HAHN-BANACH THEOREM

1. Introduction

In these notes we study geometric and analytic versions of Hahn-Banach theorem. For this we introduce topological vector spaces and study their properties over arbitrary fields with absolute value. Next we prove that all one-dimensional Hausdorff topological spaces are isomorphic. This result is used in the characterization of finite dimensional Hausdorff topological vector spaces over a complete field and it is one of the crucial ingredients of Mazur's separation theorem (also called geometric version of Hahn-Banach). Next we intoduce locally convex topological vector spaces and prove separation of convex sets for this spaces. Finally we use Mazur's theorem to deduce analytic version of Hahn-Banach theorem.
Throughout the notes $\mathbb{K}$ is a field with absolute value $|-|$. The closed disc in $\mathbb{K}$ centered in the origin and with unit radius is denoted by $\mathbb{D}$.

Definition 1.1. Suppose that every Cauchy sequence in $\mathbb{K}$ with respect to $|-|$ is convergent, then $\mathbb{K}$ is *a complete field*.

2. Preliminaries on topological vector spaces

In this section we introduce topological vector spaces and study their basic properties.

Definition 2.1. Let $\mathfrak{X}$ be a vector space over $\mathbb{K}$ together with a topology such that the multiplication by scalars $\cdot_{\mathfrak{X}} : \mathbb{K} \times \mathfrak{X} \to \mathfrak{X}$ and the addition $+_{\mathfrak{X}} : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ are continuous. Then $\mathfrak{X}$ is *a topological vector space over* $\mathbb{K}$.

Fact 2.2. *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$ and let $\mathfrak{Z}$ be its $\mathbb{K}$-subspace. Then $\mathfrak{Z}$ with subspace topology is a topological vector space over $\mathbb{K}$.*

Proof. Left for the reader as an exercise. □

Recall that $\mathbb{D}$ is the unit disc in $\mathbb{K}$ centered in the origin.

Fact 2.3. *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$ and let U be an open neighborhood of zero in $\mathfrak{X}$. Then there exists an open neighborhood W of zero in $\mathfrak{X}$ such that $W \subseteq U$ and $W = \mathbb{D} \cdot W$.*

Proof. Since the multiplication by scalars $\mathbb{K} \times \mathfrak{X} \to \mathfrak{X}$ is continuous, there exists an open neighborhood V of zero in $\mathfrak{X}$ and a positive real number r such that

$$W = \bigcup_{\alpha \in \mathbb{K},\, |\alpha| \leq r} \alpha \cdot V \subseteq U$$

Then W is an open neighborhood of zero in $\mathfrak{X}$, $W \subseteq U$ and $W = \mathbb{D} \cdot W$. □

Definition 2.4. Let $\mathfrak{X}, \mathfrak{Y}$ are topological vector spaces over $\mathbb{K}$. A map $f : \mathfrak{X} \to \mathfrak{Y}$ which is both continuous and $\mathbb{K}$-linear is *a morphism of topological vector spaces over* $\mathbb{K}$.

Theorem 2.5. *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$ and let $\mathfrak{U}$ be its $\mathbb{K}$-subspace. Consider the quotient map $q : \mathfrak{X} \twoheadrightarrow \mathfrak{X}/\mathfrak{U}$ in the category of vector spaces over $\mathbb{K}$ and equip $\mathfrak{X}/\mathfrak{U}$ with the quotient topology of $\mathfrak{X}$. Then the following assertions holds.*

(1) *q is an open map.*

(2) *$\mathfrak{X}/\mathfrak{U}$ is a topological vector space over $\mathbb{K}$ and q is a morphism of topological vector spaces.*

(3) *For every morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of topological vector spaces over $\mathbb{K}$ such that $f(\mathfrak{U}) = 0$ there exists a unique morphism $p : \mathfrak{X}/\mathfrak{U} \to \mathfrak{Y}$ of topological vector spaces over $\mathbb{K}$ which makes the triangle*

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "f"] \arrow[d, "q"'] & \mathfrak{Y} \\ \mathfrak{X}/\mathfrak{U} \arrow[ur, dotted, "p"'] & \end{tikzcd}$$

commutative.

