

FORMALLY SMOOTH AND FORMALLY UNRAMIFIED MORPHISMS

1. INTRODUCTION

This notes are devoted to notions of formally smooth and formally unramified morphisms. These two classes are defined in terms of infinitesimal lifting properties. In the first section we introduce the sheaf of infinitesimal liftings and prove that it is a pseudo-torsor over certain sheaf of groups. In the second section formally smooth and unramified morphisms are defined. Next we study their properties in terms of sequences of differentials. Finally in the last section with the aid of [Monygham, 2021] we prove Cohen's structure theorem for noetherian regular rings in equicharacteristic case and characterize formally smooth and formally unramified extension of fields.

2. INFINITESIMAL LIFTING PROPERTY

Definition 2.1. Let X be a topological space and let \mathcal{G} be a sheaf of groups on X . A \mathcal{G} -sheaf \mathcal{F} with an action $p : \mathcal{G} \times \mathcal{F} \rightarrow \mathcal{F}$ is a \mathcal{G} -pseudo-torsor if for every open subset $U \subseteq X$ such that $\mathcal{F}(U)$ is nonempty the action $p(U) : \mathcal{G}(U) \times \mathcal{F}(U) \rightarrow \mathcal{F}(U)$ is transitive and free.

Definition 2.2. Let X be a topological space and let \mathcal{G} be a sheaf of groups on X . A \mathcal{G} -pseudo-torsor $p : \mathcal{G} \times \mathcal{F} \rightarrow \mathcal{F}$ is a \mathcal{G} -torsor if there exists an open cover $X = \bigcup_{i \in I} U_i$ such that $\mathcal{F}(U_i)$ is nonempty for every $i \in I$.

Let X be a topological space and let \mathcal{G} be a sheaf of groups on X . Morphisms of \mathcal{G} -pseudo-torsors and \mathcal{G} -torsors are morphisms of corresponding \mathcal{G} -sheaves.

Proposition 2.3. Let X be a topological space and let \mathcal{G} be a sheaf of groups on X . To give a \mathcal{G} -torsor on X it is the same as to give an open cover $X = \bigcup_{i \in I} U_i$ and for every $i, j \in I$ an element $g_{ij} \in \mathcal{G}(U_i \cap U_j)$ such that

$$g_{ij} \cdot g_{jk} = g_{ik}$$

for every $i, j, k \in I$.

Proof. Suppose that \mathcal{P} is a \mathcal{G} -torsor. Then there exists an open cover $X = \bigcup_{i \in I} U_i$ such that $\mathcal{P}(U_i)$ is nonempty. Therefore, there exists a family of isomorphisms $\phi_i : \mathcal{G}|_{U_i} \rightarrow \mathcal{P}|_{U_i}$ of $\mathcal{G}|_{U_i}$ -sheaves on $U_i \cap U_j$ we have that $\phi_j^{-1} \cdot \phi_i$ are defined by right multiplication by some $g_{ij} \in \mathcal{G}(U_i \cap U_j)$. Clearly these $\{g_{ij}\}_{i,j \in I}$ satisfy cocycle the condition. \square

Corollary 2.4. Let X be a topological space and let \mathcal{G} be a sheaf of abelian groups on X . Then there is a bijection

$$\{\mathcal{G}\text{-torsors up to an isomorphisms}\} \cong \check{H}^1(X, \mathcal{G})$$

Proof. The bijection in the statement is given by the construction described in Proposition 2.3 and its proof. We left the details to the reader. \square

Definition 2.5. Suppose that X is a topological space and \mathcal{G} is a sheaf of groups on X . We say that \mathcal{G} -torsor is *trivial* if it is isomorphic to the \mathcal{G} -torsor defined by action of \mathcal{G} on itself by left multiplication.

Corollary 2.6. *Suppose that Z is an affine scheme and \mathcal{G} is a quasi-coherent \mathcal{O}_Z -module. Consider \mathcal{G} as a sheaf of abelian groups on Z . Then every \mathcal{G} -torsor is trivial.*

Proof. This follows from Corollary 2.4 and the fact that

$$\check{H}^1(Z, \mathcal{G}) = H^1(Z, \mathcal{G}) = 0$$

as Z is affine scheme and \mathcal{G} is quasi-coherent. \square

Definition 2.7. Let $f : X \rightarrow Y$ be a morphism of schemes. Consider Z a scheme over Y and a closed immersion $i : Z_{\mathfrak{I}} \rightarrow Z$ of schemes determined by the ideal $\mathfrak{I} \subseteq \mathcal{O}_Z$ such that $\mathfrak{I}^2 = 0$. Fix a morphism $u : Z_{\mathfrak{I}} \rightarrow X$. A sheaf of sets on Z given by formula

$$U \mapsto \{f \in \mathbf{Sch}_Y(U, X) \mid f \cdot i|_U = u|_U\}$$

where $U \subseteq Z$ is an open subset of Z is called *the sheaf of local infinitesimal liftings for (u, i, f)* .

