ALGEBRAIC GROUP SCHEMES OVER FIELD

1. Introduction

In these notes we group schemes over fields. For background we refer to [Mon19] and [Mon20]. Throughout these notes k is a fixed field.

2. Simple criterion for separatedness

Proposition 2.1. *Let $\mathbf{G}$ be a group scheme over k and let $e_{\mathbf{G}} : \operatorname{Spec} k \to \mathbf{G}$ be its unit. Then the following are equivalent.*

- **(i)** *$e_{\mathbf{G}}$ is a closed immersion.*
- **(ii)** *$\mathbf{G}$ is separated.*

Proof. Suppose that **(i)** holds. Consider morphism $f : \mathbf{G} \times_k \mathbf{G} \to \mathbf{G}$ given on A-points by formula

$$f(g_1, g_2) = g_1 \cdot g_2^{-1}$$

where A is a k-algebra. Note that we have a cartesian square

$$\begin{tikzcd}
\mathbf{G} \arrow[r, "\pi"] \arrow[d, "\delta_{\mathbf{G}}"'] & \operatorname{Spec} k \arrow[d, "e_{\mathbf{G}}"] \\
\mathbf{G} \times_k \mathbf{G} \arrow[r, "f"'] & \mathbf{G}
\end{tikzcd}$$

where $\delta_{\mathbf{G}}$ is a diagonal of $\mathbf{G}$ and the top horizontal arrow is the structure morphism. Since base change of a closed immersion is a closed immersion, we derive that $\delta_{\mathbf{G}}$ is a closed immersion and hence $\mathbf{G}$ is separated. This is **(ii)**.
Suppose now that **(ii)** holds. Let $\pi : \mathbf{G} \to \operatorname{Spec} k$ be the structural morphism. Then $\pi \cdot e_{\mathbf{G}} = 1_{\mathbf{G}}$. Since π is a separated morphism, we derive that (by cancellation) $e_{\mathbf{G}}$ is closed immersion. This is **(i)**. □

Definition 2.2. Let $\mathbf{G}$ be a group scheme over k. If $\mathbf{G}$ is (locally) of finite type over k, then we say that $\mathbf{G}$ is *an (a locally) algebraic group over k*.

Corollary 2.3. *Let $\mathbf{G}$ be a locally algebraic group over k. Then $\mathbf{G}$ is separated.*

Proof. Consider the unit $e_{\mathbf{G}} : \operatorname{Spec} k \to \mathbf{G}$. Since $\mathbf{G}$ is locally of finite type, we derive that each k-point is closed in $\mathbf{G}$. Thus $e_{\mathbf{G}}$ is a closed immersion. By Proposition 2.1 we derive that $\mathbf{G}$ is separated. □

3. Abelian Varieties

We start this section with the following general result.

Theorem 3.1 (Rigidity). *Let $\pi : X \to Y$ be a proper morphism of schemes such that $\pi^{\#} : \mathcal{O}_Y \to \pi_* \mathcal{O}_X$ is an isomorphism of sheaves. Let $g : X \to Z$ be a morphism of schemes. Suppose that for some point y in Y there is a point z of Z such that $\pi^{-1}(y) \subseteq g^{-1}(z)$. Then there exist an affine neighborhood V of y and an affine neighborhood W of z such that $\pi^{-1}(V) \subseteq g^{-1}(W)$. Moreover, there exists a morphism $h : V \to W$ making the diagram*

$$
\begin{tikzcd}
\pi^{-1}(V) \arrow[r, "\text{res. of } g"] \arrow[d, "\text{restriction of } \pi"'] & W \\
V \arrow[ur, "h"'] &
\end{tikzcd}
$$

commutative, where horizontal arrow is the restriction of g.

Proof. Consider an affine open neighborhood of W of z. Since π is proper and $\pi^{-1}(y)) = g^{-1}(z)$, we derive that $\pi\big(X \setminus g^{-1}(W)\big)$ is a closed subset of Y that does not contain y. Pick an open affine neighborhood V of y in Y that does not intersect with $\pi\big(X \setminus g^{-1}(W)\big)$. Then $\pi^{-1}(V) \subseteq g^{-1}(W)$. Since $\pi^{\#}$ is an isomorphism we have the composition

$$
\mathcal{O}_Z(W) \xrightarrow{g^{\#}_W} \Gamma\big(g^{-1}(W), \mathcal{O}_X\big) \xrightarrow{(-)_{|\pi^{-1}(V)}} \Gamma\big(\pi^{-1}(V), \mathcal{O}_X\big) \xrightarrow{(\pi^{\#}_V)^{-1}} \mathcal{O}_Y(V)
$$

This composition induces a morphism of affine schemes $h : V \to W$. Since a morphism from a scheme to an affine scheme is determined by the morphism on global sections of structure sheaves, we derive that h makes the triangle in the statement commutative. $\square$

Now we can apply this result to study complete algebraic groups over k. For this we need the following definition.

