GROTHENDIECK TOPOSES

1. Introduction

In this notes we study Grothendieck topologies and toposes. For prerequisites we assume familiarity with [Mon19]. As usual we work in Tarski-Grothendieck set theory and we choose base universe. We do not assume that underlying categories of our sites are small. This leads to some difficulties concerning sizes, but it pays off as the theory obtained is more general. This is especially important as possible applications we have in mind are in algebraic geometry, where so called *gross sites* are used.

2. Sites and sheaves

In this section we fix a category $\mathcal{C}$.

Definition 2.1. Let X be an object of $\mathcal{C}$. *A sieve on X* is a family S of arrows of $\mathcal{C}$ with X as a target such that for every $f : Y \to X$ in S and every morphisms $g : Z \to Y$ their composition $f \cdot g$ is in S.

Suppose now that $\mathcal{C}$ is locally small. Then every sieve S on object X of $\mathcal{C}$ corresponds to a subpresheaf of h_X given by

$$\mathcal{C} \ni Y \mapsto \{f : Y \to X \,\big|\, f \in S\} \in \mathbf{Set}$$

This identifies the collection of sieves on X with the collection of subpresheaves of h_X.

Fact 2.2. *Let X be an object of $\mathcal{C}$. The class-theoretic intersection and union of a collection of sieves on X is a sieve on X.*

Proof. Left to the reader. □

Definition 2.3. Let $\mathcal{F}$ be a collection of morphisms of $\mathcal{C}$ with codomain in X. Then the intersection of all sieves on X containing $\mathcal{F}$ is called *the sieve generated by $\mathcal{F}$*.

One can directly describe the sieve on X generated by $\mathcal{F} = \{f_i : X_i \to X\}_{i\in I}$ as a class of arrows $f : Y \to X$ in $\mathcal{C}$ such that f factors through f_i for some $i \in I$.

Definition 2.4. Let S be a sieve on X and $f : Y \to X$ be a morphism, then we define a sieve on Y by formula

$$f^*S = \left\{g \in \mathbf{Mor}(\mathcal{C}) \,\big|\, \text{target of } g \text{ is } Y \text{ and } f \cdot g \in S\right\}$$

We call f^*S *the pullback of S along f*.

Definition 2.5. For every object X in $\mathcal{C}$ the family

$$\left\{f \in \mathbf{Mor}(\mathcal{C}) \,\big|\, \text{target of } f \text{ is } X\right\}$$

is a sieve on X. We call it *the maximal sieve on X*.

Definition 2.6. *A Grothendieck topology on $\mathcal{C}$* is a collection $\mathcal{J} = \{\mathcal{J}(X)\}_{X\in\mathcal{C}}$ such that $\mathcal{J}(X)$ is a class of sieves on X and the following conditions are satisfied.

(1) The maximal sieve on X is in $\mathcal{J}(X)$.

(2) If $S \in \mathcal{J}(X)$ and $f : Y \to X$, then $f^*S \in \mathcal{J}(Y)$.

(3) Suppose that $S \in \mathcal{J}(X)$, R is a sieve on X and $f^*R \in \mathcal{J}(\mathrm{dom}(f))$ for every $f \in S$. Then $R \in \mathcal{J}(X)$.

Sieves in class

$$\bigcup_{X \in \mathcal{C}} \mathcal{J}(X)$$

are called covering sieves. A pair $(\mathcal{C}, \mathcal{J})$ consisting of a category $\mathcal{C}$ and a Grothendieck topology $\mathcal{J}$ is called *a site*.

Proposition 2.7. *Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$ and X be an object of $\mathcal{C}$. Then the following assertions hold.*

(1) *Class $\mathcal{J}(X)$ is closed under finite intersections.*

(2) *If $S \in \mathcal{J}(X)$ and R is a sieve on X such that $S \subseteq R$, then $R \in \mathcal{J}(X)$.*

Proof. We prove **(1)**. For this assume that S and T are covering sieves on X. Then $S \cap T$ is a sieve. Next pick $f : Y \to X$ in S. Note that $f^*(S \cap T) = f^*T \in \mathcal{J}(Y)$. This implies that $S \cap T \in \mathcal{J}(X)$.
We prove now **(2)**. Fix $f : Y \to X$ in S. Then f^*R is the maximal sieve on Y due to $S \subseteq R$. Hence $f^*R \in \mathcal{J}(Y)$. Since $S \in \mathcal{J}(X)$, we deduce that $R \in \mathcal{J}(X)$. □

Fact 2.8. *Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$ and X be an object of $\mathcal{C}$. Suppose that S is a covering sieve on X and for each $f : Y \to X$ in S pick a covering sieve R_f on Y. Then a family*

$$R = \bigcup_{f \in S} f \cdot R_f$$

is a covering sieve on X.

Proof. For every $f : Y \to X$ in S we have $R_f \subseteq f^*R$. By Proposition 2.7 and since R_f is in $\mathcal{J}(Y)$, we deduce that $f^*R \in \mathcal{J}(Y)$. Hence f^*R is a covering sieve for every $f \in S$. This implies that $R \in \mathcal{J}(X)$. □

Definition 2.9. Let F be a presheaf on $\mathcal{C}$. Suppose that X is an object of $\mathcal{C}$ and S is a sieve on X. We say that a family $\{x_f\}_{f \in S}$ such that $x_f \in F(\mathrm{dom}(f))$ is *a matching family for S of elements of F* if for every $f : Y \to X$ in S and $g : Z \to Y$ in $\mathcal{C}$ we have

$$F(g)(x_f) = x_{f \cdot g}$$

We say that an element $x \in F(X)$ is *an amalgamation for the matching family* $\{x_f\}_{f \in S}$ if for every $f \in S$ we have $F(f)(x) = x_f$.

Let S be an arbitrary sieve on object X in $\mathcal{C}$ and F be a presheaf on $\mathcal{C}$. In this notes we denote by $F(S)$ the class of matching families for S of elements of F.
Suppose that $\mathcal{C}$ is locally small. Note that if S is a sieve on X viewed as a subpresheaf of h_X, then a matching family for S of elements of F can be viewed as a morphisms of presheaves $S \to F$. This identifies the collection of matching families for S of elements of F with a collection of morphisms $S \to F$ of presheaves. Next suppose that $\{x_f\}_{f \in S}$ is a matching family for S of elements of F. Then amalgamations of $\{x_f\}_{f \in S}$ can be identified by means of Yoneda lemma [Mon19, Theorem 3.3] with morphisms $h_X \to F$ making the following triangle

$$\begin{tikzcd} h_X \arrow[r, dotted] & F \\ S \arrow[u, hook] \arrow[ur, "\{x_f\}_{f \in S}"'] & \end{tikzcd}$$

commutative.

Definition 2.10. Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$ and F be a presheaf on $\mathcal{C}$. We say that F is *a separated presheaf with respect to $\mathcal{J}$* if for any object X in $\mathcal{C}$, covering sieve $S \in \mathcal{J}(X)$ and for every matching family $\{x_f\}_{f \in S}$ for S of elements of F there exists at most one amalgamation $x \in F(X)$.

Definition 2.11. Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$ and F be a presheaf on $\mathcal{C}$. We say that F is *a sheaf with respect to* $\mathcal{J}$ if for any object X in $\mathcal{C}$, covering sieve $S \in \mathcal{J}(X)$ and for every matching family $\{x_f\}_{f\in S}$ for S of elements of F there exists a unique amalgamation $x \in F(X)$.

