GROTHENDIECK TOPOSES

1. Introduction

In this notes we study Grothendieck topologies and toposes. For prerequisites we assume familiarity with [Mon19].

2. Sites and sheaves

In this section we fix a category $\mathcal{C}$.

Definition 2.1. Let X be an object of $\mathcal{C}$. *A sieve on X* is a family S of arrows of $\mathcal{C}$ with X as a target such that for every $f : Y \to X$ in S and every morphisms $g : Z \to Y$ their composition $f \cdot g$ is in S.

Every sieve S on object X of $\mathcal{C}$ corresponds to a subpresheaf of h_X given by

$$\mathcal{C} \ni Y \mapsto \{f : Y \to X \,|\, f \in S\} \in \mathbf{Set}$$

This identifies the collection of sieves on X with the collection of subpresheaves of h_X.

Fact 2.2. *Let X be an object of $\mathcal{C}$. The class-theoretic intersection and union of a collection of sieves on X is a sieve on X.*

Proof. Left to the reader. □

Definition 2.3. Let $\{f_i : X_i \to X\}_{i\in I}$ be a collection of morphisms of $\mathcal{C}$ with codomain in X. Then the intersection of all sieves on X containing $\{f_i\}_{i\in I}$ is called *the sieve generated by $\{f_i\}_{i\in I}$*.

One can directly describe the sieve on X generated by $\{f_i : X_i \to X\}_{i\in I}$ as a class of arrows $f : Y \to X$ in $\mathcal{C}$ such that f factors through f_i for some $i \in I$.

Definition 2.4. Let S be a sieve on X and $f : Y \to X$ be a morphism, then we define a sieve on Y by formula

$$f^*S = \left\{g \in \mathbf{Mor}(\mathcal{C}) \,\middle|\, \text{target of } g \text{ is } Y \text{ and } f \cdot g \in S\right\}$$

We call *f^*S the pullback of S along f*.

Definition 2.5. For every object X in $\mathcal{C}$ the family

$$\left\{f \in \mathbf{Mor}(\mathcal{C}) \,\middle|\, \text{target of } f \text{ is } X\right\}$$

is a sieve on X. We call it *the maximal sieve on X*.

Definition 2.6. *A Grothendieck topology on $\mathcal{C}$* is a collection $\mathcal{J} = \{\mathcal{J}(X)\}_{X\in\mathcal{C}}$ such that $\mathcal{J}(X)$ is a class of sieves on X and the following conditions are satisfied.

(1) The maximal sieve on X is in $\mathcal{J}(X)$.

(2) If $S \in \mathcal{J}(X)$ and $f : Y \to X$, then $f^*S \in \mathcal{J}(Y)$.

(3) Suppose that $S \in \mathcal{J}(X)$, R is a sieve on X and $f^*R \in \mathcal{J}(\mathrm{dom}(f))$ for every $f \in S$. Then $R \in \mathcal{J}(X)$.

Sieves in class

$$\bigcup_{X\in\mathcal{C}} \mathcal{J}(X)$$

are called covering sieves. A pair $(\mathcal{C}, \mathcal{J})$ consisting of a category $\mathcal{C}$ and a Grothendieck topology $\mathcal{J}$ is called *a site*.

Proposition 2.7. *Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$ and X be an object of $\mathcal{C}$. Then the following assertions hold.*

(1) *Class $\mathcal{J}(X)$ is closed under finite intersections.*

(2) *If $S \in \mathcal{J}(X)$ and R is a sieve on X such that $S \subseteq R$, then $R \in \mathcal{J}(X)$.*

Proof. We prove **(1)**. For this assume that S and T are covering sieves on X. Then $S \cap T$ is a sieve. Next pick $f : Y \to X$ in S. Note that $f^*(S \cap T) = f^*T \in \mathcal{J}(Y)$. This implies that $S \cap T \in \mathcal{J}(X)$.
We prove now **(2)**. Fix $f : Y \to X$ in S. Then f^*R is the maximal sieve on Y due to $S \subseteq R$. Hence $f^*R \in \mathcal{J}(Y)$. Since $S \in \mathcal{J}(X)$, we deduce that $R \in \mathcal{J}(X)$. □

Fact 2.8. *Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$ and X be an object of $\mathcal{C}$. Suppose that S is a covering sieve on X and for each $f : Y \to X$ in S pick a covering sieve R_f on Y. Then a family*

$$R = \bigcup_{f \in S} f \cdot R_f$$

is a covering sieve on X.

Proof. For every $f : Y \to X$ in S we have $R_f \subseteq f^*R$. By Proposition 2.7 and since R_f is in $\mathcal{J}(Y)$, we deduce that $f^*R \in \mathcal{J}(Y)$. Hence f^*R is a covering sieve for every $f \in S$. This implies that $R \in \mathcal{J}(X)$. □

Definition 2.9. Let F be a presheaf on $\mathcal{C}$. Suppose that X is an object of $\mathcal{C}$ and S is a sieve on X. We say that a family $\{x_f\}_{f \in S}$ such that $x_f \in S(\mathrm{dom}(f))$ is *a matching family for S of elements of F* if for every $f : Y \to X$ in S and $g : Z \to Y$ in $\mathcal{C}$ we have

$$F(g)(x_f) = x_{f \cdot g}$$

We say that an element $x \in F(X)$ is *an amalgamation for the matching family* $\{x_f\}_{f \in S}$ if for every $f \in S$ we have $F(f)(x) = x_f$.

