ABELIAN CATEGORIES - SOME RESULTS

1. Directed classes and filtered categories

In this section we introduce the notion of directed class and then we generalize it to the categorical setting.

Definition 1.1. Let I be a class equipped with a reflexive and transitive relation $\leq$. We say that I is *a directed class* if for any $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Let I be a class equipped with a reflexive and transitive relation $\leq$. It is standard [Mac Lane, 1998, page 11] to view I as a category with at most one arrow between any two objects.

Definition 1.2. Let I be a category. Suppose that the following conditions are satisfied.

(1) For any objects $i, j \in I$ there exists an object $k \in I$ and a diagram

$$\begin{tikzcd} & k & \\ i \arrow[ur] & & j \arrow[ul] \end{tikzcd}$$

(2) For any pair of parallel morphisms in I

$$i \rightrightarrows j$$

there exist an object $k \in I$ and a morphism $j \to k$ such that, the diagram

$$i \rightrightarrows j \longrightarrow k$$

is commutative.

Then we say that I is *a filtered category*.

Fact 1.3. *Let I be a class equipped with a reflexive and transitive relation. Then I is a directed class if and only if I viewed as a category is filtered.*

Proof. Left to the reader. □

Fact 1.4. *Let I be a filtered category and J be a class equipped with a reflexive and transitive relation $\leq$. Suppose that $F : I \to J$ is a functor. Then the image of I under F is a directed subclass of J.*

Proof. Left to the reader. □

Definition 1.5. Let $\mathcal{C}$ be a category and I be a filtered category (directed class). A colimit of an I-indexed some diagram $I \to \mathcal{C}$ is called *a filtered (directed) colimit*.

2. Subobjects and objects of finite type

Definition 2.1. Let $\mathcal{C}$ be a category and X be an object in $\mathcal{C}$. Monomorphisms $X_1 \hookrightarrow X$ and $X_2 \hookrightarrow X$ are *equivalent* if there exists a commutative triangle

$$\begin{tikzcd} X_1 \arrow[rr, "\cong"] \arrow[dr, hook] & & X_2 \arrow[dl, hook'] \\ & X & \end{tikzcd}$$

in which horizontal arrow is an isomorphism. The collection $\mathrm{Sub}(X)$ of equivalence classes of monomorphisms having X as a target is called *the class of subobjects of* X.

Let X be an object of a category $\mathcal{C}$ and $X' \hookrightarrow X$ be a monomorphism. By abuse of notation we say that X' is an subobject of X and by this we understand the subobject of X represented by a monomorphism $X' \hookrightarrow X$. We also write $X' \subseteq X$. Next suppose that $X_1 \subseteq X$ and $X_2 \subseteq X$ are subobjects of X. We write $X_1 \subseteq X_2$ if there exists a commutative triangle

$$\begin{tikzcd} X_1 \arrow[rr, hook] \arrow[dr, hook] & & X_2 \arrow[dl, hook'] \\ & X & \end{tikzcd}$$

This defines a partial order on the class $\mathrm{Sub}(X)$.
Now we investigate important notion of well-powered categories. For this we introduce this it with a company of several other significant concepts.

Definition 2.2. A category $\mathcal{C}$ is called *well-powered* if $\mathrm{Sub}(X)$ is a set for every object X in $\mathcal{C}$.

Definition 2.3. Let $\mathcal{C}$ be a category such that a morphism in $\mathcal{C}$ that is simultaneously a monomorphism and an epimorphism is an isomorphism. Then we say that $\mathcal{C}$ is *balanced*.

Definition 2.4. Let $\mathcal{C}$ be a category. A class $\mathcal{G}$ of objects of $\mathcal{C}$ is called *a class of generators for* $\mathcal{C}$ if for any pair of distinct and parallel arrows

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

there exists $G \in \mathcal{G}$ and a morphism $h : G \to X$ such that $f \cdot h \neq g \cdot h$.

Proposition 2.5. *Let $\mathcal{C}$ be a balanced, locally small category that admits fiber products. Assume that $\mathcal{G}$ is a* ***set*** *of generators for $\mathcal{C}$. Then $\mathcal{C}$ is well-powered.*

Proof. Fix an object X of $\mathcal{C}$. Then every subobject X' of X gives rise to a set

$$\mathrm{Factor}(X') = \left\{ f \in \mathrm{Mor}(\mathcal{C}) \,\middle|\, \mathrm{dom}(f) \in \mathcal{G},\, \mathrm{cod}(f) = X \text{ and } f \text{ factors through } X' \right\} \subseteq \prod_{G \in \mathcal{G}} \mathrm{Mor}_{\mathcal{C}}(G, X)$$

It suffices to prove that

$$\mathrm{Sub}(X) \ni X' \mapsto \mathrm{Factor}(X') \in \mathcal{P}\left(\prod_{G \in \mathcal{G}} \mathrm{Mor}_{\mathcal{C}}(G, X) \right)$$

is injective (because a class bijective with a set is a set itself). For this assume that X_1 and X_2 are subobjects of X such that $\mathrm{Factor}(X_1) = \mathrm{Factor}(X_2)$. Since $\mathcal{C}$ admits fiber products, we derive that there exists a fiber product of $X_1 \hookrightarrow X$ and $X_2 \hookrightarrow X$. We denote it by $X_1 \cap X_2$ and consider it as a subobject of X via the canonical map $X_1 \cap X_2 \hookrightarrow X$. By universal property of fiber product we deduce that $\mathrm{Factor}(X_1) = \mathrm{Factor}(X_1 \cap X_2) = \mathrm{Factor}(X_2)$. This implies that for every object Y of $\mathcal{C}$ maps

$$\mathrm{Mor}_{\mathcal{C}}(X_1, Y) \to \mathrm{Mor}_{\mathcal{C}}(X_1 \cap X_2, Y),\, \mathrm{Mor}_{\mathcal{C}}(X_2, Y) \to \mathrm{Mor}_{\mathcal{C}}(X_1 \cap X_2, Y)$$

induced by $X_1 \cap X_2 \hookrightarrow X_1$ and $X_1 \cap X_2 \hookrightarrow X_2$ are injective. Therefore, morphisms $X_1 \cap X_2 \hookrightarrow X_1$ and $X_1 \cap X_2 \hookrightarrow X_2$ are epimorphisms. Since they are also monomorphisms and $\mathcal{C}$ is balanced, they are isomorphisms. Thus X_1, $X_1 \cap X_2$, X_2 represent the same subobject of X. $\square$

Definition 2.6. Let $\mathcal{C}$ be a category and X be an object in $\mathcal{C}$. *A filtered (directed) family of subobjects of* X is a functor $I \to \mathrm{Sub}(X)$ from a small filtered category (directed set) I.

Suppose that $\mathcal{C}$ is a category, X is an object of $\mathcal{C}$ and I is a filtered category. Let $I \to \mathrm{Sub}(X)$ be a filtered family of subobjects of X. Then it can be described as a map $I \ni i \mapsto X_i \in \mathrm{Sub}(X)$ such that for every morphism $i \to j$ in I we have $X_i \subseteq X_j$. For pragmatical reasons we usually use this more explicit description and we view filtered families of subobjects as an indexed families of the form $\{X_i\}_{i \in I}$.

Definition 2.7. Let $\mathcal{C}$ be a category and X be an object in $\mathcal{C}$. A filtered family of subobjects $\{X_i\}_{i\in I}$ of X is *complete* if $X = \mathrm{colim}_{i\in I} X_i$

Definition 2.8. Let $\mathcal{C}$ be a category and X be an object in $\mathcal{C}$. Suppose that for every complete filtered family $\{X_i\}_{i\in I}$ of subobjects of X there exists $i_0 \in I$ such that $X_{i_0} = X$ for every $i \in I$. Then we say that X is *of finite type*.

