DANIELL-KOLMOGOROV EXTENSION THEOREM

1. Introduction

In this notes we prove Daniell-Kolmogorov extension theorem, which concerns existence of probability measures on infinite products. Daniell-Kolmogorov theorem is a fundamental result of advanced probability theory. It implies the existence of various stochastic processes.

2. Daniell-Kolmogorov extension theorem

Theorem 2.1 (Daniell-Kolmogorov)**.** *Let T be a set and let $(X_t, \mathcal{F}_t)_{t\in T}$ be a set of measurable spaces. For every $t \in T$ we fix a Hausdorff topology τ_t on X_t. Suppose that the following assertions hold.*

(1) *For every finite subset F of T there exists a probability measure μ_F on the measurable space $(X_F, \mathcal{F}_F)$. The space is defined by*

$$X_F = \prod_{t\in F} X_t,\ \mathcal{F}_F = \bigotimes_{t\in F} \mathcal{F}_t$$

(2) *If $F_1 \subseteq F_2$ are finite subsets of T and $\pi_{F_2,F_1} : X_{F_2} \to X_{F_1}$ is the projection, then $\left(\pi_{F_2,F_1}\right)_* \mu_{F_2} = \mu_{F_1}$.*

(3) *The measure μ_F is inner regular with respect to the product topology $\tau_F = \prod_{t\in\mathcal{F}} \tau_t$.*

Then there exists a unique probability measure μ on the space $(X, \mathcal{F})$ where

$$X = \prod_{t\in T} X_t,\ \mathcal{F} = \bigotimes_{t\in T} \mathcal{F}_t$$

such that for every finite subset F of T we have

$$\mu_F = (\pi_F)_* \mu$$

for the projection $\pi_F : X \to X_F$.

For the proof we need some notation. First recall that $\mathcal{F} = \otimes_{t\in T} \mathcal{F}_t$ is the σ-algebra generated by subsets of X of the form $\pi_F^{-1}(B)$ where $B \in \mathcal{F}_F$. We call such sets *cylinders*. We prove now some lemma which relies on inner regularity assumption.

Lemma 2.1.1. *Let $\{F_n\}_{n\in\mathbb{N}}$ be a nondecreasing sequence of finite subsets of T and let $\{D_n\}_{n\in\mathbb{N}}$ be sets such that $D_n \in \mathcal{F}_{F_n}$ and $\{\pi_{F_n}^{-1}(D_n)\}_{n\in\mathbb{N}}$ form a nonincreasing family of cylinders in X. Assume that there exists $\epsilon > 0$ such that*

$$\mu_{F_n}(D_n) \geq \epsilon$$

for every $n \in \mathbb{N}$. Then there exists a sequence $\{K_n\}_{n\in\mathbb{N}}$ such that the following assertions are satisfied.

(1) *K_n is a subset of D_n for every $n \in \mathbb{N}$.*

(2) *K_n is compact with respect to τ_{F_n} for every $n \in \mathbb{N}$.*

(3) *$\{\pi_{F_n}^{-1}(K_n)\}_{n\in\mathbb{N}}$ is a nonincreasing sequence.*

(4) *$\mu_{F_n}(K_n) \geq \frac{\epsilon}{2}$ for every $n \in \mathbb{N}$.*

Proof of the lemma. We prove slightly stronger statement. More precisely, we prove that there exists a sequence $\{K_n\}_{n\in\mathbb{N}}$ satisfying all the assertions in the statement and the additional assertion that

$$\mu_{F_n}(D_n) \leq \mu_{F_n}(K_n) + \frac{\epsilon}{4} \cdot \left(1 + \frac{1}{2} + ... + \frac{1}{2^n}\right)$$

for all $n \in \mathbb{N}$. The proof goes by induction. For D_0 we pick its subset K_0 compact with respect to τ_{F_0} such that

$$\mu_{F_0}(D_0) \leq \mu_{F_0}(K_0) + \frac{\epsilon}{4}$$

Suppose that for some $n \in \mathbb{N}$ there is a sequence $\{K_m\}_{m \leq n}$ such that $K_m \subseteq D_m$, K_m is compact with respect to τ_{F_m}, the sequence $\{\pi_{F_m}^{-1}(K_m)\}_{m \leq n}$ is nonincreasing and

$$\mu_{F_m}(D_m) \leq \mu_{F_m}(K_m) + \frac{\epsilon}{4} \cdot \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^m}\right)$$

for every $m \leq n$. Write $\pi_{F_{n+1},F_n}^{-1}(K_n) = \tilde{K}_n$ and $\pi_{F_{n+1},F_n}^{-1}(D_n) = \tilde{D}_n$. Then

$$\mu_{F_{n+1}}(\tilde{D}_n) = \mu_{F_n}(D_n) \leq \mu_{F_n}(K_n) + \frac{\epsilon}{4} \cdot \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^n}\right) = \mu_{F_{n+1}}(\tilde{K}_n) + \frac{\epsilon}{4} \cdot \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^n}\right)$$

and $\tilde{K}_n \subseteq \tilde{D}_n$. Moreover, by assumption $D_{n+1} \subseteq \tilde{D}_n$. Thus we have

$$\mu_{F_{n+1}}\left(D_{n+1} \setminus (\tilde{K}_n \cap D_{n+1})\right) = \mu_{F_{n+1}}(D_{n+1} \setminus \tilde{K}_n) \leq \mu_{F_{n+1}}(\tilde{D}_n \setminus \tilde{K}_n) \leq \frac{\epsilon}{4} \cdot \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^n}\right)$$

and hence

$$\mu_{F_{n+1}}\left(D_{n+1}\right) - \mu_{F_{n+1}}\left(\tilde{K}_n \cap D_{n+1}\right) \leq \frac{\epsilon}{4} \cdot \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^n}\right)$$

