## 1. Introduction

Throughout this notes $k$ denote a field and $\mathbf{G}$ denote a group scheme over $k$. We also fix a $k$-scheme $X$ equipped with an action of $\mathbf{G}$ determined by morphism $a : \mathbf{G} \times_k X \to X$.

## 2. Categorical and geometric quotients

**Definition 2.1.** Let $q : X \to Y$ be a morphism of $k$-schemes such that the diagram

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel in the category of $k$-schemes. Then $q : X \to Y$ is *a categorical quotient of* $X$.

**Definition 2.2.** Consider a cokernel

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

in the category of locally ringed spaces over $k$. If $Y$ is a scheme, then $q : X \to Y$ is *a geometric quotient of* $X$.

**Fact 2.3.** *Every geometric quotient is categorical.*

*Proof.* Categorical quotient is a cokernel in the category of $k$-schemes. On the other hand geometric quotient is a cokernel in the category of locally ringed spaces and hence it also satisfies cokernel property in its full subcategory of $k$-schemes. Thus every geometric quotient is categorical. □

**Corollary 2.4.** *Let $q : X \to Y$ be a morphism of schemes. The following assertions are equivalent.*

**(i)** *The diagram*

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

*is a cokernel diagram of underlying topological spaces and the diagram*

$$\mathcal{O}_Y \xrightarrow{q^{\#}} q_*\mathcal{O}_X \underset{q_*\mathrm{pr}_X^{\#}}{\overset{q_*a^{\#}}{\rightrightarrows}} q_*\left(\mathrm{pr}_X\right)_* \mathcal{O}_{\mathbf{G}\times_k X} = q_*a_*\mathcal{O}_{\mathbf{G}\times_k X}$$

*is a kernel diagram in the category of sheaves on $Y$.*

**(ii)** *$q$ is a geometric quotient of $X$.*

*Proof.* This is a consequence of [Monygham, 2019, Theorem 2.9]. □

Let $q : X \to Y$ be a morphism of $k$-schemes such that $q \cdot \mathrm{pr}_X = q \cdot a$. For a morphism $g : Y' \to Y$ of $k$-schemes consider the cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Then there exists a unique action $a' : \mathbf{G} \times_k X' \to X'$ of $\mathbf{G}$ on $X'$ such that the square above consists of $\mathbf{G}$-equivariant morphism (we consider $Y, Y'$ as $\mathbf{G}$-schemes equipped with trivial $\mathbf{G}$-actions). Keeping this in mind we have the following.

**Definition 2.5.** A morphism $q : X \to Y$ is *a uniform categorical (geometric) quotient of* $X$ if for every flat morphism $g : Y' \to Y$ its base change $q' : X' \to Y'$ is a categorical (geometric) quotient of $X'$.

**Definition 2.6.** A morphism $q : X \to Y$ is *a universal categorical (geometric) quotient of* $X$ if for every morphism $g : Y' \to Y$ its base change $q' : X' \to Y'$ is a categorical (geometric) quotient of $X'$.

## 3. Types of actions and criterion for smoothness of universal geometric quotients

**Definition 3.1.** The action of $\mathbf{G}$ on $X$ is *separated* if the morphism $\langle a, \mathrm{pr}_X\rangle : \mathbf{G} \times_k X \to X \times_k X$ has closed set-theoretic image.

**Theorem 3.2.** *Let $q : X \to Y$ be a geometric quotient of $X$. Assume that $q$ is universally submersive. Then the following assertions are equivalent.*

- **(i)** *The action of $\mathbf{G}$ on $X$ is separated.*
- **(ii)** *$Y$ is separated.*

*Proof.* We have a cartesian square

$$\begin{tikzcd} X \times_Y X \arrow[r, hook] \arrow[d] & X \times_k X \arrow[d, "q \times_k q"] \\ Y \arrow[r, hook, "\Delta_Y"'] & Y \times_k Y \end{tikzcd}$$

It follows that $X \times_Y X \hookrightarrow X \times_k X$ is a locally closed immersion. Since $q$ is a geometric quotient, we derive that $\langle a, \mathrm{pr}_X\rangle$ factors as a surjective morphism $\mathbf{G} \times_k X \twoheadrightarrow X \times_Y X$ followed by the immersion $X \times_Y X \hookrightarrow X \times_k X$. Thus the action of $\mathbf{G}$ on $X$ is separated if and only if $X \times_Y X$ is a closed subscheme of $X \times_k X$. Since $q$ is universally submersive, we derive that $q \times_k q$ is submersive. As the square above is cartesian we derive that $\Delta_Y(Y) \subseteq Y \times_k Y$ is closed if and only if $X \times_Y X \subseteq X \times_k X$ is closed. Therefore, $Y$ is separated if and only if the action of $\mathbf{G}$ on $X$ is separated. □

The following result which concerns complete local rings is very useful.

**Definition 3.3.** Let $x$ be a $k$-point of $X$. Suppose that the morphism $\mathbf{G} \to X$ given by the composition

$$\mathbf{G} = \mathbf{G} \times_k \operatorname{Spec} k \xhookrightarrow{\text{induced by } x} \mathbf{G} \times_k X \longrightarrow X$$

is a closed immersion. Then the action of $\mathbf{G}$ on $X$ has *a closed free orbit at $x$*.

**Proposition 3.4.** *Let $(A, \mathfrak{m}, k)$ be a complete local noetherian $k$-algebra. Suppose that there exists a local morphism $\sigma : A \to A[[x_1, ..., x_n]]$ into a ring of formal power series over $A$. Assume that the composition*

$$A \xrightarrow{\sigma} A[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod (x_1, ..., x_n)} A$$

*is the identity and the composition*

$$A \xrightarrow{\sigma} A[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod \mathfrak{m}} (A/\mathfrak{m})\,[[x_1, ..., x_n]] = k[[x_1, ..., x_n]]$$

*is surjective. Consider elements $y_1, ..., y_n$ of $A$ such that $\sigma(y_i) \bmod \mathfrak{m} = x_i$ for $i = 1, ..., n$. Then the composition*

$$A \xrightarrow{\sigma} A[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod (y_1, ..., y_n)} (A/(y_1, ..., y_n))\,[[x_1, ..., x_n]]$$

*is an isomorphism.*

*Proof.* For convienience let $\phi$ denote the morphism given by the rule $a \mapsto \sigma(a) \bmod (y_1, ..., y_n)$. According to assumptions we have

$$\sigma(y_i) = x_i + y_i + \sum_{j=1}^{n} x_j \cdot \mathfrak{m}[[x_1, ..., x_n]]$$

for each $i$. Thus $\phi(y_i) = \sum_{j=1}^{n} f_{ij} \cdot x_j$ where $f_{ij} \in A/(y_1, ..., y_n)$ are elements such that

$$\det\left(\left[f_{ij}\right]_{1 \leq i,j \leq n}\right)$$

is invertible in $A/(y_1, ..., y_n)$.

□

## References

[Monygham, 2019] Monygham (2019). Locally ringed spaces. *github repository: "Monygham/Pedo-mellon-a-minno"*.