(4) *$\mathfrak{U}$ is a closed in $\mathfrak{X}$ if and ony if $\mathfrak{X}/\mathfrak{U}$ is a Hausdorff topological space.*

For the proof we need the following result.

Lemma 2.5.1. *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. Then $\mathfrak{X}$ is Hausdorff if and only if zero subspace of $\mathfrak{X}$ is closed.*

Proof of the lemma. If $\mathfrak{X}$ is Hausdorff, then each singleton subset of $\mathfrak{X}$ is closed. Hence zero subspace of $\mathfrak{X}$ is closed.
Conversely, assume that the singleton of zero in $\mathfrak{X}$ is closed. Pick two distinct points $x_1, x_2 \in \mathfrak{X}$. There exists an open neighborhood U of zero in $\mathfrak{X}$ such that $x_1 - x_2 \notin U$. Since the addition $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ is continuous, there exists an open neighborhood W of zero in $\mathfrak{X}$ such that $W + W \subseteq U$. Define V to be $W \cap (-W)$. Then V is an open neighborhood of zero such that $V + V \subseteq U$ and $V = -V$. If

$$z \in (x_1 + V) \cap (x_2 + V)$$

then $z = x_1 + z_1$ and $z = x_2 + z_2$ for some $z_1, z_2 \in V$. Hence

$$x_1 - x_2 = (z_2 - z_1) \in V + (-V) = V + V \subseteq U$$

This is a contradiction with $x_1 - x_2 \notin U$. Thus

$$\varnothing = (x_1 + V) \cap (x_2 + V)$$

and $\mathfrak{X}$ is Hausdorff. □

Proof of the theorem. Fix an open subset U of $\mathfrak{X}$, then the set

$$q^{-1}(q(U)) = \bigcup_{u \in \mathfrak{U}} (u + U)$$

is open. According to the fact that $q : \mathfrak{X} \twoheadrightarrow \mathfrak{X}/\mathfrak{U}$ is a quotient topological map, we infer that $q(U)$ is open in $\mathfrak{X}/\mathfrak{U}$. Hence q is an open map and the proof of **(1)** is completed.
Since q is open, we derive that $1_{\mathbb{K}} \times q$ and $q \times q$ are open. Since squares

$$\begin{tikzcd} \mathfrak{X} \times \mathfrak{X} \arrow[r, "+_{\mathfrak{X}}"] \arrow[d, "q \times q"'] & \mathfrak{X} \arrow[d, "q"] \\ \mathfrak{X}/\mathfrak{U} \times \mathfrak{X}/\mathfrak{U} \arrow[r, "+_{\mathfrak{X}/\mathfrak{U}}"'] & \mathfrak{X}/\mathfrak{U} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K} \times \mathfrak{X} \arrow[r, "\cdot_{\mathfrak{X}}"] \arrow[d, "1_{\mathbb{K}} \times q"'] & \mathfrak{X} \arrow[d, "q"] \\ \mathbb{K} \times \mathfrak{X}/\mathfrak{U} \arrow[r, "\cdot_{\mathfrak{X}/\mathfrak{U}}"'] & \mathfrak{X}/\mathfrak{U} \end{tikzcd}$$

are commutative, we deduce that the addition $+_{\mathfrak{X}/\mathfrak{U}} : \mathfrak{X}/\mathfrak{U} \times \mathfrak{X}/\mathfrak{U} \to \mathfrak{X}/\mathfrak{U}$ and the multiplication of scalars $\cdot_{\mathfrak{X}/\mathfrak{U}} : \mathbb{K} \times \mathfrak{X}/\mathfrak{U} \to \mathfrak{X}/\mathfrak{U}$ are continuous. Therefore, $\mathfrak{X}/\mathfrak{U}$ is a topological vector space over $\mathbb{K}$. It follows that q is a morphism of topological vector spaces over $\mathbb{K}$ and hence **(2)** holds.