Definition 2.9. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is *locally finite* if for every object X there exists a complete filtered family $\{X_i\}_{i\in I}$ of subobjects of X such that X_i is of finite type for every $i \in I$.

3. Abelian Categories - definition and statement of the embedding theorem

Definition 3.1. Let $\mathcal{C}$ be a locally small category. Suppose that the following conditions hold.

(1) $\mathcal{C}$ has a zero object (i.e. an object that is both initial and terminal).

(2) $\mathcal{C}$ has finite limits and colimits.

(3) Each epimorphism in $\mathcal{C}$ is a cokernel of some arrow and each monomorphism in $\mathcal{C}$ is a kernel of some arrow.

Then $\mathcal{C}$ is called *an abelian category*.

The definition of an abelian category is taken from [Freyd, 1964]. This differs from the more popular definition [Mac Lane, 1998, page 198] - it does not assume the existence of **Ab**-enrichment (abelian group structure on sets of morphisms). The two notions are equivalent by [Freyd, 1964, Theorem 2.39].
The next result shows that all elementary properties of categories of modules over a ring hold for general abelian categories.

Theorem 3.2 (Freyd-Mitchell embedding)**.** *Let $\mathcal{C}$ be a small abelian category. Then there exists a ring R and a full, faithful and exact functor $E : \mathcal{C} \to \mathbf{Mod}(R)$.*

This is [Freyd, 1964, Theorem 7.34].

4. Ab-conditions in categories

In this section we discuss Grothendieck's **Ab**-conditions in categories. The original source of the discussion below is the seminal work [Grothendieck, 1957]. First let us explain that Grothendieck introduces **Ab**0-*categories* as additive categories, **Ab**1-*categories* as preabelian categories and **Ab**2-*categories* as the usual abelian categories. Then he continues with more specific classes of abelian categories and recapitulation of parts of his work is our main task here.

Definition 4.1. A category $\mathcal{C}$ is *an* **Ab**3-*category* if it is an abelian category and small direct sums in $\mathcal{C}$ exists.

Let $\mathcal{C}$ be an **Ab**3-category, $\{X_i\}_{i\in I}$ be a family of objects of $\mathcal{C}$ and let $\{f_i : X_i \to Y\}_{i\in I}$ be a family of morphisms in $\mathcal{C}$. Then we denote by

$$\sum_{i\in I} f_i : \bigoplus_{i\in I} X_i \to Y$$

a unique morphism determined by requirement $(\sum_{i\in I} f_i)\cdot v_i = f_i$ for each $i \in I$, where $v_i : X_i \to \bigoplus_{i\in I} X_i$ is the canonical inclusion.

Theorem 4.2. *Let $\mathcal{C}$ be an* **Ab**3*-category and let* $\big(\{X_i\}_{i\in I}, \{u_\alpha\}_{\alpha\in \mathrm{Mor}(I)}\big)$ *be a diagram indexed by a small category I. Consider a right exact sequence*

$$\bigoplus_{\alpha\in\mathrm{Mor}(I)} X_{\mathrm{dom}(\alpha)} \xrightarrow{\sum_{\alpha\in\mathrm{Mor}(I)}\left(v_{\mathrm{cod}(\alpha)}\cdot u_\alpha - v_{\mathrm{dom}(\alpha)}\right)} \bigoplus_{i\in I} X_i \xrightarrow{\quad q \quad} X$$

where for each i morphism $v_i : X_i \to \bigoplus_{i\in I} X_i$ is canonical. Define $u_i = q\cdot v_i$ for every $i\in I$. Then $(X, \{u_i\}_{i\in I})$ is a colimiting cone of $\big(\{X_i\}_{i\in I}, \{u_\alpha\}_{\alpha\in\mathrm{Mor}(I)}\big)$.

Proof. This is a reformulation of the dual statement to [Mac Lane, 1998, page 113, Theorem 1] and we left it to the reader. □

Definition 4.3. A category $\mathcal{C}$ is *an* **Ab**4*-category* if it is **Ab**3-category and small direct sums in $\mathcal{C}$ are exact.

Fact 4.4. *Let $\mathcal{C}$ be an* **Ab**3*-category and assume that small filtered colimits in $\mathcal{C}$ are exact. Then $\mathcal{C}$ is an* **Ab**4*-category.*

Proof. Finite direct sums are exact in arbitrary abelian categories and a small direct sum is a filtered colimit of finite direct sums taken over finite subsets of the indexing set. Thus the result follows. □

Definition 4.5. Let $\mathcal{C}$ be an **Ab**3-category. If for every object X of $\mathcal{C}$, every subobject $Y\subseteq X$ and every directed family $\{X_i\}_{i\in I}$ of subobjects of X the following formula holds

$$Y\cap\sum_{i\in I} X_i = \sum_{i\in I} Y\cap X_i$$

then we say that $\mathcal{C}$ is an **Ab**5-category.

The next result is very useful.

Theorem 4.6. *Let $\mathcal{C}$ be an* **Ab**5*-category and let* $\big(\{X_i\}_{i\in I}, \{u_\alpha\}_{\alpha\in\mathrm{Mor}(I)}\big)$ *be a diagram indexed by a small filtered category I. Suppose that $(X, \{u_i\}_{i\in I})$ is a colimiting cone of* $\big(\{X_i\}_{i\in I}, \{u_\alpha\}_{\alpha\in\mathrm{Mor}(I)}\big)$. *Then*

$$\ker(u_j) = \sum_{\alpha\in\mathrm{Mor}(I),\,\mathrm{dom}(\alpha)=j} \ker(u_\alpha)$$

for every $j\in I$.

The following result is a direct consequence of definition of **Ab**5-category.

Lemma 4.6.1. *Let $\mathcal{C}$ be an* **Ab**5*-category, X be its object and $\{X_i\}_{i\in I}$ be a filtered family of subobjects of X. Assume also that $f : Y\to X$ is a monomorphism. Then*

$$f^{-1}\left(\sum_{i\in I} X_i\right) = \sum_{i\in I} f^{-1}(X_i)$$

Proof of the lemma. It suffices to prove that

$$f(Y)\cap\sum_{i\in I} X_i = \sum_{i\in I} f(Y)\cap X_i$$

From Fact 1.4 and applying factorization of $I\to\mathrm{Sub}(X)$ through its image we can view $\{X_i\}_{i\in I}$ as a directed family of subobjects of X. Thus the result follows from the fact that $\mathcal{C}$ is **Ab**5. □

Proof of the theorem. Obviously

$$\sum_{\alpha\in\mathrm{Mor}(I),\,\mathrm{dom}(\alpha)=j} \ker(u_\alpha) \subseteq \ker(u_j)$$

It suffices to prove that the reverse inclusion holds. For every i in I denote by $v_i : X_i \to \bigoplus_{i\in I} X_i$ the canonical morphism. By Theorem 4.2 we have

$$\ker(u_j) = v_j^{-1}\left(\mathrm{im}\left(\sum_{\alpha\in\mathrm{Mor}(I)} v_{\mathrm{cod}(\alpha)}\cdot u_\alpha - v_{\mathrm{dom}(\alpha)}\right)\right) = v_j^{-1}\left(\mathrm{im}\left(\sum_{F\subseteq\mathrm{Mor}(I),\,|F|\in\mathbb{N}}\ \sum_{\alpha\in F}\left(v_{\mathrm{cod}(\alpha)}\cdot u_\alpha - v_{\mathrm{dom}(\alpha)}\right)\right)\right)$$

