INTRODUCTION TO MEASURE THEORY

1. Introduction

In this notes we develop basic theory of nonnegative measures. We assume that the reader is familiar with theory of real numbers as presented in any standard analysis introductory textbook like [Fichtenholz, 1997], elementary set theory like in [Kuratowski, 2004] and has very basic understanding of metric spaces and topology. For the concept of abstract category see [Mac Lane, 1998], but we do not require the reader to be familiar with this notion except two little remarks that can be safely ignored.
First section is devoted to defining various types of families of sets. We prove here Dynkin's result on π-λ systems and related Sierpiński's lemma on monotone families. These results are indispensable in certain elementary measure-theoretic considerations.
In the second section we introduce central notions of measurable space and measure. Since we consider category theory of Eilenberg and Maclane as the main organizing framework of mathematics, we indicate the fact that measurable spaces and spaces with measures form categories with respect to certain notions of morphisms. We also give important application of Dynkin's π-λ system theorem.
Third section introduces outer measures and Carathéodory's condition. We prove theorems due to Carathéodory on generating measure out of outer measure and on extension of countably additive functions of sets.
The forth section is devoted to proof of important result due to Carathéodory on metric outer measures.

2. Families of sets

In this section we study various families of sets that are important in the development of measure theory.

Definition 2.1. Let X be a set and $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of X. We define the following types of families.

(1) $\mathcal{F}$ is *an algebra* if it contains X and is closed under finite unions, intersections and completions.

(2) $\mathcal{F}$ is *a σ-algebra* if it is an algebra and is closed under countable unions.

(3) $\mathcal{F}$ is *a monotone family* if it is closed under unions of countable non-decreasing sequences and under intersections of countable non-increasing sequences.

(4) $\mathcal{F}$ is *a π-system* if it is closed under finite intersections.

(5) $\mathcal{F}$ is *a λ-system* if it contains X and is closed under complements and countable disjoint unions.

Fact 2.2. *Let X be a set and $\{\mathcal{F}_i\}_{i\in I}$ be a class of families subsets of X. Suppose that $\mathcal{F}_i$ is an algebra (σ-algebra, monotone family, π-system, λ-system) for every $i \in I$. Then the intersection $\bigcap_{i\in I} \mathcal{F}_i$ is an algebra (σ-algebra, monotone family, π-system, λ-system).*

Proof. Left as an exercise. □

Definition 2.3. Let $\mathcal{F}$ be a family of subsets of X. We denote by $\sigma(\mathcal{F})$, $\lambda(\mathcal{F})$ and $\mathcal{M}(\mathcal{F})$ intersections of all σ-algebras, λ-systems and monotone families containing $\mathcal{F}$, respectively. We call them *σ-algebra, λ-system and monotone family generated by $\mathcal{F}$*, respectively.

Theorem 2.4 (Dynkin's π-λ lemma)**.** *Let X be a set and $\mathcal{P}$ be a π-system of its subsets. Then $\lambda(\mathcal{P}) = \sigma(\mathcal{P})$.*

For the proof we need the following result.

Lemma 2.4.1. *Let $\mathcal{L}$ be a λ-system. Then for every $A \in \mathcal{L}$ family*

$$\mathcal{L}_A = \left\{ B \in \mathcal{L} \,\middle|\, A \cap B \in \mathcal{L} \right\}$$

is a λ-system.

Proof of the lemma. Since $A \in \mathcal{L}$, we have $X \in \mathcal{L}_A$. Suppose now that $B \in \mathcal{L}_A$. Then $A \cap B \in \mathcal{L}$. Since $X \smallsetminus A \in \mathcal{L}$, we derive that also $(A \cap B) \cup (X \smallsetminus A) \in \mathcal{L}$ and hence

$$A \cap (X \smallsetminus B) = X \smallsetminus ((A \cap B) \cup (X \smallsetminus A)) \in \mathcal{L}$$

Thus $X \smallsetminus B \in \mathcal{L}_A$. Finally note that $\mathcal{L}_A$ is closed under countable disjoint unions. □