Now we pick a subset K_{n+1} of $\tilde{K}_n \cap D_{n+1}$ such that K_{n+1} is compact with respect to $\tau_{F_{n+1}}$ and

$$\mu_{F_{n+1}}(\tilde{K}_n \cap D_{n+1}) \leq \mu_{F_{n+1}}(K_{n+1}) + \frac{\epsilon}{4} \cdot \frac{1}{2^{n+1}}$$

Then

$$\mu_{F_{n+1}}\left(D_{n+1}\right) \leq \mu_{F_{n+1}}\left(\tilde{K}_n \cap D_{n+1}\right) + \frac{\epsilon}{4} \cdot \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^n}\right) \leq$$
$$\leq \mu_{F_{n+1}}(K_{n+1}) + \frac{\epsilon}{4} \cdot \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^n}\right) + \frac{\epsilon}{4} \cdot \frac{1}{2^{n+1}} = \mu_{F_{n+1}}(K_{n+1}) + \frac{\epsilon}{4} \cdot \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{n+1}}\right)$$

Hence we can extend sequence $\{K_m\}_{m \leq n}$ by adding K_{n+1} and our inductive proof is completed. □

The next lemma is purely topological.

Lemma 2.1.2. *Let $\mathcal{S}$ be a countable set. Suppose that $\{F_n\}_{n \in \mathbb{N}}$ is a nondecreasing sequence of finite subsets of $\mathcal{S}$ such that $\mathcal{S} = \bigcup_{n \in \mathbb{N}} F_n$ and assume that $\{X_s\}_{s \in \mathcal{S}}$ is a family of Hausdorff topological spaces. If $\{K_n\}_{n \in \mathbb{N}}$ is a sequence of nonempty sets such that K_n is a compact subset of $\prod_{s \in F_n} X_s$ for every $n \in \mathbb{N}$ and*

$$\left\{K_n \times \prod_{s \in \mathcal{S} \setminus F_n} X_s\right\}_{n \in \mathbb{N}}$$

is a nondecreasing sequence of subsets of $\prod_{s \in \mathcal{S}} X_s$, then

$$\bigcap_{n \in \mathbb{N}} \left(K_n \times \prod_{s \in \mathcal{S} \setminus F_n} X_s\right) \neq \emptyset$$

Proof of the lemma. For each $s \in \mathcal{S}$ we choose $n_s \in \mathbb{N}$ such that $s \in F_{n_s}$. We define

$$K_s = \pi_s\left(K_{n_s} \times \prod_{s \in \mathcal{S} \setminus F_{n_s}} X_s\right)$$

where $\pi_s : \prod_{s \in \mathcal{S}} X_s \to X_s$ is the projection. Then by assumptions K_s is a compact and nonempty subset of X_s. We define K as a product of $\prod_{s \in \mathcal{S}} K_s$. By Tychonoff's theorem K is compact and nonempty subset of $\prod_{s \in \mathcal{S}} X_s$. Fix now $n \in \mathbb{N}$. Define $m = \max\left(n, \max_{s \in F_n} n_s\right)$. Then

$$\pi_s\left(K_m \times \prod_{s \in \mathcal{S} \setminus F_m} X_s\right) \subseteq \pi_s\left(K_{n_s} \times \prod_{s \in \mathcal{S} \setminus F_{n_s}} X_s\right) \subseteq K_s$$

for every $s \in F_n$. Thus

$$K_m \times \prod_{s\in\mathcal{S}\setminus F_m} X_s \subseteq \prod_{s\in F_n} K_s \times \prod_{s\in\mathcal{S}\setminus F_n} X_s$$

Since $m \geq n$, we have

$$K_m \times \prod_{s\in\mathcal{S}\setminus F_m} X_s \subseteq \left(\prod_{s\in F_n} K_s \times \prod_{s\in\mathcal{S}\setminus F_n} X_s\right) \cap \left(K_n \times \prod_{s\in\mathcal{S}\setminus F_n} X_s\right)$$

Together with $K_m \neq \emptyset$ this implies that

$$\left(\prod_{s\in F_n} K_s \times \prod_{s\in\mathcal{S}\setminus F_n} X_s\right) \cap \left(K_n \times \prod_{s\in\mathcal{S}\setminus F_n} X_s\right) \neq \emptyset$$

We infer that

$$K \cap \left(K_n \times \prod_{s\in\mathcal{S}\setminus F_n} X_s\right) \neq \emptyset$$

