LINEARLY REDUCTIVE GROUPS

1. Motivation – linear representations of compact topological groups

In this section we fix a compact topological group $\mathbf{G}$. Assume that $\rho : \mathbf{G} \to \mathrm{GL}_n(\mathbb{C})$ is a continuous homomorphism i.e. a complex, n-dimensional linear representation of $\mathbf{G}$. For every $g \in \mathbf{G}$ we get a matrix

$$\rho(g) = \left[c_{ij}(g)\right]_{1 \leq i,j \leq n}$$

For i, j function $c_{ij} : \mathbf{G} \to \mathbb{C}$ is a continuous complex valued function. Alternatively suppose that $\{e_1, e_2, ..., e_n\}$ is the standard basis of $\mathbb{C}^n$ on which $\mathrm{GL}_n(\mathbb{C})$ act. Then c_{ij} is equal to a function

$$\mathbf{G} \ni g \mapsto \langle g \cdot e_i, e_j \rangle \in \mathbb{C}$$

Fix now $g_1, g_2 \in \mathbf{G}$ and note that

$$\left[c_{ij}(g_2 \cdot g_1)\right]_{1 \leq i,j \leq n} = \rho(g_2 \cdot g_1) = \rho(g_2) \cdot \rho(g_1) = \left[\sum_{k=1}^{n} c_{ik}(g_2) \cdot c_{kj}(g_1)\right]_{1 \leq i,j \leq n}$$

Hence

$$c_{ij}(g_2 \cdot g_1) = \sum_{k=1}^{n} c_{ik}(g_2) \cdot c_{kj}(g_1)$$

for every $1 \leq i, j \leq n$. This implies that $\sum_{1 \leq i,j \leq n} \mathbb{C} \cdot c_{ij} \subseteq \mathcal{L}^2(\mathbf{G}, \mathbb{C})$ is a linear $\mathbf{G} \times \mathbf{G}^{\mathrm{op}}$-subrepresentation of the regular representation $\mathcal{L}^2(\mathbf{G}, \mathbb{C})$. We call it *the matrix coefficients of* ρ.

2. Grothendieck toposes

Theorem 2.1. *Let $p : G \to F$ be an epimorphism of sheaves. Then*

$$\begin{tikzcd} G \times_F G \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & G \arrow[r, "p"] & F \end{tikzcd}$$

is a cokernel pair.

Proof of the theorem. Suppose that $q : G \to H$ is a morphism of sheaves such that $q \cdot pr_1 = q \cdot pr_2$. Our task is to construct and show uniqueness of a morphism $r : F \to H$ such that $q = r \cdot p$. Fix $X \in \mathcal{C}$ and $x \in F(X)$. Let S be a covering sieve S on X and for every $f : Y \to X$ in S let $s_f \in G(Y)$ be an element such that $p(s_f) = f^*x$. Existence of S and $\{s_f\}_{f \in S}$ follows from the fact that p is an epimorphism. Lemma 2.1.1 shows that there exists a unique amalgamation $y \in H(X)$ such that $q(s_f) = f^*y$ for $f \in S$ and in addition y is independent of the choice of both S and $\{s_f\}_{f \in S}$. Moreover, for any $f : Y \to X$ (not necessarily in S) f^*S is a covering sieve on Y such that for every $g : Z \to Y$ in f^*S we have $p(s_{f \cdot g}) = (f \cdot g)^*x$. Note that $\{q(s_{f \cdot g}\}_{g \in f^*S}$ is a matching family with amalgamation f^*y. The discussion above shows that $r : F \to H$ described by sending x to y is well defined morphism of sheaves and by its description we have $q = r \cdot p$. The uniqueness of r follows from the uniqueness of y. □

Lemma 2.1.1. *Let $X \in \mathcal{C}$ and $x \in F(X)$. Then the following assertions hold.*

(1) *Let S be a covering sieve on X and for every $f : Y \to X$ in S let $s_f \in G(Y)$ be an element such that $p(s_f) = f^*x$. Then $\{q(s_f)\}_{f \in S}$ is a matching family.*

(2) *Let S, T be covering sieves on X. Suppose that $\{s_f\}_{f\in S}$, $\{t_f\}_{f\in T}$ are families of sections of G such that $p(s_f) = f^*x$ and $p(t_f) = f^*x$. Then there exists a unique common amalgamation $y \in H(X)$ for $\{s_f\}_{f\in S}$ and $\{t_f\}_{f\in T}$.*

Proof of the lemma. For the proof of **(1)** fix $f : Y \to X$ in S and pick any $g : Z \to Y$ then

$$p(g^* s_f) = g^* p(s_f) = g^* f^* x = (f \cdot g)^* x = p(s_{f\cdot g})$$

Hence there exists $\xi \in (G \times_F G)(Z)$ such that $pr_1(\xi) = g^* s_f$ and $pr_2(\xi) = s_{f\cdot g}$. Thus

$$g^* q(s_f) = q(g^* s_f) = q(pr_1(\xi)) = q(pr_2(\xi)) = q(s_{f\cdot g})$$

and we deduce that $\{q(s_f)\}_{f\in S}$ is a matching family and **(1)** is proved.
Now we prove **(2)**. For every $f : Y \to X$ in $S \cap T$ we have $p(s_f) = f^*x = p(t_f)$. This implies that there exists $\xi \in (G \times_F G)(Y)$ such that $pr_1(\xi) = s_f$ and $pr_2(\xi) = t_f$. Thus

$$q(s_f) = q(pr_1(\xi)) = q(pr_2(\xi)) = q(t_f)$$

Since $S \cap T \in \mathcal{J}(X)$ and by **(1)**, we derive that $\{q(s_f)\}_{f\in S\cap T} = \{q(t_f)\}_{f\in S\cap T}$, $\{q(s_f)\}_{f\in S}$, $\{q(t_f)\}_{f\in T}$ are matching families. Now H is a sheaf, hence there exists a unique common amalgamation $y \in H(X)$ for all three families. □

3. Characterization of representable presheaves on the category of schemes

Fact 3.1. *For every k-scheme X representable presheaf h_X is a Zariski sheaf.*

Proof. Let $\{f_i : U_i \to U\}_{i\in I}$ be a Zariski covering of a k-scheme U. For every $(i,j) \in I \times I$ we denote by $f'_i : U_i \times_U U_j \to U_i$ and $f''_j : U_i \times_U U_j \to U_j$ the canonical projections. Suppose now that $\{g_i : U_i \to X\}_{i\in I}$ are morphisms of k-schemes such that $g_i \cdot f'_i = g_j \cdot f''_j$ for every $(i,j) \in I \times I$. Then one can glue morphism $\{g_i\}_{i\in I}$ to a unique morphism $g : U \to X$. This translates to the Zariski sheaf condition for h_X. □

Definition 3.2. Let F, G be presheaves on $\mathbf{Sch}_k$ and let $f : F \to G$ be their morphism. Suppose that $x \in G(X)$ for some k-scheme X. To every x of this type one can associate the cartesian square of presheaves

$$\begin{array}{ccc} h_X \times_G F & \longrightarrow & F \\ \downarrow{\scriptstyle \pi_x} & & \downarrow{\scriptstyle f} \\ h_X & \longrightarrow & G \end{array}$$

in which bottom vertical morphism $h_X \to G$ is canonically identified with x. We say that f is:

(1) *an open immersion* if for every k-scheme X and $x \in G(X)$ morphism π_x is isomorphic to the image under Yoneda embedding of some open immersion of k-schemes.

(2) *a closed immersion* if for every k-scheme X and $x \in G(X)$ morphism π_x is isomorphic to the image under Yoneda embedding of some closed immersion of k-schemes.

