LINEARLY REDUCTIVE GROUPS

1. Motivation – linear representations of compact topological groups

In this section we fix a compact topological group $\mathbf{G}$. Assume that $\rho : \mathbf{G} \to \mathrm{GL}_n(\mathbb{C})$ is a continuous homomorphism i.e. a complex, n-dimensional linear representation of $\mathbf{G}$. For every $g \in \mathbf{G}$ we get a matrix

$$\rho(g) = \left[c_{ij}(g)\right]_{1 \le i,j \le n}$$

For i, j function $c_{ij} : \mathbf{G} \to \mathbb{C}$ is a continuous complex valued function. Alternatively suppose that $\{e_1, e_2, ..., e_n\}$ is the standard basis of $\mathbb{C}^n$ on which $\mathrm{GL}_n(\mathbb{C})$ act. Then c_{ij} is equal to a function

$$\mathbf{G} \ni g \mapsto \langle g \cdot e_i, e_j \rangle \in \mathbb{C}$$

Fix now $g_1, g_2 \in \mathbf{G}$ and note that

$$\left[c_{ij}(g_2 \cdot g_1)\right]_{1 \le i,j \le n} = \rho(g_2 \cdot g_1) = \rho(g_2) \cdot \rho(g_1) = \left[\sum_{k=1}^{n} c_{ik}(g_2) \cdot c_{kj}(g_1)\right]_{1 \le i,j \le n}$$

Hence

$$c_{ij}(g_2 \cdot g_1) = \sum_{k=1}^{n} c_{ik}(g_2) \cdot c_{kj}(g_1)$$

for every $1 \le i, j \le n$. This implies that $\sum_{1 \le i,j \le n} \mathbb{C} \cdot c_{ij} \subseteq \mathcal{L}^2(\mathbf{G}, \mathbb{C})$ is a linear $\mathbf{G} \times \mathbf{G}^{\mathrm{op}}$-subrepresentation of the regular representation $\mathcal{L}^2(\mathbf{G}, \mathbb{C})$. We call it *the matrix coefficients of* ρ.

2. Matrix coefficients of a representation

Proposition 2.1. *Let $\mathfrak{X}$ be a monoid k-functor and let V be a finitely generated, projective k-module. Fix a morphism of monoids $\rho : \mathfrak{X} \to \mathcal{L}_V$. Fix k-algebra A and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. For every A-algebra B and $x \in \mathfrak{X}_A(B)$ we consider the formula*

$$c_{v,w}(x) = \langle \rho_A(x) \cdot v_B, w_B \rangle$$

Then $c_{v,w}$ defines a regular function on $\mathfrak{X}_A$ for every k-algebra A.

Proof. Suppose that $f : B \to C$ is a morphism of A-algebras and pick $x \in \mathfrak{X}_A(B)$. Since ρ_A is natural and $w : A \otimes_k V \to A$ is a morphism of A-modules, we derive that the diagram

$$\begin{tikzcd} V_B \arrow[r, "\rho_A(x)"] \arrow[d, "1_{V_A} \otimes_A f"'] & V_B \arrow[r, "w_B"] \arrow[d, "1_{V_A} \otimes_A f"] & B \arrow[d, "f"] \\ V_C \arrow[r, "\rho_A(\mathfrak{X}_A(f)(x))"'] & V_C \arrow[r, "w_C"'] & C \end{tikzcd}$$

is commutative. Hence

$$c_{v,w}\big(\mathfrak{X}_A(f)(x)\big) = \langle \rho_A\big(\mathfrak{X}_A(f)(x)\big) \cdot v_C, w_C \rangle = f\big(\langle \rho_A(x) \cdot v_B, w_B \rangle\big) = f\big(c_{v,w}(x)\big)$$

and this implies that $c_{v,w} : \mathfrak{X}_A \to \mathbb{A}^1_A$ is natural. □

Definition 2.2. Let $\mathfrak{X}$ be a monoid k-functor and let (V, ρ) be its representation with finitely generated, projective underlying k-module V. Fix k-algebra A and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. Then the regular function $c_{v,w}$ on $\mathfrak{X}_A$ is called *the matrix coefficient of v and w*.

Proposition 2.3. *Let $\mathfrak{X}$ be a monoid k-functor and let (V,ρ) be its representation with finitely generated projective underlying k-module V. Then the following assertions holds.*

(1) *For every k-algebra A map*
$$(A\otimes_k V)\times\left(A\otimes_k V^{\vee}\right)\ni(v,w)\mapsto c_{v,w}\in \mathrm{Mor}_A\left(\mathfrak{X}_A,\mathbb{A}^1_A\right)$$
is A-bilinear.