Definition 3.2. Let $\mathbf{A}$ be a geometrically integral, complete algebraic group over k. Then we say that $\mathbf{A}$ is *an abelian variety over k*.

Now we prove the following interesting result.

Theorem 3.3. *Let $\mathbf{A}$ be an abelian variety over k, let $\mathbf{G}$ be a separated group scheme over k and let $f : \mathbf{A} \to \mathbf{G}$ be a morphism of schemes over k. Suppose that the diagram*

$$
\begin{tikzcd}
\mathbf{A} \arrow[rr, "f"] & & \mathbf{G} \\
& \operatorname{Spec} k \arrow[ul, "e_{\mathbf{A}}"] \arrow[ur, "e_{\mathbf{G}}"'] &
\end{tikzcd}
$$

is commutative. Then f is a morphism of groups schemes over k.

Proof. We define a morphism $g : \mathbf{A} \times_k \mathbf{A} \to \mathbf{G}$ given by

$$
(x_1, x_2) \mapsto f(x_1) \cdot f(x_2) \cdot f(x_1 \cdot x_2)^{-1}
$$

where A is a k-algebra and x_1, x_2 are A-points of $\mathbf{A}$. It suffices to show that g factors through $\operatorname{Spec} k(e_{\mathbf{G}})$. For this we may change base to an algebraic closure of k by faitfully flat descent. So we may assume that the field k is algebraically closed and $\mathbf{A}$ is connected. Then the projection onto second factor $\pi : \mathbf{A} \times_k \mathbf{A} \to \mathbf{A}$ is proper and $k = \Gamma(\mathbf{A}, \mathcal{O}_{\mathbf{A}})$ implies that $\pi^{\#}$ is an isomorphism of sheaves on $\mathbf{A}$. Moreover, note that $\pi^{-1}(e_{\mathbf{A}}) \subseteq g^{-1}(e_{\mathbf{G}})$. Indeed, this follows from the assumption that $f(e_{\mathbf{A}}) = e_{\mathbf{G}}$. By Theorem 3.1 we deduce that there exist an affine neighborhood V of $e_{\mathbf{A}}$, an affine neighborhood W of $e_{\mathbf{G}}$ and a morphism $h : \operatorname{Spec} k \to W$ such that $\pi^{-1}(V) \subseteq g^{-1}(W)$ and the diagram

$$\begin{tikzcd} \mathbf{A} \times_k V \arrow[r, "\text{res. of } g"] \arrow[d, "\text{projection}"'] & W \\ V \arrow[ur, "h"'] & \end{tikzcd}$$

is commutative. Hence for every k-point v of V we have the restriction $g_{|\mathbf{A} \times_k \operatorname{Spec} k(v)}$ factors through $\operatorname{Spec} k\,(h(v))$. Since $g(v, e_{\mathbf{A}}) = e_{\mathbf{G}}$, we derive that $h(v) = e_{\mathbf{G}}$ and thus $g_{|\mathbf{A} \times_k \operatorname{Spec} k(v)}$ factors through $\operatorname{Spec} k(e_{\mathbf{G}})$. This holds for any k-point of V. Therefore, $g_{|\mathbf{A} \times_k V}$ factors through $\operatorname{Spec} k(e_{\mathbf{G}})$. Consider the kernel $i : Z \hookrightarrow \mathbf{A} \times_k \mathbf{A}$ of a pair consisting of g and a morphism $\mathbf{A} \times_k \mathbf{A} \to \mathbf{G}$ that factorizes through $\operatorname{Spec} k(e_{\mathbf{G}})$. Since $\mathbf{G}$ is separated, we derive that i is a closed immersion. Moreover, i dominates $\mathbf{A} \times_k V$. Since $\mathbf{A} \times_k V$ is schematically dense open subset of $\mathbf{A} \times_k \mathbf{A}$ (because $\mathbf{A} \times_k \mathbf{A}$ is integral), we derive that i is an isomorphism and hence g factors through $\operatorname{Spec} k(e_{\mathbf{G}})$. □

