ALGEBRAIZATION OF FORMAL M-SCHEMES

1. INTRODUCTION

In these notes we prove some results concerning algebraization of formal schemes in equivariant setting.

2. SOME 2-CATEGORICAL LIMITS

Consider a category $\mathcal{C}$ and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Our goal is to construct certain 2-categorical limit associated with a pair $(\mathcal{C}, T)$. Consider pairs (X, u) consisting of an object X of $\mathcal{C}$ and an isomorphism $u : T(X) \to X$ in $\mathcal{C}$. If (X, u) and (Y, w) are two such pairs, then a morphism $f : (X, u) \to (Y, u)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that the following square

$$\begin{tikzcd} T(X) \arrow[r, "u"] \arrow[d, "T(f)"'] & X \arrow[d, "f"] \\ T(Y) \arrow[r, "w"'] & Y \end{tikzcd}$$

is commutative. This data give rise to a category $\mathcal{C}(T)$. There exists a forgetful functor $\pi : \mathcal{C}(T) \to \mathcal{C}$ that sends a morphism $f : (X, u) \to (Y, w)$ to $f : X \to Y$. Moreover, there exists a natural isomorphism $\sigma : T \cdot \pi \Rightarrow \pi$ such that the component of σ on an object (X, u) of $\mathcal{C}(T)$ is u. The next result states that the data above form a certain 2-categorical limit.

Theorem 2.1. *Let $(\mathcal{C}, T)$ be a pair consiting of a category and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Suppose that $\mathcal{D}$ is a category, $P : \mathcal{D} \to \mathcal{C}$ is a functor and $\tau : T \cdot P \Rightarrow P$ is a natural isomorphisms. Then there exists a unique functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$.*

Proof. Suppose that $F : \mathcal{D} \to \mathcal{C}(T)$ is a functor such that $P = \pi \cdot F$ and $\sigma_F = \tau$. Pick an object X of $\mathcal{D}$. Then we have $\pi \cdot F(X) = P(X)$ and $\sigma_{F(X)} = \tau_X$. This implies that

$$F(X) = (P(X), \tau_X : T(P(X)) \to P(X))$$

Next if $f : X \to Y$ is a morphism in $\mathcal{D}$, then we derive that $\pi(F(f)) = P(f)$. Hence $F(f) = P(f)$. This implies that there exists at most one functor F satisfying the properties above. Note also that formulas

$$F(X) = (P(X), \tau_X : T(P(X)) \to P(X)),\ F(f) = P(f)$$

for an object X in $\mathcal{D}$ and a morphism $f : X \to Y$ in $\mathcal{D}$, give rise to a functor that satisfy $P = \pi \cdot F$ and $\sigma_F = \tau$. This establishes existence and the uniqueness of F. □

Assume now that the pair $(\mathcal{C}, T)$ consists of a monoidal category $\mathcal{C}$ and a monoidal endofunctor T. Then there exists a canonical monoidal structure on $\mathcal{C}(T)$. We define $(-) \otimes_{\mathcal{C}(T)} (-)$ by formula

$$(X, u) \otimes_{\mathcal{C}(T)} (Y, w) = \big(X \otimes_{\mathcal{C}} Y, (u \otimes_{\mathcal{C}} w) \cdot m_{X,Y}\big)$$

where

$$m_{X,Y} : T\left(X \otimes_{\mathcal{C}} Y\right) \to T(X) \otimes_{\mathcal{C}} T(Y)$$

is the tensor preserving isomorphism of T. We also define the unit

$$I_{\mathcal{C}(T)} = (I, T(I) \cong I)$$

where isomorphism $T(I) \cong I$ is precisely the unit preserving isomorphism of the monoidal functor T. The associativity natural isomorphism for $(-)\otimes_{\mathcal{C}(T)}(-)$ and right, left units for $I_{\mathcal{C}(T)}$ in $\mathcal{C}(T)$ are associavity natural isomorphism and right, left units for $\mathcal{C}$, respectively. The structure makes a functor $\pi : \mathcal{C}(T) \to \mathcal{C}$ strict monoidal and σ a monoidal natural isomorphism. The next result states that the data with these extra monoidal structure form a 2-categorical limit in the 2-category of monoidal categories.

Theorem 2.2. *Let $(\mathcal{C}, T)$ be a pair consiting of a monoidal category and its monoidal endofunctor $T : \mathcal{C} \to \mathcal{C}$. Suppose that $\mathcal{D}$ is a monoidal category, $P : \mathcal{D} \to \mathcal{C}$ is a monoidal functor and $\tau : T \cdot P \Rightarrow P$ is a monoidal natural isomorphisms. Then there exists a unique monoidal functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$ as monoidal functors and monoidal transformations.*

Proof. Note that F must be defined as it was described in the proof of Theorem 2.1. Namely we must have

$$F(X) = (P(X), \tau_X : T(P(X)) \to P(X)),\, F(f) = P(f)$$

for an object X in $\mathcal{C}$ and a morphism $f : X \to Y$ in $\mathcal{C}$.
Suppose now that F admits a structure of a monoidal functor such that $P = \pi \cdot F$ as monoidal functors. Let

$$\{m^F_{X,Y} : F(X \otimes_{\mathcal{D}} Y) \to F(X) \otimes_{\mathcal{C}(T)} F(Y)\}_{X,Y \in \mathcal{C}},\, \phi^F : F(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$$

be the data forming that structure. Since π is a strict monoidal functor and $P = \pi \cdot F$ as monoidal functors, we derive that for any objects X, Y of $\mathcal{C}$

$$\pi(m^F_{X,Y}) : P(X \otimes_{\mathcal{D}} Y) \to P(X) \otimes_{\mathcal{C}} P(Y)$$

is the tensor preserving isomorphism $m^P_{X,Y} : P(X \otimes_{\mathcal{D}} Y) \to P(X) \otimes_{\mathcal{C}} P(Y)$ of the monoidal functor P. By the same argument

$$\pi(\phi_F) : P(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$$

is the unit preserving isomorphism $\phi^P : P(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$ of P. Thus we deduce that for any objects X, Y of $\mathcal{C}$ we have $m^F_{X,Y} = m^P_{X,Y}$ and $\phi^F = \phi^P$. This implies that there exists at most one monoidal functor F such that $P = \pi \cdot F$ as monoidal functors.
On the other hand define $m^F_{X,Y} = m^P_{X,Y}$ for objects X, Y in $\mathcal{C}$ and $\phi^F = \phi^P$. We check now that F equipped with these data is a monoidal functor. Fix objects X, Y in $\mathcal{C}$. The square

$$\begin{tikzcd}
T\left(P\left(X \otimes_{\mathcal{D}} Y\right)\right) \arrow[r, "\tau_{X \otimes_{\mathcal{C}} Y}"] \arrow[d, "T\left(m^P_{X,Y}\right)"'] & P\left(X \otimes_{\mathcal{C}} Y\right) \arrow[d, "m^P_{X,Y}"] \\
T\left(P(X) \otimes_{\mathcal{C}} P(Y)\right) \arrow[r, "(\tau_X \otimes_{\mathcal{C}} \tau_Y) \cdot m^T_{P(X),P(Y)}"'] & P(X) \otimes_{\mathcal{C}} P(Y)
\end{tikzcd}$$

is commutative due to the fact that $\tau : T \cdot P \Rightarrow P$ is a monoidal natural isomorphisms. This implies that $m^F_{X,Y}$ is a morphism in $\mathcal{C}(T)$. It follows that $m^F_{X,Y}$ is a natural isomorphism and due to the definition of associativity in $\mathcal{C}(T)$, we derive its compatibility with $m^F_{X,Y}$. Similarly, since the square

$$\begin{tikzcd}
T\left(P\left(I_{\mathcal{D}}\right)\right) \arrow[r, "\tau_{I_{\mathcal{D}}}"] \arrow[d, "T\left(\phi^P\right)"'] & P\left(I_{\mathcal{D}}\right) \arrow[d, "\phi^P"] \\
T\left(I_{\mathcal{C}}\right) \arrow[r, "\phi^T"'] & I_{\mathcal{C}}
\end{tikzcd}$$

is commutative, we deduce that ϕ^F is a morphism in $\mathcal{C}(T)$. By definition of left and right unit in $\mathcal{C}(T)$, we derive their compatibility with ϕ^F. This finishes the verification of the fact that F with $\{m^F_{X,Y}\}_{X,Y\in\mathcal{C}}$ and ϕ^F is a monoidal functor. Definitions of $\{m^F_{X,Y}\}_{X,Y\in\mathcal{C}}$ and ϕ^F show that the identities $P = \pi\cdot F$ holds on the level of monoidal structures. Since the 2-forgetful functor from 2-category of monoidal categories into 2-category of categories is faithful on 2-cells, the identity $\sigma_F = \tau$ of natural isomorphisms is also the identity of monoidal natural isomorphisms. □