In other words $F \in \widehat{\mathcal{C}}$ is a separated presheaf (sheaf) with respect to a Grothendieck topology $\mathcal{J}$ on a locally small category $\mathcal{C}$ if for any $X \in \mathcal{C}$, sieve $S \in \mathcal{J}(X)$ and morphism $S \to F$ of presheaves there exists at most one (a unique) morphism $h_X \to F$ making the triangle

$$\begin{tikzcd} h_X \arrow[r, dotted] & F \\ S \arrow[u, hook] \arrow[ur, "\{x_f\}_{f\in S}"'] & \end{tikzcd}$$

commutative.

Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$. We denote by $\mathbf{PrSh}_s(\mathcal{C},\mathcal{J})$, $\mathbf{Sh}(\mathcal{C},\mathcal{J})$ full subcategories of $\widehat{\mathcal{C}}$ consisting of separated presheaves and sheaves with respect to $\mathcal{J}$, respectively.

Theorem 2.12. *Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$. Then inclusion functors $\mathbf{PrSh}_s(\mathcal{C},\mathcal{J}) \to \widehat{\mathcal{C}}$, $\mathbf{Sh}(\mathcal{C},\mathcal{J}) \to \widehat{\mathcal{C}}$ create limits.*

Proof. There exists a Grothendieck universe V containing the base universe (of our choice) U such that $\mathcal{C}$ is locally V-small. The functor $\widehat{\mathcal{C}} = \mathbf{Fun}\left(\mathcal{C},\mathbf{Set}_U\right) \hookrightarrow \mathbf{Fun}\left(\mathcal{C},\mathbf{Set}_V\right)$ preserves all limits. We embed $\mathbf{PrSh}_s(\mathcal{C},\mathcal{J})$, $\mathbf{Sh}(\mathcal{C},\mathcal{J})$ and $\widehat{\mathcal{C}}$ in $\mathbf{Fun}\left(\mathcal{C},\mathbf{Set}_V\right)$. In $\mathbf{Fun}(\mathcal{C},\mathbf{Set}_V)$ we have representables h_X for X in $\mathcal{C}$ and also we identify matching families for a sieve S of elements of a presheaf F in $\widehat{\mathcal{C}}$ with morphisms $m : S \to F$ in $\mathbf{Fun}(\mathcal{C},\mathbf{Set}_V)$. Let $D : I \to \mathbf{PrSh}_s(\mathcal{C},\mathcal{J})$ be a functor and assume that $\left(F,\left\{f_i : F \to D(i)\right\}_{i\in I}\right)$ is a limiting cone over the composition of the functor $D : I \to \mathbf{PrSh}_s(\mathcal{C},\mathcal{J})$ with the inclusion $\mathbf{PrSh}_s(\mathcal{C},\mathcal{J}) \to \widehat{\mathcal{C}}$. We show that F is a separated presheaf with respect to $\mathcal{J}$. Suppose that S is a covering sieve on X and $m : S \to F$ is a morphism that represents certain matching family for S of elements of F. Let $u : S \to h_X$ be the inclusion . Suppose that morphism $p : h_X \to F$ is an amalgamation for m. We need to show that this amalgamation is unique. For this it suffices to observe that from equality $p \cdot u = m$ we derive that $(f_i \cdot p) \cdot u = (f_i \cdot m)$ for $i \in I$. Hence for every $i \in I$ morphism $f_i \cdot p$ is an amalgamation of $f_i \cdot m$. Since $D(i)$ is a separated presheaf for every $i \in I$, this makes $f_i \cdot p$ uniquely determined for $i \in I$. Thus p is uniquely determined itself according to the fact that the cone $\left(F,\left\{f_i\right\}_{i\in I}\right)$ is limiting in $\widehat{\mathcal{C}}$. Therefore, F is a separated presheaf with respect to $\mathcal{J}$ and hence $\left(F,\left\{f_i\right\}_{i\in I}\right)$ is a limiting cone for D in the category of separated presheaves.

Now assume that $D : I \to \mathbf{Sh}(\mathcal{C},\mathcal{J})$ is a functor and $\left(F,\left\{f_i : F \to D(i)\right\}_{i\in I}\right)$ is a limiting cone over the composition of the functor $D : I \to \mathbf{Sh}(\mathcal{C},\mathcal{J})$ with the inclusion $\mathbf{Sh}(\mathcal{C},\mathcal{J}) \to \widehat{\mathcal{C}}$. We show that F is sheaf with respect to $\mathcal{J}$. From what we prove above we know that F is a separated presheaf with respect to $\mathcal{J}$. Suppose that S is a covering sieve on X and $m : S \to F$ is a morphism that represents certain matching family for S of elements of F. Let $u : S \to h_X$ be the inclusion . It suffices to construct an amalgamation $p : h_X \to F$ for m. We define $m_i = f_i \cdot m$ for $i \in I$. Now fix $i \in I$ for a moment. Then $m_i : S \to D(i)$ is a matching family for S of elements of a sheaf $D(i)$. Hence there exists a unique morphism $p_i : h_X \to D(i)$ such that the triangle

$$\begin{tikzcd} h_X \arrow[r, dotted, "p_i"] & D(i) \\ S \arrow[u, hook, "u"] \arrow[ur, "m_i"'] & \end{tikzcd}$$

is commutative. Now pick a morphism $\alpha : i \to j$ in I. Then

$$D(\alpha) \cdot p_i \cdot u = D(\alpha) \cdot m_i = m_j = p_j \cdot u$$

According to uniqueness of p_j we deduce that $D(\alpha) \cdot p_i = p_j$. Hence $\left(h_X, \{p_i\}_{i\in I}\right)$ is a cone over D. Therefore, there exists a unique morphism $p : h_X \to F$ such that $f_i \cdot p = p_i$ for every $i \in I$. Hence

$$f_i \cdot p \cdot u = p_i \cdot u = m_i = f_i \cdot m$$

for every $i \in I$. Thus $p \cdot u = m$ because the cone $\left(F, \{f_i\}_{i\in I}\right)$ is limiting. Therefore, matching family m for S of elements of F admits an amalgamation p and hence $\left(F, \{f_i\}_{i\in I}\right)$ is a limiting cone for D in the category of sheaves. □

The remaining part of this section contains some technical facts that we use in further developing material in this notes.

Definition 2.13. Let F be a presheaf on $\mathcal{C}$ and let $\mathcal{F} = \left\{f_i : X_i \to X\right\}_{i\in I}$ be a collection of morphisms in $\mathcal{C}$ with codomain X. Assume that $\{x_i\}_{i\in I}$ is a collection such that $x_i \in F(X_i)$ for every $i \in I$ and

$$F(g_i)(x_i) = F(g_j)(x_j)$$

for any morphisms $g_i : Y \to X_i$, $g_j : Y \to X_j$ in $\mathcal{C}$ satisfying $f_i \cdot g_i = f_j \cdot g_j$ for every pair $i, j \in I$. Then $\{x_i\}_{i\in I}$ is called *a matching family for $\mathcal{F}$ of elements of F.*

If F is a presheaf on $\mathcal{C}$ and $\mathcal{F} = \left\{f_i : X_i \to X\right\}_{i\in I}$ is a collection of morphisms in $\mathcal{C}$ with codomain X, then we denote the class of matching families for $\mathcal{F}$ of elements of F by $F(\mathcal{F})$. Suppose that S is a sieve generated by $\mathcal{F}$. We have canonical injective map $\mathrm{can}_{\mathcal{F}} : F(\mathcal{F}) \to \prod_{i\in I} F(X_i)$ and we denote by $\mathrm{res}_{S,\mathcal{F}} : F(S) \to F(\mathcal{F})$ a map that sends $\{x_f\}_{f\in S}$ to $\{x_{f_i}\}_{i\in I}$.

