ALGEBRAIZATION OF FORMAL M-SCHEMES

1. Some 2-categorical limits

Consider a category $\mathcal{C}$ and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Our goal is to construct certain 2-categorical limit associated with a pair $(\mathcal{C}, T)$. Consider pairs (X, u) consisting of an object X of $\mathcal{C}$ and an isomorphism $u : T(X) \to X$ in $\mathcal{C}$. If (X, u) and (Y, w) are two such pairs, then a morphism $f : (X, u) \to (Y, u)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that the following square

$$\begin{tikzcd} T(X) \arrow[r, "u"] \arrow[d, "T(f)"'] & X \arrow[d, "f"] \\ T(Y) \arrow[r, "w"'] & Y \end{tikzcd}$$

is commutative. This data give rise to a category $\mathcal{C}(T)$. There exists a forgetful functor $\pi : \mathcal{C}(T) \to \mathcal{C}$ that sends a morphism $f : (X, u) \to (Y, w)$ to $f : X \to Y$. Moreover, there exists a natural isomorphism $\sigma : T \cdot \pi \Rightarrow \pi$ such that the component of σ on an object (X, u) of $\mathcal{C}(T)$ is u. The next result states that the data above form a certain 2-categorical limit.

Theorem 1.1. *Let $(\mathcal{C}, T)$ be a pair consiting of a category and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Suppose that $\mathcal{D}$ is a category, $P : \mathcal{D} \to \mathcal{C}$ is a functor and $\tau : T \cdot P \Rightarrow P$ is a natural isomorphisms. Then there exists a unique functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$.*

Proof. Suppose that $F : \mathcal{D} \to \mathcal{C}(T)$ is a functor such that $P = \pi \cdot F$ and $\sigma_F = \tau$. Pick an object X of $\mathcal{D}$. Then we have $\pi \cdot F(X) = P(X)$ and $\sigma_{F(X)} = \tau_X$. This implies that

$$F(X) = (P(X), \tau_X : T(P(X)) \to P(X))$$

Next if $f : X \to Y$ is a morphism in $\mathcal{D}$, then we derive that $\pi(F(f)) = P(f)$. Hence $F(f) = P(f)$. This implies that there exists at most one functor F satisfying the properties above. Note also that formulas

$$F(X) = (P(X), \tau_X : T(P(X)) \to P(X)), \, F(f) = P(f)$$

for an object X in $\mathcal{D}$ and a morphism $f : X \to Y$ in $\mathcal{D}$, give rise to a functor that satisfy $P = \pi \cdot F$ and $\sigma_F = \tau$. This establishes existence and the uniqueness of F. □

Assume now that the pair $(\mathcal{C}, T)$ consists of a monoidal category $\mathcal{C}$ and a monoidal endofunctor T. Then there exists a canonical monoidal structure on $\mathcal{C}(T)$. We define $(-) \otimes_{\mathcal{C}(T)} (-)$ by formula

$$(X, u) \otimes_{\mathcal{C}(T)} (Y, w) = \big(X \otimes_{\mathcal{C}} Y, (u \otimes_{\mathcal{C}} w) \cdot m_{X,Y}\big)$$

where

$$m_{X,Y} : T\left(X \otimes_{\mathcal{C}} Y\right) \to T(X) \otimes_{\mathcal{C}} T(Y)$$

is the tensor preserving isomorphism of T. We also define the unit

$$I_{\mathcal{C}(T)} = (I, T(I) \cong I)$$

where isomorphism $T(I) \cong I$ is precisely the unit preserving isomorphism of the monoidal functor T. The associativity natural isomorphism for $(-) \otimes_{\mathcal{C}(T)} (-)$ and right, left units for $I_{\mathcal{C}(T)}$ in $\mathcal{C}(T)$ are associavity natural isomorphism and right, left units for $\mathcal{C}$, respectively. The structure makes a functor $\pi : \mathcal{C}(T) \to \mathcal{C}$ strict monoidal and σ a monoidal natural isomorphism. The next result states that the data with these extra monoidal structure form a 2-categorical limit in the 2-category of monoidal categories.

Theorem 1.2. *Let $(\mathcal{C}, T)$ be a pair consiting of a monoidal category and its monoidal endofunctor $T : \mathcal{C} \to \mathcal{C}$. Suppose that $\mathcal{D}$ is a monoidal category, $P : \mathcal{D} \to \mathcal{C}$ is a monoidal functor and $\tau : T \cdot P \Rightarrow P$ is a monoidal natural isomorphisms. Then there exists a unique monoidal functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$ as monoidal functors and monoidal transformations.*

Proof. Note that F must be defined as it was described in the proof of Theorem 1.1. Namely we must have

$$F(X) = (P(X), \tau_X : T(P(X)) \to P(X)), F(f) = P(f)$$

for an object X in $\mathcal{C}$ and a morphism $f : X \to Y$ in $\mathcal{C}$.
Suppose now that F admits a structure of a monoidal functor such that $P = \pi \cdot F$ as monoidal functors. Let

$$\left\{m^F_{X,Y} : F(X \otimes_{\mathcal{D}} Y) \to F(X) \otimes_{\mathcal{C}(T)} F(Y)\right\}_{X,Y \in \mathcal{C}}, \phi^F : F(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$$

be the data forming that structure. Since π is a strict monoidal functor and $P = \pi \cdot F$ as monoidal functors, we derive that for any objects X, Y of $\mathcal{C}$

$$\pi(m^F_{X,Y}) : P(X \otimes_{\mathcal{D}} Y) \to P(X) \otimes_{\mathcal{C}} P(Y)$$

is the tensor preserving isomorphism $m^P_{X,Y} : P(X \otimes_{\mathcal{D}} Y) \to P(X) \otimes_{\mathcal{C}} P(Y)$ of the monoidal functor P. By the same argument

$$\pi(\phi_F) : P(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$$

is the unit preserving isomorphism $\phi^P : P(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$ of P. Thus we deduce that for any objects X, Y of $\mathcal{C}$ we have $m^F_{X,Y} = m^P_{X,Y}$ and $\phi^F = \phi^P$. This implies that there exists at most one monoidal functor F such that $P = \pi \cdot F$ as monoidal functors.
On the other hand define $m^F_{X,Y} = m^P_{X,Y}$ for objects X, Y in $\mathcal{C}$ and $\phi^F = \phi^P$. We check now that F equipped with these data is a monoidal functor. Fix objects X, Y in $\mathcal{C}$. The square

$$\begin{tikzcd}
T(P(X \otimes_{\mathcal{D}} Y)) \arrow[r, "\tau_{X \otimes_{\mathcal{C}} Y}"] \arrow[d, "T(m^P_{X,Y})"'] & P(X \otimes_{\mathcal{C}} Y) \arrow[d, "m^P_{X,Y}"] \\
T(P(X) \otimes_{\mathcal{C}} P(Y)) \arrow[r, "(\tau_X \otimes_{\mathcal{C}} \tau_Y) \cdot m^T_{P(X),P(Y)}"'] & P(X) \otimes_{\mathcal{C}} P(Y)
\end{tikzcd}$$

is commutative due to the fact that $\tau : T \cdot P \Rightarrow P$ is a monoidal natural isomorphisms. This implies that $m^F_{X,Y}$ is a morphism in $\mathcal{C}(T)$. It follows that $m^F_{X,Y}$ is a natural isomorphism and due to the definition of associativity in $\mathcal{C}(T)$, we derive its compatibility with $m^F_{X,Y}$. Similarly, since the square

$$\begin{tikzcd}
T(P(I_{\mathcal{D}})) \arrow[r, "\tau_{I_{\mathcal{D}}}"] \arrow[d, "T(\phi^P)"'] & P(I_{\mathcal{D}}) \arrow[d, "\phi^P"] \\
T(I_{\mathcal{C}}) \arrow[r, "\phi^T"'] & I_{\mathcal{C}}
\end{tikzcd}$$

is commutative, we deduce that ϕ^F is a morphism in $\mathcal{C}(T)$. By definition of left and right unit in $\mathcal{C}(T)$, we derive their compatibility with ϕ^F. This finishes the verification of the fact that F with $\{m^F_{X,Y}\}_{X,Y \in \mathcal{C}}$ and ϕ^F is a monoidal functor. Definitions of $\{m^F_{X,Y}\}_{X,Y \in \mathcal{C}}$ and ϕ^F show that the identities $P = \pi \cdot F$ holds on the level of monoidal structures. Since the 2-forgetful functor from