(2) *The collection of maps*
$$\left\{(A\otimes_k V)\times\left(A\otimes_k V^{\vee}\right)\ni(v,w)\mapsto c_{v,w}\in \mathrm{Mor}_A\left(\mathfrak{X}_A,\mathbb{A}^1_A\right)\right\}_{A\in\mathbf{Alg}_k}$$
gives rise to a morphism of k-functors
$$V_{\mathrm{a}}\times V^{\vee}_{\mathrm{a}}\longrightarrow \mathcal{M}\mathrm{or}_k\left(\mathfrak{X},\mathbb{A}^1_k\right)$$

Proof. We left the proof of **(1)** to the reader.
We prove **(2)**. Consider k-algebra A and an A-algebra B with structural morphism $f:A\to B$. Fix $v\in A\otimes_k V$, $w\in A\otimes_k V^{\vee}$. We prove that restriction of $c_{v,w}:\mathfrak{X}_A\to\mathbb{A}^1_A$ to the category $\mathbf{Alg}_B$ is c_{v_B,w_B}. For this pick a B-algebra C and an element $x\in\mathfrak{X}_A(C)=\mathfrak{X}_B(C)$. Note that
$$c_{v,w}(x)=\langle\rho_A(x)\cdot v_C,w_C\rangle=\langle\rho_B(x)\cdot v_C,w_C\rangle=\langle\rho_B(x)\cdot(v_B)_C,(w_B)_C\rangle=c_{v_B,w_B}(x)$$
and hence $c_{v,w|\mathbf{Alg}_B}=c_{v_B,w_B}$. Consider the square

$$\begin{tikzcd}
V_{\mathrm{a}}(A)\times V^{\vee}_{\mathrm{a}}(A) \arrow[r] \arrow[d, "V_a(f)\times V_a^{\vee}(f)"'] & \mathcal{M}\mathrm{or}_k\left(\mathfrak{X},\mathbb{A}^1\right)(A) \arrow[d, "\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathbb{A}^1)(f)"] \\
V_{\mathrm{a}}(B)\times V^{\vee}_{\mathrm{a}}(B) \arrow[r] & \mathcal{M}\mathrm{or}_k\left(\mathfrak{X},\mathbb{A}^1\right)(B)
\end{tikzcd}$$

in which both horizontal arrows are given by formula $(v,w)\mapsto c_{v,w}$. We proved that the square commutes. Since f is an arbitrary morphism of k-algebras, we conclude the assertion. □

Corollary 2.4. *Let $\mathfrak{X}$ be a monoid k-functor and let (V,ρ) be its representation with finitely generated projective underlying k-module V. Then there exists a morphism of k-functors*
$$\left(V\otimes_k V^{\vee}\right)_{\mathrm{a}}\xrightarrow{\;c\;}\mathcal{M}\mathrm{or}_k\left(\mathfrak{X},\mathbb{A}^1_k\right)$$
given by formula
$$(A\otimes_k V)\otimes_A\left(A\otimes_k V^{\vee}\right)\ni(v,w)\mapsto c_{v,w}\in\mathrm{Mor}_A\left(\mathfrak{X}_A,\mathbb{A}^1_A\right)$$
Moreover, c is a morphism of k-functors equipped with $\mathfrak{X}\times\mathfrak{X}^{\mathrm{op}}$-actions.

Proof. The first part is an immediate consequence of Proposition 2.3. We prove that c is a morphism of k-functors equipped with $\mathfrak{X}\times\mathfrak{X}^{\mathrm{op}}$-actions. For this we fix a k-algebra k and elements $v\in A\otimes_k V$, $w\in A\otimes_k V^{\vee}$. Pick a morphism of k-algebras $f:A\to B$, $(y,z)\in\mathfrak{X}(A)\times\mathfrak{X}(A)^{\mathrm{op}}$ and $x\in\mathfrak{X}_A(B)$. Then we have
$$c_{\rho(y)\cdot v,w\cdot\rho(z)}(x)=\left\langle\rho_A(x)\cdot(\rho(y)\cdot v)_B,(w\cdot\rho(z))_B\right\rangle=$$
$$=\left\langle\rho_A(x)\cdot\rho_A((\mathfrak{X}_A(f)(y)))\cdot v_B,w_B\cdot\rho_A\left(\mathfrak{X}_A(f)(z)\right)\right\rangle=w_B\big(\rho_A\left(\mathfrak{X}_A(f)(z)\right)\cdot\rho_A(x)\cdot\rho_A\left(\mathfrak{X}_A(f)(y)\right)\cdot v_B\big)=$$
$$=w_B\big(\rho_A\left(\mathfrak{X}_A(f)(z)\cdot x\cdot\mathfrak{X}_A(f)(y)\right)\cdot v_B\big)=\big\langle\rho_A\left(\mathfrak{X}_A(f)(z)\cdot x\cdot\mathfrak{X}_A(f)(y)\right)\cdot v_B,w_B\big\rangle=$$
$$=c_{v,w}\big(\mathfrak{X}_A(f)(z)\cdot x\cdot\mathfrak{X}_A(f)(y)\big)$$
and hence c is a morphism of k-functors equipped with actions of $\mathfrak{X}\times\mathfrak{X}^{\mathrm{op}}$. □