Corollary 3.4. *Let $\mathbf{A}$ be an abelian variety over k. Then $\mathbf{A}$ is a commutative group scheme over k.*

Proof. Consider the morphism $f : \mathbf{A} \to \mathbf{A}$ given on A-points of $\mathbf{A}$ by

$$f(x) = x^{-1}$$

where A is a k-algebra. By Theorem 3.3 we derive that f is a morphism of group schemes over k. Hence $\mathbf{A}$ is a commutative group scheme. □

4. Representability of fixed points

Definition 4.1. Let $\mathfrak{G}$ be a monoid k-functor and let $\alpha : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X}$ be an action of $\mathfrak{G}$ on a k-functor. Then we define a k-subfunctor $\mathfrak{X}^{\mathfrak{G}}$ of $\mathfrak{X}$ by

$$\mathfrak{X}^{\mathfrak{G}}(A) = \left\{ x \in \mathfrak{X}(A) \,\middle|\, \text{for any } A\text{-algebra } f : A \to B \text{ and } g \in \mathfrak{G}(B) \text{ we have } \alpha\big(g, \mathfrak{X}(f)(x)\big) = \mathfrak{X}(f)(x) \right\}$$

for every k-algebra A. Then $\mathfrak{X}^{\mathfrak{G}}$ is called *the fixed point k-functor*.

Theorem 4.2. *Let $\mathbf{G}$ be a group scheme over k and let $a : \mathbf{G} \times_k X \to X$ be an action of $\mathbf{G}$ on a k-scheme X. Suppose that one of the following assertions hold.*

- **(i)** *X is separated.*
- **(ii)** *$\mathbf{G}$ is a geometrically connected, locally algebraic group.*

The following result is based on [Mon19, Theorem 6.2] and plays the fundamental role in the proof.

Lemma 4.2.1. *Let X, Y be k-schemes and let $a : Y \times_k X \to X$ be a morphism of k-schemes. Suppose that one of the following assertions hold.*

- **(1)** *X is separated.*
- **(2)** *For every open subscheme U of X we have $a\,(Y \times_k U) \subseteq U$*

Consider a k-functor given by formula

$$A \mapsto \left\{ f : \operatorname{Spec} A \to X \,\middle|\, a \cdot (1_Y \times_k f) = \mathrm{pr}_X \cdot (1_Y \times_k f) \right\}$$

where A is a k-algebra and $\mathrm{pr}_X : Y \times_k X \to X$ is the projection. Then this k-functor is representable by a closed subscheme of X.

Proof of the lemma. Assume first that X is separated. Consider a morphism $\langle a, \mathrm{pr}_X \rangle : Y \times_k X \to X \times_k X$. By [Mon20, Corollary 4.6] we deduce that $\mathfrak{P}_{\langle a, \mathrm{pr}_X \rangle}$ corresponds to a morphism $\sigma : \mathfrak{P}_X \to \mathcal{M}\mathrm{or}_k\left(\mathfrak{P}_Y, \mathfrak{P}_X \times \mathfrak{P}_X\right)$ of k-functors. Since X is separated, the diagonal $\delta_X : X \to X \times_k X$ is a closed immersion. This implies that $\mathfrak{P}_{\delta_X}$ is a closed immersion of k-functors. The fact that Y is locally free over k and [Mon19, Theorem 6.2] imply that

$$\mathcal{M}\mathrm{or}_k\left(1_{\mathfrak{P}_Y}, \mathfrak{P}_{\delta_X}\right) : \mathcal{M}\mathrm{or}_k\left(\mathfrak{P}_Y, \mathfrak{P}_X\right) \hookrightarrow \mathcal{M}\mathrm{or}_k\left(\mathfrak{P}_Y, \mathfrak{P}_X \times \mathfrak{P}_X\right)$$

is a closed immersion of k-functors. Consider now a cartesian square

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "j"] \arrow[d] & \mathcal{M}\mathrm{or}_k\left(\mathfrak{P}_Y, \mathfrak{P}_X\right) \arrow[d, "\mathcal{M}\mathrm{or}_k\left(1_{\mathfrak{P}_Y}, \mathfrak{P}_{\delta_X}\right)"] \\
\mathfrak{P}_X \arrow[r, "\sigma"'] & \mathcal{M}\mathrm{or}_k\left(\mathfrak{P}_Y, \mathfrak{P}_X \times \mathfrak{P}_X\right)
\end{tikzcd}$$