Proposition 2.14. *Fix a presheaf F on $\mathcal{C}$ and a collection $\mathcal{F} = \{f_i : X_i \to X\}_{i\in I}$ of arrows in $\mathcal{C}$ with codomain in X. Let S be a sieve generated by this family. Then $\mathrm{res}_{S,\mathcal{F}}$ is bijective. Moreover, if $\mathcal{C}$ admits fiber products, then*

$$\begin{tikzcd} F(\mathcal{F}) \arrow[r, "\mathrm{can}_{\mathcal{F}}"] & \prod_{i\in I} F(X_i) \arrow[r, shift left, "\langle F(f'_{ij}) \cdot pr_i \rangle_{(i,j)}"] \arrow[r, shift right, "\langle F(f''_{ij}) \cdot pr_j \rangle_{(i,j)}"'] & \prod_{(i,j)\in I\times I} F(X_i \times_X X_j) \end{tikzcd}$$

is a kernel of a pair of arrows, where for every $(i,j) \in I \times I$ morphisms f'_{ij} and f'_{ji} form a cartesian square

$$\begin{tikzcd} X_i \times_X X_j \arrow[r, "f''_{ij}"] \arrow[d, "f'_{ij}"'] & X_j \arrow[d, "f_j"] \\ X_i \arrow[r, "f_i"'] & X \end{tikzcd}$$

Proof. Let $\{x_i\}_{i\in I}$ be a matching family for $\mathcal{F}$ of elements of F. For every $f : Y \to X$ in S there exists $i \in I$ such that $f = f_i \cdot g_i$ for some $g_i : Y \to X_i$. Indeed, this follows from the fact that $\mathcal{F}$ generates S. We define $x_f = F(g_i)(x_i)$. Since $\{x_i\}_{i\in I}$ is a matching family for $\mathcal{F}$ of elements of F, we derive that x_f does not depend on the choice of $i \in I$ and factorization $f = f_i \cdot g_i$. This implies that $\{x_f\}_{f\in S}$ is a matching family for S of elements of F. Now correspondence $\{x_i\}_{i\in I} \mapsto \{x_f\}_{f\in S}$ is the inverse of $\mathrm{res}_{S,\mathcal{F}}$. This proves the first part of the statement.
Let $(x_i)_{i\in I}$ be an element of $\prod_{i\in I} F(X_i)$ such that $F(f'_{ij})(x_i) = F(f''_{ij})(x_j)$ for every pair $(i,j) \in I \times I$. Assume that for some $f : Y \to X$ in S we can write $f = f_i \cdot g_i$ for some $i \in I$ and $g_i : Y \to X_i$ and similarly $f = f_j \cdot g_j$ for some $j \in I$ and $g_j : Y \to X_j$. Then there exist a unique $g : Y \to X_i \times_X X_j$ such that $g_i = f'_{ij} \cdot g$ and $g_j = f''_{ij} \cdot g$. We have

$$F(g_i)(x_i) = F(f'_{ij} \cdot g)(x_i) = F(g)\left(F(f'_{ij})(x_i)\right) = F(g)\left(F(f''_{ij})(x_j)\right) = F(f''_{ij} \cdot g)(x_j) = F(g_j)(x_j)$$

It follows that $\{x_i\}_{i\in I}$ is a matching family for $\mathcal{F}$ of elements of F and $\mathrm{can}_{\mathcal{F}}(\{x_i\}_{i\in I}) = (x_i)_{i\in I}$. This proves that $\mathrm{can}_{\mathcal{F}}$ is a bijection between $F(\mathcal{F})$ and the class of elements $(x_i)_{i\in I} \in \prod_{i\in I} F(X_i)$ such that $F(f'_{ij})(x_i) = F(f''_{ij})(x_j)$ for every pair $(i,j) \in I \times I$. This finishes the proof of the second part of the statement. □

Next if $S \subseteq R$ are sieves on X and F is a presheaf on $\mathcal{C}$, then we denote by $\mathrm{res}_{R,S} : F(R) \to F(S)$ a map given by $\mathrm{res}_{R,S}(\{x_f\}_{f\in R}) = \{x_f\}_{f\in S}$. The next result is a useful technical tool.

Proposition 2.15. *Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$ and F be a separated presheaf with respect to $\mathcal{J}$. Pick X in $\mathcal{C}$. If R, S in $\mathcal{J}(X)$ satisfy $S \subseteq R$, then $\mathrm{res}_{R,S} : F(R) \to F(S)$ is injective.*

Proof. Let $\mathrm{res}_{R,S}(\{x_f\}_{f\in R}) = \{x_f\}_{f\in S}$. We show that $\{x_f\}_{f\in R}$ is uniquely determined by $\{x_f\}_{f\in S}$. For this pick $g \in R$ and consider $\{x_{g\cdot f}\}_{f\in g^*S}$. This is a subfamily of $\{x_f\}_{f\in S}$. For every $f \in g^*S$ we have $F(f)(x_g) = x_{g\cdot f}$ and hence x_g is an amalgamation for a matching family $\{x_{g\cdot f}\}_{f\in g^*S}$ for g^*S of elements of F. Since F is a separated presheaf with respect to $\mathcal{J}$, we deduce that x_g is uniquely determined with $\{x_{g\cdot f}\}_{f\in g^*S}$ and hence it is uniquely determined by $\{x_f\}_{f\in S}$. Arrow g is an arbitrary element of R. Thus $\mathrm{res}_{R,S}$ is injective. □

3. Grothendieck pretopologies

Let $\mathcal{C}$ be a category with fiber products.

Definition 3.1. For every X in $\mathcal{C}$ let $\mathcal{K}(X)$ be a class of collections $\{f_i : X_i \to X\}_{i\in I}$ of arrows in $\mathcal{C}$ with codomain in X. Assume that $\mathcal{K} = \{\mathcal{K}(X)\}_{X\in\mathcal{C}}$ satisfies the following assertions.

(1) $\{1_X : X \to X\} \in \mathcal{K}(X)$ for every object X in $\mathcal{C}$.

(2) If $\{f_i : X_i \to X\}_{i\in I} \in \mathcal{K}(X)$ for some X in $\mathcal{C}$ and $f : Y \to X$ is a morphism, then $\{f'_i : X_i \times_X Y \to Y\}_{i\in I} \in \mathcal{K}(Y)$ where f'_i are defined by cartesian squares

$$\begin{tikzcd} X_i \times_X Y \arrow[r] \arrow[d, "f'_i"'] & X_i \arrow[d, "f_i"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

(3) Suppose that $\{f_i : X_i \to X\}_{i\in I} \in \mathcal{K}(X)$ and $\{f_{ij} : X_{ij} \to X_i\}_{j\in J_i} \in \mathcal{K}(X_i)$ for every $i \in I$. Then $\{f_i \cdot f_{ij} : X_{ij} \to X\}_{i\in I, j\in J_i} \in \mathcal{K}(X)$.

Then we say that $\mathcal{K} = \{\mathcal{K}(X)\}_{x\in\mathcal{C}}$ is *a Grothendieck pretopology on $\mathcal{C}$*.

Proposition 3.2. *Suppose that $\mathcal{K} = \{\mathcal{K}(X)\}_{X\in\mathcal{C}}$ is a Grothendieck pretopology on $\mathcal{C}$. For every X in $\mathcal{C}$ define*

$$\mathcal{J}(X) = \big\{S \,\big|\, S \textit{ is a sieve on } X \textit{ and } S \textit{ contains some collection in } \mathcal{K}(X)\big\}$$

Then $\mathcal{J} = \{\mathcal{J}(X)\}_{X\in\mathcal{C}}$ is a Grothendieck topology on $\mathcal{C}$.

