HAAR MEASURE

1. Introduction

In this notes we introduce Haar measure, which is a fundamental technical tool in representation theory of locally compact topological groups. We send the interested reader to [DSS14] for excellent exposition of aspects and applications of this notion beyond our rudimentary presentation.

2. Existence of Haar measure

Definition 2.1. Let G be a topological group and let μ be a Borel measure. Then μ is *left-invariant* if $\mu(xA) = \mu(A)$ for every A in $\mathcal{B}(G)$. Similarly μ is right-invariant if $\mu(Ax) = \mu(A)$ for every A in $\mathcal{B}(G)$.

Definition 2.2. Let G be a locally compact group and μ be a Borel measure. If μ is a nonzero, left-invariant, regular Borel measure on G, then we say that μ is *a left Haar measure on* G. Similarly if μ is a nonzero, right-invariant, regular Borel measure on G, then we say that μ is *a right Haar measure on* G

Theorem 2.3. *Let G be a locally compact topological group. Then there exists a left (right) Haar measure μ on G. If in addition G is σ-compact, then μ is inner regular.*

We denote by $\mathcal{K}$ the set of all compact subsets of G and by $\mathcal{U}$ the set of all open neighborhoods of identity in G. Let U be an open nonempty subset of G and K be a compact subset of G. We define

$$(K : U) = \inf\left\{n \in \mathbb{N} \,\middle|\, \text{there exist } x_1, ..., x_n \in G \text{ such that } K \subseteq \bigcup_{i=1}^{n} x_i U\right\}$$

Throughout the proof we fix a compact subset Q of G such that $\mathbf{int}(Q) \neq \varnothing$.

Lemma 2.3.1. *Fix $U \in \mathcal{U}$. There exists a real valued function h_U on $\mathcal{K}$ such that the following assertions hold.*

(1) *For every compact subset K in $\mathcal{K}$ we have $h_U(K) \geq 0$, $h_U(\varnothing) = 0$ and $h_U(Q) = 1$.*

(2) *For every compact subset K in $\mathcal{K}$ and for every element x in G we have $h_U(xK) = h_U(K)$.*

(3) *If $K \subseteq L$ are compact subsets in $\mathcal{K}$, then $h_U(K) \subseteq h_U(L)$.*

(4) *For every compact subset K in $\mathcal{K}$ we have $h_U(K) \leq (K : \mathbf{int}(Q))$.*

(5) *If K, L are compact subsets in $\mathcal{K}$, then*
$$h_U(K \cup L) \leq h_U(K) + h_U(L)$$
and if $K \cdot U^{-1} \cap L \cdot U^{-1} = \varnothing$, then the equality holds.

Proof of the lemma. For every compact subset K of G we define

$$h_U(K) = \frac{(K : U)}{(Q : U)}$$

Now we check that h_U admits the properties above. Properties **(1)**, **(2)** and **(3)** are clear. For **(4)** note that

$$(K : U) \leq (Q : U) \cdot (K : \mathbf{int}(Q))$$

Indeed, if $K \subseteq \bigcup_{i=1}^{n} y_i \cdot \mathbf{int}(Q)$ and $Q \subseteq \bigcup_{j=1}^{m} z_j U$, then $K \subseteq \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} y_i z_j U$ and this implies the inequality above. Observe that $xU \cap K \neq \varnothing$ implies that $x \in K \cdot U^{-1}$ and similarly $xU \cap L \neq \varnothing$

implies that $x \in L \cdot U^{-1}$. Assuming that for compact subsets K, L in G we have $K \cdot U^{-1} \cap L \cdot U^{-1} = \varnothing$ we derive from this that for every $x \in G$ we have $xU \cap (K \cap L) = \varnothing$. Thus if $K \cdot U^{-1} \cap L \cdot U^{-1} = \varnothing$, then we have $(K \cup L : U) = (K : U) + (K : L)$ and hence $h_U(K \cup L) = h_U(K) + h_U(L)$. Note that in general case we have $(K \cup L : U) \leq (K : U) + (K : L)$ and hence also **(5)** holds for h_U. □

Lemma 2.3.2. *Let K, L in $\mathcal{K}$ and suppose that $K \cap L = \varnothing$. Then there exists $U \in \mathcal{U}$ such that*

$$K \cdot U^{-1} \cap L \cdot U^{-1} = \varnothing$$

Proof of the lemma. Left as an exercise. □

Lemma 2.3.3. *There exists a real valued function h on $\mathcal{K}$ such that the following assertions hold.*

(1) *For every compact subset K in $\mathcal{K}$ we have $h(K) \geq 0$, $h(\varnothing) = 0$ and $h(Q) = 1$.*

(2) *For every compact subset K in $\mathcal{K}$ and for every element x in G we have $h(xK) = h(K)$.*

(3) *If $K \subseteq L$ are compact subsets in $\mathcal{K}$, then $h(K) \subseteq h(L)$.*

(4) *For every compact subset K in $\mathcal{K}$ we have $h(K) \leq (K : \mathbf{int}(Q))$.*

(5) *If K, L are compact subsets in $\mathcal{K}$, then*

$$h(K \cup L) \leq h(K) + h(L)$$

and if $K \cap L = \varnothing$, then the equality holds.

