

MONOID k -FUNCTORS AND THEIR REPRESENTATIONS

1. INTRODUCTION AND NOTATION

In these notes we study algebraic structures in the category of k -functors with special emphasis on monoid objects.

Throughout these notes k is a fixed commutative ring and \mathbf{Alg}_k denote the category of commutative k -algebras. If A, B are k -algebras, then we denote by $\mathrm{Mor}_k(A, B)$ the set of all morphisms $A \rightarrow B$ of k -algebras. Similarly if X, Y are k -schemes (i.e. schemes together with morphism to $\mathrm{Spec} k$), then we denote by $\mathrm{Mor}_k(X, Y)$ the set of all morphisms $X \rightarrow Y$ of k -schemes (morphisms of schemes that preserve structure morphisms to $\mathrm{Spec} k$). If M is a monoid, then we denote by M^* the group of units of M . If R is a ring, then we denote by R^\times its multiplicative monoid. Let A be a k -algebra and let V be an A -module and v be an element of V . Then for A -algebra B we denote by v_B the element $1 \otimes v$ of $B \otimes_A V$. If V is an A -module, then we denote $\mathrm{Hom}_A(V, A)$ by V^\vee . Thus we have a contravariant functor

$$(-)^\vee : \mathbf{Mod}(A)^{\mathrm{op}} \rightarrow \mathbf{Mod}(A)$$

2. ALGEBRAIC STRUCTURES IN THE CATEGORY OF k -FUNCTORS

In the sequel we assume that the reader is familiar with notions of a monoid, group etc. in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 2-5].

Definition 2.1. A monoid (group, abelian group, ring) k -functor is a monoid (group, abelian group, ring) object in the category of k -functors.

Example 2.2. Let \mathfrak{X} be a k -functor such that $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ exists. Then $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ is a monoid k -functor with respect to composition of morphisms.

Example 2.3. Let \mathfrak{G} be a monoid k -functor. Then we denote by \mathfrak{G}^* the k -subfunctor of \mathfrak{G} defined by

$$\mathfrak{G}^*(A) = \mathfrak{G}(A)^*$$

for every k -algebra A . We call \mathfrak{G}^* the unit group k -functor of \mathfrak{G} .

Example 2.4. Basic example of a ring k -functor is a k -functor \mathfrak{R} given by

$$\mathfrak{R}(A) = k, \mathfrak{R}(f) = 1_k$$

for any k -algebra A and morphism f of k -algebras. It can be described as a constant k -functor ([ML98, page 67]) corresponding to k .

Definition 2.5. Let \mathfrak{R} be a ring k -functor. Then we denote by \mathfrak{R}^\times the k -subfunctor of \mathfrak{R} defined by

$$\mathfrak{R}^\times(A) = \mathfrak{R}(A)^\times$$

for every k -algebra A . We call \mathfrak{R}^\times the multiplicative monoid k -functor of \mathfrak{R} .

Definition 2.6. Let \mathfrak{A} be a commutative ring k -functor. An \mathfrak{A} -algebra is an \mathfrak{A} -algebra object in the category of k -functors.

3. GLOBAL REGULAR FUNCTIONS ON A k -FUNCTOR

Recall the ring k -functor \mathfrak{K} from Example 2.4. Note that a \mathfrak{K} -algebra \mathfrak{A} can be viewed as a functor $\mathfrak{A} : \mathbf{Alg}_k \rightarrow \mathbf{Alg}_k$.

Definition 3.1. The \mathfrak{K} -algebra \mathfrak{O}_k given by the identity functor on \mathbf{Alg}_k is called *the structure \mathfrak{K} -algebra*.

Let $|-| : \mathbf{Alg}_k \rightarrow \mathbf{Set}$ be the forgetful k -functor. Note that $|-|$ is the underlying k -functor of \mathfrak{K} -algebra \mathfrak{O}_k . Recall that the affine line \mathbb{A}_k^1 is an affine k -scheme having k -algebra of polynomials with one variable as a k -algebra of regular functions.

Fact 3.2. Let $|-| : \mathbf{Alg}_k \rightarrow \mathbf{Set}$ be the forgetful k -functor. Then we have natural isomorphism

$$\mathfrak{P}_{\mathbb{A}_k^1} \cong |-|$$

Proof. Let B be a k -algebra. We have the following chain of identifications

$$\mathfrak{P}_{\mathbb{A}_k^1}(B) = \mathrm{Mor}_k(\mathrm{Spec} B, \mathbb{A}_k^1) = \mathrm{Mor}_k(\mathrm{Spec} B, \mathrm{Spec} k[x]) = \mathrm{Mor}_k(k[x], B) = |B|$$

natural in B . □

In particular, since $|-|$ carries the structure \mathfrak{K} -algebra \mathfrak{O}_k , we derive that $\mathfrak{P}_{\mathbb{A}_k^1}$ admits a structure of \mathfrak{K} -algebra isomorphic to \mathfrak{O}_k .

No we introduce regular functions on k -functors.

Definition 3.3. Let \mathfrak{X} be a k -functor and assume that $\mathrm{Mor}_k(\mathfrak{X}, \mathfrak{O}_k)$ is a set. Then $\mathrm{Mor}_k(\mathfrak{X}, \mathfrak{O}_k)$ is a k -algebra with respect to the structure induced by \mathfrak{O}_k . We call this k -algebra *the k -algebra of global regular functions on \mathfrak{X}* . Its elements are called *global regular functions on \mathfrak{X}* .

Definition 3.4. Let \mathfrak{X} be a k -functor. Suppose that A is a k -algebra, $x \in \mathfrak{X}(A)$ and $f \in \mathrm{Mor}_k(\mathfrak{X}, \mathfrak{O}_k)$. The element $f(x) \in A$ is called *the value of f on a point x* .

For given k -functor \mathfrak{X} we describe k -algebra operations on $\mathrm{Mor}_k(\mathfrak{X}, \mathfrak{O}_k)$ in terms of values of its elements on points of \mathfrak{X} . For this consider $\alpha \in k$ and $f, g_1 \in \mathrm{Mor}_k(\mathfrak{X}, \mathfrak{O}_k)$. We have formulas

$$(f + g)(x) = f(x) + g(x), (f \cdot g)(x) = f(x) \cdot g(x), (\alpha \cdot f)(x) = \alpha \cdot f(x)$$

in which right hand side are k -algebra operations in A .

Example 3.5. Let \mathfrak{X} be a k -functor and assume that $\mathrm{Mor}_k(\mathfrak{X}, \mathfrak{O}_k)$ exists. Fix k -algebra A . Note that $\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{O}_A)$ is an A -algebra of global regular functions on \mathfrak{X}_A . Moreover, if B is an A -algebra, then

$$\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{O}_A) \ni f \mapsto f_B \in \mathrm{Mor}_B(\mathfrak{X}_B, \mathfrak{O}_B)$$

is a morphism of A -algebras. This implies that $\mathrm{Mor}_k(\mathfrak{X}, \mathfrak{O}_k)$ admits a canonical structure of an \mathfrak{O}_k -algebra k -functor.

4. INTERNAL HOM AND PRODUCT OF k -FUNCTORS

We denote by $\mathbf{1}$ a k -functor that assigns to every k -algebra a set with one element. Then for every k -algebra A the restriction $\mathbf{1}_A$ is a terminal object in the category of A -functors.

Fact 4.1. Let \mathfrak{X} be a k -functor. Suppose A is a k -algebra and $x \in \mathfrak{X}(A)$. Then x determines a morphism $\mathbf{1}_A \rightarrow \mathfrak{X}_A$ that for every A -algebra B with structural morphism $f : A \rightarrow B$ sends a unique element of $\mathbf{1}_A(B)$ to $\mathfrak{X}(f)(x) \in \mathfrak{X}_A(B)$. This gives rise to a bijection

$$\mathfrak{X}(A) \cong \mathrm{Mor}_A(\mathbf{1}_A, \mathfrak{X}_A)$$

Proof. Left to the reader as an exercise. □

The discussion below is partially an application of the main result in [Mon19a, section 6]. For reader's convenience we make our presentation self-contained.

Definition 4.2. Let $\mathfrak{U}, \mathfrak{X}, \mathfrak{Y}$ be k -functors and let $\sigma : \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of k -functors. Fix $z \in \mathfrak{U}(A)$ for some k -algebra A . We denote by $i_z : \mathbf{1}_A \rightarrow \mathfrak{U}_A$ the morphism of A -functors corresponding to z by Fact 4.1. Since $\mathbf{1}_A$ is terminal A -functor, a morphism $\sigma_A \cdot (i_z \times \mathbf{1}_{\mathfrak{X}_A})$ is isomorphic to a morphism $\sigma_z : \mathfrak{X}_A \rightarrow \mathfrak{Y}_A$ of A -functors. We call σ_z the *slice of σ along z* .

Definition 4.3. Let $\mathfrak{X}, \mathfrak{Y}$ be k -functors. Let \mathfrak{J} be a k -functor such that $\mathfrak{J}(A)$ is a subset of a class $\text{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ for every k -algebra A . Assume that for every morphism $f : A \rightarrow B$ of k -algebras and every $\sigma \in \mathfrak{J}(A)$ we have

$$\mathfrak{J}(f)(\sigma) = \sigma_B$$

where $\sigma_B \in \text{Mor}_B(\mathfrak{X}_B, \mathfrak{Y}_B)$ is the restriction of σ along f . Then we call \mathfrak{J} a *k -subfunctor of internal hom of \mathfrak{X} and \mathfrak{Y}* .

Definition 4.4. Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{U}$ be k -functors and let $\sigma : \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of k -functors. Suppose that \mathfrak{J} is a k -subfunctor of internal hom of \mathfrak{X} and \mathfrak{Y} . Assume that $\sigma_z : \mathfrak{X}_A \rightarrow \mathfrak{Y}_A$ is contained in $\mathfrak{J}(A)$ for every k -algebra A and $z \in \mathfrak{U}(A)$. Then we call σ a *family of \mathfrak{J} -morphisms parametrized by \mathfrak{U}* .