3. k-FUNCTORS OF MONOIDS AND THEIR LINEAR REPRESENTATIONS

In the sequel we assume that the reader is familiar with notions of a monoid, group etc. in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [?, pages 2-5].

Definition 3.1. *A monoid (group) k-functor* is a monoid (group) object in the category of k-functors.

Next we introduce an important notion of a linear representation of a monoid k-functor. For this we define k-functors associated with modules over k and discuss their properties.

Example 3.2. Let V be a k-module. We define a k-functor V_{a}. We set

$$V_{\mathrm{a}}(A) = A \otimes_k V,\ V_{\mathrm{a}}(f) = f \otimes_k 1_V$$

for every k-algebra A and every morphism $f : A \to B$ of k-algebras. Moreover, V_{a} admits a structure of a commutative group k-functor. Indeed, $V_{\mathrm{a}}(A)$ is a commutative group with respect to addition induced by its structure of A-module and $V_{\mathrm{a}}(f) : V_{\mathrm{a}}(A) \to V_{\mathrm{a}}(B)$ preserves the addition.

Suppose now that V, W are k-modules and $\sigma : (V_{\mathrm{a}})_A \to (W_{\mathrm{a}})_A$ is a morphism of A-functors for some k-algebra A. Then for every A-algebra B we denote by $\sigma^B : B \otimes_k V \to B \otimes_k W$ the component of σ for B.

Definition 3.3. Let V, W be k-modules and let A be a k-algebra. A morphism $\sigma : (V_{\mathrm{a}})_A \to (W_{\mathrm{a}})_A$ of A-functors is *linear* if for every A-algebra B the component $\sigma^B : B \otimes_k V \to B \otimes_k W$ is a morphism of B-modules.

Next result characterizes linear morphism.

Fact 3.4. *Let V, W be k-modules and let A be a k-algebra. Suppose that $\phi : A \otimes_k V \to A \otimes_k W$ is a morphism of A-modules. Then there exists a unique linear morphism $\sigma : (V_{\mathrm{a}})_A \to (W_{\mathrm{a}})_A$ of A-functors such that $\sigma^A = \phi$.*

Proof. Note that if such σ exists, then by requirement $\sigma^A = \phi$ for every morphism $f : A \to B$ of k-algebras the following diagram

$$\begin{tikzcd}
A \otimes_k V \arrow[r, "\phi"] \arrow[d, "f \otimes_k 1_V"'] & A \otimes_k W \arrow[d, "f \otimes_k 1_W"] \\
B \otimes_k V \arrow[r, "\sigma^B"'] & B \otimes_k W
\end{tikzcd}$$

must commute. We make this into a definition of a morphism σ^B of B-modules. It is a matter of linear algebra that this diagram uniquely determines σ^B and also that $\sigma^A = \phi$. It remains to verify that $\sigma = \{\sigma^B\}_{B \in \mathbf{Alg}_A}$ defined in such a way is a morphism of A-functors. For this suppose that $f : A \to B$ and $g : B \to C$ are morphisms of k-algebras. Then we have

$$\sigma^C \cdot (g \otimes_k 1_V) \cdot (f \otimes_k 1_V) = \sigma^C \cdot ((g \cdot f) \otimes_k 1_V) = ((g \cdot f) \otimes_k 1_W) \cdot \phi =$$
$$= (g \otimes_k 1_W) \cdot (f \otimes_k 1_V) \cdot \phi = (g \otimes_k 1_W) \cdot \sigma^B \cdot (f \otimes_k 1_V)$$

and hence $\sigma^C \cdot (g \otimes_k 1_V) = (g \otimes_k 1_W) \cdot \sigma^B$. Thus σ is a linear morphism of A-functors. □

We restate Fact 3.4 in the form of the following result.

