BIAŁYNICKI-BIRULA FUNCTORS

1. Introduction

In this notes we study Białynicki-Birula functors. In the first section we prove some results concerning the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$, where $\mathbf{M}$ is an affine monoid k-scheme and $\mathbf{G}$ is its group of units (we assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$). These results will be used in the following sections.
We assume that k is a field. In these notes we use the following notational convention.

Remark 1.1. Since the Yoneda embedding $\mathbf{Sch}_k \hookrightarrow \widehat{\mathbf{Sch}_k}$ is full and faithful, we identify $\mathbf{Sch}_k$ with the subcategory of $\widehat{\mathbf{Sch}_k}$ consisting of representable presheaves on $\mathbf{Sch}_k$. In particular, if X is a k-scheme, then we denote by the same symbol the presheaf representable by X.

2. Tannakian formalism for quotient stacks

In this section we discuss an application of the main result of [Hall and Rydh, 2019]. For this we need to briefly discuss *algebraic stacks*, although for our purposes there is no need to use any technical details of this language. We refer the interested reader to the excellent exposition [Olsson, 2016] of this subject. We note the following facts.

(1) *An algebraic stack* is a category fibered over $\mathbf{Sch}_k$ satisfying certain extra conditions described in [Olsson, 2016, Definition 4.6.1] and [Olsson, 2016, Definition 8.1.4]. By [Olsson, 2016, Definition 8.2.1, Example 8.2.3] there are well defined notions of *locally noetherian, noetherian and excellent algebraic stacks*.

(2) *A morphism of algebraic stacks* is a morphism of fibered categories over $\mathbf{Sch}_k$. If $\mathcal{X}$ and $\mathcal{Y}$ are algebraic stack, then we denote by $\mathrm{Mor}\,(\mathcal{X}, \mathcal{Y})$ the corresponding category of morphisms.

(3) For every locally noetherian algebraic stack $\mathcal{X}$ there exists an abelian monoidal category $\mathfrak{Coh}(\mathcal{X})$ of coherent sheaves on $\mathcal{X}$ ([Olsson, 2016, Definition 9.1.14]). If $\mathcal{X}$ and $\mathcal{Y}$ are locally noetherian algebraic stacks, then we denote by $\mathrm{Hom}_{r,\otimes,\cong}\left(\mathfrak{Coh}(\mathcal{X}), \mathfrak{Coh}(\mathcal{Y})\right)$ the category of right exact, monoidal functors $\mathfrak{Coh}(\mathcal{X}) \to \mathfrak{Coh}(\mathcal{Y})$ with natural isomorphism as morphisms.

(4) If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of locally noetherian algebraic stacks, then f induces the functor $f^* : \mathfrak{Coh}(\mathcal{Y}) \to \mathfrak{Coh}(\mathcal{X})$ such that $f^* \in \mathrm{Hom}_{r,\otimes,\cong}\left(\mathfrak{Coh}(\mathcal{X}), \mathfrak{Coh}(\mathcal{Y})\right)$.

(5) Let $\mathbf{G}$ be a locally algebraic group k-scheme and let X be a k-scheme equipped with an action of $\mathbf{G}$. We consider $\mathbf{Sch}_k$ as a Grothendieck site with respect to *fppf topology* ([Olsson, 2016, Example 2.1.14]). Next the quotient fibered category $[X/\mathbf{G}]$ ([Monygham, 2020c, Definition 9.5]) with respect to this topology is an algebraic stack by [Olsson, 2016, Example 8.1.12].

(6) In **(5)** if k-scheme X is locally noetherian (noetherian, excellent), then $[X/\mathbf{G}]$ is a locally noetherian (noetherian, excellent) by [Olsson, 2016, Definition 8.2.1, Example 8.2.3] and [Olsson, 2016, Example 8.1.12].

(7) In **(5)** if k-scheme X is locally noetherian, then there exists an equivalence of monoidal categories $\mathfrak{Coh}\left([X/\mathbf{G}]\right) \cong \mathfrak{Coh}_{\mathbf{G}}(X)$ ([Olsson, 2016, Exercise 9.H]). Moreover, this equivalence is functorial with respect to $\mathbf{G}$-equivariant morphism. That is if Y is another locally noetherian k-scheme with action of $\mathbf{G}$ and $f : X \to Y$ is a $\mathbf{G}$-equivariant morphism, then f induces a morphism $[f/\mathbf{G}] : [X/\mathbf{G}] \to [Y\mathbf{G}]$ by [Monygham, 2020c, Theorem 9.7] and the square

$$\begin{tikzcd} \mathfrak{Coh}([Y/\mathbf{G}]) \arrow[r, "{[f/\mathbf{G}]^*}"] \arrow[d, "\cong"'] & \mathfrak{Coh}([X/\mathbf{G}]) \arrow[d, "\cong"] \\ \mathfrak{Coh}_{\mathbf{G}}(Y) \arrow[r, "f^*"'] & \mathfrak{Coh}_{\mathbf{G}}(X) \end{tikzcd}$$

of categories and functors is commutative.

(8) If $\mathbf{G}$ is smooth and affine over k, then $[X/\mathbf{G}]$ has *affine stabilizers*.

Remark 2.1. Let $\operatorname{Spec} k$ be a point equipped with the trivial action of a smooth and affine group $\mathbf{G}$. Then **(7)** together with [Monygham, 2020b, Example 4.7] impy that $\mathfrak{Coh}([\operatorname{Spec} k/\mathbf{G}])$ can be identified with the category $\mathbf{Repf}_{\mathbf{G}}$ of finite dimensional representations of $\mathbf{G}$.

Let us state the main result of [Hall and Rydh, 2019].

Theorem 2.2 ([Hall and Rydh, 2019, Theorem 1.1])**.** *Let $\mathcal{X}$ be a noetherian algebraic stack with affine stabilizers. For every locally excellent algebraic stack $\mathcal{T}$ the functor*

$$\operatorname{Mor}(\mathcal{X}, \mathcal{T}) \xrightarrow{f \mapsto f^*} \operatorname{Hom}_{r,\otimes,\cong}(\mathfrak{Coh}(\mathcal{T}), \mathfrak{Coh}(\mathcal{X}))$$

is an equivalence of categories.

Keeping our previous remarks in mind we deduce the following result.

Corollary 2.3. *Let $\mathbf{G}$ be an smooth affine group k-scheme and let X, Z be k-schemes equipped with an action of $\mathbf{G}$. Suppose that Z is noetherian and X is locally of finite type over k. Then*

$$\operatorname{Mor}([Z/\mathbf{G}], [X/\mathbf{G}]) \xrightarrow{f \mapsto f^*} \operatorname{Hom}_{r,\otimes,\cong}\Big(\mathfrak{Coh}([X/\mathbf{G}]), \mathfrak{Coh}([Z/\mathbf{G}])\Big)$$

is an equivalence of categories.

Proof. Note that $[Z/\mathbf{G}]$ is a noetherian algebraic stack according to **(5)** and **(6)**. It has affine stabilizers according to **(8)**. Similary by **(5)** $[X/\mathbf{G}]$ is an algebraic stack. Moreover, it is locally excellent according to the fact that X is locally excellent (it is locally of finite type over k and k is a field) and **(6)**. Then by Theorem 2.2 we derive that the functor in the statement is an equivalence of categories. □

Corollary 2.4. *Let $\mathbf{G}$ be an smooth affine group k-scheme and let X, Z be k-schemes equipped with an action of $\mathbf{G}$. Suppose that Z is noetherian and X is locally of finite type over k. Then we have a bijection*

$$\big\{f : Z \to X \,\big|\, f \text{ is } \mathbf{G}\text{-equivariant}\big\} \xrightarrow{f \mapsto f^*} \big\{F \in \operatorname{Hom}_{r,\otimes,\cong}(\mathfrak{Coh}_{\mathbf{G}}(X), \mathfrak{Coh}_{\mathbf{G}}(Z)) \,\big|\, F \cdot p_X^* = p_Z^*\big\}$$

where $p_X^* : \mathbf{Repf}(\mathbf{G}) \to \mathfrak{Coh}_{\mathbf{G}}(X)$ and $p_Z^* : \mathbf{Repf}(\mathbf{G}) \to \mathfrak{Coh}_{\mathbf{G}}(Z)$ are functors induced by $\mathbf{G}$-equivariant morphisms $p_X : X \to \operatorname{Spec} k$ and $p_Z : Z \to \operatorname{Spec} k$, respectively.

