ALGEBRAIZATION OF FORMAL M-SCHEMES

1. Some 2-categorical limits

Consider a category $\mathcal{C}$ and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Our goal is to construct certain 2-categorical limit associated with a pair $(\mathcal{C}, T)$. Consider pairs (X, u) consisting of an object X of $\mathcal{C}$ and an isomorphism $u : T(X) \to X$ in $\mathcal{C}$. If (X, u) and (Y, w) are two such pairs, then a morphism $f : (X, u) \to (Y, u)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that the following square

$$\begin{tikzcd} T(X) \arrow[r, "u"] \arrow[d, "T(f)"'] & X \arrow[d, "f"] \\ T(Y) \arrow[r, "w"'] & Y \end{tikzcd}$$

is commutative. This data give rise to a category $\mathcal{C}(T)$. There exists a forgetful functor $\pi : \mathcal{C}(T) \to \mathcal{C}$ that sends a morphism $f : (X, u) \to (Y, w)$ to $f : X \to Y$. Moreover, there exists a natural isomorphism $\sigma : T \cdot \pi \Rightarrow \pi$ such that the component of σ on an object (X, u) of $\mathcal{C}(T)$ is u. The next result states that the data above form a certain 2-categorical limit.

Theorem 1.1. *Let $(\mathcal{C}, T)$ be a pair consiting of a category and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Suppose that $\mathcal{D}$ is a category, $P : \mathcal{D} \to \mathcal{C}$ is a functor and $\tau : T \cdot P \Rightarrow P$ is a natural isomorphisms. Then there exists a unique functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$.*

Proof. Suppose that $F : \mathcal{D} \to \mathcal{C}(T)$ is a functor such that $P = \pi \cdot F$ and $\sigma_F = \tau$. Pick an object X of $\mathcal{D}$. Then we have $\pi \cdot F(X) = P(X)$ and $\sigma_{F(X)} = \tau_X$. This implies that

$$F(X) = (P(X), \tau_X : T(P(X)) \to P(X))$$

Next if $f : X \to Y$ is a morphism in $\mathcal{D}$, then we derive that $\pi(F(f)) = P(f)$. Hence $F(f) = P(f)$. This implies that there exists at most one functor F satisfying the properties above. Note also that formulas

$$F(X) = (P(X), \tau_X : T(P(X)) \to P(X)), \, F(f) = P(f)$$

for an object X in $\mathcal{D}$ and a morphism $f : X \to Y$ in $\mathcal{D}$, give rise to a functor that satisfy $P = \pi \cdot F$ and $\sigma_F = \tau$. This establishes existence and the uniqueness of F. $\square$

Assume now that the pair $(\mathcal{C}, T)$ consists of a monoidal category $\mathcal{C}$ and a monoidal endofunctor T. Then there exists a canonical monoidal structure on $\mathcal{C}(T)$. We define $(-) \otimes_{\mathcal{C}(T)} (-)$ by formula

$$(X, u) \otimes_{\mathcal{C}(T)} (Y, w) = \big(X \otimes_{\mathcal{C}} Y, (u \otimes_{\mathcal{C}} w) \cdot m_{X,Y}\big)$$

where

$$m_{X,Y} : T\,(X \otimes_{\mathcal{C}} Y) \to T(X) \otimes_{\mathcal{C}} T(Y)$$

is the tensor preserving isomorphism of T. We also define the unit

$$I_{\mathcal{C}(T)} = (I, T(I) \cong I)$$

where isomorphism $T(I) \cong I$ is precisely the unit preserving isomorphism of the monoidal functor T. The associativity natural isomorphism for $(-) \otimes_{\mathcal{C}(T)} (-)$ and right, left units for $I_{\mathcal{C}(T)}$ in $\mathcal{C}(T)$ are associavity natural isomorphism and right, left units for $\mathcal{C}$, respectively. The structure makes a functor $\pi : \mathcal{C}(T) \to \mathcal{C}$ strict monoidal and σ a monoidal natural isomorphism. The next result states that the data with these extra monoidal structure form a 2-categorical limit in the 2-category of monoidal categories.

Theorem 1.2. *Let $(\mathcal{C}, T)$ be a pair consiting of a monoidal category and its monoidal endofunctor $T : \mathcal{C} \to \mathcal{C}$. Suppose that $\mathcal{D}$ is a monoidal category, $P : \mathcal{D} \to \mathcal{C}$ is a monoidal functor and $\tau : T \cdot P \Rightarrow P$ is a monoidal natural isomorphisms. Then there exists a unique monoidal functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$ as monoidal functors and monoidal transformations.*

Proof. Note that F must be defined as it was described in the proof of Theorem 1.1. Namely we must have

$$F(X) = (P(X), \tau_X : T(P(X)) \to P(X)), F(f) = P(f)$$

for an object X in $\mathcal{C}$ and a morphism $f : X \to Y$ in $\mathcal{C}$.
Suppose now that F admits a structure of a monoidal functor such that $P = \pi \cdot F$ as monoidal functors. Let

$$\left\{m^F_{X,Y} : F(X \otimes_{\mathcal{D}} Y) \to F(X) \otimes_{\mathcal{C}(T)} F(Y)\right\}_{X,Y \in \mathcal{C}}, \phi^F : F(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$$

be the data forming that structure. Since π is a strict monoidal functor and $P = \pi \cdot F$ as monoidal functors, we derive that for any objects X, Y of $\mathcal{C}$

$$\pi(m^F_{X,Y}) : P(X \otimes_{\mathcal{D}} Y) \to P(X) \otimes_{\mathcal{C}} P(Y)$$

is the tensor preserving isomorphism $m^P_{X,Y} : P(X \otimes_{\mathcal{D}} Y) \to P(X) \otimes_{\mathcal{C}} P(Y)$ of the monoidal functor P. By the same argument

$$\pi(\phi_F) : P(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$$

is the unit preserving isomorphism $\phi^P : P(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$ of P. Thus we deduce that for any objects X, Y of $\mathcal{C}$ we have $m^F_{X,Y} = m^P_{X,Y}$ and $\phi^F = \phi^P$. This implies that there exists at most one monoidal functor F such that $P = \pi \cdot F$ as monoidal functors.
On the other hand define $m^F_{X,Y} = m^P_{X,Y}$ for objects X, Y in $\mathcal{C}$ and $\phi^F = \phi^P$. We check now that F equipped with these data is a monoidal functor. Fix objects X, Y in $\mathcal{C}$. The square

$$\begin{tikzcd}
T\left(P\left(X \otimes_{\mathcal{D}} Y\right)\right) \arrow[r, "\tau_{X \otimes_{\mathcal{C}} Y}"] \arrow[d, "T\left(m^P_{X,Y}\right)"'] & P\left(X \otimes_{\mathcal{C}} Y\right) \arrow[d, "m^P_{X,Y}"] \\
T\left(P(X) \otimes_{\mathcal{C}} P(Y)\right) \arrow[r, "(\tau_X \otimes_{\mathcal{C}} \tau_Y) \cdot m^T_{P(X),P(Y)}"'] & P(X) \otimes_{\mathcal{C}} P(Y)
\end{tikzcd}$$