2-category of monoidal categories into 2-category of categories is faithful on 2-cells, the identity $\sigma_F = \tau$ of natural isomorphisms is also the identity of monoidal natural isomorphisms. □

Theorem 1.3. *Let $(\mathcal{C}, T)$ be a pair consiting of a category and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Assume that T preserves colomits. Then the following assertions hold.*

(1) *$\pi : \mathcal{C}(T) \to \mathcal{C}$ creates colimits.*

(2) *Suppose that $\mathcal{D}$ is a category, $P : \mathcal{D} \to \mathcal{C}$ a functor preserving small colimits and $\tau : T \cdot P \Rightarrow P$ a natural isomorphisms. Then the unique functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$ preserves small colimits.*

Proof. Let I be a small category and $D : I \to \mathcal{C}(T)$ be a diagram such that the composition $\pi \cdot D : I \to \mathcal{C}$ admits a colimit given by cocone $(X, \{g_i\}_{i\in I})$. Since T preserves colimits, we derive that $(T(X), \{T(u_i)\}_{i\in I})$ is a colimit of $T \cdot \pi \cdot D : I \to \mathcal{C}$. Now $\sigma_D : T \cdot \pi \cdot D \to \pi \cdot D$ is a natural isomorphism. Hence there exists a unique arrow $u : T(X) \to X$ such that $u \cdot T(g_i) = g_i \cdot \sigma_{D(i)}$ for $i \in I$. Clearly u is an isomorphism and hence (X, u) is an object of $\mathcal{C}(T)$. Moreover, the family $\{g_i\}_{i\in I}$ together with (X, u) is a colimiting cocone over D. This proves **(1)**. Now **(2)** is a consequence of **(1)**. □

Now we apply the results above to certain more general diagrams of categories.

Definition 1.4. A diagram

$$\dots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \dots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

of categories and functors is called *a telescope of categories*.

Definition 1.5. Let

$$\dots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \dots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

be a telescope of monoidal categories and monoidal cocontinuous functors. Then *a 2-categorical limit of the telescope* consists of a monoidal category $\mathcal{C}$, a family of monoidal cocontinuous functors $\{\pi_n : \mathcal{C} \to \mathcal{C}_n\}_{n\in\mathbb{N}}$ and a family of monoidal natural isomorphisms $\{\sigma_n : F_{n+1} \cdot \pi_{n+1} \Rightarrow \pi_n\}_{n\in\mathbb{N}}$ such that the following universal property holds. For any monoidal category $\mathcal{D}$, family $\{P_n : \mathcal{D} \to \mathcal{C}_n\}_{n\in\mathbb{N}}$ of cocontinuous monoidal functors and a family $\{\tau_n : F_n P_{n+1} \Rightarrow P_n\}_{n\in\mathbb{N}}$ of monoidal natural isomorphisms there exists a unique monoidal cocontinuous functor $F : \mathcal{D} \to \mathcal{C}$ satisfying $P_n = \pi_n \cdot F$ and $(\sigma_n)_F = \tau_n$ for every $n \in \mathbb{N}$.

Corollary 1.6. *Let*

$$\dots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \dots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

be a telescope of monoidal categories and monoidal cocontinuous functors. Then its 2-limit exists.

Proof. We decompose the task of constructing its 2-limit as follows. First note that one may form a product $\mathcal{C} = \prod_{n\in\mathbb{N}} \mathcal{C}_n$. Next the functors $\{F_n\}_{n\in\mathbb{N}}$ induce an endofunctor $T = \prod_{n\in\mathbb{N}} F_n \times t$, where $\mathbf{1}$ is the terminal category (it has single object and single identity arrow) and $t : \mathcal{C}_0 \to \mathbf{1}$ is the unique functor. Consider the category $\mathcal{C}(T)$. We define $\{\pi_n : \mathcal{C}(T) \to \mathcal{C}_n\}_{n\in\mathbb{N}}$ to be a family of functors given by coordinates of $\pi : \mathcal{C}(T) \to \mathcal{C}$ and $\{\sigma_n : F_n \cdot \pi_{n+1} \Rightarrow \pi_n\}_{n\in\mathbb{N}}$ to be a family of natural isomorphisms given by coordinates of $\sigma : \pi \cdot T \Rightarrow \pi$. Now this data form a 2-limit of the telescope by compilation of Theorem 1.2 and Theorem 1.3. □

2. Formal **M**-schemes

This section is devoted to introducing some notions from formal geometry that are central in this notes.

Definition 2.1. Let $\mathbf{M}$ be a monoid k-scheme. *A formal* $\mathbf{M}$*-scheme* consists of a sequence $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ of $\mathbf{M}$-schemes together with $\mathbf{M}$-equivariant closed immersions

$$Z_0 \hookrightarrow Z_1 \hookrightarrow \dots \hookrightarrow Z_n \hookrightarrow Z_{n+1} \hookrightarrow \dots$$

satisfying the following assertions.

(1) We have $Z_0 = Z_n^{\mathbf{M}}$ scheme-theoretically for every $n \in \mathbb{N}$.

(2) Let $\mathcal{I}_n$ be an ideal of $\mathcal{O}_{Z_n}$ defining Z_0. Then for every $m \leq n$ the subscheme $Z_m \subset Z_n$ is defined by $\mathcal{I}_n^{m+1}$.

Example 2.2. Let $\mathbf{M}$ be a monoid k-scheme and let Z be a $\mathbf{M}$-scheme. Consider a quasi-coherent ideal $\mathcal{I}$ of fixed point subscheme $Z^{\mathbf{M}}$ of Z. Then for every $n \in \mathbb{N}$ ideal $\mathcal{I}^n$ is $\mathbf{M}$-equivariant and hence

$$V(\mathcal{I}) \hookrightarrow V(\mathcal{I}^2) \hookrightarrow \dots \hookrightarrow V(\mathcal{I}^n) \hookrightarrow \dots$$

is a formal $\mathbf{M}$-scheme. We denote it by $\widehat{Z}$.

Definition 2.3. Let $\mathbf{M}$ be a monoid k-scheme and let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{M}$-scheme. We say that $\mathcal{Z}$ is *locally Noetherian* if for all $n \in \mathbb{N}$ scheme Z_n is locally Noetherian.

Definition 2.4. Let $\mathbf{M}$ be a monoid k-scheme. Suppose that $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ and $\mathcal{W} = \{W_n\}_{n\in\mathbb{N}}$ are formal $\mathbf{M}$-schemes. Then *a morphism* $f : \mathcal{Z} \to \mathcal{W}$ *of formal* $\mathbf{M}$*-schemes* consists of a family of $\mathbf{M}$-equivariant morphisms $f = \{f_n : Z_n \to W_n\}_{n\in\mathbb{N}}$ such that the diagram

$$\begin{tikzcd}
Z_0 \arrow[r, hook] \arrow[d, "f_0"'] & Z_1 \arrow[r, hook] \arrow[d, "f_1"'] & \dots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[d, "f_n"'] & Z_{n+1} \arrow[r, hook] \arrow[d, "f_{n+1}"'] & \dots \\
W_0 \arrow[r, hook] & W_1 \arrow[r, hook] & \dots \arrow[r, hook] & W_n \arrow[r, hook] & W_{n+1} \arrow[r, hook] & \dots
\end{tikzcd}$$

is commutative.

Since group k-scheme is also a monoid k-scheme, definitions above can be applied to group k-schemes.

Definition 2.5. Let $\mathbf{G}$ be a group k-scheme. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{G}$-scheme. Then there we have the corresponding telescope of monoidal categories

$$\dots \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_{n+1}) \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_n) \longrightarrow \dots \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_2) \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_1) \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_0)$$

and cocontinuous monoidal functors given by restricting $\mathbf{G}$-equivariant quasi-coherent sheaves to closed $\mathbf{G}$-subschemes. Then we define *a category* $\mathfrak{Qcoh}(\mathcal{Z})$ *of quasi-coherent sheaves on* $\mathcal{Z}$ as a monoidal category which is a 2-limit of the telescope above. This category is defined uniquely up to a monoidal equivalence.