Theorem 2.8. *Let $f : X \rightarrow Y$ be a morphism of schemes. Consider Z a scheme over Y and a closed immersion $i : Z_{\mathfrak{I}} \rightarrow Z$ of schemes determined by the ideal $\mathfrak{I} \subseteq \mathcal{O}_Z$ such that $\mathfrak{I}^2 = 0$. Fix a morphism $u : Z_{\mathfrak{I}} \rightarrow X$. Let $\mathcal{G} = i_* \mathcal{H}om_{\mathcal{O}_{Z_{\mathfrak{I}}}}(u^* \Omega_{X/Y}, \mathfrak{I})$ be a sheaf of abelian groups on Z . Then there exists a structure of a \mathcal{G} -pseudo-torsor on the sheaf of local infinitesimal liftings corresponding for (u, i, f) .*

Proof. We denote by \mathcal{P} the sheaf of local infinitesimal liftings for (u, i, f) . Suppose that $U \subseteq Z$, $V \subset X$ and $W \subset Y$ are open affine subsets such that $u(U) \subseteq V$ and $f(V) \subseteq W$. Write $\text{Spec } C = U \subseteq Z$, $\mathfrak{I}(U) = \mathfrak{c}$ and $V = \text{Spec } B$, $W = \text{Spec } A$. Then we have

$$\mathcal{P}(U) \subseteq \mathbf{Sch}_W(U, V) = \text{Mor}_A(B, C), \quad \Omega_{V/W} = \widetilde{\Omega_{B/A}}$$

and u is induced by some morphism $v : B \rightarrow C/\mathfrak{c}$ of A -algebras. We also denote by $q : C \twoheadrightarrow C/\mathfrak{c}$ the canonical surjection. Suppose now that $h_1, h_2 : B \rightarrow C$ correspond to sections of \mathcal{P} over U . Observe that the image of $d = h_1 - h_2$ is contained in \mathfrak{c} . Consider \mathfrak{c} as a B -module with structure induced by $v : B \rightarrow C/\mathfrak{c}$. Then we have

$$\begin{aligned} d(b_1 b_2) &= h_1(b_1 b_2) - h_2(b_1 b_2) = h_1(b_1)h_1(b_2) - h_2(b_1)h_2(b_2) = \\ &= h_1(b_1)(h_1(b_2) - h_2(b_2)) + h_2(b_2)(h_1(b_1) - h_2(b_1)) = v(b_1)d(b_2) - v(b_2)d(b_1) \end{aligned}$$

for $b_1, b_2 \in B$. Hence $d : B \rightarrow \mathfrak{c}$ is a derivation. Moreover, $d(a) = 0$ for every a in A . Therefore, $h_1 - h_2$ can be uniquely interpreted as derivation $\mathbf{Der}_A(B, v_* \mathfrak{c})$. Conversely if $d \in \mathbf{Der}_A(B, v_* \mathfrak{c})$ is a derivation, then for some $h : B \rightarrow C$ corresponding to a section \mathcal{P} over U the sum $h + d : B \rightarrow C$ also gives a section of \mathcal{P} over U . Indeed, for this we need only to verify multiplication preservice. Note that

$$\begin{aligned} (h + d)(b_1 b_2) &= h(b_1)h(b_2) + v(b_1)d(b_2) + v(b_2)d(b_1) = \\ &= h(b_1)h(b_2) + h(b_1)d(b_2) + h(b_2)d(b_1) = (h(b_1) + d(b_1))(h(b_2) + d(b_2)) \end{aligned}$$

for every $b_1, b_2 \in B$. We proved that if $\mathcal{P}(U)$ is nonempty, then there is a canonical action of $\mathbf{Der}_A(B, v_* \mathfrak{c})$ on $\mathcal{P}(U)$ that is transitive and free. Since

$$\mathbf{Der}_A(B, v_* \mathfrak{c}) \cong \text{Hom}_B(\Omega_{B/A}, v_* \mathfrak{c}) = \text{Hom}_{C/\mathfrak{c}}(C/\mathfrak{c} \otimes_B \Omega_{B/A}, \mathfrak{c}) = i_* \mathcal{H}om_{\mathcal{O}_{Z_{\mathfrak{I}}}}(u^* \Omega_{X/Y}, \mathfrak{I})(U) = \mathcal{G}(U)$$

we deduce that if $\mathcal{P}(U)$ is nonempty, then we get a canonical structure of a transitive and free $\mathcal{G}(U)$ -set on $\mathcal{P}(U)$. If $\mathcal{P}(U)$ is empty for some U , then we pick a unique map $\mathcal{G}(U) \times \mathcal{P}(U) \rightarrow \mathcal{P}(U)$. This makes \mathcal{P} into a sheaf equipped with a \mathcal{G} -action. By construction \mathcal{P} is a \mathcal{G} -pseudo-torsor with respect to this structure. \square

3. FORMALLY SMOOTH AND FORMALLY UNRAMIFIED MORPHISMS

Definition 3.1. Let $f : X \rightarrow Y$ be a morphism of schemes.

- (1) f is *formally smooth* if for every affine scheme Z over Y and every closed immersion $i : Z_{\mathfrak{I}} \rightarrow Z$ determined by the ideal $\mathfrak{I} \subseteq \mathcal{O}_Z$ such that $\mathfrak{I}^2 = 0$ the map

$$\mathrm{Mor}_Y(Z, X) \xrightarrow{\mathrm{Mor}_Y(i, 1_X)} \mathrm{Mor}_Y(Z_{\mathfrak{I}}, X)$$

is surjective.

