UNIFORM SPACES

1. INTRODUCTION

These notes are devoted to uniform spaces. In the first section we prove important result on existence of pseudometrics originally due to Weil. This result is crucial for further developments.

2. EXISTENCE OF PSEUDOMETRICS

Let X be a set. We denote

$$\Delta_X = \left\{(x,x) \in X \,\middle|\, x \in X\right\}$$

We start by introducing some set-theoretic notions concerning subsets of the square $X \times X$.

Definition 2.1. Let X be a set. Suppose that V is a subset of $X \times X$ satisfying the following assertions.

(1) If $(x,y) \in V$ for some $x,y \in X$, then $(y,x) \in V$.

(2) V contains Δ_X.

Then V *is a surrounding of* Δ_X.

Definition 2.2. Let X be a set and let V, W be subsets of $X \times X$. Consider a subset $W \cdot V$ of $X \times X$ such that $(x,z) \in W \cdot V$ for $x,z \in X$ if and only if $(x,y) \in V$ and $(y,z) \in V$ for some $y \in X$. Then $V \cdot W$ is *the composition of V and W*.

Finally we recall the notion of pseudometric.

Definition 2.3. Let X be a set. Suppose that a function $\rho : X \times X \to \mathbb{R}$ satisfies the following assertions.

(1) $\rho(x,y) \geq 0$ for all $x,y \in X$.

(2) $\rho(x,x) = 0$ for all $x \in X$.

(3) $\rho(x,y) = \rho(y,x)$ for all $x,y \in X$.

(4) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all $x,y,z \in X$.

Then ρ is *a pseudometric on X*.

Definition 2.4. Let ρ be a pseudometric on X. Suppose that $\rho(x,y) = 0$ implies $x = y$ for all $x,y \in X$. Then ρ is *a metric on X*.

Now we state and prove the fundamental result on the existence of pseudometrics.

Theorem 2.5. *Let X be a set and let $\{V_n\}_{n\in\mathbb{N}}$ be a sequence of surroundings of Δ_X such that*

$$V_{n+1} \cdot V_{n+1} \cdot V_{n+1} \subseteq V_n$$

for every $n \in \mathbb{N}$. Then there exists a pseudometric ρ on X bounded by 1 such that

$$\left\{(x,y) \in X \times X \,\middle|\, \rho(x,y) < \frac{1}{2^n}\right\} \subseteq V_n \subseteq \left\{(x,y) \in X \times X \,\middle|\, \rho(x,y) \leq \frac{1}{2^n}\right\}$$

for every $n \in \mathbb{N}$.

For the proof consider a function f defined on $X \times X$ given by formula

$$\begin{cases} 0 & \text{if } (x,y) \in V_n \text{ for each } n \in \mathbb{N} \\ \frac{1}{2^n} & \text{if } (x,y) \in V_n \setminus V_{n+1} \\ 1 & \text{if } (x,y) \notin V_0 \end{cases}$$

The proof relies on the following result.

Lemma 2.5.1. *For each $n \in \mathbb{N}$ and every finite sequence $x_0, ..., x_m$ the inequality*

$$\sum_{i=1}^{m} f(x_{i-1}, x_i) < \frac{1}{2^n}$$

implies that $(x_0, x_m) \in V_n$.

Proof of the lemma. The proof goes by induction on m. For $m = 0$ and $m = 1$ the claim is trivial. Assume that m is greater than one and suppose that the assertion holds for all numbers smaller than m. Suppose that

$$\sum_{i=1}^{m} f(x_{i-1}, x_i) < \frac{1}{2^n}$$

for some sequence $x_0, ..., x_m$ of elements in X. We have

$$\text{either } f(x_0, x_1) < \frac{1}{2^{n+1}} \text{ or } f(x_{m-1}, x_m) < \frac{1}{2^{n+1}}$$

Without loss of generality we may assume that the first inequality holds. Let k be the greatest number in $\{1, ..., m-1\}$ such that

$$\sum_{i=1}^{k} f(x_{i-1}, x_i) < \frac{1}{2^{n+1}}$$

Next we consider two cases.

- If $k < m - 1$, then we have

  $$\sum_{i=1}^{k} f(x_{i-1}, x_i) < \frac{1}{2^{n+1}}, \ f(x_k, x_{k+1}) \le \frac{1}{2^{n+1}}, \ \sum_{i=k+1}^{m} f(x_{i-1}, x_i) < \frac{1}{2^{n+1}}$$

  By induction hypothesis we have $(x_0, x_k) \in V_{n+1}, (x_{k+1}, x_m) \in V_{n+1}$ and by definition of f we have $(x_k, x_{k+1}) \in V_{n+1}$. Hence

  $$(x_0, x_m) \in V_{n+1} \cdot V_{n+1} \cdot V_{n+1} \subseteq V_n$$

  and the assertion holds.
- If $k = m - 1$. Then

  $$\sum_{i=1}^{m-1} f(x_{i-1}, x_i) < \frac{1}{2^{n+1}}, \ f(x_{m-1}, x_m) \le \frac{1}{2^{n+1}}$$

  By induction hypothesis we have $(x_0, x_{m-1}) \in V_{n+1}$ and by definition of f we have $(x_{m-1}, x_m) \in V_{n+1}$. Hence

  $$(x_0, x_m) \in V_{n+1} \cdot V_{n+1} \subseteq V_{n+1} \cdot V_{n+1} \cdot V_{n+1} \subseteq V_n$$

  and the assertion holds.