Proof. Since fppf topology is subcanonical, [Monygham, 2020c, Theorem 9.7] shows that there exists a bijection

$$\big\{f : Z \to X \,\big|\, f \text{ is } \mathbf{G}\text{-equivariant}\big\} \xrightarrow{f \mapsto [f/\mathbf{G}]} \big\{h : [Z/\mathbf{G}] \to [X/\mathbf{G}] \,\big|\, [p_X/\mathbf{G}] \cdot h = [p_Y/\mathbf{G}]\big\}$$

Corollary 2.3 implies that there exists a bijection

$$\{h : [Z/\mathbf{G}] \to [X/\mathbf{G}] \,|\, [p_X/\mathbf{G}] \cdot h = [p_Y/\mathbf{G}]\} \xrightarrow{h \mapsto h^*} \{F \in \mathrm{Hom}_{r,\otimes,\cong}(\mathfrak{Coh}([X/\mathbf{G}]), \mathfrak{Coh}([Z/\mathbf{G}])) \,|\, F \cdot [p_X/\mathbf{G}]^* = [p_Z, \mathbf{G}]^*\}$$

Next **(7)** implies that there exists a bijection

$$\{F \in \mathrm{Hom}_{r,\otimes,\cong}(\mathfrak{Coh}([X/\mathbf{G}]), \mathfrak{Coh}([Z/\mathbf{G}])) \,|\, F \cdot [p_X/\mathbf{G}]^* = [p_Z, \mathbf{G}]^*\} \longrightarrow \{F \in \mathrm{Hom}_{r,\otimes,\cong}(\mathfrak{Coh}_{\mathbf{G}}(X), \mathfrak{Coh}_{\mathbf{G}}(Z)) \,|\, F \cdot p_X^* = p_Z^*\}$$

and for a $\mathbf{G}$-equivariant morphism $f : Z \to X$ the image of $[f/\mathbf{G}]^* : \mathfrak{Coh}([X/\mathbf{G}]) \to \mathfrak{Coh}([Z/\mathbf{G}])$ under this bijection is $f^* : \mathfrak{Coh}_{\mathbf{G}}(X) \to \mathfrak{Coh}_{\mathbf{G}}(Z)$. These imply that the map of classes

$$\{f : Z \to X \,|\, f \text{ is } \mathbf{G}\text{-equivariant}\} \xrightarrow{f \mapsto f^*} \{F \in \mathrm{Hom}_{r,\otimes,\cong}(\mathfrak{Coh}_{\mathbf{G}}(X), \mathfrak{Coh}_{\mathbf{G}}(Z)) \,|\, F \cdot p_X^* = p_Z^*\}$$

is a bijection. ☐

Note that Corollary 2.4 relies on some asumptions regarding $\mathbf{G}$, X and Z. It is worth noting that Joachim Jelisiejew and the author were able to obtain a slightly more general (yet unpublished) result.

Theorem 2.5 ([Jelisiejew and Sienkiewicz, 2020, Theorem A.2])**.** *Let $\mathbf{G}$ be an affine algebraic group over k. Let Z, X be k-schemes equipped with an action of $\mathbf{G}$ and assume that X is quasi-compact and quasi-separated. Suppose that $F : \mathfrak{Qcoh}_{\mathbf{G}}(X) \to \mathfrak{Qcoh}_{\mathbf{G}}(Z)$ is a cocontinuous, monoidal functor such that $F \cdot p_X^* = p_Z^*$. Then there exists a unique $\mathbf{G}$-equivariant morphism $f : Z \to X$ such that $f^* = F$.*

3. Relations between representations of a monoid and its group of units

In this section we study the relation between the category $\mathbf{Rep}(\mathbf{M})$ of representations of an affine monoid k-scheme $\mathbf{M}$ and the category $\mathbf{Rep}(\mathbf{G})$ of representations of its group of units $\mathbf{G}$. Let $i : k[\mathbf{M}] \to k[\mathbf{G}]$ be the morphism of k-bialgebras induced by $\mathbf{G} \hookrightarrow \mathbf{M}$. Let us first note the following elementary result.

Fact 3.1. *Assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. Then i is an injective morphism of k-algebras.*

Proof. This follows from [Görtz and Wedhorn, 2010, Proposition 9.19]. ☐

Fact 3.2. *The forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$ creates colimits and finite limits.*

Proof. This follows from [Monygham, 2020e, Theorem 14.3, Theorem 14.4] and the commutative triangle

$$\begin{tikzcd} \mathbf{Rep}(\mathbf{M}) \arrow[rr] \arrow[dr] & & \mathbf{Rep}(\mathbf{G}) \arrow[dl] \\ & \mathbf{Vect}_k & \end{tikzcd}$$

of functors. ☐

The theorem below characterizes representations of $\mathbf{G}$ which are contained in the image of the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$.

Theorem 3.3. *Assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. Let V be a $\mathbf{G}$-representation. Then the following are equivalent.*

- **(i)** *V is in the image of the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$.*
- **(ii)** *The coaction $d : V \to k[\mathbf{G}] \otimes_k V$ factors through $i \otimes_k 1_V : k[\mathbf{M}] \otimes_k V \hookrightarrow k[\mathbf{G}] \otimes_k V$.*

Proof. In the proof we denote by $\Delta_{\mathbf{M}}$ and $\Delta_{\mathbf{G}}$ comultiplications of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. We also denote by $\xi_{\mathbf{M}}$ and $\xi_{\mathbf{G}}$ counits of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. According to Fact 3.1 i is an injective morphism of k-algebras.
Clearly **(i)** $\Rightarrow$ **(ii)**. We prove the converse. Suppose that **(ii)** holds. Let $c : V \to k[\mathbf{M}] \otimes_k V$ be a unique morphism such that $d = (i \otimes_k 1_V) \cdot c$. It suffices to prove that c is the coaction of the bialgebra $k[\mathbf{M}]$ on V. Observe that

$$(i \otimes_k i \otimes_k 1_V) \cdot \left(1_{k[\mathbf{M}]} \otimes_k c\right) \cdot c = (i \otimes_k d) \cdot c = \left(1_{k[\mathbf{G}]} \otimes_k d\right) \cdot d = (\Delta_{\mathbf{G}} \otimes_k 1_V) \cdot d =$$

$$= (\Delta_{\mathbf{G}} \otimes_k 1_V) \cdot \big((i \otimes_k 1_V) \cdot c \big) = ((\Delta_{\mathbf{G}} \cdot i) \otimes_k 1_V) \cdot c = (i \otimes_k i \otimes_k 1_V) \cdot (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot c$$

Since $i \otimes_k i \otimes_k 1_V$ is a monomorphism, we deduce that $\left(1_{k[\mathbf{M}]} \otimes_k c\right) \cdot c = (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot c$. Moreover, we have

$$(\xi_{\mathbf{G}} \otimes_k 1_V) \cdot d = (\xi_{\mathbf{G}} \otimes_k 1_V) \cdot ((i \otimes_k 1_V) \cdot c) = (\xi_{\mathbf{M}} \otimes_k 1_V) \cdot c$$

and hence $(\xi_{\mathbf{M}} \otimes_k 1_V) \cdot c$ is the canonical isomorphism $V \cong k \otimes_k V$. Thus c is the coaction of $k[\mathbf{M}]$ and $d = (i \otimes_k 1_V) \cdot c$. Therefore, V is in the image of $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$. □

Theorem 3.4. *Assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. Then $\mathbf{Rep}(\mathbf{M})$ is a full subcategory of $\mathbf{Rep}(\mathbf{G})$ closed under subobjects and quotients.*