of k-functors. By base change $j : \mathfrak{X} \to \mathfrak{P}_X$ is a closed immersion of k-functors. Thus we derive that $\mathfrak{X}$ is representable by a closed subscheme of $\mathfrak{X}$. It suffices to observe that $\mathfrak{X}$ is precisely the k-functor described in the statement. This proves the statement under the assumption **(1)**.
Now suppose that $a\left(Y \times_k U\right) \subseteq U$ for every open subscheme U of X. For every open subscheme denote by $a_U : Y \times_k U \to U$ the restriction of a. Let $\mathcal{U}$ be an open affine cover of X. Then functors

$$\left\{ \mathbf{Alg}_k \ni A \mapsto \left\{ f : \operatorname{Spec} A \to U \,\middle|\, a \cdot (1_Y \times_k f) = \mathrm{pr}_X \cdot (1_Y \times_k f) \right\} \in \mathbf{Set} \right\}_{U \in \mathcal{U}}$$

form an open cover ([Mon19, Definition 4.5]) of the k-functor in the statemtent. Moreover, since each U in $\mathcal{U}$ is affine and hence separated, we derive by the first part of the proof that each k-functor in the family is representable. Now [Mon19, Theorem 4.6] imply that the functor in the statement is representable. This finishes the proof in case **(2)**. □

Lemma 4.2.2. *Let $f : \mathbf{H} \to \mathbf{G}$ be a morphism of locally algebraic groups over k. Suppose that the following assertions hold.*

(1) *The morphism*

$$\widehat{\mathcal{O}_{\mathbf{G}, e_{\mathbf{G}}}} \to \widehat{\mathcal{O}_{\mathbf{H}, e_{\mathbf{H}}}}$$

induced by $f^{\#}$ is an isomorphism.

(2) *f is a monomorphism of k-schemes.*

Then f is an open immersion.

Proof of the lemma. The assertion **(1)** implies that f is étale in $e_{\mathbf{H}}$. Let K be an algebraic closure of k. Consider the étale locus U of $f_k = 1_K \otimes_k f : \mathbf{H}_K \to \mathbf{G}_K$. Then U is an open subscheme of $\mathbf{H}_K$ containing the unit. Moreover, for every K-point h of $\mathbf{H}_K$ we have a commutative square

$$\begin{tikzcd} \mathbf{H}_K \arrow[r, "f_K"] \arrow[d, "h\cdot(-)"'] & \mathbf{G}_K \arrow[d, "f_K(h)\cdot(-)"] \\ \mathbf{H}_K \arrow[r, "f_K"'] & \mathbf{G}_K \end{tikzcd}$$

where $h\cdot(-)$ and $f_K(h)\cdot(-)$ are isomorphisms of K-schemes. This proves that $h\cdot U\subseteq U$. Hence U contains all K-rational points of $\mathbf{H}_K$. Therefore, the complement of U in $\mathbf{H}_K$ is empty. Hence $U=\mathbf{H}_K$. This shows that f_K is étale and by faithfully flat descent also f is étale. Since étale monomorphisms are open immersions, we derive that f is an open immersion. □

Proof of the theorem. If **(1)** holds, then the statement follows directly from Lemma 4.2.1. Suppose now that **(2)** holds that is $\mathbf{G}$ is an algebraic group. For each $n\in\mathbb{N}$ we define

$$\mathbf{G}_n=\operatorname{Spec}\mathcal{O}_{\mathbf{G},e_{\mathbf{G}}}/\mathfrak{m}_{e_{\mathbf{G}}}^{n+1}$$

where e is the unit of $\mathbf{G}$. Then $\mathbf{G}_n$ is the n-th infinitesimal neighborhood of e in $\mathbf{G}$. Denote by $p_n:\mathbf{G}_n\times_k X\to X$ the projection on the second factor. Let $a_n:\mathbf{G}_n\times_k X\to X$ be the morphism induced by a. Note that for every open subscheme U of X we have $a_n(\mathbf{G}_n\times_k U)\subseteq U$. By Lemma 4.2.1 it follows that the k-functor given by

$$\mathbf{Alg}_k\ni A\mapsto\left\{f:\operatorname{Spec}A\to X\,\middle|\,a_n\cdot\left(1_{\mathbf{G}_n}\times_k f\right)=\mathrm{pr}_n\cdot\left(1_{\mathbf{G}_n}\times_k f\right)\right\}\in\mathbf{Set}$$