is commutative due to the fact that $\tau : T \cdot P \Rightarrow P$ is a monoidal natural isomorphisms. This implies that $m^F_{X,Y}$ is a morphism in $\mathcal{C}(T)$. It follows that $m^F_{X,Y}$ is a natural isomorphism and due to the definition of associativity in $\mathcal{C}(T)$, we derive its compatibility with $m^F_{X,Y}$. Similarly, since the square

$$\begin{tikzcd}
T\left(P\left(I_{\mathcal{D}}\right)\right) \arrow[r, "\tau_{I_{\mathcal{D}}}"] \arrow[d, "T\left(\phi^P\right)"'] & P\left(I_{\mathcal{D}}\right) \arrow[d, "\phi^P"] \\
T\left(I_{\mathcal{C}}\right) \arrow[r, "\phi^T"'] & I_{\mathcal{C}}
\end{tikzcd}$$

is commutative, we deduce that ϕ^F is a morphism in $\mathcal{C}(T)$. By definition of left and right unit in $\mathcal{C}(T)$, we derive their compatibility with ϕ^F. This finishes the verification of the fact that F with $\{m^F_{X,Y}\}_{X,Y \in \mathcal{C}}$ and ϕ^F is a monoidal functor. Definitions of $\{m^F_{X,Y}\}_{X,Y \in \mathcal{C}}$ and ϕ^F show that the identities $P = \pi \cdot F$ holds on the level of monoidal structures. Since the 2-forgetful functor from

2-category of monoidal categories into 2-category of categories is faithful on 2-cells, the identity $\sigma_F = \tau$ of natural isomorphisms is also the identity of monoidal natural isomorphisms. □

Theorem 1.3. *Let $(\mathcal{C}, T)$ be a pair consiting of a category and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Assume that T preserves colomits. Then the following assertions hold.*

(1) *$\pi : \mathcal{C}(T) \to \mathcal{C}$ creates colimits.*

(2) *Suppose that $\mathcal{D}$ is a category, $P : \mathcal{D} \to \mathcal{C}$ a functor preserving small colimits and $\tau : T \cdot P \Rightarrow P$ a natural isomorphisms. Then the unique functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$ preserves small colimits.*

Proof. Let I be a small category and $D : I \to \mathcal{C}(T)$ be a diagram such that the composition $\pi \cdot D : I \to \mathcal{C}$ admits a colimit given by cocone $(X, \{g_i\}_{i\in I})$. Since T preserves colimits, we derive that $(T(X), \{T(u_i)\}_{i\in I})$ is a colimit of $T \cdot \pi \cdot D : I \to \mathcal{C}$. Now $\sigma_D : T \cdot \pi \cdot D \to \pi \cdot D$ is a natural isomorphism. Hence there exists a unique arrow $u : T(X) \to X$ such that $u \cdot T(g_i) = g_i \cdot \sigma_{D(i)}$ for $i \in I$. Clearly u is an isomorphism and hence (X, u) is an object of $\mathcal{C}(T)$. Moreover, the family $\{g_i\}_{i\in I}$ together with (X, u) is a colimiting cocone over D. This proves **(1)**. Now **(2)** is a consequence of **(1)**. □

Now we apply the results above to certain more general diagrams of categories.

Definition 1.4. A diagram

$$\ldots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \ldots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

of categories and functors is called *a telescope of categories*.

Definition 1.5. Let

$$\ldots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \ldots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

be a telescope of monoidal categories and monoidal cocontinuous functors. Then *a 2-categorical limit of the telescope* consists of a monoidal category $\mathcal{C}$, a family of monoidal cocontinuous functors $\{\pi_n : \mathcal{C} \to \mathcal{C}_n\}_{n\in\mathbb{N}}$ and a family of monoidal natural isomorphisms $\{\sigma_n : F_{n+1} \cdot \pi_{n+1} \Rightarrow \pi_n\}_{n\in\mathbb{N}}$ such that the following universal property holds. For any monoidal category $\mathcal{D}$, family $\{P_n : \mathcal{D} \to \mathcal{C}_n\}_{n\in\mathbb{N}}$ of cocontinuous monoidal functors and a family $\{\tau_n : F_n P_{n+1} \Rightarrow P_n\}_{n\in\mathbb{N}}$ of monoidal natural isomorphisms there exists a unique monoidal cocontinuous functor $F : \mathcal{D} \to \mathcal{C}$ satisfying $P_n = \pi_n \cdot F$ and $(\sigma_n)_F = \tau_n$ for every $n \in \mathbb{N}$.

Corollary 1.6. *Let*

$$\ldots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \ldots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

be a telescope of monoidal categories and monoidal cocontinuous functors. Then its 2-limit exists.

Proof. We decompose the task of constructing its 2-limit as follows. First note that one may form a product $\mathcal{C} = \prod_{n\in\mathbb{N}} \mathcal{C}_n$. Next the functors $\{F_n\}_{n\in\mathbb{N}}$ induce an endofunctor $T = \prod_{n\in\mathbb{N}} F_n \times t$, where $\mathbf{1}$ is the terminal category (it has single object and single identity arrow) and $t : \mathcal{C}_0 \to \mathbf{1}$ is the unique functor. Consider the category $\mathcal{C}(T)$. We define $\{\pi_n : \mathcal{C}(T) \to \mathcal{C}_n\}_{n\in\mathbb{N}}$ to be a family of functors given by coordinates of $\pi : \mathcal{C}(T) \to \mathcal{C}$ and $\{\sigma_n : F_n \cdot \pi_{n+1} \Rightarrow \pi_n\}_{n\in\mathbb{N}}$ to be a family of natural isomorphisms given by coordinates of $\sigma : \pi \cdot T \Rightarrow \pi$. Now this data form a 2-limit of the telescope by compilation of Theorem 1.2 and Theorem 1.3. □

2. Formal **G**-schemes

This section is devoted to introducing some notions from formal geometry that are central in this notes. We fix a group scheme $\mathbf{G}$ over k.

Definition 2.1. *A formal* $\mathbf{G}$*-scheme* consists of a sequence $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ of $\mathbf{G}$-schemes together with $\mathbf{G}$-equivariant closed immersions

$$Z_0 \hookrightarrow Z_1 \hookrightarrow \dots \hookrightarrow Z_n \hookrightarrow Z_{n+1} \hookrightarrow \dots$$

satisfying the following assertions.

(1) We have $Z_0 = Z_n^{\mathbf{G}}$ scheme-theoretically for every $n \in \mathbb{N}$.

(2) Let $\mathcal{I}_n$ be an ideal of $\mathcal{O}_{Z_n}$ defining Z_0. Then for every $m \leq n$ the subscheme $Z_m \subset Z_n$ is defined by $\mathcal{I}_n^{m+1}$.

Example 2.2. Let Z be a $\mathbf{G}$-scheme. Consider a quasi-coherent ideal $\mathcal{I}$ of fixed point subscheme $Z^{\mathbf{G}}$ of Z. Then for every $n \in \mathbb{N}$ ideal $\mathcal{I}^n$ is $\mathbf{G}$-equivariant and hence

$$V(\mathcal{I}) \hookrightarrow V(\mathcal{I}^2) \hookrightarrow \dots \hookrightarrow V(\mathcal{I}^n) \hookrightarrow \dots$$

is a formal $\mathbf{G}$-scheme. We denote it by $\widehat{Z}$.