Theorem 2.3. *Let $(\mathcal{C}, T)$ be a pair consiting of a category and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Assume that T preserves colomits. Then the following assertions hold.*

- **(1)** *$\pi : \mathcal{C}(T) \to \mathcal{C}$ creates colimits.*
- **(2)** *Suppose that $\mathcal{D}$ is a category, $P : \mathcal{D} \to \mathcal{C}$ a functor preserving small colimits and $\tau : T\cdot P \Rightarrow P$ a natural isomorphisms. Then the unique functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi\cdot F$ and $\sigma_F = \tau$ preserves small colimits.*

Proof. Let I be a small category and $D : I \to \mathcal{C}(T)$ be a diagram such that the composition $\pi\cdot D : I \to \mathcal{C}$ admits a colimit given by cocone $(X, \{g_i\}_{i\in I})$. Since T preserves colimits, we derive that $(T(X), \{T(u_i)\}_{i\in I})$ is a colimit of $T\cdot\pi\cdot D : I \to \mathcal{C}$. Now $\sigma_D : T\cdot\pi\cdot D \to \pi\cdot D$ is a natural isomorphism. Hence there exists a unique arrow $u : T(X) \to X$ such that $u\cdot T(g_i) = g_i\cdot\sigma_{D(i)}$ for $i \in I$. Clearly u is an isomorphism and hence (X, u) is an object of $\mathcal{C}(T)$. Moreover, the family $\{g_i\}_{i\in I}$ together with (X, u) is a colimiting cocone over D. This proves **(1)**. Now **(2)** is a consequence of **(1)**. □

Now we apply the results above to certain more general diagrams of categories.

Definition 2.4. A diagram

$$... \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} ... \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

of categories and functors is called *a telescope of categories*.

Definition 2.5. Let

$$... \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} ... \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

be a telescope of monoidal categories and monoidal cocontinuous functors. Then *a 2-categorical limit of the telescope* consists of a monoidal category $\mathcal{C}$, a family of monoidal cocontinuous functors $\{\pi_n : \mathcal{C} \to \mathcal{C}_n\}_{n\in\mathbb{N}}$ and a family of monoidal natural isomorphisms $\{\sigma_n : F_{n+1}\cdot\pi_{n+1} \Rightarrow \pi_n\}_{n\in\mathbb{N}}$ such that the following universal property holds. For any monoidal category $\mathcal{D}$, family $\{P_n : \mathcal{D} \to \mathcal{C}_n\}_{n\in\mathbb{N}}$ of cocontinuous monoidal functors and a family $\{\tau_n : F_nP_{n+1} \Rightarrow P_n\}_{n\in\mathbb{N}}$ of monoidal natural isomorphisms there exists a unique monoidal cocontinuous functor $F : \mathcal{D} \to \mathcal{C}$ satisfying $P_n = \pi_n\cdot F$ and $(\sigma_n)_F = \tau_n$ for every $n \in \mathbb{N}$.

Corollary 2.6. *Let*

$$... \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} ... \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

be a telescope of monoidal categories and monoidal cocontinuous functors. Then its 2-limit exists.

Proof. We decompose the task of constructing its 2-limit as follows. First note that one may form a product $\mathcal{C} = \prod_{n\in\mathbb{N}}\mathcal{C}_n$. Next the functors $\{F_n\}_{n\in\mathbb{N}}$ induce an endofunctor $T = \prod_{n\in\mathbb{N}} F_n \times t$, where $\mathbf{1}$ is the terminal category (it has single object and single identity arrow) and $t : \mathcal{C}_0 \to \mathbf{1}$ is the unique functor. Consider the category $\mathcal{C}(T)$. We define $\{\pi_n : \mathcal{C}(T) \to \mathcal{C}_n\}_{n\in\mathbb{N}}$ to be a family of functors given by coordinates of $\pi : \mathcal{C}(T) \to \mathcal{C}$ and $\{\sigma_n : F_n\cdot\pi_{n+1} \Rightarrow \pi_n\}_{n\in\mathbb{N}}$ to be a family of natural isomorphisms given by coordinates of $\sigma : \pi \cdot T \Rightarrow \pi$. Now this data form a 2-limit of the telescope by compilation of Theorem 2.2 and Theorem 2.3. $\square$

3. Formal M-schemes

This section is devoted to introducing some notions from formal geometry that play a fundamental role in these notes.

Definition 3.1. Let $\mathbf{M}$ be a monoid k-scheme. *A formal* $\mathbf{M}$*-scheme* consists of a sequence $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ of $\mathbf{M}$-schemes together with $\mathbf{M}$-equivariant closed immersions

$$Z_0 \hookrightarrow Z_1 \hookrightarrow \dots \hookrightarrow Z_n \hookrightarrow Z_{n+1} \hookrightarrow \dots$$

satisfying the following assertions.

(1) We have $Z_0 = Z_n^{\mathbf{M}}$ scheme-theoretically for every $n \in \mathbb{N}$.

(2) Let $\mathcal{I}_n$ be an ideal of $\mathcal{O}_{Z_n}$ defining Z_0. Then for every $m \le n$ the subscheme $Z_m \subset Z_n$ is defined by $\mathcal{I}_n^{m+1}$.

Example 3.2. Let $\mathbf{M}$ be a monoid k-scheme and let Z be a $\mathbf{M}$-scheme. Consider a quasi-coherent ideal $\mathcal{I}$ of fixed point subscheme $Z^{\mathbf{M}}$ of Z. Then for every $n \in \mathbb{N}$ ideal $\mathcal{I}^n$ is $\mathbf{M}$-equivariant and hence

$$V(\mathcal{I}) \hookrightarrow V(\mathcal{I}^2) \hookrightarrow \dots \hookrightarrow V(\mathcal{I}^n) \hookrightarrow \dots$$

is a formal $\mathbf{M}$-scheme. We denote it by $\widehat{Z}$.

Definition 3.3. Let $\mathbf{M}$ be a monoid k-scheme and let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{M}$-scheme. We say that $\mathcal{Z}$ is *locally noetherian* if for all $n \in \mathbb{N}$ scheme Z_n is locally Noetherian.

Definition 3.4. Let $\mathbf{M}$ be a monoid k-scheme. Suppose that $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ and $\mathcal{W} = \{W_n\}_{n\in\mathbb{N}}$ are formal $\mathbf{M}$-schemes. Then *a morphism* $f : \mathcal{Z} \to \mathcal{W}$ *of formal* $\mathbf{M}$*-schemes* consists of a family of $\mathbf{M}$-equivariant morphisms $f = \{f_n : Z_n \to W_n\}_{n\in\mathbb{N}}$ such that the diagram

$$\begin{tikzcd}
Z_0 \arrow[r, hook] \arrow[d, "f_0"'] & Z_1 \arrow[r, hook] \arrow[d, "f_1"'] & \dots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[d, "f_n"'] & Z_{n+1} \arrow[r, hook] \arrow[d, "f_{n+1}"'] & \dots \\
W_0 \arrow[r, hook] & W_1 \arrow[r, hook] & \dots \arrow[r, hook] & W_n \arrow[r, hook] & W_{n+1} \arrow[r, hook] & \dots
\end{tikzcd}$$

is commutative.

Definition 3.5. Let $\mathbf{M}$ be a monoid k-scheme. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be locally noetherian a formal $\mathbf{M}$-scheme. Then we have the corresponding telescope of monoidal categories

$$\dots \longrightarrow \mathfrak{Coh}_{\mathbf{M}}(Z_{n+1}) \longrightarrow \mathfrak{Coh}_{\mathbf{M}}(Z_n) \longrightarrow \dots \longrightarrow \mathfrak{Coh}_{\mathbf{M}}(Z_2) \longrightarrow \mathfrak{Coh}_{\mathbf{M}}(Z_1) \longrightarrow \mathfrak{Coh}_{\mathbf{M}}(Z_0)$$

and cocontinuous monoidal functors given by restricting $\mathbf{M}$-equivariant coherent sheaves to closed $\mathbf{M}$-subschemes. Then we define *a category* $\mathfrak{Coh}_{\mathbf{M}}(\mathcal{Z})$ *of coherent* $\mathbf{M}$*-equivariant sheaves on* $\mathcal{Z}$ as a monoidal category which is a 2-limit of the telescope above. This category is defined uniquely up to a monoidal equivalence.