Proposition 3.3. *Let $f : F \to G$ be a morphism of presheaves on $\mathbf{Sch}_k$. Suppose that f is either open or closed immersion. Then f is a monomorphism of presheaves.*

Proof. Fix an element $y \in G(X)$. Consider a cartesian square

$$\begin{tikzcd} h_X \times_G F \arrow[r] \arrow[d, "\pi_y"'] & F \arrow[d, "f"] \\ h_X \arrow[r] & G \end{tikzcd}$$

in which y determines a morphism $h_X \to G$. Morphism f is either open or closed immersion. Hence there exists a monomorphism $j : Y \to X$ of k-schemes such that π_y is isomorphic with h_j. Yoneda embedding preserves monomorphisms. Thus h_j is a monomorphism of presheaves. This implies that π_y is a monomorphism of presheaves for every k-scheme X and $y \in G(X)$. In particular, there exists at most one element $x \in F(X)$ such that $f(x) = y$. Since $y \in G(X)$ is arbitrary, we deduce that f is a monomorphism of presheaves. □

Definition 3.4. Let F be a presheaf on $\mathbf{Sch}_k$ and $\{f_i : F_i \to F\}_{i\in I}$ be a family of open immersions. Then for every k-scheme X and $x \in F(X)$ we have a family of open immersions $\{f_{i,x} : U_{i,x} \to X\}_{i\in I}$ defined by cartesian squares

$$\begin{tikzcd} h_{U_{i,x}} \arrow[r] \arrow[d, "h_{f_{i,x}}"'] & F_i \arrow[d, "f_i"] \\ h_X \arrow[r] & F \end{tikzcd}$$

in which bottom vertical morphism $h_X \to G$ is canonically identified with x. We say that $\{f_i\}_{i\in I}$ is *an open cover of* F if for every k-scheme X and $x \in F(X)$ we have

$$X = \bigcup_{i\in I} f_{i,x}\left(U_{i,x}\right)$$

Theorem 3.5. *Let F be a presheaf on* $\mathbf{Sch}_k$*. Then the following are equivalent.*

(i) *$F \cong h_X$ for some k-scheme X.*

(ii) *F is a Zariski sheaf and there exists an open cover $\{v_i : h_{V_i} \to F\}_{i\in I}$ such that $\{V_i\}_{i\in I}$ are affine k-schemes.*

(iii) *F is a Zariski sheaf and there exists an open cover $\{v_i : h_{V_i} \to F\}_{i\in I}$ such that $\{V_i\}_{i\in I}$ are k-schemes.*

Proof. We prove **(i)** ⇒ **(ii)**. Since $F \cong h_X$ and properties in **(ii)** are stable under isomorphism, we deduce that we can replace F by h_X. So it suffices to show that h_X satisfies **(ii)**. By definition every k-scheme X admits an open cover $\{v_i : V_i \to X\}_{i\in I}$ by affine k-schemes. Since Yoneda embedding $h : \mathbf{Sch}_k \to \widehat{\mathbf{Sch}_k}$ preserves fiber-products, we derive that $\{h_{v_i}\}_{i\in I}$ is an open cover in the category of presheaves. Thus h_X admits an open cover by presheaves representable by affine k-schemes. Next suppose that $\{f_i : U_i \to U\}_{i\in I}$ is a Zariski covering of a k-scheme U and $\{g_i : U_i \to X\}_{i\in I}$ is a family of morphisms of k-schemes such that $g_{i|U_i\times_U U_j} = g_{j|U_i\times_U U_j}$ for every pair $(i,j) \in I \times I$. Then we can glue $\{g_i\}_{i\in I}$ into a unique morphism of k-schemes $g : U \to X$ such that $g \cdot f_i = g_i$ for every $i \in I$. This shows that h_X is a Zariski sheaf.
The implication **(ii)** ⇒ **(iii)** is a consequence of the fact that every affine k-scheme is a k-scheme.
Assume now that **(iii)** holds. Fix elements $i, j \in I$ and consider a cartesian square

$$\begin{tikzcd} h_{V_i} \times_F h_{V_j} \arrow[r, "v_j'"] \arrow[d, "v_i'"'] & h_{V_j} \arrow[d, "v_j"] \\ h_{V_i} \arrow[r, "v_i"'] & F \end{tikzcd}$$

in the category $\widehat{\mathbf{Sch}_k}$. Since v_i is an open immersion, we derive that there exists an open subscheme $V_{ij} \subseteq V_i$ and an isomorphism $p_{ij} : h_{V_{ij}} \to h_{V_i} \times_F h_{V_j}$ such that the triangle

$$\begin{tikzcd} h_{V_{ij}} \arrow[r, "p_{ij}"] \arrow[dr, hook] & h_{V_i} \times_F h_{V_j} \arrow[d, "v_i'"] \\ & h_{V_i} \end{tikzcd}$$

is commutative. Similarly since v_j is an open immersion, we derive that there exists an open subscheme $V_{ji} \subseteq V_j$ and an isomorphism $p_{ji} : h_{V_{ij}} \to h_{V_i} \times_F h_{V_j}$ such that the triangle

$$\begin{tikzcd} h_{V_{ji}} \arrow[r, "p_{ji}"] \arrow[dr, hook] & h_{V_i} \times_F h_{V_j} \arrow[d, "v_j'"] \\ & h_{V_j} \end{tikzcd}$$

is commutative. Now we define an isomorphism of k-schemes $\phi_{ij} : V_{ij} \to V_{ji}$ by requirement $h_{\phi_{ij}} = p_{ji}^{-1} \cdot p_{ij}$. Then the data consisting of families $\{V_i\}_{i \in I}$, $\{V_{ij}\}_{(i,j) \in I \times I}$ and $\{\phi_{ij}\}_{(i,j) \in I \times I}$ satisfy the following assertions.

(1) $V_{ij} \subseteq V_i$ is an open subscheme for every $i \in I$ and $j \in J$.

(2) $V_{ii} = V_i$ and $\phi_{ii} = 1_{V_i}$ for every $i \in I$.

(3) $\phi_{ij} : V_{ij} \to V_{ji}$ is an isomorphism of k-schemes for every $(i, j) \in I \times I$.

(4) For every pair $(i, j) \in I \times I$ and $k \in I$ isomorphism ϕ_{ij} restricts to an isomorphism

$$\phi'_{ij,k} : V_{ij} \cap V_{ik} \to V_{ji} \cap V_{jk}$$

of k-schemes.