Proof of the theorem. Fix $A \in \mathcal{P}$. Define $\mathcal{L}_A$ as in Lemma 2.4.1 with $\mathcal{L} = \lambda(\mathcal{P})$. Then $\mathcal{L}_A$ is a λ-system. Moreover, $\mathcal{L}_A$ contains $\mathcal{P}$. Hence $\mathcal{L}_A = \lambda(\mathcal{P})$. This shows that $\lambda(\mathcal{P})$ is closed under intersections with members of $\mathcal{P}$. Now fix $A \in \lambda(\mathcal{P})$ and define $\mathcal{L}_A$ as in Lemma 2.4.1 with $\mathcal{L} = \lambda(\mathcal{P})$. Then $\mathcal{P} \subseteq \mathcal{L}_A$ and $\mathcal{L}_A$ is a λ-system. Thus $\mathcal{L}_A = \lambda(\mathcal{P})$. This proves that $\lambda(\mathcal{P})$ is a π-system. A π-system that is simultaneously a λ-system is a σ-algebra. Thus $\sigma(\mathcal{P}) \subseteq \lambda(\mathcal{P})$. Since it is clear that $\lambda(\mathcal{P}) \subseteq \sigma(\mathcal{P})$, we derive that $\lambda(\mathcal{P}) = \sigma(\mathcal{P})$. □

Theorem 2.5 (Halmos's lemma on monotone classes)**.** *Let X be a set and $\mathcal{A}$ be an algebra of its subsets. Then $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$.*

For the proof we need the following easy results. Their proofs are left to the reader.

Lemma 2.5.1. *Let $\mathcal{M}$ be a monotone family. Then for every $A \in \mathcal{M}$ family*

$$\mathcal{M}_A = \left\{ B \in \mathcal{M} \,\middle|\, A \cap B \in \mathcal{M} \right\}$$

is monotone.

Lemma 2.5.2. *Let $\mathcal{M}$ be a monotone family. Then a family*

$$\mathcal{M}^c = \left\{ A \in \mathcal{M} \,\middle|\, X \smallsetminus A \in \mathcal{M} \right\}$$

is monotone.

Proof of the theorem. Fix $A \in \mathcal{A}$. Define $\mathcal{M}_A$ as in Lemma 2.5.1 with $\mathcal{M} = \mathcal{M}(\mathcal{A})$. Then $\mathcal{M}_A$ is a monotone family. Moreover, $\mathcal{M}_A$ contains $\mathcal{A}$. Hence $\mathcal{M}_A = \mathcal{M}(\mathcal{A})$. This shows that $\mathcal{M}(\mathcal{A})$ is closed under intersections with members of $\mathcal{A}$. Now fix $A \in \mathcal{M}(\mathcal{A})$ and define $\mathcal{M}_A$ as in Lemma 2.5.1 with $\mathcal{M} = \mathcal{M}(\mathcal{A})$. Then $\mathcal{A} \subseteq \mathcal{M}_A$ and $\mathcal{M}_A$ is a monotone family. Thus $\mathcal{M}_A = \mathcal{M}(\mathcal{A})$. This proves that $\mathcal{M}(\mathcal{A})$ is closed under finite intersections. According to Lemma 2.5.2 we derive that $\mathcal{M}(\mathcal{A})^c$ is a monotone family and contains $\mathcal{A}$. Hence $\mathcal{M}(\mathcal{A})^c = \mathcal{M}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$ is closed under complements. Therefore, $\mathcal{M}(\mathcal{A})$ is a σ-algebra. Thus $\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$. Since it is clear that $\mathcal{M}(\mathcal{A}) \subseteq \sigma(\mathcal{A})$, we derive that $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$. □

3. Measurable spaces and measures

Definition 3.1. A pair (X, Σ) consisting of a set X together with a σ-algebra Σ of its subsets is called *a measurable space*.

Definition 3.2. Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces. A function $f : X_1 \to X_2$ is called *a measurable map* if $f^{-1}(A) \in \Sigma_1$ for every $A \in \Sigma_1$.

Measurable spaces and their morphisms form a category.

Definition 3.3. Let X be a set and Σ be an algebra of its subsets. A function $\mu : \Sigma \to [0, +\infty]$ such that $\mu(\emptyset) = 0$ and

$$\mu\left(\bigcup_{n=0}^{m} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

for every family $\{A_n\}_{0\leq n\leq m}$ of pairwise disjoint subsets in Σ is called *an additive function*. If in addition μ satisfies

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

for every family $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint subsets in Σ such that $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma$, then μ is called *a σ-additive function*. Moreover, if $\mu : \Sigma \to [0, +\infty]$ is a σ-additive function and Σ is a σ-algebra, then μ is called *a measure*.

Definition 3.4. A tuple (X, Σ, μ) consisting of a measurable space (X, Σ) and a measure $\mu : \Sigma \to [0, +\infty]$ is called *a space with measure*.

Definition 3.5. Let (X, Σ, μ) be a space with measure. We say that it is *finite* if $\mu(X)$ is finite. We say that it is *σ-finite* if there exists a sequence $\{X_n\}_{n\in\mathbb{N}}$ of subsets of Σ such that $\mu(X_n)$ is finite for every $n \in \mathbb{N}$ and $X = \bigcup_{n\in\mathbb{N}} X_n$.