- (2) f is *formally unramified* if for every affine scheme Z over Y and every closed immersion $i : Z_{\mathfrak{I}} \rightarrow Z$ determined by the ideal $\mathfrak{I} \subseteq \mathcal{O}_Z$ such that $\mathfrak{I}^2 = 0$ the map

$$\mathrm{Mor}_Y(Z, X) \xrightarrow{\mathrm{Mor}_Y(i, 1_X)} \mathrm{Mor}_Y(Z_{\mathfrak{I}}, X)$$

is injective.

- (3) f is *formally étale* if it is formally smooth and formally unramified.

Fact 3.2. Open immersions are formally étale.

Proof. This is left to the reader. □

Fact 3.3. Let $f : X \rightarrow Y$ be a morphism of schemes. Then the following assertions hold.

- (1) f is formally smooth if and only if for every affine scheme Z over Y and every closed immersion $i : Z_{\mathfrak{I}} \rightarrow Z$ determined by the nilpotent ideal $\mathfrak{I} \subseteq \mathcal{O}_Z$ the map $\mathrm{Mor}_Y(i, 1_X)$ is surjective.
- (2) f is formally unramified if and only if for every affine scheme Z over Y and a closed immersion $i : Z_{\mathfrak{I}} \rightarrow Z$ determined by the nilpotent ideal $\mathfrak{I} \subseteq \mathcal{O}_Z$ the map $\mathrm{Mor}_Y(i, 1_X)$ is injective.

Proof. Suppose that $i : Z_{\mathfrak{I}} \rightarrow Z$ is a closed immersion determined by the sheaf of ideals \mathfrak{I} such that $\mathfrak{I}^n = 0$. Then we denote by $Z_k = Z_{\mathfrak{I}^k}$ the closed subscheme of Z determined by the ideal \mathfrak{I}^k . Let $i_k : Z_k \rightarrow Z_{k+1}$ be the closed immersion induced by the inclusion $\mathfrak{I}^{k+1} \subseteq \mathfrak{I}^k$. Consider a map

$$\mathrm{Mor}_Y(Z_{k+1}, X) \xrightarrow{\mathrm{Mor}_Y(i_k, 1_X)} \mathrm{Mor}_Y(Z_k, X)$$

Then

$$\mathrm{Mor}_Y(i, 1_X) = \mathrm{Mor}_Y(i_{n-1}, 1_X) \cdot \dots \cdot \mathrm{Mor}_Y(i_2, 1_X) \cdot \mathrm{Mor}_Y(i_1, 1_X)$$

Thus if f is formally smooth (unramified), then $\mathrm{Mor}_Y(i, 1_X)$ is surjective (injective). □

Proposition 3.4. Formally smooth and formally unramified morphisms are closed under compositions and base change. Moreover, the class of formally unramified morphisms is local on the base and domain.

Proof. The first part is an exercise in category theory. The second is a consequence of Fact 3.2. □

Proposition 3.5. Let $f : X \rightarrow Y$ be a formally smooth morphism. Suppose that $U \subseteq X$ and $V \subseteq Y$ are open subsets such that $f(U) \subseteq V$. Then the morphism $U \rightarrow V$ induced by f is formally smooth.

Proof. Since formally smooth morphisms are stable under base change, the morphism $f^{-1}(V) \rightarrow V$ induced by f is formally smooth. Moreover, $U \hookrightarrow X$ is formally étale by Fact 3.2. Thus $U \rightarrow V$ as the composition of two formally smooth morphisms is formally smooth. □

Proposition 3.6. *Class of formally smooth and locally of finite presentation morphisms is local on the base and target.*

Proof. Suppose that $f : X \rightarrow Y$ is a morphism locally of finite presentation. Assume that $X = \bigcup_{i \in I} U_i$ is an open cover such that $f|_{U_i}$ is formally smooth for every $i \in I$. Let Z be an affine scheme over Y and $i : Z_{\mathfrak{I}} \rightarrow Z$ be a closed immersion determined by the ideal $\mathfrak{I} \subseteq \mathcal{O}_Z$ such that $\mathfrak{I}^2 = 0$. Fix a morphism $u : Z_{\mathfrak{I}} \rightarrow X$. Denote the sheaf of infinitesimal liftings corresponding to (u, i, f) by \mathcal{P} and let $\mathcal{G} = i_* \mathcal{H}om_{\mathcal{O}_Z}(u^* \Omega_{X/Y}, \mathfrak{I})$ be a sheaf of abelian groups on Z . Then $\mathcal{P}(U_i)$ is nonempty for every $i \in I$, since $f|_{U_i}$ is formally smooth. Therefore, \mathcal{P} admits a structure of a \mathcal{G} -torsor by Theorem 2.8. The fact that f is locally of finite presentation implies that the sheaf $\Omega_{X/Y}$ is of finite presentation. Therefore, $\mathcal{G} = i_* \mathcal{H}om_{\mathcal{O}_Z}(u^* \Omega_{X/Y}, \mathfrak{I})$ is quasi-coherent on Z . Z is affine and hence by Corollary 2.6 the sheaf \mathcal{P} is the trivial \mathcal{G} -torsor. In particular, there exists a global section of \mathcal{P} . This means that there exists a global lift of u to Z . This shows that morphisms which are formally smooth and locally of finite presentation are local on the domain. It is straightforward to see that this implies that they are local on the target. \square

4. PROPERTIES OF FORMALLY SMOOTH MORPHISMS

Theorem 4.1. *Suppose that $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are morphisms of schemes. If f is formally smooth, then the differential sequence*

$$0 \longrightarrow f^* \Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

is exact. Moreover, for every x in X there exists an open neighbourhood $U \subseteq X$ of x such that the sequence is split exact on U .