(5) For every triple $(i, j, k) \in I \times I \times I$ we have

$$\phi'_{ik,j} = \phi'_{jk,i} \cdot \phi'_{ij,k}$$

Thus by [GD71, Chapitre 0, 4.1.7] family $\{V_i\}_{i \in I}$ can be considered as an open cover of a ringed k-space X in such a way that for any elements $i, j \in I$ the square

$$\begin{tikzcd} h_{V_i \cap V_j} \arrow[r, hook] \arrow[d, hook] & h_{V_j} \arrow[d, "v_j"] \\ h_{V_i} \arrow[r, "v_i"'] & F \end{tikzcd}$$

is cartesian (the intersection $V_i \cap V_j$ in the diagram is taken inside X). Since X admits an open cover by a k-schemes, it is itself a k-scheme. Next we construct a morphism $f : h_X \to F$. For this note that for each $i \in I$ morphism v_i gives rise to an element $x_i \in F(V_i)$. Since $v_{i|h_{V_i \cap V_j}} = v_{j|h_{V_i \cap V_j}}$ for any two $i, j \in I$, we deduce that $x_{i|V_i \cap V_j} = x_{j|V_i \cap V_j}$. Next we apply the fact that F is a Zariski sheaf to construct an element $x \in F(X)$ such that $x_{|V_i} = x_i$ for every $i \in I$. Now x determines a morphism $f : h_X \to F$ such that the following square

$$
\begin{tikzcd}
 & h_{V_i} \arrow[dl, hook] \arrow[dr, "v_i"] & \\
h_X \arrow[rr, "g"'] & & F
\end{tikzcd}
$$

Now let Y be a k-scheme and pick $y \in F(Y)$. Suppose that $g : h_Y \to F$ is a morphism corresponding to y. Pick $i \in I$. Since $v_i : h_{V_i} \to F$ is an open immersion, there exists open subscheme $W_i \subseteq Y$ that fits in a cartesian square

$$
\begin{tikzcd}
h_{W_i} \arrow[r, "g_i"] \arrow[d, hook] & h_{V_i} \arrow[d, "v_i"] \\
h_Y \arrow[r, "g"'] & F
\end{tikzcd}
$$

By Yoneda lemma g_i corresponds to $k_i \in h_{V_i}(W_i)$. By definition $k_i : W_i \to V_i$ is a morphism of k-schemes. Next for $i \in I$ and $j \in I$ we have

□

4. Matrix coefficients of a representation

Proposition 4.1. *Let $\mathfrak{X}$ be a monoid k-functor and let V be a finitely generated, projective k-module. Fix a morphism of monoids $\rho : \mathfrak{X} \to \mathcal{L}_V$. Fix k-algebra A and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. For every A-algebra B and $x \in \mathfrak{X}_A(B)$ we consider the formula*

$$c_{v,w}(x) = \langle \rho_A(x) \cdot v_B, w_B \rangle$$

Then $c_{v,w}$ defines a regular function on $\mathfrak{X}_A$ for every k-algebra A.

Proof. Suppose that $f : B \to C$ is a morphism of A-algebras and pick $x \in \mathfrak{X}_A(B)$. Since ρ_A is natural and $w : A \otimes_k V \to A$ is a morphism of A-modules, we derive that the diagram

$$
\begin{tikzcd}
V_B \arrow[r, "\rho_A(x)"] \arrow[d, "1_{V_A} \otimes_A f"'] & V_B \arrow[r, "w_B"] \arrow[d, "1_{V_A} \otimes_A f"] & B \arrow[d, "f"] \\
V_C \arrow[r, "\rho_A(\mathfrak{X}_A(f)(x))"'] & V_C \arrow[r, "w_C"'] & C
\end{tikzcd}
$$

is commutative. Hence

$$c_{v,w}\big(\mathfrak{X}_A(f)(x)\big) = \langle \rho_A\big(\mathfrak{X}_A(f)(x)\big) \cdot v_C, w_C \rangle = f\big(\langle \rho_A(x) \cdot v_B, w_B \rangle\big) = f\big(c_{v,w}(x)\big)$$

and this implies that $c_{v,w} : \mathfrak{X}_A \to \mathbb{A}^1_A$ is natural. □

Definition 4.2. Let $\mathfrak{X}$ be a monoid k-functor and let (V,ρ) be its representation with finitely generated, projective underlying k-module V. Fix k-algebra A and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. Then the regular function $c_{v,w}$ on $\mathfrak{X}_A$ is called *the matrix coefficient of v and w.*

Proposition 4.3. *Let $\mathfrak{X}$ be a monoid k-functor and let (V,ρ) be its representation with finitely generated projective underlying k-module V. Then the following assertions holds.*

(1) *For every k-algebra A map*

$$(A \otimes_k V) \times \left(A \otimes_k V^{\vee}\right) \ni (v,w) \mapsto c_{v,w} \in \mathrm{Mor}_A\left(\mathfrak{X}_A, \mathbb{A}^1_A\right)$$

is A-bilinear.

(2) *The collection of maps*

$$\left\{(A \otimes_k V) \times \left(A \otimes_k V^{\vee}\right) \ni (v,w) \mapsto c_{v,w} \in \mathrm{Mor}_A\left(\mathfrak{X}_A, \mathbb{A}^1_A\right)\right\}_{A \in \mathbf{Alg}_k}$$

gives rise to a morphism of k-functors

$$V_{\mathrm{a}} \times V_{\mathrm{a}}^{\vee} \longrightarrow \mathcal{M}\mathrm{or}_k\left(\mathfrak{X}, \mathbb{A}^1_k\right)$$

Proof. We left the proof of **(1)** to the reader.
We prove **(2)**. Consider k-algebra A and an A-algebra B with structural morphism $f : A \to B$. Fix $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. We prove that restriction of $c_{v,w} : \mathfrak{X}_A \to \mathbb{A}^1_A$ to the category $\mathbf{Alg}_B$ is c_{v_B,w_B}. For this pick a B-algebra C and an element $x \in \mathfrak{X}_A(C) = \mathfrak{X}_B(C)$. Note that

$$c_{v,w}(x) = \langle \rho_A(x) \cdot v_C, w_C \rangle = \langle \rho_B(x) \cdot v_C, w_C \rangle = \langle \rho_B(x) \cdot (v_B)_C, (w_B)_C \rangle = c_{v_B,w_B}(x)$$

and hence $c_{v,w|\mathbf{Alg}_B} = c_{v_B,w_B}$. Consider the square

$$\begin{tikzcd}
V_{\mathrm{a}}(A) \times V_{\mathrm{a}}^{\vee}(A) \arrow[r] \arrow[d, "V_a(f) \times V_a^{\vee}(f)"'] & \mathcal{M}\mathrm{or}_k\left(\mathfrak{X}, \mathbb{A}^1\right)(A) \arrow[d, "\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathbb{A}^1)(f)"] \\
V_{\mathrm{a}}(B) \times V_{\mathrm{a}}^{\vee}(B) \arrow[r] & \mathcal{M}\mathrm{or}_k\left(\mathfrak{X}, \mathbb{A}^1\right)(B)
\end{tikzcd}$$

in which both horizontal arrows are given by formula $(v,w) \mapsto c_{v,w}$. We proved that the square commutes. Since f is an arbitrary morphism of k-algebras, we conclude the assertion. □

Corollary 4.4. *Let $\mathfrak{X}$ be a monoid k-functor and let (V,ρ) be its representation with finitely generated projective underlying k-module V. Then there exists a morphism of k-functors*

$$\left(V \otimes_k V^{\vee}\right)_{\mathrm{a}} \xrightarrow{\;c\;} \mathcal{M}\mathrm{or}_k\left(\mathfrak{X}, \mathbb{A}^1_k\right)$$

given by formula

$$(A \otimes_k V) \otimes_A \left(A \otimes_k V^{\vee}\right) \ni (v,w) \mapsto c_{v,w} \in \mathrm{Mor}_A\left(\mathfrak{X}_A, \mathbb{A}^1_A\right)$$

Moreover, c is a morphism of k-functors equipped with $\mathfrak{X} \times \mathfrak{X}^{\mathrm{op}}$-actions.