Theorem 3.6. *Let (X, Σ) be a measurable space and μ_1, $\mu_2 : \Sigma \to [0, +\infty]$ be measures such that $\mu_1(X) = \mu_2(X)$ is finite. Suppose that $\mathcal{P}$ is a π-system of subsets of X such that $\Sigma = \sigma(\mathcal{P})$ and $\mu_1(A) = \mu_2(A)$ for every $A \in \mathcal{P}$. Then $\mu_1 = \mu_2$.*

Proof. Define $\mathcal{F} = \{A \in \Sigma \,|\, \mu_1(A) = \mu_2(A)\}$. Straightforward verification shows that $\mathcal{F}$ is a λ-system. By assumption $\mathcal{P} \subseteq \mathcal{F}$. Therefore, $\lambda(\mathcal{P}) \subseteq \mathcal{F}$. By Theorem 2.5 we deduce that $\Sigma = \sigma(\mathcal{P}) = \lambda(\mathcal{P}) \subseteq \mathcal{F} \subseteq \Sigma$. Hence $\mathcal{F} = \Sigma$. □

Definition 3.7. Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be spaces with measures. A function $f : X_1 \to X_2$ is called *a morphism of spaces with measures* if f is a morphism of measurable spaces and for every $A \in \Sigma_2$ we have equality $\mu_2(A) = \mu_1(f^{-1}(A))$.

Spaces with measures and their morphisms form a category.

4. Outer measures and Carathéodory's construction

Definition 4.1. Let X be a set and $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ be a function. Suppose that $\mu^*(\emptyset) = 0$, $\mu^*(A) \leq \mu^*(B)$ for every subset A of a set B contained in X and

$$\mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n)$$

for every family $\{A_n\}_{n\in\mathbb{N}}$ of subsets of X. Then we say that μ^* is *an outer measure on X*.

Theorem 4.2 (Carathéodory's construction)**.** *Let X be a set and $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ be an outer measure on X. We define a family of sets $\Sigma_{\mu^*} \subseteq \mathcal{P}(X)$ by condition*

$$A \in \Sigma_{\mu^*} \Leftrightarrow \forall_{E\subseteq X}\, \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$$

Then the following assertions hold.

(1) *Σ_{μ^*} is an σ-algebra of subsets of X.*

(2) *For every family $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint subsets of Σ_{μ^*} and every subset E of X we have*

$$\mu^*\left(E \cap \bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu^*(E \cap A_n)$$

In particular, $\mu^*_{|\Sigma_{\mu^*}}$ is a measure.

(3) *Every subset A of X such that $\mu^*(A) = 0$ is contained in Σ_{μ^*}. In particular, $\mu^*_{|\Sigma_{\mu^*}}$ is complete.*

The proof is encapsulated in two lemmas.

Lemma 4.2.1. *Σ_{μ^*} is an algebra of sets.*

Proof of the lemma. Clearly $\emptyset \in \Sigma_{\mu^*}$ and $A \in \Sigma_{\mu^*} \Leftrightarrow X \smallsetminus A \in \Sigma_{\mu^*}$. It suffices to prove that Σ_{μ^*} is closed under unions. For a subset B of X we denote $X \smallsetminus B$ by B^c. Now assume that $A_1, A_2 \in \Sigma_{\mu^*}$ and pick a subset E of X. Then

$$\mu^*(E) = \mu^*(E \cap A_1) + \mu^*(E \cap A_1^c) = \mu^*(E \cap A_1) + \mu^*(E \cap A_1^c \cap A_2) + \mu^*(E \cap A_1^c \cap A_2^c)$$

Since we have equalities

$$E \cap A_1 = (E \cap (A_1 \cup A_2)) \cap A_1,\ E \cap A_1^c \cap A_2 = (E \cap (A_1 \cup A_2)) \cap A_1^c$$

we derive that $\mu^*(E \cap (A_1 \cup A_2)) = \mu^*(E \cap A_1) + \mu^*(E \cap A_1^c \cap A_2)$. Similarly we have equality

$$E \cap A_1^c \cap A_2^c = E \cap (A_1 \cup A_2)^c$$

and hence $\mu^*(E \cap A_1^c \cap A_2^c) = \mu^*(E \cap (A_1 \cup A_2)^c)$. Therefore, we have

$$\mu^*(E) = \mu^*(E \cap (A_1 \cup A_2)) + \mu^*(E \cap (A_1 \cup A_2)^c)$$

Thus we proved that $A_1 \cup A_2 \in \Sigma$. Therefore, Σ_{μ^*} is a family of subsets of X closed under finite unions, complements and containing $\emptyset$. Thus Σ_{μ^*} is an algebra of sets. □

