

# KAKUTANI'S FIXED POINT THEOREM AND ITS APPLICATIONS

## 1. INTRODUCTION

In these notes we study applications of Kakutani's fixed point theorem to theory of Nash equilibria.

## 2. BROUWER FIXED POINT THEOREM

In this section we present a Milnor's analytic proof of Brouwer fixed point theorem. The proof is based on the excellent paper [Rog80], where the author vastly simplifies the original Milnor's approach.

Let  $n \in \mathbb{N}$  be a natural number. We denote by  $\mathbb{B}^n$  a closed unit euclidean ball in  $\mathbb{R}^n$  and we denote by  $S^n$  a unit euclidean sphere in  $\mathbb{R}^{n+1}$ .

**Theorem 2.1** (Brouwer's fixed point theorem). *Let  $f : \mathbb{B}^n \rightarrow \mathbb{B}^n$  be a continuous map. Then there exists  $x$  in  $\mathbb{B}^n$  such that  $f(x) = x$ .*

**Lemma 2.1.1.** *Let  $U$  be an open subset of  $\mathbb{R}^{n+1}$  containing  $\mathbb{B}^{n+1}$ . Then there is no continuously differentiable map  $f : U \rightarrow \mathbb{R}^{n+1}$  such that  $f(\mathbb{B}^{n+1}) = S^n$  and  $f(x) = x$  for  $x \in S^n$ .*

*Proof of the lemma.* Assume that such  $f$  exists. For  $t \in [0, 1]$  we define  $f_t : U \rightarrow \mathbb{R}^{n+1}$  given by formula

$$f_t(x) = x - t(f(x) - x) = (1 - t)x + tf(x)$$

for every  $x \in U$ . We have  $f_t(\mathbb{B}^{n+1}) \subseteq \mathbb{B}^{n+1}$  for every  $t \in [0, 1]$ . There exists  $T > 0$  such that the following assertions hold for every  $t \in [0, T]$ .

(1)  $f_t|_{\mathbb{B}^{n+1}}$  is injective.

(2)  $Df_t(x)$  is invertible for every  $x \in \mathbb{B}^{n+1}$ .

We explain now how to choose suitable  $T$ . For this consider a function  $g : U \rightarrow \mathbb{R}^{n+1}$  given by formula  $g(x) = x - f(x)$  for every  $x \in U$ . Let  $L = 1 + \sup_{x \in \mathbb{B}^{n+1}} \|Dg(x)\|$ . Then  $g$  is Lipschitz function on  $\mathbb{B}^{n+1}$  with constant  $L$ . Fix  $t < L^{-1}$ . If  $x_1, x_2 \in \mathbb{B}^{n+1}$  are distinct, then

$$\|f_t(x_1) - f_t(x_2)\| \geq \|x_1 - x_2\| - t \cdot \|g(x_1) - g(x_2)\| \geq \|x_1 - x_2\| - tL \cdot \|x_1 - x_2\| = (1 - tL) \cdot \|x_1 - x_2\| > 0$$

Therefore,  $f_t|_{\mathbb{B}^{n+1}}$  is injective. We have

$$t \cdot \|Dg(x)\| = tL < 1$$

for every  $x \in \mathbb{B}^{n+1}$  and hence  $Df_t(x)$  is invertible for such  $x$ . Thus it suffices to take  $T = \min(L^{-1}, 1)$ . Now we fix  $t \in [0, T]$ . Property (2) and  $f_t(\mathbb{B}^{n+1}) \subseteq \mathbb{B}^{n+1}$  imply that

$$U_t = f_t(\text{int}(\mathbb{B}^{n+1})) \subseteq \mathbb{R}^{n+1}$$

is an open subset contained in  $\mathbb{B}^{n+1}$ . If  $U_t \neq \text{int}(\mathbb{B}^{n+1})$ , then there exists

$$y \in (\text{cl}(U_t) \setminus U_t) \cap \text{int}(\mathbb{B}^{n+1})$$

Consider a sequence  $\{x_m\}_{m \in \mathbb{N}}$  of elements in  $\text{int}(\mathbb{B}^{n+1})$  such that

$$\lim_{m \rightarrow +\infty} f_t(x_m) = y$$

We may assume that the sequence  $\{x_m\}_{m \in \mathbb{N}}$  converges to some  $x$  in  $\mathbb{B}^{n+1}$ . Then  $y = f_t(x)$ . Clearly  $x \notin \text{int}(\mathbb{B}^{n+1})$  because otherwise  $y \in U_t$ . Hence  $x \in S^n$ . But then

$$\text{int}(\mathbb{B}^{n+1}) \ni y = f_t(x) = (1-t)x + tf(x) = x \in S^n$$

Thus the only possibility is that  $U_t = \text{int}(\mathbb{B}^{n+1})$ . Therefore, we have  $f_t(\mathbb{B}^{n+1}) = \mathbb{B}^{n+1}$ . Now (1) and (2) imply that  $f_t$  induces a diffeomorphism  $\mathbb{B}^{n+1} \rightarrow \mathbb{B}^{n+1}$ . Define a polynomial  $p : [0, 1] \rightarrow \mathbb{R}$  given by formula

$$p(t) = \int_{\mathbb{B}^{n+1}} |\det(Df_t(x))| dx = \int_{\mathbb{B}^{n+1}} |\det(1_{\mathbb{R}^{n+1}} + t \cdot Dg(x))| dx$$