Let Z be a $\mathbf{G}$-scheme and let $\mathcal{I}$ be a quasi-coherent ideal of $Z^{\mathbf{G}}$. We have a commutative diagram

$$\begin{tikzcd}
V(\mathcal{I}) \arrow[r, hook] \arrow[drr, hook, bend right] & V(\mathcal{I}^2) \arrow[r, hook] \arrow[dr, hook] & \dots \arrow[r, hook] & V(\mathcal{I}^n) \arrow[r, hook] \arrow[dl, hook'] & \dots \\
& & Z & &
\end{tikzcd}$$

in the category of $\mathbf{G}$-schemes. Thus restriction functors $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}_{\mathbf{G}}(V(\mathcal{I}^n))$ for $n \in \mathbb{N}$ induce a unique cocontinuous monoidal functor $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}(\widehat{Z})$.

Definition 2.6. Let Z be a $\mathbf{G}$-scheme. Then a unique cocontinuous monoidal functor $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}(\widehat{Z})$ is called *the comparison functor*.

Definition 2.7. Let $\mathbf{M}$ be a monoid k-scheme with group of units $\mathbf{G}$. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{M}$-scheme. A $\mathbf{M}$-scheme Z is called *an algebraization of* $\mathcal{Z}$ if the following two conditions are satisfied.

(1) $\mathcal{Z}$ is isomorphic to $\widehat{Z}$ in the category of formal $\mathbf{M}$-schemes.

(2) The comparison functor $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}\left(\widehat{Z}\right)$ is an equivalence of monoidal categories.

3. Locally linear M-schemes

Definition 3.1. Let $\mathbf{M}$ be a monoid k-scheme and let X be a $\mathbf{M}$-scheme. Suppose that each point of X admits an open affine $\mathbf{M}$-stable neighborhood. Then we say that X is *a locally linear* $\mathbf{M}$*-scheme*.

Proposition 3.2. *Let* $\mathbf{M}$ *be an affine monoid k-scheme and let* X *be a* $\mathbf{M}$*-scheme. Suppose that there exists a quasi-coherent* $\mathbf{M}$*-equivariant ideal* $\mathcal{I}$ *on* X *with nilpotent sections. Consider an open subset* U *of* X*. Then the following are equivalent.*

(1) *U is* $\mathbf{M}$*-stable.*

(2) $U \cap V(\mathcal{I})$ *is* $\mathbf{M}$*-stable.*

Proof. Let $\alpha : \mathbf{M} \times X \to X$ be the action of $\mathbf{M}$ on X. Fix open subset U of X. If U is $\mathbf{M}$-stable, then $U \cap V(\mathcal{I})$ is $\mathbf{M}$-stable. So suppose that $U \cap V(\mathcal{I})$ is $\mathbf{M}$-stable. Since $\mathcal{I}$ has nilpotent sections and $\mathbf{M}$ is affine, we derive that closed immersions $U \cap V(\mathcal{I}) \hookrightarrow U$ and $\mathbf{M} \times (U \cap V(\mathcal{I})) \hookrightarrow \mathbf{M} \times U$ induce homeomorphisms on topological spaces. Consider the commutative diagram

$$\begin{tikzcd}
\mathbf{M} \times U \arrow[r, "\alpha_{|U \cap V(\mathcal{I})}"] & X \\
\mathbf{M} \times (U \cap V(\mathcal{I})) \arrow[r] \arrow[u, hook] & U \cap V(\mathcal{I}) \arrow[u, hook]
\end{tikzcd}$$

where the bottom horizontal arrow is the induced action on $U \cap V(\mathcal{I})$ and vertical morphisms are homeomorphisms. The commutativity of the diagram implies that $\alpha\left(\mathbf{M} \times U\right)$ is contained set-theoretically in U. Since U is open in X, we derive that morphism of schemes $\alpha_{|\mathbf{M}\times U}$ factors through U. Hence U is $\mathbf{M}$-stable. □

Corollary 3.3. *Let* $\mathbf{M}$ *be an affine monoid k-scheme and let* X *be a* $\mathbf{M}$*-scheme. Suppose that there exists a quasi-coherent* $\mathbf{M}$*-equivariant ideal* $\mathcal{I}$ *on* X *such that* $\mathcal{I}^n = 0$ *for* $n \in \mathbb{N}$*. Consider an open subset* U *of* X*. Then the following are equivalent.*

(1) *U is* $\mathbf{M}$*-stable and affine.*

(2) $U \cap V(\mathcal{I})$ *is* $\mathbf{M}$*-stable and affine.*

Proof. Since $\mathcal{I}^n = 0$, we derive that U is affine if and only if $U \cap V(\mathcal{I})$ is affine. Combining this with Proposition 3.2, we deduce the result. □

Corollary 3.4. *Let* $\mathbf{M}$ *be an affine monoid* k*-scheme and let* X *be a* $\mathbf{M}$*-scheme. Suppose that there exists a quasi-coherent* $\mathbf{M}$*-equivariant ideal* $\mathcal{I}$ *on* X *such that* $\mathcal{I}^n = 0$ *for* $n \in \mathbb{N}$*. Then* X *is locally linear* $\mathbf{M}$*-scheme if and only if* $V(\mathcal{I})$ *is locally linear* $\mathbf{M}$*-scheme.*

Proof. This is a consequence of Corollary 3.3. □

4. Some results on formal M-schemes

Corollary 4.1. *Let* $\mathbf{M}$ *be an affine monoid* k*-scheme and let* $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ *be a formal* $\mathbf{G}$*-scheme. Then* Z_n *is locally linear* $\mathbf{G}$*-scheme for every* $n \in \mathbb{N}$*.*

Proof. Let $\mathcal{I}_n$ be an ideal defining Z_0 in Z_n. Since $\mathcal{Z}$ is a formal $\mathbf{M}$-scheme, we derive that $\mathcal{I}_n^{n+1} = 0$ and Z_0 is locally linear $\mathbf{M}$-scheme. Thus we apply Corollary 3.4 and derive that Z_n is locally linear $\mathbf{M}$-scheme. □

We are particularly interested in formal $\mathbf{M}$-schemes for monoid $\mathbf{M}$ with zero. For this we need the following elementary result.

Proposition 4.2. *Let* $\mathbf{M}$ *be a monoid* k*-scheme with zero* $\mathbf{o}$ *and let* X *be a* $\mathbf{M}$*-scheme. Then the following results hold.*

(1) *The multiplication by zero* $\mathbf{o}\cdot(-) : X \to X$ *factors through* $X^{\mathbf{M}}$ *inducing a* $\mathbf{M}$*-equivariant retraction* $\pi_{\mathbf{M}} : X \twoheadrightarrow X^{\mathbf{M}}$*.*

(2) *If* $\mathbf{N}$ *is a submonoid* k*-scheme of* $\mathbf{M}$ *and* $\mathbf{o}$ *is a* k*-point of* $\mathbf{N}$*, then* $\pi_{\mathbf{M}} = \pi_{\mathbf{N}}$*.*

(3) *If* $\mathbf{M}$ *is affine and* X *is locally linear* $\mathbf{M}$*-scheme, then* $\pi_{\mathbf{M}}$ *is affine.*

Proof. The multiplication $\mathbf{o}\cdot(-) : \mathfrak{P}_X \to \mathfrak{P}_X$ factors as an $\mathfrak{P}_{\mathbf{M}}$-equivariant epimorphism $\mathfrak{P}_X \twoheadrightarrow \mathfrak{P}_{X^{\mathbf{M}}}$ composed with a closed immersion $\mathfrak{P}_{X^{\mathbf{M}}} \hookrightarrow \mathfrak{P}_X$. The $\mathfrak{P}_{\mathbf{M}}$-equivariant epimorphism $\mathfrak{P}_X \to \mathfrak{P}_{X^{\mathbf{M}}}$ corresponds to a $\mathbf{M}$-equivariant morphism $\pi_{\mathbf{M}} : X \to X^{\mathbf{M}}$ of k-schemes such that $\pi_{\mathbf{M}}$ restricted to $X^{\mathbf{M}}$ is the identity $1_{X^{\mathbf{M}}}$. This proves **(1)**.
For the proof of **(2)** note that $\mathbf{o}\cdot(-) : \mathfrak{P}_X \to \mathfrak{P}_X$ is defined similarly for $\mathbf{M}$ and $\mathbf{N}$ (provided that $\mathbf{o}$ is a k-point of $\mathbf{N}$). Thus $\pi_{\mathbf{M}} = \pi_{\mathbf{N}}$.
Suppose now that $\mathbf{M}$ is affine and X is locally linear $\mathbf{M}$-scheme. Consider the action $\alpha : \mathbf{M}\times X \to X$ of $\mathbf{M}$ on X. Since X is locally linear and $\mathbf{M}$ is affine, we derive that α is an affine morphism of k-schemes. Now $\mathbf{o}\cdot(-) : X \to X$ is given as a composition

$$X \xrightarrow{\cong} \mathbf{o}\times X \hookrightarrow \mathbf{M}\times X \xrightarrow{\alpha} X$$

The morphism above is affine (as a composition of affine morphisms). Since the composition of π with a closed immersion $X^{\mathbf{M}} \hookrightarrow X$ is $\mathbf{o}\times(-)$ and hence an affine morphism, we derive that π is affine. This proves **(3)**. □

Let us note the immediate consequence of this result.