Proof. The first part is an immediate consequence of Proposition 4.3. We prove that c is a morphism of k-functors equipped with $\mathfrak{X} \times \mathfrak{X}^{\mathrm{op}}$-actions. For this we fix a k-algebra k and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. Pick a morphism of k-algebras $f : A \to B$, $(y,z) \in \mathfrak{X}(A) \times \mathfrak{X}(A)^{\mathrm{op}}$ and $x \in \mathfrak{X}_A(B)$. Then we have

$$c_{\rho(y)\cdot v, w \cdot \rho(z)}(x) = \left\langle \rho_A(x) \cdot \left(\rho(y) \cdot v\right)_B, \left(w \cdot \rho(z)\right)_B \right\rangle =$$

$$= \left\langle \rho_A(x) \cdot \rho_A((\mathfrak{X}_A(f)(y))) \cdot v_B, w_B \cdot \rho_A\left(\mathfrak{X}_A(f)(z)\right) \right\rangle = w_B\Big(\rho_A\left(\mathfrak{X}_A(f)(z)\right) \cdot \rho_A(x) \cdot \rho_A\left(\mathfrak{X}_A(f)(y)\right) \cdot v_B\Big) =$$

$$= w_B\Big(\rho_A\left(\mathfrak{X}_A(f)(z) \cdot x \cdot \mathfrak{X}_A(f)(y)\right) \cdot v_B\Big) = \left\langle \rho_A\left(\mathfrak{X}_A(f)(z) \cdot x \cdot \mathfrak{X}_A(f)(y)\right) \cdot v_B, w_B \right\rangle =$$

$$= c_{v,w}\big(\mathfrak{X}_A(f)(z)\cdot x\cdot \mathfrak{X}_A(f)(y)\big)$$

and hence c is a morphism of k-functors equipped with actions of $\mathfrak{X}\times\mathfrak{X}^{\mathrm{op}}$. □

5. Algebra of regular functions of a k-functor

Example 5.1. For every k-algebra A we denote by $|A|$ its underlying set. We denote by $\mathbb{A}^1_k$ a k-functor given by assignment $\mathbb{A}^1_k(A) = |A|$ for every A. We call $\mathbb{A}^1_k$ *the affine line over k*. Let $k[x]$ be a polynomial k-algebra with variable x. For every k-algebra A map of sets

$$\mathrm{Mor}_k\left(k[x], A\right) \ni f \mapsto f(x) \in |A|$$

is a bijection. The family of such maps gives rise to an isomorphism of k-functors

$$\mathrm{Mor}_k\left(\mathrm{Spec}\,(-), \mathrm{Spec}\,k[x]\right) \cong \mathrm{Mor}_k\left(k[x], -\right) \cong \mathbb{A}^1_k$$

and hence $\mathbb{A}^1_k$ is representable by an affine k-scheme $\mathrm{Spec}\,k[x]$.

Definition 5.2. Let $\mathfrak{X}$ be a k-functor. Consider $\alpha \in k$ and $f, g \in \mathrm{Mor}_k\left(\mathfrak{X}, \mathbb{A}^1_k\right)$. Then for every k-algebra A and $x \in \mathfrak{X}(A)$ formulas

$$(f+g)(x) = f(x)+g(x),\ (f\cdot g)(x) = f(x)\cdot g(x),\ (\alpha\cdot f)(x) = \alpha\cdot f(x)$$

define k-algebra operations on the class $\mathrm{Mor}_k\left(\mathfrak{X}, \mathbb{A}^1_k\right)$. We call them *pointwise k-algebra operations*. In particular, if $\mathrm{Mor}_k\left(\mathfrak{X}, \mathbb{A}^1_k\right)$ is a set, then pointwise k-algebras operations on this set give rise to *the k-algebra of regular functions on $\mathfrak{X}$*.

6. k-functors

Definition 6.1. The category $\mathbf{Fun}(\mathbf{Alg}_k, \mathbf{Set})$ of copresheaves on $\mathbf{Alg}_k$ is called *the category of k-functors*.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are k-functors, then we denote by $\mathrm{Mor}_k(\mathfrak{X},\mathfrak{Y})$ the class of morphisms $\mathfrak{X}\to\mathfrak{Y}$ of k-functors.
Since the category of k-functors is a category of copresheaves, under assumptions specified in [Mon19, section 5] for given k-functors $\mathfrak{X}$, $\mathfrak{Y}$ there exists an internal hom $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{Y})$. Let us discuss this important notion and also related ones. For details and proofs for general case we refer to [Mon19, section 5].
Let $\mathfrak{X}$ and $\mathfrak{Y}$ be A-functors for some k-algebra A. Then we denote by $\mathrm{Mor}_A\left(\mathfrak{X},\mathfrak{Y}\right)$ the class of morphisms of A-functors $\mathfrak{X}\to\mathfrak{Y}$. For every A-algebra B and a morphism $\sigma:\mathfrak{X}\to\mathfrak{Y}$ of A-functors we denote by $\mathfrak{X}_B$, $\mathfrak{Y}_B$, σ_B the restrictions $\mathfrak{X}_{|\mathbf{Alg}_B}$, $\mathfrak{Y}_{|\mathbf{Alg}_B}$, $\sigma_{|\mathbf{Alg}_B}$ of these entities to the category of B-algebras.

Fact 6.2. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be k-functors. Assume that A is a k-algebra, B is an A-algebra, C is an B-algebra. Then the composition of maps of classes*

$$\mathrm{Mor}_A\left(\mathfrak{X}_A,\mathfrak{Y}_A\right) \xrightarrow{\sigma\mapsto\sigma_B} \mathrm{Mor}_B\left(\mathfrak{X}_B,\mathfrak{Y}_B\right) \xrightarrow{\sigma\mapsto\sigma_B} \mathrm{Mor}_C\left(\mathfrak{X}_C,\mathfrak{Y}_C\right)$$

equals

$$\mathrm{Mor}_A\left(\mathfrak{X}_A,\mathfrak{Y}_A\right) \xrightarrow{\sigma\mapsto\sigma_C} \mathrm{Mor}_C\left(\mathfrak{X}_C,\mathfrak{Y}_C\right)$$

Proof. Left to the reader. □

Definition 6.3. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be k-functors and suppose that for every k-algebra A the class $\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set. We define

$$\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(A) = \mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$$

for every k-algebra A. This is a k-functor, since for every k-algebra A and A-algebra B, we can compose a morphism $\sigma : \mathfrak{X}_A \to \mathfrak{Y}_A$ of k-functors with the forgetful functor $\mathbf{Alg}_B \to \mathbf{Alg}_A$ i.e. we have a map

$$\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(A) \ni \sigma \mapsto \sigma_B \in \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(B)$$

and these according to Fact 6.2 make $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})$ a k-functor. The k-functor $\mathcal{M}\mathrm{or}_C(\mathfrak{X}, \mathfrak{Y})$ is called *a hom k-functor of $\mathfrak{X}$ and $\mathfrak{Y}$.*

We define a k-functor $\mathbf{1}$ that assigns to every k-algebra a set with one element. For every k-algebra A the restriction $\mathbf{1}_A$ is a terminal object in the category of A-functors.