Since $\mathcal{C}$ is **Ab**5-category and by Lemma 4.6.1, we deduce that

$$\ker(u_j) = \sum_{F\subseteq\mathrm{Mor}(I),\,|F|\in\mathbb{N}} v_j^{-1}\left(\mathrm{im}\left(\sum_{\alpha\in F} v_{\mathrm{cod}(\alpha)}\cdot u_\alpha - v_{\mathrm{dom}(\alpha)}\right)\right)$$

Thus it suffices to verify that

$$v_j^{-1}\left(\mathrm{im}\left(\sum_{\alpha\in F} v_{\mathrm{cod}(\alpha)}\cdot u_\alpha - v_{\mathrm{dom}(\alpha)}\right)\right) \subseteq \sum_{\alpha\in\mathrm{Mor}(I),\,\mathrm{dom}(\alpha)=j} \ker(u_\alpha)$$

for every finite subset F of $\mathrm{Mor}(I)$. Fix such F and suppose that $\{i_1,...,i_n\}$ are all objects of I, which are either domains or codomains of arrows in F. If $j\notin\{i_1,...,i_n\}$, then

$$v_j^{-1}\left(\mathrm{im}\left(\sum_{\alpha\in F} v_{\mathrm{cod}(\alpha)}\cdot u_\alpha - v_{\mathrm{dom}(\alpha)}\right)\right) = 0$$

So we may assume that $j\in\{i_1,...,i_n\}$. Consider a finite diagram in I that consists of $\{i_1,...,i_n\}$ and all arrows in F. Since I is filtered, there exists a cocone over this diagram. Hence there exist i_0 in I and a family of morphisms $\beta_{i_k} : i_k \to i_0$ for $1\le k\le n$ in I such that $\beta_{\mathrm{dom}(\alpha)} = \beta_{\mathrm{cod}(\alpha)}\cdot\alpha$ for every $\alpha\in F$. Define $f_i = 0$ for $i\in I\smallsetminus\{i_1,...,i_n\}$ and $f_i = u_{\beta_i}$ for $i\in\{i_1,...,i_n\}$. Let $f = \sum_{i\in I} f_i$. We have a commutative square

$$\begin{tikzcd}
X_j \arrow[r, "u_{\beta_j}"] \arrow[d, "v_j"'] & X_{i_0} \arrow[d, "v_{i_0}"] \\
\bigoplus_{i\in I} X_i \arrow[r, "f"'] & \bigoplus_{i\in I} X_i
\end{tikzcd}$$

and hence it follows that

$$u_{\beta_j}\left(v_j^{-1}\left(\mathrm{im}\left(\sum_{\alpha\in F} v_{\mathrm{cod}(\alpha)}\cdot u_\alpha - v_{\mathrm{dom}(\alpha)}\right)\right)\right) \subseteq v_{i_0}^{-1}\left(f\left(\mathrm{im}\left(\sum_{\alpha\in F} v_{\mathrm{cod}(\alpha)}\cdot u_\alpha - v_{\mathrm{dom}(\alpha)}\right)\right)\right) =$$

$$= v_{i_0}^{-1}\left(\mathrm{im}\left(\sum_{\alpha\in F} v_{i_0}\cdot\left(u_{\beta_{\mathrm{cod}(\alpha)}}\cdot u_\alpha - u_{\beta_{\mathrm{dom}(\alpha)}}\right)\right)\right) = 0$$

and thus

$$v_j^{-1}\left(\mathrm{im}\left(\sum_{\alpha\in F} v_{\mathrm{cod}(\alpha)}\cdot u_\alpha - v_{\mathrm{dom}(\alpha)}\right)\right) \subseteq \ker(u_{\beta_j})$$

This finishes the proof. □

Theorem 4.7. *Let $\mathcal{C}$ be an* **Ab**3*-category. Then the following are equivalent.*

(i) *$\mathcal{C}$ is an* **Ab**5*-category.*

(ii) *Small filtered colimits are exact in $\mathcal{C}$.*

Proof. The nontrivial part is **(i)** ⇒ **(ii)**. Let I be a small filtered category and

$$\left\{0 \longrightarrow X_i' \xrightarrow{r_i} X_i \xrightarrow{p_i} X_i'' \longrightarrow 0\right\}_{i\in I}$$

be a diagram of exact sequences indexed by I. We denote by $\left(\{X_i\}_{i\in I}, \{u_\alpha\}_{\alpha\in \mathrm{Mor}(I)}\right)$ and $\left(\{X'_i\}_{i\in I}, \{v_\alpha\}_{\alpha\in \mathrm{Mor}(I)}\right)$ appropriate slices of this I-indexed diagram. Consider a complex

$$0 \longrightarrow X' \xrightarrow{r} X \xrightarrow{p} X'' \longrightarrow 0$$

where $X' = \mathrm{colim}_{i\in I} X'_i$, $X = \mathrm{colim}_{i\in I} X_i$, $X'' = \mathrm{colim}_{i\in I} X''_i$, $r = \mathrm{colim}_{i\in I} r_i$ and $p = \mathrm{colim}_{i\in I} p_i$. Clearly the complex is right exact. It suffices to prove that r is a monomorphism. For $i \in I$ denote by $v_i : X'_i \to X$, $u_i : X_i \to X$ structural morphisms. Fix $j \in I$ and consider $Z = v_j^{-1}(\ker(r))$. Then $r_j(Z) \subseteq \ker(u_j)$ and hence by Theorem 4.6 we derive that

$$r_j(Z) \subseteq \sum_{\alpha\in \mathrm{Mor}(I),\, \mathrm{dom}(\alpha)=j} \ker(u_\alpha)$$

Note that for every $\alpha \in \mathrm{Mor}(I)$ with $\mathrm{dom}(\alpha) = j$ we have $r_j^{-1}\left(\ker(u_\alpha)\right) \subseteq \ker(v_\alpha)$. Indeed, this follows easily from the fact that $r_{\mathrm{cod}(\alpha)}$ is an monomorphism. Thus by Lemma 4.6.1, Theorem 4.6 and the fact that preimages preserve intersections we deduce that

$$Z = r_j^{-1}\left(r_j(Z)\right) = r_j^{-1}\left(\sum_{\alpha\in \mathrm{Mor}(I),\, \mathrm{dom}(\alpha)=j} r_j(Z)\cap \ker(u_\alpha)\right) = \sum_{\alpha\in \mathrm{Mor}(I),\, \mathrm{dom}(\alpha)=j} Z\cap r_j^{-1}\left(\ker(u_\alpha)\right) \subseteq$$
$$\subseteq \sum_{\alpha\in \mathrm{Mor}(I),\, \mathrm{dom}(\alpha)=j} Z\cap \ker(v_\alpha) = Z\cap \sum_{\alpha\in \mathrm{Mor}(I),\, \mathrm{dom}(\alpha)=j} \ker(v_\alpha) = Z\cap \ker(v_j)$$

and this implies that $Z \subseteq \ker(v_j)$. Since $Z = v_j^{-1}(\ker(r))$, we deduce that

$$0 = v_j(Z) = \ker(r)\cap \ker(v_j)$$

Now $\{\mathrm{im}(v_i)\}_{i\in I}$ is a filtered complete family of subobjects of X'. Hence by Lemma 4.6.1 we deduce that

$$\ker(r) = \ker(r)\cap \sum_{i\in I}\mathrm{im}(v_i) = \sum_{i\in I}\ker(r)\cap \mathrm{im}(v_i) = 0$$

and thus r is a monomorphism. □

Corollary 4.8. *Every* **Ab**5*-category is* **Ab**4*-category.*

Proof. This is a consequence of Fact 4.4 and Theorem 4.7. □

5. Finite type and finite presentation objects in Ab5-categories

In this section we investigate properties of finite type objects and related notion of finitely presented objects in **Ab**5-categories.

