FILTERS IN TOPOLOGY

1. Introduction

In these short notes we study filters of subsets with their applications to topological spaces. Filters were introduced in [Cartan, 1937] as an effective tool in studying general topological spaces. Here we recapitulate Cartan's approach. Our main goal is to give a concise proof of Tychonoff's theorem on compact spaces.

2. Filters

Definition 2.1. Let X be a set and let $\mathcal{F}$ be a nonempty family of subsets of X. Assume that the following assertions hold.

(1) $\mathcal{F}$ is closed under finite intersections.

(2) If F_1 and F_2 are subsets of X such that $F_1 \in \mathcal{F}$ and $F_1 \subseteq F_2$, then $F_2 \in \mathcal{F}$.

Then $\mathcal{F}$ is *a filter of subsets of* X.

We note the following fact.

Fact 2.2. *Let X be a set and let $\{\mathcal{F}_i\}_{i\in I}$ be a family of filters of subsets of X. Then*

$$\bigcap_{i\in I}\mathcal{F}_i$$

is a filter of subsets of X.

Proof. Left for the reader as an exercise. □

Definition 2.3. Let X be a set and let $\mathcal{F}$ be a filter of subsets of X. If $\emptyset \notin \mathcal{F}$, then $\mathcal{F}$ is *a proper filter*.

Filters are functorial as it is displayed in the following notion.

Definition 2.4. Let $\mathcal{F}$ be a filter of subsets of a set X and let $f : X \to Y$ be a map. Then a filter

$$f(\mathcal{F}) = \left\{G \subseteq Y \,\middle|\, \text{there exists } F \in \mathcal{F} \text{ such that } f(F) \subseteq G\right\}$$

of subsets of Y is *the image of $\mathcal{F}$ under f*.

Let us note the following results.

Fact 2.5. *Let $\mathcal{F}$ be a filter of subsets of a set X and let $f : X \to Y$ be a map. If $\mathcal{F}$ is a proper filter, then $f(\mathcal{F})$ is a proper filter.*

Proof. Left for the reader as an exercise. □

Now we introduce the notion of ultrafilter and prove by invoking axiom of choice that they exist.

Definition 2.6. Let $\mathcal{F}$ be a proper filter of subsets of a set X such that for every proper filter $\tilde{\mathcal{F}}$ if $\mathcal{F} \subseteq \tilde{\mathcal{F}}$, then $\mathcal{F} = \tilde{\mathcal{F}}$. Then $\mathcal{F}$ is *an ultrafilter of subsets of X*.

Next we describe properties of ultrafilters and prove their existence.

Proposition 2.7. *Let X be a set and let $\mathcal{F}$ be a proper filter of subsets of X. The following assertions are equivalent.*

(i) *$\mathcal{F}$ is an ultrafilter of subsets of X.*

(ii) *For each subset F of X either $F \in \mathcal{F}$ or $X \setminus F \in \mathcal{F}$.*

Proof. Assume that $\mathcal{F}$ is an ultrafilter and let F be a subset of X. Suppose that $F \notin \mathcal{F}$. Then the smallest filter (Fact 2.2) containing $\{F\} \cup \mathcal{F}$ is not a proper filter. This implies that there exists $F' \in \mathcal{F}$ such that $F \cap F' = \varnothing$. Since $F' \subseteq X \setminus F$ and $\mathcal{F}$ is a filter, we derive that $X \setminus F \in \mathcal{F}$. This proves that **(i)** $\Rightarrow$ **(ii)**.
Suppose that **(ii)** holds. Consider a filter $\tilde{\mathcal{F}}$ such that $\mathcal{F} \subsetneq \tilde{\mathcal{F}}$. If $F \in \tilde{\mathcal{F}} \setminus \mathcal{F}$, then $X \setminus F \in \mathcal{F}$ and hence $\varnothing = F \cap (X \setminus F) \in \tilde{\mathcal{F}}$. This implies that $\tilde{\mathcal{F}}$ is not a proper filter. Thus $\mathcal{F}$ is an ultrafilter of subsets of X. This completes the proof of **(ii)** $\Rightarrow$ **(i)**. □

Proposition 2.8. *Let X be a set and let $\mathcal{F}$ be a proper filter of subsets of X. Then there exists an ultrafilter $\tilde{\mathcal{F}}$ of subsets of X such that $\mathcal{F} \subseteq \tilde{\mathcal{F}}$.*

Proof. Consider the family

$$\mathrm{F} = \left\{\mathcal{G} \,\middle|\, \mathcal{G} \text{ is a proper filter of subsets of } X \text{ and } \mathcal{F} \subseteq \mathcal{G}\right\}$$

Note that F is nonempty because $\mathcal{F} \in \mathrm{F}$. The inclusion introduces partial order on F and if $\mathrm{L} \subseteq \mathrm{F}$ is a linearly ordered subset, then

$$\bigcup \mathrm{L}$$

is a proper filter. Hence each chain in $(\mathrm{F}, \subseteq)$ admits an upper bound. Zorn's lemma implies that $(\mathrm{F}, \subseteq)$ has a maximal element $\tilde{\mathcal{F}}$. Clearly $\tilde{\mathcal{F}}$ is an ultrafilter of subsets of X which contains $\mathcal{F}$. □