Fact 6.4. *Let $\mathfrak{X}$ be a k-functor. Suppose A is a k-algebra and $x \in \mathfrak{X}(A)$. Then x determines a morphism $\mathbf{1}_A \to \mathfrak{X}_A$ that for every A-algebra B with structural morphism $f : A \to B$ sends a unique element of $\mathbf{1}_A(B)$ to $\mathfrak{X}(f)(x) \in \mathfrak{X}_A(B)$. This gives rise to a bijection*

$$\mathfrak{X}(A) \cong \mathrm{Mor}_A(\mathbf{1}_A, \mathfrak{X}_A)$$

Proof. We left to the reader as an exercise. □

Definition 6.5. Let $\mathfrak{X}$ be a k-functor and A be a k-algebra. The set $\mathfrak{X}(A)$ is called *the set of A-points of $\mathfrak{X}$.*

Now let $\mathfrak{X}$, $\mathfrak{Y}$ be k-functors such that for every k-algebra A the class $\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set. Suppose next that $\mathfrak{U}$ is a k-functor and $\sigma : \mathfrak{U} \times \mathfrak{X} \to \mathfrak{Y}$ is a morphism of k-functors. Fix $x \in \mathfrak{U}(A)$. We denote by $i_x : \mathbf{1}_A \to \mathfrak{U}_A$ the morphism of A-functors corresponding to x by means of Fact 6.4. Since $\mathbf{1}_A$ is terminal A-functor, a morphism $\sigma_A \cdot (1_{\mathfrak{X}_A} \times i_x)$ is isomorphic to a morphism $\tau_x : \mathfrak{X}_A \to \mathfrak{Y}_A$ of A-functors. Next $x \mapsto \tau_x$ gives rise to a morphism $\tau : \mathfrak{U} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})$ of k-functors and hence we have a map of classes

$$\mathrm{Mor}_k(\mathfrak{U} \times \mathfrak{X}, \mathfrak{Y}) \ni \sigma \mapsto \tau \in \mathrm{Mor}_k(\mathfrak{U}, \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y}))$$

Now we have the following result [Mon19, Theorem 5.3].

Theorem 6.6. *Let $\mathfrak{X}$, $\mathfrak{Y}$ be k-functors. Assume that for every k-algebra A the class $\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set. Then the map*

$$\mathrm{Mor}_k(\mathfrak{U} \times \mathfrak{X}, \mathfrak{Y}) \to \mathrm{Mor}_k(\mathfrak{U}, \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y}))$$

described above is a bijection natural in $\mathfrak{U}$.

In the remaining part of this section we introduce some notions of geometric flavour. For every k-algebra A we denote by k_A the k-functor given by

$$k_A(B) = \mathrm{Hom}_k(A, B),\ k_A(g) = \mathrm{Hom}_k(1_A, f)$$

for every k-algebra B and for every morphism $g : B \to C$ of k-algebras. Note that if $f : A \to B$ is a morphism of k-algebras, then there exists a morphism of k-functors $k_f : k_B \to k_A$ given by formula

$$k_f(C) = \mathrm{Hom}_k(f, 1_C)$$

where C is a k-algebra. These are general definitions that make sense in any category of co-presheaves c.f. [Mon19, section 7].

Definition 6.7. Let $\mathfrak{X}$ be a k-functor. We say that $\mathfrak{X}$ is *corepresentable* if $\mathfrak{X}$ is isomorphic to k_A for some k-algebra A.

Definition 6.8. Let $\sigma : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of k-functors. Fix a k-algebra A and a morphism $\tau : k_A \to \mathfrak{Y}$ of k-functors. Consider a cartesian square

$$\begin{tikzcd} \mathfrak{U} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, "\sigma"] \\ k_A \arrow[r, "\tau"'] & \mathfrak{Y} \end{tikzcd}$$

Suppose now that $\mathfrak{U}$ is corepresentable for all choices of k-algebra A and morphism τ of k-functors. Then we say that σ is *a corepresentable morphism of k-functors*.

Definition 6.9. Let $\sigma : \mathfrak{X} \to \mathfrak{Y}$ be a corepresentable morphism of k-functors. Fix a k-algebra A and a morphism $\tau : k_A \to \mathfrak{Y}$ of k-functors. Then there exists a cartesian square of the form

$$\begin{tikzcd} k_B \arrow[r] \arrow[d, "k_f"'] & \mathfrak{X} \arrow[d, "\sigma"] \\ k_A \arrow[r, "\tau"'] & \mathfrak{Y} \end{tikzcd}$$

where $f : A \to B$ is a morphism of k-algebras. Suppose now that $\operatorname{Spec} f : \operatorname{Spec} B \to \operatorname{Spec} A$ is an open (closed) immersion of affine schemes for all choices of k-algebra A and morphism τ of k-functors. Then we say that σ is *an open (closed) immersion of k-functors*.

Fact 6.10. *The class of open (closed) immersions of k-functors is closed under base change.*

Proof. This follows since open (closed) immersions of affine k-schemes are closed under base change. □

7. k-FUNCTORS OF MONOIDS AND THEIR LINEAR REPRESENTATIONS

In the sequel we assume that the reader is familiar with notions of a monoid, group etc. in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 2-5].

Definition 7.1. *A monoid (group) k-functor* is a monoid (group) object in the category of k-functors.

Next we introduce an important notion of a linear representation of a monoid k-functor. For this we define k-functors associated with modules over k and discuss their properties.

Example 7.2. Let V be a k-module. We define a k-functor V_{a}. We set

$$V_{\mathrm{a}}(A) = A \otimes_k V,\ V_{\mathrm{a}}(f) = f \otimes_k 1_V$$

for every k-algebra A and every morphism $f : A \to B$ of k-algebras. Moreover, V_{a} admits a structure of a commutative group k-functor. Indeed, $V_{\mathrm{a}}(A)$ is a commutative group with respect to addition induced by its structure of A-module and $V_{\mathrm{a}}(f) : V_{\mathrm{a}}(A) \to V_{\mathrm{a}}(B)$ preserves the addition.

Suppose now that V, W are k-modules and $\sigma : (V_{\mathrm{a}})_A \to (W_{\mathrm{a}})_A$ is a morphism of A-functors. Then for every A-algebra B we denote by $\sigma^B : B \otimes_k V \to B \otimes_k W$ the component of σ for B.

Definition 7.3. Let V, W be k-modules and let A be a k-algebra. A morphism $\sigma : (V_{\mathrm{a}})_A \to (W_{\mathrm{a}})_A$ of A-functors is *linear* if for every A-algebra B the component $\sigma^B : B \otimes_k V \to B \otimes_k W$ is a morphism of B-modules.

Next Fact characterizes linear morphism.

Fact 7.4. *Let V, W be k-modules and let A be a k-algebra. Suppose that $\phi : A \otimes_k V \to A \otimes_k W$ is a morphism of A-modules. Then there exists a unique linear morphism $\sigma : (V_{\mathrm{a}})_A \to (W_{\mathrm{a}})_A$ of A-functors such that $\sigma^A = \phi$.*

Proof. Note that if such σ exists, then by requirement $\sigma^A = \phi$ for every morphism $f : A \to B$ of k-algebras the following diagram

$$\begin{tikzcd} A \otimes_k V \arrow[r, "\phi"] \arrow[d, "f \otimes_k 1_V"'] & A \otimes_k W \arrow[d, "f \otimes_k 1_W"] \\ B \otimes_k V \arrow[r, "\sigma^B"'] & B \otimes_k W \end{tikzcd}$$

must commute. We make this into a definition of a morphism σ^B of B-modules. It is a matter of linear algebra that this diagram uniquely determines σ^B and also that $\sigma^A = \phi$. It remains to verify that $\sigma = \{\sigma^B\}_{B \in \mathbf{Alg}_A}$ defined in such a way is a morphism of A-functors. For this suppose that $f : A \to B$ and $g : B \to C$ are morphisms of k-algebras. Then we have

$$\sigma_C \cdot (g \otimes_k 1_V) \cdot (f \otimes_k 1_V) = \sigma_C \cdot ((g \cdot f) \otimes_k 1_V) = ((g \cdot f) \otimes_k 1_W) \cdot \phi =$$
$$= (g \otimes_k 1_W) \cdot (f \otimes_k 1_V) \cdot \phi = (g \otimes_k 1_W) \cdot \sigma_B \cdot (f \otimes_k 1_V)$$

and hence $\sigma_C \cdot (g \otimes_k 1_V) = (g \otimes_k 1_W) \cdot \sigma_B$. Thus σ is a linear morphism of A-functors. $\square$

We restate Fact 7.4 in the form of the following result.