Proposition 5.1. *Let $\mathcal{C}$ be an* **Ab**5*-category and consider a short exact sequence*

$$0 \longrightarrow X'' \longrightarrow X \xrightarrow{f} X' \longrightarrow 0$$

Then the following assertions hold.

(1) *If X is of finite type, then X' is of finite type.*

(2) *If X'' and X' are of finite type, then X is of finite type.*

Proof. For the proof of **(1)** consider a complete filtered family $\{X'_i\}_{i\in I}$ of subobjects of X'. Then $\{f^{-1}(X'_i)\}_{i\in I}$ is a complete filtered family of subobjects of X. Now X is of finite type. Thus there exists $i_0 \in I$ such that $f^{-1}(X'_{i_0}) = X$. Hence $X' = X'_{i_0}$. This shows that X' is of finite type.
Now we prove **(2)**. Let $\{X_i\}_{i\in I}$ be a complete filtered family of subobjects of X. Since $\mathcal{C}$ is an **Ab**5-category, we derive that $\{X''\cap X_i\}_{i\in I}$ is a complete filtered family of subobjects of X''. Moreover,

$\{f(X_i)\}_{i\in I}$ is a complete filtered family of subobjects of X'. Since both X' and X'' are of finite type, there exists i_0 and i_1 in I such that $X'' = X'' \cap X_{i_0}$ and $X' = f(X_{i_1})$. Suppose that there are morphisms $i_0 \to i_2$ and $i_1 \to i_2$ for some $i_2 \in I$. Then $X'' = X'' \cap X_{i_2}$ and $X' = f(X_{i_2})$. This implies that $X = X_{i_2}$ and hence X is of finite type. □

Proposition 5.2. *Let $\mathcal{C}$ be an* **Ab**5*-category and X be a finite type object of $\mathcal{C}$. Then for every diagram $\big(\{X'_i\}_{i\in I}, \{u_\alpha\}_{\alpha\in \mathrm{Mor}(I)}\big)$ indexed by a small filtered category the canonical morphism*

$$\mathrm{colim}_{i\in I}\mathrm{Mor}_{\mathcal{C}}\left(X, X'_i\right) \longrightarrow \mathrm{Mor}_{\mathcal{C}}\left(X, \mathrm{colim}_{i\in I}X'_i\right)$$

is a monomorphism of abelian groups.

Proof. Denote $\mathrm{colim}_{i\in I}X'_i$ by X' and by $u_i : X'_i \to X'$ the canonical morphism for every $i \in I$. Fix morphisms $g : X \to X'_i$ for some $i \in I$. Assume that $u_i \cdot g = 0$. Then $g(X) \subseteq \ker(u_i)$ and in addition $g(X)$ is of finite type as the image of X (Proposition 5.1). We have

$$\ker(u_i) = \sum_{\alpha\in \mathrm{Mor}(I),\, \mathrm{dom}(\alpha)=i} \ker(u_\alpha)$$

by Theorem 4.6. We use the fact that $\mathcal{C}$ is **Ab**5 to derive that

$$g(X) = g(X) \cap \ker(u_i) = \sum_{\alpha\in \mathrm{Mor}(I),\, \mathrm{dom}(\alpha)=i} g(X) \cap \ker(u_\alpha)$$

Since $g(X)$ is of finite type and $\{g(X) \cap \ker(u_\alpha)\}_{\alpha\in\mathrm{Mor}(I),\,\mathrm{dom}(\alpha)=i}$ is a complete filtered family of subobjects of finite type object $g(X)$, we derive that there exists morphism $\alpha : i \to i_0$ such that

$$g(X) = g(X) \cap \ker(u_\alpha) \subseteq \ker(u_\alpha)$$

Then $u_\alpha \cdot g = 0$ and his implies that g represents zero class in $\mathrm{colim}_{i\in I}\mathrm{Mor}_{\mathcal{C}}\left(X, X'_i\right)$. □

Definition 5.3. Let $\mathcal{C}$ be an abelian category and X be an object of $\mathcal{C}$. Suppose that the following two conditions hold.

(1) X is of finite type.

(2) If $f : X' \to X$ is an epimorphism in $\mathcal{C}$ and X' is of finite type, then $\ker(f)$ is of finite type.

Then we say that X is *of finite presentation*.

Proposition 5.4. *Let $\mathcal{C}$ be an* **Ab**5*-category and X be an object of $\mathcal{C}$. Then the following assertions are equivalent.*

(i) *X is of finite presentation.*

(ii) *There exists an object of finite presentation X' in $\mathcal{C}$ and an epimorphism $f : X' \to X$ with kernel of finite type.*

Proof. For the proof of **(i)** ⇒ **(ii)** it suffices to consider an epimorphism $1_X : X \to X$ with trivial kernel.
Assume that **(ii)** holds. Fix an epimorphism $f : X' \to X$ such that X' is of finite presentation and $\ker(f)$ is of finite type. By Proposition 5.1 object X is of finite type. Next suppose that $g : X'' \to X$ is an epimorphism with X'' of finite type. Consider a commutative diagram

$$\begin{tikzcd}
 & \ker(f') \arrow[r, "\cong"] \arrow[d, hook] & \ker(f) \arrow[d, hook] \\
\ker(g') \arrow[r, hook] \arrow[d, "\cong"'] & Z \arrow[r, "g'"] \arrow[d, "f'"'] & X' \arrow[d, "f"] \\
\ker(g) \arrow[r, hook] & X'' \arrow[r, "g"'] & X
\end{tikzcd}$$

in which bottom right square is cartesian. The fact that canonical morphisms between kernels in the diagram are isomorphisms follows because $\mathcal{C}$ is an abelian category. Note also that every row and column of the diagram is a short exact sequence in $\mathcal{C}$. Since $\ker(f)$ is of finite type, we derive that $\ker(f')$ is of finite type and hence by Proposition 5.1 object Z is of finite type as an extension of finite type objects $\ker(f')$ and X''. Next according to the fact that X' is of finite presentation and $g' : Z \to X'$ is an epimorphism with Z of finite type, we deduce that $\ker(g')$ is of finite type. But $\ker(g') \cong \ker(g)$ and hence $\ker(g)$ is of finite type. This implies that X is of finite presentation. Thus **(ii)** $\Rightarrow$ **(i)**. $\square$

Theorem 5.5. *Let $\mathcal{C}$ be an* **Ab**5*-category and X be an object of $\mathcal{C}$. Consider the following statements.*

(i) *X is of finite presentation.*

(ii) *For every diagram $\left(\{X_i'\}_{i\in I}, \{u_\alpha\}_{\alpha\in \mathrm{Mor}(I)}\right)$ indexed by a small filtered category the canonical morphism*

$$\mathrm{colim}_{i\in I}\mathrm{Mor}_{\mathcal{C}}\left(X, X_i'\right) \longrightarrow \mathrm{Mor}_{\mathcal{C}}\left(X, \mathrm{colim}_{i\in I} X_i'\right)$$

is an isomorphism of abelian groups.