Corollary 3.5. *Let V, W be k-modules and A be a k-algebra. Consider the map*

$$\mathrm{Hom}_A(A\otimes_k V, A\otimes_k W)\longrightarrow \mathrm{Mor}_A\big((V_{\mathrm{a}})_A,(W_{\mathrm{a}})_A\big)$$

that sends morphism ϕ to a unique linear morphism $\sigma:(V_{\mathrm{a}})_A\to(W_{\mathrm{a}})_A$ of A-functors such that $\sigma^A=\phi$. Then this map is injective and its image consists of all linear morphisms of A-functors.

Example 3.6. Let V be a k-module. We define a k-functor $\mathcal{L}_V$. We set

$$\mathcal{L}_V(A)=\mathrm{Hom}_A(A\otimes_k V, A\otimes_k V)$$

for every k-algebra A. Next for every morphism $f:A\to B$ of k-algebras and every morphism $\phi:A\otimes_k V\to A\otimes_k V$ of A-modules we define $\mathcal{L}_V(f)(\phi)$ as a unique morphism of B-modules such that the diagram

$$\begin{tikzcd} A\otimes_k V \arrow[r, "\phi"] \arrow[d, "f\otimes_k 1_V"'] & A\otimes_k V \arrow[d, "f\otimes_k 1_V"] \\ B\otimes_k V \arrow[r, "\mathcal{L}_V(\phi)"'] & B\otimes_k V \end{tikzcd}$$

is commutative. Note also that $\mathcal{L}_V(A)$ is a monoid with respect to the usual composition of morphism of A-modules and $\mathcal{L}_V(f):\mathcal{L}_V(A)\to\mathcal{L}_V(B)$ preserves this composition. Hence $\mathcal{L}_V$ is a monoid k-functor.

Remark 3.7. Corollary 3.5 implies that there are injective maps that make the square

$$\begin{tikzcd} \mathcal{L}_V(A) \arrow[r, hook] \arrow[d, "\mathcal{L}_V(f)"'] & \mathrm{Mor}_A\big((V_{\mathrm{a}})_A,(V_{\mathrm{a}})_A\big) \arrow[d, "\sigma\mapsto\sigma_B"] \\ \mathcal{L}_V(B) \arrow[r, hook] & \mathrm{Mor}_B\big((V_{\mathrm{a}})_B,(V_{\mathrm{a}})_B\big) \end{tikzcd}$$

commutative for every morphism $f:A\to B$ of k-algebras. It also shows that for every k-algebra A this identifies $\mathcal{L}_V(A)$ with a subset of the class $\mathrm{Mor}_A\big((V_{\mathrm{a}})_A,(V_{\mathrm{a}})_A\big)$ consisting of all linear morphism of the A-functor $(V_{\mathrm{a}})_A$.

The discussion below is partially an application of the main result in [?, section 6] (Remark 3.7 shows that $\mathcal{L}_V$ is a subcopresheaf of internal endomorphisms of V_{a} and hence the machinery developed in the citation above can be applied), but for the reader's convenience we decide to include all essential details even if this requires repetition.
Let $\mathfrak{M}$ be a monoid k-functor and let be V be a k-module. Suppose that $\alpha:\mathfrak{M}\times V_{\mathrm{a}}\to V_{\mathrm{a}}$ is an action of $\mathfrak{M}$ on V_{a}. Assume that A is a k-algebra and $x\in\mathfrak{M}(A)$. We denote by $i_x:\mathbf{1}_A\to\mathfrak{M}_A$ the morphism of A-functors corresponding to x by means of [?, Fact 2.4]. Since $\mathbf{1}_A$ is terminal A-functor, a morphism $\alpha_A\cdot\left(i_x\times 1_{(V_{\mathrm{a}})_A}\right)$ is isomorphic to a morphism $\alpha_x:(V_{\mathrm{a}})_A\to(V_{\mathrm{a}})_A$ of A-functors.

