HAHN-BANACH THEOREM

1. Introduction

2. Hahn-Banach theorem

We start by introducing certain notions concerning real maps defined on $\mathbb{R}$-vector spaces.

Definition 2.1. Let V be an $\mathbb{R}$-vector space. A map $p : V \to \mathbb{R}$ is *subadditive* if

$$p(v_1 + v_2) \leq p(v_1) + p(v_2)$$

for any vectors v_1, v_2 in V.

Definition 2.2. Let V be an $\mathbb{R}$-vector space. A map $p : V \to \mathbb{R}$ is *positive homogeneous* if

$$p(\alpha \cdot v) = \alpha \cdot p(v)$$

for every $\alpha \in \mathbb{R}_+$ and every v in V.

The following is central result of these notes.

Theorem 2.3 (Hahn-Banach)**.** *Let V be an $\mathbb{R}$-vector space and let $p : V \to \mathbb{R}$ be a subadditive and positive homogeneous map. Suppose that W is an $\mathbb{R}$-subspace of V and $f : W \to \mathbb{R}$ is an $\mathbb{R}$-linear map such that*

$$f(w) \leq p(w)$$

for every w in W. Then there exists $\mathbb{R}$-linear map $\tilde{f} : V \to \mathbb{R}$ such that $\tilde{f}_{|W} = f$ and $\tilde{f}(v) \leq p(v)$ for every v in V.

The heart of the proof is the following result.

Lemma 2.3.1. *Let V be an $\mathbb{R}$-vector space and let $p : V \to \mathbb{R}$ be a subadditive and positive homogeneous map. Suppose that W is an $\mathbb{R}$-subspace of V and $f : W \to \mathbb{R}$ is an $\mathbb{R}$-linear map such that*

$$f(w) \leq p(w)$$

for every w in W. Then for every vector $\tilde{v} \in V \setminus W$ there exists $\mathbb{R}$-linear map $\tilde{f} : W + \mathbb{R} \cdot \tilde{v} \to \mathbb{R}$ such that $\tilde{f}_{|W} = f$ and $\tilde{f}(v) \leq p(v)$ for every v in $W + \mathbb{R} \cdot \tilde{v}$.

Proof of the lemma. We claim that the set of $\lambda \in \mathbb{R}$ such that for every $\gamma \in \mathbb{R}$ and every $w \in W$ the following condition is satisfied

$$f(w) + \gamma \cdot \lambda \leq p\big(w + \gamma \cdot \tilde{v}\big)$$

is nonempty. In order to prove this we analyze this condition. Note that for $\gamma = 0$ the condition holds by assumption of the theorem. Thus we may assume that $\gamma \neq 0$. Let $\alpha = |\gamma|$. Now we consider two cases.

- For $\gamma > 0$ the condition is equivalent to

  $$\lambda \leq p\left(\frac{w}{\alpha} + \tilde{v}\right) - f\left(\frac{w}{\alpha}\right)$$

  Since W is an $\mathbb{R}$-vector space, it can be equivalently stated as

  $$\lambda \leq p\left(w + \tilde{v}\right) - f\left(w\right)$$

  for every $w \in W$.

- For $\gamma < 0$ the condition is equivalent to
$$-p\left(\frac{w}{\alpha} - \tilde{v}\right) + f\left(\frac{w}{\alpha}\right) \leq \lambda$$
We invoke the fact that W is an $\mathbb{R}$-vector space one again and obtain equivalent condition
$$-p\left(w - \tilde{v}\right) + f\left(w\right) \leq \lambda$$
for every $w \in W$.

Thus in order to prove our claim it suffices to prove that
$$\sup_{w \in W} -p\left(w - \tilde{v}\right) + f\left(w\right) \leq \inf_{w \in W} p\left(w + \tilde{v}\right) - f\left(w\right)$$
Therefore, it suffices to prove that
$$p\left(w_1 - \tilde{v}\right) + f\left(w_1\right) \leq p\left(w_2 + \tilde{v}\right) - f\left(w_2\right)$$
for any $w_1, w_2 \in W$. Fix arbitrary $w_1, w_2 \in W$. The inequality
$$p\left(w_1 - \tilde{v}\right) + f\left(w_1\right) \leq p\left(w_2 + \tilde{v}\right) - f\left(w_2\right)$$
is equivalent to
$$f(w_1 + w_2) \leq p\left(w_2 + \tilde{v}\right) + p\left(w_1 - \tilde{v}\right)$$
which holds according to
$$f(w_1 + w_2) \leq p(w_1 + w_2) = p\left(w_2 + \tilde{v} + w_1 - \tilde{v}\right) \leq p\left(w_2 + \tilde{v}\right) + p\left(w_1 - \tilde{v}\right)$$
Thus the claim is proved. We infer the statement from the claim as follows. Pick $\lambda \in \mathbb{R}$ such that
$$f(w) + \gamma \cdot \lambda \leq p\big(w + \gamma \cdot \tilde{v}\big)$$
for every $\gamma \in \mathbb{R}$ and every $w \in W$. Then define $\tilde{f} : W + \mathbb{R} \cdot \tilde{v} \to \mathbb{R}$ by $\tilde{f}(w + \gamma \cdot \tilde{v}) = f(w) + \gamma \cdot \lambda$ for every $w \in W$ and $\gamma \in \mathbb{R}$. Then $\tilde{f}$ satisfies the assertion. □

Proof of the theorem. Consider the family $\mathcal{G}$ which consists of $\mathbb{R}$-linear maps $g : U \to \mathbb{R}$ such that U is a $\mathbb{R}$-subspace of V containing W, $g_{|W} = f$ and $g(u) \leq p(u)$ for every $u \in U$. For $g_1 : U_1 \to \mathbb{R}$ and $g_2 : U_2 \to \mathbb{R}$ in $\mathcal{G}$ we define $g_1 \preceq g_2$ if and only if $U_1 \subseteq U_2$ and $(g_2)_{|U_1} = g_1$. Clearly $\preceq$ is a partial order on $\mathcal{G}$. By Zorn's lemma there exists element $\tilde{f} : \tilde{V} \to \mathbb{R}$ in $\mathcal{G}$ maximal with respect to $\preceq$. If $\tilde{V} \subsetneq V$, then by Lemma 2.3.1 there exists element of $\mathcal{G}$ greater than $\tilde{f}$ with respect to $\preceq$. This is a contradiction. Hence $\tilde{V} = V$ and $\tilde{f}$ satisfies the assertion of the theorem. □

3. Normed version of Hahn-Banach theorem