Proof. Note that for every object X in $\mathcal{C}$ we have

$$\big\{f \in \mathbf{Mor}(\mathcal{C}) \,\big|\, \text{codomain of } f \text{ is } X\big\} = \text{a sieve on } X \text{ that contains } 1_X$$

According to $\{1_X : X \to X\} \in \mathcal{K}(X)$, we derive that family $\mathcal{J}(X)$ contains the maximal sieve on X. Now suppose that $S \in \mathcal{J}(X)$ and $f : Y \to X$. There exists $\{f_i : X_i \to X\}_{i\in I} \in \mathcal{K}(X)$ that is contained in S. Then f^*S contains $\{f'_i : X_i \times_X Y \to Y\}_{i\in I}$ where f'_i are defined by cartesian squares

$$\begin{tikzcd} X_i \times_X Y \arrow[r] \arrow[d, "f_i'"'] & X_i \arrow[d, "f_i"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

Since we have $\{f_i' : X_i \times_X Y \to Y\}_{i\in I} \in \mathcal{K}(Y)$, we deduce that $f^* S \in \mathcal{J}(Y)$.
Finally assume that R is a sieve on X, $S \in \mathcal{J}(X)$ and for every $f \in S$ we have $f^* R \in \mathcal{J}(\mathrm{dom}(f))$. By definition there exists $\{f_i : X_i \to X\}_{i\in I} \in \mathcal{K}(X)$ contained in S and for every $i \in I$ there exists $\{f_{ij} : X_{ij} \to X\}_{j\in J_i} \in \mathcal{K}(X_i)$ contained in $f_i^* R$. Thus R contains $\{f_i \cdot f_{ij} : X_{ij} \to X\}_{i\in I, j\in J_i}$ and this is a family in $\mathcal{K}(X)$. Hence $R \in \mathcal{J}(X)$. □

Definition 3.3. Let $\mathcal{K}$ be a Grothendieck pretopology on $\mathcal{C}$ and $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$ given by

$$\mathcal{J}(X) = \big\{ S \,\big|\, S \text{ is a sieve on } X \text{ and } S \text{ contains some collection in } \mathcal{K}(X) \big\}$$

then we say that $\mathcal{J}$ is *a Grothendieck topology generated by* $\mathcal{K}$.

Definition 3.4. Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$ and $\mathcal{K}$ be a Grothendieck pretopology on $\mathcal{C}$ that generates $\mathcal{J}$. Then we say that $\mathcal{K}$ is *a basis of the Grothendieck topology* $\mathcal{J}$.

The next result characterizes sheaves on sites for which Grothendieck topology is generated by some Grothendieck pretopology.

Theorem 3.5. *Let $\mathcal{K}$ be a Grothendieck pretopology on $\mathcal{C}$ and $\mathcal{J}$ be a topology generated by $\mathcal{K}$. Then a presheaf F on $\mathcal{C}$ is a sheaf on with respect to $\mathcal{J}$ if and only if for every $\{f_i : X_i \to X\}_{i\in I} \in \mathcal{K}(X)$ the diagram*

$$\begin{tikzcd} F(X) \arrow[r, "\langle F(f_i)\rangle_{i\in I}"] & \prod_{i\in I} F(X_i) \arrow[r, shift left, "\langle F(f_{ij}') \cdot pr_i\rangle_{(i,j)}"] \arrow[r, shift right, "\langle F(f_{ij}'') \cdot pr_j\rangle_{(i,j)}"'] & \prod_{(i,j)\in I\times I} F(X_i \times_X X_j) \end{tikzcd}$$

is a kernel of a pair of arrows, where for every $(i,j) \in I \times I$ morphisms f_{ij}' and f_{ji}' form a cartesian square

$$\begin{tikzcd} X_i \times_X X_j \arrow[r, "f_{ij}''"] \arrow[d, "f_{ij}'"'] & X_j \arrow[d, "f_j"] \\ X_i \arrow[r, "f_i"'] & X \end{tikzcd}$$

Proof. Suppose that F is a sheaf with respect to $\mathcal{J}$ and $\mathcal{F} = \{f_i : X_i \to X\}_{i\in I}$ be a collection in $\mathcal{K}(X)$. Let S be a sieve generated by $\{f_i\}_{i\in I}$. Then according to Proposition 2.14 we deduce that the diagram

$$\begin{tikzcd} F(S) \arrow[r, "\mathrm{can}_{\mathcal{F}} \cdot \mathrm{res}_{S,\mathcal{F}}^{-1}"] & \prod_{i\in I} F(X_i) \arrow[r, shift left, "\langle F(f_{ij}') \cdot pr_i\rangle_{(i,j)}"] \arrow[r, shift right, "\langle F(f_{ij}'') \cdot pr_j\rangle_{(i,j)}"'] & \prod_{(i,j)\in I\times I} F(X_i \times_X X_j) \end{tikzcd}$$

is a kernel diagram. Since F is sheaf in $\mathcal{J}$ and $S \in \mathcal{J}(X)$, we derive that the map $\mathrm{res}_S : F(X) \to F(S)$ that sends $x \in F(X)$ to $\{F(f)(x)\}_{f\in S}$ is a bijection. Hence

$$\langle F(f_i)\rangle_{i\in I} = \mathrm{can}_{\mathcal{F}} \cdot \mathrm{res}_{S,\mathcal{F}}^{-1} \cdot \mathrm{res}_S : F(X) \to \prod_{i\in I} F(X_i)$$

is a kernel of a pair consisting of $\langle F(f_{ij}') \cdot pr_i\rangle_{(i,j)}$ and $\langle F(f_{ij}'') \cdot pr_j\rangle_{(i,j)}$.
Now assume that F is a presheaf on $\mathcal{C}$ and for every collection $\{f_i : X_i \to X\}_{i\in I}$ in $\mathcal{K}(X)$ the diagram

$$F(X) \xrightarrow{\langle F(f_i)\rangle_{i\in I}} \prod_{i\in I} F(X_i) \underset{\langle F(f''_{ij})\cdot pr_j\rangle_{(i,j)}}{\overset{\langle F(f'_{ij})\cdot pr_i\rangle_{(i,j)}}{\rightrightarrows}} \prod_{(i,j)\in I\times I} F(X_i\times_X X_j)$$

is a kernel pair. Now Proposition 2.14 implies that for any object X and sieve S generated by a collection in $\mathcal{K}(X)$ every matching family for S of elements of F admits a unique amalgamation. In other words for every sieve S on X generated by some collection in $\mathcal{K}(X)$ the map $\mathrm{res}_S : F(X)\to F(S)$ that sends $x\in F(X)$ to $\{F(f)(x)\}_{f\in S}$ is bijective. Consider now any sieve R in $\mathcal{J}(X)$. Then there exists a sieve S on X generated by some collection of $\mathcal{K}(X)$ such that $S\subseteq R$. Consider a commutative triangle

$$\begin{tikzcd} F(R) \arrow[rr, "\mathrm{res}_{R,S}"] & & F(S) \\ & F(X) \arrow[ul, "\mathrm{res}_R"] \arrow[ur, "\mathrm{res}_S"'] & \end{tikzcd}$$

where $\mathrm{res}_{R,S}\left(\{x_f\}_{f\in R}\right) = \{x_f\}_{f\in S}$, $\mathrm{res}_R(x) = \{F(f)(x)\}_{f\in R}$ and $\mathrm{res}_S(x) = \{F(f)(x)\}_{f\in S}$. By what we prove above, we deduce that res_S is a bijection. Hence res_R is injective. Thus F is a separated presheaf with respect to $\mathcal{J}$. By Proposition 2.15 the map $\mathrm{res}_{R,S}$ is injective. Therefore, $\mathrm{res}_{R,S}$, res_R are injective and res_S is bijective and they form a commutative triangle. Hence they are all bijective maps of classes. In particular, res_R is bijective. We deduce that F is a sheaf with respect to $\mathcal{J}$. □