Thus the result follows from induction. □

Proof of the theorem. For $x, y \in X$ we define

$$\rho(x,y) = \inf \left\{ \sum_{i=1}^{m} f(x_{i-1}, x_i) \;\middle|\; \text{for every } m \in \mathbb{N} \text{ an every finite sequence } x_0, ..., x_m \text{ such that } x_0 = x, \ x_m = y \right\}$$

It is easy to verify that the function ρ is a pseudometric on X. It remains to prove that

$$\left\{(x,y) \in X \times X \,\middle|\, \rho(x,y) < \frac{1}{2^n}\right\} \subseteq V_n \subseteq \left\{(x,y) \in X \times X \,\middle|\, \rho(x,y) \leq \frac{1}{2^n}\right\}$$

The first inclusion follows from Lemma 2.5.1 and the second follows from the fact that $\rho(x,y) \leq f(x,y)$ for every $x,y \in X$. □

3. Uniform structures and uniform spaces

In this section we introduce main object of our study.

Definition 3.1. Let X be a set. Suppose that $\mathfrak{U}$ is a collection of surroundings of Δ_X which satisfies the following two assertions.

(1) If $U \in \mathfrak{U}$ and W is a surrounding of Δ_X such that $V \subseteq W$, then $W \in \mathfrak{U}$.

(2) If $U, W \in \mathfrak{U}$, then $U \cap W \in \mathfrak{U}$.

(3) If $U \in \mathfrak{U}$, then there exists $W \in \mathfrak{U}$ such that $W \cdot W \subseteq U$.

Then $\mathfrak{U}$ is *a uniform structure on* X.

Example 3.2. Let X be a set. Then the family $\mathfrak{D}_X$ of all surroundings of Δ_X is a uniform structure on X. It is called *the discrete uniform structure on* X.

Fact 3.3. *Let X be a set and let $\{\mathfrak{U}_i\}_{i \in I}$ be a family of uniform structures on X. Then*

$$\bigcap_{i \in I} \mathfrak{U}_i$$

is a uniform structure on X.

Proof. Left for the reader. □

Corollary 3.4. *Let X be a set and let $\mathcal{F}$ be a family of surrounding of Δ_X. Then there exists the smallest (with respect to inclusion) uniform structure $\mathfrak{U}$ on X which contain $\mathcal{F}$.*

Proof. Let $\{\mathfrak{U}_i\}_{i \in I}$ be a family of all uniform structures on X which contain $\mathcal{F}$. The family is nonempty, since it contains the discrete uniform structure on X. The intersection

$$\mathfrak{U} = \bigcap_{i \in I} \mathfrak{U}_i$$

is a uniform structure on X by Fact 3.3. Hence it is the smallest uniform structure on X which contain $\mathcal{F}$. □

Definition 3.5. A pair $(X, \mathfrak{U})$ consisting of a set X and a uniform structure $\mathfrak{U}$ on X is *a uniform space*.

Definition 3.6. Let $(X, \mathfrak{U})$ be a uniform space. A surrounding V in $\mathfrak{U}$ is called *an entourage of the diagonal in* $(X, \mathfrak{U})$.

Definition 3.7. Let $(X, \mathfrak{U}), (Y, \mathfrak{V})$ be uniform spaces and let $f : X \to Y$ be a map. Suppose that $(f \times f)^{-1}(V) \in \mathfrak{U}$ for every $V \in \mathfrak{V}$. Then f is *a morphism of uniform spaces*.

Remark 3.8. Uniform spaces and their morphisms form a category. We denote this category by **Unif**.

Now we study limits in **Unif**. For this we use the following result.

Theorem 3.9. *Let X be a set and let $\{(X_i, \mathfrak{U}_i)\}_{i\in I}$ be a family of uniform spaces. Consider a family $\{f_i : X \to X_i\}_{i\in I}$ of maps. Suppose that $\mathfrak{U}$ is the smallest uniform structure on X which makes $\{f_i\}_{i\in I}$ into a family of uniform morphisms. Then*

$$U \in \mathfrak{U}$$

if and only if there exist $n \in \mathbb{N}_+$, $i_1, ..., i_n \in I$ and $U_1 \in \mathfrak{U}_{i_1}, ..., U_n \in \mathfrak{U}_{i_n}$ such that

$$\bigcap_{k=1}^{n} \left(f_{i_k} \times f_{i_k}\right)^{-1}(U_k) \subseteq U$$

Proof. Consider the family $\mathcal{U}$ of all surrounding U of Δ_X such that there exist $n \in \mathbb{N}_+$, $i_1, ..., i_n \in I$ and $U_1 \in \mathfrak{U}_{i_1}, ..., U_n \in \mathfrak{U}_{i_n}$ satisfying

$$\bigcap_{k=1}^{n} \left(f_{i_k} \times f_{i_k}\right)^{-1}(U_k) \subseteq U$$

It is easy to verify (we left for the reader) that $\mathcal{U}$ is a uniform structure on X. Moreover, for every $n \in \mathbb{N}_+$, $i_1, ..., i_n \in I$ and $U_1 \in \mathfrak{U}_{i_1}, ..., U_n \in \mathfrak{U}_{i_n}$ we have

$$\bigcap_{k=1}^{n} \left(f_{i_k} \times f_{i_k}\right)^{-1}(U_k) \in \mathfrak{U}$$

Hence $\mathcal{U} \subseteq \mathfrak{U}$. Note also that f_i is a uniform morphism $(X, \mathcal{U}) \to (X_i, \mathfrak{U}_i)$ for each $i \in I$. Thus $\mathfrak{U} \subseteq \mathcal{U}$. Therefore, $\mathcal{U} = \mathfrak{U}$ and this proves the theorem. □

Definition 3.10. Let $(X, \mathfrak{U})$ be a uniform space and let Z be a subset of X. Then Z together with the smallest uniform structure which makes the inclusion $Z \hookrightarrow X$ into a uniform morphism is *a uniform subspace of $(X, \mathfrak{U})$ with Z as the underlying set*.