Let \mathfrak{J} be a k -subfunctor of internal hom of \mathfrak{X} and \mathfrak{Y} . Assume that $\sigma : \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ is a \mathfrak{J} -family of morphism parametrized by \mathfrak{U} . Then the family of maps

$$\mathfrak{U}(A) \ni z \mapsto \sigma_z \in \mathfrak{J}(A)$$

gives rise to a morphism $\tau : \mathfrak{U} \rightarrow \mathfrak{J}$ of k -functors. Indeed, for a morphism $f : A \rightarrow B$ of k -algebras and $z \in \mathfrak{U}(A)$ we have

$$\sigma_B \cdot (i_{\mathfrak{U}(f)(z)} \times \mathbf{1}_{\mathfrak{X}_B}) = (\sigma_A \cdot (i_z \times \mathbf{1}_{\mathfrak{X}_A}))_B$$

and hence $\sigma_{\mathfrak{U}(f)(z)} = (\sigma_z)_B$. This gives rise to a map Φ of classes

$$\left\{ \text{families } \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y} \text{ of } \mathfrak{J}\text{-morphisms parametrized by } \mathfrak{U} \right\} \ni \sigma \mapsto \tau \in \text{Mor}_k(\mathfrak{U}, \mathfrak{J})$$

Consider next a morphism $\tau : \mathfrak{U} \rightarrow \mathfrak{J}$ of k -functors and define $\sigma : \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ by formula $\sigma^A(z, x) = (\tau^A(z))^A(x)$ for every k -algebra A and points $z \in \mathfrak{U}(A)$, $x \in \mathfrak{X}(A)$. Let $f : A \rightarrow B$ be a morphism of k -algebras. Then

$$\begin{aligned} \sigma^B(\mathfrak{U}(f)(z), \mathfrak{X}(f)(x)) &= (\tau^B(\mathfrak{U}(f)(z)))^B(\mathfrak{X}(f)(x)) = \left((\tau^A(z))_B \right)^B(\mathfrak{X}(f)(x)) = \\ &= (\tau^A(z))^B(\mathfrak{X}(f)(x)) = \mathfrak{Y}(f) \left((\tau^A(z))^A(x) \right) = \mathfrak{Y}(f)(\sigma^A(z, x)) \end{aligned}$$

Thus $\sigma : \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ is a morphism of k -functors. For every k -algebra A and $z \in \mathfrak{U}(A)$ we have $\sigma_z = \tau^A(z)$. Indeed, let $f : A \rightarrow B$ be a morphism of k -algebras and x be an element in $\mathfrak{X}(B)$ then we have

$$(\sigma_z)^B(x) = \sigma^B(\mathfrak{U}(f)(z), x) = (\tau^B(\mathfrak{U}(f)(z)))^B(x) = \left((\tau^A(z))_B \right)^B(x) = (\tau^A(z))^B(x)$$

Hence σ is a family of \mathfrak{J} -morphisms parametrized by \mathfrak{U} . This gives rise to a map Ψ of classes

$$\text{Mor}_k(\mathfrak{U}, \mathfrak{J}) \ni \tau \mapsto \sigma \in \left\{ \text{families } \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y} \text{ of } \mathfrak{J}\text{-morphisms parametrized by } \mathfrak{U} \right\}$$

Now we have the following result, which is an instance [Mon19a, Theorem 6.3]. To make presentation self-contained we give a complete proof.

Theorem 4.5. Let $\mathfrak{X}, \mathfrak{Y}$ be k -functors and let \mathfrak{J} be a k -subfunctor of internal hom of \mathfrak{X} and \mathfrak{Y} . Then maps

$$\Phi : \left\{ \text{families } \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y} \text{ of } \mathfrak{J}\text{-morphisms parametrized by } \mathfrak{U} \right\} \rightarrow \text{Mor}_k(\mathfrak{U}, \mathfrak{J})$$

and

$$\Psi : \text{Mor}_k(\mathfrak{U}, \mathfrak{J}) \rightarrow \left\{ \text{families } \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y} \text{ of } \mathfrak{J}\text{-morphisms parametrized by } \mathfrak{U} \right\}$$

are mutually inverse bijections.

Proof. Pick a morphism $\tau : \mathfrak{U} \rightarrow \mathfrak{J}$ of k -functors. Let A be a k -algebra and $z \in \mathfrak{U}(A)$. In the discussion preceding the statement we showed that $\Psi(\tau)_z = \tau^A(z)$. Thus

$$(\Phi(\Psi(\tau)))^A(z) = \Psi(\tau)_z = \tau^A(z)$$

and hence $\Phi \cdot \Psi$ is the identity.

Pick a family of \mathfrak{J} -morphism $\sigma : \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ parametrized by \mathfrak{U} . Let A be a k -algebra and $z \in \mathfrak{U}(A)$, $x \in \mathfrak{X}(A)$ be points. Then

$$(\Psi(\Phi(\sigma)))^A(z, x) = (\Phi(\sigma)^A(z))^A(x) = \sigma_z^A(x) = \sigma^A(z, x)$$

Thus $\Psi \cdot \Phi$ is the identity map. \square

Now we formulate some consequences of Theorem 4.5.

Corollary 4.6. *Let $\mathfrak{X}, \mathfrak{Y}$ be k -functors. Assume that for every k -algebra A the class $\text{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set. Then there is a bijection*

$$\text{Mor}_k(\mathfrak{U} \times \mathfrak{X}, \mathfrak{Y}) \rightarrow \text{Mor}_k(\mathfrak{U}, \text{Mor}_k(\mathfrak{X}, \mathfrak{Y}))$$

of classes.

Definition 4.7. Let $\mathfrak{X}, \mathfrak{Y}$ be k -functors. If $\text{Iso}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set for every k -algebra A , then we define a k -subfunctor $\text{Iso}_k(\mathfrak{X}, \mathfrak{Y})$ of $\text{Mor}_k(\mathfrak{X}, \mathfrak{Y})$ by

$$\text{Iso}_k(\mathfrak{X}, \mathfrak{Y})(A) = \text{Iso}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$$

for every k -algebra A . We call $\text{Iso}_k(\mathfrak{X}, \mathfrak{Y})$ the k -functor of isomorphism.

Definition 4.8. Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{U}$ be k -functors and let $\sigma : \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of k -functors. Assume that $\sigma_z : \mathfrak{X}_A \rightarrow \mathfrak{Y}_A$ is an isomorphism of A -functors for every k -algebra A . Then we call σ a family of isomorphisms parametrized by \mathfrak{U} .

Corollary 4.9. *Let $\mathfrak{U}, \mathfrak{X}, \mathfrak{Y}$ be k -functors and suppose that for every k -algebra A the class $\text{Iso}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set. The the following map*

$$\left\{ \text{families } \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y} \text{ of isomorphism parametrized by } \mathfrak{U} \right\} \rightarrow \text{Mor}_k(\mathfrak{U}, \text{Iso}_k(\mathfrak{X}, \mathfrak{Y}))$$

is a bijection of classes.

5. ACTIONS OF MONOID k -FUNCTORS

In this section we assume that the reader is familiar with notion of an action of a monoid object in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 5].

Let \mathfrak{G} be a monoid k -functor and \mathfrak{X} be a k -functor together with an action $\alpha : \mathfrak{G} \times \mathfrak{X} \rightarrow \mathfrak{X}$. Next assume that k -functor $\text{Mor}_k(\mathfrak{X}, \mathfrak{X})$ exists. By Example 2.2 it is a monoid k -functor. We define a morphism $\rho : \mathfrak{G} \rightarrow \text{Mor}_k(\mathfrak{X}, \mathfrak{X})$ of k -functors by formula $\rho(g) = \alpha_g$. Note that by discussion preceding Theorem 4.5, we deduce that ρ is a well defined morphism of k -functors. We show now that ρ is a morphism of monoids. For this pick k -algebra A and $g_1, g_2 \in \mathfrak{G}(A)$. Since α is an action, we deduce that $\alpha_{g_1 g_2} = \alpha_{g_1} \cdot \alpha_{g_2}$ and hence also

$$\rho(g_1 \cdot g_2) = \alpha_{g_1 g_2} = \alpha_{g_1} \cdot \alpha_{g_2} = \rho(g_1) \cdot \rho(g_2)$$

Therefore, ρ is a morphism of monoid k -functors. This shows how to construct a morphism of monoid k -functors ρ from an action α of \mathfrak{G} .

Theorem 5.1. Let \mathfrak{G} be a monoid k -functor and let \mathfrak{X} be a k -functor such that $\text{Mor}_k(\mathfrak{X}, \mathfrak{X})$ exists. Suppose that

$$\left\{ \text{actions of } \mathfrak{G} \text{ on } \mathfrak{X} \right\} \longrightarrow \left\{ \text{Morphisms } \rho : \mathfrak{G} \rightarrow \text{Mor}_k(\mathfrak{X}, \mathfrak{X}) \text{ of monoid } k\text{-functors} \right\}$$

is a map of classes described above. Then it is bijection.

Proof. Our goal is to construct the inverse of the map. Substitute $\mathfrak{J} = \text{Mor}_k(\mathfrak{X}, \mathfrak{X})$ in Theorem 4.5. Consider maps

$$\Phi : \left\{ \text{families } \mathfrak{G} \times \mathfrak{X} \rightarrow \mathfrak{X} \text{ of morphisms} \right\} \rightarrow \text{Mor}_k(\mathfrak{G}, \text{Mor}_k(\mathfrak{X}, \mathfrak{X}))$$

and

$$\Psi : \text{Mor}_k(\mathfrak{G}, \text{Mor}_k(\mathfrak{X}, \mathfrak{X})) \rightarrow \left\{ \text{families } \mathfrak{G} \times \mathfrak{X} \rightarrow \mathfrak{X} \text{ of morphisms} \right\}$$

in that Theorem. Then the map in the statement above is the restriction of Φ to \mathfrak{G} -actions on \mathfrak{X} on the right and morphisms $\mathfrak{G} \rightarrow \text{Mor}_k(\mathfrak{X}, \mathfrak{X})$ of monoid k -functors on the left. Since by Theorem 4.5 maps Φ and Ψ are mutually inverse, it suffices to check that Ψ sends a morphism $\rho : \mathfrak{G} \rightarrow \text{Mor}_k(\mathfrak{X}, \mathfrak{X})$ of monoids to an action of \mathfrak{G} on \mathfrak{X} . For this denote $\Psi(\rho)$ by α . Consider k -algebra A and A -points $g_1, g_2 \in \mathfrak{G}(A)$, $x \in \mathfrak{X}(A)$. Then

$$\alpha(g_1, \alpha(g_2, x)) = \rho(g_1)(\rho(g_2)(x)) = (\rho(g_1) \cdot \rho(g_2))(x) = \rho(g_1 \cdot g_2)(x) = \alpha(g_1 \cdot g_2, x)$$

Therefore, α is an action of \mathfrak{G} on \mathfrak{X} . □

Proposition 5.2. Let \mathfrak{G} be a monoid k -functor and let $\mathfrak{X}_1, \mathfrak{X}_2$ be k -functors such that $\text{Mor}_k(\mathfrak{X}_1, \mathfrak{X}_1), \text{Mor}_k(\mathfrak{X}_2, \mathfrak{X}_2)$ exist. Suppose that $\alpha_1 : \mathfrak{G} \times \mathfrak{X}_1 \rightarrow \mathfrak{X}_1$, $\alpha_2 : \mathfrak{G} \times \mathfrak{X}_2 \rightarrow \mathfrak{X}_2$ are actions of \mathfrak{G} , respectively. Suppose that $\sigma : \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$ is a morphism of k -functors. Then the following assertions are equivalent.