Proof. In the proof we denote by $\Delta_{\mathbf{M}}$ and $\Delta_{\mathbf{G}}$ comultiplications of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. We also denote by $\xi_{\mathbf{M}}$ and $\xi_{\mathbf{G}}$ counits of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. According to Fact 3.1 i is an injective morphism of k-algebras.
We first prove that $\mathbf{Rep}(\mathbf{M})$ is a full subcategory of $\mathbf{Rep}(\mathbf{G})$. For this consider $\mathbf{M}$-representations V, W and a their morphism $f : V \to W$ as $\mathbf{G}$-representations. Let c_V and c_W be coactions of $k[\mathbf{M}]$ on V and W, respectively. Our goal is to prove that f is a morphism of $\mathbf{M}$-representations. Consider the diagram

$$\begin{tikzcd}
k[\mathbf{G}] \otimes_k V \arrow[r, "1_{k[\mathbf{G}]} \otimes_k f"] & k[\mathbf{G}] \otimes_k W \\
k[\mathbf{M}] \otimes_k V \arrow[u, "i \otimes_k 1_V"] \arrow[r, dotted, "1_{k[\mathbf{M}]} \otimes_k f"] & k[\mathbf{M}] \otimes_k W \arrow[u, "i \otimes_k 1_W"'] \\
V \arrow[u, "c_V"] \arrow[r, "f"'] & W \arrow[u, "c_W"']
\end{tikzcd}$$

in which the outer square is commutative. Our goal is to prove that the bottom square is commutative. We have

$$(i \otimes_k 1_W) \cdot c_W \cdot f = \left(1_{k[\mathbf{G}]} \otimes_k f\right) \cdot (i \otimes_k 1_V) \cdot c_V = (i \otimes_k 1_W) \cdot \left(1_{k[\mathbf{M}]} \otimes_k f\right) \cdot c_V$$

Since $i \otimes_k 1_W$ is a monomorphism, we deduce that $c_W \cdot f = \left(1_{k[\mathbf{M}]} \otimes_k f\right) \cdot c_V$. Hence f is a morphism of $\mathbf{M}$-representations.

Next we prove that $\mathbf{Rep}(\mathbf{M})$ is a subcategory of $\mathbf{Rep}(\mathbf{G})$ that is closed under subquotients. Consider an $\mathbf{M}$-representation V and its quotient $\mathbf{G}$-representations $q : V \twoheadrightarrow W$. We show that W is a quotient $\mathbf{M}$-representation of V. Let c_V be the coaction of $\mathbf{M}$ on V and let d_W be the coaction of $\mathbf{G}$ on W. We have a commutative diagram

$$\begin{tikzcd}
k[\mathbf{G}] \otimes_k V \arrow[r, two heads, "1_{k[\mathbf{G}]} \otimes_k q"] & k[\mathbf{G}] \otimes_k W \\
k[\mathbf{M}] \otimes_k V \arrow[u, "i \otimes_k 1_V"] & \\
V \arrow[u, "c_V"] \arrow[r, two heads, "q"'] & W \arrow[uu, "d_W"']
\end{tikzcd}$$

and hence $d_W(W) \subseteq k[\mathbf{M}] \otimes_k W$. Thus Theorem 3.3 implies that W is a representation of $\mathbf{M}$ and q is a morphism of $\mathbf{M}$-representations. This shows that $\mathbf{Rep}(\mathbf{M})$ is a subcategory of $\mathbf{Rep}(\mathbf{G})$ closed under quotients. Next let $j : U \hookrightarrow V$ be a $\mathbf{G}$-subrepresentation of a $\mathbf{M}$-representation V. By what we proved above the cokernel $q : V \twoheadrightarrow W$ of j in $\mathbf{Rep}(\mathbf{G})$ is contained in $\mathbf{Rep}(\mathbf{M})$. Since both $\mathbf{Rep}(\mathbf{M})$ and $\mathbf{Rep}(\mathbf{G})$ are abelian and the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$ is exact, we derive that the kernel of q in $\mathbf{Rep}(\mathbf{M})$ coincides with its kernel in $\mathbf{Rep}(\mathbf{G})$. Thus U is a $\mathbf{M}$-representation and $j : U \hookrightarrow V$ is a morphism of $\mathbf{M}$-representations. Hence $\mathbf{Rep}(\mathbf{M})$ is the category of $\mathbf{Rep}(\mathbf{G})$ closed under subobjects. □

Theorem 3.5. *Assume that* $\mathbf{G}$ *is open and schematically dense in* $\mathbf{M}$. *Let* V *be a* $\mathbf{G}$*-representation of* $\mathbf{G}$. *There exists an* $\mathbf{M}$*-representation* W *and a surjective morpism* $q : V \twoheadrightarrow W$ *of* $\mathbf{G}$*-representations such that for every* $\mathbf{M}$*-representation* U *and a morphism* $f : V \to U$ *of* $\mathbf{G}$*-representations there exists a unique morphism* $\tilde{f} : W \to U$ *of* $\mathbf{M}$*-representations making the triangle*

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, two heads, "q"'] & U \\
W \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

commutative.

Proof. Assume first that V is finite dimensional. Let $\mathcal{K}$ be a set of $\mathbf{G}$-subrepresentations of V that consists of all $K \subseteq V$ such that V/K carries a structure of $\mathbf{M}$-representation. Clearly $\mathcal{K} = \varnothing$ because $\{0\} \in \mathcal{K}$. Since V is finite dimensional, there exists a finite subset $\{K_1, ..., K_n\} \subseteq \mathcal{K}$ such that

$$\bigcap_{i=1}^{n} K_i = \bigcap_{K \in \mathcal{K}} K$$

Then a morphism

$$V/\left(\bigcap_{K \in \mathcal{K}} K\right) \ni v \mapsto \left(v \bmod K_i\right)_{1 \leq i \leq n} \in \bigoplus_{i=1}^{n} V/K_i$$

is a monomorphism and hence by Theorem 3.4 the quotient $W = V/\left(\bigcap_{K \in \mathcal{K}} K\right)$ is an $\mathbf{M}$-representation. Let $q : V \twoheadrightarrow W$ be the canonical epimorphism. Consider now a morphism $f : V \to U$ of $\mathbf{G}$-representations, where U is an $\mathbf{M}$-representation. Then $\mathrm{im}(f)$ is a $\mathbf{G}$-subrepresentation of U and by Theorem 3.4 we derive that $\mathrm{im}(f)$ is an $\mathbf{M}$-representation. This implies that $\ker(f)$ is in $\mathcal{K}$. Hence f factors through q. Thus there exists a unique morphism $\tilde{f} : W \to U$ of $\mathbf{G}$-representations such that $\tilde{f} \cdot q = f$. This completes the proof in case when V is finite dimensional.

Now consider the general V. Let $\mathcal{F}$ be the set of all finite dimensional **G**-representations of V. According to [Monygham, 2020e, Corollary 15.2] we deduce that $V = \mathrm{colim}_{F\in\mathcal{F}}F$. By the case considered above we deduce that for every F in $\mathcal{F}$ there exists a universal morphism $q_F : F \to W_F$ of **G**-representations into an **M**-representation W_F. Note that if $F_1 \subseteq F_2$ are two elements of $\mathcal{F}$, then

$$\begin{tikzcd} F_1 \arrow[r, two heads, "q_{F_1}"] \arrow[d, hook] & W_{F_1} \arrow[d] \\ F_2 \arrow[r, two heads, "q_{F_2}"'] & W_{F_2} \end{tikzcd}$$

Thus $\{W_F\}_{F\in\mathcal{F}}$ together with morphisms $W_{F_1} \to W_{F_2}$ for $F_1 \subseteq F_2$ in $\mathcal{F}$ form a diagram parametrized by the poset $\mathcal{F}$. The category $\mathbf{Rep}(\mathbf{M})$ has small colimits ([Monygham, 2020e, Corollary 14.5]) and we define $W = \mathrm{colim}_{F\in\mathcal{F}}W_F$. This is also a colimit of this diagram in the category $\mathbf{Rep}(\mathbf{G})$ by Fact 3.2. We also define $q = \mathrm{colim}_{F\in\mathcal{F}}q_F : V = \mathrm{colim}_{F\in\mathcal{F}}F \to W$. Since a colimit of a family of epimorphisms is an epimorphism, we derive that q is an epimorphism of **G**-representations. Suppose now that $f : V \to U$ is a morphism of **G**-representations and U is an **M**-representation. Then $f_{|F}$ uniquely factors through q_F for every F in $\mathcal{F}$. Hence by universal property of colimits we derive that f factors through q in a unique way. This completes the proof. $\square$

4. Białynicki-Birula functors

In this section we fix an affine group k-scheme **G**. Let **M** be an affine monoid k-scheme with zero **o** such that **G** is its group of units.