Note that if S is a sieve on X viewed as a subpresheaf of h_X, then a matching family for S of elements of F can be viewed as a morphisms of presheaves $S \to F$. This identifies the collection of matching families for S of elements of F with a collection of morphisms $S \to F$ of presheaves. Next suppose that $\{x_f\}_{f \in S}$ is a matching family for S of elements of F. Then amalgamations of $\{x_f\}_{f \in S}$ can be identified by means of Yoneda lemma [Mon19, Theorem 3.3] with morphisms $h_X \to F$ making the following triangle

$$\begin{tikzcd}
h_X \arrow[r, dotted] & F \\
S \arrow[u, hook] \arrow[ur, "\{x_f\}_{f \in S}"'] &
\end{tikzcd}$$

commutative.

Definition 2.10. Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$ and F be a presheaf on $\mathcal{C}$. We say that F is *a separated presheaf with respect to $\mathcal{J}$* if for any object X in $\mathcal{C}$, covering sieve $S \in \mathcal{J}(X)$ and for every matching family $\{x_f\}_{f \in S}$ for S of elements of F there exists at most one amalgamation $x \in F(X)$.

Definition 2.11. Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$ and F be a presheaf on $\mathcal{C}$. We say that F is *a sheaf with respect to $\mathcal{J}$* if for any object X in $\mathcal{C}$, covering sieve $S \in \mathcal{J}(X)$ and for every matching family $\{x_f\}_{f \in S}$ for S of elements of F there exists a unique amalgamation $x \in F(X)$.

In other words $F \in \widehat{\mathcal{C}}$ is a separated presheaf (sheaf) with respect to a Grothendieck topology $\mathcal{J}$ on $\mathcal{C}$ if for any $X \in \mathcal{C}$, sieve $S \in \mathcal{J}(X)$ and morphism $S \to F$ of presheaves there exists at most one (a unique) morphism $h_X \to F$ making the triangle

$$\begin{tikzcd} h_X \arrow[r, dotted] & F \\ S \arrow[u, hook] \arrow[ur, "\{x_f\}_{f\in S}"'] & \end{tikzcd}$$

commutative.
Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$. We denote by $\mathbf{PrSh}_s(\mathcal{C},\mathcal{J})$, $\mathbf{Sh}(\mathcal{C},\mathcal{J})$ full subcategories of $\widehat{\mathcal{C}}$ consisting of separated presheaves and sheaves with respect to $\mathcal{J}$, respectively.

Theorem 2.12. *Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$. Then inclusion functors $\mathbf{PrSh}_s(\mathcal{C},\mathcal{J}) \to \widehat{\mathcal{C}}$, $\mathbf{Sh}(\mathcal{C},\mathcal{J}) \to \widehat{\mathcal{C}}$ create limits.*

Proof. Let $D : I \to \mathbf{PrSh}_s(\mathcal{C},\mathcal{J})$ be a functor and assume that $\left(F, \left\{f_i : F \to D(i)\right\}_{i\in I}\right)$ is a limiting cone over the composition of the functor $D : I \to \mathbf{PrSh}_s(\mathcal{C},\mathcal{J})$ with the inclusion $\mathbf{PrSh}_s(\mathcal{C},\mathcal{J}) \to \widehat{\mathcal{C}}$. We show that F is a separated presheaf with respect to $\mathcal{J}$. Suppose that S is a covering sieve on X and $m : S \to F$ is a morphism that represents certain matching family for S of elements of F. Let $u : S \to h_X$ be the inclusion. Suppose that $p : h_X \to F$ is an amalgamation for m. We need to show that this amalgamation is unique. For this it suffices to observe that from equality $p \cdot u = m$ we derive that $(f_i \cdot p) \cdot u = (f_i \cdot m)$ for $i \in I$. Hence for every $i \in I$ morphism $f_i \cdot p$ is an amalgamation of $f_i \cdot m$. Since $D(i)$ is a separated presheaf for every $i \in I$, this makes $f_i \cdot p$ uniquely determined for $i \in I$. Thus p is uniquely determined itself according to the fact that the cone $\left(F, \left\{f_i\right\}_{i\in I}\right)$ is limiting in $\widehat{\mathcal{C}}$. Therefore, F is a separated presheaf with respect to $\mathcal{J}$ and hence $\left(F, \left\{f_i\right\}_{i\in I}\right)$ is a limiting cone for D in the category of separated presheaves.
Now assume that $D : I \to \mathbf{Sh}(\mathcal{C},\mathcal{J})$ is a functor and $\left(F, \left\{f_i : F \to D(i)\right\}_{i\in I}\right)$ is a limiting cone over the composition of the functor $D : I \to \mathbf{Sh}(\mathcal{C},\mathcal{J})$ with the inclusion $\mathbf{Sh}(\mathcal{C},\mathcal{J}) \to \widehat{\mathcal{C}}$. We show that F is sheaf with respect to $\mathcal{J}$. From what we prove above we know that F is a separated presheaf with respect to $\mathcal{J}$. Suppose that S is a covering sieve on X and $m : S \to F$ is a morphism that represents certain matching family for S of elements of F. Let $u : S \to h_X$ be the inclusion. It suffices to construct an amalgamation $p : h_X \to F$ for m. We define $m_i = f_i \cdot m$ for $i \in I$. Now fix $i \in I$ for a moment. Then $m_i : S \to D(i)$ is a matching family for S of elements of a sheaf $D(i)$. Hence there exists a unique morphism $p_i : h_X \to D(i)$ such that the triangle

$$\begin{tikzcd} h_X \arrow[r, dotted, "p_i"] & D(i) \\ S \arrow[u, hook, "u"] \arrow[ur, "m_i"'] & \end{tikzcd}$$

is commutative. Now pick a morphism $\alpha : i \to j$ in I. Then

$$D(\alpha) \cdot p_i \cdot u = D(\alpha) \cdot m_i = m_j = p_j \cdot u$$

According to uniqueness of p_j we deduce that $D(\alpha) \cdot p_i = p_j$. Hence $\left(h_X, \left\{p_i\right\}_{i\in I}\right)$ is a cone over D. Therefore, there exists a unique morphism $p : h_X \to F$ such that $f_i \cdot p = p_i$ for every $i \in I$. Hence

$$f_i \cdot p \cdot u = p_i \cdot u = m_i = f_i \cdot m$$

for every $i \in I$. Thus $p \cdot u = m$ because the cone $\left(F, \left\{f_i\right\}_{i\in I}\right)$ is limiting. Therefore, matching family m for S of elements of F admits an amalgamation p and hence $\left(F, \left\{f_i\right\}_{i\in I}\right)$ is a limiting cone for D in the category of sheaves. $\square$