(i) The square

$$\begin{array}{ccc} \mathfrak{G} \times \mathfrak{X}_1 & \xrightarrow{1_{\mathfrak{G}} \times \sigma} & \mathfrak{G} \times \mathfrak{X}_2 \\ \alpha_1 \downarrow & & \downarrow \alpha_2 \\ \mathfrak{X}_1 & \xrightarrow{\sigma} & \mathfrak{X}_2 \end{array}$$

is commutative.

(ii) For every k -algebra A and $g \in \mathfrak{G}(A)$ we have

$$\sigma_A \cdot \rho_1(g) = \rho_2(g) \cdot \sigma_A$$

where $\rho_1 : \mathfrak{G} \rightarrow \text{Mor}_k(\mathfrak{X}_1, \mathfrak{X}_1)$ and $\rho_2 : \mathfrak{G} \rightarrow \text{Mor}_k(\mathfrak{X}_2, \mathfrak{X}_2)$ are morphism of monoid k -functors corresponding to α_1 and α_2 , respectively.

Proof. Conditions expressed in (i) and (ii) are directly translatable to each other by virtue of the bijection in Theorem 5.1. □

Definition 5.3. Let \mathfrak{G} be a monoid k -functor and let $(\mathfrak{X}_1, \alpha_1), (\mathfrak{X}_2, \alpha_2)$ be k -functors with actions of \mathfrak{G} . Suppose that $\sigma : \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$ is a morphism k -functors such that the square

$$\begin{array}{ccc} \mathfrak{G} \times \mathfrak{X}_1 & \xrightarrow{1_{\mathfrak{G}} \times \sigma} & \mathfrak{G} \times \mathfrak{X}_2 \\ \alpha_1 \downarrow & & \downarrow \alpha_2 \\ \mathfrak{X}_1 & \xrightarrow{\sigma} & \mathfrak{X}_2 \end{array}$$

is commutative. Then σ is called an \mathfrak{G} -equivariant morphism.

6. MODULES OVER RING k -FUNCTORS

Definition 6.1. Let \mathfrak{R} be a ring k -functor. Suppose that \mathfrak{M} is an abelian group k -functor and there exists a morphism $\mathfrak{R} \times \mathfrak{M} \rightarrow \mathfrak{M}$ of k -functors that for each k -algebra A makes $\mathfrak{M}(A)$ into an $\mathfrak{R}(A)$ -module. Then we say that \mathfrak{M} is a *module k -functor over \mathfrak{R}* .

Definition 6.2. Let \mathfrak{R} be a ring k -functor and let $\mathfrak{M}_1, \mathfrak{M}_2$ be module k -functors over \mathfrak{R} . Suppose that $\sigma : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ is a morphism of abelian group k -functors such that the diagram

$$\begin{array}{ccc} \mathfrak{R} \times \mathfrak{M}_1 & \xrightarrow{1_{\mathfrak{R}} \times \sigma} & \mathfrak{R} \times \mathfrak{M}_2 \\ \alpha_1 \downarrow & & \downarrow \alpha_2 \\ \mathfrak{M}_1 & \xrightarrow{\sigma} & \mathfrak{M}_2 \end{array}$$

is commutative, where $\alpha_i : \mathfrak{R} \times \mathfrak{M}_i \rightarrow \mathfrak{M}_i$ define \mathfrak{R} -module structure on \mathfrak{M}_i for $i = 1, 2$. Then σ is a *morphism of modules over \mathfrak{R}* .

Let \mathfrak{M}_1 and \mathfrak{M}_2 be module k -functors over \mathfrak{R} . We denote by

$$\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{M}_1, \mathfrak{M}_2)$$

the class of all morphisms of modules $\mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ over \mathfrak{R} . We denote the category of \mathfrak{R} -modules by $\mathbf{Mod}(\mathfrak{R})$.

Definition 6.3. Let \mathfrak{M}_1 and \mathfrak{M}_2 be module k -functors over \mathfrak{R} . Assume that $\mathrm{Hom}_{\mathfrak{R}_A}((\mathfrak{M}_1)_A, (\mathfrak{M}_2)_A)$ is a set for every k -algebra A . Then we define a k -subfunctor $\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{M}_1, \mathfrak{M}_2)$ of internal hom of \mathfrak{M}_1 and \mathfrak{M}_2 by formula

$$\mathbf{Alg}_k \ni A \mapsto \mathrm{Hom}_{\mathfrak{R}_A}((\mathfrak{M}_1)_A, (\mathfrak{M}_2)_A) \in \mathbf{Set}$$

We call $\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{M}_1, \mathfrak{M}_2)$ a *k -functor of module morphisms of \mathfrak{M}_1 and \mathfrak{M}_2* .

If \mathfrak{M} is a module k -functor over some ring k -functor \mathfrak{R} , then we denote (if it exists) $\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{M}, \mathfrak{M})$ by $\mathrm{End}_{\mathfrak{R}}(\mathfrak{M})$.

Example 6.4. Let \mathfrak{M} be a module over a ring k -functor \mathfrak{R} . Assume that $\mathrm{End}_{\mathfrak{R}}(\mathfrak{M})$ exists. Then $\mathrm{End}_{\mathfrak{R}}(\mathfrak{M})$ is a ring k -functor with respect to composition of morphisms of modules as the multiplication and the usual addition of module morphisms. Moreover, if \mathfrak{A} is a commutative ring k -functor, then $\mathrm{End}_{\mathfrak{A}}(\mathfrak{M})$ (if exists) admits additional structure of a \mathfrak{A} -algebra k -functor induced via a unique morphism $\mathfrak{A} \rightarrow \mathrm{End}_{\mathfrak{R}}(\mathfrak{M})$ of ring k -functors that sends $1 \mapsto 1_{\mathfrak{M}}$.

Let \mathfrak{A} be a commutative ring k -functor and let \mathfrak{R} be a \mathfrak{A} -algebra k -functor. This means that there exists a morphism $\mathfrak{A} \rightarrow \mathfrak{R}$ of ring k -functors and for every k -algebra A induced morphism $\mathfrak{A}(A) \rightarrow \mathfrak{R}(A)$ sends $\mathfrak{A}(A)$ to the center of a ring $\mathfrak{R}(A)$. Fix a module \mathfrak{M} over \mathfrak{A} . Next assume that k -functor $\mathrm{End}_{\mathfrak{A}}(\mathfrak{M})$ exists. By Example 6.4 it is a ring k -functor.

Definition 6.5. In the setting above suppose that $\alpha : \mathfrak{R} \times \mathfrak{M} \rightarrow \mathfrak{M}$ is a morphism of k -functors. Suppose that α makes \mathfrak{M} into \mathfrak{R} -module and moreover, for every k -algebra A and for every point $x \in \mathfrak{R}(A)$ morphism α_x is a morphism of \mathfrak{A}_A -modules. Then α is called a *\mathfrak{A} -linear \mathfrak{R} -action on \mathfrak{M}* .

We continue the discussion. We assume that we are given an \mathfrak{A} -linear \mathfrak{R} -action $\alpha : \mathfrak{R} \times \mathfrak{M} \rightarrow \mathfrak{M}$ on \mathfrak{M} . We define a morphism $\rho : \mathfrak{R} \rightarrow \mathrm{End}_{\mathfrak{A}}(\mathfrak{M})$ of k -functors by formula $\rho(r) = \alpha_r$. As in Section 5 we can prove that ρ is a morphism of ring k -functors. Now we have the following result.

Theorem 6.6. Let \mathfrak{R} be an algebra k -functor over commutative ring \mathfrak{A} k -functor and let \mathfrak{M} be a \mathfrak{A} -module such that $\mathrm{End}_{\mathfrak{A}}(\mathfrak{M})$ exists. Suppose that

$$\left\{ \mathfrak{A} \text{ linear actions of } \mathfrak{R} \text{ on } \mathfrak{M} \right\} \longrightarrow \left\{ \text{Morphisms } \rho : \mathfrak{R} \rightarrow \text{End}_{\mathfrak{D}_k}(\mathfrak{M}) \text{ of ring } k\text{-functors} \right\}$$

is a map of classes described above. Then it is bijection.

Proof. The proof is similar to the proof of Theorem 5.1. \square

7. MONOID ALGEBRA $\mathfrak{D}_k[\mathfrak{G}]$ AND ITS MODULES

Definition 7.1. Let \mathfrak{G} be a monoid k -functor. Then we construct an \mathfrak{D}_k -algebra $\mathfrak{D}_k[\mathfrak{G}]$ as follows. For every k -algebra A we define

$$\mathfrak{D}_k[\mathfrak{G}](A) = A[\mathfrak{G}(A)]$$

where the right hand side is monoid A -algebra for the abstract monoid $\mathfrak{G}(A)$. The structure of monoid k -functor on \mathfrak{G} and \mathfrak{R} -algebra \mathfrak{D}_k makes $\mathfrak{D}_k[\mathfrak{G}]$ into a ring k -functor. Moreover, we have a morphism $\mathfrak{D}_k \rightarrow \mathfrak{D}_k[\mathfrak{G}]$ which for every k -algebra A is given by the canonical inclusion

$$A \hookrightarrow A[\mathfrak{G}(A)]$$

Thus $\mathfrak{D}_k[\mathfrak{G}]$ is \mathfrak{D}_k -algebra. We call $\mathfrak{D}_k[\mathfrak{G}]$ a monoid \mathfrak{D}_k -algebra over \mathfrak{G} .

Fact 7.2. Let \mathfrak{G} be a monoid k -functor and let \mathfrak{R} be an \mathfrak{D}_k -algebra k -functor. Then every morphism

$$\sigma : \mathfrak{G} \rightarrow \mathfrak{R}^\times$$

of monoid k -functors admits a unique extension

$$\tilde{\sigma} : \mathfrak{D}_k[\mathfrak{G}] \rightarrow \mathfrak{R}$$

to a morphism of \mathfrak{D}_k -algebras.