4. Hausdorff uniform spaces

In this short section we study uniform spaces which satisfy the following separation axiom.

Definition 4.1. Let $(X, \mathfrak{U})$ be a uniform space. Suppose that

$$\Delta_X = \bigcap_{U\in\mathfrak{U}} U$$

Then $(X, \mathfrak{U})$ is *a Hausdorff uniform space*.

The following result shows that every uniform space admits Hausdorff quotient.

Theorem 4.2. *Let $(X, \mathfrak{U})$ be a uniform space. The following assertions hold.*

(1) *The subset*

$$\Delta_{\mathfrak{U}} = \bigcap_{U\in\mathfrak{U}} U$$

of $X \times X$ is an equivalence relation.

(2) *Consider the quotient map $q : X \to X/\Delta_{\mathfrak{U}}$ and define*

$$\mathfrak{U}_{sep} = \left\{U \in \mathfrak{D}_{X/\Delta_{\mathfrak{U}}} \,\middle|\, (q \times q)^{-1}(U) \in \mathfrak{U}\right\}$$

Then $\mathfrak{U}_{sep}$ is a uniform structure on $X/\Delta_{\mathfrak{U}}$ which makes q into a morphism

$$(X, \mathfrak{U}) \to \left(X/\Delta_{\mathfrak{U}}, \mathfrak{U}_{sep}\right)$$

of uniform spaces.

(3) $\left(X/\Delta_{\mathfrak{U}}, \mathfrak{U}_{sep}\right)$ *is a Hausdorff uniform space.*

(4) *Suppose that $(Y, \mathfrak{V})$ is a Hausdorff uniform space and $f : (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ is a morhism of uniform spaces. Then there exists a unique morphism $p : (X/\Delta_{\mathfrak{U}}) \to (Y, \mathfrak{V})$ of uniform spaces which makes the triangle*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "q"'] & Y \\ X/\Delta_{\mathfrak{U}} \arrow[ur, dotted, "p"'] & \end{tikzcd}$$

commutative.

For the proof we need technical lemma.

Lemma 4.2.1. *For every U in $\mathfrak{U}$ and there exists V in $\mathfrak{U}$ such that $V \cdot V \subseteq U$ and $\Delta_{\mathfrak{U}} \cdot V \cdot \Delta_{\mathfrak{U}} \subseteq V$.*

Proof of the lemma. Pick W in $\mathfrak{U}$ such that $(W \cdot W \cdot W) \cdot (W \cdot W \cdot W) \subseteq U$. Note that for every $n \in \mathbb{N}_+$ we have

$$\underbrace{\Delta_{\mathfrak{U}} \cdot \ldots \cdot \Delta_{\mathfrak{U}}}_{n \text{ times}} \subseteq W$$

Define

$$V = \bigcup_{n \in \mathbb{N}} \underbrace{\Delta_{\mathfrak{U}} \cdot \ldots \cdot \Delta_{\mathfrak{U}}}_{n \text{ times}} \cdot W \cdot \underbrace{\Delta_{\mathfrak{U}} \cdot \ldots \cdot \Delta_{\mathfrak{U}}}_{n \text{ times}}$$

Clearly $W \subseteq V \subseteq W \cdot W \cdot W$ and $\Delta_{\mathfrak{U}} \cdot V \cdot \Delta_{\mathfrak{U}} \subseteq V$. Thus V is an element of $\mathfrak{U}$ and we have

$$V \cdot V \subseteq (W \cdot W \cdot W) \cdot (W \cdot W \cdot W) \subseteq U$$

and $\Delta_{\mathfrak{U}} \cdot V \cdot \Delta_{\mathfrak{U}} \subseteq V$. This finishes the proof. □

Proof of the theorem. Clearly $\Delta_{\mathfrak{U}}$ is reflexive and symmetric. Fix U in $\mathfrak{U}$. Then there exists $W \in \mathfrak{U}$ such that $W \cdot W \subseteq U$. Hence

$$\Delta_{\mathfrak{U}} \cdot \Delta_{\mathfrak{U}} \subseteq W \cdot W \subseteq U$$

Since U is an arbitrary element of $\mathfrak{U}$, we derive that $\Delta_{\mathfrak{U}} \cdot \Delta_{\mathfrak{U}} \subseteq \Delta_{\mathfrak{U}}$. Hence $\Delta_{\mathfrak{U}}$ is transitive. This completes the proof of **(1)**.
In order to prove that $\mathfrak{U}_{sep}$ is a uniform structure it suffices to check that for every U in $\mathfrak{U}_{sep}$ there exists $W \in \mathfrak{U}_{sep}$ such that $W \cdot W \subseteq U$. For this pick U in $\mathfrak{U}_{sep}$. By Lemma 4.2.1 there exists $V \in \mathfrak{U}$ such that $V \cdot V \subseteq (q \times q)^{-1}(U)$ and $\Delta_{\mathfrak{U}} \cdot V \cdot \Delta_{\mathfrak{U}} \subseteq V$. Since $\Delta_{\mathfrak{U}} \cdot V \cdot \Delta_{\mathfrak{U}} \subseteq V$, we have $V = (q \times q)^{-1}(W)$ where $W = (q \times q)(V)$. This implies that $W \in \mathfrak{U}_{sep}$ and $W \cdot W \subseteq U$. We proved **(2)**.
Suppose that $x, y \in X$ satisfy $q(x) \neq q(y)$ or in other words $(q(x), q(y)) \notin \Delta_{X/\Delta_{\mathfrak{U}}}$. Then $(x, y) \notin \Delta_{\mathfrak{U}}$ and there exists U in $\mathfrak{U}$ such that $(x, y) \notin U$. Invoking Lemma 4.2.1 there exists $V \in \mathfrak{U}$ such that $V \subseteq V \cdot V \subseteq U$ and $\Delta_{\mathfrak{U}} \cdot V \cdot \Delta_{\mathfrak{U}} \subseteq V$. Then $W = (q \times q)(V)$ is an element of $\mathfrak{U}_{sep}$ and $(q(x), q(y)) \notin W$. This proves **(3)**.
For the proof of **(4)** observe that $(f \times f)^{-1}(\Delta_Y)$ is an equivalence relation containing $\Delta_{\mathfrak{U}}$. Indeed, it is an equivalence relation due to the fact that f is a map and it contains $\Delta_{\mathfrak{U}}$ due to the fact that f is a uniform map and $(Y, \mathfrak{V})$ is Hausdorff. It follows that $f = p \cdot q$ for a unique map $p : X/\Delta_{\mathfrak{U}} \to Y$. Fix now an entourage $V \in \mathfrak{V}$. Then