Definition 2.3. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ and $\mathcal{W} = \{W_n\}_{n\in\mathbb{N}}$ are formal $\mathbf{G}$-schemes. Then *a morphism* $f : \mathcal{Z} \to \mathcal{W}$ *of formal* $\mathbf{G}$*-schemes* consists of a family of $\mathbf{G}$-equivariant morphisms $f = \{f_n : Z_n \to W_n\}_{n\in\mathbb{N}}$ such that the diagram

$$\begin{tikzcd}
Z_0 \arrow[r, hook] \arrow[d, "f_0"'] & Z_1 \arrow[r, hook] \arrow[d, "f_1"'] & \dots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[d, "f_n"'] & Z_{n+1} \arrow[r, hook] \arrow[d, "f_{n+1}"'] & \dots \\
W_0 \arrow[r, hook] & W_1 \arrow[r, hook] & \dots \arrow[r, hook] & W_n \arrow[r, hook] & W_{n+1} \arrow[r, hook] & \dots
\end{tikzcd}$$

is commutative.

Definition 2.4. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{G}$-scheme. Then there we have the corresponding telescope of monoidal categories

$$\dots \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_{n+1}) \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_n) \longrightarrow \dots \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_2) \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_1) \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_0)$$

and cocontinuous monoidal functors given by restricting $\mathbf{G}$-equivariant quasi-coherent sheaves to closed $\mathbf{G}$-subschemes. Then we define *a category* $\mathfrak{Qcoh}(\mathcal{Z})$ *of quasi-coherent sheaves on* $\mathcal{Z}$ as a monoidal category which is a 2-limit of the telescope above. This category is defined uniquely up to a monoidal equivalence.

Let Z be a $\mathbf{G}$-scheme and let $\mathcal{I}$ be a quasi-coherent ideal of $Z^{\mathbf{G}}$. We have a commutative diagram

$$\begin{tikzcd}
V(\mathcal{I}) \arrow[r, hook] \arrow[drr, hook, bend right] & V(\mathcal{I}^2) \arrow[r, hook] \arrow[dr, hook] & \dots \arrow[r, hook] & V(\mathcal{I}^n) \arrow[r, hook] \arrow[dl, hook] & \dots \\
 & & Z & &
\end{tikzcd}$$

in the category of $\mathbf{G}$-schemes. Thus restriction functors $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}_{\mathbf{G}}(V(\mathcal{I}^n))$ for $n \in \mathbb{N}$ induce a unique cocontinuous monoidal functor $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}(\widehat{Z})$.

Definition 2.5. Let Z be a **G**-scheme. Then a unique cocontinuous monoidal functor $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}(\widehat{Z})$ is called *the comparison functor*.

Definition 2.6. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal **G**-scheme. A **G**-scheme Z is called *an algebraization of* $\mathcal{Z}$ if the following two conditions are satisfied.

(1) $\mathcal{Z}$ is isomorphic to $\widehat{Z}$ in the category of formal **G**-schemes.

(2) The comparison functor $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}\left(\widehat{Z}\right)$ is an equivalence of monoidal categories.

3. Preliminaries

LS TODO: Większość z wyników, które tutaj są, powinna być w teoretyczym wstępie. Idea jest taka, by tutaj w zasadzie tylko przygotować notację do dowodu głównego twierdzenia.

3.1. Toruses.

Definition 3.1. Let T be an affine algebraic group over k. Suppose that there exists $n \in \mathbb{N}$ such that for every algebraically closed extension K of k there exists an isomorphism

$$T_K \cong \underbrace{\mathbb{G}_{m,K} \times \mathbb{G}_{m,K} \times \ldots \times \mathbb{G}_{m,K}}_{n\text{ times}}$$

of group schemes over k. Then T is called *a torus over* k.

Example 3.2. If $T \cong \underbrace{\mathbb{G}_{m,K} \times \mathbb{G}_{m,K} \times \ldots \times \mathbb{G}_{m,K}}_{n\text{ times}}$, then T is a torus. We call toruses T of this form *split toruses*.

Example 3.3. Assume that the characteristic of k is not 2. Define

$$\mathbf{S}^1 = \operatorname{Spec} k[x,y]/(x^2+y^2-1)$$

a scheme over k and let $\mathfrak{P}_{\mathbf{S}^1}$ be its functor of points. Then for every k-algebra A we have

$$\mathfrak{P}_{\mathbf{S}^1}(A) = \left\{(u,v) \in A \times A \,\middle|\, u^2+v^2 = 1\right\}$$

There is also a morphism $\mathfrak{P}_{\mathbf{S}^1} \times \mathfrak{P}_{\mathbf{S}^1} \to \mathfrak{P}_{\mathbf{S}^1}$ of k-functors given by

$$\mathfrak{P}_{\mathbf{S}^1}(A) \times \mathfrak{P}_{\mathbf{S}^1}(A) \to \mathfrak{P}_{\mathbf{S}^1} \ni ((u_1,v_1),(u_2,v_2)) \mapsto (u_1u_2 - v_1v_2, u_1v_2 + u_2v_1) \in \mathfrak{P}_{\mathbf{S}^1}(A)$$

for every k-algebra A. This makes $\mathfrak{P}_{\mathbf{S}^1}$ into a group k-functor. Thus $\mathbf{S}^1$ with the group structure described above is an affine algebraic group over k. We call it *the circle group over* k.
Now suppose that K is an algebraically closed extension of k. Consider an element $i \in K$ such that $i^2 = -1$. For every K-algebra A we have a map

$$\mathfrak{P}_{\mathbf{S}^1}(A) \ni (u,v) \mapsto u + iv \in A^*$$

First note that this map is bijective. Indeed, its inverse is given by

$$A^* \ni a \mapsto \left(\frac{1}{2}(a+a^{-1}), \frac{1}{2i}(a-a^{-1})\right) \in \mathfrak{P}_{\mathbf{S}^1}(A)$$

Moreover, the map $\mathfrak{P}_{\mathbf{S}^1}(A) \to A^*$ is a homomorphism of abstract groups. Thus $\mathfrak{P}_{\mathbf{S}^1}$ resricted to the category $\mathbf{Alg}_K$ of K-algebras is isomorphic with $\mathfrak{P}_{\mathbb{G}_{m,K}}$ as a group k-functor. Hence

$$\mathbf{S}^1_K \cong \mathbb{G}_{m,K}$$

as algebraic group schemes over K. Hence $\mathbf{S}^1$ is a torus over k.
Now assume that $k = \mathbb{R}$. Then abstract groups

$$\mathfrak{P}_{\mathbf{S}^1}(\mathbb{R}) = \left\{z \in \mathbb{C} \,\middle|\, |z| = 1\right\} \subseteq \mathbb{C}^*,\ \mathbb{R}^*$$

are not isomorphic. Indeed, the left hand side group has infinite torsion subgroup and the right hand side group has torsion subgroup equal to $\{-1,1\}$. This imlies that over $\mathbb{R}$ algebraic groups $\mathbf{S}^1$ and $\mathbb{G}_m$ are not isomorphic. Hence $\mathbf{S}^1$ is not a split torus over $\mathbb{R}$.