Definition 4.1. Let X be a k-scheme equipped with an action of **G**. For every k-scheme Y (considered as **G**-scheme with the trivial **G**-action) we define

$$\mathcal{D}_X(Y) = \left\{\gamma : \mathbf{M}\times_k Y \to X \,\middle|\, \gamma \text{ is } \mathbf{G}\text{-equivariant}\right\}$$

This gives gives rise to a subfunctor $\mathcal{D}_X$ of $\mathrm{Mor}_k(\mathbf{M}\times_k(-), X) : \mathbf{Sch}_k^{op} \to \mathbf{Set}$. We call it *the Białynicki-Birula functor of* X.

Fact 4.2. *Let X be a scheme equipped with an action of* **G**. *Then $\mathcal{D}_X$ is a Zariski sheaf.*

Proof. This is a consequence of the fact that $\mathrm{Mor}_k(\mathbf{M}\times_k(-), X)$ is a Zariski sheaf and if we glue **G**-equivariant morphisms, then the result is **G**-equivariant. Indeed, this shows that $\mathcal{D}_X$ is a Zariski subsheaf of $\mathrm{Mor}_k(\mathbf{M}\times_k(-), X)$. $\square$

Remark 4.3. Let X be a k-scheme equipped with an action of **G**. Then there are canonical morphism of functors

$$\begin{tikzcd} \mathcal{D}_X \arrow[r, "i_X"] \arrow[d, "r_X"] & X \\ X^{\mathbf{G}} \arrow[u, bend left, "s_X"] & \end{tikzcd}$$

which we define now. First let us explain that in the diagram X stands for the presheaf representable by the k-scheme X (Remark 1.1) and $X^{\mathbf{G}}$ denotes the functor of fixed points of X ([Monygham, 2020d, Definition 7.1]). Now fix k-scheme Y and $\gamma \in \mathcal{D}_X(Y)$, then we define

$$i_X(\gamma) = \gamma_{|\{e\}\times_k X} = \gamma\cdot\langle e, 1_X\rangle,\; r_X(\gamma) = \gamma_{|\{o\}\times_k X} = \gamma\cdot\langle o, 1_X\rangle$$

where $e : \operatorname{Spec} k \to \mathbf{M}$ is the unit of $\mathbf{M}$ and $\mathbf{o} : \operatorname{Spec} k \to \mathbf{M}$ is the zero. Next if $f : Y \to X$ is a morphism in $X^{\mathbf{G}}(Y)$, then we define

$$s_X(f) = f \cdot pr_Y$$

where $pr_Y : \mathbf{M} \times_k Y \to Y$ is the projection. Finally note that $r_X \cdot s_X = 1_{X^{\mathbf{G}}}$.

Remark 4.4. Let X be a k-scheme equipped with an action of $\mathbf{G}$. Then $\mathbf{M}$ (actually the presheaf of monoids represented by $\mathbf{M}$) acts on $\mathcal{D}_X$. Indeed, fix k-scheme Y, $\gamma \in \mathcal{D}_X(Y)$ and $m : Y \to \mathbf{M}$. Then we define the product

$$m\gamma = \gamma \cdot \langle m, 1_Y \rangle$$

and this determines an action of $\mathbf{M}$ on $\mathcal{D}_X$. Moreover, with respect to this action i_X is $\mathbf{G}$-equivariant and r_X, s_X are $\mathbf{M}$-equivariant ($X^{\mathbf{G}}$ is equipped with trivial action of $\mathbf{M}$).

Remark 4.5. Let X, Y be k-schemes equipped with actions of $\mathbf{G}$ and let $f : X \to Y$ be a $\mathbf{G}$-equivariant morphism, then there exists a morphism of functors $\mathcal{D}_f : \mathcal{D}_X \to \mathcal{D}_Y$ given by

$$\mathcal{D}_f(\gamma) = f \cdot \gamma$$

for every element γ of the functor $\mathcal{D}_X$. Moreover, $\mathcal{D}_f$ preserves the action of $\mathbf{M}$ described in Remark 4.4 above.

Let X be a k-scheme equipped with an action of $\mathbf{G}$. It is useful to discuss subfunctors of $\mathcal{D}_X$ defined by closed $\mathbf{G}$-stable subschemes of X.

Theorem 4.6. *Let X be a k-scheme equipped with an action of the group $\mathbf{G}$. Suppose that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. If $j : Z \hookrightarrow X$ is a closed $\mathbf{G}$-stable subscheme of X, then the square*

$$\begin{tikzcd} \mathcal{D}_Z \arrow[r, "\mathcal{D}_j"] \arrow[d, "i_Z"'] & \mathcal{D}_X \arrow[d, "i_X"] \\ Z \arrow[r, "j"'] & X \end{tikzcd}$$

is cartesian in the category of presheaves on $\mathbf{Sch}_k$.

Proof. The fact that the square is commutative follows by examination of definitions in Remarks 4.3 and 4.5. Pick k-scheme Y, $f : Y \to Z$ and $\gamma \in \mathcal{D}_X(Y)$ such that $j \cdot f = i_X(\gamma)$. This is depicted in the diagram

$$\begin{tikzcd} & \gamma \arrow[d, mapsto, "i_X"] \\ f \arrow[r, mapsto, "j"'] & j \cdot f = \gamma_{|\{e\} \times_k X} \end{tikzcd}$$

Our goal is to show that there exists a unique $\mathbf{G}$-equivariant morphism $\eta : \mathbf{M} \times_k Y \to U$ such that $\mathcal{D}_j(\eta) = \gamma$ and $i_Z(\eta) = f$. This is depicted by the diagram

$$\begin{tikzcd} \eta \arrow[r, mapsto, "\mathcal{D}_j"] \arrow[d, mapsto, "r_U"'] & \gamma = j \cdot \eta \\ f = \eta_{|\{e\} \times_k X} & \end{tikzcd}$$

It suffices to prove that γ factors through j. First note that the assumption $\gamma_{|\{e\}\times_k Y} = j\cdot f$ implies that

$$\gamma_{|\mathbf{G}\times_k Y} = j\cdot f\cdot pr_Y$$

where $pr_Y : \mathbf{G}\times_k Y \to Y$ is the projection. This implies that $\gamma_{|\mathbf{G}\times_k}$ factors through j. Consider scheme-theoretic preimage $\gamma^{-1}(Z)$. Then $\gamma^{-1}(Z)$ is a closed $\mathbf{G}$-stable (as an inverse image of a $\mathbf{G}$-stable closed subscheme under the $\mathbf{G}$-equivariant morphism) subscheme of $\mathbf{M}\times_k Y$, which contains $\mathbf{G}\times_k Y$. Since $\mathbf{G}$ is open, schematically dense in $\mathbf{M}$ and k is a field, we derive that $\mathbf{G}\times_k Y$ is open and schematically dense in $\mathbf{M}\times_k Y$. Thus $\gamma^{-1}(Z) = \mathbf{M}\times_k Y$ and hence γ factors through j. □

In order to prove interesting result in the spirit of Theorem 4.6 which concerns open $\mathbf{G}$-stable subschemes, we need to assume that $\mathbf{M}$ is a Kempf monoid.

Theorem 4.7. *Let X be a k-scheme equipped with an action of the group $\mathbf{G}$ of units of a Kempf monoid $\mathbf{M}$. If $j : U \hookrightarrow X$ is an open $\mathbf{G}$-stable subscheme of X, then the square*

$$\begin{tikzcd} \mathcal{D}_U \arrow[r, "\mathcal{D}_j"] \arrow[d, "r_U"'] & \mathcal{D}_X \arrow[d, "r_X"] \\ U^{\mathbf{G}} \arrow[r, "j^{\mathbf{G}}"'] & X^{\mathbf{G}} \end{tikzcd}$$

is cartesian in the category of presheaves on $\mathbf{Sch}_k$.