Proof. Observe that the question is local. Therefore, we may assume that all schemes occurring in the statement are affine. Assume further that we have a morphism of A -algebras $f : B \rightarrow C$ such that $\text{Spec } f : \text{Spec } C \rightarrow \text{Spec } B$ is formally smooth. Our aim is to prove that the natural map:

$$\mathbf{Der}_A(C, M) \rightarrow \mathbf{Der}_A(B, f_* M)$$

induced by f is surjective for every C -module M . For this we define C -algebra $C \ltimes M$. The underlying C -module of $C \ltimes M$ is $C \oplus M$ and the multiplication is given by formula

$$(c_1, m_1) \cdot (c_2, m_2) = (c_1 c_2, c_2 m_1 + c_1 m_2)$$

Then the embedding $i : C \hookrightarrow C \ltimes M$ of C on the first summand of $C \oplus M$ induces C -algebra structure on $C \ltimes M$. Moreover, there exists a surjective morphism $q : C \ltimes M \rightarrow C$ given by $q(c, m) = c$. Its kernel is a nilpotent ideal. A derivation $d \in \mathbf{Der}_A(B, f_* M)$ gives rise to a morphism $g : B \rightarrow C \ltimes M$ given by $b \mapsto (f(b), d(b))$. Hence there exists a commutative square

$$\begin{array}{ccc} \text{Spec } C & \xrightarrow{\text{Spec } 1_C} & \text{Spec } C \\ \text{Spec } q \downarrow & & \downarrow \text{Spec } f \\ \text{Spec } C \ltimes M & \xrightarrow{\text{Spec } g} & \text{Spec } B \end{array}$$

and since $\text{Spec } f$ is formally smooth, there exists a morphism $s : C \rightarrow C \ltimes M$ of B -algebras such that the diagram

$$\begin{array}{ccc}
 \mathrm{Spec} C & \xrightarrow{\mathrm{Spec} 1_C} & \mathrm{Spec} C \\
 \mathrm{Spec} q \downarrow & \nearrow \mathrm{Spec} s & \downarrow \mathrm{Spec} f \\
 \mathrm{Spec} C \times M & \xrightarrow{\mathrm{Spec} g} & \mathrm{Spec} B
 \end{array}$$

is commutative. One can easily verify that the additive morphism $\pi_M \cdot s : C \rightarrow M$ is an A -derivation of C in M such that $\pi_M \cdot s \cdot f = d$. This shows that the map

$$\mathbf{Der}_A(C, M) \rightarrow \mathbf{Der}_A(B, f_* M)$$

induced by f is surjective for every C -module M . \square

Theorem 4.2. *Suppose that $f : X \rightarrow Y$ is a morphism of schemes. Let $i : Z \rightarrow X$ be a closed immersion of schemes such that fi is formally smooth. Then the conormal sequence*

$$0 \longrightarrow \mathcal{I}_Z / \mathcal{I}_Z^2 \longrightarrow i^* \Omega_{X/Y} \longrightarrow \Omega_{Z/Y} \longrightarrow 0$$

is exact. Moreover, for every z in Z there exists an open neighbourhood $U \subseteq Z$ of z such that the sequence is split exact on U .

Proof. Observe that the question is local. Therefore, we may assume that all schemes occurring in the statement are affine. Assume that we have A -algebra B and an ideal $\mathfrak{b} \subseteq B$. Consider the surjection $w : B/\mathfrak{b}^2 \twoheadrightarrow B/\mathfrak{b}$ and suppose that B/\mathfrak{b} is formally smooth over A . Then we have a lift

$$\begin{array}{ccc}
 \mathrm{Spec} B/\mathfrak{b} & \xrightarrow{1_{\mathrm{Spec} B/\mathfrak{b}}} & \mathrm{Spec} B/\mathfrak{b} \\
 \mathrm{Spec} w \downarrow & \nearrow \mathrm{Spec} s & \\
 \mathrm{Spec} B/\mathfrak{b}^2 & &
 \end{array}$$

Next consider a diagram

$$\begin{array}{ccc}
 \mathrm{Spec} B/\mathfrak{b} & \xrightarrow{\mathrm{Spec} w} & \mathrm{Spec} B/\mathfrak{b}^2 \\
 \mathrm{Spec} w \downarrow & \nearrow & \\
 \mathrm{Spec} B/\mathfrak{b}^2 & &
 \end{array}$$