3.2. Locally linear schemes.

Definition 3.4. Let $\mathbf{M}$ be a monoid k-scheme and let X be a $\mathbf{M}$-scheme. Suppose that each point of X admits an open affine $\mathbf{M}$-stable neighborhood. Then we say that X is *a locally linear* $\mathbf{M}$*-scheme*.

Proposition 3.5. *Let $\mathbf{M}$ be an affine monoid k-scheme and let X be a $\mathbf{M}$-scheme. Suppose that there exists a quasi-coherent $\mathbf{M}$-equivariant ideal $\mathcal{I}$ on X with nilpotent sections. Consider an open subset U of X. Then the following are equivalent.*

(1) *U is $\mathbf{M}$-stable.*

(2) *$U \cap V(\mathcal{I})$ is $\mathbf{M}$-stable.*

Proof. Let $\alpha : \mathbf{M} \times X \to X$ be the action of $\mathbf{M}$ on X. Fix open subset U of X. If U is $\mathbf{M}$-stable, then $U \cap V(\mathcal{I})$ is $\mathbf{M}$-stable. So suppose that $U \cap V(\mathcal{I})$ is $\mathbf{M}$-stable. Since $\mathcal{I}$ has nilpotent sections and $\mathbf{M}$ is affine, we derive that closed immersions $U \cap V(\mathcal{I}) \hookrightarrow U$ and $\mathbf{M} \times (U \cap V(\mathcal{I})) \hookrightarrow \mathbf{M} \times U$ induce homeomorphisms on topological spaces. Consider the commutative diagram

$$\begin{tikzcd}
\mathbf{M} \times U \arrow[r, "\alpha_{|U \cap V(\mathcal{I})}"] & X \\
\mathbf{M} \times (U \cap V(\mathcal{I})) \arrow[r] \arrow[u, hook] & U \cap V(\mathcal{I}) \arrow[u, hook]
\end{tikzcd}$$

where the bottom horizontal arrow is the induced action on $U \cap V(\mathcal{I})$ and vertical morphisms are homeomorphisms. The commutativity of the diagram implies that $\alpha\left(\mathbf{M} \times U\right)$ is contained set-theoretically in U. Since U is open in X, we derive that morphism of schemes $\alpha_{|\mathbf{M} \times U}$ factors through U. Hence U is $\mathbf{M}$-stable. □

Corollary 3.6. *Let $\mathbf{M}$ be an affine monoid k-scheme and let X be a $\mathbf{M}$-scheme. Suppose that there exists a quasi-coherent $\mathbf{M}$-equivariant ideal $\mathcal{I}$ on X such that $\mathcal{I}^n = 0$ for $n \in \mathbb{N}$. Consider an open subset U of X. Then the following are equivalent.*

(1) *U is $\mathbf{M}$-stable and affine.*

(2) *$U \cap V(\mathcal{I})$ is $\mathbf{M}$-stable and affine.*

Proof. Since $\mathcal{I}^n = 0$, we derive that U is affine if and only if $U \cap V(\mathcal{I})$ is affine. Combining this with Proposition 3.5, we deduce the result. □

Corollary 3.7. *Let $\mathbf{M}$ be an affine monoid k-scheme and let X be a $\mathbf{M}$-scheme. Suppose that there exists a quasi-coherent $\mathbf{M}$-equivariant ideal $\mathcal{I}$ on X such that $\mathcal{I}^n = 0$ for $n \in \mathbb{N}$. Then X is locally linear $\mathbf{M}$-scheme if and only if $V(\mathcal{I})$ is locally linear $\mathbf{M}$-scheme.*

Proof. This is a consequence of Corollary 3.6. □

3.3. Affine monoid schemes with zero.

Proposition 3.8. *Let $\mathbf{M}$ be an affine monoid k-scheme with zero and let X be a locally linear $\mathbf{M}$-scheme. Then there exists an affine $\mathbf{M}$-equivariant morphism*

$$X \xrightarrow{r} X^{\mathbf{M}}$$

such that $r_{|X^{\mathbf{M}}} = 1_{X^{\mathbf{M}}}$.

Proof. Consider the action $\alpha : \mathbf{M} \times X \to X$ of $\mathbf{M}$ on X. Since X is locally linear and $\mathbf{M}$ is affine, we derive that α is an affine morphism of k-schemes. Now if $\mathbf{o}$ is a zero of $\mathbf{M}$, then we define a morphism

$$X \xrightarrow{\cong} \mathbf{o} \times X \hookrightarrow \mathbf{M} \times X \xrightarrow{\alpha} X$$

The morphism above is affine (as a composition of affine morphisms) and induces multiplication by $\mathbf{o}$ on functors of points $\mathbf{o} \cdot (-) : \mathfrak{P}_X \to \mathfrak{P}_X$. Now $\mathbf{o} \cdot (-) : \mathfrak{P}_X \to \mathfrak{P}_X$ factors as an $fP_{\mathbf{M}}$-equivariant epimorphism $\mathfrak{P}_X \twoheadrightarrow \mathfrak{P}_{X^{\mathbf{M}}}$ composed with a closed immersion $\mathfrak{P}_{X^{\mathbf{M}}} \hookrightarrow \mathfrak{P}_X$. The $\mathfrak{P}_{\mathbf{M}}$-equivariant epimorphism $\mathfrak{P}_X \to \mathfrak{P}_{X^{\mathbf{M}}}$ corresponds to a $\mathbf{M}$-equivariant morphism $r : X \to X^{\mathbf{M}}$ of k-schemes such that $r_{|X^{\mathbf{M}}} = 1_{X^{\mathbf{M}}}$. Moreover, the composition of r with a closed immersion $X^{\mathbf{M}} \hookrightarrow X$ is an affine morphism. Thus r is affine. $\square$

3.4. **Results on linear representations.**

Proposition 3.9. *Let $\mathbf{M}$ be an affine monoid k-scheme and let V be a representation of $\mathbf{M}$. Then for every k-algebra A the natural morphism of A-modules*

$$V^{\mathbf{M}} \otimes_k A \to (A \otimes_k V)^{\mathbf{M}_A}$$

is an isomorphism.

Proof. Note that we have a left exact sequence of k-vector spaces defining invariants

$$0 \longrightarrow V^{\mathbf{M}} \longrightarrow V \xrightarrow{\Delta - p} \Gamma(\mathbf{M}, \mathcal{O}_{\mathbf{M}}) \otimes_k V$$

where $\Delta : V \to \Gamma(\mathbf{M}, \mathcal{O}_{\mathbf{M}}) \otimes_k V$ is the coaction and $p : V \to \Gamma(\mathbf{M}, \mathcal{O}_{\mathbf{M}}) \otimes_k V$ is the trivial coaction defined by formula $p(v) = 1 \otimes v$ for every v in V. Now tensoring the sequence with k-algebra A yields a left exact sequence

$$0 \longrightarrow V^{\mathbf{M}} \otimes_k A \longrightarrow A \otimes_k V \xrightarrow{\Delta_A - p_A} \Gamma(\mathbf{M}_A, \mathcal{O}_{\mathbf{M}_A}) \otimes_A (A \otimes_k V)$$

where Δ_A is the coaction on $A \otimes_k V$ induced by Δ and p_A is the trivial coaction on $A \otimes_k V$. This shows that $V^{\mathbf{M}} \otimes_k A \to (A \otimes_k V)^{\mathbf{M}_A}$ is an isomorphism. $\square$

Proposition 3.10. *Let $\mathbf{G}$ be an affine group k-scheme and let V, W be representations of $\mathbf{G}$. If V is finite dimensional, then for every k-algebra A the canonical morphism*

$$A \otimes_k \mathrm{Hom}_{\mathbf{G}}(V, W) \longrightarrow \mathrm{Hom}_{\mathbf{G}_A}(A \otimes_k V, A \otimes_k W)$$

is an isomorphism of A-modules.

