BIAŁYNICKI-BIRULA FUNCTORS

1. Introduction

In this notes we study Białynicki-Birula functors. In the first section we prove some results concerning the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$, where $\mathbf{M}$ is an affine monoid k-scheme and $\mathbf{G}$ is its group of units (we assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$). These results will be used in following sections.
We assume that k is a field.

2. Relations between representations of a monoid and its group of units

In this section we study the relation between the category $\mathbf{Rep}(\mathbf{M})$ of representations of an affine monoid k-scheme $\mathbf{M}$ and the category $\mathbf{Rep}(\mathbf{G})$ of representations of its group of units $\mathbf{G}$. Let $i : k[\mathbf{M}] \to k[\mathbf{G}]$ be the morphism of k-bialgebras induced by $\mathbf{G} \hookrightarrow \mathbf{M}$. Let us first note the following elementary result.

Fact 2.1. *Let $\mathbf{M}$ be an affine monoid k-scheme and let $\mathbf{G}$ be its group of units. Assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. Then i is an injective morphism of k-algebras.*

Proof. This follows from [Görtz and Wedhorn, 2010, Proposition 9.19]. □

Fact 2.2. *Let $\mathbf{M}$ be an affine monoid k-scheme and let $\mathbf{G}$ be its group of units. Then the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$ creates colimits and finite limits.*

Proof. This follows from [Monygham, 2020, Theorem 14.3, Theorem 14.4] and the commutative triangle

$$\begin{tikzcd} \mathbf{Rep}(\mathbf{M}) \arrow[rr] \arrow[dr] & & \mathbf{Rep}(\mathbf{G}) \arrow[dl] \\ & \mathbf{Vect}_k & \end{tikzcd}$$

of functors. □

The theorem below characterizes representations of $\mathbf{G}$ which are contained in the image of the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$.

Theorem 2.3. *Let $\mathbf{M}$ be an affine monoid k-scheme and let $\mathbf{G}$ be its group of units. Assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. Let V be a $\mathbf{G}$-representation. Then the following are equivalent.*

- **(i)** *V is in the image of the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$.*
- **(ii)** *The coaction $d : V \to k[\mathbf{G}] \otimes_k V$ factors through $i \otimes_k 1_V : k[\mathbf{M}] \otimes_k V \hookrightarrow k[\mathbf{G}] \otimes_k V$.*

Proof. In the proof we denote by $\Delta_{\mathbf{M}}$ and $\Delta_{\mathbf{G}}$ comultiplications of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. We also denote by $\xi_{\mathbf{M}}$ and $\xi_{\mathbf{G}}$ counits of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. According to Fact 2.1 i is an injective morphism of k-algebras.
Clearly **(i)** $\Rightarrow$ **(ii)**. We prove the converse. Suppose that **(ii)** holds. Let $c : V \to k[\mathbf{M}] \otimes_k V$ be

a unique morphism such that $d = (i \otimes_k 1_V) \cdot c$. It suffices to prove that c is the coaction of the bialgebra $k[\mathbf{M}]$ on V. Observe that

$$(i \otimes_k i \otimes_k 1_V) \cdot \left(1_{k[\mathbf{M}]} \otimes_k c\right) \cdot c = (i \otimes_k d) \cdot c = \left(1_{k[\mathbf{G}]} \otimes_k d\right) \cdot d = (\Delta_{\mathbf{G}} \otimes_k 1_V) \cdot d =$$

$$= (\Delta_{\mathbf{G}} \otimes_k 1_V) \cdot \big((i \otimes_k 1_V) \cdot c\big) = ((\Delta_{\mathbf{G}} \cdot i) \otimes_k 1_V) \cdot c = (i \otimes_k i \otimes_k 1_V) \cdot (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot c$$

Since $i \otimes_k i \otimes_k 1_V$ is a monomorphism, we deduce that $\left(1_{k[\mathbf{M}]} \otimes_k c\right) \cdot c = (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot c$. Moreover, we have

$$(\xi_{\mathbf{G}} \otimes_k 1_V) \cdot d = (\xi_{\mathbf{G}} \otimes_k 1_V) \cdot ((i \otimes_k 1_V) \cdot c) = (\xi_{\mathbf{M}} \otimes_k 1_V) \cdot c$$

and hence $(\xi_{\mathbf{M}} \otimes_k 1_V) \cdot c$ is the canonical isomorphism $V \cong k \otimes_k V$. Thus c is the coaction of $k[\mathbf{M}]$ and $d = (i \otimes_k 1_V) \cdot c$. Therefore, V is in the image of $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$. □