Let \mathcal{P} be the sheaf of local infinitesimal liftings for $\mathrm{Spec} w$ and the morphism $\mathrm{Spec} B/\mathfrak{b} \rightarrow \mathrm{Spec} A$. Its global sections are presented in the diagram above by the dashed arrow. Then $1_{\mathrm{Spec} B/\mathfrak{b}^2}$ and $\mathrm{Spec} (s \cdot w)$ are global sections of \mathcal{P} . Then \mathcal{P} is a \mathcal{G} -torsor, where

$$\mathcal{G} = (\mathrm{Spec} w)^* \mathcal{H}om_{\mathcal{O}_{\mathrm{Spec} B/\mathfrak{b}^2}}((\mathrm{Spec} w)^* \Omega_{(\mathrm{Spec} B/\mathfrak{b}^2)/\mathrm{Spec} A}, \widetilde{\mathfrak{b}/\mathfrak{b}^2})$$

according to Theorem 2.8. We have

$$\mathcal{G}(\mathrm{Spec} B/\mathfrak{b}^2) = \mathrm{Hom}_{B/\mathfrak{b}^2}(\Omega_{(B/\mathfrak{b}^2)/A}, w_* \mathfrak{b}/\mathfrak{b}^2) = \mathbf{Der}_A(B/\mathfrak{b}^2, w_* \mathfrak{b}/\mathfrak{b}^2)$$

Hence there exists an A -derivation $B/\mathfrak{b}^2 \rightarrow \mathfrak{b}/\mathfrak{b}^2$ given by $1_{B/\mathfrak{b}^2} - s \cdot w$. Let $q : B \twoheadrightarrow B/\mathfrak{b}^2$ be the canonical surjection and define $d = (1_{B/\mathfrak{b}^2} - s \cdot w) \cdot q : B \rightarrow \mathfrak{b}/\mathfrak{b}^2$. Thus $d \in \mathbf{Der}_A(B, w_* \mathfrak{b}/\mathfrak{b}^2)$. Since d

induces an identity on $\mathfrak{b}/\mathfrak{b}^2$, the corresponding morphism in $\mathrm{Hom}_C(C \otimes_B \Omega_{B/A}, \mathfrak{b}/\mathfrak{b}^2)$ is a splitting for the sequence

$$\mathfrak{b}/\mathfrak{b}^2 \longrightarrow C \otimes_B \Omega_{B/A} \longrightarrow \Omega_{C/A} \longrightarrow 0$$

□

Theorem 4.3. *Suppose that $f : \mathrm{Spec} B \rightarrow \mathrm{Spec} A$ is a formally smooth morphism. Let $\mathfrak{b} \subseteq B$ be an ideal and let $i : \mathrm{Spec} B/\mathfrak{b} \rightarrow \mathrm{Spec} B$ be a closed immersion corresponding to this ideal. Suppose that the conormal sequence*

$$\widetilde{\mathfrak{b}/\mathfrak{b}^2} \longrightarrow i^* \Omega_{\mathrm{Spec} B/\mathrm{Spec} A} \longrightarrow \Omega_{\mathrm{Spec} (B/\mathfrak{b})/\mathrm{Spec} A} \longrightarrow 0$$

is split exact. Then $f \cdot i$ is formally smooth.

Proof. Let $q : B \twoheadrightarrow B/\mathfrak{b}$ be a surjection giving i and denote by $w : B/\mathfrak{b}^2 \twoheadrightarrow B/\mathfrak{b}$ the canonical surjection. Since the conormal sequence in the statement splits, we have an A -derivation $d : B \rightarrow w_* \mathfrak{b}/\mathfrak{b}^2$ that induces the identity on $\mathfrak{b}/\mathfrak{b}^2$. There exists an A -derivation $\delta : B/\mathfrak{b}^2 \rightarrow w_* \mathfrak{b}/\mathfrak{b}^2$ induced by d . Consider the extension problem

$$\begin{array}{ccc} \mathrm{Spec} B/\mathfrak{b} & \xrightarrow{\mathrm{Spec} w} & \mathrm{Spec} B/\mathfrak{b}^2 \\ \mathrm{Spec} w \downarrow & \nearrow & \\ \mathrm{Spec} B/\mathfrak{b}^2 & & \end{array}$$

and suppose that \mathcal{P} is the sheaf of local infinitesimal liftings corresponding to $\mathrm{Spec} w$. Then $1_{\mathrm{Spec} B/\mathfrak{b}^2} \in \mathcal{P}(\mathrm{Spec} B/\mathfrak{b}^2)$ and again \mathcal{P} is a torsor for

$$\mathcal{G} = (\mathrm{Spec} w)^* \mathrm{Hom}_{\mathcal{O}_{\mathrm{Spec} B/\mathfrak{b}^2}}((\mathrm{Spec} w)^* \Omega_{(\mathrm{Spec} B/\mathfrak{b}^2)/\mathrm{Spec} A}, \widetilde{\mathfrak{b}/\mathfrak{b}^2})$$

by Theorem 2.8. According the fact that

$$\mathcal{G}(\mathrm{Spec} B/\mathfrak{b}^2) = \mathrm{Der}_A(B/\mathfrak{b}^2, w_* \mathfrak{b}/\mathfrak{b}^2)$$

we deduce that $h : B/\mathfrak{b}^2 \rightarrow B/\mathfrak{b}^2$ given by $1_{B/\mathfrak{b}^2} - \delta$ is in $\mathcal{P}(\mathrm{Spec} B/\mathfrak{b}^2)$. Then h admits a factorization $h = s \cdot w$ for some morphism $s : B/\mathfrak{b} \rightarrow B/\mathfrak{b}^2$. Moreover, $w = w \cdot h = w \cdot s \cdot w$ and w is surjective. Hence $1_{B/\mathfrak{b}} = w \cdot s$. Now we are ready to prove that $f \cdot i$ is formally smooth. Consider a diagram of A -algebras