Proof. Fix a k-algebra A. Since V is finite dimensional, for every k-algebra B there exists an isomorphism $B \otimes_k \mathrm{Hom}_k(V, W) \to \mathrm{Hom}_B(B \otimes_k V, B \otimes_k W)$ of B-modules natural in B. This implies that $\mathrm{Hom}_k(V, W)$ is a representation of $\mathbf{G}$ via the action given by formula

$$(g \cdot f)(v) = g \cdot f(g^{-1} \cdot v)$$

where $f \in \mathrm{Hom}_B(B \otimes_k V, B \otimes_k W)$, $v \in B \otimes_k V$ and $g \in \mathfrak{P}_{\mathbf{G}}(B)$. Similarly $\mathrm{Hom}_A(A \otimes_k V, A \otimes_k W)$ is a representation of $\mathbf{G}_K$ and the canonical isomorphism $A \otimes_k \mathrm{Hom}_k(V, W) \to \mathrm{Hom}_A(A \otimes_k V, A \otimes_k W)$

of A-modules is $\mathbf{G}_A$-equivariant. Now we apply Proposition 3.9 to derive a chain of isomorphisms

$$\mathrm{Hom}_A\left(A\otimes_k V, A\otimes_k W\right)^{\mathbf{G}_A} \cong \left(A\otimes_k \mathrm{Hom}_k(V,W)\right)^{\mathbf{G}_A} \cong A\otimes_k \mathrm{Hom}_k(V,W)^{\mathbf{G}}$$

of A-modules. Since we have identifications

$$\mathrm{Hom}_{\mathbf{G}_A}\left(A\otimes_k V, A\otimes_k W\right) \cong \mathrm{Hom}_A\left(A\otimes_k V, A\otimes_k W\right)^{\mathbf{G}_A},\ \mathrm{Hom}_{\mathbf{G}}\left(V,W\right) \cong \mathrm{Hom}_k\left(V,W\right)^{\mathbf{G}}$$

we deduce the statement. □

Proposition 3.11. *Let $\mathbf{G}$ be an affine group scheme over k and let $\mathfrak{G}$ be a monoid k-functor. Denote by Λ the set of isomorphism classes of irreducible $\mathbf{G}$-representations. Suppose that V is a representation of both $\mathbf{G}$ and $\mathfrak{G}$ and assume that their actions on V commute. Assume that V is completely reducible as a $\mathbf{G}$-representation and consider the decomposition*

$$V = \bigoplus_{\lambda\in\Lambda} V_\lambda$$

onto isotypic components with respect to the action of $\mathbf{G}$. Then for every λ in Λ the subspace V_λ is a $\mathfrak{G}$-subrepresentation of V.

LS TODO: Uzupełnić dowód w oparciu o Stwierdzenie 3.10.

Proof. □

3.5. M-equivariant quasi-coherent sheaves.

LS TODO: Tu trzeba zdefiniować i następnie opisać przypadek schematu z trywialnym działaniem, bo on jest najważniejszy

3.6. Kempf monoids.

LS TODO: Tutaj trzeba zdefiniować monoidy Kempfa. Najpierw trzeba porządnie spisać dowód algebraizacji, żeby mieć poprawną definicję

Definition 3.12. Let $\mathbf{M}$ be a monoid k-scheme. Suppose that the following conditions hold.

(1) $\mathbf{M}$ is affine, geometrically connected and geometrically normal.

(2) There exists zero $\mathbf{o}$ in $\mathbf{M}$.

(3) There exists a torus T over k contained in the center of $\mathbf{M}$ such that the closure $\mathbf{cl}(T)$ of T in $\mathbf{M}$ contains $\mathbf{o}$.

Then $\mathbf{M}$ is called *Kempf monoid*.

Let $\mathbf{M}$ be a Kempf monoid and let $\mathbf{G}$ be its group of units. If V is a representation of $\mathbf{G}$ and λ is a class in Λ, then we denote by $V[\lambda] \subseteq V$ the sum of all irreducible T-subpresentations of V of isomorphism type λ. Since T is a central subgroup of $\mathbf{G}$, we derive by Proposition 3.11 that $V[\lambda]$ is a $\mathbf{G}$-representation of V.
Suppose that Z is a k-scheme with trivial action of $\mathbf{M}$. If $\mathcal{F}$ is a quasi-coherent sheaf on Z equipped with $\mathbf{G}$-action, then we denote by $\mathcal{F}[\lambda]$ a sheaf given by

$$U \mapsto \mathcal{F}(U)[\lambda]$$

for every open affine subset U of Z. Then $\mathcal{F}[\lambda] \subseteq \mathcal{F}$ is a $\mathbf{G}$-quasi-coherent subsheaf of $\mathcal{F}$.

3.7. Formal M-schemes.

LS TODO: (or rather Jelisiejew :D)

Definition 3.13. Let $\mathbf{M}$ be a monoid k-scheme having $\mathbf{G}$ as the group of units. *A formal* $\mathbf{M}$*-scheme* is a formal $\mathbf{G}$-scheme $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ such that for each $n\in\mathbb{N}$ scheme Z_n is $\mathbf{M}$-scheme and the sequence of closed immersions

$$Z_0 \hookrightarrow Z_1 \hookrightarrow \dots \hookrightarrow Z_n \hookrightarrow Z_{n+1} \hookrightarrow \dots$$

consists of $\mathbf{M}$-equivariant morphisms.