Proof. The fact that the square is commutative follows by examination of definitions in Remarks 4.3 and 4.5. Pick k-scheme Y, $f \in U^{\mathbf{G}}(Y)$ and $\gamma \in \mathcal{D}_X(Y)$ such that $j^{\mathbf{G}}(f) = r_X(\gamma)$. This is depicted in the diagram

$$\begin{tikzcd} & \gamma \arrow[d, mapsto, "r_X"] \\ f \arrow[r, mapsto, "j^{\mathbf{G}}"'] & j\cdot f = \gamma_{|\{\mathbf{o}\}\times_k Y} \end{tikzcd}$$

Our goal is to show that there exists a unique $\mathbf{G}$-equivariant morphism $\eta : \mathbf{M}\times_k Y \to U$ such that $\mathcal{D}_j(\eta) = \gamma$ and $r_U(\eta) = f$. This is depicted by the diagram

$$\begin{tikzcd} \eta \arrow[r, mapsto, "\mathcal{D}_j"] \arrow[d, mapsto, "r_U"'] & \gamma = j\cdot\eta \\ f = \eta_{|\{\mathbf{o}\}\times_k Y} & \end{tikzcd}$$

Fir this it suffices to prove that γ factors through j. Consider $W = \gamma^{-1}(U)$. Note that W is an open $\mathbf{G}$-stable (as an inverse image of a $\mathbf{G}$-stable open subscheme under the $\mathbf{G}$-equivariant morphism) subscheme of $\mathbf{M}\times_k Y$, which contains $\{\mathbf{o}\}\times_k Y$. [Monygham, 2020a, Theorem 3.8] asserts that for every geometric point $\overline{y}$ of Y we have $W_{\overline{y}} = \mathbf{M}_{k(\overline{y})}$, where $W_{\overline{y}}$ is the fiber over $\overline{y}$ of the projection $\mathbf{M}\times_k Y \to Y$ restricted to W. Since W is an open subscheme of $\mathbf{M}\times_k Y$, this implies that $W = \mathbf{M}\times_k Y$ and hence γ factors through j. □

As we shall see below both Theorems are extremely useful properties of Białynicki-Birula functors.

5. Formal Białynicki-Birula functors

We introduce a formal version of the Białynicki-Birula functor. We fix an affine group k-scheme $\mathbf{G}$. Let $\mathbf{M}$ be an affine monoid k-scheme with zero $\mathbf{o}$ such that $\mathbf{G}$ is its group of units.

Definition 5.1. Let $\mathbf{M}$ be an affine monoid k-scheme with zero $\mathbf{o}$ and let $\mathbf{G}$ be its group of units. For every $n \in \mathbb{N}$ let $\mathbf{M}_n \hookrightarrow \mathbf{M}$ be an n-th infinitesimal neighborhood of $\mathbf{o}$ in $\mathbf{M}$. Let X be a k-scheme equipped with an action of $\mathbf{G}$. For every k-scheme Y (considered as $\mathbf{G}$-scheme with the trivial $\mathbf{G}$-action) we define

$$\widehat{\mathcal{D}}_X(Y) = \left\{ \{\gamma_n : \mathbf{M}_n \times_k Y \to X\}_{n\in\mathbb{N}} \,\middle|\, \forall_{n\in\mathbb{N}}\, \gamma_n \text{ is } \mathbf{G}\text{-equivariant and } \gamma_{n+1|\mathbf{M}_n \times_k Y} = \gamma_n \right\}$$

This gives gives rise to a functor $\widehat{\mathcal{D}}_X$. We call it *the formal Białynicki-Birula functor of X.*

Remark 5.2. Let X, Y be k-schemes equipped with actions of $\mathbf{G}$ and let $f : X \to Y$ be a $\mathbf{G}$-equivariant morphism, then there exists a morphism of functors $\widehat{\mathcal{D}}_f : \widehat{\mathcal{D}}_X \to \widehat{\mathcal{D}}_Y$ given by

$$\widehat{\mathcal{D}}_f\left(\{\gamma_n\}_{n\in\mathbb{N}}\right) = \{f \cdot \gamma_n\}_{n\in\mathbb{N}}$$

for every element γ of the functor $\widehat{\mathcal{D}}_X$.

Remark 5.3. Let $\mathbf{M}$ be an affine monoid k-scheme with zero $\mathbf{o}$ and let $\mathbf{G}$ be its group of units. Let X be a k-scheme equipped with an action of $\mathbf{G}$. Then there exists a canonical morphism of functors $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ given by

$$\gamma \mapsto \{\gamma_{|\mathbf{M}_n \times_k Y}\}_{n\in\mathbb{N}}$$

for every $\gamma \in \mathcal{D}_X(Y)$ and every k-scheme Y.

Remark 5.4. Let X be a k-scheme equipped with an action of $\mathbf{G}$. We define a morphism $\widehat{r}_X : \widehat{\mathcal{D}}_X \to X^{\mathbf{G}}$ by formula

$$\widehat{\mathcal{D}}_X(Y) \ni \{\gamma_n\}_{n\in\mathbb{N}} \mapsto \gamma_0 \in X^{\mathbf{G}}(Y)$$

for every k-scheme Y. This morphism fits into a commutative triangle

$$\begin{tikzcd}
\mathcal{D}_X \arrow[rr] \arrow[dr, "r_X"'] & & \widehat{\mathcal{D}}_X \arrow[dl, "\widehat{r}_X"] \\
& X^{\mathbf{G}} &
\end{tikzcd}$$

where horizontal morphism is described in Remark 5.3.

The next result is analogous to Theorem 4.7, although its proof is much simpler.

Proposition 5.5. *Let X be a k-scheme equipped with an action of the group $\mathbf{G}$. If $j : U \hookrightarrow X$ is an open $\mathbf{G}$-stable subscheme of X, then the square*

$$\begin{tikzcd}
\widehat{\mathcal{D}}_U \arrow[r, "\widehat{\mathcal{D}}_j"] \arrow[d, "\widehat{r}_U"'] & \widehat{\mathcal{D}}_X \arrow[d, "\widehat{r}_X"] \\
U^{\mathbf{G}} \arrow[r, "j^{\mathbf{G}}"'] & X^{\mathbf{G}}
\end{tikzcd}$$

is cartesian in the category of presheaves on $\mathbf{Sch}_k$.

Proof. The fact that the square is commutative follows by examination of definitions in Remarks 5.2 and 5.2. Pick k-scheme Y, $f \in U^{\mathbf{G}}(Y)$ and $\{\gamma_n\}_{n\in\mathbb{N}} \in \widehat{\mathcal{D}}_X(Y)$ such that $j^{\mathbf{G}}(f) = \widehat{r}_X\left(\{\gamma_n\}_{n\in\mathbb{N}}\right)$. This is depicted in the diagram

$$\begin{tikzcd} & \{\gamma_n\}_{n\in\mathbb{N}} \arrow[d, mapsto, "\widehat{r}_X"] \\ f \arrow[r, mapsto, "j^{\mathbf{G}}"'] & j\cdot f = \gamma_0 \end{tikzcd}$$

Our goal is to show that there exists a unique family of $\mathbf{G}$-equivariant morphism $\eta_n : \mathbf{M}_n \times_k Y \to U$ for $n \in \mathbb{N}$ such that $\widehat{\mathcal{D}}_j\big(\{\eta_n\}_{n\in\mathbb{N}}\big) = \{\gamma_n\}_{n\in\mathbb{N}}$ and $\widehat{r}_U\big(\{\eta_n\}_{n\in\mathbb{N}}\big) = f$. This is depicted by the diagram

$$\begin{tikzcd} \{\eta_n\}_{n\in\mathbb{N}} \arrow[r, mapsto, "\mathcal{D}_j"] \arrow[d, mapsto, "r_U"'] & \{\gamma_n\}_{n\in\mathbb{N}} = \{j\cdot\eta_n\}_{n\in\mathbb{N}} \\ f = \eta_0 & \end{tikzcd}$$

Fir this it suffices to prove that γ_n factors through j for every $n \in \mathbb{N}$. Note that all maps $\{\gamma_n\}_{n\in\mathbb{N}}$ are equal set-theoretically and $\gamma_0 = j\cdot f$ factors through j. Thus γ_n factors through j for every $n \in \mathbb{N}$. □

Theorem 5.6. *Let* $\mathbf{G}$ *be a group k-scheme and* $\mathbf{M}$ *be a Kempf monoid having* $\mathbf{G}$ *as a group of units. Suppose that* X *is a k-scheme equipped with an action of* $\mathbf{G}$*. Then the canonical morphism* $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ *is a monomorphism of functors.*

For the proof it is useful to make the following observation (essentially the same observation was made in the proof of Theorem 4.6).

