1. Introduction

Throughout this notes k denote a field and $\mathbf{G}$ denote a group scheme over k. We also fix a k-scheme X equipped with an action of $\mathbf{G}$ determined by morphism $a : \mathbf{G} \times_k X \to X$.

2. Categorical and geometric quotients

Definition 2.1. Let $q : X \to Y$ be a morphism of k-schemes such that the diagram

$$\mathbf{G} \times_k X \overset{a}{\underset{\mathrm{pr}_X}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel in the category of k-schemes. Then $q : X \to Y$ is *a categorical quotient of* X.

Definition 2.2. Consider a cokernel

$$\mathbf{G} \times_k X \overset{a}{\underset{\mathrm{pr}_X}{\rightrightarrows}} X \xrightarrow{q} Y$$

in the category of locally ringed spaces over k. If Y is a scheme, then $q : X \to Y$ is *a geometric quotient of* X.

Fact 2.3. *Every geometric quotient is categorical.*

Proof. Categorical quotient is a cokernel in the category of k-schemes. On the other hand geometric quotient is a cokernel in the category of locally ringed spaces and hence it also satisfies cokernel property in its full subcategory of k-schemes. Thus every geometric quotient is categorical. □

Corollary 2.4. *Let $q : X \to Y$ be a morphism of schemes. The following assertions are equivalent.*

(i) *The diagram*

$$\mathbf{G} \times_k X \overset{a}{\underset{\mathrm{pr}_X}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel diagram of underlying topological spaces and the diagram

$$\mathcal{O}_Y \xrightarrow{q^{\#}} q_*\mathcal{O}_X \overset{q_*a^{\#}}{\underset{q_*\mathrm{pr}_X^{\#}}{\rightrightarrows}} q_* \left(\mathrm{pr}_X\right)_* \mathcal{O}_{\mathbf{G}\times_k X} = q_* a_* \mathcal{O}_{\mathbf{G}\times_k X}$$

is a kernel diagram in the category of sheaves on Y.

(ii) *q is a geometric quotient of X.*

Proof. This is a consequence of [Monygham, 2019, Theorem 2.9]. □

Let $q : X \to Y$ be a morphism of k-schemes such that $q \cdot \mathrm{pr}_X = q \cdot a$. For a morphism $g : Y' \to Y$ of k-schemes consider the cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Then there exists a unique action $a' : \mathbf{G} \times_k X' \to X'$ of $\mathbf{G}$ on X' such that the square above consists of $\mathbf{G}$-equivariant morphism (we consider Y, Y' as $\mathbf{G}$-schemes equipped with trivial $\mathbf{G}$-actions). Keeping this in mind we have the following.

Definition 2.5. A morphism $q : X \to Y$ is *a uniform categorical (geometric) quotient of* X if for every flat morphism $g : Y' \to Y$ its base change $q' : X' \to Y'$ is a categorical (geometric) quotient of X'.

Definition 2.6. A morphism $q : X \to Y$ is *a universal categorical (geometric) quotient of* X if for every morphism $g : Y' \to Y$ its base change $q' : X' \to Y'$ is a categorical (geometric) quotient of X'.

3. Types of actions and criterion for smoothness of universal geometric quotients

Definition 3.1. The action of $\mathbf{G}$ on X is *separated* if the morphism $\langle a, \mathrm{pr}_X \rangle : \mathbf{G} \times_k X \to X \times_k X$ has closed set-theoretic image.

Theorem 3.2. *Let $q : X \to Y$ be a geometric quotient of X. Assume that q is universally submersive. Then the following assertions are equivalent.*

- **(i)** *The action of* $\mathbf{G}$ *on X is separated.*
- **(ii)** *Y is separated.*

Proof. We have a cartesian square

$$\begin{tikzcd} X \times_Y X \arrow[r, hook] \arrow[d] & X \times_k X \arrow[d, "q \times_k q"] \\ Y \arrow[r, hook, "\Delta_Y"'] & Y \times_k Y \end{tikzcd}$$

It follows that $X \times_Y X \hookrightarrow X \times_k X$ is a locally closed immersion. Since q is a geometric quotient, we derive that $\langle a, \mathrm{pr}_X \rangle$ factors as a surjective morphism $\mathbf{G} \times_k X \twoheadrightarrow X \times_Y X$ followed by the immersion $X \times_Y X \hookrightarrow X \times_k X$. Thus the action of $\mathbf{G}$ on X is separated if and only if $X \times_Y X$ is a closed subscheme of $X \times_k X$. Since q is universally submersive, we derive that $q \times_k q$ is submersive. As the square above is cartesian we derive that $\Delta_Y(Y) \subseteq Y \times_k Y$ is closed if and only if $X \times_Y X \subseteq X \times_k X$ is closed. Therefore, Y is separated if and only if the action of $\mathbf{G}$ on X is separated. □

The following result which concerns complete local rings is very useful.