is representable by a closed subscheme Z_n of X. Consider now the quasi-coherent ideal $\mathcal{I}_n$ of Z_n inside X. Define

$$\mathcal{I}=\sum_{n\in\mathbb{N}}\mathcal{I}_n$$

Let $i:Z\hookrightarrow X$ be a closed subscheme of X determined by $\mathcal{I}$. This means that Z is the scheme-theoretic intersection inside X of closed subschemes Z_n for $n\in\mathbb{N}$. We show that Z represents the fixed point functor. For this assume that A is a k-algebra and $f:\operatorname{Spec}A\to X$ is a morphism of k-schemes such that f is an A-point of the fixed point functor. This is equivalent with

$$a\cdot(1_{\mathbf{G}}\times_k f)=\mathrm{pr}_X\cdot(1_{\mathbf{G}}\times_k f)$$

From this equality we deduce that

$$a_n\cdot\left(1_{\mathbf{G}_n}\times_k f\right)=\mathrm{pr}_n\cdot\left(1_{\mathbf{G}_n}\times_k f\right)$$

for every $n\in\mathbb{N}$ and hence f factors through Z_n for every $n\in\mathbb{N}$. We derive that f factors through Z. This proves that the fixed point functor is a k-subfunctor of the functor of points of Z. It suffices to prove that Z is invariant with respect to $\mathbf{G}$-action. For this consider the morphism $b:\mathbf{G}\times_k Z\to X$ induced by a. By [Mon20, Corollary 4.6] morphism b corresponds to a morphism $\sigma:\mathfrak{P}_{\mathbf{G}}\to\mathcal{M}\mathrm{or}_k(\mathfrak{P}_Z,\mathfrak{P}_X)$ of k-functors. Consider the cartesian square

$$\begin{tikzcd} \mathfrak{H} \arrow[r, "j"] \arrow[d] & \mathcal{M}\mathrm{or}_k(\mathfrak{P}_Z,\mathfrak{P}_Z) \arrow[d, "\mathcal{M}\mathrm{or}_k(1_{\mathfrak{P}_Z},\mathfrak{P}_i)"] \\ \mathfrak{P}_{\mathbf{G}} \arrow[r, "\sigma"'] & \mathcal{M}\mathrm{or}_k(\mathfrak{P}_Z,\mathfrak{P}_X) \end{tikzcd}$$

The fact that Z is locally free over k and [Mon19, Theorem 6.2] imply that $\mathcal{M}\mathrm{or}_k(\mathfrak{P}_Z,\mathfrak{P}_i)$ is a closed immersion of k-functors. Hence $j:\mathfrak{H}\hookrightarrow\mathfrak{P}_{\mathbf{G}}$ is a closed immersion. Moreover, $\mathfrak{H}$ is a subgroup k-functor of $\mathfrak{P}_{\mathbf{G}}$. Thus we deduce that j is induced by a closed immersion of an algebraic groups $f:\mathbf{H}\hookrightarrow\mathbf{G}$. By definition of $i:Z\hookrightarrow X$, we derive that morphism of local k-algebras

$$\widehat{\mathcal{O}_{\mathbf{G},e_{\mathbf{G}}}}\to\widehat{\mathcal{O}_{\mathbf{H},e_{\mathbf{H}}}}$$

induced by $f^{\#}$ is an isomorphism. Hence by Lemma 4.2.2 f is an open immersion of locally algebraic groups. Since $\mathbf{G}$ is geometrically connected, we deduce that f is an isomorphism. Thus j is an isomorphism and this means that $b : \mathbf{G} \times_k Z \to X$ factors through $i : Z \hookrightarrow X$. □

5. Transporters

Definition 5.1. Let $\mathfrak{G}$ be a monoid k-functor and let $\alpha : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X}$ be an action of $\mathfrak{G}$ on a k-functor $\mathfrak{X}$. Suppose that $\mathfrak{Y}_1, \mathfrak{Y}_2$ are k-subfunctors of $\mathfrak{X}$. For every k-algebra A we define

$$\mathrm{Transp}_{\mathfrak{G}}\left(\mathfrak{Y}_1, \mathfrak{Y}_2\right)(A) = \left\{g \in \mathfrak{G}(A) \,\middle|\, \alpha_g\left(\mathfrak{Y}_1(A)\right) \subseteq \mathfrak{Y}_2(A)\right\}$$

where as usual α_g is a slice of α along g. Then $\mathrm{Transp}_{\mathfrak{G}}\left(\mathfrak{Y}_1, \mathfrak{Y}_2\right)$ is a k-subfunctor of $\mathfrak{G}$. It is called *the tranporter of $\mathfrak{Y}_1$ into $\mathfrak{Y}_2$ with respect to α.*