$$(q \times q)^{-1}\left((p \times p)^{-1}(V)\right) = (f \times f)^{-1}(V) \in \mathfrak{U}$$

Hence $(p \times p)^{-1}(V) \in \mathfrak{U}_{sep}$ by definition. Thus p is a unique uniform morphism such that $f = p \cdot q$. □

Definition 4.3. Let $(X, \mathfrak{U})$ be a uniform space. Then a morphism $q : (X, \mathfrak{U}) \twoheadrightarrow (X/\Delta_{\mathfrak{U}}, \mathfrak{U}_{sep})$ described in Theorem 4.2 is *the universal Hausdorff quotient of* $(X, \mathfrak{U})$.

5. Topology induced by uniform structure

We start by introducing the notion of a ball with respect to surrounding of the diagonal.

Definition 5.1. Let X be a set. For every x in X and U in $\mathfrak{D}_X$ the set

$$B(x,U) = \{y \in X \,|\, (x,y) \in U\}$$

is *the ball with center x and radius U*.

Fact 5.2. *Let X be a set and let $\mathfrak{U}$ be a uniform structure on X. The family*

$$\tau_{\mathfrak{U}} = \{\mathcal{O} \subseteq X \,|\, \text{for each } x \in \mathcal{O} \text{ there exists } U \in \mathfrak{U} \text{ such that } B(x,U) \subseteq \mathcal{O}\}$$

is a topology on X.

Proof. We left the proof for the reader as an exercise. □

Definition 5.3. Let X be a set and let $\mathfrak{U}$ be a uniform structure on X. Then the topology $\tau_{\mathfrak{U}}$ is *the topology on X induced by $\mathfrak{U}$*.

The following result is a useful property of a topology induced by a uniform structure.

Proposition 5.4. *Let $(X,\mathfrak{U})$ be a uniform space and let $U \in \mathfrak{U}$. Then there exists $W \in \mathfrak{U}$ contained in U such that for every x in X the ball $B(x,W)$ is open with respect to the topology induced by $\mathfrak{U}$ on X.*

Proof. We pick $U_1 \in \mathfrak{U}$ such that $U_1 \cdot U_1 \subseteq U$. Next suppose that U_n is defined for some $n \in \mathbb{N}_+$. Then there exists $U_{n+1} \in \mathfrak{U}$ such that $U_{n+1} \cdot U_{n+1} \subseteq U_n$. Thus by recursive method we construct a sequence $\{U_n\}_{n\in\mathbb{N}}$ of elements of $\mathfrak{U}$. Easy induction shows that

$$U_1 \cdot U_2 \cdot \ldots \cdot U_n \subseteq U$$

for each $n \in \mathbb{N}_+$. Then

$$W = \bigcup_{n\in\mathbb{N}_+} U_1 \cdot U_2 \cdot \ldots \cdot U_n$$

is in $\mathfrak{U}$ and is contained in U. Moreover, for every x in X the ball $B(x,W) \in \tau_{\mathfrak{U}}$. □

Example 5.5. The interval $[0,1]$ admits a uniform structure given by

$$\{U \subseteq \mathfrak{D}_{[0,1]} \,|\, \text{there exists } \epsilon > 0 \text{ such that } |x-y| < \epsilon \text{ for some } x,y \in [0,1] \text{ implies } (x,y) \in U\}$$

Now the topology induced by this uniform structure coincides with the natural topology on $[0,1]$.

Fact 5.6. *Let $(X,\mathfrak{U})$ and $(Y,\mathfrak{V})$ be uniform spaces and let $f : X \to Y$ be a morphism of uniform spaces. Then f is a continuous map $(X,\tau_{\mathfrak{U}}) \to (Y,\tau_{\mathfrak{V}})$.*

Proof. Pick open subset $\mathcal{O}$ with respect to the topology induced by $\mathfrak{V}$ on Y. Suppose that $f(x) \in \mathcal{O}$ for some x in X. Then there exists $V_x \in \mathfrak{V}$ such that $B(f(x),V_x) \subseteq \mathcal{O}$. Note that the image of $B\big(x,(f\times f)^{-1}(V_x)\big)$ under f is contained in $\mathcal{O}$. Therefore,

$$f^{-1}(\mathcal{O}) = \bigcup_{x\in f^{-1}(\mathcal{O})} B\big(x,(f\times f)^{-1}(V_x)\big)$$

is open in the topology induced by $\mathfrak{U}$. □

Fact 5.2 and Fact 5.6 imply the existence of the functor

$$\mathbf{Unif} \ni (X,\mathfrak{U}) \mapsto (X,\tau_{\mathfrak{U}}) \in \mathbf{Top}$$

In the remaining part of this section we shall investigate the properties of this functor. We start by describing the image of the functor.