Definition 3.3. Let x be a k-point of X. Suppose that the morphism $\mathbf{G} \to X$ given by the composition

$$\mathbf{G} = \mathbf{G} \times_k \operatorname{Spec} k \xhookrightarrow{\text{induced by } x} \mathbf{G} \times_k X \longrightarrow X$$

is a closed immersion. Then the action of $\mathbf{G}$ on X has *a closed free orbit at x*.

Proposition 3.4. *Let $(A, \mathfrak{m}, k)$ be a complete local noetherian k-algebra and let $\sigma : A \to A[[x_1, ..., x_n]]$ be a local morphism into a ring of formal power series over A. Assume that the composition*

$$A \xrightarrow{\sigma} A[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod (x_1, ..., x_n)} A$$

is the identity and the composition

$$A \xrightarrow{\sigma} A[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod \mathfrak{m}} (A/\mathfrak{m})[[x_1, ..., x_n]] = k[[x_1, ..., x_n]]$$

is surjective. Consider elements $y_1, ..., y_n$ of A such that $\sigma(y_i) \bmod \mathfrak{m} = x_i$ for $i = 1, ..., n$. Then the composition

$$A \xrightarrow{\sigma} A[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod (y_1, ..., y_n)} (A/(y_1, ..., y_n))[[x_1, ..., x_n]]$$

is an isomorphism.

Proof. For convienience let ϕ denote the morphism given by the rule $a \mapsto \sigma(a) \bmod (y_1, ..., y_n)$. According to assumptions we have

$$\sigma(y_i) = x_i + y_i + \sum_{j=1}^{n} x_j \cdot \mathfrak{m}[[x_1, ..., x_n]]$$

for each i. Thus $\phi(y_i) = \sum_{j=1}^{n} f_{ij} \cdot x_j$ where $f_{ij} \in A/(y_1, ..., y_n)$ are elements such that the matrix $\left[f_{ij}\right]_{1 \leq i,j \leq n}$ is invertible in $A/(y_1, ..., y_n)$. Hence

$$(A/(y_1, ..., y_n))[[x_1, ..., x_n]] = (A/(y_1, ..., y_n))[[\phi(y_1), ..., \phi(y_n)]]$$

and ϕ composed with $(A/(y_1, ..., y_n))[[\phi(y_1), ..., \phi(y_n)]] \twoheadrightarrow A/(y_1, ..., y_n)$ is the quotient morphism $A \twoheadrightarrow A/(y_1, ..., y_n)$. From this observations we derive that ϕ is surjective. It remains to prove that it is injective. Consider z in A such that $\phi(z) = 0$. Suppose that $z \in (y_1, ..., y_n)^m$ for some $m \in \mathbb{N}$. Write

$$z = \sum_{\alpha \in \Lambda} c_\alpha \cdot y_1^{\alpha_1} ... y_n^{\alpha_n}$$

for some $c_\alpha \in A$ where $\Lambda = \{\alpha \in \mathbb{N}^n \,|\, \alpha_1 + ... + \alpha_n = m\}$. Hence

$$0 = \phi(z) = \sum_{\alpha \in \Lambda} \phi(c_\alpha) \cdot \phi(y_1)^{\alpha_1} ... \phi(y_n)^{\alpha_n}$$

Thus $\phi(c_\alpha) \in (\phi(y_1), ..., \phi(y_n))$ for every $\alpha \in \Lambda$. Since ϕ composed with $(A/(y_1, ..., y_n))[[\phi(y_1), ..., \phi(y_n)]] \twoheadrightarrow A/(y_1, ..., y_n)$ is the quotient morphism $A \twoheadrightarrow A/(y_1, ..., y_n)$, we derive that

$$c_\alpha \bmod (y_1, ..., y_n) = \phi(c_\alpha) \bmod (\phi(y_1), ..., \phi(y_n)) = 0$$

for every $\alpha \in \Lambda$. Thus $c_\alpha \in (y_1, ..., y_n)$ for every $\alpha \in \Lambda$, which implies that $z \in (y_1, ..., y_n)^{m+1}$. Thus we proved that

$$\phi(z) = 0 \text{ and } z \in (y_1, ..., y_n)^m \Rightarrow z \in (y_1, ..., y_n)^{m+1}$$

By $\mathfrak{m}$-adic completeness of A this implies that $\phi(z) = 0$ if and only if $z = 0$. Hence ϕ is also injective. □

References

[Monygham, 2019] Monygham (2019). Locally ringed spaces. *github repository: "Monygham/Pedo-mellon-a-minno"*.