Lemma 5.6.1. *Let* X *be a k-scheme equipped with an action of a monoid k-scheme* $\mathbf{M}$*. Suppose that* $j : Z \hookrightarrow X$ *is closed* $\mathbf{G}$*-equivariant immersion, where* $\mathbf{G}$ *is a group of units of* $\mathbf{M}$*. If* $\mathbf{G}$ *is schematically dense in* $\mathbf{M}$*, then the action of* $\mathbf{G}$ *on* Z *extends to the action of* $\mathbf{M}$ *in such a way that* j *becomes* $\mathbf{M}$*-equivariant.*

Proof of the lemma. Let $a : \mathbf{M}\times_k X \to X$ be the action of $\mathbf{M}$ on X. Since j is $\mathbf{G}$-equivariant, we derive that $\mathbf{G}\times_k Z \subseteq a^{-1}(Z)$. Moreover, $\mathbf{G}\times_k Z$ is open and schematically dense in $\mathbf{M}\times_k Z$. Hence $\mathbf{M}\times_k Z \subseteq a^{-1}(Z)$ and thus $a_{|\mathbf{M}\times_k Z}$ factors through $j : Z \hookrightarrow X$. □

Proof of the theorem. Let Y be a k-scheme and let $\gamma, \eta : \mathbf{M}\times_k Y \to X$ be $\mathbf{G}$-equivariant morphisms. Suppose that $\gamma_{|\mathbf{M}_n\times_k Y} = \eta_{|\mathbf{M}_n\times_k Y}$ for every $n \in \mathbb{N}$. Consider the kernel (equalizer) $j : E \hookrightarrow \mathbf{M}\times_k Y$ of the pair (γ, η). Then E admits an action of $\mathbf{G}$ such that i is $\mathbf{G}$-equivariant locally closed immersion and $\mathbf{M}_n \times_k Y \subseteq E$ for every $n \in \mathbb{N}$. Fix a point y in Y. Let $\mathbf{M}_y$ and E_y be fibers of the projection $\mathrm{pr} : \mathbf{M}\times_k Y \to Y$ and $\mathrm{pr}\cdot j$, respectively. Then $E_y \subseteq \mathbf{M}_y$ is a locally closed $\mathbf{G}_y$-equivariant subscheme, where $\mathbf{G}_y = \mathbf{G}\times_k \operatorname{Spec} k(y)$. Since $\mathbf{M}_y = \mathbf{M}\times_k \operatorname{Spec} k(y)$ is a Kempf monoid over $k(y)$ with group of units $\mathbf{G}_y$ and moreover, E_y contains all infinitesimal neighborhoods of the zero in $\mathbf{M}_y$, we deduce by [Monygham, 2020a, Theorem 3.8] that $E_y = \mathbf{M}_y$. This implies that a locally closed immersion $j : E \hookrightarrow \mathbf{M}\times_k Y$ is bijective. Hence it is a closed immersion. Now Lemma 5.6.1 implies that E is a locally linear $\mathbf{M}$-scheme and j is $\mathbf{M}$-equivariant. Note that j induces an isomorphism $\widehat{E} \cong \widehat{\mathbf{M}\times_k Y}$ of formal $\mathbf{M}$-schemes. Hence according to [Monygham, 2020b, Corollary 7.4] we infer that j is an isomorphism. This proves that $\gamma = \eta$. Therefore, the map

$$\mathcal{D}_X(Y) \to \widehat{\mathcal{D}}_X(Y)$$

is injective. As Y is arbitrary we infer that the canonical morphism $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ of Remark 5.3 is a monomorphism of functors. □

6. Representability of Białynicki-Birula functor for Kempf monoids

In this section we prove various results concerning representability of Białynicki-Birula functors.

Theorem 6.1. *Let $\mathbf{M}$ be an affine monoid k-scheme with open and schematically dense group of units $\mathbf{G}$. Suppose that X is an affine k-scheme equipped with an ation of $\mathbf{G}$. Then $\mathcal{D}_X$ is representable and i_X is a closed immersion of k-schemes.*

Proof. Since X is an affine k-scheme, the action of $\mathbf{G}$ on X corresponds to the coaction of $k[\mathbf{G}]$ by $c : \Gamma(X, \mathcal{O}_X) \to k[\mathbf{G}] \otimes_k \Gamma(X, \mathcal{O}_X)$. Note that c is a morphism of k-algebras. By Theorem 3.5 there exists a universal morphism $q : \Gamma(X, \mathcal{O}_X) \twoheadrightarrow W$ of $\mathbf{G}$-representations into a $\mathbf{M}$-representation. Let $I \subseteq \Gamma(X, \mathcal{O}_X)$ be the ideal generated by $\ker(q)$. Fix f in I. Then

$$f = \sum_{i=1}^{n} g_i \cdot f_i$$

where $g_i \in k[\mathbf{G}]$ and $f_i \in \ker(q)$ for $1 \le i \le n$. Then

$$c(f) = c\left(\sum_{i=1}^{n} g_i \cdot f_i\right) = \sum_{i=1}^{n} c(g_i) \cdot c(f_i) \subseteq \Big(k[\mathbf{G}] \otimes_k \Gamma(X, \mathcal{O}_X)\Big) \cdot \Big(k[\mathbf{G}] \otimes_k \ker(q)\Big) \subseteq k[\mathbf{G}] \otimes_k I$$

Thus $c(I) \subseteq k[\mathbf{G}] \otimes_k I$ and hence I is a $\mathbf{G}$-representation. Consider

$$X^+ = V(I) = \operatorname{Spec} \Gamma(X, \mathcal{O}_X)/I \hookrightarrow X$$

Since $\Gamma(X, \mathcal{O}_X)/I$ is the quotient $\mathbf{G}$-representation of W, we deduce by Theorem 3.5 that $\Gamma(X, \mathcal{O}_X)/I$ is a $\mathbf{M}$-representation. Hence X^+ is a k-scheme equipped with action of $\mathbf{M}$ and $X^+ \hookrightarrow X$ is $\mathbf{G}$-equivariant. Suppose now that Y is an affine k-scheme. Then $\mathbf{M} \times_k Y$ is a $\mathbf{M}$-scheme with respect to the left-hand side action of $\mathbf{M}$ and hence $\Gamma(\mathbf{M} \times_k Y, \mathcal{O}_{\mathbf{M} \times_k Y})$ is a $\mathbf{M}$-representation. Now Theorem 3.5 implies that if $\gamma : \mathbf{M} \times_k Y \to X$ is a $\mathbf{G}$-equivariant morphism, then a morphism $\gamma^\# : \Gamma(X, \mathcal{O}_X) \to \Gamma(\mathbf{M} \times_k Y, \mathcal{O}_{\mathbf{M}} \times_k Y)$ of k-algebras and $\mathbf{G}$-representations factors through $q : \Gamma(X, \mathcal{O}_X) \twoheadrightarrow W$ and thus by construction of I we have

$$\begin{tikzcd} \Gamma(X, \mathcal{O}_X) \arrow[r, two heads] \arrow[rr, bend left, "\gamma^\#"] & \Gamma(X, \mathcal{O}_X)/I \arrow[r, "f"] & \Gamma(\mathbf{M} \times_k Y, \mathcal{O}_{\mathbf{M}} \times_k Y) \end{tikzcd}$$

for some morphism f of k-algebras and $\mathbf{G}$-representations. Since both $\Gamma(X, \mathcal{O}_X)/I$ and $\Gamma(\mathbf{M} \times_k Y, \mathcal{O}_{\mathbf{M}} \times_k Y)$ are $\mathbf{M}$-representations and by Theorem 3.4 the subcategory $\mathbf{Rep}(\mathbf{M}) \subseteq \mathbf{Rep}(\mathbf{G})$ is full, we derive that f is a morphism of $\mathbf{M}$-representations. Thus f corresponds to a unique $\mathbf{M}$-equivariant morphism $\eta : \mathbf{M} \times_k Y \to X^+$ such that the diagram

$$\begin{tikzcd} & X^+ \arrow[d, hook] \\ \mathbf{M} \times_k Y \arrow[ur, bend left, "\eta"] \arrow[r, "\gamma"'] & X \end{tikzcd}$$

is commutative. Now this result can be extended to an arbitrary k-scheme Y, since $\operatorname{Mor}_k(\mathbf{M} \times_k (-), X^+)$ is a Zariski sheaf and a morphism that is $\mathbf{M}$-equivariant locally on the domain is $\mathbf{M}$-equivariant. Thus for every k-scheme Y we have a bijection

$$\mathcal{D}_X(Y) \ni \gamma \mapsto \eta \in \left\{\mathbf{M}\text{-equivariant morphisms } \mathbf{M} \times_k Y \to X^+\right\}$$