Proof. This follows from the analogical universal property of algebras over abstract monoids. \square

Definition 7.3. Let \mathfrak{G} be a monoid k -functor and let \mathfrak{M} be a module over \mathfrak{D}_k . Suppose that $\alpha : \mathfrak{G} \times \mathfrak{M} \rightarrow \mathfrak{M}$ is an action of \mathfrak{G} such that for any k -algebra A and point $g \in \mathfrak{G}(A)$ morphism $\alpha_g : \mathfrak{M}_A \rightarrow \mathfrak{M}_A$ is a morphism of \mathfrak{D}_A -modules. Then α is called a linear \mathfrak{G} -action on \mathfrak{M} .

Suppose now that \mathfrak{G} is a monoid k -functor and \mathfrak{M} is a module \mathfrak{D}_k . Note that every linear \mathfrak{G} -action $\alpha : \mathfrak{G} \times \mathfrak{M} \rightarrow \mathfrak{M}$ extends uniquely to a \mathfrak{D}_k -linear action $\mathfrak{D}_k[\mathfrak{G}] \times \mathfrak{M} \rightarrow \mathfrak{M}$ of monoid \mathfrak{D}_k -algebra. This gives a bijection

$$\left\{ \text{Linear actions of } \mathfrak{G} \text{ on } \mathfrak{M} \right\} \longrightarrow \left\{ \mathfrak{D}_k\text{-linear actions } \mathfrak{D}_k[\mathfrak{G}] \times \mathfrak{M} \rightarrow \mathfrak{M} \right\}$$

Next assume that k -functor $\text{End}_{\mathfrak{D}_k}(\mathfrak{M})$ exists. By Example 6.4 it is an \mathfrak{D}_k -algebra k -functor. Next by Theorem 6.6 we have a bijection

$$\left\{ \mathfrak{D}_k\text{-linear actions of } \mathfrak{D}_k[\mathfrak{G}] \times \mathfrak{M} \rightarrow \mathfrak{M} \right\} \longrightarrow \left\{ \text{Morphisms } \mathfrak{D}_k[\mathfrak{G}] \rightarrow \text{End}_{\mathfrak{D}_k}(\mathfrak{M}) \text{ of } \mathfrak{D}_k\text{-algebras} \right\}$$

Finally Fact 7.2 implies that we have a bijection

$$\left\{ \text{Morphisms } \mathfrak{D}_k[\mathfrak{G}] \rightarrow \text{End}_{\mathfrak{D}_k}(\mathfrak{M}) \text{ of } \mathfrak{D}_k\text{-algebras} \right\} \longrightarrow \left\{ \text{Morphisms } \mathfrak{G} \rightarrow \text{End}_{\mathfrak{D}_k}(\mathfrak{M}) \text{ of monoids} \right\}$$

This chain of bijections sends a linear action $\alpha : \mathfrak{G} \times \mathfrak{M} \rightarrow \mathfrak{M}$ of \mathfrak{G} to a morphism $\rho : \mathfrak{G} \rightarrow \text{End}_{\mathfrak{D}_k}(\mathfrak{M})$ of monoid k -functors given by $\rho(g) = \alpha_g$ for every $g \in \mathfrak{G}(A)$ and every k -algebra A . We proved the following result.

Proposition 7.4. Let \mathfrak{G} be a monoid k -functor and let \mathfrak{M} be a \mathfrak{D}_k -module such that $\text{End}_{\mathfrak{D}_k}(\mathfrak{M})$ exists. Then the following classes are in canonical bijections described above.

- (1) Linear actions of \mathfrak{G} on \mathfrak{M} .
- (2) \mathfrak{D}_k -linear actions $\mathfrak{D}_k[\mathfrak{G}] \times \mathfrak{M} \rightarrow \mathfrak{M}$. These are precisely $\mathfrak{D}_k[\mathfrak{G}]$ -modules.
- (3) Morphisms $\mathfrak{D}_k[\mathfrak{G}] \rightarrow \mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M})$ of \mathfrak{D}_k -algebras.
- (4) Morphisms $\mathfrak{G} \rightarrow \mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M})$ of monoids.

Moreover, the bijection between class (1) and (2) does not require the existence of $\mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M})$.

Now in a similar manner we can describe morphisms.

Proposition 7.5. *Let \mathfrak{G} be a monoid k -functor and let $\mathfrak{M}_1, \mathfrak{M}_2$ be k -functors of \mathfrak{D}_k -modules such that $\mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M}_1), \mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M}_2)$ exist. Suppose that $\alpha_1 : \mathfrak{G} \times \mathfrak{M}_1 \rightarrow \mathfrak{M}_1, \alpha_2 : \mathfrak{G} \times \mathfrak{M}_2 \rightarrow \mathfrak{M}_2$ are linear actions of \mathfrak{G} . Suppose that $\sigma : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ is a morphism of modules over \mathfrak{D}_k . Then the following assertions are equivalent.*

- (i) The square

$$\begin{array}{ccc} \mathfrak{G} \times \mathfrak{M}_1 & \xrightarrow{1_{\mathfrak{G}} \times \sigma} & \mathfrak{G} \times \mathfrak{M}_2 \\ \alpha_1 \downarrow & & \downarrow \alpha_2 \\ \mathfrak{M}_1 & \xrightarrow{\sigma} & \mathfrak{M}_2 \end{array}$$

is commutative.

- (ii) The square

$$\begin{array}{ccc} \mathfrak{D}_k[\mathfrak{G}] \times \mathfrak{M}_1 & \xrightarrow{1_{\mathfrak{D}_k[\mathfrak{G}]} \times \sigma} & \mathfrak{D}_k[\mathfrak{G}] \times \mathfrak{M}_2 \\ \tilde{\alpha}_1 \downarrow & & \downarrow \tilde{\alpha}_2 \\ \mathfrak{M}_1 & \xrightarrow{\sigma} & \mathfrak{M}_2 \end{array}$$

is commutative, where $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are \mathfrak{D}_k -linear actions of $\mathfrak{D}_k[\mathfrak{G}]$ corresponding to α_1 and α_2 , respectively. This states that σ is a morphism of $\mathfrak{D}_k[\mathfrak{G}]$ -modules.

- (iii) For every k -algebra A and $g \in \mathfrak{G}(A)$ we have

$$\sigma_A \cdot \tilde{\rho}_1(g) = \tilde{\rho}_2(g) \cdot \sigma_A$$

where $\tilde{\rho}_1 : \mathfrak{D}_k[\mathfrak{G}] \rightarrow \mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M}_1)$ and $\tilde{\rho}_2 : \mathfrak{D}_k[\mathfrak{G}] \rightarrow \mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M}_2)$ are morphism of \mathfrak{D}_k -algebras corresponding to $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, respectively.

- (iv) For every k -algebra A and $g \in \mathfrak{G}(A)$ we have

$$\sigma_A \cdot \rho_1(g) = \rho_2(g) \cdot \sigma_A$$

where $\rho_1 : \mathfrak{G} \rightarrow \mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M}_1)$ and $\rho_2 : \mathfrak{G} \rightarrow \mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M}_2)$ are restrictions of $\tilde{\rho}_1$ and $\tilde{\rho}_2$, respectively.

The equivalence of (i) and (ii) does not require the existence of $\mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M}_1)$ and $\mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M}_2)$.

Proof. Conditions expressed in (i)-(iv) are directly translatable to each other by virtue of bijections in Proposition 7.4. \square

Let \mathfrak{G} be a monoid k -functor. We denote by $\mathbf{Mod}(\mathfrak{D}_k[\mathfrak{G}])$ the category of $\mathfrak{D}_k[\mathfrak{G}]$ -modules.

8. LINEAR REPRESENTATIONS OF A MONOID k -FUNCTORS

We start the discussion with some results that relates categories $\mathbf{Mod}(k)$ and $\mathbf{Mod}(\mathfrak{D}_k)$.

Example 8.1. Let V be a k -module. We define a k -functor V_a . We set

$$V_a(A) = A \otimes_k V, \quad V_a(f) = f \otimes_k 1_V$$

for every k -algebra A and every morphism $f : A \rightarrow B$ of k -algebras. Note that V_a is \mathfrak{D}_k -module. Suppose that $\phi : V \rightarrow W$ is a morphism of k -modules, then we define $\phi_a : V_a \rightarrow W_a$ by formula

$$\phi_a^A = 1_A \otimes_k \phi$$

for every k -algebra. Then ϕ_a is a morphism of \mathfrak{D}_k -modules.

Remark 8.2. Let V be a finitely generated, projective k -module. Then for each k -algebra A we have an isomorphism

$$\mathfrak{P}_{\text{SpecSym}(V^\vee)}(A) = \text{Mor}_k(\text{Sym}(V^\vee), A) = \text{Hom}_k(V^\vee, A) \cong A \otimes_k V$$

Clearly this isomorphism is natural in A . Thus V_a is representable by a k -scheme $\text{SpecSym}(V^\vee)$.

Proposition 8.3. The functor $(-)_a : \mathbf{Mod}(k) \rightarrow \mathbf{Mod}(\mathfrak{D}_k)$ is full and faithful.

Proof. Fix k -modules V, W . Then

$$\text{Hom}_{\mathfrak{D}_k}(V_a, W_a) \ni \sigma \mapsto \sigma^k \in \text{Hom}_k(V, W)$$

and

$$\text{Hom}_k(V, W) \ni \phi \mapsto \phi_a \in \text{Hom}_{\mathfrak{D}_k}(V_a, W_a)$$

are mutually inverse bijections. Hence the functor is full and faithful. \square

Example 8.4. Let V be a k -module. We define a k -functor \mathcal{L}_V . We set

$$\mathcal{L}_V(A) = \text{Hom}_A(A \otimes_k V, A \otimes_k V)$$

for every k -algebra A . Next for every morphism $f : A \rightarrow B$ of k -algebras and every morphism $\phi : A \otimes_k V \rightarrow A \otimes_k V$ of A -modules we define $\mathcal{L}_V(f)(\phi)$ as a unique morphism of B -modules such that the diagram

$$\begin{array}{ccc} A \otimes_k V & \xrightarrow{\phi} & A \otimes_k V \\ f \otimes_k 1_V \downarrow & & \downarrow f \otimes_k 1_V \\ B \otimes_k V & \xrightarrow{\mathcal{L}_V(\phi)} & B \otimes_k V \end{array}$$

is commutative. Note also that $\mathcal{L}_V(A)$ is an A -algebra for every k -algebra A . Hence \mathcal{L}_V is a monoid \mathfrak{D}_k -algebra. Note that we have natural identification

$$\mathcal{L}_V(A) = \text{Hom}_k(V, A \otimes_k V)$$

for every k -algebra. One can describe \mathfrak{D}_k -algebra structure on \mathcal{L}_V in terms of this identification as follows. Since $\text{Hom}_k(V, A \otimes_k V)$ carries canonical structure of A -module it suffices to describe the multiplication. For this suppose that $d_1, d_2 \in \text{Hom}_k(V, A \otimes_k V)$. Then their product is given by

$$(\mu_A \otimes_k 1_V) \cdot (1_A \otimes_k d_2) \cdot d_1$$

where $\mu_A : A \otimes_k A \rightarrow A$ is the multiplication on A .