Definition 3.8. Let $\mathfrak{M}$ be a monoid k-functor and let V be a k-module. An action $\alpha:\mathfrak{M}\times V_{\mathrm{a}}\to V_{\mathrm{a}}$ of $\mathfrak{M}$ such that for any k-algebra A and point $x\in\mathfrak{M}(A)$ morphism α_x is linear is called *a linear action of $\mathfrak{M}$ on V.*

Now we continue the discussion assuming that $\alpha:\mathfrak{M}\times V_{\mathrm{a}}\to V_{\mathrm{a}}$ is a linear action. We define a morphism $\rho:\mathfrak{M}\to\mathcal{L}_V$ of k-functors by formula $\rho(x)=\alpha_x^A$. We first check that ρ really is a morphism of k-functors. For this fix morphism $f:A\to B$ of k-algebras and $x\in\mathfrak{M}(A)$. Then

$\alpha_{\mathfrak{M}(f)(x)}$ is a morphism of B-functors isomorphic with $\alpha_B \cdot \left(i_{\mathfrak{M}(f)(x)} \times 1_{(V_{\mathrm{a}})_B}\right)$ and since

$$\alpha_B \cdot \left(i_{\mathfrak{M}(f)(x)} \times 1_{(V_{\mathrm{a}})_B}\right) = \alpha_B \cdot \left(i_x \times 1_{(V_{\mathrm{a}})_A}\right)_B = \left(\alpha_A \cdot \left(i_x \times 1_{(V_{\mathrm{a}})_A}\right)\right)_B$$

we derive that $\alpha_{\mathfrak{M}(f)(x)} = (\alpha_x)_B$. This implies that

$$\rho\left(\mathfrak{M}(f)(x)\right) = \alpha^B_{\mathfrak{M}(f)(x)} = \left((\alpha_x)_B\right)^B = \alpha^B_x = \mathcal{L}_V(f)\left(\alpha^A_x\right) = \mathcal{L}_V(f)(\rho(x))$$

and thus ρ is a morphism of k-functors. Now we show that ρ is a morphism of monoids. For this pick k-algebra A and $x, y \in \mathfrak{M}(A)$. Since α is an action, we deduce that $\alpha_{x\cdot y} = \alpha_x \cdot \alpha_y$ and hence also

$$\rho(x \cdot y) = \alpha^A_{x\cdot y} = \alpha^A_x \cdot \alpha^A_y = \rho(x) \cdot \rho(y)$$

Therefore, ρ is a morphism of monoid k-functors. This shows how to construct a morphism of monoid k-functors ρ from linear action α.

Theorem 3.9. *Let $\mathfrak{M}$ be a monoid k-functor and let V be a k-module. Suppose that*

$$\left\{\textit{Linear actions of } \mathfrak{M} \textit{ on } V\right\} \xrightarrow{\Phi} \left\{\textit{Morphisms } \rho : \mathfrak{M} \to \mathcal{L}_V \textit{ of monoid k-functors}\right\}$$

is a map of classes described above. Then Φ is bijection.

Proof. We may refer to [**?**, Theorem 6.3], but for self-containment of the presentation let us give a direct proof of this important result.
Our goal is to construct the inverse of Φ. We construct a map

$$\left\{\text{Morphisms } \rho : \mathfrak{M} \to \mathcal{L}_V \text{ of monoid } k\text{-functors}\right\} \xrightarrow{\Psi} \left\{\text{Linear actions of } \mathfrak{M} \text{ on } V\right\}$$

of classes. For this we start with morphism of k-monoids $\rho : \mathfrak{M} \to \mathcal{L}_V$ and we construct a linear action $\alpha : \mathfrak{M} \times V_{\mathrm{a}} \to V_{\mathrm{a}}$ out of ρ. Consider k-algebra A and A-points $x \in \mathfrak{M}(A)$, $v \in V_{\mathrm{a}}(A)$ we set $\alpha^A(x, v) = \rho(x)(v)$. We verify that α is a morphism of k-functors. For this we pick morphism $f : A \to B$ of k-algebras. Note that we have (Example 3.6) a commutative square

$$\begin{tikzcd}
A \otimes_k V \arrow[r, "\rho(x)"] \arrow[d, "f \otimes_k 1_V"'] & A \otimes_k V \arrow[d, "f \otimes_k 1_V"] \\
B \otimes_k V \arrow[r, "\mathcal{L}_V(\rho(x))"'] & B \otimes_k V
\end{tikzcd}$$

Hence

$$\alpha^B\left(\mathfrak{M}(f)(x), V_{\mathrm{a}}(f)(v)\right) = \rho\left(\mathfrak{M}(f)(x)\right)\left(V_{\mathrm{a}}(f)(v)\right) = \mathcal{L}_V(f)\left(\rho(x)\right)\left(V_{\mathrm{a}}(f)(v)\right) =$$

$$= V_{\mathrm{a}}(f)\left(\rho(x)(v)\right) = V_{\mathrm{a}}(f)\left(\alpha^A(x, v)\right)$$