Definition 5.7. Let (X,τ) be a topological space. Suppose that for every closed subset F of X and for every point x in $X\smallsetminus F$ there exists a continuous function $f : X \to [0,1]$ such that $f(F)\subseteq\{1\}$ and $f(x)=0$. Then X is *a completely regular space.*

Theorem 5.8. *The image of the object part of the functor*

$$\mathbf{Unif} \ni (X,\mathfrak{U}) \mapsto (X,\tau_{\mathfrak{U}}) \in \mathbf{Top}$$

consists of the class of completely regular spaces.

Proof. Let $(X,\mathfrak{U})$ be a uniform space. Consider a closed set F with respect to $\tau_{\mathfrak{U}}$. Let x be a point in $X\smallsetminus F$. Since $x\notin F$ and F is closed in $\tau_{\mathfrak{U}}$, we derive that there exists $U\in\mathfrak{U}$ such that $B(x,U)\cap F=\emptyset$. Next we define a sequence $\{V_n\}_{n\in\mathbb{N}}$ of elements in $\mathfrak{U}$ by recursion. We set $V_0=U$ and if $V_0,...,V_n$ are defined for some $n\in\mathbb{N}$, then we pick an element V_{n+1} of $\mathfrak{U}$ such that

$$V_{n+1}\cdot V_{n+1}\cdot V_{n+1}\subseteq V_n$$

According to Theorem 2.5 there exists a pseudometric ρ on X bounded by 1 such that

$$\left\{(x,y)\in X\times X\,\middle|\,\rho(x,y)<\frac{1}{2^n}\right\}\subseteq V_n\subseteq\left\{(x,y)\in X\times X\,\middle|\,\rho(x,y)\le\frac{1}{2^n}\right\}$$

for every $n\in\mathbb{N}$. Note that

$$|\rho(x,y_1)-\rho(x,y_2)|\le\rho(y_1,y_2)$$

for any pair $y_1,y_2\in X$. Indeed, this is the triangle inequality for ρ. Thus if $(y_1,y_2)\in V_n$ for some $n\in\mathbb{N}$, then

$$|\rho(x,y_1)-\rho(x,y_2)|\le\rho(y_1,y_2)\le\frac{1}{2^n}$$

Hence the map $f : X\to[0,1]$ given by formula $f(y)=\rho(x,y)\in[0,1]$ is a morphism of uniform spaces, where X is a uniform space with respect to $\mathfrak{U}$ and $[0,1]$ is considered with uniform structure described in Example 5.5. This implies (by Fact 5.6) that f is a continuous map, where X carries topology $\tau_{\mathfrak{U}}$ and $[0,1]$ is considered with natural topology. Pick $y\in F$. Then $y\notin B(x,U)$ and hence $(x,y)\notin U$. Since $V_0=U$ and ρ is bounded by 1, we derive that $f(y)=\rho(x,y)=1$. On the other hand $f(x)=\rho(x,x)=0$. Therefore, $f(F)\subseteq\{1\}$ and $f(x)=0$. Thus $(X,\tau_{\mathfrak{U}})$ is a completely regular space.

Suppose now that (X,τ) is a completely regular space. Consider the set $C(\tau,\mathbb{R})$ of all continuous real valued functions on (X,τ). For $m\in\mathbb{N}_+$ and set of m functions $f_1,...,f_m\in C(\tau,\mathbb{R})$ define

$$\rho_{f_1,...,f_m}(x,y)=\max\left\{|f_1(x)-f_1(y)|,...,|f_m(x)-f_m(y)|\right\}$$

where $x,y\in X$. Clearly $\rho_{f_1,...,f_m}$ is a pseudometric on X. Next consider a family $\mathfrak{U}$ of all $U\in\mathfrak{D}_X$ such that there exist a finite subset $\{f_1,...,f_m\}\subseteq C(\tau,\mathbb{R})$ for some $m\in\mathbb{N}_+$ and $\epsilon>0$ such that

$$\left\{(x,y)\in X\times X\,\middle|\,\rho_{f_1,...,f_m}(x,y)<\epsilon\right\}\subseteq U$$

Clearly $\mathfrak{U}$ is a uniform structure on X. Suppose that $\mathcal{O}\in\tau_{\mathfrak{U}}$. Then for each point z in $\mathcal{O}$ there exists U in $\mathfrak{U}$ such that $B(z,U)\subseteq\mathcal{O}$. By definition there exist a finite subset $\{f_1,...,f_m\}\subseteq C(\tau,\mathbb{R})$ for some $m\in\mathbb{N}_+$ and $\epsilon>0$ such that

$$\left\{(x,y)\in X\times X\,\middle|\,\rho_{f_1,...,f_m}(x,y)<\epsilon\right\}\subseteq U$$

Thus

$$\bigcap_{i=1}^{m}f_i^{-1}\Big(\big(f_i(z)-\epsilon,f_i(z)+\epsilon\big)\Big)=\left\{y\in X\,\middle|\,\rho_{f_1,...,f_m}(z,y)<\epsilon\right\}\subseteq B(z,U)\subseteq\mathcal{O}$$

