1. Introduction

Throughout this notes k denote a field and $\mathbf{G}$ denote a group scheme over k. We also fix a k-scheme X equipped with an action of $\mathbf{G}$ determined by morphism $a : \mathbf{G} \times_k X \to X$.

2. Categorical and geometric quotients

Definition 2.1. Let $q : X \to Y$ be a morphism of k-schemes such that the diagram

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel in the category of k-schemes. Then $q : X \to Y$ is *a categorical quotient of* X.

Definition 2.2. Consider a cokernel

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

in the category of locally ringed spaces over k. If Y is a scheme, then $q : X \to Y$ is *a geometric quotient of* X.

Fact 2.3. *Every geometric quotient is categorical.*

Proof. Categorical quotient is a cokernel in the category of k-schemes. On the other hand geometric quotient is a cokernel in the category of locally ringed spaces and hence it also satisfies cokernel property in its full subcategory of k-schemes. Thus every geometric quotient is categorical. $\square$

Corollary 2.4. *Let $q : X \to Y$ be a morphism of schemes. The following assertions are equivalent.*

(i) *The diagram*

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel diagram of underlying topological spaces and the diagram

$$\mathcal{O}_Y \xrightarrow{q^\#} q_*\mathcal{O}_X \underset{q_*\mathrm{pr}_X^\#}{\overset{q_*a^\#}{\rightrightarrows}} q_*\left(\mathrm{pr}_X\right)_*\mathcal{O}_{\mathbf{G}\times_k X} = q_*a_*\mathcal{O}_{\mathbf{G}\times_k X}$$

is a kernel diagram in the category of sheaves on Y.

(ii) *q is a geometric quotient of X.*

Proof. This is a consequence of [Monygham, 2019, Theorem 2.9]. $\square$

Let $q : X \to Y$ be a morphism of k-schemes such that $q \cdot \mathrm{pr}_X = q \cdot a$. For a morphism $g : Y' \to Y$ of k-schemes consider the cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Then there exists a unique action $a' : \mathbf{G} \times_k X' \to X'$ of $\mathbf{G}$ on X' such that the square above consists of $\mathbf{G}$-equivariant morphism (we consider Y, Y' as $\mathbf{G}$-schemes equipped with trivial $\mathbf{G}$-actions). Keeping this in mind we have the following.

Definition 2.5. A morphism $q : X \to Y$ is *a uniform categorical (geometric) quotient of* X if for every flat morphism $g : Y' \to Y$ its base change $q' : X' \to Y'$ is a categorical (geometric) quotient of X'.

Definition 2.6. A morphism $q : X \to Y$ is *a universal categorical (geometric) quotient of* X if for every morphism $g : Y' \to Y$ its base change $q' : X' \to Y'$ is a categorical (geometric) quotient of X'.

3. Types of actions and criteria for smoothness of quotients

Definition 3.1. The action of $\mathbf{G}$ on X is *separated* if the morphism $\langle a, \mathrm{pr}_X \rangle : \mathbf{G} \times_k X \to X \times_k X$ has closed set-theoretic image.

Theorem 3.2. *Let $q : X \to Y$ be a geometric quotient of X. Assume that q is universally submersive. Then the following assertions are equivalent.*

- **(i)** *The action of* $\mathbf{G}$ *on X is separated.*
- **(ii)** *Y is separated.*

Proof. We have a cartesian square

$$\begin{tikzcd} X \times_Y X \arrow[r, hook] \arrow[d] & X \times_k X \arrow[d, "q \times_k q"] \\ Y \arrow[r, hook, "\Delta_Y"'] & Y \times_k Y \end{tikzcd}$$

It follows that $X \times_Y X \hookrightarrow X \times_k X$ is a locally closed immersion. Since q is a geometric quotient, we derive that $\langle a, \mathrm{pr}_X \rangle$ factors as a surjective morphism $\mathbf{G} \times_k X \twoheadrightarrow X \times_Y X$ followed by the immersion $X \times_Y X \hookrightarrow X \times_k X$. Thus the action of $\mathbf{G}$ on X is separated if and only if $X \times_Y X$ is a closed subscheme of $X \times_k X$. Since q is universally submersive, we derive that $q \times_k q$ is submersive. As the square above is cartesian we derive that $\Delta_Y(Y) \subseteq Y \times_k Y$ is closed if and only if $X \times_Y X \subseteq X \times_k X$ is closed. Therefore, Y is separated if and only if the action of $\mathbf{G}$ on X is separated. $\square$

Definition 3.3. The action of $\mathbf{G}$ on X is *free* if the morphism $\langle a, \mathrm{pr}_X \rangle : \mathbf{G} \times_k X \to X \times_k X$ is a closed immersion.