Proof. We prove that **(i)** $\Rightarrow$ **(ii)**. Denote $\mathrm{colim}_{i\in I}X_i'$ by X' and suppose that $u_i : X_i' \to X'$ is the canonical morphism for every $i \in I$. Fix a morphism $f : X \to X'$ in $\mathcal{C}$. Let $\left(\{X_i\}_{i\in I}, \{v_\alpha\}_{\alpha\in\mathrm{Mor}(I)}\right)$ be a diagram obtained by pulling back diagram $\left(\{X_i'\}_{i\in I}, \{u_\alpha\}_{\alpha\in\mathrm{Mor}(I)}\right)$ along f. This means that for every $i \in I$ we have a cartesian square

$$\begin{tikzcd}
X_i \arrow[r, "f_i"] \arrow[d, "v_i"'] & X_i' \arrow[d, "u_i"] \\
X \arrow[r, "f"'] & X'
\end{tikzcd}$$

and if $\alpha : i \to j$ is a morphism in I, then $f_j \cdot v_\alpha = u_\alpha \cdot f_i$. In **Ab**5-category filtered colimits commute with pullbacks (Theorem 4.7). Hence $X = \mathrm{colim}_{i\in I}X_i$. In particular, we have $X = \sum_{i\in I} v_i(X_i)$ and $\{v_i(X_i)\}_{i\in I}$ is a filtered family of subobjects of X. Next as X is of finite type, we deduce that there exists $i_0 \in I$ such that $X = v_{i_0}(X_{i_0})$. Let $\{Z_k\}_{k\in K}$ be a complete filtered family of subobjects of X_{i_0} that consists of objects of finite type ($\mathcal{C}$ is locally finite). Then $\{v_{i_0}(Z_k)\}_{k\in K}$ is a complete filtered family of subobjects of X. Thus there exists $k_0 \in K$ such that $v_{i_0}(Z_{k_0}) = X$. This implies that there exists finite type subobject X'' of X_{i_0} such that $X = v_{i_0}(X'')$. Since X is of finite presentation, we derive that $X'' \cap \ker(v_{i_0})$ is of finite type. On the other hand

$$\ker(v_{i_0}) = \sum_{\alpha\in\mathrm{Mor}(I),\, \mathrm{dom}(\alpha)=i_0} \ker(v_\alpha)$$

by Theorem 4.6. Again we use the fact that $\mathcal{C}$ is **Ab**5 to derive that

$$X'' \cap \ker(v_{i_0}) = \sum_{\alpha \in \mathrm{Mor}(I),\, \mathrm{dom}(\alpha)=i_0} X'' \cap \ker(v_\alpha)$$

Since $X'' \cap \ker(v_{i_0})$ is of finite type and $\{X'' \cap \ker(v_\alpha)\}_{\alpha \in \mathrm{Mor}(I),\, \mathrm{dom}(\alpha)=i_0}$ is a complete filtered family of subobjects of $X'' \cap \ker(v_{i_0})$, we derive that there exists morphism $\alpha : i_0 \to i_1$ such that

$$X'' \cap \ker(v_{i_0}) = X'' \cap \ker(v_\alpha)$$

Let $X''' = v_\alpha(X'')$ be a subobject of X_{i_1}. Since $v_{i_0} = v_{i_1} \cdot v_\alpha$, we derive that v_{i_1} induces an isomorphism $X''' \cong X$. Therefore, $v_{i_1} : X_{i_1} \to X$ admits a section $s : X \to X_{i_1}$. Let $g = f_{i_1} \cdot s$. Then $g : X \to X'_{i_1}$ is a morphism such that $u_{i_1} \cdot g = f$ and this implies that the canonical morphism

$$\mathrm{colim}_{i \in I} \mathrm{Mor}_{\mathcal{C}}\left(X, X'_i\right) \longrightarrow \mathrm{Mor}_{\mathcal{C}}\left(X, X'\right)$$

is surjective. The injectivity follows from Proposition 5.2.
Now let us prove that **(ii)** $\Rightarrow$ **(i)**. Since $\mathcal{C}$ is locally finite, there exists a complete filtered family $\{X_i\}_{i \in I}$ of subobjects of X and these family consists of objects of finite type in $\mathcal{C}$. Applying **(ii)** to this particular filtered diagram we deduce that there exists i_0 in I and a morphism $g : X \to X_{i_0}$ such that g composed with $X_{i_0} \hookrightarrow X$ is 1_X. This implies that g is an isomorphism and $X = X_{i_0}$. Hence X is of finite type. Thus in order to prove that X is of finite presentation it suffices to check that an epimorphism $f : X' \to X$ with X' of finite type has the kernel of finite type. For this let $\{K_i\}_{i \in I}$ be a complete filtered family of subobjects of $\ker(f)$ that consists of objects of finite type in $\mathcal{C}$. We define $X_i = X'/K_i$ for every $i \in I$. Then $\{X_i\}_{i \in I}$ gives rise to a canonical I-indexed diagram such that we have an identification

$$X'/\ker(f) = \mathrm{colim}_{i \in I} X_i$$

We apply **(ii)** to this I-indexed diagram and obtain i_0 in I together with a morphism $s : X \to X_{i_0}$ such that s composed with the canonical epimorphism $X_{i_0} \to X'/\ker(f)$ yields and isomorphism $X \cong X'/\ker(f)$ induced by f. This implies that $X_{i_0} \cong X \oplus \left(\ker(f)/K_{i_0}\right)$. Since both X_{i_0} and X are of finite type, we deduce that $\ker(f)/K_{i_0}$ is of finite type. Moreover, K_{i_0} is of finite type. This proves that $\ker(f)$ is of finite type. □

6. Applications to modules over a ring

Let R be a ring. We denote by $\mathbf{Mod}(R)$ the category of left R-modules.

Fact 6.1. *Let M be a left R-module and $\{M_i\}_{i \in I}$ be a filtered family of submodules of M. Then*

$$\sum_{i \in I} M_i = \bigcup_{i \in I} M_i$$

Proof. Left to the reader. □

Proposition 6.2. $\mathbf{Mod}(R)$ *is a locally finite category and an object M of* $\mathbf{Mod}(R)$ *is of finite type if and only if M is finitely generated left R-module.*

Proof. Suppose that M is a finitely generated left R-module and $\{M_i\}_{i \in I}$ is a complete filtered family of submodules of M. By Fact 6.1 we derive that

$$M = \bigcup_{i \in I} M_i$$

Let $m_1,...,m_n$ be generators of M. Then for every $1 \le j \le n$ there exists $i_j \in I$ such that $m_j \in M_{i_j}$. According to the fact that I is filtered there exists $i_0 \in I$ and morphisms $i_j \to i_0$ for every $1 \le j \le n$. This implies that $m_j \in M_{i_0}$ for every $1 \le j \le n$ and hence $M = M_{i_0}$. Therefore, M is an object of finite type in $\mathbf{Mod}(R)$.

Let M be an arbitrary left R-module and $\mathcal{F}$ be a family of its finitely generated submodules. Since sum of two finitely generated submodules of a given module is finitely generated, we deduce that $\mathcal{F}$ is directed. Moreover, M together with embeddings $N \hookrightarrow M$ for $N \in \mathcal{F}$ is the colimit of $\mathcal{F}$. Therefore, every object in $\mathbf{Mod}(R)$ admits a complete filtered family of subobjects of finite type. Now suppose that M itself is of finite type in $\mathbf{Mod}(R)$. Then $M = N$ for some $N \in \mathcal{F}$ and hence M is finitely generated. □

Proposition 6.3. *$\mathbf{Mod}(R)$ is an $\mathbf{Ab}5$-category.*

Proof. Fix an R-module M. Let $\{M_i\}_{i\in I}$ be a directed family of its submodules and $N \subseteq M$ be a submodule. By Fact 6.1 we have

$$N \cap \sum_{i\in I} M_i = N \cap \bigcup_{i\in I} M_i = \bigcup_{i\in I} (N \cap M_i) = \sum_{i\in I} N \cap M_i$$

□

Corollary 6.4. *Let M be a left R-module. Then the following statements are equivalent.*