Corollary 4.3. *Let* $\mathbf{M}$ *be an affine monoid* k*-scheme with zero and let* $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ *be a formal* $\mathbf{M}$*-scheme. Then* $\mathcal{Z}$ *is a part of the commutative diagram*

$$\begin{tikzcd} Z_0 \arrow[r, hook] \arrow[drr, bend right, two heads, "\pi_0 = 1_{Z_0}"'] & Z_1 \arrow[r, hook] \arrow[dr, two heads, "\pi_1"] & \dots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[dl, two heads, "\pi_n"] & \dots \\ & & Z_0 & & \end{tikzcd}$$

in which vertical morphisms $\pi_n : Z_n \twoheadrightarrow Z_0$ are affine **M***-equivariant morphisms such that $\pi_{n|Z_0} = 1_{Z_0}$. Moreover, if* **N** *is a submonoid k-scheme of* **M** *containing the zero of* **M**, *then $\mathcal{Z}$ is a formal* **N***-scheme.*

Proof. This is an immediate consequence of Corollary 4.1 and Proposition 4.2. □

5. TORUSES AND TORIC MONOID k-SCHEMES

Definition 5.1. Let T be an affine algebraic group over k. Suppose that there exists $n \in \mathbb{N}$ such that for every algebraically closed extension K of k there exists an isomorphism

$$T_K \cong \operatorname{Spec} K \times \underbrace{\mathbb{G}_m \times \mathbb{G}_m \times \dots \times \mathbb{G}_m}_{n \text{ times}}$$

of group schemes over K. Then T is called *a torus over k.*

Example 5.2. If $T \cong \underbrace{\mathbb{G}_m \times \mathbb{G}_m \times \dots \times \mathbb{G}_m}_{n \text{ times}}$, then T is a torus. We call toruses T of this form *split toruses*.

Example 5.3. Define

$$\mathbf{S}^1 = \operatorname{Spec} k[x,y]/(x^2+y^2-1)$$

a scheme over k and let $\mathfrak{P}_{\mathbf{S}^1}$ be its functor of points. Then for every k-algebra A we have

$$\mathfrak{P}_{\mathbf{S}^1}(A) = \left\{(u,v) \in A \times A \,\middle|\, u^2+v^2 = 1\right\}$$

There is also a morphism $\mathfrak{P}_{\mathbf{S}^1} \times \mathfrak{P}_{\mathbf{S}^1} \to \mathfrak{P}_{\mathbf{S}^1}$ of k-functors given by

$$\mathfrak{P}_{\mathbf{S}^1}(A) \times \mathfrak{P}_{\mathbf{S}^1}(A) \to \mathfrak{P}_{\mathbf{S}^1} \ni ((u_1,v_1),(u_2,v_2)) \mapsto (u_1u_2 - v_1v_2, u_1v_2 + u_2v_1) \in \mathfrak{P}_{\mathbf{S}^1}(A)$$

for every k-algebra A. This makes $\mathfrak{P}_{\mathbf{S}^1}$ into a group k-functor. Thus $\mathbf{S}^1$ with the group structure described above is an affine algebraic group over k. We call it *the circle group over k.*
Now suppose that $\mathrm{char}(k) = 2$ and K is an algebraically closed extension of k. Consider an element $i \in K$ such that $i^2 = -1$. For every K-algebra A we have a map

$$\mathfrak{P}_{\mathbf{S}^1}(A) \ni (u,v) \mapsto u + iv \in A^*$$

First note that this map is bijective. Indeed, its inverse is given by

$$A^* \ni a \mapsto \left(\frac{1}{2}(a+a^{-1}), \frac{1}{2i}(a-a^{-1})\right) \in \mathfrak{P}_{\mathbf{S}^1}(A)$$

Moreover, the map $\mathfrak{P}_{\mathbf{S}^1}(A) \to A^*$ is a homomorphism of abstract groups. Thus $\mathfrak{P}_{\mathbf{S}^1}$ resricted to the category $\mathbf{Alg}_K$ of K-algebras is isomorphic with $\mathfrak{P}_{\operatorname{Spec} K \times \mathbb{G}_m}$ as a group k-functor. Hence

$$\mathbf{S}^1_K \cong \operatorname{Spec} K \times \mathbb{G}_m$$

as algebraic group schemes over K. Hence $\mathbf{S}^1$ is a torus over k.
Now assume that $k = \mathbb{R}$. Then abstract groups

$$\mathfrak{P}_{\mathbf{S}^1}(\mathbb{R}) = \left\{z \in \mathbb{C} \,\middle|\, |z| = 1\right\} \subseteq \mathbb{C}^*, \ \mathbb{R}^*$$

are not isomorphic. Indeed, the left hand side group has infinite torsion subgroup and the right hand side group has torsion subgroup equal to $\{-1,1\}$. This implies that over $\mathbb{R}$ algebraic groups $\mathbf{S}^1$ and $\mathbb{G}_m$ are not isomorphic. Hence $\mathbf{S}^1$ is not a split torus over $\mathbb{R}$.

Corollary 5.4. *Let T be a torus over k. Then T is a linearly reductive algebraic group.*

Definition 5.5. Let T be a torus over k and let $\overline{T}$ be a linearly reductive monoid having T as the group of units. Then $\overline{T}$ is *a toric monoid over k*

Theorem 5.6. *Let $\overline{T}$ be a toric monoid over k with group of units T and let K be an algebraically closed extension of k. Suppose that N is a dimension of T.*

(1) *The group of characters of T_K is isomorphic to $\mathbb{Z}^N$ and there exists an abstract submonoid S of $\mathbb{Z}^N$ such that the open immersion*

$$T_K = \operatorname{Spec}\left(\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m\right) \hookrightarrow \operatorname{Spec}\left(\bigoplus_{m\in S} K\cdot\chi^m\right) = \overline{T}_K$$

is induced by the inclusion $S \hookrightarrow \mathbb{Z}^N$.

(2) *Let $\{V_\lambda\}_{\lambda\in\mathbf{Irr}(T)}$ be a set of irreducible representation of T such that V_λ is in isomorphism class λ. For every λ there exists a finite subset A_λ of $\mathbb{Z}^N$ such that*

$$K\otimes_k V_\lambda = \bigoplus_{m\in A_\lambda} K\cdot\chi^m$$

If λ is in $\mathbf{Irr}(\overline{T})$, then A_λ is a subset of S. Moreover, we have

$$\mathbb{Z}^N = \coprod_{\lambda\in\mathbf{Irr}(T)} A_\lambda$$

and $A_{\lambda_0} = \{0\}$, where λ_0 is the class of the trivial representation of T.

(3) *If $\overline{T}$ has a zero, then there exists a homomorphism $f:\mathbb{Z}^N\to\mathbb{Z}$ of abelian groups such that $f_{|S\smallsetminus\{0\}} > 0$. In particular, f induces a closed immersion*

$$\operatorname{Spec} K\times\mathbb{G}_m = \operatorname{Spec} K[\mathbb{Z}] \hookrightarrow \operatorname{Spec}\left(\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m\right) = T_K$$

of group K-schemes that extends to a zero preserving closed immersion $\mathbb{A}^1_K \hookrightarrow \overline{T}_K$ of monoid K-schemes.