3. Some results on presheaves and matching families

This section contains some technical facts that we use in development of theory in this notes. Let S be an arbitrary sieve on object X in $\mathcal{C}$ and F be a presheaf on $\mathcal{C}$. We denote by $F(S)$ the class of matching families for S of elements of F. Suppose now that S is generated by a collection $\mathcal{F} = \left\{f_i : X_i \to X\right\}_{i \in I}$. Assume that $\{x_i\}_{i \in I}$ is a collection such that $x_i \in F(X_i)$ for every $i \in I$ and

$$F(g_i)(x_i) = F(g_j)(x_j)$$

for any morphisms $g_i : Y \to X_i$, $g_j : Y \to X_j$ in $\mathcal{C}$ satisfying $f_i \cdot g_i = f_j \cdot g_j$ for every pair $i, j \in I$. Then $\{x_i\}_{i \in I}$ is called a matching family for $\mathcal{F}$ of elements of F. The class of matching families for $\mathcal{F}$ of elements of $\mathcal{F}$ is denoted by $F(\mathcal{F})$. We have canonical injective map $\mathrm{can}_{\mathcal{F}} : F(\mathcal{F}) \to \prod_{i \in I} F(X_i)$ and we denote by $\mathrm{res}_{S,\mathcal{F}} : F(S) \to F(\mathcal{F})$ a map that sends $\{x_f\}_{f \in S}$ to $\{x_{f_i}\}_{i \in I}$.

Proposition 3.1. *Fix a presheaf F on $\mathcal{C}$ and a collection $\mathcal{F} = \{f_i : X_i \to X\}_{i \in I}$ of arrows in $\mathcal{C}$ with codomain in X. Let S be a sieve generated by this family. Then $\mathrm{res}_{S,\mathcal{F}}$ is bijective. Moreover, if $\mathcal{C}$ admits fiber products, then*

$$\begin{tikzcd} F(\mathcal{F}) \arrow[r, "\mathrm{can}_{\mathcal{F}}"] & \prod_{i \in I} F(X_i) \arrow[r, shift left, "\langle F(f'_{ij}) \cdot pr_i \rangle_{(i,j)}"] \arrow[r, shift right, "\langle F(f''_{ij}) \cdot pr_j \rangle_{(i,j)}"'] & \prod_{(i,j) \in I \times I} F(X_i \times_X X_j) \end{tikzcd}$$

is a kernel of a pair of arrows, where for every $(i, j) \in I \times I$ morphisms f'_{ij} and f'_{ji} form a cartesian square

$$\begin{tikzcd} X_i \times_X X_j \arrow[r, "f''_{ij}"] \arrow[d, "f'_{ij}"'] & X_j \arrow[d, "f_j"] \\ X_i \arrow[r, "f_i"'] & X \end{tikzcd}$$

Proof. Let $\{x_i\}_{i \in I}$ be a matching family for $\mathcal{F}$ of elements of F. For every $f : Y \to X$ in S there exists $i \in I$ such that $f = f_i \cdot g_i$ for some $g_i : Y \to X_i$. Indeed, this follows from the fact that $\mathcal{F}$ generates S. We define $x_f = F(g_i)(x_i)$. Since $\{x_i\}_{i \in I}$ is a matching family for $\mathcal{F}$ of elements of F, we derive that x_f does not depend on the choice of $i \in I$ and factorization $f = f_i \cdot g_i$. This implies that $\{x_f\}_{f \in S}$ is a matching family for S of elements of F. Now correspondence $\{x_i\}_{i \in I} \mapsto \{x_f\}_{f \in S}$ is the inverse of $\mathrm{res}_{S,\mathcal{F}}$. This proves the first part of the statement.
Let $(x_i)_{i \in I}$ be an element of $\prod_{i \in I} F(X_i)$ such that $F(f'_{ij})(x_i) = F(f''_{ij})(x_j)$ for every pair $(i, j) \in I \times I$. Assume that for some $f : Y \to X$ in S we can write $f = f_i \cdot g_i$ for some $i \in I$ and $g_i : Y \to X_i$ and similarly $f = f_j \cdot g_j$ for some $j \in I$ and $g_j : Y \to X_j$. Then there exist a unique $g : Y \to X_i \times_X X_j$ such that $g_i = f'_{ij} \cdot g$ and $g_j = f''_{ij} \cdot g$. We have

$$F(g_i)(x_i) = F(f'_{ij} \cdot g)(x_i) = F(g)\left(F(f'_{ij})(x_i)\right) = F(g)\left(F(f''_{ij})(x_j)\right) = F(f''_{ij} \cdot g)(x_j) = F(g_j)(x_j)$$

It follows that $\{x_i\}_{i \in I}$ is a matching family for $\mathcal{F}$ of elements of F and $\mathrm{can}_{\mathcal{F}}\left(\{x_i\}_{i \in I}\right) = (x_i)_{i \in I}$. This proves that $\mathrm{can}_{\mathcal{F}}$ is a bijection between $F(\mathcal{F})$ and the class of elements $(x_i)_{i \in I} \in \prod_{i \in I} F(X_i)$ such that $F(f'_{ij})(x_i) = F(f''_{ij})(x_j)$ for every pair $(i, j) \in I \times I$. This finishes the proof of the second part of the statement. $\square$

Next if $S \subseteq R$ are sieves on X and F is a presheaf, then we denote by $\mathrm{res}_{R,S} : F(R) \to F(S)$ a map given by $\mathrm{res}_{R,S}(\{x_f\}_{f \in R}) = \{x_f\}_{f \in S}$. The next result is a useful technical tool.