Thus α is a morphism of k-functors. Next suppose that $y \in \mathfrak{M}(A)$. Then

$$\alpha^A\left(y, \alpha^A(x, v)\right) = \rho(y)\left(\rho(x)(v)\right) = \left(\rho(y) \cdot \rho(x)\right)(v) = \rho\left(x \cdot y\right)(v) = \alpha^A\left(x \cdot y, v\right)$$

Therefore, α is an action of $\mathfrak{M}$ on V_{a}. It is also linear action and we define $\Psi(\rho) = \alpha$.
Finally routine verification shows that $\Psi \cdot \Phi$ and $\Phi \cdot \Psi$ are identity maps of classes. Thus Φ is a bijection. □

Theorem 3.9 states that for every monoid k-functor $\mathfrak{M}$ and every k-module V linear actions $\alpha : \mathfrak{M} \times V_{\mathrm{a}} \to V_{\mathrm{a}}$ and morphisms $\rho : \mathfrak{M} \to \mathcal{L}_V$ of k-monoids are in bijective correspondence. This shows that the formal machinery developed so far works as expected. Now we introduce the following notion.

Definition 3.10. Let $\mathfrak{M}$ be a monoid k-functor. A pair (V, ρ) consisting of a k-module V and a morphism $\rho : \mathfrak{M} \to \mathcal{L}_V$ of k-monoids is called *a linear representation of* $\mathfrak{M}$.

Proposition 3.11. *Let $\mathfrak{M}$ be a monoid k-functor. Suppose that $\alpha : \mathfrak{M} \times V_{\mathrm{a}} \to V_{\mathrm{a}}$, $\beta : \mathfrak{M} \times W_{\mathrm{a}} \to W_{\mathrm{a}}$ are k-linear actions on k-modules V and W, respectively. Suppose that $\sigma : V_{\mathrm{a}} \to W_{\mathrm{a}}$ is a linear morphism of k-functors and $\phi = \sigma^k$ is the corresponding morphism of k-modules. Then the following assertions are equivalent.*

(i) *The square*

$$\begin{tikzcd} \mathfrak{M} \times V_{\mathrm{a}} \arrow[r, "1_{\mathfrak{M}} \times \sigma"] \arrow[d, "\beta"'] & \mathfrak{M} \times W_{\mathrm{a}} \arrow[d, "\alpha"] \\ V_{\mathrm{a}} \arrow[r, "\sigma"'] & W_{\mathrm{a}} \end{tikzcd}$$

is commutative.

(ii) *For every k-algebra A and $x \in \mathfrak{M}(A)$ we have*

$$(1_A \otimes_k \phi) \cdot \rho(x) = \delta(x) \cdot (1_A \otimes_k \phi)$$

where $\rho : \mathfrak{M} \to \mathcal{L}_V$ and $\delta : \mathfrak{M} \to \mathcal{L}_W$ are morphism of monoid k-functors corresponding to α and β, respectively.

Proof. Indeed, conditions expressed in **(i)** and **(ii)** are directly translatable to each other by virtue of Fact 3.4 and the bijection in Theorem 3.9. □

Definition 3.12. Let $\mathfrak{M}$ be a monoid k-functor and let (V, ρ), (W, δ) be its linear representations. A morphism $\phi : V \to W$ of k-modules such that for every k-algebra A and $x \in \mathfrak{M}(A)$ we have

$$(1_A \otimes_k \phi) \cdot \rho(x) = \delta(x) \cdot (1_A \otimes_k \phi)$$

is called *a morphism of linear representations of* $\mathfrak{M}$.

Let $\mathfrak{M}$ be a monoid k-functor. We denote by $\mathbf{Rep}(\mathfrak{M})$ its category of linear representations.

4. The category of linear representations

In this section we fix a monoid k-functor $\mathfrak{M}$. Note that there exists the forgetful functor $\mathbf{Rep}(\mathfrak{M}) \to \mathbf{Mod}(k)$ that sends each linear representation to its underlying k-module.

Theorem 4.1. *The forgetful functor*

$$\mathbf{Rep}(\mathfrak{M}) \longrightarrow \mathbf{Mod}(k)$$

creates small colimits.