Fix now a monoid k-scheme $\mathbf{M}$. Let Z be a locally noetherian $\mathbf{M}$-scheme and suppose that $Z^{\mathbf{M}}$ exists. Suppose that $\mathcal{I}$ is a coherent ideal of $Z^{\mathbf{M}}$. We have a commutative diagram

$$\begin{tikzcd} V(\mathcal{I}) \arrow[r, hook] \arrow[drr, hook, bend right] & V(\mathcal{I}^2) \arrow[r, hook] \arrow[dr, hook] & \ldots \arrow[r, hook] & V(\mathcal{I}^n) \arrow[r, hook] \arrow[dl, hook] & \ldots \\ & & Z & & \end{tikzcd}$$

in the category of $\mathbf{M}$-schemes. Thus restriction functors $\mathfrak{Coh}_{\mathbf{M}}(Z) \to \mathfrak{Coh}_{\mathbf{M}}(V(\mathcal{I}^n))$ for $n \in \mathbb{N}$ induce a unique cocontinuous monoidal functor $\mathfrak{Coh}_{\mathbf{M}}(Z) \to \mathfrak{Coh}_{\mathbf{M}}(\widehat{Z})$.

Definition 3.6. Let Z be a locally noetherian $\mathbf{M}$-scheme such that $Z^{\mathbf{M}}$ exists. Then a unique cocontinuous monoidal functor $\mathfrak{Coh}_{\mathbf{M}}(Z) \to \mathfrak{Coh}_{\mathbf{M}}(\widehat{Z})$ is called *the comparison functor*.

Since group k-scheme is also a monoid k-scheme, definitions above can be applied to group k-schemes.

Definition 3.7. Let $\mathbf{M}$ be a monoid k-scheme with group of units $\mathbf{G}$. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{M}$-scheme. A $\mathbf{M}$-scheme Z is called *an algebraization of* $\mathcal{Z}$ if the following two conditions are satisfied.

- **(1)** $\mathcal{Z}$ is isomorphic to $\widehat{Z}$ in the category of formal $\mathbf{M}$-schemes.
- **(2)** The comparison functor $\mathfrak{Coh}_{\mathbf{G}}(Z) \to \mathfrak{Coh}_{\mathbf{G}}\left(\widehat{Z}\right)$ is an equivalence of monoidal categories.

4. Locally linear M-schemes

Definition 4.1. Let $\mathbf{M}$ be a monoid k-scheme and let X be a $\mathbf{M}$-scheme. Suppose that each point of X admits an open affine $\mathbf{M}$-stable neighborhood. Then we say that X is *a locally linear* $\mathbf{M}$*-scheme*.

Proposition 4.2. *Let* $\mathbf{M}$ *be an affine monoid k-scheme and let X be a* $\mathbf{M}$*-scheme. Suppose that there exists a quasi-coherent* $\mathbf{M}$*-equivariant ideal* $\mathcal{I}$ *on X with nilpotent sections. Consider an open subset U of X. Then the following are equivalent.*

- **(1)** *U is* $\mathbf{M}$*-stable.*
- **(2)** *$U \cap V(\mathcal{I})$ is* $\mathbf{M}$*-stable.*

Proof. Let $\alpha : \mathbf{M} \times_k X \to X$ be the action of $\mathbf{M}$ on X. Fix open subset U of X. If U is $\mathbf{M}$-stable, then $U \cap V(\mathcal{I})$ is $\mathbf{M}$-stable. So suppose that $U \cap V(\mathcal{I})$ is $\mathbf{M}$-stable. Since $\mathcal{I}$ has nilpotent sections and $\mathbf{M}$ is affine, we derive that closed immersions $U \cap V(\mathcal{I}) \hookrightarrow U$ and $\mathbf{M} \times_k (U \cap V(\mathcal{I})) \hookrightarrow \mathbf{M} \times_k U$ induce homeomorphisms on topological spaces. Consider the commutative diagram

$$\begin{tikzcd} \mathbf{M} \times_k U \arrow[r, "\alpha_{|U\cap V(\mathcal{I})}"] & X \\ \mathbf{M} \times_k (U \cap V(\mathcal{I})) \arrow[u, hook] \arrow[r] & U \cap V(\mathcal{I}) \arrow[u, hook] \end{tikzcd}$$

where the bottom horizontal arrow is the induced action on $U \cap V(\mathcal{I})$ and vertical morphisms are homeomorphisms. The commutativity of the diagram implies that $\alpha\left(\mathbf{M} \times_k U\right)$ is contained set-theoretically in U. Since U is open in X, we derive that morphism of schemes $\alpha_{|\mathbf{M}\times_k U}$ factors through U. Hence U is $\mathbf{M}$-stable. □

Corollary 4.3. *Let* $\mathbf{M}$ *be an affine monoid k-scheme and let X be a* $\mathbf{M}$*-scheme. Suppose that there exists a quasi-coherent* $\mathbf{M}$*-equivariant ideal* $\mathcal{I}$ *on X such that* $\mathcal{I}^n = 0$ *for* $n \in \mathbb{N}$*. Consider an open subset U of X. Then the following are equivalent.*

- **(1)** *U is* $\mathbf{M}$*-stable and affine.*

(2) *$U \cap V(\mathcal{I})$ is $\mathbf{M}$-stable and affine.*

Proof. Since $\mathcal{I}^n = 0$, we derive that U is affine if and only if $U \cap V(\mathcal{I})$ is affine. Combining this with Proposition 4.2, we deduce the result. □

Corollary 4.4. *Let $\mathbf{M}$ be an affine monoid k-scheme and let X be a $\mathbf{M}$-scheme. Suppose that there exists a quasi-coherent $\mathbf{M}$-equivariant ideal $\mathcal{I}$ on X such that $\mathcal{I}^n = 0$ for $n \in \mathbb{N}$. Then X is locally linear $\mathbf{M}$-scheme if and only if $V(\mathcal{I})$ is locally linear $\mathbf{M}$-scheme.*

Proof. This is a consequence of Corollary 4.3. □

5. Some results on formal $\mathbf{M}$-schemes

Corollary 5.1. *Let $\mathbf{M}$ be an affine monoid k-scheme and let $\mathcal{Z} = \{Z_n\}_{n \in \mathbb{N}}$ be a formal $\mathbf{G}$-scheme. Then Z_n is locally linear $\mathbf{G}$-scheme for every $n \in \mathbb{N}$.*

Proof. Let $\mathcal{I}_n$ be an ideal defining Z_0 in Z_n. Since $\mathcal{Z}$ is a formal $\mathbf{M}$-scheme, we derive that $\mathcal{I}_n^{n+1} = 0$ and Z_0 is locally linear $\mathbf{M}$-scheme. Thus we apply Corollary 4.4 and derive that Z_n is locally linear $\mathbf{M}$-scheme. □

We are particularly interested in formal $\mathbf{M}$-schemes for monoid $\mathbf{M}$ with zero. For this we need the following elementary result.

Proposition 5.2. *Let $\mathbf{M}$ be a monoid k-scheme with zero $\mathbf{o}$ and let X be a $\mathbf{M}$-scheme. Then the following results hold.*

(1) *The multiplication by zero $\mathbf{o} \cdot (-) : X \to X$ factors through $X^{\mathbf{M}}$ inducing a $\mathbf{M}$-equivariant retraction $\pi_{\mathbf{M}} : X \twoheadrightarrow X^{\mathbf{M}}$.*

(2) *If $\mathbf{N}$ is a submonoid k-scheme of $\mathbf{M}$ and $\mathbf{o}$ is a k-point of $\mathbf{N}$, then $\pi_{\mathbf{M}} = \pi_{\mathbf{N}}$.*

(3) *If $\mathbf{M}$ is affine and X is locally linear $\mathbf{M}$-scheme, then $\pi_{\mathbf{M}}$ is affine.*

Proof. The multiplication $\mathbf{o} \cdot (-) : \mathfrak{P}_X \to \mathfrak{P}_X$ factors as an $\mathfrak{P}_{\mathbf{M}}$-equivariant epimorphism $\mathfrak{P}_X \twoheadrightarrow \mathfrak{P}_{X^{\mathbf{M}}}$ composed with a closed immersion $\mathfrak{P}_{X^{\mathbf{M}}} \hookrightarrow \mathfrak{P}_X$. The $\mathfrak{P}_{\mathbf{M}}$-equivariant epimorphism $\mathfrak{P}_X \to \mathfrak{P}_{X^{\mathbf{M}}}$ corresponds to a $\mathbf{M}$-equivariant morphism $\pi_{\mathbf{M}} : X \to X^{\mathbf{M}}$ of k-schemes such that $\pi_{\mathbf{M}}$ restricted to $X^{\mathbf{M}}$ is the identity $1_{X^{\mathbf{M}}}$. This proves **(1)**.
For the proof of **(2)** note that $\mathbf{o} \cdot (-) : \mathfrak{P}_X \to \mathfrak{P}_X$ is defined similarly for $\mathbf{M}$ and $\mathbf{N}$ (provided that $\mathbf{o}$ is a k-point of $\mathbf{N}$). Thus $\pi_{\mathbf{M}} = \pi_{\mathbf{N}}$.
Suppose now that $\mathbf{M}$ is affine and X is locally linear $\mathbf{M}$-scheme. Consider the action $\alpha : \mathbf{M} \times_k X \to X$ of $\mathbf{M}$ on X. Since X is locally linear and $\mathbf{M}$ is affine, we derive that α is an affine morphism of k-schemes. Now $\mathbf{o} \cdot (-) : X \to X$ is given as a composition

$$X \xrightarrow{\cong} \mathbf{o} \times_k X \hookrightarrow \mathbf{M} \times_k X \xrightarrow{\alpha} X$$

The morphism above is affine (as a composition of affine morphisms). Since the composition of π with a closed immersion $X^{\mathbf{M}} \hookrightarrow X$ is $\mathbf{o} \times_k (-)$ and hence an affine morphism, we derive that π is affine. This proves **(3)**. □

Let us note the immediate consequence of this result.