Lemma 4.2.2. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a family of pairwise disjoint subsets of Σ_{μ^*}. Then $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma_{\mu^*}$ and for every subset E of X there is an equality*

$$\mu^*\left(E \cap \bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu^*(E \cap A_n)$$

Proof of the lemma. We prove that $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma_{\mu^*}$. For this observe that we have

$$\mu^*(E) \le \mu^*\left(E \cap \bigcup_{n\in\mathbb{N}} A_n\right) + \mu^*\left(E \smallsetminus \bigcup_{n\in\mathbb{N}} A_n\right) \le \sum_{n\in\mathbb{N}} \mu^*(E \cap A_n) + \mu^*\left(E \smallsetminus \bigcup_{n\in\mathbb{N}} A_n\right) =$$

$$= \lim_{m\to+\infty}\left(\mu^*\left(E \cap \bigcup_{n=0}^{m} A_n\right) + \mu^*\left(E \smallsetminus \bigcup_{n\in\mathbb{N}} A_n\right)\right) \le \lim_{m\to+\infty}\left(\mu^*\left(E \cap \bigcup_{n=0}^{m} A_n\right) + \mu^*\left(E \smallsetminus \bigcup_{n=0}^{m} A_n\right)\right) = \mu^*(E)$$

and the last equality holds, since $\bigcup_{n=0}^{m} A_n \in \Sigma_{\mu^*}$ by Lemma 4.2.1. This implies that we have equalities everywhere above. Hence

$$\mu^*\left(E \cap \bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu^*(E \cap A_n)$$

and $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma_{\mu^*}$. □

Proof of the theorem. Lemma 4.2.1 and Lemma 4.2.2 imply that Σ_{μ^*} is a σ-algebra and statement **(2)** holds. It suffices to verify that statement **(3)** holds. For this pick a subset A of X such that $\mu^*(A) = 0$. Then for every subset E of X we have

$$\mu^*(E) \le \mu^*(E \cap A) + \mu^*(E \smallsetminus A) = \mu^*(E \smallsetminus A) \le \mu^*(E)$$

Hence $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \smallsetminus A)$ and thus $A \in \Sigma_{\mu^*}$. □

Next result is a general tool of constructing measures.

Theorem 4.3 (Carathéodory extension)**.** *Let X be a set and Σ be some algebra of its subsets. Suppose that $\mu : \Sigma \to [0, +\infty]$ is a σ-additive function. Now for every subset A in X we define*

$$\mu^*(A) = \inf\left\{\sum_{n\in\mathbb{N}} \mu(A_n) \,\middle|\, A_n \in \Sigma \text{ for every } n \in \mathbb{N} \text{ and } A \subseteq \bigcup_{n\in\mathbb{N}} A_n\right\}$$

Then $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ is an outer measure, $\Sigma \subseteq \Sigma_{\mu^*}$ and $\mu^*_{|\Sigma} = \mu$. Moreover, if $\mu(X)$ is finite, then $\mu^*_{|\sigma(\Sigma)}$ is a unique extension of Σ to a measure on $\sigma(\Sigma)$.

Proof. Standard verification shows that μ^* is an outer measure. Note that

$$\mu^*(A) = \inf\left\{ \sum_{n\in\mathbb{N}} \mu(A_n) \,\middle|\, \{A_n\}_{n\in\mathbb{N}} \text{ is a family of pairwise disjoint subsets of } \Sigma \text{ and } A \subseteq \bigcup_{n\in\mathbb{N}} A_n \right\}$$

for every subset A of X. Let A be element of A and let E be an arbitrary subset of X. Fix $\epsilon > 0$. By the remark above there exists a family $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint elements of Σ such that

$$E \subseteq \bigcup_{n\in\mathbb{N}} A_n,\ \sum_{n\in\mathbb{N}} \mu(A_n) \leq \mu^*(E) + \epsilon$$

By definition of μ^* we have $\mu^*(E \cap A) \leq \sum_{n\in\mathbb{N}} \mu(A_n \cap A)$, $\mu^*(E \setminus A) \leq \sum_{n\in\mathbb{N}} \mu(A_n \setminus A)$ and hence

$$\mu^*(E) \leq \mu^*(E\cap A) + \mu^*(E \setminus A) \leq \sum_{n\in\mathbb{N}} \mu(A_n \cap A) + \sum_{n\in\mathbb{N}} \mu(A_n \setminus A) =$$

$$= \sum_{n\in\mathbb{N}} \left(\mu(A_n \cap A) + \mu(A_n \setminus A)\right) = \sum_{n\in\mathbb{N}} \mu(A_n) \leq \mu^*(E) + \epsilon$$