Proof. Suppose that $I \ni i \mapsto (V_i, \rho_i) \in \mathbf{Rep}(\mathfrak{M})$ is a diagram of linear representations of $\mathfrak{M}$ indexed by some category I. Let V together with $u_i : V_i \to V$ for $i \in I$ be a colimit of the diagram $I \ni i \mapsto V_i \in \mathbf{Mod}(k)$.
Assume first that (V, ρ) is a structure of the linear representation of $\mathfrak{M}$ on V such that $u_i : V_i \to V$ for $i \in I$ becomes a cocone over the diagram $I \ni i \mapsto (V_i, \rho_i) \in \mathbf{Rep}(\mathfrak{M})$. For every k-algebra A

the functor $A \otimes_k (-)$ preserves colimits and hence $1_A \otimes_k u_i$ for $i \in I$ is a colimit of the diagram $I \ni i \mapsto 1_A \otimes_k V_i \in \mathbf{Mod}(A)$. For each $i \in I$ we have an action $\rho_i^A : \mathfrak{M}(A) \to \mathrm{Hom}_A\left(A \otimes_k V_i, A \otimes_k V_i\right)$ of $\mathfrak{M}(A)$ on $A \otimes_k V_i$ and we may view the diagram $I \ni i \mapsto 1_A \otimes_k V_i \in \mathbf{Mod}(A)$ as a diagram in the category of A-modules equipped with $\mathfrak{M}(A)$-actions given by A-module morphisms. We refer to this category as to category of A-linear $\mathfrak{M}(A)$-actions. Now the forgetful functor

$$\left\{\text{the category of } A\text{-linear } \mathfrak{M}(A)\text{-actions}\right\} \longrightarrow \mathbf{Mod}(A)$$

creates small limits. Indeed, the category on the right hand side is isomorphic with the category $\mathbf{Mod}\left(A[\mathfrak{M}(A)]\right)$ of left modules over the monoid A-algebra $A[\mathfrak{M}(A)]$ and the forgetful functor

$$\mathbf{Mod}\left(A[\mathfrak{M}(A)]\right) \longrightarrow \mathbf{Mod}(A)$$

creates small colimits. This implies that $\rho^A : \mathfrak{M}(A) \to \mathrm{Hom}_A(A \otimes_k V, A \otimes_k V)$ must be a unique morphism of monoids such that $1_A \otimes_k u_i$ for every $i \in I$ is a morphism of A-modules with A-linear action of $\mathfrak{M}(A)$. This implies that ρ is unique and hence (V, ρ) is unique lift of $(V, \{u_i\}_{i \in I})$ to $\mathbf{Rep}(\mathfrak{M})$. This shows the uniqueness of a lift.
For the existence assume for given k-algebra A that $\rho^A : \mathfrak{M}(A) \to \mathrm{Hom}_A(A \otimes_k V, A \otimes_k V)$ is a unique morphism of monoids such that $1_A \otimes_k u_i$ for every $i \in I$ is a morphism of A-modules with A-linear action of $\mathfrak{M}(A)$. Note that ρ^A exists because the forgetful functor

$$\left\{\text{the category of } A\text{-linear } \mathfrak{M}(A)\text{-actions}\right\} \longrightarrow \mathbf{Mod}(A)$$

creates small colimits. Denote $\rho = \{\rho^A\}_{A \in \mathbf{Alg}_k}$. We verify that ρ is a morphism of k-functors $\rho : \mathfrak{M} \to \mathcal{L}_V$. For this consider morphism $f : A \to B$ of k-algebras and the commutative square

$$\begin{tikzcd}
A \otimes_k V_i \arrow[r, "1_A \otimes_k u_i"] \arrow[d, "f \otimes_k 1_{V_i}"'] & A \otimes_k V \arrow[d, "f \otimes_k 1_V"] \\
B \otimes_k V_i \arrow[r, "1_B \otimes_k u_i"'] & B \otimes_k V
\end{tikzcd}$$

defined for every $i \in I$. Note that the top row of the square is a morphism of A-modules with A-linear $\mathfrak{M}(A)$-actions. Similarly interpreting $B \otimes_k V_i$ and $B \otimes_k V$ as A-modules with A-linear actions of $\mathfrak{M}(A)$ given by $\rho_i^B \cdot \mathfrak{M}(f)$ and $\rho^B \cdot \mathfrak{M}(f)$, respectively, we derive that the square consists of A-modules with A-linear actions of $\mathfrak{M}(A)$ and all maps preserve actions except possibly $f \otimes_k 1_V$. Since $A \otimes_k V$ together with $1_A \otimes_k u_i$ for $i \in I$ is a colimit of $I \ni i \mapsto 1_A \otimes_k V_i \in \mathbf{Mod}(A)$ in the category of A-modules, we deduce that $f \otimes_k 1_V$ is the only morphism of A-modules making squares commutative for all $i \in I$. Since $A \otimes_k V$ with ρ^A and $1_A \otimes_k u_i$ for $i \in I$ is a colimit of the same diagram, but interpreted as a diagram of A-modules with A-linear action of $\mathfrak{M}(A)$-modules, we derive from uniqueness of $f \otimes_k 1_V$ that it must also preserve $\mathfrak{M}(A)$-action. Hence $(f \otimes_k 1_V) \cdot \rho^A = \rho^B \cdot \mathfrak{M}(f)$. Thus ρ is a morphism of k-functors. By definition of ρ^A for each k-algebra A, we derive that it is a morphism of monoid k-functors. Hence (V, ρ) is a linear representation of $\mathfrak{M}$ and again by componentwise definition of ρ we deduce that (V, ρ) is a colimit of the diagram $I \ni i \mapsto (V_i, \rho_i) \in \mathbf{Rep}(\mathfrak{M})$. □