4. Dense subsites

Proposition 4.1. *Let $(\mathcal{C},\mathcal{J})$ be a site and $\mathcal{K}$ be its full subcategory. Then the following are equivalent.*

- **(i)** *For every object X of $\mathcal{C}$ and every S covering sieve in $\mathcal{J}(X)$ there exists a sieve R in $\mathcal{J}(X)$ generated by a collection of morphisms with domains in $\mathcal{K}$ and contained in S.*
- **(ii)** *For every object X of $\mathcal{C}$ there exists a covering sieve S of X generated by a collection of morphisms in $\mathcal{C}$ with domains in $\mathcal{K}$.*

Proof. The implication **(i)** $\Rightarrow$ **(ii)** is obvious. We prove **(ii)** $\Rightarrow$ **(i)**. Let $f: Y\to X$ be a morphism in S. Since $\mathcal{K}$ is dense subcategory of the site $(\mathcal{C},\mathcal{J})$, we derive that there exists a covering sieve R_f in $\mathcal{J}(Y)$ generated by a collection of morphisms with domains in $\mathcal{K}$. Now a collection

$$R = \bigcup_{f\in S} f\cdot R_f$$

is a covering sieve on X by Fact 2.8. It is also contained in S and is generated by morphisms with domains in $\mathcal{K}$. □

Definition 4.2. Let $(\mathcal{C},\mathcal{J})$ be a site and $\mathcal{K}$ be a full subcategory of $\mathcal{C}$ satisfying equivalent condition of Proposition 4.1. Then $\mathcal{K}$ is called *a dense subcategory of a site* $(\mathcal{C},\mathcal{J})$.

Corollary 4.3. *Let $(\mathcal{C},\mathcal{J})$ be a site and $\mathcal{K}$ be its dense subcategory. Fix an object X of $\mathcal{K}$ and a sieve T in $\mathcal{J}(X)$. Then $T\cap\mathcal{K}$ generates a sieve in $\mathcal{C}$ contained in $\mathcal{J}(X)$.*

Proof. By Proposition 4.1 we derive that there exists a sieve R in $\mathcal{J}(X)$ contained in T and generated by morphisms in $\mathcal{K}$. Now a sieve in $\mathcal{C}$ generated by $T\cap\mathcal{K}$ contains R and hence is an element of $\mathcal{J}(X)$ according to Proposition 2.7. □

Corollary 4.4. *Let $(\mathcal{C},\mathcal{J})$ be a site and $\mathcal{K}$ be its dense subcategory. For an object X of $\mathcal{K}$ we define $\mathcal{J}_{\mathcal{K}}(X)$ as a collection of all sieves on X of the form $T\cap\mathcal{K}$ for T in $\mathcal{J}(X)$. Then $\mathcal{J}_{\mathcal{K}}$ is a Grothendieck topology on $\mathcal{K}$.*

Proof. Let X be an object of $\mathcal{K}$. The maximal sieve on X in $\mathcal{K}$ is the intersection of the maximal sieve on X in $\mathcal{C}$ and $\mathcal{K}$. Hence the former is an element of $\mathcal{J}_{\mathcal{K}}(X)$.
Suppose next that T is a sieve in $\mathcal{J}(X)$ for some object X of $\mathcal{K}$ and let $f : Y \to X$ be a morphism in $\mathcal{K}$. Then $f^*T \in \mathcal{J}(Y)$ and since we have $f^*(T \cap \mathcal{K}) \subseteq f^*T \cap \mathcal{K}$, we deduce that $f^*(T \cap \mathcal{K}) \in \mathcal{J}_{\mathcal{K}}(Y)$. Thus pullback of an element of $\mathcal{J}_{\mathcal{K}}(X)$ by f is in $\mathcal{J}_{\mathcal{K}}(Y)$.
Finally suppose that X is an object of $\mathcal{K}$ and S, R are sieves on X in $\mathcal{K}$. Assume that $S \in \mathcal{J}_{\mathcal{K}}(X)$ and $f^*R \in \mathcal{J}_{\mathcal{K}}(\mathrm{dom}(f))$ for every $f \in S$. Let T be a sieve in $\mathcal{C}$ generated by R. Then for every $f \in S$ we have $f^*R \subseteq f^*T$. Since $f^*R \in \mathcal{J}_{\mathcal{K}}(\mathrm{dom}(f))$, we deduce by Corollary 4.3 that sieve in $\mathcal{C}$ generated by f^*R is in $\mathcal{J}(\mathrm{dom}(f))$. This also shows $f^*T \in \mathcal{J}(\mathrm{dom}(f))$. Therefore, f^*T is a covering sieve in $\mathcal{C}$ for every $f \in S$. Since S generates a covering sieve in $\mathcal{C}$ by Corollary 4.3, we deduce that $T \in \mathcal{J}(X)$. Note that $R = T \cap \mathcal{K}$ and hence $R \in \mathcal{J}_{\mathcal{K}}(X)$. □

Definition 4.5. Let $(\mathcal{C}, \mathcal{J})$ be a site and $\mathcal{K}$ be its dense subcategory. Then the Grothendieck topology $\mathcal{J}_{\mathcal{K}}$ on $\mathcal{K}$ described in Corollary 4.4 is called *the induced topology on* $\mathcal{K}$ and a pair $(\mathcal{K}, \mathcal{J}_{\mathcal{K}})$ is called *a dense subsite of* $(\mathcal{C}, \mathcal{J})$.

Theorem 4.6. *Let $(\mathcal{C}, \mathcal{J})$ be a site and $\mathcal{K} \subseteq \mathcal{C}$ be a dense subcategory. Then the embedding $\mathcal{K} \hookrightarrow \mathcal{C}$ induces a full and faithful functor*

$$\mathbf{Sh}(\mathcal{C}, \mathcal{J}) \to \mathbf{Sh}(\mathcal{K}, \mathcal{J}_{\mathcal{K}})$$

Moreover, if for every object X of $\mathcal{C}$ there exists a covering sieve S in $\mathcal{J}(X)$ generated by a set of morphisms with domains in $\mathcal{K}$, then this functor is an equivalence of categories.

The proof is a bit technical and for clarity we divide it into lemmas that encapsulate main steps of the argument. First we need to introduce some notation. The functor $\mathbf{Sh}(\mathcal{C}, \mathcal{J}) \to \mathbf{Sh}(\mathcal{K}, \mathcal{J}_{\mathcal{K}})$ is the restriction of the functor $\widehat{\mathcal{C}} \to \widehat{\mathcal{K}}$ induced by the inclusion $\mathcal{K} \hookrightarrow \mathcal{C}$. We denote values of $\widehat{\mathcal{C}} \to \widehat{\mathcal{K}}$ by $(-)_{|\mathcal{K}}$, where placeholder $-$ stands either for a presheaf in $\mathcal{C}$ or for a morphism of such presheaves.

Lemma 4.6.1. *The functor*

$$\mathbf{Sh}(\mathcal{C}, \mathcal{J}) \to \mathbf{Sh}(\mathcal{K}, \mathcal{J}_{\mathcal{K}})$$

induced by the embedding $\mathcal{K} \hookrightarrow \mathcal{C}$ is full and faitful.