The assertion **(3)** describes the universal property which follows easily from definition and **(2)**. For **(4)** observe that

$$\mathfrak{U} \text{ is closed subset of } \mathfrak{X} \Leftrightarrow \text{ zero subspace of } \mathfrak{X}/\mathfrak{U} \text{ is closed}$$

Thus it suffices to prove that

$$\text{zero subspace of } \mathfrak{X}/\mathfrak{U} \text{ is closed } \Leftrightarrow \mathfrak{X}/\mathfrak{U} \text{ is a Hausdorff topological space}$$

but this is a consequence of Lemma 2.5.1. □

3. Complete topological vector spaces

We need some basic results on complete topological vector spaces. We start by defining this important notion.

Definition 3.1. Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. Suppose that $\mathcal{F}$ is a proper filter of subsets of $\mathfrak{X}$ such that for every open neighborhood U of zero in $\mathfrak{X}$ there exists $F \in \mathcal{F}$ such that

$$F - F \subseteq U$$

Then $\mathcal{F}$ is *a Cauchy filter in* $\mathfrak{X}$.

Definition 3.2. A topological vector space $\mathfrak{X}$ over $\mathbb{K}$ is *complete* if every Cauchy filter in $\mathfrak{X}$ is convergent.

Theorem 3.3. *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$ and let $\mathfrak{Z}$ be its $\mathbb{K}$-subspace. Consider $\mathfrak{Z}$ as a topological vector space over $\mathbb{K}$ with subspace topology. Then the following assertions hold.*

(1) *If $\mathfrak{X}$ is complete and $\mathfrak{Z}$ is a closed in $\mathfrak{X}$, then $\mathfrak{Z}$ is complete.*

(2) *If $\mathfrak{Z}$ is complete and $\mathfrak{X}$ is Hausdorff, then $\mathfrak{Z}$ is closed in $\mathfrak{X}$.*

Proof. Consider a Cauchy filter $\mathcal{F}$ in $\mathfrak{Z}$. We define

$$\tilde{\mathcal{F}} = \left\{\tilde{F} \subseteq \mathfrak{X} \,\middle|\, \text{there exists } F \in \mathcal{F} \text{ such that } F \subseteq \tilde{F}\right\}$$

Clearly $\tilde{\mathcal{F}}$ is a Cauchy filter in $\mathfrak{X}$. Since $\mathfrak{X}$ is complete, we derive that $\tilde{\mathcal{F}}$ is convergent to some x in $\mathfrak{X}$. This together with definition of $\tilde{\mathcal{F}}$ show that for every open neighborhood U of zero in $\mathfrak{X}$ there exists $F \in \mathcal{F}$ such that $F \subseteq x + U$. In particular, for every open neighborhood U of zero in $\mathfrak{X}$ intersection $(x + U) \cap \mathfrak{Z}$ is nonempty. Since $\mathfrak{Z}$ is closed in $\mathfrak{X}$, it follows that $x \in \mathfrak{Z}$ and $\mathcal{F}$ is convergent to x. Thus $\mathfrak{Z}$ is complete.
Suppose now that $\mathfrak{Z}$ is complete. Assume that for some point x in $\mathfrak{X}$ and for every open neighborhood of zero U in $\mathfrak{X}$ intersection $(x + U) \cap \mathfrak{Z}$ is nonempty. Define

$$\mathcal{F} = \left\{F \subseteq \mathfrak{Z} \,\middle|\, \text{there exists open neighborhood } U \text{ of zero in } \mathfrak{X} \text{ such that } (x + U) \cap \mathfrak{Z} \subseteq F\right\}$$

Then $\mathcal{F}$ is a Cauchy filter in $\mathfrak{Z}$. Since $\mathfrak{Z}$ is complete, $\mathcal{F}$ is convergent to some point z in $\mathfrak{Z}$. By definition of $\mathcal{F}$ we have $z \in x + U$ for every open neighborhood U of zero x. Since $\mathfrak{X}$ is Hausdorff, it follows that z is identical to x. This proves that $\mathfrak{Z}$ is closed in $\mathfrak{X}$. □

Theorem 3.4. *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. Suppose that there exists a pseudometric $\rho : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}_+ \cup \{0\}$ which induces topology on $\mathfrak{X}$. Then the following assertions hold.*