Let $\mathbf{M}$ be an affine monoid k-scheme with zero and let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{M}$-scheme. Suppose that $\mathbf{M}$ is a monoid with zero. Then by Proposition 3.8, we derive that $\mathcal{Z}$ is a part of the commutative diagram

$$\begin{tikzcd} Z_0 \arrow[r, hook] \arrow[drr, bend right, two heads, "r_0 = 1_{Z_0}"'] & Z_1 \arrow[r, hook] \arrow[dr, two heads, "r_1"] & \dots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[dl, bend left, two heads, "r_n"] & \dots \\ & & Z_0 & & \end{tikzcd}$$

in which vertical morphisms $r_n : Z_n \twoheadrightarrow Z_0$ are affine morphisms such that $r_{n|Z_0} = 1_{Z_0}$. This implies that Z_n is affine over Z_0 for each $n \in \mathbb{N}$ and hence we write $\mathrm{Spec}_{Z_0}\mathcal{A}_n$ for $n \in \mathbb{N}$, where $\mathcal{A}_n$ is a quasi-coherent Z_0-algebra equipped with the action of $\mathbf{M}$. Moreover, we have $\mathcal{A}_0 = \mathcal{O}_{Z_0}$. The diagram above induces the following sequence of epimorphisms

$$\dots \twoheadrightarrow \mathcal{A}_{n+1} \twoheadrightarrow \mathcal{A}_n \twoheadrightarrow \dots \twoheadrightarrow \mathcal{A}_1 \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$$

of quasi-coherent $\mathcal{O}_{Z_0}$-algebras with $\mathbf{M}$-action. Denote by $\mathbf{G}$ the group of units of $\mathbf{M}$. If $\mathbf{G}$ is schematically dense in $\mathbf{M}$ (for instance if $\mathbf{M}$ is integral), then we have $Z_0 = Z_n^{\mathbf{G}} = Z_n^{\mathbf{M}}$ and hence Z_0 admits trivial $\mathbf{M}$-action. This alternative description of formal $\mathbf{M}$-schemes will be used in the proof of the main theorem.

4. Formal functors and representability - OLD

Theorem 4.1 (Algebraization of a formal $\overline{\mathbf{G}}$-scheme)**.** *Let $\mathcal{Z} = \{Z_n\}$ be a formal $\overline{\mathbf{G}}$-scheme. Then there exists a colimit*

$$Z = \mathrm{colim}_n Z_n$$

in the category of locally linear $\overline{\mathbf{G}}$-schemes and Z is the unique algebraization of $\mathcal{Z}$. If in addition $\mathcal{Z}$ is locally Noetherian, then π_Z is of finite type. If $\mathcal{Z}$ is locally Noetherian and Z_0 is of finite type, then also Z is of finite type.

Now we spell out the main idea of the proof: the $\overline{\mathbf{G}}$-scheme Z required in Theorem 4.1 is equal to $\mathrm{Spec}_{Z_0}\mathcal{A}$, where $\mathcal{A}$ is the limit of $\mathcal{A}_n$ *in the category of* $\overline{\mathbf{G}}$*-algebras*; in other words each isotypic component of $\mathcal{A}$ is the limit of isotypic components of $\mathcal{A}_n$. Our first goal is to prove a stabilization result. We denote by $\mathrm{Irr}(\mathbf{G})$ the set of isomorphism types of irreducible $\mathbf{G}$-representations and by $\mathrm{Irr}(\overline{\mathbf{G}}) \subset \mathrm{Irr}(\mathbf{G})$ the subset of $\overline{\mathbf{G}}$-representations. For $\lambda \in \mathrm{Irr}(\mathbf{G})$ and a quasi-coherent $\overline{\mathbf{G}}$-module $\mathcal{C}$ on Z_0 we denote by $\mathcal{C}[\lambda] \subset \mathcal{C}$ the $\overline{\mathbf{G}}$-submodule such that $H^0(U, \mathcal{C}[\lambda]) \subset H^0(U, \mathcal{C})$ is the union of all $\mathbf{G}$-subrepresentations of $H^0(U, \mathcal{C})$ isomorphic to λ (i.e., the isotypic component of λ).

Lemma 4.1.1 (stabilization on an isotypic component)**.** *Let $\lambda \in \mathrm{Irr}(\overline{\mathbf{G}})$. Then there exists a number $n_\lambda \in \mathbb{N}$ such that the following holds. Let $\mathcal{Z} = \{Z_n\}$ be a formal $\overline{\mathbf{G}}$-scheme and $\{\mathcal{A}_{n+1} \twoheadrightarrow \mathcal{A}_n\}$ be the associated sequence of quasi-coherent $\overline{\mathbf{G}}$-algebras. Then for every $n > n_\lambda$ the surjection*

$$\mathcal{A}_n[\lambda] \twoheadrightarrow \mathcal{A}_{n-1}[\lambda]$$

is an isomorphism. If $\lambda_0 \in \mathrm{Irr}(\overline{\mathbf{G}})$ is the trivial representation, then we may take $n_{\lambda_0} = 0$.

Proof of Lemma 4.1.1. The claims are preserved under field extension, so we may assume our field is algebraically closed (hence perfect) so we may use the Kempf's torus. Fix a grading on $k[\overline{\mathbf{G}}]$ induced by a Kempf's torus for k as in Corollary **??**. Denote by $A_\lambda \subseteq \mathbb{N}$ the set of weights which appear in $k[\mathbf{G}]_\lambda$. Since $\dim_k k[\mathbf{G}]_\lambda$ is finite by Proposition **??**, the set A_λ is finite. Put

$$n_\lambda = \sup A_\lambda.$$

Fix $n > n_\lambda$ and let $\mathcal{I}_n = \ker(\mathcal{A}_n \to \mathcal{A}_0)$. Then we have a decomposition with respect to the chosen torus

$$\mathcal{A}_n = \bigoplus_{i \geq 0} (\mathcal{A}_n)[i],$$

By Corollary **??**, we have $\mathcal{I}_n = \bigoplus_{i\geq 1}(\mathcal{A}_n)[i]$. Since $n > n_\lambda$ we have

$$\mathcal{I}_n^n \subset \bigoplus_{i\geq n}(\mathcal{A}_n)[i] \subseteq \bigoplus_{i\notin A_\lambda}(\mathcal{A}_n)[i]$$

Hence, $\mathcal{I}_n^n[\lambda] = 0$. But $\mathcal{I}_n^n[\lambda] = \ker(\mathcal{A}_n[\lambda] \to \mathcal{A}_{n-1}[\lambda])$, thus $\mathcal{A}_n[\lambda] \to \mathcal{A}_{n-1}[\lambda]$ is an isomorphism. Finally note that $A_{\lambda_0} = \{0\}$. This implies that $n_{\lambda_0} = 0$. □

Proof of Theorem 4.1. Let $\mathcal{A}_n$ be the quasi-coherent $\overline{\mathbf{G}}$-algebras as in (**??**). For $\lambda \in \mathrm{Irr}(\overline{\mathbf{G}})$ we define $\mathcal{A}[\lambda] := \mathcal{A}_n[\lambda]$, where $n \geq n_\lambda$ as in Lemma 4.1.1.

$$\mathcal{A} = \bigoplus_{\lambda\in\mathrm{Irr}(\overline{\mathbf{G}})} \mathcal{A}[\lambda] = \bigoplus_{\lambda\in\mathrm{Irr}(\overline{\mathbf{G}})} \mathcal{A}_{n_\lambda}[\lambda].$$

