LINEARLY REDUCTIVE GROUPS

Theorem 0.1 (Rigidity). *Let* $\mathbf{G}$ *be an anti-affine algebraic group, let* Z *be a separated* k*-scheme with* $\mathbf{G}$*-action and let* $f : \mathbf{G} \times_k Y \to Z$ *be a* $\mathbf{G}$*-equivariant morphism of* k*-schemes. Suppose that* e *is the unit of* $\mathbf{G}$ *and* y *is a* k*-point of* Y*. Denote the* k*-point* $f(e, y)$ *of* Z *by* z *and suppose that the restriction* $f_{|\mathbf{G} \times_k \operatorname{Spec} k(y)}$ *factors through the inclusion* $\operatorname{Spec} k(z) \hookrightarrow Z$*. Then* f *equals*

$$\mathbf{G} \times_k Y \xrightarrow{\mathrm{pr}_Y} Y \xrightarrow{\cong} \operatorname{Spec} k(e) \times_k Y \xrightarrow{f_{|\operatorname{Spec} k(e) \times_k Y}} Z$$

Proof. Let $\mathcal{J} \subseteq \mathcal{O}_Z$ be the quasi-coherent ideal determining the closed immersion $\operatorname{Spec} k(z) \hookrightarrow Z$ and let $\mathcal{I}$ be the quasi-coherent ideal determining $\mathbf{G} \times_k \operatorname{Spec} k(y)$ as a closed subscheme of $\mathbf{G} \times_k Y$. Then $f^{-1}\mathcal{J} \cdot \mathcal{O}_Z \subseteq \mathcal{I}$ and hence $f^{-1}\mathcal{J}^n \cdot \mathcal{O}_Z \subseteq \mathcal{I}^n$ for every positive integer n. This implies that for each positive integer n the morphism $f_{|\mathbf{G} \times_k \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n}$ factors through a morphism

$$h_n : \mathbf{G} \times_k \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n \to \operatorname{Spec} \mathcal{O}_{Z,z}/\mathfrak{m}_z^n$$

Since h_n is a morphism into an affine k-scheme, it is uniquely determined by k-algebra morphism induced on global sections

$$\Gamma(h_n^{\#}, \operatorname{Spec} \mathcal{O}_{Z,z}/\mathfrak{m}_z^n) : \mathcal{O}_{Z,z}/\mathfrak{m}_z^n \to \Gamma\left(\mathbf{G} \times_k \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n, \mathcal{O}_{\mathbf{G} \times_k \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n}\right)$$

By Theorem **??** we have canonical identification

$$\Gamma\left(\mathbf{G} \times_k \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n, \mathcal{O}_{\mathbf{G} \times_k \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n}\right) = \Gamma\left(\mathbf{G}, \mathcal{O}_{\mathbf{G}}\right) \otimes_k \mathcal{O}_{Y,y}/\mathfrak{m}_y^n = \mathcal{O}_{Y,y}/\mathfrak{m}_y^n$$

This implies that $\Gamma(h_n^{\#}, \operatorname{Spec} \mathcal{O}_{Z,z}/\mathfrak{m}_z^n)$ factors through $\mathcal{O}_{Y,y}/\mathfrak{m}_y^n$ and hence $f_{|\mathbf{G} \times_k \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n}$ equals

$$\mathbf{G} \times_k \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n \xrightarrow{\mathrm{pr}_n} \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n \xrightarrow{\cong} \operatorname{Spec} k(e) \times_k \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n \xrightarrow{f_{|\operatorname{Spec} k(e) \times_k \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n}} Z$$

for every positive integer n. Now consider a closed subscheme $i : E \hookrightarrow \mathbf{G} \times_k Y$ that is a kernel of a pair consisting of f and the morphism

$$\mathbf{G} \times_k Y \xrightarrow{\mathrm{pr}_Y} Y \xrightarrow{\cong} \operatorname{Spec} k(e) \times_k Y \xrightarrow{f_{|\operatorname{Spec} k(e) \times_k Y}} Z$$

Then by what we proved above we deduce that $\mathbf{G} \times_k \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n$ is a subscheme of E for every positive integer n. This implies that $\mathbf{G} \times_k \operatorname{Spec} \mathcal{O}_{Y,y}$ is a subscheme of E.

□

References

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