Proposition 3.2. *Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$ and F be a separated presheaf with respect to $\mathcal{J}$. Pick X in $\mathcal{C}$. If R, S in $\mathcal{J}(X)$ satisfy $S \subseteq R$, then $\mathrm{res}_{R,S} : F(R) \to F(S)$ is injective.*

Proof. Let $\mathrm{res}_{R,S}(\{x_f\}_{f\in R}) = \{x_f\}_{f\in S}$. We show that $\{x_f\}_{f\in R}$ is uniquely determined by $\{x_f\}_{f\in S}$. For this pick $g \in R$ and consider $\{x_{g\cdot f}\}_{f\in g^*S}$. This is a subfamily of $\{x_f\}_{f\in S}$. For every $f \in g^*S$ we have $F(f)(x_g) = x_{g\cdot f}$ and hence x_g is an amalgamation for a matching family $\{x_{g\cdot f}\}_{f\in g^*S}$ for g^*S of elements of F. Since F is a separated presheaf with respect to $\mathcal{J}$, we deduce that x_g is uniquely determined with $\{x_{g\cdot f}\}_{f\in g^*S}$ and hence it is uniquely determined by $\{x_f\}_{f\in S}$. Arrow g is an arbitrary element of R. Thus $\mathrm{res}_{R,S}$ is injective. □

4. Grothendieck pretopologies

Let $\mathcal{C}$ be a category with fiber products.

Definition 4.1. For every X in $\mathcal{C}$ let $\mathcal{K}(X)$ be a class of collections $\{f_i : X_i \to X\}_{i\in I}$ of arrows in $\mathcal{C}$ with codomain in X. Assume that $\mathcal{K} = \{\mathcal{K}(X)\}_{X\in\mathcal{C}}$ satisfies the following assertions.

(1) $\{1_X : X \to X\} \in \mathcal{K}(X)$ for every object X in $\mathcal{C}$.

(2) If $\{f_i : X_i \to X\}_{i\in I} \in \mathcal{K}(X)$ for some X in $\mathcal{C}$ and $f : Y \to X$ is a morphism, then $\{f_i' : X_i \times_X Y \to Y\}_{i\in I} \in \mathcal{K}(Y)$ where f_i' are defined by cartesian squares

$$\begin{tikzcd} X_i \times_X Y \arrow[r] \arrow[d, "f_i'"'] & X_i \arrow[d, "f_i"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

(3) Suppose that $\{f_i : X_i \to X\}_{i\in I} \in \mathcal{K}(X)$ and $\{f_{ij} : X_{ij} \to X_i\}_{j\in J_i} \in \mathcal{K}(X_i)$ for every $i \in I$. Then $\{f_i \cdot f_{ij} : X_{ij} \to X\}_{i\in I, j\in J_i} \in \mathcal{K}(X)$.

Then we say that $\mathcal{K} = \{\mathcal{K}(X)\}_{x\in\mathcal{C}}$ is *a Grothendieck pretopology on* $\mathcal{C}$.

Proposition 4.2. *Suppose that* $\mathcal{K} = \{\mathcal{K}(X)\}_{X\in\mathcal{C}}$ *is a Grothendieck pretopology on* $\mathcal{C}$. *For every* X *in* $\mathcal{C}$ *define*

$$\mathcal{J}(X) = \big\{S \,\big|\, S \text{ is a sieve on } X \text{ and } S \text{ contains some collection in } \mathcal{K}(X)\big\}$$

Then $\mathcal{J} = \{\mathcal{J}(X)\}_{X\in\mathcal{C}}$ *is a Grothendieck topology on* $\mathcal{C}$.

Proof. Note that for every object X in $\mathcal{C}$ we have

$$\big\{f \in \mathbf{Mor}(\mathcal{C}) \,\big|\, \text{codomain of } f \text{ is } X\big\} = \text{a sieve on } X \text{ that contains } 1_X$$

According to $\{1_X : X \to X\} \in \mathcal{K}(X)$, we derive that family $\mathcal{J}(X)$ contains the maximal sieve on X. Now suppose that $S \in \mathcal{J}(X)$ and $f : Y \to X$. There exists $\{f_i : X_i \to X\}_{i\in I} \in \mathcal{K}(X)$ that is contained in S. Then f^*S contains $\{f_i' : X_i \times_X Y \to Y\}_{i\in I}$ where f_i' are defined by cartesian squares

$$\begin{tikzcd} X_i \times_X Y \arrow[r] \arrow[d, "f_i'"'] & X_i \arrow[d, "f_i"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

Since we have $\{f_i' : X_i \times_X Y \to Y\}_{i\in I} \in \mathcal{K}(Y)$, we deduce that $f^*S \in \mathcal{J}(Y)$.
Finally assume that R is a sieve on X, $S \in \mathcal{J}(X)$ and for every $f \in S$ we have $f^*R \in \mathcal{J}(\mathrm{dom}(f))$. By definition there exists $\{f_i : X_i \to X\}_{i\in I} \in \mathcal{K}(X)$ contained in S and for every $i \in I$ there exists $\{f_{ij} : X_{ij} \to X\}_{j\in J_i} \in \mathcal{K}(X_i)$ contained in f_i^*R. Thus R contains $\{f_i \cdot f_{ij} : X_{ij} \to X\}_{i\in I, j\in J_i}$ and this is a family in $\mathcal{K}(X)$. Hence $R \in \mathcal{J}(X)$. □

Definition 4.3. Let $\mathcal{K}$ be a Grothendieck pretopology on $\mathcal{C}$ and $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$ given by

$$\mathcal{J}(X) = \big\{ S \,\big|\, S \text{ is a sieve on } X \text{ and } S \text{ contains some collection in } \mathcal{K}(X) \big\}$$

then we say that $\mathcal{J}$ is *a Grothendieck topology generated by* $\mathcal{K}$.