Corollary 7.5. *Let V, W be k-modules and A be a k-algebra. Consider the map*

$$\mathrm{Hom}_A(A \otimes_k V, A \otimes_k W) \longrightarrow \mathrm{Mor}_A\left((V_{\mathrm{a}})_A, (W_{\mathrm{a}})_A\right)$$

that sends morphism ϕ to a unique linear morphism $\sigma : (V_{\mathrm{a}})_A \to (W_{\mathrm{a}})_A$ of A-functors such that $\sigma^A = \phi$. Then this map is injective and its image consists of all linear morphisms of A-functors.

Example 7.6. Let V be a k-module. We define a k-functor $\mathcal{L}_V$. We set

$$\mathcal{L}_V(A) = \mathrm{Hom}_A(A \otimes_k V, A \otimes_k V)$$

for every k-algebra A. Next for every morphism $f : A \to B$ of k-algebras and a morphism $\phi : A \otimes_k V \to A \otimes_k V$ of A-modules we define $\mathcal{L}_V(f)(\phi)$ as a unique morphism of B-modules such that the diagram

$$\begin{tikzcd} A \otimes_k V \arrow[r, "\phi"] \arrow[d, "f \otimes_k 1_V"'] & A \otimes_k W \arrow[d, "f \otimes_k 1_W"] \\ B \otimes_k V \arrow[r, "\mathcal{L}_V(\phi)"'] & B \otimes_k W \end{tikzcd}$$

is commutative. Note also that $\mathcal{L}_v(A)$ is a monoid k-functor with respect to the usual composition of morphism of A-modules and $\mathcal{L}_V(f) : \mathcal{L}_V(A) \to \mathcal{L}_V(B)$ preserves this composition.

Remark 7.7. Corollary 7.5 implies that there are injective maps that make the square

$$\begin{tikzcd} \mathcal{L}_V(A) \arrow[r, hook] \arrow[d, "\mathcal{L}_V(f)"'] & \mathrm{Mor}_A\left((V_{\mathrm{a}})_A, (V_{\mathrm{a}})_A\right) \arrow[d, "\sigma \mapsto \sigma_B"] \\ \mathcal{L}_V(B) \arrow[r, hook] & \mathrm{Mor}_B\left((V_{\mathrm{a}})_B, (V_{\mathrm{a}})_B\right) \end{tikzcd}$$

commutative for every morphism $f : A \to B$ of k-algebras. Also Corollary 7.5 shows that for every k-algebra A this identifies $\mathcal{L}_V(A)$ with a subset of the class $\mathrm{Mor}_A\left((V_{\mathrm{a}})_A, (V_{\mathrm{a}})\right)$ consisting of all linear morphism of A-functor.

The discussion below is partially an application of the main result in [Mon19, section 6] (Remark 7.7 shows that $\mathcal{L}_V$ is a subcopresheaf of internal endomorphisms of V_{a} and hence the machinery developed in the citation above can be applied), but for the reader's convenience we decide to include all essential details even if this requires repetition.
Let $\mathfrak{X}$ be a monoid k-functor and let be V be a k-module. Suppose that $\alpha : \mathfrak{X} \times V_{\mathrm{a}} \to V_{\mathrm{a}}$ is an action of $\mathfrak{X}$ on V_{a}. Assume that A is a k-algebra and $x \in \mathfrak{X}(A)$. We denote by $i_x : \mathbf{1}_A \to \mathfrak{X}_A$ the morphism of A-functors corresponding to x by means of Fact 6.4. Since $\mathbf{1}_A$ is terminal A-functor, a morphism $\alpha_A \cdot \left(i_x \times 1_{(V_{\mathrm{a}})_A}\right)$ is isomorphic to a morphism $\alpha_x : (V_{\mathrm{a}})_A \to (V_{\mathrm{a}})_A$ of A-functors. Suppose now that for any k-algebra A and point $x \in \mathfrak{X}(A)$ morphism α_x is linear. Then we define a morphism $\rho : \mathfrak{X} \to \mathcal{L}_V$ of k-functors by formula $\rho(x) = \alpha_x^A$. We first check that ρ really is a morphism of k-functors. For this fix morphism $f : A \to B$ of k-algebras and $x \in \mathfrak{X}(A)$. Then $\alpha_{\mathfrak{X}(f)(x)}$ is a morphism of B-functors isomorphic with $\alpha_B \cdot \left(i_{\mathfrak{X}(f)(x)} \times 1_{(V_{\mathrm{a}})_B}\right)$ and since

$$\alpha_B \cdot \left(i_{\mathfrak{X}(f)(x)} \times 1_{(V_{\mathrm{a}})_B}\right) = \alpha_B \cdot \left(i_x \times 1_{(V_{\mathrm{a}})_A}\right)_B = \left(\alpha_A \cdot \left(i_x \times 1_{(V_{\mathrm{a}})_A}\right)\right)_B$$

we derive that $\alpha_{\mathfrak{X}(f)(x)} = (\alpha_x)_B$. This implies that

$$\rho\left(\mathfrak{X}(f)(x)\right) = \alpha^B_{\mathfrak{X}(f)(x)} = \left((\alpha_x)_B\right)^B = \alpha_x^B = \mathcal{L}_V(f)(\rho(x))$$

and thus ρ is a morphism of k-functors. Now we show that ρ is a morphism of monoids. For this pick k-algebra A and $x, y \in \mathfrak{X}(A)$. Since α is an action, we deduce that $\alpha_{x\cdot y} = \alpha_x \cdot \alpha_y$ and hence also

$$\rho(x \cdot y) = \alpha^A_{x\cdot y} = \alpha^A_x \cdot \alpha^A_y = \rho(x) \cdot \rho(y)$$

Therefore, ρ is a morphism of monoid k-functors.

Theorem 7.8. *Let $\mathfrak{X}$ be a monoid k-functor and let V be a k-module. Consider the following classes.*

(1) *The class of actions $\alpha : \mathfrak{X} \times V_{\mathrm{a}} \to V_{\mathrm{a}}$ of $\mathfrak{X}$ such that for any k-algebra A and point $x \in \mathfrak{X}(A)$ morphism α_x is linear.*

(2) *The class of morphisms $\rho : \mathfrak{X} \to \mathcal{L}_V$ of monoid k-functors.*

Let α be an element of **(1)** *and $\rho : \mathfrak{X} \to \mathcal{L}_V$ be the element of* **(2)** *such that $\rho(x) = \alpha_x^A$ for any k-algebra A and $x \in \mathfrak{X}(A)$. Then the correspondence $\alpha \mapsto \rho$ is a bijection between these classes.*

Proof. We may refer to [Mon19, Theorem 6.3], but for self-containment of the presentation let us give a direct proof of this important result. □

8. Transporters

Definition 8.1. Let X be a k-scheme. Suppose that there exists an open affine cover $X = \bigcup_{i\in I} X_i$ such that k-algebra $\Gamma(X_i, \mathcal{O}_{X_i})$ is free as a k-module. Then we say that X is *a locally free k-scheme*.

Next theorem is the main result of this section.

Theorem 8.2. *Let $j : \mathfrak{Y}' \to \mathfrak{Y}$ be a closed immersion of k-functors and X be a locally free k-scheme. Suppose that classes $\mathrm{Mor}_A\left(X_A, \mathfrak{Y}_A\right)$ are sets for every k-algebra A. Then classes $\mathrm{Mor}_A\left(X_A, \mathfrak{Y}'_A\right)$ are sets for every k-algebra A and the morphism*

$$\mathcal{M}\mathrm{or}_k\left(1_X, j\right) : \mathcal{M}\mathrm{or}_k\left(X, \mathfrak{Y}'\right) \to \mathcal{M}\mathrm{or}_k\left(X, \mathfrak{Y}\right)$$

is a closed immersion of k-functors.

