Theorem 4.3** (Carathéodory). Let  $(X, d)$  be a metric space and  $\mu^* : \mathcal{P}(X) \rightarrow [0, +\infty]$  be an outer metric measure on  $X$ . Then the  $\sigma$ -algebra  $\mathcal{B}_X$  of Borel subsets of  $X$  is contained in  $\Sigma_{\mu^*}$ .

*Proof.* Let  $U$  be an open subset of  $X$ . Define  $F = X \setminus U$  and  $U_n = \{x \in X \mid \text{dist}(x, F) > \frac{1}{2^n}\}$  for  $n \in \mathbb{N}$ . Then  $\{U_n\}_{n \in \mathbb{N}}$  form an ascending family of open sets and  $U = \bigcup_{n \in \mathbb{N}} U_n$ . Fix now a subset  $E$  of  $X$  such that  $\mu^*(E) \in \mathbb{R}$ . We define  $E_n = E \cap U_n$  for every  $n \in \mathbb{N}$ . Since  $\mu^*$  is an outer metric measure, we derive that

$$\mu^*\left(\bigcup_{n=0}^m E_{2n+1} \setminus E_{2n}\right) = \sum_{n=0}^m \mu^*(E_{2n+1} \setminus E_{2n}), \quad \mu^*\left(\bigcup_{n=1}^m E_{2n} \setminus E_{2n-1}\right) = \sum_{n=1}^m \mu^*(E_{2n} \setminus E_{2n-1})$$

for every positive integer  $m$ . Thus we derive

$$\sum_{n \in \mathbb{N}} \mu^*(E_{2n+1} \setminus E_{2n}) \leq \mu^*(E) \in \mathbb{R}, \quad \sum_{n \in \mathbb{N}} \mu^*(E_{2n} \setminus E_{2n-1}) \leq \mu^*(E) \in \mathbb{R}$$

Hence we have  $\sum_{n \in \mathbb{N}} \mu^*(E_{n+1} \setminus E_n) \leq 2 \cdot \mu^*(E) \in \mathbb{R}$ . Using the fact that  $\mu^*$  is an outer measure, we derive that

$$\mu(E_m) \leq \mu^*(E \cap U) \leq \mu^*(E_m) + \sum_{n \geq m} \mu^*(E_{n+1} \setminus E_n)$$

for every  $m \in \mathbb{N}$ . Hence these inequalities yield  $\lim_{m \rightarrow +\infty} \mu^*(E_m) = \mu^*(E \cap U)$ . Now we have  $\mu^*(E_m) + \mu^*(E \setminus U) \leq \mu^*(E) \leq \mu^*(E \cap U) + \mu^*(E \setminus U)$  for every  $m \in \mathbb{N}$ . The first inequality holds due to the fact that  $\mu^*$  is an outer metric measure. We derive that  $\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$ . Note that if  $\mu^*(E) = +\infty$ , then inequality  $\mu^*(E) \leq \mu^*(E \cap U) + \mu^*(E \setminus U)$  must be equality. Hence for every subset  $E$  of  $X$  we have  $\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$ . This implies that  $U \in \Sigma_{\mu^*}$ . Since  $U$  is an arbitrary open subset of  $X$ , we deduce that  $\mathcal{B}_X \subseteq \Sigma_{\mu^*}$ .  $\square$