Definition 4.4. Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$ and $\mathcal{K}$ be a Grothendieck pretopology on $\mathcal{C}$ that generates $\mathcal{J}$. Then we say that $\mathcal{K}$ is *a basis of the Grothendieck topology* $\mathcal{J}$.

The next result characterizes sheaves on sites for which Grothendieck topology is generated by some Grothendieck pretopology.

Theorem 4.5. *Let $\mathcal{K}$ be a Grothendieck pretopology on $\mathcal{C}$ and $\mathcal{J}$ be a topology generated by $\mathcal{K}$. Then a presheaf F on $\mathcal{C}$ is a sheaf on with respect to $\mathcal{J}$ if and only if for every $\{f_i : X_i \to X\}_{i\in I} \in \mathcal{K}(X)$ the diagram*

$$\begin{tikzcd} F(X) \arrow[r, "\langle F(f_i)\rangle_{i\in I}"] & \prod_{i\in I} F(X_i) \arrow[r, shift left, "\langle F(f'_{ij})\cdot pr_i\rangle_{(i,j)}"] \arrow[r, shift right, "\langle F(f''_{ij})\cdot pr_j\rangle_{(i,j)}"'] & \prod_{(i,j)\in I\times I} F(X_i\times_X X_j) \end{tikzcd}$$

is a kernel of a pair of arrows, where for every $(i,j)\in I\times I$ morphisms f'_{ij} and f'_{ji} form a cartesian square

$$\begin{tikzcd} X_i\times_X X_j \arrow[r, "f''_{ij}"] \arrow[d, "f'_{ij}"'] & X_j \arrow[d, "f_j"] \\ X_i \arrow[r, "f_i"'] & X \end{tikzcd}$$

Proof. Suppose that F is a sheaf with respect to $\mathcal{J}$ and $\mathcal{F} = \{f_i : X_i \to X\}_{i\in I}$ be a collection in $\mathcal{K}(X)$. Let S be a sieve generated by $\{f_i\}_{i\in I}$. Then according to Proposition 3.1 we deduce that the diagram

$$\begin{tikzcd} F(S) \arrow[r, "\mathrm{can}_{\mathcal{F}}\cdot \mathrm{res}^{-1}_{S,\mathcal{F}}"] & \prod_{i\in I} F(X_i) \arrow[r, shift left, "\langle F(f'_{ij})\cdot pr_i\rangle_{(i,j)}"] \arrow[r, shift right, "\langle F(f''_{ij})\cdot pr_j\rangle_{(i,j)}"'] & \prod_{(i,j)\in I\times I} F(X_i\times_X X_j) \end{tikzcd}$$

is a kernel diagram. Since F is sheaf in $\mathcal{J}$ and $S\in\mathcal{J}(X)$, we derive that the map $\mathrm{res}_S : F(X)\to F(S)$ that sends $x\in F(X)$ to $\{F(f)(x)\}_{f\in S}$ is a bijection. Hence

$$\langle F(f_i)\rangle_{i\in I} = \mathrm{can}_{\mathcal{F}}\cdot \mathrm{res}^{-1}_{S,\mathcal{F}}\cdot \mathrm{res}_S : F(X)\to \prod_{i\in I} F(X_i)$$

is a kernel of a pair consisting of $\langle F(f'_{ij})\cdot pr_i\rangle_{(i,j)}$ and $\langle F(f''_{ij})\cdot pr_j\rangle_{(i,j)}$.
Now assume that F is a presheaf on $\mathcal{C}$ and for every collection $\{f_i : X_i\to X\}_{i\in I}$ in $\mathcal{K}(X)$ the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\langle F(f_i)\rangle_{i\in I}"] & \prod_{i\in I} F(X_i) \arrow[r, shift left, "\langle F(f'_{ij})\cdot pr_i\rangle_{(i,j)}"] \arrow[r, shift right, "\langle F(f''_{ij})\cdot pr_j\rangle_{(i,j)}"'] & \prod_{(i,j)\in I\times I} F(X_i\times_X X_j) \end{tikzcd}$$

is a kernel pair. Now Proposition 3.1 implies that for any object X and sieve S generated by a collection in $\mathcal{K}(X)$ every matching family for S of elements of F admits a unique amalgamation. In other words for every sieve S on X generated by some collection in $\mathcal{K}(X)$ the map $\mathrm{res}_S : F(X)\to F(S)$ that sends $x\in F(X)$ to $\{F(f)(x)\}_{f\in S}$ is bijective. Consider now any sieve R in $\mathcal{J}(X)$. Then there exists a sieve S on X generated by some collection of $\mathcal{K}(X)$ such that $S\subseteq R$. Consider a commutative triangle

$$\begin{tikzcd} F(R) \arrow[rr, "\mathrm{res}_{R,S}"] & & F(S) \\ & F(X) \arrow[ul, "\mathrm{res}_R"] \arrow[ur, "\mathrm{res}_S"'] & \end{tikzcd}$$

where $\mathrm{res}_{R,S}\left(\{x_f\}_{f\in R}\right) = \{x_f\}_{f\in S}$, $\mathrm{res}_R(x) = \{F(f)(x)\}_{f\in R}$ and $\mathrm{res}_S(x) = \{F(f)(x)\}_{f\in S}$. By what we prove above, we deduce that res_S is a bijection. Hence res_R is injective. Thus F is a separated presheaf with respect to $\mathcal{J}$. By Proposition 3.2 the map $\mathrm{res}_{R,S}$ is injective. Therefore, $\mathrm{res}_{R,S}$, res_R are injective and res_S is bijective and they form a commutative triangle. Hence they are all bijective maps of classes. In particular, res_R is bijective. We deduce that F is a sheaf with respect to $\mathcal{J}$. □

The next result deals with certain set-theoretic issues and is important from the point of view of the next section.