(i) *$\mathfrak{X}$ is complete.*

(ii) *Every Cauchy sequence with respect to ρ is convergent.*

Proof. Assume that $\mathfrak{X}$ is complete and $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to ρ. Define

$$F_n = \left\{x_k \,\middle|\, k \geq n\right\}$$

for every $n \in \mathbb{N}$ and let

$$\mathcal{F} = \left\{F \subseteq \mathfrak{X} \,\middle|\, F_n \subseteq F \text{ for some } n \in \mathbb{N}\right\}$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to ρ and this pseudometric induces topology on $\mathfrak{X}$, we derive that $\mathcal{F}$ is a Cauchy filter in $\mathfrak{X}$. Hence $\mathcal{F}$ is convergent to some point of $\mathfrak{X}$. This proves that $\{x_n\}_{n\in\mathbb{N}}$ is convergent to some point of $\mathfrak{X}$. Hence $\{x_n\}_{n\in\mathbb{N}}$ is convergent with respect to ρ. This completes the proof of **(i)** $\Rightarrow$ **(ii)**.
Suppose that every Cauchy sequence with respect to ρ is convergent in $\mathfrak{X}$. Consider a Cauchy filter $\mathcal{F}$ in $\mathfrak{X}$. Since topology of $\mathfrak{X}$ is pseudometrizable, we derive that there exists a countable basis $\{U_n\}_{n\in\mathbb{N}}$ of open neighborhoods of zero in $\mathfrak{X}$. There exists a decreasing sequence $\{F_n\}$ of elements of $\mathcal{F}$ such that

$$F_n - F_n \subseteq U_n$$

for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let $x_n \in F_n$. Then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to ρ. Hence it is convergent to some point x in $\mathfrak{X}$. Pick an open neighborhood U of zero in $\mathfrak{X}$. Consider open neighborhood W of zero in $\mathfrak{X}$ such that $W + W \subseteq U$. For sufficiently large $n \in \mathbb{N}$ we have

$$F_n - F_n \subseteq W,\ x_n - x \in W$$

If $z \in F_n$, then

$$x - z = (x - x_n) + (x_n - z) \in W + (F_n - F_n) \subseteq W + W \subseteq U$$

Hence $F_n \subseteq x + U$. This proves that $\mathcal{F}$ is convergent to x. The implication **(ii)** $\Rightarrow$ **(i)** holds. □

Theorem 3.5. *Let $\{\mathfrak{X}_i\}_{i\in I}$ be a family of topological vector space over $\mathbb{K}$. Then the following assertions are equivalent.*

- **(i)** *$\mathfrak{X}_i$ is complete for every $i \in I$.*
- **(ii)** *$\prod_{i\in I}\mathfrak{X}_i$ is complete topological vector space over $\mathbb{K}$.*

Proof. We denote $\prod_{i\in I}\mathfrak{X}_i$ by $\mathfrak{X}$ and let $pr_i : \mathfrak{X} \to \mathfrak{X}_i$ be canonical projection on i-th axis.
Assume that $\mathfrak{X}_i$ is complete for every $i \in I$. Suppose that $\mathcal{F}$ is a Cauchy filter in $\mathfrak{X}$. Then $pr_i(\mathcal{F})$ is a Cauchy filter in $\mathfrak{X}_i$ for each i. Since $\mathfrak{X}_i$ is complete, we derive that $pr_i(\mathcal{F})$ is convergent to some point x_i in $\mathfrak{X}_i$. Define $x \in \mathfrak{X}$ by condition $pr_i(x) = x_i$ for each $i \in I$. Then $\mathcal{F}$ is convergent to x. Thus $\mathfrak{X}$ is a complete topological vector space over $\mathbb{K}$.
Suppose now that $\mathfrak{X}$ is complete. Fix i_0 in I and consider a Cauchy filter $\mathcal{F}$ in $\mathfrak{X}_{i_0}$. Define

$$\tilde{\mathcal{F}} = \Big\{ \underbrace{F}_{i_0} \times \underbrace{\{0\}}_{i\neq i_0} \subseteq \mathfrak{X} \,\Big|\, F \in \mathcal{F} \Big\}$$