Corollary 5.3. *Let $\mathbf{M}$ be an affine monoid k-scheme with zero and let $\mathcal{Z} = \{Z_n\}_{n \in \mathbb{N}}$ be a formal $\mathbf{M}$-scheme. Then $\mathcal{Z}$ is a part of the commutative diagram*

$$\begin{tikzcd}
Z_0 \arrow[r, hook] \arrow[drr, bend right, two heads, "\pi_0 = 1_{Z_0}"'] & Z_1 \arrow[r, hook] \arrow[dr, two heads, "\pi_1"] & \ldots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[dl, bend left, two heads, "\pi_n"'] & \ldots \\
 & & Z_0 & &
\end{tikzcd}$$

in which vertical morphisms $\pi_n : Z_n \twoheadrightarrow Z_0$ are affine $\mathbf{M}$*-equivariant morphisms such that $\pi_{n|Z_0} = 1_{Z_0}$. Moreover, if* $\mathbf{N}$ *is a submonoid k-scheme of* $\mathbf{M}$ *containing the zero of* $\mathbf{M}$*, then $\mathcal{Z}$ is a formal* $\mathbf{N}$*-scheme.*

Proof. This is an immediate consequence of Corollary 5.1 and Proposition 5.2. □

6. Toruses and toric monoid k-schemes

Definition 6.1. Let T be an affine algebraic group over k. Suppose that there exists $n \in \mathbb{N}$ such that for every algebraically closed extension K of k there exists an isomorphism

$$T_K \cong \mathrm{Spec}\, K \times_k \underbrace{\mathbb{G}_m \times_k \mathbb{G}_m \times_k \ldots \times_k \mathbb{G}_m}_{n \text{ times}}$$

of group schemes over K. Then T is called *a torus over k*.

Example 6.2. If $T \cong \underbrace{\mathbb{G}_m \times_k \mathbb{G}_m \times_k \ldots \times_k \mathbb{G}_m}_{n \text{ times}}$, then T is a torus. We call toruses T of this form *split toruses*.

Example 6.3. Define

$$\mathbf{S}^1 = \mathrm{Spec}\, k[x,y]/(x^2 + y^2 - 1)$$

a scheme over k and let $\mathfrak{P}_{\mathbf{S}^1}$ be its functor of points. Then for every k-algebra A we have

$$\mathfrak{P}_{\mathbf{S}^1}(A) = \left\{(u,v) \in A \times_k A \,\middle|\, u^2 + v^2 = 1\right\}$$

There is also a morphism $\mathfrak{P}_{\mathbf{S}^1} \times_k \mathfrak{P}_{\mathbf{S}^1} \to \mathfrak{P}_{\mathbf{S}^1}$ of k-functors given by

$$\mathfrak{P}_{\mathbf{S}^1}(A) \times_k \mathfrak{P}_{\mathbf{S}^1}(A) \to \mathfrak{P}_{\mathbf{S}^1} \ni \big((u_1,v_1),(u_2,v_2)\big) \mapsto (u_1u_2 - v_1v_2, u_1v_2 + u_2v_1) \in \mathfrak{P}_{\mathbf{S}^1}(A)$$

for every k-algebra A. This makes $\mathfrak{P}_{\mathbf{S}^1}$ into a group k-functor. Thus $\mathbf{S}^1$ with the group structure described above is an affine algebraic group over k. We call it *the circle group over k*.
Now suppose that $\mathrm{char}(k) = 2$ and K is an algebraically closed extension of k. Consider an element $i \in K$ such that $i^2 = -1$. For every K-algebra A we have a map

$$\mathfrak{P}_{\mathbf{S}^1}(A) \ni (u,v) \mapsto u + iv \in A^*$$

First note that this map is bijective. Indeed, its inverse is given by

$$A^* \ni a \mapsto \left(\frac{1}{2}(a + a^{-1}), \frac{1}{2i}(a - a^{-1})\right) \in \mathfrak{P}_{\mathbf{S}^1}(A)$$

Moreover, the map $\mathfrak{P}_{\mathbf{S}^1}(A) \to A^*$ is a homomorphism of abstract groups. Thus $\mathfrak{P}_{\mathbf{S}^1}$ resricted to the category $\mathbf{Alg}_K$ of K-algebras is isomorphic with $\mathfrak{P}_{\mathrm{Spec}\, K \times_k \mathbb{G}_m}$ as a group k-functor. Hence

$$\mathbf{S}^1_K \cong \mathrm{Spec}\, K \times_k \mathbb{G}_m$$

as algebraic group schemes over K. Hence $\mathbf{S}^1$ is a torus over k.
Now assume that $k = \mathbb{R}$. Then abstract groups

$$\mathfrak{P}_{\mathbf{S}^1}(\mathbb{R}) = \left\{z \in \mathbb{C} \,\middle|\, |z| = 1\right\} \subseteq \mathbb{C}^*,\ \mathbb{R}^*$$

are not isomorphic. Indeed, the left hand side group has infinite torsion subgroup and the right hand side group has torsion subgroup equal to $\{-1,1\}$. This implies that over $\mathbb{R}$ algebraic groups $\mathbf{S}^1$ and $\mathbb{G}_m$ are not isomorphic. Hence $\mathbf{S}^1$ is not a split torus over $\mathbb{R}$.

Corollary 6.4. *Let T be a torus over k. Then T is a linearly reductive algebraic group.*

Definition 6.5. Let T be a torus over k and let $\overline{T}$ be a linearly reductive monoid having T as the group of units. Then $\overline{T}$ is *a toric monoid over k*

Theorem 6.6. *Let $\overline{T}$ be a toric monoid over k with group of units T and let K be an algebraically closed extension of k. Suppose that N is a dimension of T.*

(1) *The group of characters of T_K is isomorphic to $\mathbb{Z}^N$ and there exists an abstract submonoid S of $\mathbb{Z}^N$ such that the open immersion*

$$T_K = \mathrm{Spec}\left(\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m\right) \hookrightarrow \mathrm{Spec}\left(\bigoplus_{m\in S} K\cdot\chi^m\right) = \overline{T}_K$$

is induced by the inclusion $S \hookrightarrow \mathbb{Z}^N$.

(2) *Let $\{V_\lambda\}_{\lambda\in\mathbf{Irr}(T)}$ be a set of irreducible representation of T such that V_λ is in isomorphism class λ. For every λ there exists a finite subset A_λ of $\mathbb{Z}^N$ such that*

$$K\otimes_k V_\lambda = \bigoplus_{m\in A_\lambda} K\cdot\chi^m$$

If λ is in $\mathbf{Irr}(\overline{T})$, then A_λ is a subset of S. Moreover, we have

$$\mathbb{Z}^N = \coprod_{\lambda\in\mathbf{Irr}(T)} A_\lambda$$

and $A_{\lambda_0} = \{0\}$, where λ_0 is the class of the trivial representation of T.

(3) *If $\overline{T}$ has a zero, then there exists a homomorphism $f : \mathbb{Z}^N \to \mathbb{Z}$ of abelian groups such that $f_{|S\smallsetminus\{0\}} > 0$. In particular, f induces a closed immersion*

$$\mathrm{Spec}\, K\times_k \mathbb{G}_m = \mathrm{Spec}\, K[\mathbb{Z}] \hookrightarrow \mathrm{Spec}\left(\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m\right) = T_K$$

of group K-schemes that extends to a zero preserving closed immersion $\mathbb{A}^1_K \hookrightarrow \overline{T}_K$ of monoid K-schemes.