Remark 8.5. Let V be a k -module. Proposition 8.3 implies that there are bijective maps that make the square

$$\begin{array}{ccc}
\mathcal{L}_V(A) & \xrightarrow{\cong} & \mathcal{E}nd_{\mathfrak{D}_A}((V_a)_A, (V_a)_A) \\
\mathcal{L}_V(f) \downarrow & & \downarrow \sigma \mapsto \sigma_B \\
\mathcal{L}_V(B) & \xrightarrow{\cong} & \mathcal{E}nd_{\mathfrak{D}_B}((V_a)_B, (V_a)_B)
\end{array}$$

commutative for every morphism $f : A \rightarrow B$ of k -algebras. This induces an identification $\mathcal{L}_V = \mathcal{E}nd_{\mathfrak{D}_k}(V_a)$ of \mathfrak{D}_k -algebras.

Remark 8.6. Suppose that V is a finitely generated, projective k -module. Then for each k -algebra A we have an isomorphism

$$\mathcal{L}_V(A) = \text{Hom}_A(V, A \otimes_k V) \cong A \otimes_k V^\vee \otimes_k V$$

Clearly this isomorphism is natural in A . Hence \mathcal{L}_V is isomorphic with $(V^\vee \otimes_k V)_a$ and thus (Remark 8.2) it is representable by a k -scheme $\text{SpecSym}(V \otimes_k V^\vee)$.

Definition 8.7. Let \mathfrak{G} be a monoid k -functor. A pair (V, ρ) consisting of a k -module V and a morphism $\rho : \mathfrak{G} \rightarrow \mathcal{L}_V$ of k -monoids is called a *linear representation of \mathfrak{G}* .

Next result characterizes linear representations of monoid k -functors.

Corollary 8.8. Let \mathfrak{G} be a monoid k -functor and let V be a k -module. Then the following classes are in canonical bijections.

- (1) Linear actions of \mathfrak{G} on V_a .
- (2) \mathfrak{D}_k -linear actions $\mathfrak{D}_k[\mathfrak{G}] \times V_a \rightarrow V_a$. These are precisely $\mathfrak{D}_k[\mathfrak{G}]$ -modules.
- (3) Morphisms $\mathfrak{D}_k[\mathfrak{G}] \rightarrow \mathcal{L}_V$ of \mathfrak{D}_k -algebras.
- (4) Morphisms $\mathfrak{G} \rightarrow \mathcal{L}_V$ of monoids.

Proof. This follows from Proposition 7.4. □

Definition 8.9. Let \mathfrak{G} be a monoid k -functor and let $(V, \rho), (W, \delta)$ be its linear representations. A morphism $\phi : V \rightarrow W$ of k -modules such that

$$\phi_a^A \cdot \rho(g) = \delta(g) \cdot \phi_a^A$$

for every k -algebra A and $g \in \mathfrak{G}(A)$ is called a *morphism of linear representations of \mathfrak{G}* .

Next result characterizes morphisms of linear representations of monoid k -functor.

Corollary 8.10. Let \mathfrak{G} be a monoid k -functor and let V, W be k -modules. Suppose that $\alpha_1 : \mathfrak{G} \times V_a \rightarrow V_a, \alpha_2 : \mathfrak{G} \times W_a \rightarrow W_a$ are linear actions of \mathfrak{G} . Suppose that $\phi : V \rightarrow W$ is a morphism of k -modules. Then the following assertions are equivalent.

- (i) The square

$$\begin{array}{ccc}
\mathfrak{G} \times V_a & \xrightarrow{1_{\mathfrak{G}} \times \phi_a} & \mathfrak{G} \times W_a \\
\alpha_1 \downarrow & & \downarrow \alpha_2 \\
V_a & \xrightarrow{\phi_a} & W_a
\end{array}$$

is commutative.

- (ii) The square

$$\begin{array}{ccc}
\mathfrak{D}_k[\mathfrak{G}] \times V_a & \xrightarrow{1_{\mathfrak{D}_k[\mathfrak{G}]} \times \phi_a} & \mathfrak{D}_k[\mathfrak{G}] \times W_a \\
\downarrow \tilde{\alpha}_1 & & \downarrow \tilde{\alpha}_2 \\
V_a & \xrightarrow{\phi_a} & W_a
\end{array}$$

is commutative, where $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are \mathfrak{D}_k -linear actions of $\mathfrak{D}_k[\mathfrak{G}]$ corresponding to α_1 and α_2 , respectively.

(iii) For every k -algebra A and $g \in \mathfrak{G}(A)$ we have

$$\phi_a^A \cdot \tilde{\rho}_1(g) = \tilde{\rho}_2(g) \cdot \phi_a^A$$

where $\tilde{\rho}_1 : \mathfrak{D}_k[\mathfrak{G}] \rightarrow \mathcal{L}_V$ and $\tilde{\rho}_2 : \mathfrak{D}_k[\mathfrak{G}] \rightarrow \mathcal{L}_W$ are morphism of \mathfrak{D}_k -algebras corresponding to $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, respectively.

(iv) For every k -algebra A and $g \in \mathfrak{G}(A)$ we have

$$\phi_a^A \cdot \rho_1(g) = \rho_2(g) \cdot \phi_a^A$$

where $\rho_1 : \mathfrak{G} \rightarrow \mathcal{L}_V$ and $\rho_2 : \mathfrak{G} \rightarrow \mathcal{L}_W$ are restrictions of $\tilde{\rho}_1$ and $\tilde{\rho}_2$, respectively. This states that ϕ is a morphism of linear representations of \mathfrak{G} .

Proof. This follows from Proposition 7.5. \square

Let \mathfrak{G} be a monoid k -functor. We denote by $\mathbf{Rep}(\mathfrak{G})$ its category of linear representations. Note that $\mathbf{Rep}(\mathfrak{G})$ is a full subcategory of $\mathbf{Mod}(\mathfrak{D}_k[\mathfrak{G}])$.

9. CONSTRUCTIONS OF LINEAR REPRESENTATIONS

Example 9.1 (Outer tensor product of representations). Let (V_1, ρ_1) and (V_2, ρ_2) are linear representations of monoid k -functors \mathfrak{G}_1 and \mathfrak{G}_2 , respectively. Then we define a linear representation of $\mathfrak{G}_1 \times \mathfrak{G}_2$ with $V_1 \otimes_k V_2$ as the underlying k -module that corresponds to a morphism $\rho : \mathfrak{G}_1 \times \mathfrak{G}_2 \rightarrow \mathcal{L}_{V_1 \otimes_k V_2}$ of monoid k -functors given by

$$\rho(g_1, g_2) = \rho_1(g_1) \otimes_A \rho_2(g_2) : A \otimes_k V_1 \otimes_k V_2 \rightarrow A \otimes_k V_1 \otimes_k V_2$$

for $(g_1, g_2) \in \mathfrak{G}_1(A) \times \mathfrak{G}_2(A)$, where A is a k -algebra.

Example 9.2 (Tensor product of representations). Let (V_1, ρ_1) and (V_2, ρ_2) are linear representations of monoid k -functor \mathfrak{G} . Then we define a linear representation of \mathfrak{G} with $V_1 \otimes_k V_2$ as the underlying k -module given as the composition of the outer tensor product of (V_1, ρ_1) and (V_2, ρ_2) with the diagonal $\mathfrak{G} \hookrightarrow \mathfrak{G} \times \mathfrak{G}$.

Example 9.3 (Tensor operations). Let \mathfrak{G} be a monoid k -functor, let V be k -module and let $\rho : \mathfrak{G} \rightarrow \mathcal{L}_V$ be a morphism of monoid k -functors. Then both $\wedge^n V$ and $\text{Sym}^n(V)$ for $n \in \mathbb{N}$ carry canonical structure of linear representation of \mathfrak{G} .

Note that if V is a finitely generated, projective k -module, then there is a canonical isomorphism of A -modules $(V^\vee)_A(A) \cong (A \otimes_k V)^\vee$ natural in k -algebra A . Under these assumptions on V there exists an anti-isomorphism of A -algebras

$$\text{Hom}_A(A \otimes_k V, A \otimes_k V) \ni \phi \mapsto \phi^\vee \in \text{Hom}_A((A \otimes_k V)^\vee, (A \otimes_k V)^\vee)$$

natural in k -algebra A . This proves the following result.

Fact 9.4. Let V be a finitely generated, projective k -module. Then we have an identification of k -functors of \mathfrak{D}_k -algebras

$$\mathcal{L}_V^{\text{op}} = \mathcal{L}_{V^\vee}$$

Example 9.5 (Dual representation). Let \mathfrak{G} be a monoid k -functor, let V be k -module and let $\rho : \mathfrak{G} \rightarrow \mathcal{L}_V$ be a morphism of monoid k -functors. Suppose that V is a projective and finitely generated k -module. Fact 9.4 implies that morphism of a monoid k -functors $\rho^{\text{op}} : \mathfrak{G}^{\text{op}} \rightarrow \mathcal{L}_V^{\text{op}}$ can be identified with $\rho^\vee : \mathfrak{G}^{\text{op}} \rightarrow \mathcal{L}_{V^\vee}$. Hence a pair (V^\vee, ρ^\vee) is a linear representation of \mathfrak{G}^{op} .

Example 9.6 (Hom representation). Let (V_1, ρ_1) and (V_2, ρ_2) are linear representations of monoid k -functor \mathfrak{G} . Suppose that V_1, V_2 are finitely generated, projective k -module. Then we have an identification

$$\text{Hom}_k(V_1, V_2)_a = (V_1^\vee \otimes_k V_2)_a$$

of \mathfrak{D}_k -modules. By Examples 9.1 and 9.5 this isomorphism makes $\text{Hom}_k(V_1, V_2)$ into linear representation of $\mathfrak{G} \times \mathfrak{G}^{\text{op}}$.

10. EXAMPLE OF \mathfrak{G} -ACTION: REGULAR FUNCTIONS k -FUNCTOR

First we need the following notion.