Since $f_1,...,f_m$ are continuous on X with respect to τ, we derive from the inclusion above that there exists an open neighborhood of z with respect to τ contained in $\mathcal{O}$. According to the fact that z is an arbitrary point in $\mathcal{O}$ it follows that $\mathcal{O}\in\tau$. This proves that $\tau_{\mathfrak{U}}\subseteq\tau$. Now we prove the converse. For this assume that $\mathcal{O}\in\tau$. We claim that $\mathcal{O}$ is also open in the topology induced by $\mathfrak{U}$. For this pick $z\in\mathcal{O}$. Since (X,τ) is completely regular, there exists a function $f_z : X\to\mathbb{R}$

continuous with respect to τ such that $f_z(X \setminus \mathcal{O}) \subseteq \{1\}$ and $f_z(z) = 0$. Let U_z be a set consisting of all pairs in $X \times X$ for which ρ_{f_z} is smaller than 1. Then $U_z \in \mathfrak{U}$ and obviously $B(z, U_z) \subseteq \mathcal{O}$. Thus

$$\mathcal{O} = \bigcup_{z \in \mathcal{O}} B(z, U_z)$$

and this proves the claim that $\mathcal{O}$ is in $\tau_{\mathfrak{U}}$. Hence $\tau \subseteq \tau_{\mathfrak{U}}$. This completes the proof. $\square$

Next we prove the following important fact.

Proposition 5.9. *Let $(X, \mathfrak{U})$ be a uniform space and let Z be a subset of X. Let $\mathfrak{U}_Z$ be the subspace uniform structure on Z. Then $\tau_{\mathfrak{U}_Z}$ coincide with the subspace topology on Z induced by $\tau_{\mathfrak{U}}$.*

Proof. Let $\mathcal{O}$ be a set in $\tau_{\mathfrak{U}}$. Then for each x in $\mathcal{O}$ there exists $U_x \in \mathfrak{U}$ such that $B(x, U_x) \subseteq \mathcal{O}$. Thus

$$\mathcal{O} \cap Z = \bigcup_{z \in \mathcal{O} \cap Z} B\big(z, U_z \cap (Z \times Z)\big)$$

Since $U_z \cap (Z \times Z) \in \mathfrak{U}_Z$ for every $z \in \mathcal{O} \cap Z$, it follows that $\mathcal{O} \cap Z$ is open with respect to $\tau_{\mathfrak{U}_Z}$. Hence

$$\big\{Z \cap \mathcal{O} \,\big|\, \mathcal{O} \in \tau_{\mathfrak{U}}\big\} \subseteq \tau_{\mathfrak{U}_Z}$$

Suppose now that $\mathcal{O}_Z \in \tau_{\mathfrak{U}_Z}$. Then for each $z \in \mathcal{O}_Z$ there exists $U_z \in \mathfrak{U}$ such that $B\big(z, U_z \cap (Z \times Z)\big) \subseteq \mathcal{O}_Z$. By Proposition 5.4 there exists $\mathcal{O}_z \in \tau_{\mathfrak{U}}$ such that $z \in \mathcal{O}_z \subseteq B(z, U_z)$. Thus

$$\mathcal{O} = \bigcup_{z \in \mathcal{O}_Z} \mathcal{O}_z$$

is an element of $\tau_{\mathfrak{U}}$ and

$$\mathcal{O} \subseteq \bigcup_{z \in \mathcal{O}_Z} B(z, U_z)$$

and hence $Z \cap \mathcal{O} = \mathcal{O}_Z$. Therefore, $\mathcal{O}_Z$ is an open subset in the subspace topology induced on Z by $\tau_{\mathfrak{U}}$. This completes the proof. $\square$

Theorem 5.10. *Let $\mathcal{I}$ be a small category and let $F : \mathcal{I} \to$ **Unif** be a functor given by*

$$F(i) = (X_i, \mathfrak{U}_i)$$

for $i \in \mathcal{I}$. Let $\big\{f_i : X \to X_i\big\}_{i \in \mathcal{I}}$ be a limiting cone of the composition of F with the functor **Unif** $\to$ **Set** *which sends each uniform space to its underlying set. Consider the smallest uniform structure $\mathfrak{U}$ on X which makes $\{f_i\}_{i \in \mathcal{I}}$ into a family of uniform morphisms. Then $(X, \mathfrak{U})$ together with $\{f_i\}_{i \in \mathcal{I}}$ is a limiting cone of F.*

Proof. We may equivalently describe $\mathfrak{U}$ as the smallest uniform structure on X such that

$$(f_i \times f_i)^{-1}(U) \in \mathfrak{U}$$

for every $i \in \mathcal{I}$ and every $U \in \mathfrak{U}_i$. Suppose that $\big\{g_i : (Y, \mathfrak{V}) \to (X_i, \mathfrak{U}_i)\big\}_{i \in \mathcal{I}}$ is some cone over F. Then there exists a unique map $h : Y \to X$ such that $h \cdot f_i = g_i$ for every $i \in \mathcal{I}$. It is easy to verify that

$$\big\{U \in \mathfrak{U} \,\big|\, (h \times h)^{-1}(U) \text{ is an entourage of the diagonal in } \mathfrak{V}\big\}$$

is a uniform structure on X. Moreover, it contains $(f_i \times f_i)^{-1}(U)$ for every $i \in \mathcal{I}$ and every $U \in \mathfrak{U}_i$. Since $\mathfrak{U}$ is the smallest such uniform structure, we derive that $\mathfrak{U}$ and

$$\big\{U \in \mathfrak{U} \,\big|\, (h \times h)^{-1}(U) \text{ is an entourage of the diagonal in } \mathfrak{B}\big\}$$

coincide and hence h is a morphism of uniform spaces $(Y, \mathfrak{B}) \to (X, \mathfrak{U})$. This shows that $(X, \mathfrak{U})$ together with $\big\{f_i : X \to X_i\big\}_{i \in \mathcal{I}}$ is a limiting cone of F. $\square$

Theorem 5.11. *The functor*

$$\mathbf{Unif} \ni (X, \mathfrak{U}) \mapsto (X, \tau_{\mathfrak{U}}) \in \mathbf{Top}$$

preserves small limits.