Clearly $\mathcal{A}[\lambda_0] = \mathcal{A}_0 = \mathcal{O}_{Z_0}$ canonically (where λ_0 is the trivial representation), hence $\mathcal{A}$ is an $\mathcal{O}_{Z_0}$-module. Actually $\mathcal{A} = \lim_n \mathcal{A}_n$ in the category of quasi-coherent $\overline{\mathbf{G}}$-modules on Z_0. We construct the algebra structure on $\mathcal{A}$. For this pick $\eta_1, \eta_2 \in \mathrm{Irr}(\overline{\mathbf{G}})$. Fix the finite set $\{\lambda_1, \ldots, \lambda_s\} \subseteq \mathrm{Irr}(\overline{\mathbf{G}})$ of representations which appear in $k[\overline{\mathbf{G}}]_{\eta_1} \otimes_k k[\overline{\mathbf{G}}]_{\eta_2}$. Then, for every $n \in \mathbb{N}$, we have the multiplication

$$\mathcal{A}_n[\eta_1] \otimes_k \mathcal{A}_n[\eta_2] \to \mathcal{A}_n[\eta_1]\cdot\mathcal{A}_n[\eta_2] \subseteq \bigoplus_{i=1}^{s}\mathcal{A}_n[\lambda_i]$$

and by Lemma 4.1.1 these morphisms can be identified for $n \geq \sup\{n_{\eta_1}, n_{\eta_2}, n_{\lambda_1}, ..., n_{\lambda_s}\}$. We define

$$\mathcal{A}[\eta_1] \otimes_k \mathcal{A}[\eta_2] \to \bigoplus_{i=1}^{s}\mathcal{A}[\lambda_i] \subseteq \mathcal{A}$$

as a morphism induced by the multiplication morphism for any $n \geq \sup\{n_{\eta_1}, n_{\eta_2}, n_{\lambda_1}, \ldots, n_{\lambda_s}\}$. This gives an $\mathcal{O}_{Z_0}$-algebra structure on $\mathcal{A}$, so $\mathcal{A}$ is in fact the limit of $\mathcal{A}_n$ is the category of $\overline{\mathbf{G}}$-algebras. Note that from the description of $\mathcal{A}$ it follows that for every $n \in \mathbb{N}$ we have a surjective morphism $p_n : \mathcal{A} \twoheadrightarrow \mathcal{A}_n$ of $\overline{\mathbf{G}}$-algebras. We denote its kernel by $\mathcal{J}_n$ and we put $\mathcal{J} := \mathcal{J}_0$. The natural injection $\mathcal{O}_{Z_0} = \mathcal{A}_0 \to \mathcal{A}$ is a section of p_0, so that we have

$$\mathcal{J} = \bigoplus_{\lambda\in\mathrm{Irr}(\overline{\mathbf{G}})\setminus\{\lambda_0\}} \mathcal{A}[\lambda].$$

We also denote by $\mathcal{I}_n$ the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$ for $n \in \mathbb{N}$. Then $\mathcal{I}_n = \mathcal{J}/\mathcal{J}_n$. Fix $m \in \mathbb{N}$ and consider $n \in \mathbb{N}$ such that $n \geq m$. Since $\mathcal{Z}$ is a formal $\overline{\mathbf{G}}$-scheme, the sheaf $\mathcal{I}_n^{m+1}$ is the kernel of the morphism $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_m$. Thus

$$\mathcal{J}_m/\mathcal{J}_n = \mathcal{I}_n^{m+1} = (\mathcal{J}^{m+1} + \mathcal{J}_n)/\mathcal{J}_n.$$

Both $\mathcal{J}_m$ and $\mathcal{J}^{m+1}$ are $\mathrm{Irr}(\overline{\mathbf{G}})$-graded and for given $\lambda \in \mathrm{Irr}(\overline{\mathbf{G}})$ and $n \gg 0$ the isotypic component $\mathcal{J}_n[\lambda]$ is zero by Lemma 4.1.1. Hence $\mathcal{J}_m = \mathcal{J}^{m+1}$ for every $m \in \mathbb{N}$. We define

$$Z = \mathrm{Spec}_{Z_0}(\mathcal{A})$$

and we denote by $\pi : Z \to Z_0$ the structural morphism. The scheme Z inherits a $\overline{\mathbf{G}}$-action from $\mathcal{A}$. For every $n \in \mathbb{N}$ the zero-set of $\mathcal{J}^{n+1} \subseteq \mathcal{A}$ is a $\overline{\mathbf{G}}$-scheme isomorphic to Z_n. Hence $\mathcal{Z}$ is isomorphic to $\widehat{Z}$. Thus Z is an algebraization of $\mathcal{Z}$. Since $\mathcal{A} = \lim \mathcal{A}_n$, we have $Z = \mathrm{colim}\, Z_n$ in the category of locally linear $\overline{\mathbf{G}}$-schemes.

It remains to prove uniqueness of algebraization. Let $Z' = \mathrm{Spec}_{Z_0}\mathcal{A}'$ be an algebraization of $\mathcal{Z} = \{Z_n\}$. Then $Z_n \hookrightarrow Z'$, so by the universal property of colimit, we obtain a $\overline{\mathbf{G}}$-morphism $Z \to Z'$, corresponding to $\mathcal{A}' \to \mathcal{A}$. It induces epimorphisms $\mathcal{A}' \twoheadrightarrow \mathcal{A}_n$ for all n. For each $\lambda \in \mathrm{Irr}(\overline{\mathbf{G}})$, the composition

$$\mathcal{A}'[\lambda] \to \mathcal{A}[\lambda] \simeq \mathcal{A}_{n_\lambda}[\lambda]$$

is an epimorphism, hence $\mathcal{A}' \to \mathcal{A}$ is an epimorphism. The kernel of $\mathcal{A}' \to \mathcal{A}$ is equal to

$$\bigcap_n \ker(\mathcal{A}' \to \mathcal{A}_n) = \bigcap_n \ker(\mathcal{A}' \to \mathcal{A}_0)^n.$$

To prove that this kernel is zero, we may enlarge the field to an algebraically closed field, so the result follows from Corollary **??**.

Assume that each scheme Z_n is locally Noetherian over k. Then $\mathcal{I}_n$ is a coherent $\mathcal{A}_n$-module, thus $\mathcal{I}_n^i/\mathcal{I}^{i+1}$ is a coherent $\mathcal{A}_0$-module for all i. The series

$$0 = \mathcal{I}_n^{n+1} \subset \mathcal{I}^n \subset \ldots \subset \mathcal{I} \subset \mathcal{A}_n$$

has coherent subquotients, hence $\mathcal{A}_n$ is a coherent $\mathcal{O}_{Z_n}$-algebra. Thus $\mathcal{A}[\lambda]$ is a coherent $\mathcal{O}_{Z_0}$-module for every $\lambda \in \mathrm{Irr}(\overline{\mathbf{G}})$. The claim that π is of finite type is local on $Z^{\mathbf{G}}$, hence we may assume that $Z^{\mathbf{G}}$ is quasi-compact. The sheaf $\mathcal{J}/\mathcal{J}^2 \subseteq \mathcal{A}_1$ is coherent so there exists a finite set $\lambda_1, \ldots, \lambda_r \in \mathrm{Irr}(\overline{\mathbf{G}}) \smallsetminus \{\lambda_0\}$ such that the morphism