Since we also have a bijection

$$\{\mathbf{M}\text{-equivariant morphisms } \mathbf{M}\times_k Y \to X^+\} \ni \eta \mapsto \eta\cdot\langle e, 1_{X^+}\rangle \in \mathrm{Mor}_k(Y, X^+)$$

and both this bijections are natural, we derive that $\mathcal{D}_X$ is represented by X^+ and moreover, $i_X : \mathcal{D}_X \to X$ is a closed immersion $X^+ \hookrightarrow X$. □

Corollary 6.2. *Let $\mathbf{G}$ be a group k-scheme and $\mathbf{M}$ be a Kempf monoid having $\mathbf{G}$ as a group of units. Suppose that X is a k-scheme equipped with an action of $\mathbf{G}$ such that there exists a family $\mathcal{U}$ of open affine $\mathbf{G}$-stable open subschemes of X such that functors $\{U^{\mathbf{G}}\}_{U\in\mathcal{U}}$ form an open cover of $X^{\mathbf{G}}$. Then $\mathcal{D}_X$ is representable.*

Proof. Note that $\mathbf{G}$ is affine group k-scheme as a unit group of an affine monoid $\mathbf{M}$ ([Monygham, 2020e, Proposition 12.4]). Moreover, $\mathbf{M}$ is a Kempf monoid and hence $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. By Theorem 6.1 each $\mathcal{D}_U$ is representable by a k-scheme. On the other hand by Theorem 4.7 for each $U \in \mathcal{U}$ we have a cartesian square

$$\begin{tikzcd} \mathcal{D}_U \arrow[r] \arrow[d, "r_U"'] & \mathcal{D}_X \arrow[d, "r_X"] \\ U^{\mathbf{G}} \arrow[r] & X^{\mathbf{G}} \end{tikzcd}$$

of functors. This implies that $\{\mathcal{D}_U \hookrightarrow \mathcal{D}_X\}_{U\in\mathcal{U}}$ is an open cover of $\mathcal{D}_X$ as a pullback of an open cover $\{U^{\mathbf{G}} \hookrightarrow X^{\mathbf{G}}\}_{U\in\mathcal{U}}$. Hence Fact 4.2 and [Görtz and Wedhorn, 2010, Theorem 8.9] (or if you like [Monygham, 2019, Theorem 4.6]) imply that $\mathcal{D}_X$ is representable. □

Corollary 6.3. *Let $\mathbf{G}$ be group k-scheme and $\mathbf{M}$ be a Kempf monoid having $\mathbf{G}$ as a group of units. Suppose that X is a locally linear $\mathbf{G}$-scheme. Then $\mathcal{D}_X$ is representable.*

Proof. This is a consequence of Corollary 6.2. Indeed, X admits a cover $\mathcal{U}$ by open $\mathbf{G}$-stable affine subschemes. Then $\{U^{\mathbf{G}}\}_{U\in\mathcal{U}}$ is an open cover of $X^{\mathbf{G}}$. □

Now we prove our main result.

Theorem 6.4. *Let $\mathbf{G}$ be a group k-scheme and $\mathbf{M}$ be a Kempf monoid having $\mathbf{G}$ as a group of units. Suppose that X is a k-scheme equipped with an action of $\mathbf{G}$. Then the following results hold.*

- **(1)** *$\widehat{\mathcal{D}}_X$ is representable. Moreover, the morphism $\widehat{r}_X : \widehat{\mathcal{D}}_X \to X^{\mathbf{G}}$ is affine and if X is locally noetherian, then it is of finite type.*
- **(2)** *If X is of finite type over k, then the canonical morphism $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ is an isomorphism of functors.*

Proof. Consider the ideal $\mathcal{I}$ in $\mathcal{O}_X$ corresponding to a closed subscheme $X^{\mathbf{G}}$ of X. We define X_n as a closed subscheme of X determined by the ideal $\mathcal{I}^n$ and we denote by $\mathcal{I}_n$ the ideal of X_0 in X_n. Then $\widehat{X} = \{X_n\}_{n\in\mathbb{N}}$ is a formal $\mathbf{G}$-scheme. Moreover, by [Monygham, 2020b, Corollary 5.1] each X_n is a locally linear $\mathbf{G}$-scheme and hence by Corollary 6.3 there exists a k-scheme Z_n equipped with $\mathbf{M}$-action that represents $\mathcal{D}_{X_n}$. Note that the square

$$\begin{tikzcd} Z_n \arrow[r, hook] \arrow[d, "i_n"'] & Z_{n+1} \arrow[d, "i_{n+1}"] \\ X_n \arrow[r, hook] & X_{n+1} \end{tikzcd}$$

is cartesian according to Theorem 4.6 for each $n \in \mathbb{N}$. This implies that the vanishing closed subscheme of $i_{n+1}^{-1}\mathcal{I}_{n+1}^{n} \cdot \mathcal{O}_{Z_{n+1}}$ in Z_{n+1} is Z_n. Since the square

$$\begin{tikzcd} Z_0 \arrow[r, hook] \arrow[d, "i_0"'] & Z_{n+1} \arrow[d, "i_{n+1}"] \\ X_0 \arrow[r, hook] & X_{n+1} \end{tikzcd}$$

is cartesian as a combination of cartesian squares, we derive that the vanishing closed subscheme of $i_{n+1}^{-1}\mathcal{I}_{n+1} \cdot \mathcal{O}_{Z_{n+1}}$ in Z_{n+1} is Z_0. Note that

$$\left(i_{n+1}\mathcal{I}_{n+1} \cdot \mathcal{O}_{Z_{n+1}}\right)^{n} = i_{n+1}^{-1}\mathcal{I}_{n+1}^{n}\mathcal{O}_{Z_{n+1}}$$

Thus $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ is a formal $\mathbf{G}$-scheme. According to Remarks 4.4 and 4.5, we derive that it is a formal $\mathbf{M}$-scheme. Now the commutative diagram

$$\begin{tikzcd} Z_0 \arrow[r, hook] \arrow[d, "i_0"'] & \ldots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[d, "i_n"'] & Z_{n+1} \arrow[r, hook] \arrow[d, "i_{n+1}"'] & \ldots & \\ X_0 \arrow[r, hook] & \ldots \arrow[r, hook] & X_n \arrow[r, hook] & X_{n+1} \arrow[r, hook] & \ldots \arrow[r, hook] & X \end{tikzcd}$$

shows that $\{i_n\}_{n\in\mathbb{N}}$ is a morphism of formal $\mathbf{G}$-schemes. Since $\mathbf{M}$ is a Kempf monoid, [Monygham, 2020b, Theorem 7.1] implies that there exists a locally linear $\mathbf{M}$-scheme Z such that $\widehat{Z} = \{Z_n\}_{n\in\mathbb{N}}$. Here our argument ramifies. We first provide the proof of **(1)** and later deal with **(2)**.

- Consider a k-scheme Y and a family $\{\gamma_n : \mathbf{M}_n \times_k Y \to X\}_{n\in\mathbb{N}} \in \widehat{\mathcal{D}}_X(Y)$. Note that γ_n uniquely factors through X_n and hence there exists a unique $\mathbf{M}$-equivariant morphism $\delta_n : \mathbf{M}_n \times_k Y \to Z_n$. Hence the family $\{\delta_n\}_{n\in\mathbb{N}}$ is a morphism

$$\begin{tikzcd} \mathbf{M}_0 \times_k Y \arrow[r, hook] \arrow[d, "\delta_0"'] & \ldots \arrow[r, hook] & \mathbf{M}_n \times_k Y \arrow[r, hook] \arrow[d, "\delta_n"'] & \mathbf{M}_{n+1} \times_k Y \arrow[r, hook] \arrow[d, "\delta_{n+1}"'] & \ldots \\ Z_0 \arrow[r, hook] & \ldots \arrow[r, hook] & Z_n \arrow[r, hook] & Z_{n+1} \arrow[r, hook] & \ldots \end{tikzcd}$$

  of a formal $\mathbf{M}$-schemes. According to [Monygham, 2020b, Example 7.3] and [Monygham, 2020b, Corollary 7.4] there exists a unique $\mathbf{M}$-equivariant morphism $\delta : \mathbf{M} \times_k Y \to Z$ such that $\delta_{|\mathbf{M}_n \times_k Y}$ induces $\delta_n : \mathbf{M}_n \times_k Y \to Z_n$ for every $n \in \mathbb{N}$. Note that δ as a $\mathbf{M}$-equivariant morphism is uniquely determined by a morphism $\eta = \delta \cdot \langle e, 1_Y \rangle$ of k-schemes, where $e : \operatorname{Spec} k \to \mathbf{M}$ is the unit of $\mathbf{M}$. This proves that