Proof. Let $\mathcal{I}$ be a set and let $\{(X_i, \mathfrak{U}_i)\}_{i\in\mathcal{I}}$ be a family of uniform spaces parametrized by $\mathcal{I}$. Consider the cartesian product $X = \prod_{i\in\mathcal{I}} X_i$ and let $pr_i : X \to X_i$ be the projection for $i \in \mathcal{I}$. Let $\mathfrak{U}$ be the smallest uniform structure on X which makes $\left\{pr_i : (X, \mathfrak{U}) \to (X_i, \mathfrak{U}_i)\right\}_{i\in\mathcal{I}}$ into a family of morphisms of uniform spaces. By Theorem 3.9 family $\mathfrak{U}$ consists of all surrounding U of Δ_X such that there exist $n \in \mathbb{N}_+$, $i_1, ..., i_n \in \mathcal{I}$ and $U_1 \in \mathfrak{U}_{i_1}, ..., U_n \in \mathfrak{U}_{i_n}$ satisfying

$$\bigcap_{k=1}^{n} \left(pr_{i_k} \times pr_{i_k}\right)^{-1}(U_k) \subseteq U$$

Note that we have

$$B\left(x, \bigcap_{k=1}^{n} \left(pr_{i_k} \times pr_{i_k}\right)^{-1}(U_k)\right) = \prod_{k=1}^{n} B\left(pr_{i_k}(x), U_k\right) \times \prod_{i\in\mathcal{I}\smallsetminus\{i_1,...,i_k\}} X_i$$

for every $x \in X$. By Proposition 5.4 there exist $\mathcal{O}_1 \in \tau_{\mathfrak{U}_{i_1}}, ..., \mathcal{O}_n \in \tau_{\mathfrak{U}_{i_n}}$ such that

$$pr_{i_k}(x) \in \mathcal{O}_k \subseteq \left(pr_{i_k}(x), U_k\right)$$

for each k. Thus

$$\prod_{k=1}^{n} \mathcal{O}_k \times \prod_{i\in\mathcal{I}\smallsetminus\{i_1,...,i_k\}} X_i \subseteq B\left(x, \bigcap_{k=1}^{n} \left(pr_{i_k} \times pr_{i_k}\right)^{-1}(U_k)\right) \subseteq B(x, U)$$

Therefore, each ball centered in some point x of X and with radius U in $\mathfrak{U}$ contains open neighborhood of x with respect to the product of topologies $\{\tau_{\mathfrak{U}_i}\}_{i\in\mathcal{I}}$. This implies that $\tau_{\mathfrak{U}}$ is contained in the product of topologies $\{\tau_{\mathfrak{U}_i}\}_{i\in\mathcal{I}}$. On the other hand the fact that $pr_i : (X, \tau_{\mathfrak{U}}) \to (X_i, \tau_{\mathfrak{U}_i})$ is continuous for every $i \in \mathcal{I}$ implies that the product of topologies $\{\tau_{\mathfrak{U}_i}\}_{i\in\mathcal{I}}$ is contained in $\tau_{\mathfrak{U}}$. Thus $\tau_{\mathfrak{U}}$ is the product topology determined by $\{\tau_{\mathfrak{U}_i}\}_{i\in\mathcal{I}}$. Hence $(X, \tau_{\mathfrak{U}})$ together with $\{pr_i\}_{i\in\mathcal{I}}$ is a product of topological spaces $\{(X_i, \tau_{\mathfrak{U}_i})\}_{i\in\mathcal{I}}$. By Theorem 5.11 it follows that

$$\mathbf{Unif} \ni (X, \mathfrak{U}) \mapsto (X, \tau_{\mathfrak{U}}) \in \mathbf{Top}$$

preserves small products. Since every small limit is a combination of small product and kernel pair, it remains to show that the functor above preserves kernel pairs. Suppose that

$$\begin{tikzcd} (X, \mathfrak{W}) \arrow[r, hook] & (Y_1, \mathfrak{V}_1) \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & (Y_1, \mathfrak{V}_2) \end{tikzcd}$$

is a kernel pair of f_1 and f_2 in **Unif**. Then Theorem 5.11 shows that

$$Z = \left\{y \in Y_1 \,\middle|\, f_1(y) = f_2(y)\right\} \subseteq Y_1$$

and $\mathfrak{W}$ is the subspace uniform structure on Z induced by $\mathfrak{V}_1$. Now Proposition 5.9 implies that

$$\begin{tikzcd} (z, \tau_{\mathfrak{W}}) \arrow[r, hook] & \left(Y_1, \tau_{\mathfrak{V}_1}\right) \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & \left(Y_1, \tau_{\mathfrak{V}_2}\right) \end{tikzcd}$$

is a kernel pair in the category **Top**. Therefore, the functor

$$\mathbf{Unif} \ni (X, \mathfrak{U}) \mapsto (X, \tau_{\mathfrak{U}}) \in \mathbf{Top}$$

preserves kernel pairs. The proof is complete. □

6. Kolmogorov spaces and quotients

In this section we introduce certain class of topological spaces which is relevant to uniform spaces.

Definition 6.1. Let (X, τ) be a topological space. Suppose that for any pair of points x, y in X there exists set $\mathcal{O}$ in τ such that either $x \in \mathcal{O}$ and $y \notin \mathcal{O}$ or $y \in \mathcal{O}$ and $x \notin \mathcal{O}$. Then (X, τ) is *a Kolmogorov space*.

The following result shows that every topological space admits Hausdorff quotient.