Definition 10.1. Let $(-)^{\text{op}} : \mathbf{Mon} \rightarrow \mathbf{Mon}$ be the opposite monoid functor and let \mathfrak{G} be a monoid k -functor. Then the composition $\mathfrak{G}^{\text{op}} = (-)^{\text{op}} \cdot \mathfrak{G}$ is called *the opposite monoid k -functor of \mathfrak{G}* .

Let \mathfrak{G} be a monoid k -functor. In this section we discuss important example of a $\mathfrak{D}_k[\mathfrak{G}]$ -module. Fix a k -functor \mathfrak{X} for which $\text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$ exists. Recall that by Example 3.5 $\text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$ is \mathfrak{D}_k -algebra k -functor. Let $\alpha : \mathfrak{G} \times \mathfrak{X} \rightarrow \mathfrak{X}$ be an action of \mathfrak{G} on \mathfrak{X} . For every k -algebra A we have a map of sets

$$\text{Mor}_A(\mathfrak{X}_A, (\mathfrak{D}_k)_A) \ni f \mapsto f \cdot \alpha_g \in \text{Mor}_A(\mathfrak{X}_A, (\mathfrak{D}_k)_A)$$

where $g \in \mathfrak{G}(A)$. From this description it follows that the map $f \mapsto f \cdot \alpha_g$ is a morphism of A -algebras. Moreover, note that if $g_1, g_2 \in \mathfrak{G}(A)$, then $(f \cdot \alpha_{g_1}) \cdot \alpha_{g_2} = f \cdot \alpha_{g_1 \cdot g_2}$, where $g_1 \cdot g_2 \in \mathfrak{G}(A)$ is a product of g_1 and g_2 . Thus the opposite monoid $\mathfrak{G}^{\text{op}}(A)$ acts on the A -algebra $\text{Mor}_A(\mathfrak{X}_A, (\mathfrak{D}_k)_A)$ by morphism of A -algebras. Next for every A -algebra B and every point $x \in \mathfrak{X}(B)$ we have

$$(f \cdot \alpha_g)(x) = f(\alpha_g(x))$$

where $g \in \mathfrak{G}(A)$. This proves the following result.

Proposition 10.2. Let \mathfrak{X} be a k -functor and let $\alpha : \mathfrak{G} \times \mathfrak{X} \rightarrow \mathfrak{X}$ be an action of a monoid k -functor \mathfrak{G} . Suppose that $\text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$ exists. Then \mathfrak{G}^{op} acts canonically on \mathfrak{D}_k -algebra k -functor $\text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$ by morphisms of \mathfrak{D}_k -algebras.

Let us note one important consequence of this result.

Corollary 10.3. Let \mathfrak{G} be a monoid k -functor. The action of $\mathfrak{G} \times \mathfrak{G}^{\text{op}}$ on \mathfrak{G} induces the action of $\mathfrak{G}^{\text{op}} \times \mathfrak{G}$ on \mathfrak{D}_k -algebra k -functor $\text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$ by morphisms of \mathfrak{D}_k -algebras.

11. MATRIX COEFFICIENTS OF A REPRESENTATION

Proposition 11.1. Let \mathfrak{G} be a monoid k -functor and let V be a finitely generated, projective k -module. Fix a morphism $\rho : \mathfrak{G} \rightarrow \mathcal{L}_V$ of monoid k -functors. Fix k -algebra A and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^\vee$. For every A -algebra B and $g \in \mathfrak{G}(B)$ we consider the formula

$$c_{v,w}(g) = \langle \rho_A(g) \cdot v_B, w_B \rangle$$

Then $c_{v,w}$ defines a regular function on \mathfrak{G}_A for every k -algebra A .

Proof. Suppose that $f : B \rightarrow C$ is a morphism of A -algebras and pick $g \in \mathfrak{G}(B)$. Since ρ_A is natural and $w : A \otimes_k V \rightarrow A$ is a morphism of A -modules, we derive that the diagram

$$\begin{array}{ccccc}
B \otimes_k V & \xrightarrow{\rho_A(g)} & B \otimes_k V & \xrightarrow{w_B} & B \\
\downarrow f \otimes_A 1_{A \otimes_k V} & & \downarrow f \otimes_A 1_{A \otimes_k V} & & \downarrow f \\
C \otimes_k V & \xrightarrow{\rho_A(\mathfrak{G}_A(f)(g))} & C \otimes_k V & \xrightarrow{w_C} & C
\end{array}$$

is commutative. Hence

$$c_{v,w}(\mathfrak{G}_A(f)(g)) = \langle \rho_A(\mathfrak{G}_A(f)(g)) \cdot v_C, w_C \rangle = f(\langle \rho_A(g) \cdot v_B, w_B \rangle) = f(c_{v,w}(g))$$

and this implies that $c_{v,w} : \mathfrak{G}_A \rightarrow \mathfrak{D}_A$ is a morphism of A -functors. \square

Definition 11.2. Let \mathfrak{G} be a monoid k -functor and let (V, ρ) be its representation with finitely generated, projective underlying k -module V . Fix k -algebra A and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^\vee$. Then the regular function $c_{v,w}$ on \mathfrak{G}_A is called *the matrix coefficient of v and w* .

Proposition 11.3. Let \mathfrak{G} be a monoid k -functor and let (V, ρ) be its representation with finitely generated projective underlying k -module V . Then the following assertions holds.

(1) For every k -algebra A map

$$(A \otimes_k V) \times (A \otimes_k V^\vee) \ni (v, w) \mapsto c_{v,w} \in \text{Mor}_A(\mathfrak{G}_A, \mathfrak{D}_A)$$

is A -bilinear.

(2) Suppose that $\text{Mor}_k(\mathfrak{G}, \mathfrak{D}_k)$ exists. Then the collection of maps

$$\left\{ (A \otimes_k V) \times (A \otimes_k V^\vee) \ni (v, w) \mapsto c_{v,w} \in \text{Mor}_A(\mathfrak{G}_A, \mathfrak{D}_A) \right\}_{A \in \mathbf{Alg}_k}$$

gives rise to a morphism of k -functors

$$V_a \times V_a^\vee \longrightarrow \text{Mor}_k(\mathfrak{G}, \mathfrak{D}_k)$$

Proof. We left the proof of (1) to the reader.

We prove (2). Consider k -algebra A and an A -algebra B with structural morphism $f : A \rightarrow B$. Fix $v \in A \otimes_k V$, $w \in A \otimes_k V^\vee$. We prove that restriction of $c_{v,w} : \mathfrak{G}_A \rightarrow \mathfrak{D}_A$ to the category \mathbf{Alg}_B is c_{v_B, w_B} . For this pick a B -algebra C and an element $g \in \mathfrak{G}(C)$. Note that

$$c_{v,w}(g) = \langle \rho_A(g) \cdot v_C, w_C \rangle = \langle \rho_B(g) \cdot v_C, w_C \rangle = \langle \rho_B(g) \cdot (v_B)_C, (w_B)_C \rangle = c_{v_B, w_B}(g)$$

and hence $c_{v,w}|_{\mathbf{Alg}_B} = c_{v_B, w_B}$. Consider the square

$$\begin{array}{ccc}
V_a(A) \times V_a^\vee(A) & \longrightarrow & \text{Mor}_k(\mathfrak{G}, \mathfrak{D}_A)(A) \\
\downarrow V_a(f) \times V_a^\vee(f) & & \downarrow \text{Mor}_k(\mathfrak{G}, \mathfrak{D}_k)(f) \\
V_a(B) \times V_a^\vee(B) & \longrightarrow & \text{Mor}_k(\mathfrak{G}, \mathfrak{D}_B)(B)
\end{array}$$

in which both horizontal arrows are given by formula $(v, w) \mapsto c_{v,w}$. We proved that the square commutes. Since f is an arbitrary morphism of k -algebras, we conclude the assertion. \square

Corollary 11.4. Let \mathfrak{G} be a monoid k -functor and let (V, ρ) be its representation with finitely generated projective underlying k -module V . Suppose that $\text{Mor}_k(\mathfrak{G}, \mathfrak{D}_k)$ exists. Then there exists a morphism of k -functors

$$(V \otimes_k V^\vee)_a \xrightarrow{c} \text{Mor}_k(\mathfrak{G}, \mathfrak{D}_k)$$

given by formula

$$(A \otimes_k V) \otimes_A (A \otimes_k V^\vee) \ni (v, w) \mapsto c_{v,w} \in \text{Mor}_A(\mathfrak{G}_A, \mathfrak{D}_A)$$

Moreover, c is a morphism of k -functors equipped with $\mathfrak{G} \times \mathfrak{G}^{\text{op}}$ -actions.

Proof. The first part is an immediate consequence of Proposition 11.3. We prove that c is a morphism of k -functors equipped with $\mathfrak{G} \times \mathfrak{G}^{\text{op}}$ -actions. For this we fix a k -algebra k and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^\vee$. Pick a morphism of k -algebras $f : A \rightarrow B$, $(g_1, g_2) \in \mathfrak{G}(A) \times \mathfrak{G}(A)^{\text{op}}$ and $g \in \mathfrak{G}(B)$. Then we have

$$\begin{aligned} c_{\rho(g_1) \cdot v, w \cdot \rho(g_2)}(g) &= \langle \rho_A(g) \cdot (\rho(g_1) \cdot v)_B, (w \cdot \rho(g_2))_B \rangle = \\ &= \langle \rho_A(g) \cdot \rho_A((\mathfrak{G}_A(f)(g_1))) \cdot v_B, w_B \cdot \rho_A(\mathfrak{G}_A(f)(g_2)) \rangle = w_B(\rho_A(\mathfrak{G}_A(f)(g_2)) \cdot \rho_A(g) \cdot \rho_A(\mathfrak{G}_A(f)(g_1)) \cdot v_B) = \\ &= w_B(\rho_A(\mathfrak{G}_A(f)(g_2)) \cdot g \cdot \mathfrak{G}_A(f)(g_1)) \cdot v_B = \langle \rho_A(\mathfrak{G}_A(f)(g_2)) \cdot g \cdot \mathfrak{G}_A(f)(g_1) \cdot v_B, w_B \rangle = \\ &= c_{v,w}(\mathfrak{G}_A(f)(g_2) \cdot g \cdot \mathfrak{G}_A(f)(g_1)) \end{aligned}$$

and hence c is a morphism of k -functors equipped with actions of $\mathfrak{G} \times \mathfrak{G}^{\text{op}}$. \square

12. MONOID k -SCHEMES

Definition 12.1. A monoid k -scheme \mathbf{M} is a monoid object in the category of k -schemes. If \mathbf{M} is affine, then we say that \mathbf{M} is an affine monoid k -scheme.

