ALGEBRAIC GROUP SCHEMES OVER FIELD

1. Introduction

In these notes we group schemes over fields. For background we refer to [Mon19] and [Mon20]. Throughout these notes k is a fixed field.

Definition 1.1. Let $\mathbf{G}$ be a group scheme over k. If $\mathbf{G}$ is of finite type over k, then we say that $\mathbf{G}$ is *an algebraic group over* k.

2. Simple criterion for separatedness

Proposition 2.1. *Let* $\mathbf{G}$ *be a group scheme over* k *and let* $e_{\mathbf{G}} : \mathrm{Spec}\, k \rightarrow \mathbf{G}$ *be its unit. Then the following are equivalent.*

- **(i)** $e_{\mathbf{G}}$ *is a closed immersion.*
- **(ii)** $\mathbf{G}$ *is separated.*

Proof. Suppose that **(i)** holds. Consider morphism $f : \mathbf{G} \times_k \mathbf{G} \rightarrow \mathbf{G}$ given on A-points by formula

$$f(g_1, g_2) = g_1 \cdot g_2^{-1}$$

where A is a k-algebra. Note that we have a cartesian square

$$\begin{tikzcd}
\mathbf{G} \arrow[r, "\mathrm{can.}"] \arrow[d, "\delta_{\mathbf{G}}"'] & \mathrm{Spec}\, k \arrow[d, "e_{\mathbf{G}}"] \\
\mathbf{G} \times_k \mathbf{G} \arrow[r, "f"'] & \mathbf{G}
\end{tikzcd}$$

where $\delta_{\mathbf{G}}$ is a diagonal of $\mathbf{G}$. Since base change of a closed immersion is a closed immersion, we derive that $\delta_{\mathbf{G}}$ is a closed immersion and hence $\mathbf{G}$ is separated. This is **(ii)**.
Suppose now that **(ii)** holds. Let $\pi : \mathbf{G} \rightarrow \mathrm{Spec}\, k$ be the structural morphism. Then $\pi \cdot e_{\mathbf{G}} = 1_{\mathbf{G}}$. Since π is a separated morphism, we derive that (by cancellation) $e_{\mathbf{G}}$ is closed immersion. This is **(i)**. $\square$

3. Complete group schemes

We start this section with the following general result.

Theorem 3.1 (Rigidity)**.** *Let* $\pi : X \rightarrow Y$ *be a proper morphism of schemes such that* $\pi^{\#} : \mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_X$ *is an isomorphism of sheaves. Let* $g : X \rightarrow Z$ *be a morphism of schemes. Suppose that for some point* y *in* Y *there is a point* z *of* Z *such that* $\pi^{-1}(y) \subseteq g^{-1}(z)$*. Then there exist an affine neighborhood* V *of* y *and an affine neighborhood* W *of* z *such that* $\pi^{-1}(V) \subseteq g^{-1}(W)$*. Moreover, there exists a morphism* $h : V \rightarrow W$ *making the diagram*

$$\begin{tikzcd}
\pi^{-1}(V) \arrow[r, "\text{res. of } g"] \arrow[d, "\text{restriction of } \pi"'] & W \\
V \arrow[ur, "h"'] &
\end{tikzcd}$$

commutative, where horizontal arrow is the restriction of g.

Proof. Consider an affine open neighborhood of W of z. Since π is proper and $\pi^{-1}(y)) = g^{-1}(z)$, we derive that $\pi\big(X \smallsetminus g^{-1}(W)\big)$ is a closed subset of Y that does not contain y. Pick an open affine neighborhood V of y in Y that does not intersect with $\pi\big(X \smallsetminus g^{-1}(W)\big)$. Then $\pi^{-1}(V) \subseteq g^{-1}(W)$. Since $\pi^{\#}$ is an isomorphism we have the composition

$$\mathcal{O}_Z(W) \xrightarrow{g^{\#}_W} \Gamma\big(g^{-1}(W), \mathcal{O}_X\big) \xrightarrow{(-)_{|\pi^{-1}(V)}} \Gamma\big(\pi^{-1}(V), \mathcal{O}_X\big) \xrightarrow{(\pi^{\#}_V)^{-1}} \mathcal{O}_Y(V)$$

This composition induces a morphism of affine schemes $h : V \to W$. Since a morphism from a scheme to an affine scheme is determined by the morphism on global sections of structure sheaves, we derive that h makes the triangle in the statement commutative. □

Now we can apply this result to study complete algebraic groups over k. For this we need the following definition.

Definition 3.2. Let $\mathbf{A}$ be a geometrically integral, complete algebraic group over k. Then we say that $\mathbf{A}$ is *an abelian variety over k.*

Now we prove the following interesting result.

Theorem 3.3. *Let $\mathbf{A}$ be an abelian variety over k, let $\mathbf{G}$ be a separated group scheme over k and let $f : \mathbf{A} \to \mathbf{G}$ be a morphism of schemes over k. Suppose that the diagram*

$$\begin{tikzcd}
\mathbf{A} \arrow[rr, "f"] & & \mathbf{G} \\
& \operatorname{Spec} k \arrow[ul, "e_{\mathbf{A}}"] \arrow[ur, "e_{\mathbf{G}}"'] &
\end{tikzcd}$$

is commutative. Then f is a morphism of groups schemes over k.