It is useful to isolate crucial steps in the argument. For this we proceed by proving some lemmas.

Lemma 8.2.1. *Suppose that A is a commutative ring. Let $j : \mathfrak{Y}' \to \mathfrak{Y}$ be a closed immersion of A-functors and X be an affine A-scheme such that $\Gamma(X, \mathcal{O}_X)$ is a free A-module. Assume that $\tau : X \to \mathfrak{Y}$ is a morphism of A-functors. Then there exists an ideal $\mathfrak{a} \subseteq A$ such that for every A-algebra B the restriction τ_B factors through j_B if and only if the structure morphism $f : A \to B$ of B satisfies $\mathfrak{a} \subseteq \ker(f)$.*

Proof of the lemma. Consider a cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "j'"'] & \mathfrak{Y}' \arrow[d, "j"] \\ X \arrow[r, "\tau"'] & \mathfrak{Y} \end{tikzcd}$$

Since j is a closed immersion of A-functors, we derive by Fact 6.10 that j' is a closed immersion. By assumption X is affine. Hence X' is a functor of points of some A-scheme and $j' : X' \to X$ is (induced by) a closed immersion of A-schemes. Next let B be an A-algebra with the structure morphism $f : A \to B$. Then τ_B factors through j_B if and only if the projection $\operatorname{Spec} B \times_{\operatorname{Spec} A} X \to X$ induced by f factors through X'. Let $A[X]$ be the A-algebra of global regular functions on X and let $\mathfrak{J}$ be an ideal in $A[X]$ such that $A[X]/\mathfrak{J} = A[X']$ is the A-algebra of global regular functions of X'. With this notation we derive that the projection $\operatorname{Spec} B \times_{\operatorname{Spec} A} X \to X$ induced by f factors through X' if and only if the morphism $A[X] \to B \otimes_A A[X]$ induced by f sends every element of $\mathfrak{J}$ to zero. Since $A[X]$ is a free A-module, we write $A[X] = A^{\oplus I}$ for some index set I. Then the morphism $A[X] \to B \otimes_A A[X]$ induced by f is just $f^{\oplus I} : A^{\oplus I} \to B^{\oplus I}$. We have $f^{\oplus I}(\mathfrak{J}) = 0$ if and only if $\left(pr_i^B \cdot f^{\oplus I}\right)(\mathfrak{J}) =$ for every $i \in I$, where $pr_i^B : B^{\oplus I} \to B$ is the projection on i-th component. Pick $i \in I$ and consider the commutative diagram

$$\begin{tikzcd} A^{\oplus I} \arrow[r, "f^{\oplus I}"] \arrow[d, "pr_i^A"'] & B^{\oplus I} \arrow[d, "pr_i^B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

In the diagram pr_i^A is the projection on i-th component. Diagram implies that $\left(pr_i^B \cdot f^{\oplus I}\right)(\mathfrak{J}) =$ for every $i \in I$ if and only if $\left(f \cdot pr_i^A\right)(\mathfrak{J}) = 0$ for every $i \in I$. This is equivalent with the condition that $f(\mathfrak{a}) = 0$ for ideal $\mathfrak{a}$ in A generated by $\sum_{i \in I} pr_i^A(\mathfrak{J})$. Thus the lemma is proved. □

Lemma 8.2.2. *Suppose that A is a commutative ring. Let $j : \mathfrak{Y}' \to \mathfrak{Y}$ be a closed immersion of A-functors and X be an A-scheme with open cover*

$$X = \bigcup_{i \in I} X_i$$

Assume that $\tau : X \to \mathfrak{Y}$ is a morphism of A-functors. Fix an A-algebra B. Then τ_B factors through j_B if and only if $\left(\tau_{|X_i}\right)_B$ factors through j_B for every $i \in I$.

Proof of the lemma. If τ_B factors through j_B, then also $\left(\tau_{|X_i}\right)_B$ factors through j_B for every $i \in I$. It suffices to prove the converse. So suppose that $\left(\tau_{|X_i}\right)_B$ factors through j_B for every $i \in I$. Since j is a closed immersion of A-functors and X is an A-scheme, there exists a cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "j'"'] & \mathfrak{Y}' \arrow[d, "j"] \\ X \arrow[r, "\tau"'] & \mathfrak{Y} \end{tikzcd}$$

where $j' : X' \to X$ is (induced by) a closed immersion of A-schemes (this follows from Fact 6.10 and Fact 9.2). For each $i \in I$ let $j'_i : j'^{-1}(X_i) \to X_i$ be the restriction of j'. We have the induced cartesian square

$$\begin{tikzcd} j'^{-1}(X_i) \arrow[r] \arrow[d, "j'_i"'] & \mathfrak{Y}' \arrow[d, "j"] \\ X_i \arrow[r, "\tau_{|X_i}"'] & \mathfrak{Y} \end{tikzcd}$$

Now $\left(\tau_{|X_i}\right)_B$ factors through j_B. Together with Fact 9.2 this shows that $\left(j'_i\right)_B$ is an isomorphism of B-schemes. This holds for every $i \in I$. Hence j'_B is an isomorphism of B-schemes (again by application of Fact 9.2). Therefore, τ_B factors through j_B. □

Proof of the theorem. Let A be a k-algebra. The restriction functor $(-)_{|\mathbf{Alg}_A} = (-)_A$ preserves all closed immersions. Thus j_A is a closed immersion of A-functors and hence we derive that $j_A : \mathfrak{Y}'_A \to \mathfrak{Y}_A$ is a monomorphism of A-functors. Thus we have an injective map of classes

$$\mathrm{Mor}_A\left(1_{X_A}, j_A\right) : \mathrm{Mor}_A\left(X_A, \mathfrak{Y}'_A\right) \hookrightarrow \mathrm{Mor}_A\left(X_A, \mathfrak{Y}_A\right)$$

Hence if $\mathrm{Mor}_A\left(X_A, \mathfrak{Y}_A\right)$ is a set, then $\mathrm{Mor}_A\left(X_A, \mathfrak{Y}'_A\right)$ is a set. All these facts imply that both internal homs

$$\mathcal{M}\mathrm{or}_k\left(X, \mathfrak{Y}'\right), \mathcal{M}\mathrm{or}_k\left(X, \mathfrak{Y}\right)$$

exist and morphism $\mathcal{M}\mathrm{or}_k(1_X, j)$ of k-functors is a monomorphism. Our task is to prove that it is a closed immersion. For this consider a k-algebra A and a morphism $\sigma : k_A \to \mathcal{M}\mathrm{or}_k\left(X, \mathfrak{Y}\right)$ of k-functors that sends 1_A to some morphism $\tau : X_A \to \mathfrak{Y}_A$ of A-functors. Consider a cartesian square

$$\begin{tikzcd} \mathfrak{U} \arrow[r] \arrow[d] & \mathcal{M}\mathrm{or}_k\left(X, \mathfrak{Y}'\right) \arrow[d, "\mathcal{M}\mathrm{or}_k\left(1_X, j\right)"] \\ k_A \arrow[r, "\sigma"'] & \mathcal{M}\mathrm{or}_k\left(X, \mathfrak{Y}\right) \end{tikzcd}$$