Proof. Since T is a torus, we derive that

$$T_K = \mathrm{Spec}\, K\times_k \underbrace{\mathbb{G}_m\times_k\mathbb{G}_m\times_k \ldots\times_k\mathbb{G}_m}_{N\text{ times}} = \mathrm{Spec}\left(\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m\right)$$

and hence

$$\overline{T}_K = \mathrm{Spec}\left(\bigoplus_{s\in S} K\cdot\chi^s\right)$$

for some abstract submonoid S of $\mathbb{Z}^N$. Moreover, the open immersion $T_K \hookrightarrow \overline{T}_K$ is induced by the inclusion $S\hookrightarrow\mathbb{Z}^N$. This proves **(1)**.
We have identification

$$k[T] = \bigoplus_{\lambda\in\mathbf{Irr}(T)} V_\lambda^{n_\lambda}$$

of T-representations, where $n_\lambda \in \mathbb{N}\smallsetminus\{0\}$ for each λ. Thus

$$\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m = K\otimes_k k[T] = \bigoplus_{\lambda\in\mathbf{Irr}(T)} (K\otimes_k V_\lambda)^{n_\lambda}$$

This implies that $n_\lambda = 1$ for every λ and moreover, we derive that

$$K\otimes_k V_\lambda = \bigoplus_{m\in A_\lambda} K\cdot\chi^m$$

for some finite set $A_\lambda \subseteq \mathbb{Z}^N$. We also have $A_{\lambda_0} = \{0\}$ and $A_\lambda \subseteq S \setminus \{0\}$ for $\lambda \in \mathbf{Irr}(\overline{T})$. This proves **(2)**.
Since $\overline{T}$ admits a zero, we derive that

$$\mathfrak{m} = \bigoplus_{m \in S \setminus \{0\}} K \cdot \chi^s \subseteq \bigoplus_{m \in \mathbb{Z}^N} K \cdot \chi^m$$

is an ideal. This implies that $S \setminus \{0\}$ is closed under addition. In particular, there exists a homomorphism of abelian groups $f : \mathbb{Z}^N \to \mathbb{Z}$ such that $f_{|S \setminus \{0\}} > 0$. This implies **(3)**. □

7. Commuting actions

Corollary 7.1. *Let $\mathbf{G}$ be an affine group scheme over k and let $\mathfrak{G}$ be a monoid k-functor. Denote by Λ the set of isomorphism classes of irreducible $\mathbf{G}$-representations. Suppose that V is a representation of both $\mathbf{G}$ and $\mathfrak{G}$ and assume that their actions on V commute. Assume that V is completely reducible as a $\mathbf{G}$-representation and consider the decomposition*

$$V = \bigoplus_{\lambda \in \Lambda} V[\lambda]$$

onto isotypic components with respect to the action of $\mathbf{G}$. Then for every λ in Λ the subspace $V[\lambda]$ is a $\mathfrak{G}$-subrepresentation of V.

Proof. Part of the structure V as the $\mathfrak{G}$-representation is the morphism $\rho : \mathfrak{G} \to \mathcal{L}_V$ of k-monoids. Fix k-algebra A and $g \in \mathfrak{G}(A)$. Since actions of $\mathbf{G}$ and $\mathfrak{G}$ on V commute, morphism $\rho(g) : A \otimes_k V \to A \otimes_k V$ of A-modules is a morphism of $\mathbf{G}_A$-representation. According to Proposition **??** we derive that

$$\mathrm{Hom}_{\mathbf{G}_A}\left(A \otimes_k V[\lambda_1], A \otimes_k V[\lambda_2]\right) = 0$$

for distinct $\lambda_1, \lambda_2 \in \Lambda$. Thus

$$\rho(g)\left(A \otimes_k V[\lambda]\right) \subseteq A \otimes_k V[\lambda]$$

for every λ in Λ. This holds for every k-algebra A and $g \in \mathfrak{G}(A)$. Hence $V[\lambda]$ is $\mathfrak{G}$-subrepresentation of V. □

8. Algebraization of formal M-schemes

This section proves some results in equivariant formal geometry.

Theorem 8.1. *Let $\mathbf{M}$ be a Kempf monoid and let $\mathcal{Z} = \{Z_n\}_{n \in \mathbb{N}}$ be a formal $\mathbf{M}$-scheme. Then there exists a locally linear $\mathbf{M}$-scheme Z equipped with an action of $\mathbf{M}$ such that $\widehat{Z}$ is isomorphic to $\mathcal{Z}$.*

Setup. Monoid $\mathbf{M}$ is affine and admits zero $\mathbf{o}$. Hence by Corollary 5.3 formal $\mathbf{M}$-scheme $\mathcal{Z}$ corresponds to a sequence of surjections

$$\ldots \twoheadrightarrow \mathcal{A}_{n+1} \twoheadrightarrow \mathcal{A}_n \twoheadrightarrow \ldots \twoheadrightarrow \mathcal{A}_1 \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$$

of quasi-coherent $\mathcal{O}_{Z_0}$-algebras with $\mathbf{M}$-linearization such that $\mathcal{A}_n^{\mathbf{M}} = \mathcal{A}_0$ for every $n \in \mathbb{N}$ and if $\mathcal{I}_n$ is the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_0$ in $\mathcal{A}_n$, then $\mathcal{I}_n^{m+1}$ is the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_m$ for $m \leq n$ and $n \in \mathbb{N}$. Since $\mathbf{M}$ is a Kempf monoid, there exists a closed subgroup T of the center $Z(\mathbf{G})$ of the unit group $\mathbf{G}$ of $\mathbf{M}$ such that T is a torus and the scheme-theoretic closure $\overline{T}$ of T in $\mathbf{M}$ contains the zero $\mathbf{o}$ of $\mathbf{M}$. We derive by Corollary 5.3 that $\mathcal{A}_n^{\overline{\mathbf{T}}} = \mathcal{A}_0$ for every $n \in \mathbb{N}$. By definition $\overline{T}$ is a toric monoid k-scheme with T as a group of units. Let $\{V_\lambda\}_{\lambda \in \mathbf{Irr}(T)}$ be a set of irreducible representations of T such that V_λ is contained in λ. □

Lemma 8.1.1. *Let λ be in $\mathbf{Irr}(\overline{T})$. Then there exists $n_\lambda \in \mathbb{N}$ such that for each $n > n_\lambda$ and any $\lambda_1, ..., \lambda_n \in \mathbf{Irr}(\overline{T}) \setminus \{\lambda_0\}$ the representation*

$$\bigotimes_{i=1}^{n} V_{\lambda_i}$$

has trivial isotypic component of type λ. We have $n_{\lambda_0} = 0$, where λ_0 is an isomorphism type of the trivial representation of T.

Proof of the lemma. Let K be an algebraically closed extension of k. Pick A_λ and f as in Theorem 6.6 and define

$$n_\lambda = \sup_{m \in A_\lambda} f(m)$$

We have

$$K \otimes_k V_{\lambda_1} \otimes_k ... \otimes_k V_{\lambda_n} = \bigoplus_{(m_1,...,m_n) \in A_{\lambda_1} \times_k ... \times_k A_{\lambda_n}} K \cdot \chi^{m_1 + ... + m_n}$$

and since $m_1, ...m_n \in A_{\lambda_1} \cup ... \cup A_{\lambda_n} \subseteq S \setminus \{0\}$ we derive that

$$f(m_1 + ... + m_n) = f(m_1) + ... + f(m_n) \geq n > n_\lambda = \sup_{m \in A_\lambda} f(m)$$

This implies that V_λ is not an isotypic component of $V_{\lambda_1} \otimes_k ... \otimes_k V_{\lambda_n}$. □

Lemma 8.1.2. *Fix λ in $\mathbf{Irr}(\overline{T})$. Then $\mathcal{A}_{n+1}[\lambda] \twoheadrightarrow \mathcal{A}_n[\lambda]$ is an isomorphism for $n \geq n_\lambda$.*

Proof of the lemma. Since $\mathcal{A}_n^{\overline{T}} = \mathcal{A}_0$ and $\overline{T}$ is linearly reductive monoid, we derive that $\mathcal{I}_n[\lambda] = 0$ for $\lambda \notin \mathbf{Irr}(\overline{T}) \setminus \{\lambda_0\}$. Fix $\lambda \in \mathbf{Irr}(\overline{T})$. By Lemma 8.1.1 we derive that

$$\left(\underbrace{\mathcal{I}_{n+1} \otimes_k \mathcal{I}_{n+1} \otimes_k ... \otimes_k \mathcal{I}_{n+1}}_{n+1 \text{ times}}\right)[\lambda] = 0$$

for $n \geq n_\lambda$. Note also that the image of the composition

$$\underbrace{\mathcal{I}_{n+1} \otimes_k \mathcal{I}_{n+1} \otimes_k ... \otimes_k \mathcal{I}_{n+1}}_{n \text{ times}} \twoheadrightarrow \underbrace{\mathcal{I}_{n+1} \otimes_{\mathcal{O}_{Z_0}} \mathcal{I}_{n+1} \otimes_{\mathcal{O}_{Z_0}} ... \otimes_{\mathcal{O}_{Z_0}} \mathcal{I}_{n+1}}_{n \text{ times}} \longrightarrow \mathcal{A}_{n+1}$$

is $\mathcal{I}_{n+1}^{n+1}$. Since the composition above is a morphism of sheaves with $\overline{T}$-linearization, we derive that $\mathcal{I}_{n+1}^{n+1}[\lambda] = 0$ for $n \geq n_\lambda$. Hence the kernel of $\mathcal{A}_{n+1}[\lambda] \twoheadrightarrow \mathcal{A}_n[\lambda]$ is trivial. □