(i) *There exist $n, m \in \mathbb{N}$ and a right exact sequence*

$$R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0$$

(ii) *M is a finitely presented object of $\mathbf{Mod}(R)$.*

Proof. Note that $\mathrm{Hom}_R(R^{\oplus n}, -) : \mathbf{Mod}(R) \to \mathbf{Ab}$ is naturally isomorphic with the functor $|-|^{\oplus n}$: $\mathbf{Mod}(R) \to \mathbf{Ab}$ that sends each left R-module to a direct sum of n-copies of its underlying abelian group. By Propositions 6.2 and 6.3, Theorem 5.5 and the fact that $|-|^{\oplus n}$ preserves all colimits, we derive that left R-module $R^{\oplus n}$ is finitely presented in $\mathbf{Mod}(R)$. Now according to Propositions 6.3 and 5.4 we deduce that **(i)** $\Leftrightarrow$ **(ii)**. The converse **(ii)** $\Rightarrow$ **(i)** holds by Proposition 6.2 and definition of finitely presented object in abelian category. □

Example 6.5 (Finitely generated module that is not finitely presented)**.** Let A be a commutative ring and $R = A[x_0, x_1, ...]$ be a polynomial A-algebra with infinitely many free variables. Denote by $\mathfrak{a}$ the ideal $\sum_{i\in\mathbb{N}} R \cdot x_i$ of R. Then $M = R/\mathfrak{a}$ is a finitely generated (even cyclic) R-module. On the other hand the kernel of the canonical epimorphism $R \to R/\mathfrak{a} = M$ is $\mathfrak{a}$ and hence it is not a finitely generated R-module. Thus M is not finitely presented but finitely generated.

7. Noetherian and Artinian objects

Definition 7.1. Let $\mathcal{C}$ be an abelian category. An object X of $\mathcal{C}$ is called *noetherian* (*artinian*) if for every ascending (descending) chain $\{X_n\}_{n\in\mathbb{N}}$ of its subobjects, there exists $n_0 \in \mathbb{N}$ such that $X_n = X_{n_0}$ for $n \geq n_0$.

First we note the following elementary result.

Fact 7.2. *Let $\mathcal{C}$ be an abelian category. Then noetherian objects in $\mathcal{C}^{\mathrm{op}}$ are artinian in $\mathcal{C}$ and vice versa.*

The next result is useful for certain applications.

Proposition 7.3. *Let $\mathcal{C}$ be an abelian category and let X be a noetherian (artinian) object of $\mathcal{C}$. Then for every nonempty class $\mathcal{F}$ of subobjects of X there exists a maximal (minimal) element contained in this family.*

Proof. We present the proof for noetherian case (the proof for artinian case can be obtained by passing to the opposite category). Let $\mathcal{F}$ be a class of subobjects of a noetherian object X. We construct an ascending chain $\{X_n\}_{n\in\mathbb{N}}$ of subobjects of $\mathcal{F}$ as follows. We pick $X_0 \in \mathcal{F}$. Suppose that $X_0 \subseteq ... \subseteq X_n$ are defined. If X_n is a maximal element of $\mathcal{F}$, then we set $X_{n+1} = X_n$. Otherwise we pick $X_{n+1} \in \mathcal{F}$ such that $X_n \subsetneq X_{n+1}$. Since X is noetherian, there exists $n_0 \in \mathbb{N}$ such that $X_n = X_{n_0}$ for $n \geq n_0$. This implies that X_{n_0} is maximal element in $\mathcal{F}$. □

Definition 7.4. Let $\mathcal{C}$ be an abelian category and let $\mathcal{S}$ be its full subcategory. Suppose that for every exact sequence in $\mathcal{C}$

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

we have $X \in \mathcal{S}$ if and only if $X', X'' \in \mathcal{S}$. Then $\mathcal{S}$ is called *a thick subcategory of* $\mathcal{C}$.

Proposition 7.5. *Let $\mathcal{C}$ be an abelian category. Then subcategory of $\mathcal{C}$ consisting of noetherian (artinian) objects is thick.*

Proof. We present the proof for noetherian case (the proof for artinian case can be obtained by passing to the opposite category). Consider a short exact sequence

$$0 \longrightarrow X' \xrightarrow{\ i\ } X \xrightarrow{\ p\ } X'' \longrightarrow 0$$

Suppose first that X is noetherian. Consider an ascending chain $\{X'_n\}_{n\in\mathbb{N}}$ of subobjects of X'. Then it is also an ascending chain of subobjects of X. Hence there exists $n_0 \in \mathbb{N}$ such that $X'_n = X'_{n_0}$ for $n \geq n_0$. This implies that X' is noetherian. Now suppose that $\{X''_n\}_{n\in\mathbb{N}}$ is an ascending chain of subobjects of X''. Then $\{p^{-1}(X''_n)\}_{n\in\mathbb{N}}$ is an ascending chain of subobjects of X. Hence there exists $n_0 \in \mathbb{N}$ such that $p^{-1}(X''_n) = p^{-1}(X''_{n_0})$ for $n \geq n_0$. This shows that $X''_n = X''_{n_0}$ for $n \geq n_0$. Thus X'' is noetherian.
Next suppose that both X' and X'' are noetherian. Let $\{X_n\}_{n\in\mathbb{N}}$ be an ascending chain of subobjects of X. Then $\{i^{-1}(X_n)\}_{n\in\mathbb{N}}$ and $\{p(X_n)\}_{n\in\mathbb{N}}$ are ascending chains of subobjects of X' and X'', respectively. Since X' and X'' are noetherian, we derive that there exists $n_0 \in \mathbb{N}$ such that $i^{-1}(X_n) = i^{-1}(X_{n_0})$ and $p(X_n) = p(X_{n_0})$ for $n \geq n_0$. This implies that $X_n = X_{n_0}$ for $n \geq n_0$ and hence X is noetherian. □

8. Objects of finite length

It is interesting to investigate the case when object is both noetherian and artinian. We need the following definition first.

Definition 8.1. Let $\mathcal{C}$ be an abelian category. An nonzero object X of $\mathcal{C}$ is *irreducible* if the only subobjects of X are zero and X itself.

Proposition 8.2. *Let $\mathcal{C}$ be an abelian category and let X be an object of $\mathcal{C}$. Then the following assertions are equivalent.*

(i) *X is both noetherian and artinian.*

(ii) *There exists a chain*

$$0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq ... \subseteq X_n = X$$

of subobjects of X such that for each $0 \leq i \leq n-1$ the quotient X_{i+1}/X_i is an irreducible object of $\mathcal{C}$.

For the proof we will need the following.

Lemma 8.2.1. *Let $\mathcal{C}$ be an abelian category and let X be an artinian object of $\mathcal{C}$. Suppose that $X' \subsetneq X$ is a subobject. Then there exists a minimal subobject X'' of X such that $X' \subsetneq X''$.*

Proof of the lemma. It suffices to apply Proposition 7.3 to a nonempty class

$$\mathcal{F} = \left\{Z \,\middle|\, Z \text{ is a subobject of } X \text{ strictly containing } X'\right\}$$

□

Proof of the proposition. We prove **(i)** $\Rightarrow$ **(ii)**. Suppose that X is both noetherian and artinian. Since X is artinian, by Lemma 8.2.1 we construct a chain

$$0 = X_0 \subsetneq X_1 \subsetneq \ldots$$

such that for each i the quotient X_{i+1}/X_i is irreducible object of $\mathcal{C}$. Indeed, starting from $X_0 = 0$ we apply the following procedure. If X_i was constructed and $X_i \subsetneq X$, then by Lemma 8.2.1 we pick X_{i+1} as a minimal subobject of X such that $X_i \subsetneq X_{i+1}$. Note that the construction of this ascending chain of subobjects of X cannot go indefinitely, because X is noetherian. Hence we deduce that

$$0 = X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n = X$$

for some $n \in \mathbb{N}$.
Now we prove that **(ii)** $\Rightarrow$ **(i)**. Assume that there exists a chain

$$0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X_n = X$$

of subobjects of X such that for each $0 \leq i \leq n-1$ the quotient X_{i+1}/X_i is an irreducible object of $\mathcal{C}$. Then we prove by finite induction that each X_i is both noetherian and artinian. This clearly holds for X_0. Suppose that it holds for X_i for some $i < n$. Then there exists a canonical short exact chain

$$0 \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow X_{i+1}/X_i \longrightarrow 0$$