Proof. Since T is a torus, we derive that

$$T_K = \operatorname{Spec} K\times\underbrace{\mathbb{G}_m\times\mathbb{G}_m\times\ldots\times\mathbb{G}_m}_{N\text{ times}} = \operatorname{Spec}\left(\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m\right)$$

and hence

$$\overline{T}_K = \operatorname{Spec}\left(\bigoplus_{s\in S} K\cdot\chi^s\right)$$

for some abstract submonoid S of $\mathbb{Z}^N$. Moreover, the open immersion $T_K \hookrightarrow \overline{T}_K$ is induced by the inclusion $S\hookrightarrow\mathbb{Z}^N$. This proves **(1)**.
We have identification

$$k[T] = \bigoplus_{\lambda\in\mathbf{Irr}(T)} V_\lambda^{n_\lambda}$$

of T-representations, where $n_\lambda\in\mathbb{N}\smallsetminus\{0\}$ for each λ. Thus

$$\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m = K\otimes_k k[T] = \bigoplus_{\lambda\in\mathbf{Irr}(T)} (K\otimes_k V_\lambda)^{n_\lambda}$$

This implies that $n_\lambda = 1$ for every λ and moreover, we derive that

$$K\otimes_k V_\lambda = \bigoplus_{m\in A_\lambda} K\cdot\chi^m$$

for some finite set $A_\lambda \subseteq \mathbb{Z}^N$. We also have $A_{\lambda_0} = \{0\}$ and $A_\lambda \subseteq S \setminus \{0\}$ for $\lambda \in \mathbf{Irr}(\overline{T})$. This proves **(2)**.
Since $\overline{T}$ admits a zero, we derive that

$$\mathfrak{m} = \bigoplus_{m \in S \setminus \{0\}} K \cdot \chi^s \subseteq \bigoplus_{m \in \mathbb{Z}^N} K \cdot \chi^m$$

is an ideal. This implies that $S \setminus \{0\}$ is closed under addition. In particular, there exists a homomorphism of abelian groups $f : \mathbb{Z}^N \to \mathbb{Z}$ such that $f_{|S \setminus \{0\}} > 0$. This implies **(3)**. □

6. Commuting actions

Corollary 6.1. *Let $\mathbf{G}$ be an affine group scheme over k and let $\mathfrak{G}$ be a monoid k-functor. Denote by Λ the set of isomorphism classes of irreducible $\mathbf{G}$-representations. Suppose that V is a representation of both $\mathbf{G}$ and $\mathfrak{G}$ and assume that their actions on V commute. Assume that V is completely reducible as a $\mathbf{G}$-representation and consider the decomposition*

$$V = \bigoplus_{\lambda \in \Lambda} V[\lambda]$$

onto isotypic components with respect to the action of $\mathbf{G}$. Then for every λ in Λ the subspace $V[\lambda]$ is a $\mathfrak{G}$-subrepresentation of V.

Proof. Part of the structure V as the $\mathfrak{G}$-representation is the morphism $\rho : \mathfrak{G} \to \mathcal{L}_V$ of k-monoids. Fix k-algebra A and $g \in \mathfrak{G}(A)$. Since actions of $\mathbf{G}$ and $\mathfrak{G}$ on V commute, morphism $\rho(g) : A \otimes_k V \to A \otimes_k V$ of A-modules is a morphism of $\mathbf{G}_A$-representation. According to Proposition **??** we derive that

$$\mathrm{Hom}_{\mathbf{G}_A}\left(A \otimes_k V[\lambda_1], A \otimes_k V[\lambda_2]\right) = 0$$

for distinct $\lambda_1, \lambda_2 \in \Lambda$. Thus

$$\rho(g)\left(A \otimes_k V[\lambda]\right) \subseteq A \otimes_k V[\lambda]$$

for every λ in Λ. This holds for every k-algebra A and $g \in \mathfrak{G}(A)$. Hence $V[\lambda]$ is $\mathfrak{G}$-subrepresentation of V. □

7. Algebraization of formal M-schemes

This section proves some results in equivariant formal geometry.

Theorem 7.1. *Let $\mathbf{M}$ be a Kempf monoid and let $\mathcal{Z} = \{Z_n\}_{n \in \mathbb{N}}$ be a formal $\mathbf{M}$-scheme. Then there exists a locally linear $\mathbf{M}$-scheme Z equipped with an action of $\mathbf{M}$ such that $\widehat{Z}$ is isomorphic to $\mathcal{Z}$.*

Monoid $\mathbf{M}$ is affine and admits zero $\mathbf{o}$. Hence by Corollary 4.3 formal $\mathbf{M}$-scheme $\mathcal{Z}$ corresponds to a sequence of surjections

$$\ldots \twoheadrightarrow \mathcal{A}_{n+1} \twoheadrightarrow \mathcal{A}_n \twoheadrightarrow \ldots \twoheadrightarrow \mathcal{A}_1 \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$$

of quasi-coherent $\mathcal{O}_{Z_0}$-algebras with $\mathbf{M}$-linearization such that $\mathcal{A}_n^{\mathbf{M}} = \mathcal{A}_0$ for every $n \in \mathbb{N}$ and if $\mathcal{I}_n$ is the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_0$ in $\mathcal{A}_n$, then $\mathcal{I}_n^{m+1}$ is the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_m$ for $m \leq n$ and $n \in \mathbb{N}$. Since $\mathbf{M}$ is a Kempf monoid, there exists a closed subgroup T of the center $Z(\mathbf{G})$ of the unit group $\mathbf{G}$ of $\mathbf{M}$ such that T is a torus and the scheme-theoretic closure $\overline{T}$ of T in $\mathbf{M}$ contains the zero $\mathbf{o}$ of $\mathbf{M}$. We derive by Corollary 4.3 that $\mathcal{A}_n^{\overline{\mathbf{T}}} = \mathcal{A}_0$ for every $n \in \mathbb{N}$. By definition $\overline{T}$ is a toric monoid k-scheme with T as a group of units. Let $\{V_\lambda\}_{\lambda \in \mathbf{Irr}(T)}$ be a set of irreducible representations of T such that V_λ is contained in λ. We start with some lemmas.

Lemma 7.1.1. *Let λ be in $\mathbf{Irr}(\overline{T})$. Then there exists $n_\lambda \in \mathbb{N}$ such that for each $n > n_\lambda$ and any $\lambda_1, ..., \lambda_n \in \mathbf{Irr}(\overline{T}) \smallsetminus \{\lambda_0\}$ the representation*

$$\bigotimes_{i=1}^{n} V_{\lambda_i}$$

has trivial isotypic component of type λ. We have $n_{\lambda_0} = 0$, where λ_0 is an isomorphism type of the trivial representation of T.

Proof of the lemma. Let K be an algebraically closed extension of k. Pick A_λ and f as in Theorem 5.6 and define

$$n_\lambda = \sup_{m \in A_\lambda} f(m)$$

We have

$$K \otimes_k V_{\lambda_1} \otimes_k ... \otimes_k V_{\lambda_n} = \bigoplus_{(m_1,...,m_n) \in A_{\lambda_1} \times ... \times A_{\lambda_n}} K \cdot \chi^{m_1 + ... + m_n}$$

and since $m_1, ... m_n \in A_{\lambda_1} \cup ... \cup A_{\lambda_n} \subseteq S \smallsetminus \{0\}$ we derive that

$$f(m_1 + ... + m_n) = f(m_1) + ... + f(m_n) \geq n > n_\lambda = \sup_{m \in A_\lambda} f(m)$$

This implies that V_λ is not an isotypic component of $V_{\lambda_1} \otimes_k ... \otimes_k V_{\lambda_n}$. □

Lemma 7.1.2. *Fix λ in $\mathbf{Irr}(\overline{T})$. Then $\mathcal{A}_{n+1}[\lambda] \twoheadrightarrow \mathcal{A}_n[\lambda]$ is an isomorphism for $n \geq n_\lambda$.*