Proof of the lemma. For an object X of $\mathcal{C}$ and a covering sieve S in $\mathcal{J}(X)$ we denote by $S_{|\mathcal{K}}$ the class of all morphisms in S with domains in $\mathcal{K}$. Suppose that S is generated by $S_{|\mathcal{K}}$. Let F, G be sheaves on $(\mathcal{C}, \mathcal{J})$ and let $\tau : F_{|\mathcal{K}} \to G_{|\mathcal{K}}$ be a morphism of presheaves. Then τ induces map $\tau_S : F(S_{|\mathcal{K}}) \to G(S_{|\mathcal{K}})$. Indeed, if $S_{|\mathcal{K}} = \left\{f_i : X_i \to X\right\}_{i \in I}$, then

$$\begin{tikzcd} \prod_{i\in I} F(X_i) \arrow[r, "\prod_{i\in I} \tau_{X_i}"] & \prod_{i\in I} G(X_i) \\ F(S_{|\mathcal{K}}) \arrow[u, hook] \arrow[r, "\tau_S"'] & G(S_{|\mathcal{K}}) \arrow[u, hook] \end{tikzcd}$$

where vertical injections are canonical. Since F, G are sheaves on $(\mathcal{C}, \mathcal{J})$, we derive by Proposition 2.14 that $F(S_{|\mathcal{K}})$ and $G(S_{|\mathcal{K}})$ can be identified with $F(X)$ and $G(X)$, respectively. Thus τ induces a map $\tau_X : F(X) \to G(X)$. Morphism τ_X does not depend on a covering sieve S on X generated by $S_{|\mathcal{K}}$. This is a consequence of Proposition 4.1. Thus the meaning of the symbol τ_X for an object X already in $\mathcal{K}$ is unambiguous, which follows from applying the definiton above to the maximal sieve on X. Now suppose that $f : X \to Y$ is a morphism in $\mathcal{C}$ and S, R are covering sieves in $\mathcal{C}$ on X, Y, respectively. Assume that S, R are generated by $S_{|\mathcal{K}}, R_{|\mathcal{K}}$, respectively and $S \subseteq f^*R$. We have a commutative diagram

$$\begin{tikzcd}
F(R_{|\mathcal{K}}) \arrow[r, "\tau_R"] \arrow[d] & G(R_{|\mathcal{K}}) \arrow[d] \\
F\left((f^*R)_{|\mathcal{K}}\right) \arrow[r, "\text{induced by } \tau"'] & G\left((f^*R)_{|\mathcal{K}}\right) \\
F(S_{|\mathcal{K}}) \arrow[r, "\tau_S"'] \arrow[u, hook, "\text{inclusion}"] & G(S_{|\mathcal{K}}) \arrow[u, hook', "\text{inclusion}"']
\end{tikzcd}$$

in which left hand side vertical arrow with no labels is given by

$$F(R_{|\mathcal{K}}) \ni \{x_h\}_{h \in R_{|\mathcal{K}}} \mapsto \{y_g\}_{g \in (f^*R)_{|\mathcal{K}}} \in F\left((f^*R)_{|\mathcal{K}}\right)$$

where $y_g = x_{f \cdot g}$ for $g \in (f^*R)_{|\mathcal{K}}$ and analogically we define right hand side vertical arrow with no label in the diagram. This diagram implies that $\tau_X \cdot F(f) = F(f) \cdot \tau_Y$ by Proposition 2.14. This extends $\tau : F_{|\mathcal{K}} \to G_{|\mathcal{K}}$ to a morphism of sheaves $F \to G$. This extension is unique by its definition. □

In our argument below we apply the usual trick of enlarging the universe. This was used in the proof of Theorem 2.12. This makes possible to speak about representables h_X for X in $\mathcal{C}$ and k_X for X in $\mathcal{K}$. Suppose that X is an object of $\mathcal{C}$. Since a sieve S on X in $\mathcal{C}$ can be identified with a subpresheaf (in the sens of enlarged universe) of h_X, the restriction $S_{|\mathcal{K}}$ makes sense (it also has the same meaning as the one introduced in the proof of Lemma 4.6.1). Note also that if X is an object of $\mathcal{K}$, then $(h_X)_{|\mathcal{K}} = k_X$.

Lemma 4.6.2. *Let F be a sheaf on $(\mathcal{K}, \mathcal{J}_{\mathcal{K}})$. Consider an object X of $\mathcal{C}$ and a covering sieve S in $\mathcal{J}(X)$. Then for every morphism $\sigma : S_{|\mathcal{K}} \to F$ there exists a unique morphism $\tilde{\sigma} : (h_X)_{|\mathcal{K}} \to F$ making the following diagram commutative.*

$$\begin{tikzcd}
(h_X)_{|\mathcal{K}} \arrow[r, dotted, "\tilde{\sigma}"] & F \\
S_{|\mathcal{K}} \arrow[u, hook] \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

Proof of the lemma. Fix $\sigma : S_{|\mathcal{K}} \to F$. Next fix a morphism $f : Y \to X$ in $\mathcal{C}$ with Y in $\mathcal{K}$. Consider a sieve $R_f = f^*S \cap \mathcal{K}$ in $\mathcal{K}$ as a presheaf on $\mathcal{K}$. Then we have a morphism

$$R_f \ni g \mapsto f \cdot g \in S_{|\mathcal{K}}$$

of presheaves on $\mathcal{K}$. We denote the composition of this morphism with σ by σ_f. Thus $\sigma_f : R_f \to F$ is a morphism of presheaves. Since F is a sheaf on $(\mathcal{K}, \mathcal{J}_{\mathcal{K}})$ and $R_f \in \mathcal{J}_{\mathcal{K}}(Y)$, we derive that there exists a unique morphism $\tilde{\sigma}_f : k_Y \to F$ such that the triangle

$$\begin{tikzcd}
k_Y \arrow[r, dotted, "\tilde{\sigma}_f"] & F \\
R_f \arrow[u, hook] \arrow[ur, "\sigma_f"'] &
\end{tikzcd}$$

is commutative. We define $\tilde{\sigma} : (h_X)_{|\mathcal{K}} \to F$ by formula $\tilde{\sigma}(f) = \tilde{\sigma}_f(1_Y)$. Then $\tilde{\sigma}$ satisfies conditions in the statement. □

Let F be a presheaf on $\mathcal{K}$. Suppose that for every object X in $\mathcal{C}$ the class $\mathrm{Mor}_{\mathcal{K}}\left((h_X)_{|\mathcal{K}}, F\right)$ is a set. Then we denote the presheaf $X \mapsto \mathrm{Mor}_{\mathcal{K}}\left((h_X)_{|\mathcal{K}}, F\right)$ on $\mathcal{C}$ by $\tilde{F}$. If X is in $\mathcal{K}$, then we have a bijection

$$\mathrm{Mor}_{\mathcal{K}}\left((h_X)_{|\mathcal{K}}, F\right) = \mathrm{Mor}_{\mathcal{K}}\left(k_X, F\right) \ni \tau \mapsto \tau(1_X) \in F(X)$$

This bijection is natural in object X of $\mathcal{K}$ and hence $\tilde{F}_{|\mathcal{K}}$ can be identified with F. This defines an isomorphism $\xi_F : \tilde{F}_{|\mathcal{K}} \to F$.