Since X_i is both noetherian and artinian and moreover, X_{i+1}/X_i as an irreducible object is both noetherian and artinian, we derive by Proposition 7.5 that X_{i+1} is both noetherian and artinian. □

Definition 8.3. Let $\mathcal{C}$ be an abelian category and let X be an object of $\mathcal{C}$. A chain

$$0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = X$$

of subobjects such that X_{i+1}/X_i is an irreducible object for $0 \leq i \leq n-1$ is called *a composition series of X*. Irreducible objects

$$X_1/X_0, X_2/X_1, \ldots, X_n/X_{n-1}$$

are called *factors of the composition series*.

Let X be an object of a category $\mathcal{C}$. Then we denote by $[X]_{\cong}$ the isomorphism class of X.

Theorem 8.4 (Jordan-Hölder)**.** *Let $\mathcal{C}$ be an abelian category and let X be its object. Suppose that*

$$0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = X$$

and

$$0 = X_0' \subseteq X_1' \subseteq \ldots \subseteq X_m' = X$$

are composition series of X. Then multisets (sets which allow repetitions)

$$\{[X_1/X_0]_{\cong}, [X_2/X_1]_{\cong}, \ldots, [X_n/X_{n-1}]_{\cong}\},\ \{[X_1'/X_0']_{\cong}, [X_2'/X_1']_{\cong}, \ldots, [X_m'/X_{m-1}']_{\cong}\}$$

are equal.

Proof. The proof goes by induction on n. If $n = 1$, then X is irreducible and the result holds. Suppose now $n \geq 1$. Consider object $Z = X/X_1$. Then $Z_i = X_{i+1}/X_1$ for $0 \leq i \leq n-1$. Clearly

$$0 = Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_{n-1} = Z$$

is a composition series of Z and we have

$$\{[Z_1/Z_0]_{\cong}, [Z_2/Z_1]_{\cong}, \ldots, [Z_{n-1}/Z_{n-2}]_{\cong}\} = \{[X_2/X_1]_{\cong}, [X_3/X_2]_{\cong}, \ldots, [X_n/X_{n-1}]_{\cong}\}$$

Consider now objects $W_i = X_i'/(X_i' \cap X_1)$ for $0 \leq i \leq m$. Then we have a chain of subobjects

$$0 = W_0 \subseteq W_1 \subseteq \ldots \subseteq W_m = Z$$

of Z. Pick $i = 0, 1, \ldots, m-1$. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X'_i \arrow[r] \arrow[d, "p_i"'] & X'_{i+1} \arrow[r] \arrow[d, "p_{i+1}"'] & X'_{i+1}/X'_i \arrow[r] \arrow[d, "q_{i+1}"] & 0 \\ 0 \arrow[r] & W_i \arrow[r] & W_{i+1} \arrow[r] & W_{i+1}/W_i \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{C}$ with exact rows and canonical epimorphism as vertical arrows. By Snake Lemma ([Weibel, 1995, page 11]) we derive that there exists an exact sequence

$$0 \to \ker(p_i) \to \ker(p_{i+1}) \to \ker(q_{i+1}) \to \operatorname{coker}(p_i) \to \operatorname{coker}(p_{i+1}) \to \operatorname{coker}(q_{i+1}) \to 0$$

Since $\ker(p_i) = X'_i \cap X_1$, $\ker(p_{i+1}) = X'_{i+1} \cap X_1$ and p_i, p_{i+1}, q_{i+1} are epimorphism, we deduce that this exact sequence induces a short exact sequence

$$0 \longrightarrow X'_i \cap X_1 \longrightarrow X'_{i+1} \cap X_1 \longrightarrow \ker(q_{i+1}) \longrightarrow 0$$

Pick now k such that $X'_i \cap X_1 = 0$ for $i \leq k$ and $X'_i \cap X_1 = X_1$ for $k < i$. Thus for $i \neq k$ we have $\ker(q_{i+1}) = 0$ and hence X'_{i+1}/X'_i is isomorphic to W_{i+1}/W_i. Moreover, for $i = k$ we have $X_1 \cong \ker(q_{k+1})$. Hence $X_1 \cong X'_{k+1}/X'_k$ and $W_{k+1}/W_k = 0$. We define

$$Z'_i = \begin{cases} W_i & \text{for } i \leq k \\ W_{i+1} & \text{for } k < i \end{cases}$$

for $0 \leq i \leq m-1$. Then

$$0 = Z'_0 \subseteq Z'_1 \subseteq \ldots \subseteq Z'_{m-1} = Z$$

is a composition series and

$$\{[Z'_1/Z'_0]_{\cong}, [Z'_2/Z'_1]_{\cong} ..., [Z'_m/Z'_{m-1}]_{\cong}\} = \{[X'_2/X'_1]_{\cong}, [X'_3/X'_2]_{\cong}, ..., [X'_m/X'_{m-1}]_{\cong}\}$$

Now by induction we have

$$\{[Z'_1/Z'_0]_{\cong}, [Z'_2/Z'_1]_{\cong} ..., [Z'_m/Z'_{m-1}]_{\cong}\} = \{[Z_1/Z_0]_{\cong}, [Z_2/Z_1]_{\cong} ..., [Z_m/Z_{m-1}]_{\cong}\}$$

Since $X'_{k+1}/X'_k \cong X_1$, we deduce that

$$\{[X'_1/X'_0]_{\cong}, [X'_2/X'_1]_{\cong}, ..., [X'_m/X'_{m-1}]_{\cong}\} = \{[Z'_1/Z'_0]_{\cong}, [Z'_2/Z'_1]_{\cong} ..., [Z'_m/Z'_{m-1}]_{\cong}\} \cup \{[X'_{k+1}/X'_k]_{\cong}\} =$$
$$= \{[Z_1/Z_0]_{\cong}, [Z_2/Z_1]_{\cong} ..., [Z_m/Z_{m-1}]_{\cong}\} \cup \{[X_1]_{\cong}\} = \{[X_1/X_0]_{\cong}, [X_2/X_1]_{\cong}, ..., [X_n/X_{n-1}]_{\cong}\}$$

□

Definition 8.5. Let $\mathcal{C}$ be an abelian category and let X be an object of $\mathcal{C}$ that has a composition series (or equivalently that X is both noetherian and artinian). Then we say that X is *an object of finite length*.

Corollary 8.6. *Let X be an object of finite length in an abelian category $\mathcal{C}$ and let λ be an isomorphism type of some irreducible object in $\mathcal{C}$. Suppose that*

$$0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = X$$

is a composition series. Let $l_\lambda(X)$ be the number of elements of the multiset

$$\{[X_1/X_0]_{\cong}, [X_2/X_1]_{\cong}, ..., [X_n/X_{n-1}]_{\cong}\}$$

equal to λ. Then $l_\lambda(X)$ does not depend on the composition series of X.