Proof of Theorem. According to Corollary 7.1 and the fact that T is central in $\mathbf{M}$ we derive that $\mathcal{A}_n[\lambda]$ is a quasi-coherent sheaf with $\mathbf{M}$-linearization. For $\lambda \in \mathbf{Irr}(\overline{T})$ we define

$$\mathcal{A}[\lambda] = \mathcal{A}_n[\lambda]$$

where $n \geq n_\lambda$ as in Lemma 8.1.2. We set

$$\mathcal{A} = \bigoplus_{\lambda \in \mathbf{Irr}(\overline{T})} \mathcal{A}[\lambda]$$

Clearly $\mathcal{A}[\lambda_0] = \mathcal{A}_0 = \mathcal{O}_{Z_0}$ canonically (where λ_0 is the trivial T-representation), hence $\mathcal{A}$ is a quasi-coherent sheaf on Z_0 with $\mathbf{M}$-linearization. Actually $\mathcal{A} = \lim_{n \in \mathbb{N}} \mathcal{A}_n$ in the category of quasi-coherent sheaves with $\mathbf{M}$-linearization on Z_0. We construct the $\mathcal{O}_{Z_0}$-algebra structure on $\mathcal{A}$. For this pick $\lambda_1, \lambda_2 \in \mathbf{Irr}(\overline{T})$. Consider the irreducible representations V_{λ_1} and V_{λ_1} in classes λ_1 and λ_2, respectively. Suppose that $\eta_1, ..., \eta_s$ are finitely many classes in $\mathbf{Irr}(\overline{T})$ such that $V_{\lambda_1} \otimes_k V_{\lambda_2}$ can be completely decomposed onto irreducible representation in these classes. Since the image of the multiplication $\mathcal{A}_n[\lambda_1] \otimes_{\mathcal{O}_{Z_0}} \mathcal{A}_n[\lambda_2] \to \mathcal{A}_n$ on $\mathcal{A}_n$ is also the image of a morphism

$$\mathcal{A}_n[\lambda_1] \otimes_k \mathcal{A}_n[\lambda_2] \twoheadrightarrow \mathcal{A}_n[\lambda_1] \otimes_{\mathcal{O}_{Z_0}} \mathcal{A}_n[\lambda_2] \longrightarrow \mathcal{A}_n$$

we deduce that it is contained in $\bigoplus_{i=1}^{s} \mathcal{A}_n[\eta_i]$. By Lemma 8.1.2 all these multiplications for $n \geq \sup\{n_{\lambda_1}, n_{\lambda_2}, n_{\eta_1}, ..., n_{\eta_s}\}$ can be identified. Now we define

$$\mathcal{A}[\lambda_1] \otimes_{\mathcal{O}_{Z_0}} \mathcal{A}[\lambda_2] \to \bigoplus_{i=1}^{s} \mathcal{A}[\eta_i] \subseteq \mathcal{A}$$

as a morphism induced by the multiplication morphism for any $n \geq \sup\{n_{\lambda_1}, n_{\lambda_2}, n_{\eta_1}, ..., n_{\eta_s}\}$. This gives an $\mathcal{O}_{Z_0}$-algebra structure on $\mathcal{A}$, so $\mathcal{A}$ is in fact the limit of $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ is the category of quasi-coherent algebras with **M**-linearization. Note that from the description of $\mathcal{A}$ it follows that for every $n \in \mathbb{N}$ we have a surjective morphism $p_n : \mathcal{A} \twoheadrightarrow \mathcal{A}_n$ of algebras. We denote its kernel by $\mathcal{J}_n$ and we put $\mathcal{J} = \mathcal{J}_0$. The natural injection $\mathcal{O}_{Z_0} = \mathcal{A}_0 \to \mathcal{A}$ is a section of p_0, so that we have

$$\mathcal{J} = \bigoplus_{\lambda\in\mathbf{Irr}(\overline{T})\smallsetminus\{\lambda_0\}} \mathcal{A}[\lambda]$$

Recall that we denote by $\mathcal{I}_n$ the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$ for $n \in \mathbb{N}$. Then $\mathcal{I}_n = \mathcal{J}/\mathcal{J}_n$. Fix $m \in \mathbb{N}$ and consider $n \in \mathbb{N}$ such that $n \geq m$. Since $\mathcal{Z}$ is a formal **M**-scheme, the sheaf $\mathcal{I}_n^{m+1}$ is the kernel of the morphism $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_m$. Thus

$$\mathcal{J}_m/\mathcal{J}_n = \mathcal{I}_n^{m+1} = (\mathcal{J}^{m+1} + \mathcal{J}_n)/\mathcal{J}_n$$

Both $\mathcal{J}_m$ and $\mathcal{J}^{m+1}$ are $\mathbf{Irr}(\overline{T})$-graded and for given $\lambda \in \mathbf{Irr}(\overline{T})$ and for $n \geq n_\lambda$ the isotypic component $\mathcal{J}_n[\lambda]$ is zero by Lemma 8.1.2. Hence $\mathcal{J}_m = \mathcal{J}^{m+1}$ for every $m \in \mathbb{N}$. We define

$$Z = \mathrm{Spec}_{Z_0}\mathcal{A}$$

and we denote by $\pi : Z \to Z_0$ the structural morphism. The scheme Z inherits a **M**-action from $\mathcal{A}$. For every $n \in \mathbb{N}$ the zero-set of $\mathcal{J}^{n+1}$ in $\mathcal{A}$ is a **M**-scheme isomorphic to $Z_n = \mathrm{Spec}_{Z_0}\mathcal{A}_n$. Hence $\mathcal{Z}$ is isomorphic to $\widehat{Z}$ and this proves the theorem.

□

Theorem 8.2. *Let* **M** *be a Kempf monoid and let Z be a locally linear* **M***-scheme. Suppose that $\pi : Z \to Z^{\mathbf{M}}$ is the canonical retraction. If the formal* **M***-scheme $\widehat{Z}$ is locally noetherian, then $\pi : Z \to Z^{\mathbf{M}}$ is of finite type.*

Proof. Since π is affine (Proposition 5.2), we derive that $\mathcal{A} = \pi_*\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-algebra with **M**-linearization. We denote by $\mathcal{J}$ the ideal of $\mathcal{A}$ that corresponds to the closed immersion $Z^{\mathbf{M}} \hookrightarrow Z$. We know that the formal **M**-scheme

$$Z^{\mathbf{M}} = \mathrm{Spec}_{Z^{\mathbf{M}}}\mathcal{A}/\mathcal{J} \hookrightarrow ... \hookrightarrow \mathrm{Spec}_{Z^{\mathbf{M}}}\mathcal{A}/\mathcal{J}^{n+1} \hookrightarrow \mathrm{Spec}_{Z^{\mathbf{M}}}\mathcal{A}/\mathcal{J}^{n+2} \hookrightarrow ...$$

is locally noetherian. Hence $\mathcal{J}/\mathcal{J}^{n+1}$ is $\mathcal{A}/\mathcal{J}^{n+1}$-module of finite type. Thus $\{\mathcal{J}^i/\mathcal{J}^{i+1}\}_{1\leq i\leq n}$ are finite type $\mathcal{A}/\mathcal{J}$-modules. The series

$$0 \subseteq \mathcal{J}^n/\mathcal{J}^{n+1} \subseteq ... \subseteq \mathcal{J}/\mathcal{J}^{n+1} \subseteq \mathcal{A}/\mathcal{J}^{n+1}$$

has subquotients that are of finite type over $\mathcal{O}_{Z^{\mathbf{M}}} = \mathcal{A}/\mathcal{J}$. This implies that $\mathcal{A}/\mathcal{J}^{n+1}$ is a coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-algebra for every $n \in \mathbb{N}$. The claim that π is of finite type is local on $Z^{\mathbf{M}}$, hence we may assume that $Z^{\mathbf{M}}$ is quasi-compact. This reduces the question to the noetherian $Z^{\mathbf{M}}$. The sheaf $\mathcal{J}/\mathcal{J}^2 \subseteq \mathcal{A}/\mathcal{J}$ is coherent over $\mathcal{O}_{Z^{\mathbf{M}}}$. Since $Z^{\mathbf{M}}$ is noetherian, there exists coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-subsheaf $\mathcal{M} \subseteq \mathcal{J}$ such that the morphism $\mathcal{M} \twoheadrightarrow \mathcal{J}/\mathcal{J}^2$ is surjective. Fix an algebraically closed K extension of k and denote

$$\mathcal{A}_K = K \otimes_k \mathcal{A}, \mathcal{J}_K = K \otimes_k \mathcal{J}, \mathcal{M}_K = K \otimes_k \mathcal{M}$$