Theorem 2.4. *Let $\mathbf{M}$ be an affine monoid k-scheme and let $\mathbf{G}$ be its group of units. Assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. Then $\mathbf{Rep}(\mathbf{M})$ is a full subcategory of $\mathbf{Rep}(\mathbf{G})$ closed under subobjects and quotients.*

Proof. In the proof we denote by $\Delta_{\mathbf{M}}$ and $\Delta_{\mathbf{G}}$ comultiplications of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. We also denote by $\xi_{\mathbf{M}}$ and $\xi_{\mathbf{G}}$ counits of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. According to Fact 2.1 i is an injective morphism of k-algebras.
We first prove that $\mathbf{Rep}(\mathbf{M})$ is a full subcategory of $\mathbf{Rep}(\mathbf{G})$. For this consider $\mathbf{M}$-representations V, W and a their morphism $f : V \to W$ as $\mathbf{G}$-representations. Let c_V and c_W be coactions of $k[\mathbf{M}]$ on V and W, respectively. Our goal is to prove that f is a morphism of $\mathbf{M}$-representations. Consider the diagram

$$\begin{tikzcd}
k[\mathbf{G}] \otimes_k V \arrow[r, "1_{k[\mathbf{G}]} \otimes_k f"] & k[\mathbf{G}] \otimes_k W \\
k[\mathbf{M}] \otimes_k V \arrow[u, "i \otimes_k 1_V"] \arrow[r, dotted, "1_{k[\mathbf{M}]} \otimes_k f"] & k[\mathbf{M}] \otimes_k W \arrow[u, "i \otimes_k 1_W"'] \\
V \arrow[u, "c_V"] \arrow[r, "f"'] & W \arrow[u, "c_W"']
\end{tikzcd}$$

in which the outer square is commutative. Our goal is to prove that the bottom square is commutative. We have

$$(i \otimes_k 1_W) \cdot c_W \cdot f = \left(1_{k[\mathbf{G}]} \otimes_k f\right) \cdot (i \otimes_k 1_V) \cdot c_V = (i \otimes_k 1_W) \cdot \left(1_{k[\mathbf{M}]} \otimes_k f\right) \cdot c_V$$

Since $i \otimes_k 1_W$ is a monomorphism, we deduce that $c_W \cdot f = \left(1_{k[\mathbf{M}]} \otimes_k f\right) \cdot c_V$. Hence f is a morphism of $\mathbf{M}$-representations.
Next we prove that $\mathbf{Rep}(\mathbf{M})$ is a subcategory of $\mathbf{Rep}(\mathbf{G})$ that is closed under subquotients. Consider an $\mathbf{M}$-representation V and its quotient $\mathbf{G}$-representations $q : V \twoheadrightarrow W$. We show that W is a quotient $\mathbf{M}$-representation of V. Let c_V be the coaction of $\mathbf{M}$ on V and let d_W be the coaction of $\mathbf{G}$ on W. We have a commutative diagram

$$
\begin{tikzcd}
k[\mathbf{G}] \otimes_k V \arrow[rr, two heads, "1_{k[\mathbf{G}]} \otimes_k q"] & & k[\mathbf{G}] \otimes_k W \\
k[\mathbf{M}] \otimes_k V \arrow[u, "i \otimes_k 1_V"] & & \\
V \arrow[u, "c_V"] \arrow[rr, two heads, "q"'] & & W \arrow[uu, "d_W"']
\end{tikzcd}
$$

and hence $d_W(W) \subseteq k[\mathbf{M}] \otimes_k W$. Thus Theorem 2.3 implies that W is a representation of $\mathbf{M}$ and q is a morphism of $\mathbf{M}$-representations. This shows that $\mathbf{Rep}(\mathbf{M})$ is a subcategory of $\mathbf{Rep}(\mathbf{G})$ closed under quotients. Next let $j : U \hookrightarrow V$ be a $\mathbf{G}$-subrepresentation of a $\mathbf{M}$-representation V. By what we proved above the cokernel $q : V \twoheadrightarrow W$ of j in $\mathbf{Rep}(\mathbf{G})$ is contained in $\mathbf{Rep}(\mathbf{M})$. Since both $\mathbf{Rep}(\mathbf{M})$ and $\mathbf{Rep}(\mathbf{G})$ are abelian and the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$ is exact, we derive that the kernel of q in $\mathbf{Rep}(\mathbf{M})$ coincides with its kernel in $\mathbf{Rep}(\mathbf{G})$. Thus U is a $\mathbf{M}$-representation and $j : U \hookrightarrow V$ is a morphism of $\mathbf{M}$-representations. Hence $\mathbf{Rep}(\mathbf{M})$ is the category of $\mathbf{Rep}(\mathbf{G})$ closed under subobjects. □

Theorem 2.5. *Let $\mathbf{M}$ be an affine monoid k-scheme and let $\mathbf{G}$ be its group of units. Assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. Let V be a $\mathbf{G}$-representation of $\mathbf{G}$. There exists an $\mathbf{M}$-representation W and a surjective morpism $q : V \twoheadrightarrow W$ of $\mathbf{G}$-representations such that for every $\mathbf{M}$-representation U and a morphism $f : V \to U$ of $\mathbf{G}$-representations there exists a unique morphism $\tilde{f} : W \to U$ of $\mathbf{M}$-representations making the triangle*

$$
\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, two heads, "q"'] & U \\
W \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}
$$

commutative.