Theorem 4.2. *Let A be a commutative ring. The following assertions are equivalent.*

(i) Spec *A is a Hausdorff space.*

(ii) *Every prime ideal of A is maximal.*

(iii) *Every $A/\mathcal{N}$-module is flat, where $\mathcal{N}$ is a nilradical of A.*

(iv) *Every finitely generated ideal of A is generated by an idempotent.*

Lemma 4.2.1. *Let A be a commutative ring and M be an A-module. Then M is flat if and only if $M_{\mathfrak{p}}$ is flat for all $\mathfrak{p} \in \operatorname{Spec} A$.*

Proof of the lemma. For every $\mathfrak{p} \in \operatorname{Spec} A$ we have a natural isomorphism

$$M_{\mathfrak{p}} \otimes_A (-) \cong (M \otimes_A (-))_{\mathfrak{p}}$$

Now the statement follows from the fact that a chain complex of A-modules is exact if and only if it is exact after localization in every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ □

Lemma 4.2.2. *Let A be a local ring such that each A-module is flat. Then A is a field.*

Proof of the lemma. Let $\mathfrak{m}$ be a maximal ideal of A and k be a residue field. Pick finitely generated ideal $\mathfrak{a} \subseteq \mathfrak{m}$. Consider the canonical exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow A \xrightarrow{a \mapsto a \bmod \mathfrak{a}} A/\mathfrak{a} \longrightarrow 0$$

Since k is a flat A-module, we derive that the sequence

$$0 \longrightarrow k \otimes_A \mathfrak{a} \longrightarrow k \xrightarrow{\alpha \mapsto \alpha \bmod \mathfrak{a}k} k/\mathfrak{a}k \longrightarrow 0$$

is exact. Since $\mathfrak{a}k = 0$ because $\mathfrak{a} \subseteq \mathfrak{m}$, we deduce from the short exact sequence that $k \otimes_A \mathfrak{a} = 0$. By Nakayama lemma this implies that $\mathfrak{a} = 0$ ($\mathfrak{a}$ is finitely generated over A). Thus every finitely generated A-submodule of $\mathfrak{m}$ is trivial. Thus $\mathfrak{m} = 0$ and hence A is a field. □

Proof. □

shows that $\mathcal{L}_V$ is a subcopresheaf of internal endomorphisms of V_{a} and hence the machinery developed in the citation above can be applied), but for the reader's convenience we decide to include all essential details even if this requires repetition.

5. k-SUBFUNCTORS OF INTERNAL HOM

The discussion below is partially an application of the main result in [**?**, section 6]. For reader's convenience we decide not to use the general machinery developed there. Instead we give complete proofs.

Definition 5.1. Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{U}$ be k-functors and let $\sigma : \mathfrak{U} \times \mathfrak{X} \to \mathfrak{Y}$ be a morphism of k-functors. Assume that $\sigma_z : \mathfrak{X}_A \to \mathfrak{Y}_A$ is an isomorphism of A-functors for every k-algebra A and $z \in \mathfrak{U}(A)$. Then we call σ *a family of isomorphism parametrized by* $\mathfrak{U}$.

Definition 5.2. Let $\mathfrak{X}, \mathfrak{Y}$ be k-functors. For every k-algebra A we consider a subclass $\operatorname{Iso}_A(\mathfrak{X}_A, \mathfrak{Y}_A) \subseteq \operatorname{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ consisting of all morphism of A-functors that are isomorphism. If $\operatorname{Iso}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set for every k-algebra A, then we define a k-subfunctor $\mathcal{I}\mathrm{so}_k(\mathfrak{X}, \mathfrak{Y})$ of $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})$ such that

$$\mathcal{I}\mathrm{so}_k(\mathfrak{X}, \mathfrak{Y})(A) = \operatorname{Iso}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$$