$$\bigoplus_{i=1}^{r} \mathcal{A}[\lambda_i] \to \mathcal{J}/\mathcal{J}^2$$

induced by $\mathcal{A} \twoheadrightarrow \mathcal{A}_2$ is surjective. Let $\mathcal{B} \subset \mathcal{A}$ be the quasi-coherent $\mathcal{O}_{Z_0}$-subalgebra generated by the coherent subsheaf $\mathcal{M} := \bigoplus_{i=1}^{r} \mathcal{A}[\lambda_i] \subseteq \mathcal{A}$. Let $\overline{k}$ be an algebraic closure of k and let $\mathcal{A}' = \mathcal{A} \otimes \overline{k}$. Fix a Kempf's torus over $\overline{k}$ and the associated grading $\mathcal{A}' = \bigoplus_{i\geq 0} \mathcal{A}'[i]$ as in Corollary **??**. Then $\mathcal{J} = \bigoplus_{i\geq 1} \mathcal{A}'[i]$ is a graded ideal and $\mathcal{J}/\mathcal{J}^2$ is generated by the graded (coherent) subsheaf $\mathcal{M}' = \bigoplus_{i=1}^{r} \mathcal{A}'[\lambda_i]$. By graded Nakayama's lemma, the ideal $\mathcal{J}$ itself is generated by (the elements of) $\mathcal{M}'$. Then by induction on the degree, $\mathcal{A}'$ is generated by $\mathcal{M}'$ as an algebra. In other words, $\mathcal{A}' = \mathcal{B} \otimes \overline{k}$. Thus also $\mathcal{A} = \mathcal{B}$ and so $\mathcal{A}$ is of finite type over $\mathcal{O}_{Z_0}$. □

5. ALGEBRAIZATION OF FORMAL **M**-SCHEMES

Now we prove the main result.

Theorem 5.1. *Let* $\mathbf{M}$ *be a Kempf monoid with unit group* $\mathbf{G}$ *and let* $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ *be a formal* $\mathbf{M}$*-scheme. Then there exists an algebraization* Z *of* $\mathcal{Z}$*. Moreover, the following assertions hold.*

(1) *Z is* $\mathbf{M}$*-scheme.*

(2) *The canonical morphism* $\pi : Z \to Z_0$ *is an affine morphism.*

Moreover, if $\mathcal{Z}$ *is locally noetherian, then* π *is of finite type.*

Let $\mathbf{G}$ be the group of units of $\mathbf{M}$. According to the fact that $\mathbf{M}$ is integral, we derive that $\mathbf{G}$ is schematically dense in $\mathbf{M}$ and hence Z_0 admits trivial $\mathbf{M}$-action. Since $\mathbf{M}$ has zero, formal $\mathbf{M}$-scheme $\mathcal{Z}$ corresponds to a sequence

$$\ldots \twoheadrightarrow \mathcal{A}_{n+1} \twoheadrightarrow \mathcal{A}_n \twoheadrightarrow \ldots \twoheadrightarrow \mathcal{A}_1 \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$$

of $\mathbf{M}$-quasi-coherent $\mathcal{O}_{Z_0}$-algebras such that $Z_n = \mathrm{Spec}_{Z_0}\mathcal{A}_n$ for every $n \in \mathbb{N}$. Next since $\mathbf{M}$ is a Kempf monoid, there exists a closed subgroup T of the center $Z(\mathbf{G})$ such that T is a torus and the scheme-theoretic closure $\overline{T}$ of T in $\mathbf{M}$ contains the zero $\mathbf{o}$ of $\mathbf{M}$. Denote by Λ the isomorphism classes of irreducible representations of T. The proof of the theorem is based on the following results.

Lemma 5.1.1. *Let* K *be an algebraically closed field over* k*. Then the following assertions hold.*

(1) $\overline{T}_K = \mathrm{Spec}\, K \times_{\mathrm{Spec}\, k} \overline{T}$ *is an affine toric variety with torus* T_K*.*

(2) *There exists an abstract submonoid* S *of the character group* $\mathcal{X}(T_K)$ *such that*

$$\overline{T}_K = \mathrm{Spec}\, K[S]$$

as T_K*-varieties, where* $K[S]$ *is monoid* K*-algebra of* S*.*

(3) *K-subspace*

$$\mathfrak{m} = \bigoplus_{\chi \in S \setminus \{0\}} K \cdot \chi \subseteq \bigoplus_{\chi \in S} K \cdot \chi = K[S]$$

is an ideal of $K[S]$ and for every finite subset A of $\mathcal{X}(T_K)$ there exists $n_A \in \mathbb{N}$ such that for all $n \geq n_A$ we have

$$K \cdot \chi \notin \mathfrak{m}^n$$

for every $\chi \in A$.

Proof of the lemma. Note that $T \hookrightarrow \overline{T}$ is schematically dense open T-equivariant immersion of affine k-schemes. Since K is flat over k, we derive that $T_K \hookrightarrow \overline{T}_K$ is schematically dense open T_K-immersion. Thus $\overline{T}_K$ is integral affine T_K-scheme and hence $\overline{T}_K$ is a affine toric variety having T_K as a torus. Moreover, zero of **M** is contained in $\overline{T}$. Thus $\overline{T}_K$ admits a fixed K-point with respect to T_K-action. Now the lemma follows by basic affine toric geometry. □

LS TODO: dokładniej ten dowód, lub odniesienie do KOKSA. Nie ma Kempfa HAHA!

Lemma 5.1.2 (Stabilization)**.** *Fix λ in Λ. Then there exists a number $n_\lambda \in \mathbb{N}$ such that the following holds. Let $\mathcal{Z} = \{Z_n\}$ be a formal* **M***-scheme and $\{\mathcal{A}_{n+1} \twoheadrightarrow \mathcal{A}_n\}_{n \in \mathbb{N}}$ be the associated sequence of quasi-coherent* **M***-algebras. Then for every $n > n_\lambda$ the surjection*

$$\mathcal{A}_{n+1}[\lambda] \longrightarrow\!\!\!\!\!\rightarrow \mathcal{A}_n[\lambda]$$

is an isomorphism. If λ_0 is the isomorphism type of trivial representation of **G***, then $n_{\lambda_0} = 0$.*

Proof of the lemma. Fix an algebraically closed field K over k. Consider an irreducible representation V of T in λ. Then $V_K = K \otimes_k V$ is an irreducible representation of T_K. Since V_K is finite dimensional over K, it follows that V_K as a T_K-representation decomposes onto finitely many character spaces of T_K. Say

$$V_K = \bigoplus_{\chi \in A_\lambda} V_K[\chi]$$

for some finite subset $A_\lambda \subseteq \mathcal{X}(T_K)$, where $\mathcal{X}(T_K)$ is the character group of T_K. Next we use Lemma 5.1.1. We can write

$$\overline{T}_K = \operatorname{Spec} K[S], \mathfrak{m} = \bigoplus_{\chi \in S \setminus \{0\}} K \cdot \chi \subseteq \bigoplus_{\chi \in S} K \cdot \chi = K[S]$$

and moreover, there exists $n_\lambda \in \mathbb{N}$ such that for every $n \geq n_\lambda$ we have $K \cdot \chi \notin \mathfrak{m}^n$ for every $\chi \in A_\lambda$. Next $\mathcal{A}_n$ is a quasi-coherent $\overline{T}$-sheaf on Z_0 for every $n \in \mathbb{N}$. Hence

$$K \otimes_k \mathcal{A}_n = \bigoplus_{\chi \in S} (K \otimes_k \mathcal{A}_n)[\chi]$$

for every $n \in \mathbb{N}$. □

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