Definition 12.2. A group k -scheme \mathbf{G} is a group object in the category of k -schemes. If \mathbf{G} is affine, then we say that \mathbf{G} is an affine group k -scheme.

Corollary 12.3. The functor

$$\text{Sch}_k \xrightarrow{\mathfrak{P}} \text{the category of } k\text{-functors}$$

induces an equivalence of categories

the category of monoid k -schemes \cong monoid k -functors representable by k -schemes

Similarly for categories of groups.

Proof. Follows from [Mon19b, Fact 4.1]. \square

Recall that by Example 2.3 each monoid k -functor \mathfrak{G} has its group k -functor \mathfrak{G}^* of units.

Proposition 12.4. Let \mathbf{M} be an affine monoid k -scheme. Then the k -functor of units $\mathfrak{P}_{\mathbf{M}}^*$ is representable (by an affine k -scheme).

Proof. Note that $\mathfrak{P}_{\mathbf{M}}^*$ fits into a cartesian square

$$\begin{array}{ccc} \mathfrak{P}_{\mathbf{M}}^* & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \mathfrak{P}_e \\ \mathfrak{P}_{\mathbf{M}} \times \mathfrak{P}_{\mathbf{M}} & \xrightarrow{\mathfrak{P}_m} & \mathfrak{P}_{\mathbf{M}} \end{array}$$

where $m : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{M}$ is the multiplication and $e : \text{Spec } k \rightarrow \mathbf{M}$ is the unit. By [Mon19b, Fact 4.1] the functor \mathfrak{P} preserves finite products and hence it preserves fiber-products. This implies that $\mathfrak{P}_{\mathbf{M}}^*$ is represented by a unique (up to an isomorphism) k -scheme \mathbf{M}^* that fit into a cartesian square below.

$$\begin{array}{ccc}
\mathbf{M}^* & \longrightarrow & \operatorname{Spec} k \\
\downarrow & & \downarrow e \\
\mathbf{M} \times \mathbf{M} & \xrightarrow{m} & \mathbf{M}
\end{array}$$

Note that if \mathbf{M} is affine, then also \mathbf{M}^* is affine. \square

Definition 12.5. Let \mathbf{M} be a monoid k -scheme. Then the group k -scheme \mathbf{M}^* representing $\mathfrak{P}_{\mathbf{M}}^*$ is called *the group of units of \mathbf{M}* .

Remark 12.6. Under the embedding given in Corollary 12.3 notions defined for monoid k -functors can be translated to monoid k -schemes.

We give two instances of the use of Remark 12.6 below.

Definition 12.7. Let \mathbf{M} be a monoid k -scheme. Then *the category of linear representations of \mathbf{M}* is the category of linear representations of the monoid k -functor $\mathfrak{P}_{\mathbf{M}}$. We denote this category by $\operatorname{Rep}(\mathbf{M})$.

Definition 12.8. Let \mathbf{M} be a monoid k -functor and let $\alpha : \mathfrak{P}_{\mathbf{M}} \times \mathfrak{X} \rightarrow \mathfrak{X}$ be an action of $\mathfrak{P}_{\mathbf{M}}$ on a k -functor \mathfrak{X} . Then we say that α is *an action of \mathbf{M} on \mathfrak{X}* .

13. BIALGEBRAS AND AFFINE MONOID k -SCHEMES

We start here with a general notion of k -coalgebras.

Definition 13.1. Let (C, Δ, ζ) be a triple consisting of a module C over k and morphisms

$$\Delta : C \rightarrow C \otimes_k C, \zeta : C \rightarrow k$$

of k -modules such that the following diagrams are commutative.

$$\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes_k C \\
\Delta \downarrow & & \downarrow 1_C \otimes \Delta \\
C \otimes_k C & \xrightarrow{\Delta \otimes_k 1_C} & C \otimes_k C \otimes_k C
\end{array}
\quad
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes_k C \\
\searrow \cong & & \downarrow 1 \otimes \zeta \\
& & C \otimes_k k
\end{array}
\quad
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes_k C \\
\searrow \cong & & \downarrow \zeta \otimes_k 1_C \\
& & k \otimes_k C
\end{array}$$

Then (C, Δ, ζ) is called a k -coalgebra. Morphisms Δ, ζ are called a *comultiplication* and a *counit*, respectively.

Definition 13.2. Let (C_1, Δ_1, ζ_1) and (C_2, Δ_2, ζ_2) are k -coalgebras. Then a morphism $f : C_1 \rightarrow C_2$ of k -modules is a *morphism of k -coalgebras* if the following diagrams are commutative.

$$\begin{array}{ccc}
C_1 \otimes_k C_1 & \xrightarrow{f \otimes_k f} & C_2 \otimes_k C_2 \\
\Delta_1 \uparrow & & \uparrow \Delta_2 \\
C_1 & \xrightarrow{f} & C_2
\end{array}
\quad
\begin{array}{ccc}
C_1 & \xrightarrow{f} & C_2 \\
\searrow \zeta_1 & & \swarrow \zeta_2 \\
& k &
\end{array}$$

By k -algebra we mean commutative and unital k -algebra.

Definition 13.3. Let B be a k -module with structures of both k -algebra and k -coalgebra. Assume that the comultiplication and the counit of B are morphisms of k -algebras with respect to k -algebra structure of B . Then we say that B with these structures is a *k -bialgebra*.

Definition 13.4. Let B_1, B_2 be k -bialgebras and let $f : B_1 \rightarrow B_2$ be a morphism of k -modules. We say that f is a *morphism of k -bialgebras* if it is simultaneously morphism of k -algebras and k -coalgebras.

Theorem 13.5. The functor $\text{Spec} : \mathbf{Alg}_k \rightarrow \mathbf{Sch}_k$ induces an equivalence of categories
 k -bialgebras \cong the category of affine monoid k -schemes

Proof. This is an exercise in translation. For details see [DG70, II, 1.6]. \square

Let \mathbf{M} be an affine monoid k -scheme. Then we denote by $k[\mathbf{M}]$ its coordinate k -bialgebra, by $\Delta_{\mathbf{M}}$ its comultiplication and by $\zeta_{\mathbf{M}}$ its counit. This is a notation that we consistently use in these notes.

14. COMODULES OVER k -COALGEBRAS

Definition 14.1. Let C be a k -coalgebra with the comultiplication Δ and the counit ζ . A pair (V, d) consisting of a k -module V and a morphism $d : V \rightarrow C \otimes_k V$ of k -modules such that the following diagrams are commutative

$$\begin{array}{ccc} V & \xrightarrow{d} & C \otimes_k V \\ d \downarrow & & \downarrow 1_C \otimes_k d \\ C \otimes_k V & \xrightarrow{\Delta \otimes_k 1_V} & C \otimes_k C \otimes_k V \end{array} \quad \begin{array}{ccc} V & \xrightarrow{d} & C \otimes_k V \\ & \searrow \cong & \downarrow \zeta \otimes_k 1_V \\ & & k \otimes_k V \end{array}$$

is called a *C -comodule*. Morphism d is called a *coaction of C on V* .

Definition 14.2. Let C be a k -coalgebra and let $(V_1, d_1), (V_2, d_2)$ be two comodules over C . A morphism of k -modules $f : V_1 \rightarrow V_2$ is a *morphism of C -comodules* if the diagram

$$\begin{array}{ccc} C \otimes_k V_1 & \xrightarrow{1_C \otimes_k f} & C \otimes_k V_2 \\ d_1 \uparrow & & \uparrow d_2 \\ V_1 & \xrightarrow{f} & V_2 \end{array}$$

is commutative.

We denote by $\mathbf{coMod}(C)$ the category of C -comodules for a k -coalgebra C .

Theorem 14.3. Let C be a k -coalgebra. Then the forgetful functor $\mathbf{coMod}(C) \rightarrow \mathbf{Mod}(k)$ creates colimits.

Proof. Let Δ, ζ be the comultiplication and the counit of C , respectively. Suppose that $I \ni i \mapsto (V_i, d_i) \in \mathbf{coMod}(C)$ is a diagram of C -comodules indexed by some category I . Let V together with $u_i : V_i \rightarrow V$ for $i \in I$ be a colimit of the diagram $I \ni i \mapsto V_i \in \mathbf{Mod}(k)$. By the universal property of colimit we deduce that there exists a unique morphism $d : V \rightarrow C \otimes_k V$ such that diagrams

$$\begin{array}{ccc} C \otimes_k V_i & \xrightarrow{1_C \otimes_k u_i} & C \otimes_k V \\ d_i \uparrow & & \uparrow d \\ V_i & \xrightarrow{u_i} & V \end{array}$$

are commutative for every $i \in I$. In order to verify that diagrams

$$\begin{array}{ccc} V & \xrightarrow{d} & C \otimes_k V \\ d \downarrow & & \downarrow 1_C \otimes_k d \\ C \otimes_k V & \xrightarrow{\Delta \otimes_k 1_V} & C \otimes_k C \otimes_k V \end{array} \quad \begin{array}{ccc} V & \xrightarrow{d} & C \otimes_k V \\ \cong \searrow & & \downarrow \xi \otimes_k 1_V \\ & & k \otimes_k V \end{array}$$

are commutative it suffices to note that for every $i \in I$ we have chains of equalities

$$(1_C \otimes_k d) \cdot d \cdot u_i = (1_C \otimes_k 1_C \otimes_k u_i) \cdot (1_C \otimes_k 1_C \otimes_k d_i) \cdot d_i = (1_C \otimes_k 1_C \otimes_k u_i) \cdot (\Delta \otimes_k 1_{V_i}) \cdot d_i = (\Delta \otimes_k 1_V) \cdot d \cdot u_i$$

and

$$(\xi \otimes_k 1_V) \cdot d \cdot u_i = (1_k \otimes_k u_i) \cdot (\xi \otimes_k 1_{V_i}) \cdot d_i = (1_k \otimes_k u_i) \cdot j_{V_i} = j_V \cdot u_i$$

where $j_W : W \rightarrow k \otimes_k W$ is the natural isomorphism for every k -module W . Hence (V, d) is a C -comodule. Suppose now that (W, e) is a C -comodule and $w_i : V_i \rightarrow W$ for $i \in I$ is a family of C -comodule morphisms compatible with the diagram $I \ni i \mapsto (V_i, d_i) \in \mathbf{coMod}(C)$. Since $\{u_i : V_i \rightarrow V\}_{i \in I}$ form a colimiting cocone for $I \ni i \mapsto V_i \in \mathbf{Mod}(k)$, there exists a unique morphism of k -modules $f : V \rightarrow W$ such that $f \cdot u_i = w_i$. Note that

$$e \cdot f \cdot u_i = e \cdot w_i = (1_C \otimes_k w_i) \cdot d_i = (1_C \otimes_k f) \cdot (1_C \otimes_k u_i) \cdot d_i = (1_C \otimes_k f) \cdot d \cdot u_i$$

for every $i \in I$. Hence $e \cdot f = (1_C \otimes_k f) \cdot d$. Thus f is a morphism of C -comodules. Thus (V, d) together with family $\{u_i : (V_i, d_i) \rightarrow (V, d)\}_{i \in I}$ is a colimit of the diagram $I \ni i \mapsto (V_i, d_i) \in \mathbf{coMod}(C)$ of C -comodules. This implies that the forgetful functor $\mathbf{coMod}(C) \rightarrow \mathbf{Mod}(k)$ creates colimits. \square

Theorem 14.4. *Let C be a k -coalgebra such that C is a flat k -module. Then the forgetful functor $\mathbf{coMod}(C) \rightarrow \mathbf{Mod}(k)$ creates finite limits.*

Proof. The proof is similar to the proof of Theorem 14.3. \square

Corollary 14.5. *Let C be a coalgebra over k and assume that C is flat as a k -module. Then $\mathbf{coMod}(C)$ is an abelian category with small colimits.*

Proof. This follows from Theorems 14.3 and 14.4. \square

The next result is of fundamental importance.