Definition 3.4. Let x be a k-point of X. Suppose that *the orbit morphism* $\mathbf{G} \to X$ *of* x given by the composition

$$\mathbf{G} = \mathbf{G} \times_k \operatorname{Spec} k \xhookrightarrow{\text{induced by } x} \mathbf{G} \times_k X \longrightarrow X$$

is a closed immersion. Then the action of $\mathbf{G}$ on X has *a closed free orbit at* x.

Fact 3.5. *If the action of* $\mathbf{G}$ *on X is free, then every k-point of X has a closed free orbit.*

The following is important result concerning smoothness of geometric quotients.

Theorem 3.6. *Suppose that* $\mathbf{G}$ *is a smooth locally algebraic group over* k*. Let* $q : X \to Y$ *be a geometric quotient locally of finite type and assume that* Y *is the spectrum of a complete local noetherian* k*-algebra such that the residue field of the closed point of* Y *is* k*. Then the following assertions hold.*

(1) *Suppose that* x *is a* k*-point of* X *which has a closed free orbit. Then there exists a* $\mathbf{G}$*-equivariant, étale and surjective morphism* $f : \mathbf{G} \times_k Y \to X$ *such that the triangle*

$$\begin{tikzcd} \mathbf{G} \times_k Y \arrow[rr, "f"] \arrow[dr, "\mathrm{pr}_Y"'] & & X \arrow[dl, "q"] \\ & Y & \end{tikzcd}$$

is commutative.

(2) *If the action of* $\mathbf{G}$ *on* X *is free, then* f *is an isomorphism.*

The proof relies on two algebraic lemmas. The first describe free actions in context of complete rings and the second is a version of Hensel's lemma.

Lemma 3.6.1. *Let* $(R, \mathfrak{m}, k)$ *be a complete local noetherian* k*-algebra and let* $\sigma : R \to R[[x_1, ..., x_n]]$ *be a local morphism into a ring of formal power series over* R*. Assume that the composition*

$$R \xrightarrow{\sigma} R[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod (x_1, ..., x_n)} R$$

is the identity and the composition

$$R \xrightarrow{\sigma} R[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod \mathfrak{m}} (R/\mathfrak{m})\,[[x_1, ..., x_n]] = k[[x_1, ..., x_n]]$$

is surjective. Consider elements $y_1, ..., y_n$ *of* R *such that* $\sigma(y_i) \bmod \mathfrak{m} = x_i$ *for* $i = 1, ..., n$*. Then the composition*

$$R \xrightarrow{\sigma} R[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod (y_1, ..., y_n)} (R/(y_1, ..., y_n))\,[[x_1, ..., x_n]]$$

is an isomorphism.

Proof of the lemma. For convienience let ϕ denote the morphism given by the rule $r \mapsto \sigma(r) \bmod (y_1, ..., y_n)$. Also denote $R/(y_1, ..., y_n)$ by S. According to assumptions we have

$$\sigma(y_i) = x_i + y_i + \sum_{j=1}^{n} x_j \cdot \mathfrak{m}[[x_1, ..., x_n]]$$

for each i. Thus $\phi(y_i) = \sum_{j=1}^{n} f_{ij} \cdot x_j$ where $f_{ij} \in S$ are elements such that the matrix $\left[f_{ij}\right]_{1 \le i,j \le n}$ is invertible in S. Hence

$$S[[x_1, ..., x_n]] = S[[\phi(y_1), ..., \phi(y_n)]]$$

and ϕ composed with $S[[\phi(y_1), ..., \phi(y_n)]] \twoheadrightarrow S$ is the quotient morphism $R \twoheadrightarrow S$. From this observations we derive that ϕ is surjective. It remains to prove that it is injective. Consider z in R such that $\phi(z) = 0$. Suppose that $z \in (y_1, ..., y_n)^m$ for some $m \in \mathbb{N}$. Write

$$z = \sum_{\alpha \in \Lambda} c_\alpha \cdot y_1^{\alpha_1} ... y_n^{\alpha_n}$$

for some $c_\alpha \in R$ where $\Lambda = \{\alpha \in \mathbb{N}^n \,|\, \alpha_1 + ... + \alpha_n = m\}$. Hence

$$0 = \phi(z) = \sum_{\alpha \in \Lambda} \phi(c_\alpha) \cdot \phi(y_1)^{\alpha_1} ... \phi(y_n)^{\alpha_n}$$