Proof of the lemma. Consider a topological space

$$X = \prod_{K \in \mathcal{K}} \big[0, (K : \mathbf{int}(Q)\big]$$

By Tichonoff's theorem X is compact. For every $U \in \mathcal{U}$ we define a subset $F_U \subseteq X$ that consists of tuples $\{a_K\}_{K \in \mathcal{K}}$ such that $a_{\varnothing} = 0$, $a_Q = 1$, $a_{xK} = a_K$ for $x \in G$ and K in $\mathcal{K}$, $a_K \leq a_L$ for $K \subseteq L$ in $\mathcal{K}$, $a_{K \cup L} \leq a_K + a_L$ for K, L in $\mathcal{K}$ and the equality holds if $K \cdot U^{-1} \cap L \cdot U^{-1} = \varnothing$. Conditions imposed on tuples in F_U imply that F_U is a closed subset. Note that $\{h_U(K)\}_{K \in \mathcal{K}} \in F_U$ for every $U \in \mathcal{U}$. Moreover, we have

$$F_{U_1 \cap U_2 \cap \ldots \cap U_n} \subseteq F_{U_1} \cap F_{U_2} \cap \ldots \cap F_{U_n}$$

for $U_1, U_2, \ldots, U_n \in \mathcal{U}$. This implies that $\big\{F_U\big\}_{U \in \mathcal{U}}$ is a centered family of nonempty closed subsets of a compact space X. Thus

$$\bigcap_{U \in \mathcal{U}} F_U \neq \varnothing$$

by compactness of X. Hence there exists $\{c_K\}_{K \in \mathcal{K}}$ in the intersection. We define a real function h on $\mathcal{K}$ by $h(K) = c_K$ for K in $\mathcal{K}$. The fact that properties **(1)**, **(2)**, **(3)** and **(4)** hold for h follows by definition of F_U for $U \in \mathcal{U}$. Since $\{c_K\}_{K \in \mathcal{K}}$ is an element in F_U for every $U \in \mathcal{U}$ we derive that

$$c_{K \cup L} \leq c_K + c_L$$

for K, L in $\mathcal{K}$. This implies $h(K \cup L) \leq h(K) + h(L)$ for $K, L \in \mathcal{K}$. Moreover, $c_{K \cup L} = c_K + c_L$ if $K \cdot U^{-1} \cap L \cdot U^{-1} = \varnothing$ for some $U \in \mathcal{U}$. This implies that $c_{K \cup L} = c_K + c_L$ if $K \cap L = \varnothing$ by Lemma 2.3.2. Thus h admits **(4)**. □

Proof of the theorem. We fix h as in Lemma 2.3.3 and we define $\mu^* : \mathcal{P}(G) \to [0, +\infty]$. First if U is an open subset of G, then we define

$$\mu^*(U) = \sup_{K \in \mathcal{K}, K \subseteq U} h(K)$$

Note that if U, V are open subsets of G and $U \subseteq V$, then $\mu^*(U) \leq \mu^*(V)$. Thus it makes sense to define

$$\mu^*(A) = \inf\big\{\mu^*(U) \,\big|\, U \text{ is an open subset of } G \text{ containing } A\big\}$$

for arbitrary subset $A \subseteq G$. Note that $\mu^*(xA) = \mu^*(A)$ by definition of μ^* and the corresponding property of h. By [Mon18a, Theorem 1.3] we have that Borel sets $\mathcal{B}(G)$ are μ^*-measurable,

$\mu^*_{|\mathcal{B}(G)} = \mu$ is a regular Borel measure on G. According to this result if G is σ-compact, then μ is inner regular. Clearly μ is left-invariant and since

$$1 = h(Q) \leq \mu(Q)$$

we derive that it is nonzero measure. $\square$

3. Uniqueness of Haar measure

Theorem 3.1. *Let G be a locally compact group. If μ_1 and μ_2 are left (right) Haar measures on G, then there exists positive constant $a \in \mathbb{R}$ such that*

$$\mu_1 = a \cdot \mu_2$$

For the proof we need the following result.