Therefore, the sequence

$$\left\{K \cap \left(K_n \times \prod_{s\in\mathcal{S}\setminus F_n} X_s\right)\right\}_{n\in\mathbb{N}}$$

is nondecreasing and consists of closed nonempty subsets of a compact topological space K. Hence it has nonempty intersection. This completes the proof that

$$\bigcap_{n\in\mathbb{N}} \left(K_n \times \prod_{s\in\mathcal{S}\setminus F_n} X_s\right) \neq \emptyset$$

□

Proof of the theorem. Fix a finite subset F of T and for B in $\mathcal{F}_F$ we set

$$\mu\left(\pi_F^{-1}(B)\right) = \mu_F(B)$$

Note that this makes μ into a function defined on the family of all cylinders. Moreover, it is clear that the family of all cylinders is an algebra of subsets of X and μ defined in such a way is an additive function. We claim that μ is also σ-additive. For this pick a nondecreasing sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite subsets of T and a sequence $\{B_n\}_{n\in\mathbb{N}}$ of sets such that $B_n \in \mathcal{F}_{F_n}$ for every $n \in \mathbb{N}$ and the sequence $\{\pi_{F_n}^{-1}(B_n)\}_{n\in\mathbb{N}}$ is nondecreasing. Moreover, we assume that there exists a finite subset F of T and a set $B \in \mathcal{F}_F$ such that

$$\pi_F^{-1}(B) = \bigcup_{n\in\mathbb{N}} \pi_{F_n}^{-1}(B_n)$$

We define $D_n = \pi_{F\cup F_n,F}^{-1}(B) \setminus \pi_{F\cup F_n,F_n}^{-1}(B_n)$ for every $n \in \mathbb{N}$. Then $D_n \in \mathcal{F}_{F\cup F_n}$ satisfies

$$\pi_{F\cup F_n}^{-1}(D_n) = \pi_F^{-1}(B) \setminus \pi_{F_n}^{-1}(B_n)$$

Hence the sequence $\{\pi_{F\cup F_n}^{-1}(D_n)\}_{n\in\mathbb{N}}$ is nonincreasing and

$$\emptyset = \bigcap_{n\in\mathbb{N}} \pi_{F\cup F_n}^{-1}(D_n)$$

Now suppose that there exists $\epsilon > 0$ such that $\mu\left(\pi_{F\cup F_n}^{-1}(D_n)\right) \geq \epsilon$ for every $n \in \mathbb{N}$. Then by Lemma 2.1.1 there exists a sequence $\{K_n\}_{n\in\mathbb{N}}$ such that the following assertions are satisfied.

(1) K_n is a subset of D_n for every $n \in \mathbb{N}$.

(2) K_n is compact with respect to $\tau_{F\cup F_n}$ for every $n \in \mathbb{N}$.

(3) $\{\pi_{F\cup F_n}^{-1}(K_n)\}_{n\in\mathbb{N}}$ is a nonincreasing sequence.

(4) $\mu_{F\cup F_n}(K_n) \geq \frac{\epsilon}{2}$ for every $n \in \mathbb{N}$.

In particular, assertion **(4)** implies that K_n is nonempty for every $n \in \mathbb{N}$. We set $\mathcal{S} = \bigcup_{n\in\mathbb{N}} (F \cup F_n)$ and apply Lemma 2.1.2 to $\mathcal{S}$, $\{(X_s, \tau_s)\}_{s\in\mathcal{S}}$ and $\{K_n\}_{n\in\mathbb{N}}$. We obtain

$$\bigcap_{n\in\mathbb{N}} \left(K_n \times \prod_{s\in\mathcal{S}\setminus F_n} X_s \right) \neq \emptyset$$

which implies that

$$\bigcap_{n\in\mathbb{N}} \pi^{-1}_{F\cup F_n}(K_n) \neq \emptyset$$

This is contradiction, since

$$\bigcap_{n\in\mathbb{N}} \pi^{-1}_{F\cup F_n}(K_n) \subseteq \bigcap_{n\in\mathbb{N}} \pi^{-1}_{F\cup F_n}(D_n) = \emptyset$$

Therefore, we proved that

$$\lim_{n\to+\infty} \mu(\pi^{-1}_{F\cup F_n}(D_n)) = 0$$

and hence

$$\mu(\pi^{-1}_F(B)) = \lim_{n\to+\infty} \mu(\pi^{-1}_{F_n}(B_n))$$

This proves the claim that μ is σ-additive. According to [Monygham, 2018, Theorem 3.3] additive function μ defined on the algebra of cylinders can be extended to a unique probability measure on $\mathcal{F}$. Since no confusion can arise, we denote this probability measure by μ. Then $\mu : \mathcal{F} \to [0,1]$ is a unique probability measure such that

$$\mu\left(\pi^{-1}_F(B)\right) = \mu_F(B)$$

for every finite subset F of T and $B \in \mathcal{F}_F$. □

References

[Monygham, 2018] Monygham (2018). Introduction to measure theory. *github repository: "Monygham/Pedo-mellon-a-minno"*.