Proposition 4.6. *Let $\mathcal{K}$ be a Grothendieck pretopology on $\mathcal{C}$ and let $\mathcal{J}$ be a Grothendieck topology generated by $\mathcal{K}$. Assume that the following assertions hold.*

(1) *For every object X in $\mathcal{C}$ the class $\mathcal{K}(X)$ is a set.*

(2) *For every object X in $\mathcal{C}$ every collection $\{f_i : X_i \to X\}_{i\in I} \in \mathcal{K}(X)$ is a set.*

Then for every presheaf $F \in \widehat{\mathcal{C}}$ and every object X in $\mathcal{C}$ a colimit

$$F^+(X) = \mathrm{colim}_{S\in\mathcal{J}(X)} F(S)$$

is a set.

5. Sheaf associated to a presheaf

Let $\mathcal{C}$ be a category. Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$. Let us formulate certain technical set-theoretic assumption on $\mathcal{J}$.

($\star$) For any presheaf F on $\mathcal{C}$, object X in $\mathcal{C}$ and a covering sieve S there the class $F(S)$ of matching families for S of elements of F form a set and the class

Theorem 5.1. *Let F be a presheaf on a Grothendieck site $(\mathcal{C}, \mathcal{J})$. There exists a sheaf $a(F)$ and a morphism $\eta_F : F \to a(F)$ of presheaves such that for every sheaf G and every morphism of presheaves $p : F \to G$ there exists a unique morphism $r : a(F) \to G$ making the diagram*

$$\begin{tikzcd} a(F) \arrow[r, "r"] & G \\ F \arrow[u, "\eta_F"] \arrow[ur, "p"'] & \end{tikzcd}$$

commutative.

First we construct a separated presheaf F^+ out of F. Fix an object X of $\mathcal{C}$. Suppose that S is a covering sieve on X. Denote by $F(S)$ the set of all matching families for S of elements of F. If $S_1 \subseteq S_2$ are covering sieves on X, then we have a function $F(S_2) \to F(S_1)$ given by restriction. Thus $\{F(S)\}_{S\in\mathcal{J}(X)}$ is a diagram indexed by a directed set $\mathcal{J}(X)$ and we define

$$F^+(X) = \mathrm{colim}_{S\in\mathcal{J}(X)} F(S)$$

Note that for every morphism $f : X_1 \to X_2$ in $\mathcal{C}$ and for every sieve $S \in \mathcal{J}(X_2)$ we have a function $F(S) \to F(f^*S)$ given by $F(S) \ni \{s_g\}_{g\in S} \mapsto \{s_{f\cdot g}\}_{g\in f^*S} \in F(f^*S)$. These functions for all $S \in \mathcal{J}(X_2)$

induce a map

$$F^+(X_2) \to F^+(X_1)$$

and this defines a presheaf F^+. We also have a morphism of presheaves $i_F^+ : F \to F^+$ that sends $x \in F(X)$ to a class in $F^+(X)$ represented by a matching family of the form $\{F(f)(x)\}_{f\in S}$ for every covering sieve S on X.

Lemma 5.1.1. *The following assertions hold.*

(1) *F^+ is a separated presheaf.*

(2) *If F is separated presheaf, then F^+ is a sheaf.*

Proof of the lemma. We prove **(1)**. Fix an object $X \in \mathcal{C}$ and a covering sieve S on X. Suppose that $\{x_f\}_{f\in S}$ is a matching family for S of elements of F^+. Assume that $y, z \in F^+(X)$ are amalgamations of $\{x_f\}_{f\in S}$. Then there exists a covering sieve T on X such that y is represented by some matching family $\{s_f\}_{f\in T}$ for T of elements of F and z is represented by some matching family $\{t_f\}_{f\in T}$ for T of elements of F. Fix a morphism $f : Y \to X$ in S. Then $F^+(f)(y)$ is represented by $\{s_{f\cdot g}\}_{g\in f^*T}$ and $F^+(f)(z)$ is represented by $\{t_{f\cdot g}\}_{g\in f^*T}$. Moreover, $F^+(f)(y) = x_f = F^+(f)(z)$ and hence there exists a covering sieve R_f on Y such that $R_f \subseteq f^*T$ and $s_{f\cdot g} = t_{f\cdot g}$ for every $g \in R_f$. Now we know that

$$R = \bigcup_{f\in T} f\cdot R_f \subseteq S$$

is a covering sieve on X and matching families $\{s_f\}_{f\in R}$, $\{t_f\}_{f\in R}$ for R of elements of F represent respectively y and z. Since these families are equal, we derive that $y = z$. This implies that F^+ is separated.

Let us prove **(2)**. Fix an object $X \in \mathcal{C}$ and a covering sieve S on X. Suppose that $\{x_f\}_{f\in S}$ is a matching family for S of elements of F^+. For every $f : Y \to X$ in S there exists a covering sieve R_f on Y and a matching family $\{s(f)_g\}_{g\in R_f}$ for R_f of elements of F that represents x_f. Formula

$$R = \bigcup_{f\in S} f\cdot R_f$$

defines a covering sieve on X contained in S. We set $r_{f\cdot g} = s(f)_g$ for every $f \in S$ and $g \in R_f$. We check now that this definition is independent of choices of $f \in S$ and $g \in R_f$. For this suppose that $f_1, f_2 \in S$ and $g_1 \in R_{f_1}$, $g_2 \in R_{f_2}$ satisfy $f_1\cdot g_1 = f_2\cdot g_2$. Let $Z \in \mathcal{C}$ denote a common domain of morphisms g_1, g_2. Now $F^+(g_1)(x_{f_1})$ is represented by a matching family $\{s(f_1)_{g_1\cdot g}\}_{\mathrm{cod}(g)=Z}$ and $F^+(g_2)(x_{f_2})$ is represented by a matching family $\{s(f_2)_{g_2\cdot g}\}_{\mathrm{cod}(g)=Z}$. According to equality

