HAAR MEASURE

1. Existence of Haar measure

For a topological space X we denote by $\mathcal{B}(X)$ the σ-algebra of all open subsets of X.

Definition 1.1. Let X be a locally compact space and let $\mu : \mathcal{B}(X) \to [0, +\infty]$ be a measure. If $\mu(K) \in \mathbb{R}$ for every compact subset K of X, then μ is *finite on compact sets*. Suppose that for every open subset U of X we have

$$\mu(U) = \sup\left\{\mu(K) \,\middle|\, K \text{ compact subset of } X \text{ contained in } U\right\}$$

then μ is *inner regular*. We say that μ is *outer regular* if for every A in $\mathcal{B}(X)$ we have

$$\mu(A) = \inf\left\{\mu(U) \,\middle|\, U \text{ is open in } X \text{ and contains } A\right\}$$

Finally μ is *a Radon measure* if it is finite on compact sets, inner regular and outer regular.

Definition 1.2. Let G be a topological group and let $\mu : \mathcal{B}(G) \to [0, +\infty]$ be a measure. Then μ is *left-invariant* if $\mu(xA) = \mu(A)$ for every A in $\mathcal{B}(G)$. Similarly μ is right-invariant if $\mu(Ax) = \mu(A)$ for every A in $\mathcal{B}(G)$.

Theorem 1.3. *Let G be a locally compact topological group. Then there exists a nonzero, left-invariant Radon measure μ on G.*

We denote by $\mathcal{K}$ the set of all compact subsets of G and by $\mathcal{U}$ the set of all open neighborhoods of identity in G. Let U be an open nonempty subset of G and K be a compact subset of G. We define

$$(K : U) = \inf\left\{n \in \mathbb{N} \,\middle|\, \text{there exist } x_1, ..., x_n \in G \text{ such that } K \subseteq \bigcup_{i=1}^{n} x_i U\right\}$$

Throughout the proof we fix a compact subset Q of G such that $\mathbf{int}(Q) \neq \varnothing$.

Lemma 1.3.1. *Fix $U \in \mathcal{U}$. There exists a real valued function h_U on $\mathcal{K}$ such that the following assertions hold.*

(1) *For every compact subset K in $\mathcal{K}$ we have $h_U(K) \geq 0$, $h_U(\varnothing) = 0$ and $h_U(Q) = 1$.*

(2) *For every compact subset K in $\mathcal{K}$ and for every element x in G we have $h_U(xK) = h_U(K)$.*

(3) *If $K \subseteq L$ are compact subsets in $\mathcal{K}$, then $h_U(K) \subseteq h_U(L)$.*

(4) *For every compact subset K in $\mathcal{K}$ we have $h_U(K) \leq (K : \mathbf{int}(Q))$.*

(5) *If K, L are compact subsets in $\mathcal{K}$, then*

$$h_U(K \cup L) \leq h_U(K) + h_U(L)$$

and if $K \cdot U^{-1} \cap L \cdot U^{-1} = \varnothing$, then the equality holds.

Proof of the lemma. For every compact subset K of G we define

$$h_U(K) = \frac{(K : U)}{(Q : U)}$$

Now we check that h_U admits the properties above. Properties **(1)**, **(2)** and **(3)** are clear. For **(4)** note that

$$(K : U) \leq (Q : U) \cdot (K : \mathbf{int}(Q))$$

Indeed, if $K \subseteq \bigcup_{i=1}^{n} y_i \cdot \mathbf{int}(Q)$ and $Q \subseteq \bigcup_{j=1}^{m} z_j U$, then $K \subseteq \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} y_i z_j U$ and this implies the inequality above. Observe that $xU \cap K \neq \varnothing$ implies that $x \in K \cdot U^{-1}$ and similarly $xU \cap L \neq \varnothing$

implies that $x \in L \cdot U^{-1}$. Assuming that for compact subsets K, L in G we have $K \cdot U^{-1} \cap L \cdot U^{-1} = \varnothing$ we derive from this that for every $x \in G$ we have $xU \cap (K \cap L) = \varnothing$. Thus if $K \cdot U^{-1} \cap L \cdot U^{-1} = \varnothing$, then we have $(K \cup L : U) = (K : U) + (K : L)$ and hence $h_U(K \cup L) = h_U(K) + h_U(L)$. Note that in general case we have $(K \cup L : U) \leq (K : U) + (K : L)$ and hence also **(5)** holds for h_U. □

Lemma 1.3.2. *Let K, L in $\mathcal{K}$ and suppose that $K \cap L = \varnothing$. Then there exists $U \in \mathcal{U}$ such that $K \cdot U^{-1} \cap L \cdot U^{-1} = \varnothing$.*

Proof of the lemma. □

Lemma 1.3.3. *There exists a real valued function h on $\mathcal{K}$ such that*

(1) *For every compact subset K in $\mathcal{K}$ we have $h(K) \geq 0$, $h(\varnothing) = 0$ and $h(Q) = 1$.*

(2) *For every compact subset K in $\mathcal{K}$ and for every element x in G we have $h(xK) = h(K)$.*

(3) *If $K \subseteq L$ are compact subsets in $\mathcal{K}$, then $h(K) \subseteq h(L)$.*

(4) *For every compact subset K in $\mathcal{K}$ we have $h(K) \leq (K : \mathbf{int}(Q))$.*

(5) *If K, L are compact subsets in $\mathcal{K}$, then*

$$h(K \cup L) \leq h(K) + h(L)$$

and if $K \cap L = \varnothing$, then the equality holds.