Proof. Assume first that V is finite dimensional. Let $\mathcal{K}$ be a set of $\mathbf{G}$-subrepresentations of V that consists of all $K \subseteq V$ such that V/K carries a structure of $\mathbf{M}$-representation. Clearly $\mathcal{K} = \varnothing$ because $\{0\} \in \mathcal{K}$. Since V is finite dimensional, there exists a finite subset $\{K_1, ..., K_n\} \subseteq \mathcal{K}$ such that

$$\bigcap_{i=1}^{n} K_i = \bigcap_{K \in \mathcal{K}} K$$

Then a morphism

$$V/\left(\bigcap_{K \in \mathcal{K}} K\right) \ni v \mapsto \left(v \operatorname{mod} K_i\right)_{1 \leq i \leq n} \in \bigoplus_{i=1}^{n} V/K_i$$

is a monomorphism and hence by Theorem 2.4 the quotient $W = V/\left(\bigcap_{K \in \mathcal{K}} K\right)$ is an $\mathbf{M}$-representation. Let $q : V \twoheadrightarrow W$ be the canonical epimorphism. Consider now a morphism $f : V \to U$ of $\mathbf{G}$-representations, where U is an $\mathbf{M}$-representation. Then $\operatorname{im}(f)$ is a $\mathbf{G}$-subrepresentation of U and by Theorem 2.4 we derive that $\operatorname{im}(f)$ is an $\mathbf{M}$-representation. This implies that $\ker(f)$ is in $\mathcal{K}$. Hence f factors through q. Thus there exists a unique morphism $\tilde{f} : W \to U$ of $\mathbf{G}$-representations such that $\tilde{f} \cdot q = f$. This completes the proof in case when V is finite dimensional.
Now consider the general V. Let $\mathcal{F}$ be the set of all finite dimensional $\mathbf{G}$-representations of V. According to [Monygham, 2020, Corollary 15.2] we deduce that $V = \operatorname{colim}_{F \in \mathcal{F}} F$. By the case considered above we deduce that for every F in $\mathcal{F}$ there exists a universal morphism $q_F : F \to W_F$ of $\mathbf{G}$-representations into an $\mathbf{M}$-representation W_F. Note that if $F_1 \subseteq F_2$ are two elements of $\mathcal{F}$, then

$$\begin{tikzcd} F_1 \arrow[r, two heads, "q_{F_1}"] \arrow[d, hook] & W_{F_1} \arrow[d] \\ F_2 \arrow[r, two heads, "q_{F_2}"'] & W_{F_2} \end{tikzcd}$$

Thus $\{W_F\}_{F\in\mathcal{F}}$ together with morphisms $W_{F_1} \to W_{F_2}$ for $F_1 \subseteq F_2$ in $\mathcal{F}$ form a diagram parametrized by the poset $\mathcal{F}$. The category $\mathbf{Rep}(\mathbf{M})$ has small colimits ([Monygham, 2020, Corollary 14.5]) and we define $W = \operatorname{colim}_{F\in\mathcal{F}} W_F$. This is also a colimit of this diagram in the category $\mathbf{Rep}(\mathbf{G})$ by Fact 2.2. We also define $q = \operatorname{colim}_{F\in\mathcal{F}} q_F : V = \operatorname{colim}_{F\in\mathcal{F}} F \to W$. Since a colimit of a family of epimorphisms is an epimorphism, we derive that q is an epimorphism of $\mathbf{G}$-representations. Suppose now that $f : V \to U$ is a morphism of $\mathbf{G}$-representations and U is an $\mathbf{M}$-representation. Then $f_{|F}$ uniquely factors through q_F for every F in $\mathcal{F}$. Hence by universal property of colimits we derive that f factors through q in a unique way. This completes the proof. □

3. Białynicki-Birula functors and its representability for locally linear schemes

References

[Görtz and Wedhorn, 2010] Görtz, U. and Wedhorn, T. (2010). *Algebraic Geometry: Part I: Schemes. With Examples and Exercises*. Advanced Lectures in Mathematics.

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