Theorem 6.2. *Let (X, τ) be a topological space. The following assertions hold.*

(1) *The subset*
$$\Delta_{Kol} = \left\{(x, y) \in X \times X \,\middle|\, x \in \mathcal{O} \textit{ if and only if } y \in \mathcal{O} \textit{ for every set } \mathcal{O} \textit{ in } \tau\right\}$$
is an equivalence relation on X.

(2) *Consider the quotient topology τ_{Kol} on X/Δ_{Kol}. Then $(X/\Delta_{Kol}, \tau_{Kol})$ is a Komogorov space.*

(3) *Suppose that (Y, θ) is a Kolmogorov space and $f : (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ is a morhism of uniform spaces. Then there exists a unique morphism $p : (X/\Delta_{\mathfrak{U}}) \to (Y, \mathfrak{V})$ of uniform spaces which makes the triangle*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "q"'] & Y \\ X/\Delta_{\mathfrak{U}} \arrow[ur, dotted, "p"'] & \end{tikzcd}$$

commutative.

Definition 6.3. Let (X, τ) be a topological space. Consider the equivalence relation
$$\Delta_{Kol} = \left\{(x, y) \in X \times X \,\middle|\, x \in \mathcal{O} \text{ if and only if } y \in \mathcal{O} \text{ for every set } \mathcal{O} \text{ in } \tau\right\}$$
and let $q : (X, \tau) \twoheadrightarrow (X/\Delta_{Kol},)$

7. Pseudometrizable and Hausdorff uniform spaces

In this section we use Theorem 2.5 to prove certain structure theorems concerning uniform spaces.

Definition 7.1. A uniform space $(X, \mathfrak{U})$ is *pseudometrizable* if there exists a pseudometric ρ on X such that the uniform structure
$$\left\{U \in \mathfrak{D}_X \,\middle|\, \text{ there exists } \epsilon > 0 \text{ such that for all } x, y \in X \text{ if } \rho(x, y) \leq \epsilon \text{ then } (x, y) \in U\right\}$$
coincides with $\mathfrak{U}$.

Theorem 7.2. *Let $(X, \mathfrak{U})$ be a uniform space. The following assertions are equivalent.*

(i) *$(X, \mathfrak{U})$ is a pseudometrizable uniform space.*

(ii) *There exists a sequence $\{U_n\}_{n \in \mathbb{N}}$ of elements in $\mathfrak{U}$ such that the family*
$$\left\{U \in \mathfrak{D}_X \,\middle|\, \exists_{n \in \mathbb{N}}\, U_n \subseteq U\right\}$$
coincides with $\mathfrak{U}$.

Proof. For **(i)** $\Rightarrow$ **(ii)** observe that if ρ is a pseudometric on X such that

$$\left\{U \in \mathfrak{D}_X \,\middle|\, \text{there exists } \epsilon > 0 \text{ such that for all } x, y \in X \text{ if } \rho(x,y) \leq \epsilon \text{ then } (x,y) \in U\right\}$$

coincides with $\mathfrak{U}$, then the sequence $\{U_n\}_{n\in\mathbb{N}}$ given by formula

$$U_n = \left\{(x,y) \in X \times X \,\middle|\, \rho(x,y) \leq \frac{1}{2^n}\right\}$$

satisfies **(ii)**.

Suppose now that **(ii)** holds. We define a sequence $\{V_n\}_{n\in\mathbb{N}}$ of elements in $\mathfrak{U}$ by recursion. We set $V_0 = U_0$ and if $V_0, ..., V_n$ are defined for some $n \in \mathbb{N}$, then we pick an element W of $\mathfrak{U}$ such that

$$W \cdot W \cdot W \subseteq V_n$$

and define $V_{n+1} = W \cap U_{n+1}$. Note that $\{V_n\}_{n\in\mathbb{N}}$ satisfies

$$V_{n+1} \cdot V_{n+1} \cdot V_{n+1} \subseteq V_n$$

for each $n \in \mathbb{N}$. Moreover, we have

$$\mathfrak{U} = \left\{U \in \mathfrak{D}_X \,\middle|\, \exists_{n\in\mathbb{N}}\, V_n \subseteq U\right\}$$

By Theorem 2.5 there exists a pseudometric ρ on X such that

$$\left\{(x,y) \in X \times X \,\middle|\, \rho(x,y) < \frac{1}{2^n}\right\} \subseteq V_n \subseteq \left\{(x,y) \in X \times X \,\middle|\, \rho(x,y) \leq \frac{1}{2^n}\right\}$$

for every $n \in \mathbb{N}$. This implies that

$$\left\{U \in \mathfrak{D}_X \,\middle|\, \text{there exists } \epsilon > 0 \text{ such that for all } x, y \in X \text{ if } \rho(x,y) \leq \epsilon \text{ then } (x,y) \in U\right\}$$

coincides with $\mathfrak{U}$. Hence **(ii)** $\Rightarrow$ **(i)**. □

Corollary 7.3. *Every uniform space is a uniform subspace of a product of pseudometrizable uniform spaces.*

Proof. Let $(X, \mathfrak{U})$ be a uniform space. For each U in $\mathfrak{U}$ we construct a nonincreasing sequence $\{U_n\}_{n\in\mathbb{N}}$ of elements of $\mathfrak{U}$ such that $U_0 = U$ and $U_{n+1} \cdot U_{n+1} \cdot U_{n+1} \subseteq U_n$ for each $n \in \mathbb{N}$. Next define $\mathfrak{V}_U$ as a family

$$\left\{W \in \mathfrak{D}_X \,\middle|\, \exists_{n\in\mathbb{N}}\, U_n \subseteq W\right\}$$

Then $\mathfrak{V}_U$ is a uniform structure on X and $U \in \mathfrak{V}_U \subseteq \mathfrak{U}$. Moreover, by Theorem 7.2

□