$$\widehat{\mathcal{D}}_X(Y) \ni \{\gamma_n : \mathbf{M}_n \times_k Y \to X\}_{n\in\mathbb{N}} \mapsto \eta \in \operatorname{Mor}_k(Y, Z)$$

  is a bijection natural in Y. Thus $\widehat{\mathcal{D}}_X$ is representable by Z. Note that $\widehat{r}_X : \widehat{\mathcal{D}}_X \to X^{\mathbf{G}}$ is representable by the canonical retraction $r_Z : Z \to Z^{\mathbf{M}} = X^{\mathbf{G}}$. Hence $\widehat{r}_X$ is affine and if X is locally noetherian, then $\widehat{Z} = \mathcal{Z}$ is a locally noetherian formal $\mathbf{M}$-scheme and hence by [Monygham, 2020b, Theorem 7.5] we derive that $\widehat{r}_X$ is of finite type.

- Assume That X is of finite type over k. Then $\mathcal{Z}$ is locally noetherian formal $\mathbf{M}$-scheme and [Monygham, 2020b, Theorem 7.5] implies that the canonical retraction ([Monygham, 2020b, Proposition 5.2]) $r : Z \to Z^{\mathbf{M}} = X^{\mathbf{G}}$ is of finite type. Since $X^{\mathbf{G}}$ is closed subscheme of X, we dervie that Z is of finite type over k. Next [Monygham, 2020b, Theorem 7.6] implies that the comparison functor $\mathfrak{Coh}_{\mathbf{G}}(Z) \to \mathfrak{Coh}_{\mathbf{G}}(\mathcal{Z})$ is an equivalence of categories. Therefore, we derive that there exists a unique monoidal and finitely cocontinuous functor $F : \mathfrak{Coh}_{\mathbf{G}}(X) \to \mathfrak{Coh}_{\mathbf{G}}(Z)$ such that for every $n \in \mathbb{N}$ we have the commutative square of monoidal, finitely cocontinuous functors

$$\begin{tikzcd}
\mathfrak{Coh}_{\mathbf{G}}(Z) \arrow[r] & \mathfrak{Coh}_{\mathbf{G}}(Z_n) \\
\mathfrak{Coh}_{\mathbf{G}}(X) \arrow[r] \arrow[u, "F"] & \mathfrak{Coh}_{\mathbf{G}}(X_n) \arrow[u, "i_n^*"']
\end{tikzcd}$$

where horizontal functors are pullbacks along closed immersions $X_n \hookrightarrow X$ and $Z_n \hookrightarrow Z$. In particular, it follows that $F \cdot p_X^* = p_Z^*$, where $p_X : X \to \operatorname{Spec} k$, $p_Z : Z \to \operatorname{Spec} k$ are structural morphism and $p_X^* : \mathbf{Repf}(\mathbf{G}) \to \mathfrak{Coh}_{\mathbf{G}}(X), p_Z^* : \mathbf{Repf}(\mathbf{G}) \to \mathfrak{Coh}_{\mathbf{G}}(Z)$ are pullbacks of coherent $\mathbf{G}$-sheaves (i.e. finite dimensional $\mathbf{G}$-representations by [Monygham, 2020b, Example 4.7]) from $\operatorname{Spec} k$ (eqipped with the trivial $\mathbf{G}$-action). Corollary 2.4 implies that there exists a unique $\mathbf{G}$-equivariant morphism $f : Z \to X$ such that for every $n \in \mathbb{N}$ we have a commutative square

$$\begin{tikzcd}
Z_n \arrow[r, hook] \arrow[d, "i_n"'] & Z \arrow[d, "f"] \\
X_n \arrow[r, hook] & X
\end{tikzcd}$$

Consider a k-scheme Y and a family $\{\gamma_n : \mathbf{M}_n \times_k Y \to X\}_{n\in\mathbb{N}} \in \widehat{\mathcal{D}}_X(Y)$. Then γ_n factors through the composition of $i_n : Z_n \to X_n$ and the closed immersion $X_n \hookrightarrow X$ for every $n \in \mathbb{N}$. Thus a family $\{\gamma_n\}_{n\in\mathbb{N}}$ determines and is determined by a unique family $\{\delta_n : \mathbf{M}_n \times_k Y \to Z_n\}_{n\in\mathbb{N}}$ of $\mathbf{M}$-equivariant morphisms. As above [Monygham, 2020b, Example 7.3] and [Monygham, 2020b, Corollary 7.4] show that there is a $\mathbf{M}$-equivariant morphism $\delta : \mathbf{M} \times_k Y \to Z$ such that $\delta_{|\mathbf{M}_n \times_k Y}$ induces δ_n for every $n \in \mathbb{N}$. Define $\gamma = f \cdot \delta$. Then $\gamma : \mathbf{M} \times_k Y \to X$ is a $\mathbf{G}$-equivariant morphism and $\gamma_{|\mathbf{M}_n \times_k Y} = \gamma_n$ for every $n \in \mathbb{N}$. This shows that the map

$$\mathcal{D}_X(Y) \to \widehat{\mathcal{D}}_X(Y)$$

is surjective for every k-scheme Y. By Theorem 5.6 we derive that it is injective and hence the canonical morphism $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ is an isomorphism.

□

It is easy to strengthen **(2)** in Theorem 6.4.

Corollary 6.5. *Let $\mathbf{G}$ be a group k-scheme and $\mathbf{M}$ be a Kempf monoid having $\mathbf{G}$ as a group of units. Suppose that X is a scheme locally of finite type over k equipped with an action of $\mathbf{G}$. Then the canonical morphism $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ is an isomorphism. In particular, $\mathcal{D}_X$ is representable and $r_X : \mathcal{D}_X \to X^{\mathbf{G}}$ is affine and of finite type.*

Proof. Let $a : \mathbf{G} \times_k X \to X$ be an action of $\mathbf{G}$ on X. Consider an open affine subscheme V of X. Then a induces a surjective morphism $a_V : \mathbf{G} \times_k V \twoheadrightarrow a(\mathbf{G} \times_k V) = \mathbf{G} \cdot V$. Since $\mathbf{G} \times_k V$ is affine k-scheme, it is quasi-compact. The image of a quasi-compact topological space under a continuous map is quasi-compact. Thus $\mathbf{G} \cdot V$ is quasi-compact. Since X is locally of finite type over k, we derive that $\mathbf{G} \cdot V$ is of finite type over k. It is also $\mathbf{G}$-stable. This proves that X admits an open cover $\mathcal{U}$ by an open $\mathbf{G}$-stable subschemes of finite type over k. By Remark 5.4 we have a commutative triangle

$$\begin{tikzcd} \mathcal{D}_X \arrow[rr] \arrow[dr, "r_X"'] & & \widehat{\mathcal{D}}_X \arrow[dl, "\widehat{r}_X"] \\ & X^{\mathbf{G}} & \end{tikzcd}$$

and according to Theorem 4.7 and Proposition 5.5 for every U in $\mathcal{U}$ base change of the triangle above along open immersion $U^{\mathbf{G}} \hookrightarrow X^{\mathbf{G}}$ yields a triangle

$$\begin{tikzcd} \mathcal{D}_U \arrow[rr] \arrow[dr, "r_U"'] & & \widehat{\mathcal{D}}_U \arrow[dl, "\widehat{r}_U"] \\ & U^{\mathbf{G}} & \end{tikzcd}$$

in which the horizontal morphism $\mathcal{D}_U \rightarrow \widehat{\mathcal{D}}_U$ is an isomorphism by **(2)** in Theorem 6.4 and the fact that U is $\mathbf{G}$-scheme of finite type over k. Since $\widehat{\mathcal{D}}_X$ is representable by **(1)** in Theorem 6.4, it follows that $\mathcal{D}_X$ is representable and the canonical morphism $\mathcal{D}_X \rightarrow \widehat{\mathcal{D}}_X$ is an isomorphism of functors. Thus r_X and $\widehat{r}_X$ are isomorphic and this completes the proof. □

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