$$\begin{array}{ccc} \mathrm{Spec} R/\mathfrak{I} & \longrightarrow & \mathrm{Spec} B/\mathfrak{b} \\ \downarrow j & & \downarrow \mathrm{Spec} q \\ & \nearrow \mathrm{Spec} g & \mathrm{Spec} B \\ \mathrm{Spec} R & \longrightarrow & \mathrm{Spec} A \\ & & \downarrow \mathrm{Spec} f \end{array}$$

where $j : \mathrm{Spec} R/\mathfrak{I} \rightarrow \mathrm{Spec} R$ is a closed immersion and $\mathfrak{I}^2 = 0$. Since B is formally smooth over A , we have an extension $\mathrm{Spec} g$ for some $g : B \rightarrow R$. Clearly $g(\mathfrak{b}) \subseteq \mathfrak{I}$. Thus we can write a diagram:

$$\begin{array}{ccc}
 \mathrm{Spec} R/\mathfrak{I} & \longrightarrow & \mathrm{Spec} B/\mathfrak{b} \\
 \downarrow j & \nearrow \mathrm{Spec} w & \downarrow \mathrm{Spec} w \\
 & & \mathrm{Spec} B/\mathfrak{b}^2 \\
 \mathrm{Spec} R & \xrightarrow{\quad} & \mathrm{Spec} A \\
 & \nearrow \mathrm{Spec} \bar{g} & \downarrow \\
 & & \mathrm{Spec} A
 \end{array}$$

where $\mathrm{Spec} \bar{g}$ is a morphism induced by g and the dashed arrow is $\mathrm{Spec} (\bar{g} \cdot s)$. Thus the upper-left triangle commutes and this shows that $f \cdot i : \mathrm{Spec} B/\mathfrak{b} \rightarrow \mathrm{Spec} A$ is formally smooth. \square

5. PROPERTIES OF FORMALLY UNRAMIFIED MORPHISMS

Theorem 5.1. *Suppose that $f : X \rightarrow Y$ is a morphism of schemes. Then f is formally unramified if and only if $\Omega_{X/Y} = 0$.*

Proof. The question is local on base and domain. Thus we may assume that $X = \mathrm{Spec} B$, $Y = \mathrm{Spec} A$ and $f = \mathrm{Spec} g$ for some morphism of rings $g : A \rightarrow B$.

Suppose that f is formally unramified. Let M be a B -module. Consider an A -derivation $d : B \rightarrow M$ and suppose that $s : B \rightarrow B \ltimes M$ is a morphism given by $s(b) = (b, 0)$. We have a commutative diagram

$$\begin{array}{ccc}
 \mathrm{Spec} B & \xrightarrow{\mathrm{Spec} 1_B} & \mathrm{Spec} B \\
 \downarrow \mathrm{Spec} q & \nearrow \mathrm{Spec} s & \downarrow \mathrm{Spec} g \\
 \mathrm{Spec} B \ltimes M & \longrightarrow & \mathrm{Spec} A
 \end{array}$$

in which $q : B \ltimes M \rightarrow B$ is the quotient morphism. Similarly we have a commutative diagram

$$\begin{array}{ccc}
 \mathrm{Spec} B & \xrightarrow{\mathrm{Spec} 1_B} & \mathrm{Spec} B \\
 \downarrow \mathrm{Spec} q & \nearrow \mathrm{Spec} \tilde{s} & \downarrow \mathrm{Spec} g \\
 \mathrm{Spec} B \ltimes M & \longrightarrow & \mathrm{Spec} A
 \end{array}$$

where $\tilde{s}(b) = (b, db)$. Since f is formally unramified, we derive that $s = \tilde{s}$ and hence $d = 0$. Thus $\mathrm{Der}_A(B, M) = 0$ for every B -module M . This shows that

$$\Omega_{\mathrm{Spec} B / \mathrm{Spec} A} = \Omega_{B/A} = 0$$

Suppose now that $\Omega_{\mathrm{Spec} B / \mathrm{Spec} A}$ is zero. Let C be an A -algebra and $\mathfrak{c} \subseteq C$ be an ideal such that $\mathfrak{c}^2 = 0$. Assume that we have morphisms $h : B \rightarrow C/\mathfrak{c}$, $s_1, s_2 : B \rightarrow C$ of A -algebras such that diagrams

$$\begin{array}{ccc}
\mathrm{Spec} C/\mathfrak{c} & \xrightarrow{\mathrm{Spec} h} & \mathrm{Spec} B \\
\mathrm{Spec} q \downarrow & \nearrow \mathrm{Spec} s_i & \downarrow \mathrm{Spec} g \\
\mathrm{Spec} C & \longrightarrow & \mathrm{Spec} A
\end{array}$$

are commutative for $i = 1, 2$. Then $d = s_1 - s_2$ is a derivation in $\mathbf{Der}_A(B, h_* \mathfrak{c})$. Since

$$\Omega_{B/A} = \Omega_{\mathrm{Spec} B / \mathrm{Spec} A} = 0$$

we derive that $d = 0$. Thus $s_1 = s_2$. This shows that f is formally unramified. \square

Corollary 5.2. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms of schemes. Then f is formally unramified if and only if the morphism:*

$$f^* \Omega_{Y/Z} \longrightarrow \Omega_{X/Z}$$

is an epimorphism.