$$F^+(g_1)(x_{f_1}) = x_{f_1\cdot g_1} = x_{f_2\cdot g_2} = F^+(g_2)(x_{f_2})$$

these families represent the same element of $F^+(Z)$. Hence we deduce that there exists a covering sieve T on Z such that $\{s(f_1)_{g_1\cdot g}\}_{g\in T} = \{s(f_2)_{g_2\cdot g}\}_{g\in T}$. Next $s(f_1)_{g_1}$ is an amalgamation for $\{s(f_1)_{g_1\cdot g}\}_{g\in T}$ and $s(f_2)_{g_2}$ is an amalgamation for $\{s(f_2)_{g_2\cdot g}\}_{g\in T}$. By separatedness of F, we derive that $s(f_1)_{g_1} = s(f_2)_{g_2}$. Thus family $\{r_f\}_{f\in R}$ is well defined. By definition it is a matching family for R of elements of F. Hence it defines an element of $F(R)$ and this element represents some $x \in F^+(X)$. Fix now $f \in S$. By definition of F^+ we deduce that $F^+(f)(x)$ is represented by $\{r_{f\cdot g}\}_{g\in f^*R}$. This family contains $\{r_{f\cdot g}\}_{g\in R_f} = \{s(f)_g\}_{g\in R_f}$ and thus $F^+(f)(x) = x_f$. This proves that $\{x_f\}_{f\in S}$ admits an amalgamation. By **(1)** presheaf F is separated. Hence amalgamation of $\{x_f\}_{f\in S}$ is unique. $\square$

Lemma 5.1.2. *Let $p : F \to G$ be a morphism of presheaves and assume that G is a sheaf. Then there exists a unique morphism $q : F^+ \to G$ such that the diagram*

$$\begin{tikzcd} F^+ \arrow[r, "q"] & G \\ F \arrow[u, "i_F^+"] \arrow[ur, "p"'] & \end{tikzcd}$$

is commutative.

Proof of the lemma. Fix $X \in \mathcal{C}$ and $x \in F^+(X)$. Then there exists a covering sieve S on X and a matching family $\{s_f\}_{f\in S}$ for S of elements of F that represents x. By definitions of F^+ and i_F^+ we have matching family $\{i_F^+(s_f)\}_{f\in S}$ for S of elements of F^+ with x as its amalgamation.
Assume that $q : F^+ \to G$ is a morphism such that $p = q \cdot i_F^+$. We have $p(s_f) = q(i_F^+(s_f))$ for every $f \in S$. Therefore, $q(x)$ must be an amalgamation of a matching family $\{p(s_f)\}_{f\in S} = \{q(i_F^+(s_f))\}_{f\in S}$ for S of elements of G. Since G is a separated presheaf, there exists at most one such amalgamation. This proves uniqueness of q.
The existence of such a q is also evident. As G is a sheaf, one picks $q(x)$ to be the amalgamation of a matching family $\{p(s_f)\}_{f\in S}$ for S of elements of G. Verification that uses definitions of F^+ and i_F^+ shows that this gives rise to a morphism $q : F^+ \to G$ which satisfies $p = q \cdot i_F^+$. □

Proof of the theorem. We define $a(F) = (F^+)^+$ and $\eta_F = i_{F^+}^+ \cdot i_F^+$. By Lemma 5.1.1 presheaf $a(F)$ is a sheaf. Now suppose that $p : F \to G$ is a morphism of presheaves and G is a sheaf. We apply Lemma 5.1.2 twice to obtain a unique morphism $r : a(F) \to G$ such that $p = r \cdot \eta_F$. □

6. Dense subsites

Proposition 6.1. *Let $(\mathcal{C}, \mathcal{J})$ be a site and $\mathcal{K}$ be its full subcategory. Then the following are equivalent.*

(i) *For every object X of $\mathcal{C}$ and every S covering sieve in $\mathcal{J}(X)$ there exists a sieve R in $\mathcal{J}(X)$ generated by a collection of morphisms with domains in $\mathcal{K}$ and contained in S.*

(ii) *For every object X of $\mathcal{C}$ there exists a covering sieve S of X generated by a collection of morphisms in $\mathcal{C}$ with domains in $\mathcal{K}$.*

Proof. The implication **(i)** ⇒ **(ii)** is obvious. We prove **(ii)** ⇒ **(i)**. Let $f : Y \to X$ be a morphism in S. Since $\mathcal{K}$ is dense subcategory of the site $(\mathcal{C}, \mathcal{J})$, we derive that there exists a covering sieve R_f in $\mathcal{J}(Y)$ generated by a collection of morphisms with domains in $\mathcal{K}$. Now a collection

$$R = \bigcup_{f\in S} f \cdot R_f$$

is a covering sieve on X by Fact 2.8. It is also contained in S and is generated by morphisms with domains in $\mathcal{K}$. □

Definition 6.2. Let $(\mathcal{C}, \mathcal{J})$ be a site and $\mathcal{K}$ be a full subcategory of $\mathcal{C}$ satisfying equivalent condition of Proposition 6.1. Then $\mathcal{K}$ is called *a dense subcategory of a site* $(\mathcal{C}, \mathcal{J})$.