Then $\tilde{\mathcal{F}}$ is a Cauchy filter in $\mathfrak{X}$. Hence $\tilde{\mathcal{F}}$ is convergent to some point x in $\mathfrak{X}$. Then $\mathcal{F} = pr_{i_0}(\tilde{\mathcal{F}})$ is convergent to $pr_{i_0}(x)$. Thus $\mathfrak{X}_{i_0}$ is complete. Since i_0 is arbitrary, we derive that $\mathfrak{X}_i$ is complete for every $i \in I$. □

Corollary 3.6. *Let $\mathbb{K}$ be a complete field. Topological vector spaces $\mathbb{K}^n$ over $\mathbb{K}$ are complete for each $n \in \mathbb{N}$.*

Proof. This is a direct consequence of Theorems 3.4 and 3.5. □

4. Finite dimensional topological vector spaces

Fact 4.1. *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. Suppose that $f : \mathbb{K}^n \to \mathfrak{X}$ is a $\mathbb{K}$-linear map for some $n \in \mathbb{N}$. Then f is continuous.*

Proof. Let $\{e_1, ..., e_n\}$ be the canonical basis of $\mathbb{K}^n$. For every i let $pr_i : \mathbb{K}^n \to \mathbb{K}$ be the projection onto i-th axis and let $m_i : \mathbb{K} \to \mathfrak{X}$ be the composition of the multiplication of scalars $\mathbb{K} \times \mathfrak{X} \to \mathfrak{X}$ with the continuous embedding $\mathbb{K} \ni \alpha \mapsto (\alpha, f(e_i)) \in \mathbb{K} \times \mathfrak{X}$. Since pr_i and m_i are continuous for each i, we derive that their compositions $m_i \cdot pr_i$ are also continuous. According to the fact that the addition $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ is continuous, we infer that the sum

$$\sum_{i=1}^{n} m_i \cdot pr_i$$

is continuous. This sum is equal to f. Thus f is continuous. □

Theorem 4.2. *Let $\mathfrak{X}$ be a one-dimensional topological vector space over $\mathbb{K}$. Then the following assertions hold.*

(1) *If $\mathfrak{X}$ is Hausdorff, then every $\mathbb{K}$-linear isomorphisn $\mathfrak{X} \to \mathbb{K}$ is a homeomorphism.*

(2) *If $\mathfrak{X}$ is not Hausdorff, then the topology on $\mathfrak{X}$ is indiscrete.*

Proof. Assume that $\mathfrak{X}$ is Hausdorff. Let $f : \mathfrak{X} \to \mathbb{K}$ be $\mathbb{K}$-linear isomorphism. If the topology of $\mathbb{K}$ is discrete, then f is a homeomorphism. Hence without loss of generality we may assume that the topology on $\mathbb{K}$ is not discrete. In particular, for each positive real number r there exists nonzero $\gamma \in \mathbb{K}$ such that $|\gamma| < r$. Consider x_γ in $\mathfrak{X}$ such that $f(x_\gamma) = \gamma$. It is unique element of $\mathfrak{X}$. Since $\mathfrak{X}$ is Hausdorff, by Fact 2.3 there exists open neighborhood W of zero in $\mathfrak{X}$ such that $\mathbb{D} \cdot W = W$ and $x_\gamma \notin W$. Then $\mathbb{D} \cdot f(W) = f(W)$ and $\gamma \notin f(W)$. This proves that $f(W)$ is a subset of

$$\left\{\alpha \in \mathbb{K} \,\middle|\, |\alpha| < r\right\}$$

Therefore, f is continuous at zero and hence f is continuous. On the other hand map $f^{-1} : \mathbb{K} \to \mathfrak{X}$ is continuous by Fact 4.1. This means that f is a homeomorphism.
Suppose now that $\mathfrak{X}$ is not Hausdorff. Theorem 2.5 implies that zero subspace is not closed in $\mathfrak{X}$. Since in every topological vector space closure of a subspace is a subspace, we derive that $\mathfrak{X}$ is the closure of its zero subspace. This shows that $\mathfrak{X}$ is indiscrete. □