Proof of the lemma. Consider a topological space

$$X = \prod_{K \in \mathcal{K}} \left[0, (K : \mathbf{int}(Q)\right]$$

By Tichonoff's theorem X is compact. For every $U \in \mathcal{U}$ we define a subset $F_U \subseteq X$ that consists of tuples $\{a_K\}_{K \in \mathcal{K}}$ such that $a_{\varnothing} = 0$, $a_Q = 1$, $a_{xK} = a_K$ for $x \in G$ and K in $\mathcal{K}$, $a_K \leq a_L$ for $K \subseteq L$ in $\mathcal{K}$, $a_{K \cup L} \leq a_K + a_L$ for K, L in $\mathcal{K}$ and the equality holds if $K \cdot U^{-1} \cap L \cdot U^{-1} = \varnothing$. Conditions imposed on tuples in F_U imply that F_U is a closed subset. Note that $\{h_U(K)\}_{K \in \mathcal{K}} \in F_U$ for every $U \in \mathcal{U}$. Moreover, we have

$$F_{U_1 \cap U_2 \cap \ldots \cap U_n} \subseteq F_{U_1} \cap F_{U_2} \cap \ldots \cap F_{U_n}$$

for $U_1, U_2, \ldots, U_n \in \mathcal{U}$. This implies that $\left\{F_U\right\}_{U \in \mathcal{U}}$ is a centered family of nonempty closed subsets of a compact space X. Thus

$$\bigcap_{U \in \mathcal{U}} F_U \neq \varnothing$$

by compactness of X. Hence there exists $\{c_K\}_{K \in \mathcal{K}}$ in the intersection. We define a real function h on $\mathcal{K}$ by $h(K) = c_K$ for K in $\mathcal{K}$. The fact that properties **(1)**, **(2)**, **(3)** and **(4)** hold for h follows by definition of F_U for $U \in \mathcal{U}$. Since $\{c_K\}_{K \in \mathcal{K}}$ is an element in F_U for every $U \in \mathcal{U}$ we derive that

$$c_{K \cup L} \leq c_K + c_L$$

for K, L in $\mathcal{K}$. This implies $h(K \cup L) \leq h(K) + h(L)$ for $K, L \in \mathcal{K}$. Moreover, $c_{K \cup L} = c_K + c_L$ if $K \cdot U^{-1} \cap L \cdot U^{-1} = \varnothing$ for some $U \in \mathcal{U}$. This implies that $c_{K \cup L} = c_K + c_L$ if $K \cap L = \varnothing$ by Lemma 1.3.2. Thus h admits **(4)**. □

Proof of the theorem. We fix h as in Lemma 1.3.3 and we define $\mu^* : \mathcal{P}(G) \to [0, +\infty]$. First if U is an open subset of G, then we define

$$\mu^*(U) = \sup_{K \in \mathcal{K}, K \subseteq U} h(K)$$

Note that if U, V are open subsets of G and $U \subseteq V$, then $\mu^*(U) \leq \mu^*(V)$. Thus it makes sense to define

$$\mu^*(A) = \inf\left\{\mu^*(U) \,\middle|\, U \text{ is an open subset of } G \text{ containing } A\right\}$$

for arbitrary subset $A \subseteq G$. Note that $\mu^*(xA) = \mu^*(A)$ by definition of μ^* and the corresponding property of h. We check that μ^* is an outer measure. By definition and corresponding properties of h we have $\mu^*(\varnothing)=0$ and μ^* is monotone. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of G such that $\mu^*(A_n)\in\mathbb{R}$ for every $n\in\mathbb{N}$. Fix $\epsilon>0$ and for each $n\in\mathbb{N}$ we pick an open subset U_n such that $A_n\subseteq U_n$ and

$$\mu^*(U_n)\le\mu^*(A_n)+\frac{\epsilon}{2^{n+1}}$$

There exists a compact subset K of $\bigcup_{n\in\mathbb{N}}U_n$ such that

$$\mu^*\left(\bigcup_{n\in\mathbb{N}}U_n\right)\le h(K)+\frac{\epsilon}{2}$$

Since K is compact, there exists $k\in\mathbb{N}$ such that $K\subseteq\bigcup_{n=0}^{k}U_n$. Since G is locally compact, there exist compact sets $K_0,K_1,...,K_k$ such that $K_n\subseteq U_n$ and $K=\bigcup_{n=0}^{k}K_n$. Thus we have