Theorem 14.6. *Let C be a k -coalgebra that is free as a k -module. Suppose that V is a C -comodule over C . Then for every finitely generated k -submodule $U \subseteq V$ there exists a C -subcomodule W of V such that $U \subseteq W$ and W is a finitely generated k -module.*

The theorem follows from the following simple lemma.

Lemma 14.6.1. *Let C be a k -coalgebra over k that is free as a k -module. Suppose that V is a C -comodule over C and fix an element $v \in V$. Then there exists a C -subcomodule W of V such that $v \in W$ and W is a finitely generated k -module.*

Proof of the lemma. Let $\{e_j\}_{j \in J}$ be a free basis of C over k and let $d : V \rightarrow C \otimes_k V$ be a left coaction of C on V . Denote by $\Delta : C \rightarrow C \otimes_k C$ the comultiplication of C . Then we have

$$d(v) = \sum_{j \in J} e_j \otimes v_j$$

where $v_j \in V$ are zero for almost all $j \in J$. Next according to

$$(\Delta \otimes_k 1_V) \cdot d = (1_C \otimes_k d) \cdot d$$

we derive that equality

$$\sum_{j \in J} e_j \otimes d(v_j) = (1_C \otimes_k d)(d(v)) = (\Delta \otimes_k 1_V)(d(v)) = \sum_{j \in J} \Delta(e_j) \otimes v_j \subseteq \sum_{j \in J} C \otimes_k C \otimes_k k \cdot v_j$$

holds. This implies that $d(v_j) \subseteq C \otimes_k (\sum_{j \in J} k \cdot v_j)$. Hence k -submodule W of V generated by v and $\{v_j\}_{j \in J}$ is C -subcomodule of V . It is finitely generated as a k -module and $v \in W$. \square

Proof of the theorem. Suppose that U is generated by $\{v_1, \dots, v_n\}$ as a k -module. For each i pick C -subcomodule W_i of V such that W_i is finitely generated as a k -module and $v_i \in W_i$. This can be done by Lemma 14.6.1. Next

$$W = W_1 + \dots + W_n$$

is a C -subcomodule of V that is finitely generated as a k -module and contains U . \square

15. LINEAR REPRESENTATIONS AND COMODULES

Let \mathbf{M} be an affine monoid k -scheme and let $\rho : \mathfrak{P}_{\mathbf{M}} \rightarrow \mathcal{L}_V$ be a morphism of functors of sets, where V is a k -module. Yoneda Lemma implies that ρ is determined by some element (Example 8.4)

$$d_\rho \in \text{Hom}_k(V, k[\mathbf{M}] \otimes_k V)$$

Theorem 15.1. *Let \mathbf{M} be an affine monoid k -scheme. Then the correspondence*

$$\mathbf{Rep}(\mathbf{M}) \ni (V, \rho) \mapsto (V, d_\rho) \in \mathbf{coMod}(k[\mathbf{M}])$$

is an isomorphism of categories over $\mathbf{Mod}(k)$.

Proof. We fix notation in the proof. We denote by $\mu_A : A \otimes_k A \rightarrow A$ the multiplication and by $\eta_A : k \rightarrow A$ the unit for every k -algebra A . If A is a k -algebra, then we denote by e_A the composition $\eta_A \cdot \xi_{\mathbf{M}} : k[\mathbf{M}] \rightarrow A$. Note that $e_A \in \mathfrak{P}_{\mathbf{M}}(A)$ is the neutral element.

We start the proof with some useful remarks. If V is a k -module, then

$$\mathcal{L}_V(A) = \text{Hom}_k(V, A \otimes_k V)$$

for every k -algebra A with \mathfrak{D}_k -algebra structure discussed in Example 8.4. Moreover, if $\rho : \mathfrak{P}_{\mathbf{M}} \rightarrow \mathcal{L}_V$ is a morphism of k -functors corresponding to $d_\rho : V \rightarrow k[\mathbf{M}] \otimes_k V$, then for every k -algebra A and a morphism $f : k[\mathbf{M}] \rightarrow A$ of k -algebras we have

$$\rho(f) = (f \otimes_k 1_V) \cdot d_\rho$$

Our discussion in Example 8.4 and Yoneda Lemma show that the following assertions hold.

(1) For k -algebra A and $f_1, f_2 \in \text{Hom}_k(k[\mathbf{M}], A) = \mathfrak{P}_{\mathbf{M}}(A)$ we have

$$\rho(f_1) \cdot \rho(f_2) = (\mu_A \otimes_k 1_V) \cdot (f_2 \otimes_k f_1 \otimes_k 1_V) \cdot (1_{k[\mathbf{M}]} \otimes_k d_\rho) \cdot d_\rho$$

and

$$\rho(f_1 \cdot f_2) = (\mu_A \otimes_k 1_V) \cdot (f_2 \otimes_k f_1 \otimes_k 1_V) \cdot (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho$$

(2) For k -algebra A we have

$$\rho(e_A) = (\eta_A \otimes_k 1_V) \cdot (\xi_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho$$

Now (1) imply that if $(\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho = (1_{O_{\mathbf{M}}} \otimes_k d_\rho) \cdot d_\rho$ then $\rho(f_1 \cdot f_2) = \rho(f_1) \cdot \rho(f_2)$. On the other hand suppose that $\rho(f_1 \cdot f_2) = \rho(f_1) \cdot \rho(f_2)$ for any two $f_1, f_2 : k[\mathbf{M}] \rightarrow A$ morphism of k -algebras and for every k -algebra A . Pick inclusions $f_1, f_2 : k[\mathbf{M}] \rightarrow k[\mathbf{M}] \otimes_k k[\mathbf{M}]$ onto first and second component, respectively. Then

$$(\mu_{k[\mathbf{M}] \otimes_k k[\mathbf{M}]} \otimes_k 1_V) \cdot (f_2 \otimes_k f_1 \otimes_k 1_V) = 1_{k[\mathbf{M}]} \otimes_k 1_{k[\mathbf{M}]} \otimes_k 1_V$$

and hence $(\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho = (1_{O_{\mathbf{M}}} \otimes_k d_\rho) \cdot d_\rho$ by (1).

Now if $(\xi_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho$ is the canonical isomorphism $V \cong k \otimes_k V$. Then by (2) we derive that $\rho(e_A)$

is the canonical morphism $V \rightarrow A \otimes_k V$. On the other hand if $\rho(e_A)$ is $V \rightarrow A \otimes_k V$ for every k -algebra A , then substituting k for A we deduce by (2) that $\rho(e_k) = (\xi_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho$ is the canonical isomorphism $V \cong k \otimes_k V$.

These considerations prove that ρ is a morphism of monoid k -functors if and only if d_ρ is a coaction of $k[\mathbf{M}]$ on V .

Now suppose that V_1, V_2 are k -modules and $\rho_1 : \mathfrak{P}_{\mathbf{M}} \rightarrow \mathcal{L}_V, \rho_2 : \mathfrak{P}_{\mathbf{M}} \rightarrow \mathcal{L}_W$ are morphisms of k -functors. Suppose that $\phi : V_1 \rightarrow V_2$ is a morphism of k -modules. Pick a k -algebra A and a morphism $f : k[\mathbf{M}] \rightarrow A$ of k -algebras. Assume that the diagram

$$\begin{array}{ccc} k[\mathbf{M}] \otimes_k V_1 & \xrightarrow{1_{k[\mathbf{M}]} \otimes_k \phi} & k[\mathbf{M}] \otimes_k V_2 \\ d_{\rho_1} \uparrow & & \uparrow d_{\rho_2} \\ V_1 & \xrightarrow{\phi} & V_2 \end{array}$$

is commutative. Since the square

$$\begin{array}{ccc} A \otimes_k V_1 & \xrightarrow{1_A \otimes_k \phi} & A \otimes_k V_2 \\ f \otimes_k 1_V \uparrow & & \uparrow f \otimes_k 1_W \\ k[\mathbf{M}] \otimes_k V_1 & \xrightarrow{1_{k[\mathbf{M}]} \otimes_k \phi} & k[\mathbf{M}] \otimes_k V_2 \end{array}$$

is commutative, we derive that

$$\begin{array}{ccc} A \otimes_k V_1 & \xrightarrow{1_A \otimes_k \phi} & A \otimes_k V_2 \\ \rho_1(f) \uparrow & & \uparrow \rho_2(f) \\ V_1 & \xrightarrow{\phi} & V_2 \end{array}$$

Moreover, if the square above commutes for every k -algebra A , then it also commutes for $A = k[\mathbf{M}]$ and this recovers the commutativity of the first square. Suppose now that (V, ρ_1) and (W, ρ_2) are linear representations of \mathbf{M} , then the discussion above implies that ϕ is a morphism of linear representations if and only if ϕ is a morphism of $k[\mathbf{M}]$ -comodules (V, d_{ρ_1}) and (W, d_{ρ_2}) . \square

We obtain immediate consequence.

Corollary 15.2. *Let k be a field. Let (V, ρ) be a linear representation of an affine monoid k -scheme \mathbf{M} . Then for every finitely generated k -subspace $U \subseteq V$ there exists a subrepresentation W of (V, ρ) such that $U \subseteq W$ and W is a finitely generated k -module.*

Proof. This follows from Theorems 15.1 and 14.6. \square

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