Lemma 4.6.3. *Let F be a presheaf on $\mathcal{K}$ and let X be an object of $\mathcal{K}$. Suppose that for every object X in $\mathcal{C}$ the class $\mathrm{Mor}_{\mathcal{K}}\left((h_X)_{|\mathcal{K}}, F\right)$ is a set. Fix a morphism $\phi : k_X \to \tilde{F}_{|\mathcal{K}}$ of presheaves. Then $\xi_F \cdot \phi = \phi(1_X)$.*

Proof of the lemma. We denote $\phi(1_X)$ by τ. Then for every morphism $f : Y \to X$ in $\mathcal{K}$ we have

$$(\xi_F \cdot \phi)(f) = \xi_F(\phi(f)) = \xi_F\left(\tau \cdot (h_f)_{|\mathcal{K}}\right) = \left(\tau \cdot (h_f)_{|\mathcal{K}}\right)(1_Y) = \tau(f)$$

and hence $\xi_F \cdot \phi = \tau$. □

Lemma 4.6.4. *Let F be a sheaf on $(\mathcal{K}, \mathcal{J}_{\mathcal{K}})$. Suppose that for every object X in $\mathcal{C}$ the class $\mathrm{Mor}_{\mathcal{K}}\left((h_X)_{|\mathcal{K}}, F\right)$ is a set. Then the presheaf $\tilde{F}$ on $\mathcal{C}$ is a sheaf on $(\mathcal{C}, \mathcal{J})$.*

Proof of the lemma. Fix object X in $\mathcal{C}$ and a sieve S in $\mathcal{J}(X)$. Assume that S is generated by morphisms with domain in $\mathcal{K}$. Consider a morphism $\sigma : S \to \tilde{F}$. Then $\sigma_{|\mathcal{K}} : S_{|\mathcal{K}} \to \tilde{F}_{|\mathcal{K}}$ composed with the identification $\xi_F : \tilde{F}_{|\mathcal{K}} \to F$ gives rise to a morphism $\tau : S_{|\mathcal{K}} \to F$. According to Lemma 4.6.2 we derive that there exists $\tilde{\tau} : (h_X)_{|\mathcal{K}} \to F$ that extends τ. Now suppose that $f : Y \to X$ is a morphism in S with Y being object of $\mathcal{K}$. Let $i_f : k_Y \to S_{|\mathcal{K}}$ be a morphism determined by f and let $i : S \hookrightarrow h_X$ be the inclusion. Note that $(h_f)_{|\mathcal{K}} = i_{|\mathcal{K}} \cdot i_f$. We have (one step in the chain below requires Lemma 4.6.3)

$$\tilde{F}(f)(\tilde{\tau}) = \tilde{\tau} \cdot (h_f)_{|\mathcal{K}} = \tilde{\tau} \cdot i_{|\mathcal{K}} \cdot i_f = \tau \cdot i_f = \xi_F \cdot \sigma_{|\mathcal{K}} \cdot i_f = \left(\sigma_{|\mathcal{K}} \cdot i_f\right)(1_Y) = \sigma_{|\mathcal{K}}(f) = \sigma(f)$$

This equality implies that $\tilde{\tau}$ is an amalgamation of a matching family σ and moreover, it shows that it is a unique amalgamation.
Now we proceed as in the proof of Theorem 3.5. Consider any sieve R in $\mathcal{J}(X)$. Then by Proposition 4.1 there exists a sieve S on X generated by morphisms with domains in $\mathcal{K}$ such that $S \subseteq R$. Consider a commutative triangle

$$\begin{tikzcd}
\tilde{F}(R) \arrow[rr, "\mathrm{res}_{R,S}"] & & \tilde{F}(S) \\
& \tilde{F}(X) \arrow[ul, "\mathrm{res}_R"] \arrow[ur, "\mathrm{res}_S"'] &
\end{tikzcd}$$

where $\mathrm{res}_{R,S}$ is the restriction of matching families, res_R and res_S send elements of $\tilde{F}(X)$ to matching families on R and S, respectively. By what we prove above, we deduce that res_S is a bijection. Hence res_R is injective. Thus $\tilde{F}$ is a separated presheaf with respect to $\mathcal{J}$. By Proposition 2.15 the map $\mathrm{res}_{R,S}$ is injective. Therefore, $\mathrm{res}_{R,S}$, res_R are injective and res_S is bijective and they form a commutative triangle. Hence they are all bijective maps of classes. In particular, res_R is bijective. We deduce that $\tilde{F}$ is a sheaf with respect to $\mathcal{J}$. □

Proof of the theorem. Lemma 4.6.1 implies that the functor in question is full and faithful. Now suppose that every object X in $\mathcal{C}$ admits a covering sieve S in $\mathcal{J}(X)$ generated by a set of morphisms with domains in $\mathcal{K}$. Fix X in $\mathcal{C}$ and a sheaf F on $(\mathcal{K}, \mathcal{J}_{\mathcal{K}})$. Let $\mathcal{F} = \left\{f_i : X_i \to X\right\}_{i \in I}$ be a set of morphisms generating some covering sieve $S \in \mathcal{J}(X)$. Then $\mathrm{Mor}_{\mathcal{K}}\left(S_{|\mathcal{K}}, F\right)$ is a set and by

Lemma 4.6.2 we deduce that $\mathrm{Mor}_{\mathcal{K}}\left((h_X)_{|\mathcal{K}}, F\right)$ is a set. Therefore, for every sheaf F on $(\mathcal{K}, \mathcal{J}_{\mathcal{K}})$ presheaf $\tilde{F}$ exists. By Lemma 4.6.4 it is a sheaf on $(\mathcal{C}, \mathcal{J})$ such that $\xi_F : \tilde{F}_{|\mathcal{K}} \to F$ is an isomorphism. Thus the functor

$$\mathbf{Sh}(\mathcal{C}, \mathcal{J}) \to \mathbf{Sh}(\mathcal{K}, \mathcal{J}_{\mathcal{K}})$$

is essentially surjective. Since it is full and faithful as it was previously established, we deduce that it is an equivalence of categories. □

5. Something stupid

The next result deals with certain set-theoretic issues and is important from the point of view of the next section.

Proposition 5.1. *Let $\mathcal{K}$ be a Grothendieck pretopology on $\mathcal{C}$ and let $\mathcal{J}$ be a Grothendieck topology generated by $\mathcal{K}$. Assume that the following assertions hold.*

(1) *For every object X in $\mathcal{C}$ the class $\mathcal{K}(X)$ is a set.*

(2) *For every object X in $\mathcal{C}$ every collection $\{f_i : X_i \to X\}_{i\in I} \in \mathcal{K}(X)$ is a set.*

Then for every presheaf $F \in \widehat{\mathcal{C}}$ and every object X in $\mathcal{C}$ a colimit

$$F^+(X) = \mathrm{colim}_{S\in\mathcal{J}(X)} F(S)$$

is a set.

6. Sheaf associated to a presheaf

Let $\mathcal{C}$ be a category. Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$. Let us formulate certain technical set-theoretic assumption on $\mathcal{J}$.

($\star$) For any presheaf F on $\mathcal{C}$, object X in $\mathcal{C}$ and a covering sieve S there the class $F(S)$ of matching families for S of elements of F form a set and the class

Theorem 6.1. *Let F be a presheaf on a Grothendieck site $(\mathcal{C}, \mathcal{J})$. There exists a sheaf $a(F)$ and a morphism $\eta_F : F \to a(F)$ of presheaves such that for every sheaf G and every morphism of presheaves $p : F \to G$ there exists a unique morphism $r : a(F) \to G$ making the diagram*

$$\begin{tikzcd}
a(F) \arrow[r, "r"] & G \\
F \arrow[u, "\eta_F"] \arrow[ur, "p"'] &
\end{tikzcd}$$

commutative.