Proof of the lemma. Since $\mathcal{A}_n^{\overline{T}} = \mathcal{A}_0$ and $\overline{T}$ is linearly reductive monoid, we derive that $\mathcal{I}_n[\lambda] = 0$ for $\lambda \notin \mathbf{Irr}(\overline{T}) \smallsetminus \{\lambda_0\}$. Fix $\lambda \in \mathbf{Irr}(\overline{T})$. By Lemma 7.1.1 we derive that

$$\left(\underbrace{\mathcal{I}_{n+1} \otimes_k \mathcal{I}_{n+1} \otimes_k ... \otimes_k \mathcal{I}_{n+1}}_{n+1 \text{ times}}\right)[\lambda] = 0$$

for $n \geq n_\lambda$. Note also that the image of the composition

$$\underbrace{\mathcal{I}_{n+1} \otimes_k \mathcal{I}_{n+1} \otimes_k ... \otimes_k \mathcal{I}_{n+1}}_{n \text{ times}} \twoheadrightarrow \underbrace{\mathcal{I}_{n+1} \otimes_{\mathcal{O}_{Z_0}} \mathcal{I}_{n+1} \otimes_{\mathcal{O}_{Z_0}} ... \otimes_{\mathcal{O}_{Z_0}} \mathcal{I}_{n+1}}_{n \text{ times}} \longrightarrow \mathcal{A}_{n+1}$$

is $\mathcal{I}_{n+1}^{n+1}$. Since the composition above is a morphism of sheaves with $\overline{T}$-linearization, we derive that $\mathcal{I}_{n+1}^{n+1}[\lambda] = 0$ for $n \geq n_\lambda$. Hence the kernel of $\mathcal{A}_{n+1}[\lambda] \twoheadrightarrow \mathcal{A}_n[\lambda]$ is trivial. □

Proof of Theorem. According to Corollary 6.1 and the fact that T is central in $\mathbf{M}$ we derive that $\mathcal{A}_n[\lambda]$ is a quasi-coherent sheaf with $\mathbf{M}$-linearization. For $\lambda \in \mathbf{Irr}(\overline{T})$ we define

$$\mathcal{A}[\lambda] = \mathcal{A}_n[\lambda]$$

where $n \geq n_\lambda$ as in Lemma 7.1.2. We set

$$\mathcal{A} = \bigoplus_{\lambda \in \mathbf{Irr}(\overline{T})} \mathcal{A}[\lambda]$$

Clearly $\mathcal{A}[\lambda_0] = \mathcal{A}_0 = \mathcal{O}_{Z_0}$ canonically (where λ_0 is the trivial T-representation), hence $\mathcal{A}$ is a quasi-coherent sheaf on Z_0 with $\mathbf{M}$-linearization. Actually $\mathcal{A} = \lim_{n \in \mathbb{N}} \mathcal{A}_n$ in the category of quasi-coherent sheaves with $\mathbf{M}$-linearization on Z_0. We construct the $\mathcal{O}_{Z_0}$-algebra structure on $\mathcal{A}$. For this pick $\lambda_1, \lambda_2 \in \mathbf{Irr}(\overline{T})$. Consider the irreducible representations V_{λ_1} and V_{λ_1} in classes λ_1 and λ_2, respectively. Suppose that $\eta_1, ..., \eta_s$ are finitely many classes in $\mathbf{Irr}(\overline{T})$ such that $V_{\lambda_1} \otimes_k V_{\lambda_2}$ can be completely decomposed onto irreducible representation in these classes. Since the image of the multiplication $\mathcal{A}_n[\lambda_1] \otimes_{\mathcal{O}_{Z_0}} \mathcal{A}_n[\lambda_2] \to \mathcal{A}_n$ on $\mathcal{A}_n$ is also the image of a morphism

$$\mathcal{A}_n[\lambda_1] \otimes_k \mathcal{A}_n[\lambda_2] \twoheadrightarrow \mathcal{A}_n[\lambda_1] \otimes_{\mathcal{O}_{Z_0}} \mathcal{A}_n[\lambda_2] \longrightarrow \mathcal{A}_n$$

we deduce that it is contained in $\bigoplus_{i=1}^{s} \mathcal{A}_n[\eta_i]$. By Lemma 7.1.2 all these multiplications for $n \geq \sup\{n_{\lambda_1}, n_{\lambda_2}, n_{\eta_1}, ..., n_{\eta_s}\}$ can be identified. Now we define

$$\mathcal{A}[\lambda_1] \otimes_{\mathcal{O}_{Z_0}} \mathcal{A}[\lambda_2] \to \bigoplus_{i=1}^{s} \mathcal{A}[\eta_i] \subseteq \mathcal{A}$$

as a morphism induced by the multiplication morphism for any $n \geq \sup\{n_{\lambda_1}, n_{\lambda_2}, n_{\eta_1}, ..., n_{\eta_s}\}$. This gives an $\mathcal{O}_{Z_0}$-algebra structure on $\mathcal{A}$, so $\mathcal{A}$ is in fact the limit of $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$ is the category of quasi-coherent algebras with **M**-linearization. Note that from the description of $\mathcal{A}$ it follows that for every $n \in \mathbb{N}$ we have a surjective morphism $p_n : \mathcal{A} \twoheadrightarrow \mathcal{A}_n$ of algebras. We denote its kernel by $\mathcal{J}_n$ and we put $\mathcal{J} = \mathcal{J}_0$. The natural injection $\mathcal{O}_{Z_0} = \mathcal{A}_0 \to \mathcal{A}$ is a section of p_0, so that we have

$$\mathcal{J} = \bigoplus_{\lambda \in \mathbf{Irr}(\overline{T}) \setminus \{\lambda_0\}} \mathcal{A}[\lambda]$$

Recall that we denote by $\mathcal{I}_n$ the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$ for $n \in \mathbb{N}$. Then $\mathcal{I}_n = \mathcal{J}/\mathcal{J}_n$. Fix $m \in \mathbb{N}$ and consider $n \in \mathbb{N}$ such that $n \geq m$. Since $\mathcal{Z}$ is a formal **M**-scheme, the sheaf $\mathcal{I}_n^{m+1}$ is the kernel of the morphism $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_m$. Thus

$$\mathcal{J}_m/\mathcal{J}_n = \mathcal{I}_n^{m+1} = (\mathcal{J}^{m+1} + \mathcal{J}_n)/\mathcal{J}_n$$

Both $\mathcal{J}_m$ and $\mathcal{J}^{m+1}$ are $\mathbf{Irr}(\overline{T})$-graded and for given $\lambda \in \mathbf{Irr}(\overline{T})$ and for $n \geq n_\lambda$ the isotypic component $\mathcal{J}_n[\lambda]$ is zero by Lemma 7.1.2. Hence $\mathcal{J}_m = \mathcal{J}^{m+1}$ for every $m \in \mathbb{N}$. We define

$$Z = \mathrm{Spec}_{\,Z_0}\mathcal{A}$$

and we denote by $\pi : Z \to Z_0$ the structural morphism. The scheme Z inherits a **M**-action from $\mathcal{A}$. For every $n \in \mathbb{N}$ the zero-set of $\mathcal{J}^{n+1}$ in $\mathcal{A}$ is a **M**-scheme isomorphic to $Z_n = \mathrm{Spec}_{\,Z_0}\mathcal{A}_n$. Hence $\mathcal{Z}$ is isomorphic to $\widehat{Z}$ and this proves the theorem.

□

Theorem 7.2. *Let* **M** *be a Kempf monoid and let* Z *be a locally linear* **M***-scheme. Suppose that* $\pi : Z \to Z^{\mathbf{M}}$ *is the canonical retraction. If the formal* **M***-scheme* $\widehat{Z}$ *is locally noetherian, then* $\pi : Z \to Z^{\mathbf{M}}$ *is of finite type.*

Proof. Since π is affine (Proposition 4.2), we derive that $\mathcal{A} = \pi_*\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-algebra with **M**-linearization. We denote by $\mathcal{J}$ the ideal of $\mathcal{A}$ that corresponds to the closed immersion $Z^{\mathbf{M}} \hookrightarrow Z$. We know that the formal **M**-scheme

$$Z^{\mathbf{M}} = \mathrm{Spec}_{\,Z^{\mathbf{M}}}\mathcal{A}/\mathcal{J} \hookrightarrow ... \hookrightarrow \mathrm{Spec}_{\,Z^{\mathbf{M}}}\mathcal{A}/\mathcal{J}^{n+1} \hookrightarrow \mathrm{Spec}_{\,Z^{\mathbf{M}}}\mathcal{A}/\mathcal{J}^{n+2} \hookrightarrow ...$$

is locally noetherian. Hence $\mathcal{J}/\mathcal{J}^{n+1}$ is $\mathcal{A}/\mathcal{J}^{n+1}$-module of finite type. Thus $\{\mathcal{J}^i/\mathcal{J}^{i+1}\}_{1 \leq i \leq n}$ are finite type $\mathcal{A}/\mathcal{J}$-modules. The series

$$0 \subseteq \mathcal{J}^n/\mathcal{J}^{n+1} \subseteq ... \subseteq \mathcal{J}/\mathcal{J}^{n+1} \subseteq \mathcal{A}/\mathcal{J}^{n+1}$$