Since $\mathbf{M}$ is a Kempf torus and by **(3)** Theorem 6.6 there exists a closed immersion $\mathbb{A}^1_K \hookrightarrow \mathbf{M}_K$ of monoid K-schemes that preserve zero. This implies that we have $\mathbb{N}$-grading $\mathcal{A}_K = \bigoplus_{i\geq 0}\mathcal{A}_K[i]$ that gives rise to the action of $\mathbb{A}^1_K$. Moreover, by Propostion 5.2 we deduce that

$$\operatorname{Spec} K \times_k Z^{\mathbf{M}} = (\operatorname{Spec} K \times_k Z)^{\mathbf{M}_K} = (\operatorname{Spec} K \times_k Z)^{\mathbb{A}^1_K}$$

as K-schemes. This shows that $\mathcal{J}_K = K \otimes_k \mathcal{J} = \bigoplus_{i\geq 1}\mathcal{A}_K[i]$ is an ideal with positive grading. We have surjection $\mathcal{M}_K \twoheadrightarrow \mathcal{J}_K/\mathcal{J}_K^2$. By graded Nakayama's lemma, the ideal $\mathcal{J}_K$ is generated by $\mathcal{M}_K$. Then by induction on degrees we deduce that $\mathcal{A}_K$ is generated by $\mathcal{M}_K$ as a $K \otimes_k \mathcal{O}_{Z^{\mathbf{M}}}$-algebra. Thus $1_{\operatorname{Spec} K} \times_k \pi$ is of finite type and by faitfully flat descent also π is of finite type. $\square$

Theorem 8.3. *Let $\mathbf{M}$ be a Kempf monoid with group of unit $\mathbf{G}$ and let Z be a locally linear $\mathbf{M}$-scheme. Suppose that $\pi : Z \to Z^{\mathbf{M}}$ is the canonical retraction. If Z is locally noetherian, then the comparison functor*

$$\mathfrak{Coh}_{\mathbf{G}}(Z) \to \mathfrak{Coh}_{\mathbf{G}}(\widehat{Z})$$

is an equivalence of monoidal categories.

Setup. Since $\mathbf{M}$ is a Kempf torus, there exists a central closed torus T in $\mathbf{G}$ such that the scheme-theoretic closure $\overline{T}$ of T in $\mathbf{M}$ contains the zero. As above we note that π is affine (Proposition 5.2) and we pick a quasi-coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-algebra $\mathcal{A} = \pi_*\mathcal{O}_Z$ with $\mathbf{M}$-linearization. We denote by $\mathcal{J}$ the ideal of $\mathcal{A}$ that corresponds to the closed immersion $Z^{\mathbf{M}} \hookrightarrow Z$. Then $\mathcal{O}_{Z^{\mathbf{M}}} = \mathcal{A}/\mathcal{J}$ and since π is a retraction, we derive that $\mathcal{A} = \mathcal{O}_{Z^{\mathbf{M}}} \oplus \mathcal{J}$. Next $\widehat{Z}$ is locally noetherian (this follows from the fact that Z is locally noetherian). Hence an object of $\mathfrak{Coh}_{\mathbf{G}}(\widehat{Z})$ corresponds to a sequence of surjections

$$\ldots \twoheadrightarrow \mathcal{M}_{n+1} \twoheadrightarrow \mathcal{M}_n \twoheadrightarrow \ldots \twoheadrightarrow \mathcal{M}_1 \twoheadrightarrow \mathcal{M}_0$$

of coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-modules with $\mathbf{G}$-linearizations such that the following assertions hold.

(1) For each $n \in \mathbb{N}$ sheaf $\mathcal{M}_n$ is a module over $\mathcal{A}/\mathcal{J}^{n+1}$.

(2) For each $n \in \mathbb{N}$ the kernel of the surjection $\mathcal{M}_{n+1} \twoheadrightarrow \mathcal{M}_n$ is $\mathcal{J}^{n+1}\mathcal{M}_{n+1}$.

We fix an algebraically closed field K containing k. By **(3)** of Theorem 6.6 there exists a closed immersion $\operatorname{Spec} K \times_k \mathbb{G}_m \hookrightarrow T_K$ of group K-schemes that induces zero preserving closed immersion $\mathbb{A}^1_K \hookrightarrow \overline{T}_K$ of monoid K-schemes. By Proposition 5.2 we have

$$\operatorname{Spec} K \times_k Z^{\mathbf{M}} = (\operatorname{Spec} K \times_k Z)^{\mathbf{M}_K} = (\operatorname{Spec} K \times_k Z)^{\mathbb{A}^1_K}$$

This implies that

$$\mathcal{A}_K = K \otimes_k \mathcal{A} = \bigoplus_{i\geq 0}\mathcal{A}_K[i],\ \mathcal{J}_K = K \otimes_k \mathcal{J} = \bigoplus_{i\geq 1}\mathcal{A}_K[i]$$

where gradation is induced by the action of $\mathbb{A}^1_K$. For every $n \in \mathbb{N}$ the action of $\operatorname{Spec} K \times_k \mathbb{G}_m$ on $K \otimes_k \mathcal{M}_n$ induced by the closed immersion $\operatorname{Spec} K \times_k \mathbb{G}_m \hookrightarrow \overline{T}_K \hookrightarrow \mathbf{M}_K$ of group K-schemes gives rise to a gradation

$$K \otimes_k \mathcal{M}_n = \bigoplus_{i\in\mathbb{Z}}(K \otimes_k \mathcal{M}_n)[i]$$

$\square$

Lemma 8.3.1. *The following assertions hold.*

(1) *There exists $i_0 \in \mathbb{Z}$ such that for every $n \in \mathbb{N}$ we have $(K \otimes_k \mathcal{M}_n)[i] = 0$ for $i < i_0$.*

(2) *For every $i \in \mathbb{Z}$ there exists $n_i \in \mathbb{N}$ such that for all $n \geq n_i$ the surjection $(K \otimes_k \mathcal{M}_{n+1})[i] \twoheadrightarrow (K \otimes_k \mathcal{M}_n)[i]$ is an isomorphisms.*

(3) *For every λ in $\mathbf{Irr}(T)$ there exists $n_\lambda \in \mathbb{N}$ such that for all $n \geq n_\lambda$ the surjection $\mathcal{M}_{n+1}[\lambda] \twoheadrightarrow \mathcal{M}_n[\lambda]$ is an isomorphisms.*

Proof of the lemma. Fix $n \in \mathbb{N}$ and consider the decomposition $K \otimes_k \mathcal{M}_n = \bigoplus_{i \in \mathbb{Z}} (K \otimes_k \mathcal{M}_0)[i]$. Since $K \otimes_k \mathcal{M}_n$ is a coherent $K \otimes_k \mathcal{O}_{Z\mathbf{M}}$-module and the decomposition consists of modules over $K \otimes_k \mathcal{O}_{Z\mathbf{M}}$, we derive that there are only finitely many $i \in \mathbb{Z}$ such that $(K \otimes_k \mathcal{M}_0)[i] \neq 0$. Hence we may write $K \otimes_k \mathcal{M}_n = \bigoplus_{i \geq i_n} (K \otimes_k \mathcal{M}_n)[i]$ for some $i_n \in \mathbb{Z}$ such that $(K \otimes_k \mathcal{M}_n)[i_n] \neq 0$. Moreover, we know that the kernel of the surjection

$$K \otimes_k \mathcal{M}_{n+1} = \bigoplus_{i \geq i_{n+1}} (K \otimes_k \mathcal{M}_{n+1})[i] \twoheadrightarrow \bigoplus_{i \geq i_n} (K \otimes_k \mathcal{M}_n)[i] = K \otimes_k \mathcal{M}_n$$

is $\mathcal{J}_K^{n+1} \cdot (K \otimes_k \mathcal{M}_{n+1})$ and hence is contained in $\bigoplus_{i \geq (i_{n+1}+n+1)} (K \otimes_k \mathcal{M}_{n+1})[i]$ This implies that $(K \otimes_k \mathcal{M}_n)[i] = (K \otimes_k \mathcal{M}_{n+1})[i]$ for $i_{n+1} \leq i \leq i_{n+1} + n$. In particular, we have $(K \otimes_k \mathcal{M}_n)[i] = (K \otimes_k \mathcal{M}_{n+1})[i] \neq 0$ and thus $i_{n+1} \geq i_n$. This shows that $i_n \geq i_0$ for every $n \in \mathbb{N}$ and **(1)** is proved. Now the surjection

$$K \otimes_k \mathcal{M}_{n+1} = \bigoplus_{i \geq i_0} (K \otimes_k \mathcal{M}_{n+1})[i] \twoheadrightarrow \bigoplus_{i \geq i_0} (K \otimes_k \mathcal{M}_n)[i] = K \otimes_k \mathcal{M}_n$$

induces an isomorphism for i-th graded component, where $i_0 \leq i \leq i_0 + n$. Hence for fixed $i \in \mathbb{Z}$ there exists $n_i \in \mathbb{N}$ such that for all $n \geq n_i$ the surjection $(K \otimes_k \mathcal{M}_{n+1})[i] \twoheadrightarrow (K \otimes_k \mathcal{M}_n)[i]$ is an isomorphism. Thus we proved **(2)**. Fix now λ in $\mathbf{Irr}(T)$ and let V_λ be an irreducible representation in class λ. There exists finite subset $B_\lambda \subseteq \mathbb{Z}$ such that for $(K \otimes_k V_\lambda)[i] \neq 0$ if $i \in B_\lambda$. Now define $n_\lambda = \sup_{i \in B_\lambda} n_i$ the surjection $K \otimes_k \mathcal{M}_{n+1} \twoheadrightarrow K \otimes_k \mathcal{M}_n$ induces an isomorphism $(K \otimes_k \mathcal{M}_{n+1})[i] \cong (K \otimes_k \mathcal{M}_n)[i]$ for every i in B_λ. Thus for $n \geq n_\lambda$ the surjection $\mathcal{M}_{n+1} \twoheadrightarrow \mathcal{M}_n$ induces an isomorphism $\mathcal{M}_{n+1}[\lambda] \cong \mathcal{M}_n[\lambda]$. This shows **(3)**. □