First we construct a separated presheaf F^+ out of F. Fix an object X of $\mathcal{C}$. Suppose that S is a covering sieve on X. Denote by $F(S)$ the set of all matching families for S of elements of F. If $S_1 \subseteq S_2$ are covering sieves on X, then we have a function $F(S_2) \to F(S_1)$ given by restriction. Thus $\{F(S)\}_{S\in\mathcal{J}(X)}$ is a diagram indexed by a directed set $\mathcal{J}(X)$ and we define

$$F^+(X) = \mathrm{colim}_{S\in\mathcal{J}(X)} F(S)$$

Note that for every morphism $f : X_1 \to X_2$ in $\mathcal{C}$ and for every sieve $S \in \mathcal{J}(X_2)$ we have a function $F(S) \to F(f^*S)$ given by $F(S) \ni \{s_g\}_{g\in S} \mapsto \{s_{f\cdot g}\}_{g\in f^*S} \in F(f^*S)$. These functions for all $S \in \mathcal{J}(X_2)$ induce a map

$$F^+(X_2) \to F^+(X_1)$$

and this defines a presheaf F^+. We also have a morphism of presheaves $i_F^+ : F \to F^+$ that sends $x \in F(X)$ to a class in $F^+(X)$ represented by a matching family of the form $\{F(f)(x)\}_{f\in S}$ for every covering sieve S on X.

Lemma 6.1.1. *The following assertions hold.*

(1) *F^+ is a separated presheaf.*

(2) *If F is separated presheaf, then F^+ is a sheaf.*

Proof of the lemma. We prove **(1)**. Fix an object $X \in \mathcal{C}$ and a covering sieve S on X. Suppose that $\{x_f\}_{f\in S}$ is a matching family for S of elements of F^+. Assume that $y, z \in F^+(X)$ are amalgamations of $\{x_f\}_{f\in S}$. Then there exists a covering sieve T on X such that y is represented by some matching family $\{s_f\}_{f\in T}$ for T of elements of F and z is represented by some matching family $\{t_f\}_{f\in T}$ for T of elements of F. Fix a morphism $f : Y \to X$ in S. Then $F^+(f)(y)$ is represented by $\{s_{f\cdot g}\}_{g\in f^*T}$ and $F^+(f)(z)$ is represented by $\{t_{f\cdot g}\}_{g\in f^*T}$. Moreover, $F^+(f)(y) = x_f = F^+(f)(z)$ and hence there exists a covering sieve R_f on Y such that $R_f \subseteq f^*T$ and $s_{f\cdot g} = t_{f\cdot g}$ for every $g \in R_f$. Now we know that

$$R = \bigcup_{f\in T} f \cdot R_f \subseteq S$$

is a covering sieve on X and matching families $\{s_f\}_{f\in R}$, $\{t_f\}_{f\in R}$ for R of elements of F represent respectively y and z. Since these families are equal, we derive that $y = z$. This implies that F^+ is separated.

Let us prove **(2)**. Fix an object $X \in \mathcal{C}$ and a covering sieve S on X. Suppose that $\{x_f\}_{f\in S}$ is a matching family for S of elements of F^+. For every $f : Y \to X$ in S there exists a covering sieve R_f on Y and a matching family $\{s(f)_g\}_{g\in R_f}$ for R_f of elements of F that represents x_f. Formula

$$R = \bigcup_{f\in S} f \cdot R_f$$

defines a covering sieve on X contained in S. We set $r_{f\cdot g} = s(f)_g$ for every $f \in S$ and $g \in R_f$. We check now that this definition is independent of choices of $f \in S$ and $g \in R_f$. For this suppose that $f_1, f_2 \in S$ and $g_1 \in R_{f_1}$, $g_2 \in R_{f_2}$ satisfy $f_1 \cdot g_1 = f_2 \cdot g_2$. Let $Z \in \mathcal{C}$ denote a common domain of morphisms g_1, g_2. Now $F^+(g_1)(x_{f_1})$ is represented by a matching family $\{s(f_1)_{g_1\cdot g}\}_{\mathrm{cod}(g)=Z}$ and $F^+(g_2)(x_{f_2})$ is represented by a matching family $\{s(f_2)_{g_2\cdot g}\}_{\mathrm{cod}(g)=Z}$. According to equality

$$F^+(g_1)(x_{f_1}) = x_{f_1\cdot g_1} = x_{f_2\cdot g_2} = F^+(g_2)(x_{f_2})$$

these families represent the same element of $F^+(Z)$. Hence we deduce that there exists a covering sieve T on Z such that $\{s(f_1)_{g_1\cdot g}\}_{g\in T} = \{s(f_2)_{g_2\cdot g}\}_{g\in T}$. Next $s(f_1)_{g_1}$ is an amalgamation for $\{s(f_1)_{g_1\cdot g}\}_{g\in T}$ and $s(f_2)_{g_2}$ is an amalgamation for $\{s(f_2)_{g_2\cdot g}\}_{g\in T}$. By separatedness of F, we derive that $s(f_1)_{g_1} = s(f_2)_{g_2}$. Thus family $\{r_f\}_{f\in R}$ is well defined. By definition it is a matching family for R of elements of F. Hence it defines an element of $F(R)$ and this element represents some $x \in F^+(X)$. Fix now $f \in S$. By definition of F^+ we deduce that $F^+(f)(x)$ is represented by $\{r_{f\cdot g}\}_{g\in f^*R}$. This family contains $\{r_{f\cdot g}\}_{g\in R_f} = \{s(f)_g\}_{g\in R_f}$ and thus $F^+(f)(x) = x_f$. This proves that $\{x_f\}_{f\in S}$ admits an amalgamation. By **(1)** presheaf F is separated. Hence amalgamation of $\{x_f\}_{f\in S}$ is unique. □

Lemma 6.1.2. *Let $p : F \to G$ be a morphism of presheaves and assume that G is a sheaf. Then there exists a unique morphism $q : F^+ \to G$ such that the diagram*

$$\begin{tikzcd} F^+ \arrow[r, "q"] & G \\ F \arrow[u, "i_F^+"] \arrow[ur, "p"'] & \end{tikzcd}$$

is commutative.

Proof of the lemma. Fix $X \in \mathcal{C}$ and $x \in F^+(X)$. Then there exists a covering sieve S on X and a matching family $\{s_f\}_{f \in S}$ for S of elements of F that represents x. By definitions of F^+ and i^+_F we have matching family $\{i^+_F(s_f)\}_{f \in S}$ for S of elements of F^+ with x as its amalgamation.
Assume that $q : F^+ \to G$ is a morphism such that $p = q \cdot i^+_F$. We have $p(s_f) = q(i^+_F(s_f))$ for every $f \in S$. Therefore, $q(x)$ must be an amalgamation of a matching family $\{p(s_f)\}_{f \in S} = \{q(i^+_F(s_f))\}_{f \in S}$ for S of elements of G. Since G is a separated presheaf, there exists at most one such amalgamation. This proves uniqueness of q.
The existence of such a q is also evident. As G is a sheaf, one picks $q(x)$ to be the amalgamation of a matching family $\{p(s_f)\}_{f \in S}$ for S of elements of G. Verification that uses definitions of F^+ and i^+_F shows that this gives rise to a morphism $q : F^+ \to G$ which satisfies $p = q \cdot i^+_F$. □

Proof of the theorem. We define $a(F) = (F^+)^+$ and $\eta_F = i^+_{F^+} \cdot i^+_F$. By Lemma 6.1.1 presheaf $a(F)$ is a sheaf. Now suppose that $p : F \to G$ is a morphism of presheaves and G is a sheaf. We apply Lemma 6.1.2 twice to obtain a unique morphism $r : a(F) \to G$ such that $p = r \cdot \eta_F$. □

References

[Mon19] Monygham. Categories of presheaves. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2019.