Corollary 4.3. *Let $f : \mathfrak{X} \to \mathbb{K}$ be a $\mathbb{K}$-linear map between topological vector spaces over $\mathbb{K}$. Then the following are equivalent.*

(i) *f is continuous.*

(ii) *$\ker(f)$ is a closed subspace of $\mathfrak{X}$.*

Proof. Follows immediately from Theorems 2.5 and 4.2. □

Theorem 4.4. *Let $\mathbb{K}$ be a complete field and let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. If $\mathfrak{X}$ is Hausdorff and of dimension n over $\mathbb{K}$ for some $n \in \mathbb{N}$, then $\mathfrak{X}$ is isomorphic with $\mathbb{K}^n$.*

Proof. The proof goes on induction by $n \in \mathbb{N}$. For $n = 0$ it is clear. Suppose that the result holds for $n \in \mathbb{N}$. Assume that $\mathfrak{X}$ is a Hausdorff topological vector space over $\mathbb{K}$ of dimension $n + 1$. By induction each n-dimensional subspace of $\mathfrak{X}$ is isomorphic to $\mathbb{K}^n$ and hence by Corollary 3.6 it is complete. Thus Theorem 3.3 asserts that all n-dimensional subspaces are closed in $\mathfrak{X}$. Corollary 4.3 implies that each $\mathbb{K}$-linear map $f : \mathfrak{X} \to \mathbb{K}$ is continuous. Therefore, every $\mathbb{K}$-linear map $\Phi : \mathfrak{X} \to \mathbb{K}^{n+1}$ is continuous. Next Φ^{-1} is continuous according to Fact 4.1. Therefore, $\mathfrak{X}$ is isomorphic to $\mathbb{K}^{n+1}$ as a topological vector space over $\mathbb{K}$. The proof is completed. □

5. Mazur's theorem

In this section assume that $\mathbb{K}$ is either real numbers field $\mathbb{R}$ of complex numbers field $\mathbb{C}$.

Theorem 5.1 (Mazur)**.** *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$ and let U be an open and convex subset of $\mathfrak{X}$. Suppose that $\mathfrak{U}$ is a $\mathbb{K}$-subspace of $\mathfrak{X}$ such that $\mathfrak{U}$ does not intersect with U. Then there exists a $\mathbb{K}$-linear continuous map $f : \mathfrak{X} \to \mathbb{K}$ such that $\mathfrak{U} \subseteq \ker(f)$ and $0 \notin f(U)$.*

For the proof we need the following result.

Lemma 5.1.1. *Let $\mathfrak{X}$ be a two-dimensional Hausdorff topological vector space over $\mathbb{R}$ and let U be an open and convex subset which does not contain zero of $\mathfrak{X}$. Then there exists one-dimensional subspace L of $\mathfrak{X}$ which does not intersect U.*

Proof of the lemma. Theorem 4.4 implies that we may assume that $\mathfrak{X}$ is $\mathbb{R}^2$. Consider

$$S^1 = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1\right\}$$

and a retraction $r : \mathbb{R}^2 \setminus \{0\} \to S^1$ given by formula

$$r(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

Note that r is a continuous open map. Thus $\tilde{U} = r(U)$ is an open subset of S^1. Let $i : S^1 \to S^1$ be a homeomorphism given by formula $i(x,y) = (-x,-y)$. Since U is convex and does not contain zero, sets $i(\tilde{U})$ and $\tilde{U}$ have empty intersection. According to the fact that S^1 is connected, we deduce that $i(\tilde{U}) \cup \tilde{U}$ is a proper subset of S^1. This is the case if and only if there exists $(x,y) \in S^1$ such that $(x,y) \notin \tilde{U}$ and $(-x,-y) \notin \tilde{U}$. Then one-dimensional subspace $\mathbb{R} \cdot (x,y)$ of $\mathfrak{X}$ does not intersect U. □