has subquotients that are of finite type over $\mathcal{O}_{Z^{\mathbf{M}}} = \mathcal{A}/\mathcal{J}$. This implies that $\mathcal{A}/\mathcal{J}^{n+1}$ is a coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-algebra for every $n \in \mathbb{N}$. The claim that π is of finite type is local on $Z^{\mathbf{M}}$, hence we may assume that $Z^{\mathbf{M}}$ is quasi-compact. This reduces the question to the noetherian $Z^{\mathbf{M}}$. The sheaf $\mathcal{J}/\mathcal{J}^2 \subseteq \mathcal{A}/\mathcal{J}$ is coherent over $\mathcal{O}_{Z^{\mathbf{M}}}$. Since $Z^{\mathbf{M}}$ is noetherian, there exists coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-subsheaf $\mathcal{M} \subseteq \mathcal{J}$ such that the morphism $\mathcal{M} \twoheadrightarrow \mathcal{J}/\mathcal{J}^2$ is surjective. Fix an algebraically closed K extension of k and denote

$$\mathcal{A}_K = K \otimes_k \mathcal{A}, \mathcal{J}_K = K \otimes_k \mathcal{J}, \mathcal{M}_K = K \otimes_k \mathcal{M}$$

Since **M** is a Kempf torus and by **(3)** Theorem 5.6 there exists a closed immersion $\mathbb{A}^1_K \hookrightarrow \mathbf{M}_K$ of monoid K-schemes that preserve zero. This implies that we have $\mathbb{N}$-grading $\mathcal{A}_K = \bigoplus_{i\geq 0}\mathcal{A}_K[i]$ that gives rise to the action of $\mathbb{A}^1_K$. Moreover, by Propostion 4.2 we deduce that

$$\mathrm{Spec}\,K \times Z^{\mathbf{M}} = (\mathrm{Spec}\,K \times Z)^{\mathbf{M}_K} = (\mathrm{Spec}\,K \times Z)^{\mathbb{A}^1_K}$$

as K-schemes. This shows that $\mathcal{J}_K = K \otimes_k \mathcal{J} = \bigoplus_{i\geq 1}\mathcal{A}_K[i]$ is an ideal with positive grading. We have surjection $\mathcal{M}_K \twoheadrightarrow \mathcal{J}_K/\mathcal{J}_K^2$. By graded Nakayama's lemma, the ideal $\mathcal{J}_K$ is generated by $\mathcal{M}_K$. Then by induction on degrees we deduce that $\mathcal{A}_K$ is generated by $\mathcal{M}_K$ as a $K\otimes_k \mathcal{O}_{Z^{\mathbf{M}}}$-algebra. Thus $1_{\mathrm{Spec}\,K}\times \pi$ is of finite type and by faitfully flat descent also π is of finite type. □

Theorem 7.3. *Let* **M** *be a Kempf monoid with group of unit* **G** *and let* Z *be a locally linear* **M***-scheme. Suppose that* $\pi : Z \to Z^{\mathbf{M}}$ *is the canonical retraction. If* Z *is locally noetherian, then the comparison functor*

$$\mathfrak{Coh}_{\mathbf{G}}(Z) \to \mathfrak{Coh}_{\mathbf{G}}(\widehat{Z})$$

is an equivalence of monoidal categories.

Proof. Since **M** is a Kempf torus, there exists the central torus T closed subgroup k-scheme in **G** such that the scheme-theoretic closure $\overline{T}$ of T in **M** contains the zero of **M**. As above we note that π is affine (Proposition 4.2) and we pick a quasi-coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-algebra $\mathcal{A} = \pi_*\mathcal{O}_Z$ with **M**-linearization. We denote by $\mathcal{J}$ the ideal of $\mathcal{A}$ that corresponds to the closed immersion $Z^{\mathbf{M}} \hookrightarrow Z$. Then $\mathcal{O}_{Z^{\mathbf{M}}} = \mathcal{A}/\mathcal{J}$ and since π is a retraction, we derive that $\mathcal{A} = \mathcal{O}_{Z^{\mathbf{M}}} \oplus \mathcal{J}$. Next $\widehat{Z}$ is locally noetherian (this follows from the fact that Z is locally noetherian). Hence an object of $\mathfrak{Coh}_{\mathbf{G}}(\widehat{Z})$ corresponds to a sequence of surjections

$$\ldots \twoheadrightarrow \mathcal{M}_{n+1} \twoheadrightarrow \mathcal{M}_n \twoheadrightarrow \ldots \twoheadrightarrow \mathcal{M}_1 \twoheadrightarrow \mathcal{M}_0$$

of coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-modules such that the following assertions hold.

(1) For each $n\in\mathbb{N}$ sheaf $\mathcal{M}_n$ is a module over $\mathcal{A}/\mathcal{J}^{n+1}$.

(2) For each $n\in\mathbb{N}$ the kernel of surjection $\mathcal{M}_{n+1}\twoheadrightarrow\mathcal{M}_n$ is $\mathcal{J}\mathcal{M}_{n+1}$.

Out goal is to show that there are only finitely many λ in $\mathbf{Irr}(T)$ such that $\mathcal{M}_n[\lambda]\neq 0$ for all $n\in\mathbb{N}$ and moreover, that for each such λ the sequence

$$\ldots \twoheadrightarrow \mathcal{M}_{n+1}[\lambda] \twoheadrightarrow \mathcal{M}_n[\lambda] \twoheadrightarrow \ldots \twoheadrightarrow \mathcal{M}_1[\lambda] \twoheadrightarrow \mathcal{M}_0[\lambda]$$

We also fix an algebraically closed field K containing k. By **(3)** of Theorem 5.6 there exists a closed immersion $\mathbb{A}^1_K \hookrightarrow \overline{T}_K$ such that □

LS TODO: Tutaj szanowni państwo kończy się uporządkowany świat i zaczyna się...

8. Nibylandia

Now a **G**-coherent sheaf on $\mathcal{Z}$ corresponds to a sequence

such that $\mathcal{M}_n$ is a coherent sheaf on $\mathcal{A}_0$ with **G**-action and the kernel of $\mathcal{M}_{n+1}\twoheadrightarrow\mathcal{M}_n$ is $\mathcal{I}_{n+1,n}\cdot\mathcal{M}_{n+1}$ for every $n\in\mathbb{N}$.

8.1. Kempf monoids.

LS TODO: Tutaj trzeba zdefiniować monoidy Kempfa. Najpierw trzeba porządnie spisać dowód algebraizacji, żeby mieć poprawną definicję

Definition 8.1. Let $\mathbf{M}$ be a monoid k-scheme. Suppose that the following conditions hold.

(1) $\mathbf{M}$ is affine, geometrically connected and geometrically normal.

(2) There exists zero $\mathbf{o}$ in $\mathbf{M}$.

(3) There exists a torus T over k contained in the center of $\mathbf{M}$ such that the closure $\mathbf{cl}(T)$ of T in $\mathbf{M}$ contains $\mathbf{o}$.

Then $\mathbf{M}$ is called *Kempf monoid*.

LS TODO: (or rather Jelisiejew :D)

Let $\mathbf{M}$ be a Kempf monoid and let $\mathbf{G}$ be its group of units. If V is a representation of $\mathbf{G}$ and λ is a class in Λ, then we denote by $V[\lambda] \subseteq V$ the sum of all irreducible T-subpresentations of V of isomorphism type λ. Since T is a central subgroup of $\mathbf{G}$, we derive by Proposition **??** that $V[\lambda]$ is a $\mathbf{G}$-representation of V.
Suppose that Z is a k-scheme with trivial action of $\mathbf{M}$. If $\mathcal{F}$ is a quasi-coherent sheaf on Z equipped with $\mathbf{G}$-action, then we denote by $\mathcal{F}[\lambda]$ a sheaf given by

$$U \mapsto \mathcal{F}(U)[\lambda]$$

for every open affine subset U of Z. Then $\mathcal{F}[\lambda] \subseteq \mathcal{F}$ is a $\mathbf{G}$-quasi-coherent subsheaf of $\mathcal{F}$.