Since $\mathcal{M}\mathrm{or}_k\left(1_X, j\right)$ is a monomorphism, we may consider $\mathfrak{U}$ as a k-subfunctor of k_A. For every k-algebra B subset $\mathfrak{U}(B) \subseteq \mathrm{Mor}_k(A, B) = k_A(B)$ consists of A-algebras B with structure morphisms $f : A \to B$ such that τ_B factors through $j_B : \mathfrak{Y}'_B \to \mathfrak{Y}_B$. Since X is a locally free k-scheme, we deduce that X_A is (a functor of points of) a locally free A-scheme. Pick an open affine cover $X_A = \bigcup_{i \in I} X_i$ such that $\Gamma(X_i, \mathcal{O}_X)$ is a free A-module. Now Lemma 8.2.2 implies that τ_B factors through j_B if and only if $\left(\tau_{|X_i}\right)_B$ factors through j_B for every $i \in I$. Next by Lemma 8.2.1 we deduce that $\left(\tau_{|X_i}\right)_B$ factors through j_B for given $i \in I$ if and only if $f(\mathfrak{a}_i) = 0$ for some ideal $\mathfrak{a}_i \subseteq A$ independent of f. Thus $\mathfrak{U}$ consists of all morphisms $f : A \to B$ of k-algebras such that $f(\mathfrak{a}) = 0$ where $\mathfrak{a} = \sum_{i \in I} \mathfrak{a}_i$. Therefore, $\mathfrak{U} \hookrightarrow k_A$ is isomorphic with $k_{A/\mathfrak{a}} \hookrightarrow k_A$ and hence $\mathcal{M}\mathrm{or}_k(1_X, j)$ is a closed immersion of k-functors. □

The Theorem 8.2 is a simple yet powerful result. Before giving any interesting applications we state its immediate consequence.

9. Zariski local k-functors and k-schemes

Definition 9.1. Let X be a k-scheme. We define a k-functor out of X. First let A be a k-algebra. The set

$$X(A) = \{\text{morphisms } \operatorname{Spec} A \to X \text{ of } k\text{-schemes}\}$$

is called *the set of A-points of X*. Also if $f : A \to B$ is a morphism of k-algebras, then $X(f)$ is defined as the precomposition with $\operatorname{Spec} f$. This makes X into a k-functor called *the k-functor of points of X*.

For every k-scheme X we denote its k-functor of points just by X. Consider a functor defined on the category of k-schemes with values in the category of k-functors that sends a k-scheme X to its functor of points and also sends a morphism $f : X \to Y$ of k-schemes to a map that is given by composition with f.

Fact 9.2. *The functor described above is full and faithful.*

Proof. This follows from the fact that morphisms of k-schemes are defined locally with respect to Zariski topology. □

In particular, we may consider the category $\mathbf{Sch}_k$ as a full subcategory of the category of k-functors.

Definition 9.3. Let $\{f_i : X_i \to X\}_{i\in I}$ be a family of morphisms of k-schemes. We say that $\{f_i\}_{i\in I}$ is *a Zariski covering of X* if the following conditions are satisfied.

(1) For every $i \in I$ morphism f_i is an open immersion of schemes.

(2) Morphism $\coprod_{i\in I} X_i \to X$ induced by $\{f_i\}_{i\in I}$ is surjective.

Definition 9.4. Let $\mathfrak{X}$ be a presheaf on the category of k-schemes. Suppose that for every k-scheme X and for every Zariski covering $\{f_i : X_i \to X\}$ of X the diagram

$$\mathfrak{X}(X) \xrightarrow{\langle \mathfrak{X}(f_i)\rangle_{i\in I}} \prod_{i\in I} \mathfrak{X}(X_i) \overset{\langle \mathfrak{X}(f'_{ij})\cdot pr_i\rangle_{(i,j)}}{\underset{\langle \mathfrak{X}(f''_{ij})\cdot pr_j\rangle_{(i,j)}}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathfrak{X}(X_i \times_X X_j)$$

is a kernel of a pair of arrows, where for every $(i,j) \in I \times I$ morphisms f'_{ij} and f'_{ji} form a cartesian square

$$\begin{tikzcd} X_i \times_X X_j \arrow[r, "f''_{ij}"] \arrow[d, "f'_{ij}"'] & X_j \arrow[d, "f_j"] \\ X_i \arrow[r, "f_i"'] & X \end{tikzcd}$$

Then we call $\mathfrak{X}$ *a Zariski sheaf on* $\mathbf{Sch}_k$.

Now we repeat this definitions for k-algebras and k-functors.

Definition 9.5. Let $\{f_i : A \to A_i\}_{i\in I}$ be a family of morphisms of k-algebras. We say that $\{f_i\}_{i\in I}$ is *a Zariski covering of A* if the following conditions are satisfied.

(1) For every $i \in I$ morphism $\operatorname{Spec} f_i$ is an open immersion of schemes.

(2) Morphism $\coprod_{i\in I} \operatorname{Spec} A_i \to \operatorname{Spec} A$ induced by $\{\operatorname{Spec} f_i\}_{i\in I}$ is surjective.

Definition 9.6. Let $\mathfrak{X}$ be a k-functor. Suppose that for every k-algebra A and for every Zariski covering $\{f_i : A \to A_i\}$ of A the diagram

$$\mathfrak{X}(A) \xrightarrow{\langle \mathfrak{X}(f_i)\rangle_{i\in I}} \prod_{i\in I}\mathfrak{X}(A_i) \overset{\langle \mathfrak{X}(f'_{ij})\cdot pr_i\rangle_{(i,j)}}{\underset{\langle \mathfrak{X}(f''_{ij})\cdot pr_j\rangle_{(i,j)}}{\rightrightarrows}} \prod_{(i,j)\in I\times I}\mathfrak{X}(A_i\otimes_A A_j)$$

is a kernel of a pair of arrows, where for every $(i,j)\in I\times I$ morphisms f'_{ij} and f'_{ji} form a cocartesian square

$$\begin{tikzcd} A \arrow[r, "f_j"] \arrow[d, "f_i"'] & A_j \arrow[d, "f'_{ji}"] \\ A_i \arrow[r, "f'_{ij}"'] & A_i\otimes_A A_j \end{tikzcd}$$

Then we call $\mathfrak{X}$ *a Zariski local k-functor*.

Theorem 9.7. *Let*

$$\widehat{\mathbf{Sch}_k} \longrightarrow \textit{the category of k-functors}$$

be the restriction of presheaves on $\mathbf{Sch}_k$ *to copresheaves on* $\mathbf{Alg}_k$ *(k-functors) induced by the contravariant functor* $\operatorname{Spec}:\mathbf{Alg}_k\to\mathbf{Sch}_k$*. Then it induces an equivalence of categories between Zariski sheaves on* $\mathbf{Sch}_k$ *and Zariski local k-functors.*

Let $\mathfrak{X}$ be a k-functor. Then using the fact that Spec induces an equivalence between $\mathbf{Alg}_k$ and dual category of $\mathbf{Aff}_k$, we may consider $\mathfrak{X}$ as a presheaf on $\mathbf{Aff}_k$. Suppose that X is a k-scheme. Let $\mathcal{U}_X$ be the set of all open affine subsets of X. For each element U in $\mathcal{U}_X$ we denote by $f_U:U\to X$ the corresponding open immersion. Now the collection $\{f_U:U\to X\}_{U\in\mathcal{U}_X}$ is a Zariski cover of X. Next let $\mathcal{U}\subseteq\mathcal{U}_X$ be any subset that is also a Zariski cover of X.

Lemma 9.7.1. *Let* $\mathfrak{X}$ *be a Zariski sheaf on* $\mathbf{Sch}_k$*. Fix a k-scheme X and let*

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