Proof of the theorem. Assume first that $\mathbb{K}$ is $\mathbb{R}$. By Zorn's lemma there exists maximal $\mathbb{R}$-subspace $\mathfrak{Z}$ such that $\mathfrak{U} \subseteq \mathfrak{Z}$ and $\mathfrak{Z}$ does not intersect U. Since U is open, we derive that $\mathbf{cl}(\mathfrak{Z})$ does not intersect U. This shows that $\mathfrak{Z}$ is a closed subspace of $\mathfrak{X}$. Now consider the quotient map $q : \mathfrak{X} \twoheadrightarrow \mathfrak{X}/\mathfrak{Z}$. By Theorem 2.5 space $\mathfrak{X}/\mathfrak{Z}$ is Hausdorff and $q(U)$ is an open set. Moreover, $q(U)$ does not intersect zero and is convex. Suppose that there exists two-dimensional $\mathbb{R}$-subspace $\mathfrak{Y}$ of $\mathfrak{X}/\mathfrak{Z}$. Applying Lemma 5.1.1 to $\mathfrak{Y}$ and $\mathfrak{Y} \cap q(U)$ we deduce that there exists one-dimensional $\mathbb{R}$-subspace L of $\mathfrak{X}/\mathfrak{Z}$ such that L does not intersect $q(U)$. Then $q^{-1}(L)$ is $\mathbb{R}$-subspace of $\mathfrak{X}$ strictly containing $\mathfrak{Z}$ which does not intersect U. This is contradiction with maximality of $\mathfrak{Z}$. Thus $\mathfrak{X}/\mathfrak{Z}$ contains no two-dimensional subspaces and hence it is one-dimensional. According to Theorem 4.4 we have isomorphism $\phi : \mathfrak{X}/\mathfrak{Z} \to \mathbb{R}$ of topological vector spaces over $\mathbb{R}$. The composition $f = \phi \cdot q$ satisfies the assertion of the theorem and this completes the proof for $\mathbb{R}$.
Next assume that $\mathbb{K}$ is $\mathbb{C}$. Since $\mathfrak{X}$ is a topological vector space over $\mathbb{C}$, it is also topological vector space over $\mathbb{R}$. Hence there exists an $\mathbb{R}$-linear continuous map $\tilde{f} : \mathfrak{X} \to \mathbb{R}$ such that $\mathfrak{U} \subseteq \ker(\tilde{f})$ and $0 \notin \tilde{f}(U)$. Consider $f : \mathfrak{X} \to \mathbb{C}$ given by formula

$$f(x) = \tilde{f}(x) - \sqrt{-1} \cdot \tilde{f}\left(\sqrt{-1} \cdot x\right)$$

for x in $\mathfrak{X}$. Then f is a $\mathbb{C}$-linear continuous map such that $\mathfrak{U} \subseteq \ker(f)$ and $0 \notin f(U)$. □

The result above is often called geometric Hahn-Banach theorem.

6. Analytic Hahn-Banach theorem

Definition 6.1. Let $\mathfrak{X}$ be a vector space over $\mathbb{R}$ and let $p : \mathfrak{X} \to \mathbb{R}$ be a map. Suppose that

$$p(x_1 + x_2) \leq p(x_1) + p(x_2)$$

for all $x_1, x_2 \in \mathfrak{X}$ and

$$p(r \cdot x) = r \cdot p(x)$$

for each $x \in \mathfrak{X}$ and each $r \in \mathbb{R}_+$. Then p is *a sublinear map*.

Theorem 6.2 (Hahn-Banach)**.** *Let $\mathfrak{X}$ be a vector space over $\mathbb{R}$ and let $p : \mathfrak{X} \to \mathbb{R}$ be a sublinear map. Suppose that $\mathfrak{U}$ is an $\mathbb{R}$-subspace of $\mathfrak{X}$ and $g : \mathfrak{U} \to \mathbb{R}$ is an $\mathbb{R}$-linear map such that $f(x) \leq p(x)$ for every x in $\mathfrak{U}$. Then there exists an $\mathbb{R}$-linear map $\tilde{f} : \mathfrak{X} \to \mathbb{R}$ such that $\tilde{f}(x) \leq p(x)$ and $\tilde{f}_{|\mathfrak{U}} = f$.*

We need the following result.