Proof. This is a consequence of the fact that multisets of isomorphism types of factors are equal for any two composition series of X (Theorem 8.4). □

Definition 8.7. Let X be an object of finite length in an abelian category $\mathcal{C}$. Suppose that Λ is a class of isomorphism types of irreducible objects in $\mathcal{C}$. For a given $\lambda \in \Lambda$ we define $l_\lambda(X)$ as in Corollary 8.6 above. We call it *the multiplicity of* λ *in* X. Next we define

$$l(X) = \sum_{\lambda\in\Lambda} l_\lambda(X) \in \mathbb{N}$$

(note that the components of the sum on the right hand side are almost all equal to zero) and we call this number *the length of* X.

Proposition 8.8. *Suppose that*

$$0 \longrightarrow X' \xrightarrow{i} X \xrightarrow{p} X'' \longrightarrow 0$$

is a short exact sequence of finite length objects in an abelian category $\mathcal{C}$. *Then for every isomorphism type* λ *of some irreducible object* X *in* $\mathcal{C}$ *we have*

$$l_\lambda(X) = l_\lambda(X') + l_\lambda(X'')$$

Proof. Since X' and X'' are of finite length, we fix composition series

$$0 = X'_0 \subseteq X'_1 \subseteq ... \subseteq X'_m = X',\ 0 = X''_0 \subseteq X''_1 \subseteq ... \subseteq X''_k = X''$$

Then

$$0 = i(X'_0) \subseteq ... \subseteq i(X'_m) \subseteq p^{-1}(X''_1) \subseteq ... \subseteq p^{-1}(X''_k) = X$$

is a composition series for X. By Corollar 8.6 we can calculate $l_\lambda(X)$ by means of this compositon series. This implies that

$$l_\lambda(X) = l_\lambda(X') + l_\lambda(X'')$$

□

9. Semisimple objects

Proposition 9.1. *Let* $\mathcal{C}$ *be a well-powered* **Ab**3*-category and let* X *be a nonzero object of* $\mathcal{C}$ *of finite type. Then there exists a subobject* X' *of* X *such that* X/X' *is irreducible.*

Proof. Let $\mathcal{F}$ be the class of proper subobjects of X. That is $\mathcal{F} = \mathrm{Sub}(X) \setminus \{X\}$. This class is nonempty, because X is nonzero. Since $\mathcal{C}$ is well-powered, we derive that $\mathcal{F}$ is a set. Consider a linearly ordered set I and a set $\{X_i\}_{i\in I}$ of elements of $\mathcal{F}$ such that $X_{i_1} \subseteq X_{i_2}$ for $i_1 \leq i_2$ in I. If $X = \sum_{i\in I} X_i$, then there exists $i_0 \in I$ such that $X = X_{i_0}$. This is a contradiction with the fact that X_{i_0} in an element of $\mathcal{F}$. This implies that $\sum_{i\in I} X_i$ is an element of $\mathcal{F}$. Hence chains of elements of $\mathcal{F}$ admit upper bounds. By Zorn's lemma we deduce that there exists a maximal element X' in $\mathcal{F}$. Then X' is a proper and maximal subobject of X. This is equivalent with X/X' being irreducible. □

Definition 9.2. Let $\mathcal{C}$ be an abelian category and let X be an object of $\mathcal{C}$. If every subobject X' of X is a direct summand of X, then we say that X is *completely reducible*.

Proposition 9.3. *Let* $\mathcal{C}$ *be an abelian category. Then the class of completely reducible objects of* $\mathcal{C}$ *is closed under subobjects.*

Proof. Let X be a completely reducible object of $\mathcal{C}$. Consider its subobject Y. Assume that Y' is a subobject of Y. Since X is completely reducible, we derive that there exists a subobject X' such that $X = Y' + X'$ and $X' \cap Y' = 0$. Since subobject lattices in abelian categories are modular (it is a consequence of Theorem 3.2), we derive that

$$Y = Y \cap X = Y \cap (Y' + X') = Y' + Y \cap X'$$

Moreover, $Y' \cap (Y \cap X') = 0$. Thus Y' is a direct summand of Y. This proves that Y is completely reducible. □

Definition 9.4. Let $\mathcal{C}$ be an abelian category and let X be an object of $\mathcal{C}$. If X is the sum of its irreducible subobjects, then we say that X is *semisimple*.

Proposition 9.5. *Let $\mathcal{C}$ be an abelian category. Then the class of semisimple objects of $\mathcal{C}$ is closed under quotient objects.*

Proof. Let X be a semisimple object of $\mathcal{C}$. Consider an epimorphism $q : X \to X'$. Since q preserves sums of subobjects and an epimorphic image of irreducible subobject is either zero or irreducible, we derive that X' is semisimple. □

Theorem 9.6. *Let $\mathcal{C}$ be an abelian category and let X be an object of $\mathcal{C}$. Consider the following statements.*

(i) *X is semisimple.*

(ii) *X is completely reducible.*

If $\mathcal{C}$ is a well-powered **Ab**5*-category, then* **(i)** ⇒ **(ii)**. *If in addition X is locally finite, then* **(ii)** ⇒ **(i)**.

Proof. Assume that $\mathcal{C}$ is a well-powered **Ab**5-category. We prove **(i)** ⇒ **(ii)**. Consider a subobject X' of X. Consider a class

$$\mathcal{F} = \left\{ Y \,\middle|\, Y \text{ is a subobject of } X \text{ and } Y \cap X' = 0 \right\}$$

Since $\mathcal{C}$ is well-powered, the class $\mathcal{F}$ is a set. Since $\mathcal{C}$ is an **Ab**5-category, Zorn's lemma implies that the set $\mathcal{F}$ ordered by inclusion of subobjects has a maximal element X''. Now consider an irreducible subobject K of X. If $K \cap (X' + X'') = 0$, then $Y = X'' + K$ is a subobject of X containing X'' and satisfying $Y \cap X' = 0$. In particular, Y is an element of $\mathcal{F}$. By maximality of X'' in $\mathcal{F}$, we derive that $Y = X''$ and hence K is a contained in X''. This is a contradiction because then $K = K \cap (X' + X'') = 0$ and irreducible objects are nonzero. Thus for every irreducible subobject K of X we have $K \cap (X' + X'') \neq 0$. This implies that $K = K \cap (X' + X'')$ and hence K is contained in $X' + X''$. Since X is a sum of its irreducible subobjects, we derive that $X = X' + X''$. Therefore, X' is a direct summand of X.
Suppose that in addition X is locally finite. Now we prove that **(ii)** ⇒ **(i)**. Suppose that X' is a sum of all irreducible subobjects of X. Since X' is a direct summand of X, we derive that there exists a subobject X'' of X such that $X' \cap X'' = 0$ and $X = X' + X''$. Assume that X'' is nonzero. Since X is locally finite and X'' is isomorphic with X/X', we deduce that X'' is locally finite. Consider a nonzero subobject Y of X'' of finite type. By Proposition 9.1 we deduce that there exists a subobject Y' of Y such that Y/Y' is irreducible. According to Proposition 9.3 we deduce that Y is completely reducible as a subobject of X. Hence there exists a subobject Y'' of Y such that $Y = Y' + Y''$ and $Y' \cap Y'' = 0$. Clearly $Y'' \cong Y/Y'$ and hence Y'' is irreducible subobject of Y. Since Y is contained in X'', we deduce that X'' admits an irreducible subobject. This is contradiction with $X' \cap X'' = 0$. This proves that X'' must be zero and hence X is a sum of its irreducible subobjects. □

References

[Freyd, 1964] Freyd, P. J. (1964). *Abelian categories*, volume 1964. Harper & Row New York.

[Grothendieck, 1957] Grothendieck, A. (1957). Sur quelques points d'algèbre homologique, i. *Tohoku Math. J. (2)*, 9(2):119–221.

[Mac Lane, 1998] Mac Lane, S. (1998). *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition.

[Weibel, 1995] Weibel, C. (1995). *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press.