6. Morphisms of locally algebraic groups

Theorem 6.1. *Let $f : \mathbf{H} \to \mathbf{G}$ be a morphism of algebraic groups over k. Then the following are equivalent.*

- **(i)** *f is a monomorphism of k-schemes.*
- **(ii)** *f is a locally closed immersion of k-schemes.*
- **(iii)** *f is a closed immersion of k-schemes.*

Theorem 6.2. *Let $f : X \to Y$ be a monomorphism of finite type with X, Y noetherian. Then there exists open dense subscheme V of Y such that the morphism $f^{-1}(V) \to V$ induced by f is a locally closed immersion.*

The proof is based on a sequence of results.

Lemma 6.2.1. *Let K and L be a fields. If $\mathrm{Spec}\, L \hookrightarrow \mathrm{Spec}\, K$ is a monomorphism of schemes, then it is an isomorphism.*

Proof of the lemma. Since the diagonal of a monomorphism is an isomorphism, we deduce that the multiplication map $L \otimes_K L \to L$ is an isomorphism. This implies that $\dim_k(L) = \dim_L\left(L \otimes_k L\right) = 1$. Hence $k \hookrightarrow L$ is an isomorphism of fields. □

Lemma 6.2.2. *There exists an open dense subset U of X such that $f_{|U}$ is a locally closed immersion.*

Proof of the lemma. Suppose that x is a generic point of an irreducible component of X. Let $y = f(x)$. Pick $f_y : X_y \to \mathrm{Spec}\, k(y)$. Then f_y is a monomorphism. Since $\mathrm{Spec}\, k(x)$ is a closed subscheme of X_y, we derive that the composition of $\mathrm{Spec}\, k(x) \hookrightarrow X_y$ and f_y is a monomorphism $\mathrm{Spec}\, k(x) \hookrightarrow \mathrm{Spec}\, k(y)$ of schemes. By Lemma 6.2.1 we deduce that is an isomorphism. We derive that f_y is a retraction. A retraction that is a monomorphism is an isomorphism. Hence f_y is an isomorphism. This shows that $k(y) \cong \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$. Thus we have

$$\mathcal{O}_{X,x} = f^{\#}\left(\mathcal{O}_{Y,y}\right) + \mathfrak{m}_y\mathcal{O}_{X,x}$$

Since x is a generic point of an irreducible component of X, we derive that $\mathcal{O}_{X,x}$ is artinian. Thus $\mathfrak{m}_y\mathcal{O}_{X,x} \subseteq \mathfrak{m}_x$ is a nilpotent ideal. Thus $f^{\#} : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is onto. □

Lemma 6.2.3. *Let $f : A \to B$ be a morphism of finite type between noetherian rings. Suppose that $\mathfrak{p} \in \mathrm{Spec}\, A$ and $\mathfrak{q} \in \mathrm{Spec}\, B$ are prime ideals such that $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. Assume that f induces a surjective morphism $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$. Then there exists $s \in B \setminus \mathfrak{q}$ such that f induces a surjective morphism $A \to B_s$.*

Proof of the Lemma. First assume that $A_\mathfrak{p} \to B_\mathfrak{q}$ is bijective and let $\phi : B_\mathfrak{q} \to A_\mathfrak{p}$. Since $f : A \to B$ is morphism of finite type between noetherian rings, we have $B \cong A[x_1, ..., x_n]/I$ for some finitely generated ideal $I \subseteq A[x_1, .., x_n]$ and free variables $x_1, ..., x_n$. Let $\overline{x}_i = x_i \bmod I$ for $1 \leq i \leq n$. Suppose that $a_1, ..., a_n$ are elements in A such that

$$\frac{a_i}{1} = \phi\left(\frac{\overline{x}_i}{1}\right)$$

for $1 \leq i \leq n$. □

References

[DG70] Michel Demazure and Pierre Gabriel. *Groupes algébriques. Tome I. Géométrie algébrique généralités. Groupes commutatifs*. North-Holland, 1970.

[Mon19] Monygham. Geometry of k-functors. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2019.

[Mon20] Monygham. Monoid k-functors and their representations. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2020.