Lemma 6.2.1. *Let $\mathfrak{X}$ be a vector space over $\mathbb{R}$ and let $p : \mathfrak{X} \to \mathbb{R}$ be a sublinear map. Consider $q : \mathfrak{X} \to \mathbb{R}$ given by formula*

$$q(x) = \max\{p(x), p(-x)\}$$

for $x \in \mathfrak{X}$. Then q is a seminorm on $\mathfrak{X}$ and p is continuous with respect to q.

Proof of the lemma. Note that q is a sublinear map. Since

$$0 \le p(x) + p(-x)$$

for $x \in \mathfrak{X}$, we derive that the image of q is $\mathbb{R}_+ \cup \{0\}$. Moreover, $q(x) = q(-x)$ for each x in $\mathfrak{X}$. Therefore, q is a seminorm on $\mathfrak{X}$. Observe that

$$|p(x_1) - p(x_2)| \le q(x_1 - x_2)$$

and hence p is continuous with respect to topology induced by q on $\mathfrak{X}$. □

Proof of the theorem. By Lemma 6.2.1 we may assume that $\mathfrak{X}$ is a topological vector space over $\mathbb{R}$ and p is continuous map. Define

$$U = \{(x, r) \in \mathfrak{X} \times \mathbb{R} \,|\, p(x) < r\},\ \mathfrak{Z} = \{(x, f(x)) \in \mathfrak{X} \times \mathbb{R} \,|\, x \in \mathfrak{U}\}$$

It follows that U is a convex open subset of $\mathfrak{X} \times \mathbb{R}$ and $\mathfrak{Z}$ is an $\mathbb{R}$-subspace of $\mathfrak{X} \times \mathbb{R}$ such that $U \cap \mathfrak{Z} = \varnothing$. By Theorem 5.1 there exists an $\mathbb{R}$-linear continuous map $\tilde{g} : \mathfrak{X} \times \mathbb{R} \to \mathbb{R}$ such that $\mathfrak{Z} \subseteq \ker(\tilde{g})$ and $0 \notin \tilde{f}(U)$. Since U is convex, without loss of generality we may assume that $\tilde{g}(U) \subseteq \mathbb{R}_+$. There exists $u \in \mathbb{R}$ and $\mathbb{R}$-linear map $g : \mathfrak{X} \to \mathbb{R}$ such that

$$\tilde{g}(x, r) = g(x) + u \cdot r$$

for every $x \in \mathfrak{X}$ and $r \in \mathbb{R}$. Suppose now that $u \le 0$. We have

$$g(x) + u \cdot r = \tilde{g}(x, r) > 0$$

for each $(x, r) \in U$. Hence $g(x) > (-u) \cdot r$ for every $(x, r) \in U$. Fix now $x \in \mathfrak{X}$ and pick $r \in \mathbb{R}_+$ such that $r > p(x)$. Then

$$g(x) > (-u) \cdot r \ge 0$$

and this shows that $g(x) > 0$ for $x \in \mathfrak{X}$ and this contradicts the fact that g is an $\mathbb{R}$-linear map. Thus $u > 0$. We define $\tilde{f} : \mathfrak{X} \to \mathbb{R}$ by formula $\tilde{f}(x) = -\frac{1}{u} \cdot g(x)$. Then f is an $\mathbb{R}$-linear map and

$$\tilde{g}(x, r) = u \cdot (r - \tilde{f}(x))$$

for every $(x, r) \in \mathfrak{X} \times \mathbb{R}$. For each $x \in \mathfrak{U}$ we have

$$0 = \tilde{g}(x, f(x)) = u \cdot (f(x) - \tilde{f}(x))$$

Hence $\tilde{f}_{|\mathfrak{U}} = f$. Moreover, for $(x, r) \in U$ we have

$$u \cdot (r - \tilde{f}(x)) = \tilde{g}(x, r) > 0$$

and hence

$$r > \tilde{f}(x)$$

for every $(x, r) \in U$. We deduce that $\tilde{f}(x) \le p(x)$ for all $x \in \mathfrak{X}$.

□