Corollary 6.3. *Let $(\mathcal{C}, \mathcal{J})$ be a site and $\mathcal{K}$ be its dense subcategory. Fix an object X of $\mathcal{K}$ and a sieve T in $\mathcal{J}(X)$. Then $T \cap \mathcal{K}$ generates a sieve in $\mathcal{C}$ contained in $\mathcal{J}(X)$.*

Proof. By Proposition 6.1 we derive that there exists a sieve R in $\mathcal{J}(X)$ contained in T and generated by morphisms in $\mathcal{K}$. Now a sieve in $\mathcal{C}$ generated by $T \cap \mathcal{K}$ contains R and hence is an element of $\mathcal{J}(X)$ according to Proposition 2.7. □

Corollary 6.4. *Let $(\mathcal{C}, \mathcal{J})$ be a site and $\mathcal{K}$ be its dense subcategory. For an object X of $\mathcal{K}$ we define $\mathcal{J}_{\mathcal{K}}(X)$ as a collection of all sieves on X of the form $T \cap \mathcal{K}$ for T in $\mathcal{J}(X)$. Then $\mathcal{J}_{\mathcal{K}}$ is a Grothendieck topology on $\mathcal{K}$.*

Proof. Let X be an object of $\mathcal{K}$. The maximal sieve on X in $\mathcal{K}$ is the intersection of the maximal sieve on X in $\mathcal{C}$ and $\mathcal{K}$. Hence the former is an element of $\mathcal{J}_{\mathcal{K}}(X)$.
Suppose next that T is a sieve in $\mathcal{J}(X)$ for some object X of $\mathcal{K}$ and let $f : Y \to X$ be a morphism in $\mathcal{K}$. Then $f^*T \in \mathcal{J}(Y)$ and since we have $f^*(T \cap \mathcal{K}) \subseteq f^*T \cap \mathcal{K}$, we deduce that $f^*(T \cap \mathcal{K}) \in \mathcal{J}_{\mathcal{K}}(Y)$. Thus pullback of an element of $\mathcal{J}_{\mathcal{K}}(X)$ by f is in $\mathcal{J}_{\mathcal{K}}(Y)$.
Finally suppose that X is an object of $\mathcal{K}$ and S, R are sieves on X in $\mathcal{K}$. Assume that $S \in \mathcal{J}_{\mathcal{K}}(X)$ and $f^*R \in \mathcal{J}_{\mathcal{K}}(\mathrm{dom}(f))$ for every $f \in S$. Let T be a sieve in $\mathcal{C}$ generated by R. Then for every $f \in S$ we have $f^*R \subseteq f^*T$. Since $f^*R \in \mathcal{J}_{\mathcal{K}}(\mathrm{dom}(f))$, we deduce by Corollary 6.3 that sieve in $\mathcal{C}$ generated by f^*R is in $\mathcal{J}(\mathrm{dom}(f))$. This also shows $f^*T \in \mathcal{J}(\mathrm{dom}(f))$. Therefore, f^*T is a covering sieve in $\mathcal{C}$ for every $f \in S$. Since S generates a covering sieve in $\mathcal{C}$ by Corollary 6.3, we deduce that $T \in \mathcal{J}(X)$. Note that $R = T \cap \mathcal{K}$ and hence $R \in \mathcal{J}_{\mathcal{K}}(X)$. □

Definition 6.5. Let $(\mathcal{C}, \mathcal{J})$ be a site and $\mathcal{K}$ be its dense subcategory. Then the Grothendieck topology $\mathcal{J}_{\mathcal{K}}$ on $\mathcal{K}$ described in Corollary 6.4 is called *the induced topology on* $\mathcal{K}$ and a pair $(\mathcal{K}, \mathcal{J}_{\mathcal{K}})$ is called *a dense subsite of* $(\mathcal{C}, \mathcal{J})$.

Theorem 6.6. *Let $(\mathcal{C}, \mathcal{J})$ be a site and $\mathcal{K} \subseteq \mathcal{C}$ be a dense subcategory. Then the embedding $\mathcal{K} \hookrightarrow \mathcal{C}$ induces a full and faithful functor*

$$\mathbf{Sh}(\mathcal{C}, \mathcal{J}) \to \mathbf{Sh}(\mathcal{K}, \mathcal{J}_{\mathcal{K}})$$

If in addition for every object X of $\mathcal{C}$ there exists a covering sieve S in $\mathcal{J}(X)$ generated by a set of morphisms with domains in $\mathcal{K}$, then this functor is an equivalence.

Let F be a presheaf on $\mathcal{C}$ and let X be an object of $\mathcal{C}$

Lemma 6.6.1. *Let F be a presheaf on $\mathcal{C}$ and let X be an object of $\mathcal{C}$. Suppose that S is a sieve on X generated by a collection of morphisms $\mathcal{F} = \left\{f_i : X_i \to X\right\}_{i \in I}$. We denote by $F(\mathcal{F})$ the collection of all tuples $\{x_i\}_{i \in I}$ such that $x_i \in F(X_i)$ and for any $i, j \in I$ and morphisms g_i, g_j.*

Lemma 6.6.2. *Let F be a sheaf on $(\mathcal{C}, \mathcal{J})$. Fix an object X of $\mathcal{C}$ and a sieve S in $\mathcal{J}(X)$ generated by a collection $\left\{f_i : X_i \to X\right\}_{i \in I}$ of morphisms.*

$$F(S) \xrightarrow{\quad \mathrm{res} \quad} \prod_{i \in I} F(X_i) \underset{\langle F(f''_{ij}) \cdot pr_j \rangle_{(i,j)}}{\overset{\langle F(f'_{ij}) \cdot pr_i \rangle_{(i,j)}}{\rightrightarrows}} \prod_{(i,j) \in I \times I} F(X_i \times_X X_j)$$

Proof of the lemma. □

Proof. First we prove that this functor is full and faithful. Suppose that $\sigma : F \to G$ is a morphism of sheaves on $(\mathcal{K}, \mathcal{J}_{\mathcal{K}})$. □

References

[Mon19] Monygham. Categories of presheaves. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2019.