Lemma 3.1.1. *Let G be a locally compact group. Then there exists a σ-compact, open subgroup H of G.*

Proof of the lemma. Let U be an open neighborhood of identity in G such that $\mathbf{cl}(U)$ is compact. Consider $V = U \cap U^{-1}$. Then V is open neighborhood of identity in G such that $V = V^{-1}$ and $\mathbf{cl}(V)$ is compact. We define $H = \bigcup_{n \in \mathbb{N}} V^n$. Then H is an open subgroup of G. We have

$$H = G \setminus \left(\bigcup_{g \in G \setminus H} gH \right)$$

and hence H is also a closed subgroup of G. Moreover, for every $n \in \mathbb{N}$ set $\mathbf{cl}\left(V^n\right)$ is compact in G. Since

$$H = \bigcup_{n \in \mathbb{N}} \left(H \cap \mathbf{cl}\left(V^n\right)\right)$$

we derive that H is σ-compact. $\square$

Proof of the theorem. By Lemma 3.1.1 there exists an open subgroup H of G that is σ-compact. We prove now that there exists $a \in \mathbb{R}$ such that

$$\mu_{1|\mathcal{B}(H)} = a \cdot \mu_{2|\mathcal{B}(H)}$$

For this consider $\mu = \mu_{1|\mathcal{B}(H)} + \mu_{2|\mathcal{B}(H)}$ and denote $\mu_{2|\mathcal{B}(H)}$ by ν. Measures μ, ν are σ-finite as they are finite on compact subsets of H and H is σ-compact space. Moreover, $\nu \ll \mu$ and hence by [Mon18b, Theorem 5.3] there exists a Borel function $f : H \to \mathbb{C}$ such that

$$\nu(A) = \int_A f d\mu$$

for every Borel subset A in H. Since μ and ν are nonnegative measures, we derive that f is real and nonnegative μ-almost everywhere. Hence we may assume that f takes only nonnegative real values. We define

$$E = \left\{(x, y) \in H \times H \,\middle|\, f(xy) - f(y) \neq 0\right\}$$

Next as ν, μ are left-invariant, we deduce that

$$0 = \nu\big(l_x(A)\big) - \nu(A) = \int_{l_x(A)} f d\mu - \int_A f d\mu = \int_A (f \cdot l_x - f)\, d\mu$$

for every $x \in H$ and $A \in \mathcal{B}(H)$, where $l_x : H \to H$ is a continuous map given by left multiplication by x. This implies that for all $x \in H$ we have

$$\mu(E_x) = 0$$

By [Mon19, Theorem 7.5] applied to measure $\mu \otimes \mu$ on $H \times H$, we deduce that there exists $y \in H$ such that the set

$$E_y = \left\{x \in H \,\middle|\, f(xy) - f(y) \neq 0\right\}$$

has measure μ zero. This implies that f is constant almost everywhere with respect to μ and thus there exists nonzero $b \in \mathbb{R}$ such that $\nu = b \cdot \mu$. Hence we have

$$\mu_{1|\mathcal{B}(H)} = a \cdot \mu_{2|\mathcal{B}(H)}$$

for $a = (1-b)b^{-1}$. Let K be a compact subset of G. Since H is an open subgroup of G, there exists $x_1, ..., x_n \in G$ such that

$$K \subseteq x_1 H \cup ... \cup x_n H$$

and the sum is disjoint. Therefore, we have

$$\mu_1(K) = \sum_{i=1}^{n} \mu_1 (K \cap x_i H) = \sum_{i=1}^{n} \mu_1 \left(x_i^{-1} K \cap H \right) = a \cdot \sum_{i=1}^{n} \mu_2 \left(x_i^{-1} K \cap H \right) = a \cdot \sum_{i=1}^{n} \mu_2 (K \cap x_i H) = a \cdot \mu_2(K)$$

This implies that $\mu_1 = a \cdot \mu_2$ because μ_1, μ_2 are regular Borel measures. □

4. Modular function and invariance of Haar measure on compact groups

References

[DSS14] J. Diestel, A. Spalsbury, and American Mathematical Society. *The Joys of Haar Measure*. Graduate Studies in Mathematics. American Mathematical Society, 2014.

[Mon18a] Monygham. Measures on locally compact spaces. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2018.

[Mon18b] Monygham. Radon-nikodym theorem, hahn-jordan decomposition and lebesgue decomposition. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2018.

[Mon19] Monygham. Integration. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2019.