6. CASE OF THE FIELD AND COHEN'S THEOREM IN EQUICARACTERISTIC CASE

Proposition 6.1. *Let k be a field and K be its separable extension. Then K is formally smooth k -algebra.*

Proof. First write $K = \mathrm{colim}_{i \in I} K_i$, where K_i are finitely generated over k and I is a directed set. Since K is separable over k , by [Monygham, 2021, Theorem 4.3] each K_i admits separating transcendence basis over k . It is easy to verify that fields with separating transcendence basis over k are formally smooth as k -algebras. Let

$$\mathfrak{a}_i \hookrightarrow k[K_i] \xrightarrow{q_i} K_i$$

be the functorial presentation of K_i i.e. $k[K_i]$ is a free k -algebra on the set K_i , $q_i(x) = x$ for every x in K_i and $\ker(q_i) = \mathfrak{a}_i$. Similarly we can write the functorial presentation for K

$$\mathfrak{a} \hookrightarrow k[K] \xrightarrow{q} K$$

Now it is easy to verify that the conormal sequence

$$\mathfrak{a}/\mathfrak{a}^2 \longrightarrow K \otimes_k \Omega_{k[K]/k} \longrightarrow \Omega_{K/k} \longrightarrow 0$$

is a colimit over $i \in I$ of conormal sequences

$$\mathfrak{a}_i/\mathfrak{a}_i^2 \longrightarrow K_i \otimes_k \Omega_{k[K_i]/k} \longrightarrow \Omega_{K_i/k} \longrightarrow 0$$

For every $i \in I$ conormal sequences for $q_i : k[K_i] \twoheadrightarrow K_i$ are exact. Hence the conormal sequence for $q : k[K] \twoheadrightarrow K$ is exact. It is then split exact as a short exact sequence of vector spaces over K . By Theorem 4.3 we deduce that K is formally smooth algebra over k . \square

Theorem 6.2 (Cohen's structure theorem in equicharacteristic case). *Let (A, \mathfrak{m}, K) be a local noetherian ring over a field k . Then the following are equivalent:*

(i) The completion \hat{A} of A in \mathfrak{m} -adic topology is the algebra of formal power series over K . That is

$$\hat{A} = K[[x_1, \dots, x_n]]$$

(ii) A is a regular local ring

Proof. The implication (i) \Rightarrow (ii) is obvious according to the fact that:

$$\mathrm{gr}_{\mathfrak{m}}(A) \cong \mathrm{gr}_{\hat{\mathfrak{m}}}(\hat{A}) = K[\bar{x}_1, \dots, \bar{x}_n]$$

where $\bar{x}_1, \dots, \bar{x}_n$ are classes of x_1, \dots, x_n in $\hat{\mathfrak{m}}/\hat{\mathfrak{m}}^2$.

Suppose that (ii) holds. Using similar argument as above we derive that $(\hat{A}, \hat{\mathfrak{m}})$ is a regular local k -algebra. Without loss of generality we assume that k is a prime field. Then k is perfect and K is separable over k by Proposition 6.1. We have a commutative diagram

$$\begin{array}{ccc} K & \xleftarrow{1_K} & K \\ q \uparrow & \swarrow s & \downarrow \\ \hat{A} & \xrightarrow{\quad} & k \end{array}$$

Hence we obtain a section $s : K \rightarrow \hat{A}$. In particular, \hat{A} is a K -algebra. Now pick some system $y_1, \dots, y_n \in \hat{\mathfrak{m}}$ of parameters for \hat{A} . Fix a polynomial ring $K[x_1, \dots, x_n]$ and define the morphism of K -algebras $\theta : K[x_1, \dots, x_n] \rightarrow \hat{A}$ by $\theta(x_i) = y_i$ for $1 \leq i \leq n$. By the universal property of completion, we have a local morphism $\bar{\theta} : K[[x_1, \dots, x_n]] \rightarrow \hat{A}$ of local K -algebras. Now this morphism induces an isomorphism on associated graded rings. Thus $\bar{\theta}$ is an isomorphism. \square

Theorem 6.3. *Let (A, \mathfrak{m}) be local noetherian k -algebra. If A is formally smooth over k , then A is a regular local ring.*

Proof. Composition of formally smooth morphisms is formally smooth. Moreover, every field extension of prime field is formally smooth according to the fact that prime fields are perfect and by virtue of Proposition 6.1. Using this two properties we may assume that k is a prime field. Let K be the residue field of A and \hat{A} be its completion with respect to \mathfrak{m} -adic topology. Then K is formally smooth over k and we have a commutative diagram

$$\begin{array}{ccc} K & \xleftarrow{1_K} & K \\ q \uparrow & \swarrow s & \downarrow \\ \hat{A} & \xrightarrow{\quad} & k \end{array}$$