$$\mu^*\left(\bigcup_{n\in\mathbb{N}}A_n\right)\le\mu^*\left(\bigcup_{n\in\mathbb{N}}U_n\right)\le h(K)+\frac{\epsilon}{2}\le\frac{\epsilon}{2}+\sum_{n=0}^{k}h(K_n)\le$$

$$\le\frac{\epsilon}{2}+\sum_{n\in\mathbb{N}}\mu^*(U_n)\le\sum_{n\in\mathbb{N}}\mu^*(A_n)+\frac{\epsilon}{2}+\sum_{n\in\mathbb{N}}\frac{\epsilon}{2^n}=\sum_{n\in\mathbb{N}}\mu^*(A_n)+\epsilon$$

Since ϵ is an arbitrary positive number, we derive that

$$\mu^*\left(\bigcup_{n\in\mathbb{N}}A_n\right)\le\sum_{n\in\mathbb{N}}\mu^*(A_n)$$

Note that this inequality is obvious when there exists $n\in\mathbb{N}$ such that $\mu^*(A_n)=+\infty$. Thus the inequality above holds for arbitrary countable family of subsets of G. Therefore, μ^* is an outer measure. Now we use Carathéodory construction [Mon18, Theorem 3.2] in order to obtain a σ-algebra Σ_{μ^*} such that $\mu^*_{|\Sigma_{\mu^*}}$ is a measure. Now we show that σ-algebra of Borel sets $\mathcal{B}(G)$ is contained in Σ_{μ^*}. For this consider a set E of G and let U be an open subset of G. We show that

$$\mu^*(E)=\mu^*(E\cap U)+\mu^*(E\setminus U)$$

Clearly the inequality $\le$ holds and hence if $\mu^*(E)=+\infty$, then the equality holds regardless of U. Thus we may assume that $\mu^*(E)\in\mathbb{R}$. Fix $\epsilon>0$ and consider open subset V such that $E\subseteq V$ and $\mu^*(V)\le\mu^*(E)+\frac{\epsilon}{2}$. Next let $K\subseteq U\cap V$ be a compact subset such that $\mu^*(U\cap V)\le h(K)+\frac{\epsilon}{4}$. Let L be a compact subset of $V\setminus K$ such that $\mu^*(V\setminus K)\le\mu^*(L)+\frac{\epsilon}{4}$. We have

$$\mu^*(E)\le\mu^*(E\cap U)+\mu^*(E\setminus U)\le\mu^*(V\cap U)+\mu^*(V\setminus U)\le\mu^*(V\cap U)+\mu^*(V\setminus K)\le$$

$$\le\left(h(K)+\frac{\epsilon}{4}\right)+\left(h(L)+\frac{\epsilon}{4}\right)=h(K)+h(L)+\frac{\epsilon}{2}=h(K\cap L)+\frac{\epsilon}{2}\le\mu^*(V)+\frac{\epsilon}{2}\le\mu^*(E)+\epsilon$$

and since $\epsilon>0$ was arbitrary, we derive that

$$\mu^*(E)=\mu^*(E\cap U)+\mu^*(E\setminus U)$$

Hence this equality holds for every subset E of G and every open subset U of G. Thus open subsets of G are members of Σ_{μ^*}. Hence $\mathcal{B}(G)\subseteq\Sigma_{\mu^*}$. Let μ be a restriction of μ^* to $\mathcal{B}(G)$. Then μ is a left-invariant measure on $\mathcal{B}(G)$. By definition of μ^* we derive that μ is outer regular. Now we show that $\mu(K)\in\mathbb{R}$ for every compact subset of G. We pick an open subset U of G containing K and such that $\mathbf{cl}(U)$ is compact. Then we have $h(L)\le h(\mathbf{cl}(U))$ for every compact $L\subseteq U$ and hence $\mu(U)\le h(\mathbf{cl}(U))$. Thus $\mu(U)\in\mathbb{R}$ and hence also $\mu(K)\in\mathbb{R}$. Thus μ is finite on compact subsets of G. Moreover, $1=h(Q)\le\mu(Q)$. This implies that μ is nontrivial. Finally for every open subset U of G we have

$$\mu(U)=\sup_{K\in\mathcal{K},K\subseteq U}h(K)\le\sup_{K\in\mathcal{K},K\subseteq U}\mu(K)\le\mu(U)$$

and thus μ is inner regular.

We proved that μ is nonzero, left-invariant Radon measure on G. This finishes the proof. □

Definition 1.4. Let G be a locally compact group and $\mu : \mathcal{B}(G) \to [0, +\infty]$ be a measure. If μ is left-invariant, nontrivial Radon measure on G, then we say that μ is *a (left) Haar measure on* G. Similarly if μ is right-invariant, nontrivial Radon measure on G, then we say that μ is *a (right) Haar measure on* G

REFERENCES

[Mon18] Monygham. Introduction to measure theory. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2018.