9. M-EQUIVARIANT QUASI-COHERENT SHEAVES

Definition 9.1. Let $\mathbf{M}$ be a monoid k-scheme and X be a k-scheme together with $\mathbf{M}$-action $a : \mathbf{M} \times X \to X$. Fix a quasi-coherent sheaf $\mathcal{F}$ on X. A $\mathbf{M}$-linearization of $\mathcal{F}$ is an isomorphism $\phi : \mathbf{pr}_X^*\mathcal{F} \to a^*\mathcal{F}$ such that, the following condition holds:

$$(\mu \times_S 1_X)^*\phi = (1_G \times_S a)^*\phi\, \mathbf{pr}^*_{(G\times_S X)}\phi$$

where $\mu : G \times_S G \to G$ is the multiplication and $\mathbf{pr}_{(G\times_S X)} : G \times_S (G \times_S X) \to G \times_S X$ is a projection. A pair $(\mathcal{F}, \phi)$ is called a G-sheaf.
Suppose that $(\mathcal{F}_1, \phi_1)$ and $(\mathcal{F}_2, \phi_2)$ are G-sheaves. A morphism $f : (\mathcal{F}_1, \phi_1) \to (\mathcal{F}_2, \phi_2)$ is a morphism $f : \mathcal{F}_1 \to \mathcal{F}_2$ of $\mathcal{O}_X$-modules that commutes with G-linearizations. Thus we have a category $\mathbf{Mod}_G(\mathcal{O}_X)$ of G-sheaves.

Proposition 9.2 (compatibility with identity)**.** *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules and $\phi : \mathbf{pr}_X^*\mathcal{F} \to a^*\mathcal{F}$ be a G-linearization. Assume that $e : S \to G$ is an identity of G. Let $i_e : X \to G \times_S X$ be a morphism induced by e and 1_X. Then $i_e^*\phi = 1_{\mathcal{F}}$.*

Proof. Let $j : X \to G \times_S G \times_S X$ be a morphism induced by e and 1_X. Observe that:

$$i_e^*\phi = j^*(\mu \times_S 1_X)^*\phi = j^*((1_G \times_S a)^*\phi\mathbf{pr}^*_{(G\times_S X)}\phi) = i_e^*\phi i_e^*\phi$$

Thus $i_e^*\phi = 1_{\mathcal{F}}$. □

Proposition 9.3 (canonical linearization of the structure sheaf)**.** *Let $\mathbf{pr}_{X\#} : \mathbf{pr}_X^*\mathcal{O}_X \to \mathcal{O}_{G\times_S X}$, $a_\# : a^*\mathcal{O}_X \to \mathcal{O}_{G\times_S X}$ be adjoints of sheaf parts $\mathbf{pr}_X^\# : \mathcal{O}_X \to \mathbf{pr}_{X*}\mathcal{O}_{G\times_S X}$, $a^\# : \mathcal{O}_X \to a_*\mathcal{O}_{G\times_S X}$ of morphisms $\mathbf{pr}_X$ and a respectively. Then $\phi_{\mathcal{O}_X} = a_\#^{-1}\mathbf{pr}_{X\#}$ is a G-linearization of the structure sheaf $\mathcal{O}_X$.*

Proof. First we introduce convienient notation. If $(f, f^\#) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ is a morphism of ringed spaces, then we denote $f_\# : f^*\mathcal{O}_Z \to \mathcal{O}_Y$ adjoint of $f^\# : \mathcal{O}_Z \to f_*\mathcal{O}_Y$. This is consistent with notation from the statement. Note that $f_\#$ is always an isomorphism.
We have:

$$(1_G \times_S a)^*(\mathbf{pr}_{X\#})\mathbf{pr}^*_{(G\times_S X)}(a_\#)^{-1} =$$
$$= (1_G \times_S a)^{-1}_\#(1_G \times_S a)_\#(1_G \times_S a)^*(\mathbf{pr}_{X\#})(\mathbf{pr}_{(G\times_S X)_\#}^{-1}\mathbf{pr}_{(G\times_S X)_\#}\mathbf{pr}^*_{(G\times_S X)}(a_\#))^{-1} =$$
$$= (1_G \times_S a)^{-1}_\#(\mathbf{pr}_X(1_G \times_S a))_\#(a\mathbf{pr}_{(G\times_S X)})^{-1}_\#\mathbf{pr}_{(G\times_S X)_\#} = (1_G \times_S a)^{-1}_\#\mathbf{pr}_{(G\times_S X)_\#}$$

Hence:

$$(1_G \times_S a)^*(a^{-1}_\#\mathbf{pr}_{X\#})\mathbf{pr}^*_{(G\times_S X)}(a^{-1}_\#\mathbf{pr}_{X\#}) =$$
$$= (1_G \times_S a)^*(a^{-1}_\#)(1_G \times_S a)^{-1}_\#\mathbf{pr}_{(G\times_S X)_\#}\mathbf{pr}^*_{(G\times_S X)}(\mathbf{pr}_{X\#}) = (a(1_G \times_S a))^{-1}_\#(\mathbf{pr}_X\mathbf{pr}_{(G\times_S X)})_\# =$$
$$= (a(\mu \times_S 1_X))^{-1}_\#(\mathbf{pr}_X(\mu \times_S 1_X))_\# = (\mu \times_S 1_X)^*(a_\#)^{-1}(\mu \times_S 1_X)^{-1}_\#(\mu \times_S 1_X)_\#(\mu \times_S 1_X)^*(\mathbf{pr}_{X\#}) =$$
$$= (\mu \times_S 1_X)^*(a_\#)^{-1}(\mu \times_S 1_X)^*(\mathbf{pr}_{X\#}) = (\mu \times_S 1_X)^*(a^{-1}_\#\mathbf{pr}_{X\#})$$

This means that $(\mu \times_S 1_X)^*\phi_{\mathcal{O}_X} = (1_G \times a)^*\phi_{\mathcal{O}_X}\mathbf{pr}^*_{(G\times_S X)}\phi_{\mathcal{O}_X}$ □

Proposition 9.4. *The forgetful functor $\mathbf{Mod}_G(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_X)$ creates finite products and colimits. Moreover, if $\mathbf{pr}^*_X$ and a^* are exact, then the functor above creates all finite limits.*

Proof. The first assertion is a straighforward consequence of the fact that $\mathbf{pr}^*_X$ and a^* preserve finite products and colimits. The second assertion follows by the same argument using additional assumptions. □

Corollary 9.5. *Suppose that $\mathbf{pr}^*_X$ and a^* are exact. Then $\mathbf{Mod}_G(\mathcal{O}_X)$ is an* **Ab**5*-category.*

Proof. The result is a consequence of the general rule. If a category admits a functor to some **Ab**5-category and this functor creates colimits and finite limits, then the category itself is an **Ab**5-category. □

Proposition 9.6. *Let $f : X \to Y$ be a G-equivariant morphism of locally ringed spaces over S admiting action of some S-group G. For a G-sheaf $\mathcal{G}$ of $\mathcal{O}_Y$-modules together with a linearization ϕ there exists a linearization of $f^*\mathcal{G}$ given by $(1_G \times f)^*\phi$. This gives a lift $f^* : \mathbf{Mod}_G(\mathcal{O}_Y) \to \mathbf{Mod}_G(\mathcal{O}_X)$ of a pullback $f^* : \mathbf{Mod}(\mathcal{O}_Y) \to \mathbf{Mod}(\mathcal{O}_X)$.*

Proof. Let $b : G \times_S Y \to Y$ denote an action of G on Y.
Observe that:

$$(1_G \times_S a)^*((1_G \times_S f)^*\phi))\mathbf{pr}^*_{(G\times_S X)}((1_G \times_S f)^*\phi) = (1_G \times_S fa)^*\phi(\mathbf{pr}_{(G\times_S Y)}(1_{(G\times_S G)} \times_S f))^*\phi =$$
$$= (1_{(G\times_S G)} \times_S f)^*((1_G \times_S b)^*\phi\mathbf{pr}^*_{(G\times_S Y)}\phi) = (1_{(G\times_S G)} \times_S f)^*(\mu \times_S 1_Y)^*\phi = (\mu \times_S 1_X)^*(1_G \times_S f)^*\phi$$

Thus indeed $(1_G \times_S f)^*\phi$ is a G-linearization on $f^*\mathcal{F}$. □

Remark 9.7. Exactly the same results are derived in the category of complex analytic spaces, smooth manifolds and (not necessarly locally) ringed topological spaces. Definitions and proofs are the same after replacing fiber product over S by product in the corresponding category.

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