Proof of the theorem. For fixed λ in $\mathbf{Irr}(T)$ we define $\mathcal{M}[\lambda] = \mathcal{M}_n[\lambda]$ for any $n \geq n_\lambda$, where $n_\lambda \in \mathbb{N}$ is as in **(3)** of Lemma 8.3.1 (in particular, this does not depend on $n \geq n_\lambda$). Next we define

$$\mathcal{M} = \bigoplus_{\lambda \in \mathbf{Irr}} \mathcal{M}[\lambda]$$

Since by Corollary 7.1 for every $n \in \mathbb{N}$ and $\lambda \in \mathbf{Irr}(T)$ sheaf $\mathcal{M}_n[\lambda]$ admits $\mathbf{G}$-linearization. Therefore, $\mathcal{M}$ is a quasi-coherent sheaf of $\mathcal{O}_{Z\mathbf{M}}$-modules with $\mathbf{G}$-linearization. We now show that $\mathcal{M}$ admits a canonical structure of $\mathcal{A}$-module. For this pick λ_1 and λ_2 in $\mathbf{Irr}(T)$. Consider the irreducible representations V_{λ_1} and V_{λ_1} in classes λ_1 and λ_2, respectively. Suppose that $\eta_1, ..., \eta_s$ are finitely many classes in $\mathbf{Irr}(T)$ such that $V_{\lambda_1} \otimes_k V_{\lambda_2}$ can be completely decomposed into irreducible representation in $\eta_1, ..., \eta_s$. Since the image of the multiplication $\mathcal{A}[\lambda_1] \otimes_{\mathcal{O}_{Z_0}} \mathcal{M}_n[\lambda_2] \to \mathcal{M}_n$ is also the image of a morphism

$$\mathcal{A}[\lambda_1] \otimes_k \mathcal{M}_n[\lambda_2] \twoheadrightarrow \mathcal{A}[\lambda_1] \otimes_{\mathcal{O}_{Z_0}} \mathcal{M}_n[\lambda_2] \longrightarrow \mathcal{M}_n$$

we deduce that it is contained in $\bigoplus_{i=1}^s \mathcal{M}_n[\eta_i]$. By **(3)** of Lemma 8.3.1 all these multiplications for $n \geq \sup\{n_{\lambda_1}, n_{\lambda_2}, n_{\eta_1}, ..., n_{\eta_s}\}$ can be identified. Now we define

$$\mathcal{A}[\lambda_1] \otimes_{\mathcal{O}_{Z_0}} \mathcal{M}[\lambda_2] \to \bigoplus_{i=1}^s \mathcal{M}[\eta_i] \subseteq \mathcal{M}$$

as a morphism induced by the multiplication morphism for any $n \geq \sup\{n_{\lambda_1}, n_{\lambda_2}, n_{\eta_1}, ..., n_{\eta_s}\}$. This gives an $\mathcal{A}$-module structure on $\mathcal{M}$. Next we prove that $\mathcal{M}$ is $\mathcal{A}$-module of finite type. Denote $K \otimes_k \mathcal{M}$ by $\mathcal{M}_K$. Note that the combination of **(2)** and **(3)** of Lemma 8.3.1 show that

$$\mathcal{M}_K[i] = (K \otimes_k \mathcal{M}_n)[i]$$

for $n \geq n_i$. Hence by **(1)** of Lemma 8.3.1 we have

$$\bigoplus_{\lambda \in \mathbf{Irr}(T)} \mathcal{M}[\lambda]_K = \mathcal{M}_K = \bigoplus_{i \geq i_0} \mathcal{M}_K[i]$$

Since each $\mathcal{M}_n$ is a coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-module, we derive that $\mathcal{M}_K[i]$ is a coherent $K \otimes_k \mathcal{O}_{Z^{\mathbf{M}}}$-module for every $i \in \mathbb{Z}$. Now we may pick $\lambda_1, ..., \lambda_r$ in $\mathbf{Irr}(T)$ such that we have a surjection

$$\bigoplus_{j=1}^{r} \mathcal{M}[\lambda_j]_K \twoheadrightarrow \bigoplus_{i_0 \leq i \leq 1} \mathcal{M}_K[i]$$

induced by the projection $\mathcal{M}_K = \bigoplus_{i \geq i_0} \mathcal{M}_K[i] \twoheadrightarrow \bigoplus_{i_0 \leq i \leq 1} \mathcal{M}_K[i]$. Let

$$\mathcal{G} = \bigoplus_{j=1}^{r} \mathcal{M}[\lambda_j]$$

be a $\mathcal{O}_{Z^{\mathbf{M}}}$-submodule of $\mathcal{M}$. Clearly each $\mathcal{M}[\lambda]$ is a coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-module. Hence $\mathcal{G}$ is a coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-module. Since $\mathcal{J}_K = \bigoplus_{i \geq 1} \mathcal{A}_K[i]$, we derive that

$$\mathcal{M}_K = \sum_{j \geq 1} \mathcal{J}_K^j \cdot \mathcal{G}_K$$

and hence $\mathcal{G}_K$ generates $\mathcal{M}_K$ as an $\mathcal{A}_K$-module. By faithfully flat descent we deduce that $\mathcal{G}$ generates $\mathcal{M}$ as an $\mathcal{A}$-module. Since $\mathcal{G}$ is a coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-module, we derive that $\mathcal{M}$ is $\mathcal{A}$-module of finite type. Moreover, by construction of $\mathcal{M}$ we have $\mathcal{M}/\mathcal{J}^{n+1}\mathcal{M} = \mathcal{M}_n$ for every $n \in \mathbb{N}$.
All these facts imply that $\mathcal{M}$ corresponds to a coherent sheaf on Z with $\mathbf{G}$-linearization such that its image under the comparison functor $\mathfrak{Coh}_{\mathbf{G}}(Z) \to \mathfrak{Coh}_{\mathbf{G}}(\widehat{Z})$ is a coherent sheaf on $\widehat{Z}$ with $\mathbf{G}$-linearization described by $\{\mathcal{M}_n\}_{n \in \mathbb{N}}$. Hence the comparison functor is essentially surjective. We now prove that it is full and faithful. For this let

$$... \longrightarrow\!\!\!\!\!\rightarrow \mathcal{N}_{n+1} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{N}_n \longrightarrow\!\!\!\!\!\rightarrow ... \longrightarrow\!\!\!\!\!\rightarrow \mathcal{N}_1 \longrightarrow\!\!\!\!\!\rightarrow \mathcal{N}_0$$

represents some other object of $\mathfrak{Coh}_{\mathbf{G}}(\widehat{Z})$. As for $\{\mathcal{M}_n\}_{n \in \mathbb{N}}$ we can construct finite type $\mathcal{A}$-module $\mathcal{N}$ with $\mathbf{G}$-linearization such that $\mathcal{N}/\mathcal{J}^{n+1}\mathcal{N} = \mathcal{N}_n$ for every $n \in \mathbb{N}$. Pick a morphism $f : \mathcal{M} \to \mathcal{N}$ of $\mathcal{A}$-modules with $\mathbf{G}$-linearization. For every λ in $\mathbf{Irr}(T)$ morphism $f[\lambda] : \mathcal{M}[\lambda] \to \mathcal{N}[\lambda]$ is equal (by virtue of constructions of $\mathcal{N}$ and $\mathcal{M}$) to a morphism $\left(1_{\mathcal{A}/\mathcal{J}^{n+1}} \otimes_{\mathcal{A}} f\right)[\lambda]$ for sufficiently large $n \in \mathbb{N}$. This implies that the comparison functor is full and faithful. □

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