Since  $f_t$  induces a diffeomorphism  $\mathbb{B}^{n+1} \rightarrow \mathbb{B}^{n+1}$  for  $t \in [0, T]$ , we deduce that  $p(t) = \text{vol}(\mathbb{B}^{n+1})$  for  $t \in [0, T]$ . Next  $p(t)$  is a polynomial and hence we deduce that  $p(t) = \text{vol}(\mathbb{B}^{n+1})$  for every  $t \in [0, 1]$ . On the other hand we have

$$p(1) = \int_{\mathbb{B}^{n+1}} |\det(Df(x))| dx$$

Since  $f(\mathbb{B}^{n+1}) = S^n$ , we deduce that  $\det(Df(x)) = 0$  for every  $x \in \text{int}(\mathbb{B}^{n+1})$  and hence  $p(1) = 0$ . This is contradiction, because  $\text{vol}(\mathbb{B}^{n+1}) = p(1) \neq 0$ .  $\square$

**Lemma 2.1.2.** *Let  $U$  be an open subset of  $\mathbb{R}^{n+1}$  containing  $\mathbb{B}^{n+1}$ . Suppose that  $f : U \rightarrow \mathbb{R}^{n+1}$  is a continuously differentiable map such that  $f(\mathbb{B}^{n+1}) \subseteq \mathbb{B}^{n+1}$ . Then there exists a fixed point of  $f$  in  $\mathbb{B}^{n+1}$ .*

*Proof of the lemma.* Assume that  $f$  does not have fixed points in  $\mathbb{B}^{n+1}$ . Consider an open subset  $W$  of  $U$  defined by  $f(x) \neq x$ . Then  $W$  contains  $\mathbb{B}^{n+1}$ . For every  $x$  in  $W$  we define a point  $r(x) \in S^n$  as the intersection of a line

$$\{f(x) + t \cdot (x - f(x)) \in \mathbb{R}^{n+1} \mid t \in \mathbb{R}_+\}$$

with  $S^n$ . Then  $r : W \rightarrow \mathbb{R}^{n+1}$  is continuously differentiable,  $r(W) = S^n$  and  $r(x) = x$  for every  $x \in S^n$ . This is a contradiction with Lemma 2.1.1.  $\square$

*Proof of the theorem.* Suppose that  $f : \mathbb{B}^{n+1} \rightarrow \mathbb{B}^{n+1}$  is a continuous map without fixed points. We consider  $f$  as a map  $\tilde{f} : \mathbb{B}^{n+1} \rightarrow \mathbb{R}^{n+1}$ . By Stone-Weierstrass theorem there exists a sequence  $\{p_m : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}\}_{m \in \mathbb{N}}$  of polynomials such that the sequence  $\{p_m|_{\mathbb{B}^{n+1}}\}_{m \in \mathbb{N}}$  is uniformly convergent to  $\tilde{f}$ . Let

$$\alpha_m = \sup_{x \in \mathbb{B}^{n+1}} \|\tilde{f}(x) - p_m(x)\|$$

and consider an open subset  $U_m$  such that  $\|p_m(x)\| < 1 + \alpha_m$  for every  $x \in U_m$ . Clearly  $U_m$  is an open subset of  $\mathbb{R}^{n+1}$  containing  $\mathbb{B}^{n+1}$ . Define a sequence  $\{q_m : U_m \rightarrow \mathbb{R}^{n+1}\}_{m \in \mathbb{N}}$  by formula  $q_m(x) = (1 + \alpha_m)^{-1} \cdot p_m(x)$  for  $x \in U_m$ . Then  $q_m(\mathbb{B}^{n+1}) \subseteq \mathbb{B}^{m+1}$  and  $q_m$  is continuously differentiable for every  $m \in \mathbb{N}$ . By Lemma 2.1.2 we derive that there exists  $x_m \in \mathbb{B}^{m+1}$  such that  $q_m(x_m) = x_m$ . Since  $\mathbb{B}^{n+1}$  is compact, we may assume that the sequence  $\{x_m\}_{m \in \mathbb{N}}$  converges to some  $x \in \mathbb{B}^{n+1}$ . Note also that  $\{q_m|_{\mathbb{B}^{n+1}}\}_{m \in \mathbb{N}}$  is uniformly convergent to  $\tilde{f}$ . Thus we have

$$x = \lim_{m \rightarrow +\infty} x_m = \lim_{m \rightarrow +\infty} q_m(x_m) = \tilde{f}(x)$$

and hence  $f(x) = x$ . This proves the theorem in the case  $n \geq 1$ . For  $n = 0$  the set  $\mathbb{B}^n$  consists of a point and hence the theorem holds trivially.  $\square$

#### REFERENCES

- [Rog80] Claude Ambrose Rogers. A less strange version of milnor's proof of brouwer's fixed-point theorem. *The American Mathematical Monthly*, 87(7):525–527, 1980.