Thus $\phi(c_\alpha) \in (\phi(y_1), ..., \phi(y_n))$ for every $\alpha \in \Lambda$. Since ϕ composed with $S[[\phi(y_1), ..., \phi(y_n)]] \twoheadrightarrow S$ is the quotient morphism $R \twoheadrightarrow S$, we derive that

$$c_\alpha \bmod (y_1, ..., y_n) = \phi(c_\alpha) \bmod (\phi(y_1), ..., \phi(y_n)) = 0$$

for every $\alpha \in \Lambda$. Thus $c_\alpha \in (y_1, ..., y_n)$ for every $\alpha \in \Lambda$, which implies that $z \in (y_1, ..., y_n)^{m+1}$. Thus we proved that

$$\phi(z) = 0 \text{ and } z \in (y_1, ..., y_n)^m \Rightarrow z \in (y_1, ..., y_n)^{m+1}$$

By $\mathfrak{m}$-adic completeness of R this implies that $\phi(z) = 0$ if and only if $z = 0$. Hence ϕ is also injective. □

Lemma 3.6.2. *Let $(R, \mathfrak{m})$ be a complete local noetherian ring and let $R \to S$ be a local morphism into a local noetherian ring. Suppose that there exists a finitely generated R-submodule N of S such that*

$$S = N + \mathfrak{m}S$$

Then $S = N$.

Proof of the lemma. Pick s in S. Since $S = N + \mathfrak{m}S$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $x_n \in \mathfrak{m}^n N$ and

$$s - \sum_{i \le n} x_i \in \mathfrak{m}^{n+1} S$$

According to the assumption that $(R, \mathfrak{m})$ is complete with respect to $\mathfrak{m}$-adic topology and N is finitely generated over R, we deduce that N is complete with respect to $\mathfrak{m}$-adic topology. Hence there exists a unique element x in N such that

$$x = \sum_{n \in \mathbb{N}} x_n$$

where above series is convergent with respect to $\mathfrak{m}$-adic topology. Note also that

$$x - \sum_{i \le n} x_i \in \mathfrak{m}^{n+1} N$$

for every $n \in \mathbb{N}$. Thus we have

$$s - x = \left(s - \sum_{i \le n} x_i\right) - \left(x - \sum_{i \le n} x_i\right) \in \mathfrak{m}^{n+1} S + \mathfrak{m}^{n+1} N = \mathfrak{m}^{n+1} S$$

for every $n \in \mathbb{N}$. Hence

$$x - s \in \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n S$$

Since $R \to S$ is local morphism and S is a local ring, we deduce that $\mathfrak{m}S$ is contained in the maximal ideal of S. By assumptions S is noetherian. Therefore, S is separated with respect to $\mathfrak{m}$-adic topology. This implies that

$$0 = \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n S$$

Thus $s - x = 0$ and we infer that s is an element of N. This completes the proof that $S = N$. □

In what follows we shall denote by $\mathbf{G}x$ the closed subscheme determined by the orbit morphism $\mathbf{G} \to X$ of a k-point x of X which has a closed free orbit. For readers convienience we include the following lemma of topological flavour.

Lemma 3.6.3. *Let $q : X \to Y$ be a geometric quotient and assume that Y is the spectrum of a local k-algebra such that the residue field of the closed point o of Y is k. Let x be a k-point of X with free closed orbit, then $q^{-1}(o) = \mathbf{G}x$ as closed subschemes of X.*

Proof of the lemma. Morphism q induces the morphism of residue fields $k(q(x)) \hookrightarrow k(x) = k$ over k. This implies that $k(q(x)) = k$ and hence $q(x)$ is a k-point of Y. Note that o is the unique k-point of Y. Thus $q(x) = o$. Clearly $q^{-1}(o)$ is a closed $\mathbf{G}$-stable subscheme of X (it is the preimage of o under $\mathbf{G}$-equivariant q), that contains x. Since $\mathbf{G}x$ is the smallest closed $\mathbf{G}$-stable subscheme of X containing x, we deduce that $\mathbf{G}x \subseteq q^{-1}(o)$ scheme-theoretically. Consider the pair of arrows

$$\mathbf{G} \times_k X \overset{a}{\underset{\mathrm{pr}_X}{\rightrightarrows}} X$$