Thus we obtain a section $s : K \rightarrow \hat{A}$. Fix elements $x_1, \dots, x_n \in \mathfrak{m}$ that modulo \mathfrak{m}^2 form a basis of $\mathfrak{m}/\mathfrak{m}^2$ over K . Consider the ring of power series $B = K[[t_1, \dots, t_n]]$ over K and denote by \mathfrak{n} its unique maximal ideal (t_1, \dots, t_n) . Obviously s induces an isomorphism $\phi : B/\mathfrak{n}^2 \rightarrow A/\mathfrak{m}^2$. Recall that A was formally smooth over k . By this assumption we derive a commutative diagram

$$\begin{array}{ccc}
 A/m^2 & \xleftarrow{\quad} & A \\
 \uparrow p & \swarrow \theta & \downarrow \\
 B & \xrightarrow{\quad} & k
 \end{array}$$

In the diagram above p is a surjective morphism given as the composition of the canonical morphism $p : B \rightarrow B/m^2$ and ϕ . Now θ is a local ring morphism. Since B is complete with respect to n -adic topology, we deduce that there exists a morphism $\bar{\theta} : \hat{A} \rightarrow B$ induced by θ . Next the morphism $\text{gr}_{\hat{m}}(\hat{A}) \rightarrow \text{gr}_n(B)$ induced by $\bar{\theta}$ is a surjective morphism of graded K -algebras. Moreover, $\text{gr}_{\hat{m}}(\hat{A})$ is generated by n -elements in its first gradation and $\text{gr}_n(B)$ is a polynomial ring on n -variables in the first gradation. Thus we derive that $\bar{\theta}$ induces an isomorphism on associated graded rings. Hence it is an isomorphism by completeness of \hat{A} and B . Thus \hat{A} is regular. By Theorem 6.2 we infer that A is regular. \square

Corollary 6.4. *Let k be a field and let K be its extension. Then K is formally smooth as a k -algebra if and only if K is separable over k .*

Proof. Suppose that K is formally smooth over k . Let $k \subseteq k'$ be a finite extension contained in $k^{\frac{1}{p}}$. Then the base change $B = K \otimes_k k'$ is formally smooth k' -algebra. As it is finite over K it is noetherian. Since k' is purely inseparable over k , by [Monygham, 2021, Proposition 3.2] it has only one prime ideal. Thus it is formally smooth local noetherian algebra over some field. By Proposition 6.3 it is regular and in particular reduced. Since $K \otimes_k k^{\frac{1}{p}}$ is a colimit of its subalgebras of the form $K \otimes_k k'$, we derive that $K \otimes_k k^{\frac{1}{p}}$ is reduced. By [Monygham, 2021, Theorem 4.3] this implies that K is separable over k . \square

Theorem 6.5. *Let k be a field and let K its extension. Then the following assertions are equivalent.*

- (i) K is formally unramified over k .
- (ii) K is separable algebraic over k if $\text{char}(k) = 0$ or $k(K^p) = K$ if $\text{char}(k) = p > 0$.

Proof. Assume (i). This is equivalent with $\Omega_{K/k} = 0$.

Suppose that $\text{char}(k) = 0$, then every extension of k has separating transcendence basis and image of this basis under universal derivation $d_{K/k} : K \rightarrow \Omega_{K/k}$ will be the basis of K -vector space $\Omega_{K/k}$. Thus in this case K must be algebraic over k . In characteristic zero every extension is separable. Thus K is separable algebraic.

Assume now that $\text{char}(k) = p > 0$. Fix $x \in K \setminus k(K^p)$ and let L_x be the maximal subfield of K containing $k(K^p)$ and not containing x . Then $L_x(x) = K$. Moreover, $x^p \in L_x$ and $x \notin L_x$. Hence $\Omega_{K/L_x} = K \cdot dx$ and in particular there exists a derivation $d_x : K \rightarrow K$ such that $d_x(x) = 1$ and $d_x(L_x) = 0$. This derivation is a nontrivial derivation of K over k due to the fact that $k \subseteq L_x$. This is a contradiction with $\Omega_{K/k} = 0$.

Suppose that (ii) holds. If $\text{char}(k) = 0$ and K is separable algebraic, then $\Omega_{K/k} = 0$ by easy computation and an argument involving colimit. If $\text{char}(k) = p > 0$, then $\Omega_{K/k} = \Omega_{K/k(K^p)}$ and since $\Omega_{K/k(K^p)} = 0$ as $K = k(K^p)$, we derive that $\Omega_{K/k} = 0$. \square

Example 6.6. Let k be a field of characteristic $p > 0$. Consider a variable x and an infinite tower

$$k(x) \hookrightarrow k(x^{\frac{1}{p}}) \hookrightarrow \dots \hookrightarrow k(x^{\frac{1}{p^n}}) \hookrightarrow k(x^{\frac{1}{p^{n+1}}}) \hookrightarrow \dots$$

of rational function fields over k . Let K be its colimit. Then $K = k(K^p)$ and hence by Theorem 6.5 we derive that K is formally unramified over k . On the other hand K is neither algebraic nor separable extension of k . This shows that situation in positive characteristic is dramatically different from characteristic zero.

REFERENCES

[Monygham, 2021] Monygham (2021). Topics in theory of fields, *github repository: "Monygham/Pedo-mellon-a-minno"*.