Since $\epsilon > 0$ is arbitrary, we derive that $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$ and hence $A \in \Sigma_{\mu^*}$. Thus $\Sigma \subseteq \Sigma_{\mu^*}$. Once again fix $A \in \Sigma$. Then for every family $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint elements of Σ such that $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$ we have $\mu(A) = \sum_{n\in\mathbb{N}} \mu(A_n \cap A) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$ and thus $\mu(A) \leq \mu^*(A)$. Obviously $\mu^*(A) \leq \mu(A)$. Therefore, for every $A \in \Sigma$ we have $\mu(A) = \mu^*(A)$. Together with $\Sigma \subseteq \Sigma_{\mu^*}$ this implies that $\mu^*_{|\sigma(\Sigma)}$ is a measure that extends μ. Now we prove the uniqueness of extension under the assumption that $\mu(X)$ is finite. This follows immediately from Theorem 3.6. □

5. Outer metric measures

Definition 5.1. Let X be a topological space. The σ-algebra $\mathcal{B}_X$ generated by all open sets of X is called *the σ-algebra of Borel subsets of X.*

Definition 5.2. Let (X, d) be a metric space and $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ be an outer measure. We say that μ^* is *a metric outer measure* if

$$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$$

for any two subsets E_1, E_2 of X with $\mathrm{dist}(E_1, E_2) = \inf_{x_1\in E_1,\, x_2\in E_2} d(x_1, x_2) > 0$.

Theorem 5.3 (Carathéodory)**.** *Let (X, d) be a metric space and $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ be an outer metric measure on X. Then the σ-algebra $\mathcal{B}_X$ of Borel subsets of X is contained in Σ_{μ^*}.*

Proof. Let U be an open subset of X. Define $F = X \setminus U$ and $U_n = \{x \in X \mid \mathrm{dist}(x, F) > \frac{1}{2^n}\}$ for $n \in \mathbb{N}$. Then $\{U_n\}_{n\in\mathbb{N}}$ form an ascending family of open sets and $U = \bigcup_{n\in\mathbb{N}} U_n$. Fix now a subset E of X such that $\mu^*(E) \in \mathbb{R}$. We define $E_n = E \cap U_n$ for every $n \in \mathbb{N}$. Since μ^* is an outer metric measure, we derive that

$$\mu^*\left(\bigcup_{n=0}^{m} E_{2n+1} \setminus E_{2n}\right) = \sum_{n=0}^{m} \mu^*(E_{2n+1} \setminus E_{2n}),\ \mu^*\left(\bigcup_{n=1}^{m} E_{2n} \setminus E_{2n-1}\right) = \sum_{n=1}^{m} \mu^*(E_{2n} \setminus E_{2n-1})$$

for every positive integer m. Thus we derive

$$\sum_{n\in\mathbb{N}} \mu^*(E_{2n+1} \setminus E_{2n}) \leq \mu^*(E) \in \mathbb{R},\ \sum_{n\in\mathbb{N}} \mu^*(E_{2n} \setminus E_{2n-1}) \leq \mu^*(E) \in \mathbb{R}$$

Hence we have $\sum_{n\in\mathbb{N}} \mu^*(E_{n+1} \setminus E_n) \leq 2 \cdot \mu^*(E) \in \mathbb{R}$. Using the fact that μ^* is an outer measure, we derive that

$$\mu(E_m) \leq \mu^*(E \cap U) \leq \mu^*(E_m) + \sum_{n\geq m} \mu^*(E_{n+1} \setminus E_n)$$

for every $m \in \mathbb{N}$. Hence these inequalities yield $\lim_{m\to+\infty} \mu^*(E_m) = \mu^*(E \cap U)$. Now we have $\mu^*(E_m) + \mu^*(E \setminus U) \leq \mu^*(E) \leq \mu^*(E \cap U) + \mu^*(E \setminus U)$ for every $m \in \mathbb{N}$. The first inequality holds due to the fact that μ^* is an outer metric measure. We derive that $\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$. Note that if $\mu^*(E) = +\infty$, then inequality $\mu^*(E) \leq \mu^*(E \cap U) + \mu^*(E \setminus U)$ must be equality. Hence for every subset E of X we have $\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$. This implies that $U \in \Sigma_{\mu^*}$. Since U is an arbitrary open subset of X, we deduce that $\mathcal{B}_X \subseteq \Sigma_{\mu^*}$. □

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