Proof. We define a morphism $g : \mathbf{A} \times_k \mathbf{A} \to \mathbf{G}$ given by

$$(x_1, x_2) \mapsto f(x_1) \cdot f(x_2) \cdot f(x_1 \cdot x_2)^{-1}$$

where A is a k-algebra and x_1, x_2 are A-points of $\mathbf{A}$. It suffices to show that g factors through $\operatorname{Spec} k(e_{\mathbf{G}})$. For this we may change base to an algebraic closure of k by faitfully flat descent. So we may assume that the field k is algebraically closed and $\mathbf{A}$ is connected. Then the projection onto second factor $\pi : \mathbf{A} \times_k \mathbf{A} \to \mathbf{A}$ is proper and $k = \Gamma\big(\mathbf{A}, \mathcal{O}_{\mathbf{A}}\big)$ implies that $\pi^{\#}$ is an isomorphism of sheaves on $\mathbf{A}$. Moreover, note that $\pi^{-1}(e_{\mathbf{A}}) \subseteq g^{-1}(e_{\mathbf{G}})$. Indeed, this follows from the assumption that $f(e_{\mathbf{A}}) = e_{\mathbf{G}}$. By Theorem 3.1 we deduce that there exist an affine neighborhood V of $e_{\mathbf{A}}$, an affine neighborhood W of $e_{\mathbf{G}}$ and a morphism $h : \operatorname{Spec} k \to W$ such that $\pi^{-1}(V) \subseteq g^{-1}(W)$ and the diagram

$$\begin{tikzcd} \mathbf{A} \times_k V \arrow[r, "\text{res. of } g"] \arrow[d, "\text{projection}"'] & W \\ V \arrow[ur, "h"'] & \end{tikzcd}$$

is commutative. Hence for every k-point v of V we have the restiction $g_{|\mathbf{A}\times_k \mathrm{Spec}\, k(v)}$ factors through $\mathrm{Spec}\, k\,(h(v))$. Since $g(v, e_{\mathbf{A}}) = e_{\mathbf{G}}$, we derive that $h(v) = e_{\mathbf{G}}$ and thus $g_{|\mathbf{A}\times_k \mathrm{Spec}\, k(v)}$ factors through $\mathrm{Spec}\, k(e_{\mathbf{G}})$. This holds for any k-point of V. Therefore, $g_{|\mathbf{A}\times_k V}$ factors through $\mathrm{Spec}\, k(e_{\mathbf{G}})$. Consider the kernel $i : Z \hookrightarrow \mathbf{A} \times_k \mathbf{A}$ of a pair consisting of g and a morphism $\mathbf{A} \times_k \mathbf{A} \to \mathbf{G}$ that factorizes through $\mathrm{Spec}\, k(e_{\mathbf{G}})$. Since $\mathbf{G}$ is separated, we derive that i is a closed immersion. Moreover, i dominates $\mathbf{A} \times_k V$. Since $\mathbf{A} \times_k V$ is schematically dense open subset of $\mathbf{A} \times_k \mathbf{A}$ (because $\mathbf{A} \times_k \mathbf{A}$ is integral), we derive that i is an isomorphism and hence g factors through $\mathrm{Spec}\, k(e_{\mathbf{G}})$. □

Corollary 3.4. *Let $\mathbf{A}$ be an abelian variety over k. Then $\mathbf{A}$ is a commutative group scheme over k.*

Proof. Consider the morphism $(-)^{-1} : \mathbf{A} \to \mathbf{A}$. By Theorem 3.3 we derive $(-)^{-1}$ is a morphism of group schemes over k. Hence $\mathbf{A}$ is a commutative group scheme. □

4. Transporters

Definition 4.1. Let $\mathfrak{G}$ be a monoid k-functor and let $\alpha : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X}$ be an action of $\mathfrak{G}$ on a k-functor $\mathfrak{X}$. Suppose that $\mathfrak{Y}_1, \mathfrak{Y}_2$ are k-subfunctors of $\mathfrak{X}$. For every k-algebra A we define

$$\mathrm{Transp}_{\mathfrak{G}}\,(\mathfrak{Y}_1, \mathfrak{Y}_2)\,(A) = \left\{g \in \mathfrak{G}(A) \,\middle|\, \alpha_g\,(\mathfrak{Y}_1(A)) \subseteq \mathfrak{Y}_2(A)\right\}$$

where as usual α_g is a slice of α along g. Then $\mathrm{Transp}_{\mathfrak{G}}\,(\mathfrak{Y}_1, \mathfrak{Y}_2)$ is a k-subfunctor of $\mathfrak{G}$. It is called *the tranporter of $\mathfrak{Y}_1$ into $\mathfrak{Y}_2$ with respect to α.*

References

[Mon19] Monygham. Geometry of k-functors. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2019.
[Mon20] Monygham. Monoid k-functors and their representations. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2020.