Passing to functors of points we obtain that $a^{-1}(\mathbf{G}x) = \mathrm{pr}_X(\mathbf{G}.x)$. Since q is the cokernel of the pair (a, pr_X) in the category of topological spaces, we deduce that there exists a closed subset Z of Y such that $q^{-1}(Z) = \mathbf{G}x$. Clearly $o \in Z$ and hence $q^{-1}(o) \subseteq \mathbf{G}x$ set-theoretically. On the other hand above we proved that $\mathbf{G}x \subseteq q^{-1}(o)$ scheme-theoretically. This can only happen if $q^{-1}(o) = \mathbf{G}x$ as closed subschemes of X. □

Proof of the theorem. Denote by o the closed point of Y and by e the unit of $\mathbf{G}$. We also denote $Y = \operatorname{Spec} R$ where $(R, \mathfrak{m}, k)$ is a complete local noetherian k-algebra. We first prove **(1)**. Assume that x is a k-point of X which has a closed free orbit. Consider the surjective morphism $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{\mathbf{G},e}$ induced by the orbit morphism $\mathbf{G} \hookrightarrow X$ of x. Since $\mathbf{G}$ is smooth over k, the ring $\mathcal{O}_{\mathbf{G},e}$ is regular. Pick a system of parameters $x_1, ..., x_n$ of $\mathcal{O}_{\mathbf{G},e}$ and let $y_1, ..., y_n$ be elements of $\mathcal{O}_{X,x}$ such that y_i is send to x_i by the morphism $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{\mathbf{G},e}$ for $1 \leq i \leq n$. Define S to be the quotient ring $\mathcal{O}_{X,x}/(y_1, ..., y_n)$. Let $f : \mathbf{G} \times_k \operatorname{Spec} S \to X$ be the unique $\mathbf{G}$-equivariant morphism induced by the surjection $\mathcal{O}_{X,x} \twoheadrightarrow S$. The morphism q induces the morphism $q^{\#} : \mathcal{O}_{Y,o} \to \mathcal{O}_{X,x}$ and hence the morphism $\mathcal{O}_{Y,o} \to S$. By Lemma 3.6.3 we have

$$S/\mathfrak{m}_o S = k$$

where $\mathfrak{m}_o$ is the maximal ideal of $\mathcal{O}_{Y,o}$. Next by Lemma 3.6.2 we derive that $\mathcal{O}_{Y,o} \to S$ is surjective. Let $f : \mathbf{G} \times_k \operatorname{Spec} S \to X$ be the morphism induced by the surjection $\mathcal{O}_{X,x} \twoheadrightarrow S$. We have a commutative square

$$\begin{tikzcd}
\mathbf{G} \times_k \operatorname{Spec} S \arrow[r, "f"] \arrow[d, "\mathrm{pr}_{\operatorname{Spec} S}"'] & X \arrow[d, "q"] \\
\operatorname{Spec} S \arrow[r, hook] & Y
\end{tikzcd}$$

where bottom arrow is a closed immersion induced by $\mathcal{O}_{Y,o} \twoheadrightarrow S$. According to assumptions q is locally of finite type. Moreover, $\mathbf{G}$ is locally algebraic group over k and hence $\mathrm{pr}_{\operatorname{Spec} S}$ is locally of finite type. These two facts together with the fact that $\operatorname{Spec} S \hookrightarrow Y$ is a closed immersion (and thus is of finite type) imply that f is locally of finite type. Then by Lemma 3.6.1 we deduce that f induces an isomorphism

$$\widehat{\mathcal{O}_{X,x}} \longrightarrow \hat{S}[[x_1, ..., x_n]] = \widehat{S \otimes_k \mathcal{O}_{\mathbf{G},e}}$$

of completions. Since f is locally of finite type, it follows that f is étale at point $(e, u) \in \mathbf{G} \times_k \operatorname{Spec} S$, where u is the unique closed point of $\operatorname{Spec} S$.

□

References

[Monygham, 2019] Monygham (2019). Locally ringed spaces. *github repository: "Monygham/Pedo-mellon-a-minno"*.