3. Filters and convergence in topological spaces

Definition 3.1. Let (X, τ) be a topological space and let $\mathcal{F}$ be a proper filter of subsets of X. Consider a point x in X. Suppose that for every open neighborhood U of x we have $U \in \mathcal{F}$. Then filter $\mathcal{F}$ *converges to x with respect to τ.*

Proposition 3.2. *Let $(X, \tau), (Y, \theta)$ be topological spaces and let $f : X \to Y$ be a map. Then the following assertions are equivalent.*

(i) *f is a continuous map $(X, \tau) \to (Y, \theta)$.*

(ii) *If $\mathcal{F}$ is a proper filter of subsets of X convergent to some point x with respect to τ, then $f(\mathcal{F})$ converges to $f(x)$ with respect to θ.*

Proof. Suppose that f is a continuous map $(X, \tau) \to (Y, \theta)$. Fix a proper filter $\mathcal{F}$ of subsets of X convergent to x with respect to τ. Fix an open neighborhood V of $f(x)$ with respect to θ. By continuity of f we have $f^{-1}(V) \in \tau$. Thus $f^{-1}(V)$ is an open neighborhood of x with respect to τ. Hence $f^{-1}(V) \in \mathcal{F}$ and we infer that $V \in f(\mathcal{F})$. Since V is arbitrary open neighborhood of $f(x)$ with respect to θ, we derive that $f(\mathcal{F})$ converges to $f(x)$ with respect to θ. This proves the implication **(i)** $\Rightarrow$ **(ii)**.
Suppose now that **(ii)** holds. Fix a point x in X and consider an open neighborhood V of $f(x)$ with respect to θ. Define

$$\mathcal{F} = \left\{F \subseteq X \,\middle|\, U \setminus f^{-1}(V) \subseteq F \text{ for some open neighborhood } U \text{ of } x \text{ with respect to } \tau\right\}$$

Then $\mathcal{F}$ is a filter of subsets of X. Note that

$$Y \setminus V = f\left(X \setminus f^{-1}(V)\right) \in f(\mathcal{F})$$

This implies that $V \notin f(\mathcal{F})$. If $\mathcal{F}$ is a proper filter, then it converges to x with respect τ and thus $f(\mathcal{F})$ converges to $f(x)$ with respect to θ. Since $V \notin f(\mathcal{F})$, the filter $f(\mathcal{F})$ cannot converge to $f(x)$ with respect to θ. Therefore, $\mathcal{F}$ is not a proper filter. This means that there exists an open neighborhood U of x with respect to τ such that $U \subseteq f^{-1}(V)$. This proves that f is continuous at x as a map $(X, \tau) \to (Y, \theta)$. Since $x \in X$ is arbitrary, we derive the implication **(ii)** $\Rightarrow$ **(i)**. □

Theorem 3.3. *Let (X, τ) be a topological space. Then the following assertions are equivalent.*

(i) *Each ultrafilter of subsets of X is convergent to some point of X with respect to τ.*

(ii) *(X, τ) is a quasi-compact topological space.*

Proof. Suppose that **(i)** holds. Pick a family $\{F_i\}_{i \in I}$ of closed and nonempty subsets of (X, τ) which is closed under finite intersections. Then the family

$$\left\{F \subseteq X \,\middle|\, F_i \subseteq F \text{ for some } i \in I\right\}$$

is a proper filter of subsets of X. By Proposition 2.8 there exists an ultrafilter $\mathcal{F}$ of subsets of X which contains the filter defined above. According to **(i)** ultrafilter $\mathcal{F}$ is convergent to some point x in X with respect to τ. Then for every open neighborhood U of x with respect to τ we have $U \in \mathcal{F}$. In particular, $U \cap F_i \neq \varnothing$ for every $i \in I$ and for every open neighborhood U of x with respect to τ. Since F_i is closed for each $i \in I$, this implies that $x \in F_i$ for every $i \in I$. Thus

$$x \in \bigcap_{i \in I} F_i$$

and this concludes that (X, τ) is quasi-compact.

Assume that (X, τ) is quasi-compact and suppose that $\mathcal{F}$ is an ultrafilter of subsets of X. Suppose that $\mathcal{F}$ is not convergent. Then for every $x \in X$ there exists open neighborhood U_x of x with respect to τ such that $U_x \notin \mathcal{F}$. Since (X, τ) is quasi-compact, we deduce that there exist finite subset $\{x_1, ..., x_n\} \in X$ such that

$$X = \bigcup_{i=1}^{n} U_{x_i}$$

According to Proposition 2.7 we derive that $X \smallsetminus U_x \in \mathcal{F}$ for every $x \in X$. Hence

$$\bigcap_{i=1}^{n} \left(X \smallsetminus U_{x_i}\right) \in \mathcal{F}$$

On the other hand we have

$$\bigcap_{i=1}^{n} \left(X \smallsetminus U_{x_i}\right) = X \smallsetminus \bigcup_{i=1}^{n} U_{x_i} = \varnothing$$

This is contradiction. Thus the implication **(ii)** $\Rightarrow$ **(i)** holds. □

4. Tychonoff's theorem

Theorem 4.1. *Let $\left\{(X_i, \tau_i)\right\}_{i \in I}$ be a family of nonempty topological spaces. Then*

$$\prod_{i \in I} (X_i, \tau_i)$$

is quasi-compact if and only if (X_i, τ_i) is quasi-compact for each $i \in I$.

References

[Cartan, 1937] Cartan, H. (1937). Théorie des filtres. *CR Acad. Sci. Paris*, 205:595–598.