ALGEBRAIC GROUP SCHEMES OVER FIELD

1. Introduction

In these notes we group schemes over fields. For background we refer to [Mon19] and [Mon20]. Throughout these notes k is a fixed field.

Definition 1.1. Let **G** be a group scheme over *k*. If **G** is of finite type over *k*, then we say that **G** is *an algebraic group over k*.

2. SIMPLE CRITERION FOR SEPARATEDNESS

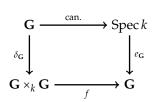
Proposition 2.1. Let **G** be a group scheme over k and let $e_{\mathbf{G}}: \operatorname{Spec} k \to \mathbf{G}$ be its unit. Then the following are equivalent.

- (i) $e_{\mathbf{G}}$ is a closed immersion.
- (ii) **G** is separated.

Proof. Suppose that (i) holds. Consider morphism $f : \mathbf{G} \times_k \mathbf{G} \to \mathbf{G}$ given on *A*-points by formula

$$f(g_1,g_2) = g_1 \cdot g_2^{-1}$$

where *A* is a *k*-algebra. Note that we have a cartesian square



where δ_G is a diagonal of G. Since base change of a closed immersion is a closed immersion, we derive that δ_G is a closed immersion and hence G is separated. This is (ii).

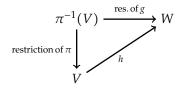
Suppose now that (ii) holds. Let $\pi : \mathbf{G} \to \operatorname{Spec} k$ be the structural morphism. Then $\pi \cdot e_{\mathbf{G}} = 1_{\mathbf{G}}$. Since π is a separated morphism, we derive that (by cancellation) $e_{\mathbf{G}}$ is closed immersion. This is (i).

3. Complete group schemes

We start this section with the following general result.

Theorem 3.1 (Rigidity). Let $\pi: X \to Y$ be a proper morphism of schemes such that $\pi^{\sharp}: \mathcal{O}_Y \to \pi_* \mathcal{O}_X$ is an isomorphism of sheaves. Let $g: X \to Z$ be a morphism of schemes. Suppose that for some point y in Y there is a point z of Z such that $\pi^{-1}(y) \subseteq g^{-1}(z)$. Then there exist an affine neighborhood V of y and an affine neighborhood W of z such that $\pi^{-1}(V) \subseteq g^{-1}(W)$. Moreover, there exists a morphism $h: V \to W$ making the diagram

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commutative, where horizontal arrow is the restriction of g.

Proof. Consider an affine open neighborhood of W of z. Since π is proper and $\pi^{-1}(y) = g^{-1}(z)$, we derive that $\pi(X \setminus g^{-1}(W))$ is a closed subset of Y that does not contain y. Pick an open affine neighborhood V of y in Y that does not intersect with $\pi(X \setminus g^{-1}(W))$. Then $\pi^{-1}(V) \subseteq g^{-1}(W)$. Since $\pi^{\#}$ is an isomorphism we have the composition

$$\mathcal{O}_{Z}(W) \xrightarrow{g_{W}^{\#}} \Gamma(g^{-1}(W), \mathcal{O}_{X}) \xrightarrow{(-)_{\mid \pi^{-1}(V)}} \Gamma(\pi^{-1}(V), \mathcal{O}_{X}) \xrightarrow{(\pi_{V}^{\#})^{-1}} \mathcal{O}_{Y}(V)$$

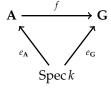
This composition induces a morphism of affine schemes $h: V \to W$. Since a morphism from a scheme to an affine scheme is determined by the morphism on global sections of structure sheaves, we derive that h makes the triangle in the statement commutative.

Now we can apply this result to study complete algebraic groups over *k*. For this we need the following definition.

Definition 3.2. Let **A** be a geometrically integral, complete algebraic group over k. Then we say that **A** is an abelian variety over k.

Now we prove the following interesting result.

Theorem 3.3. Let **A** be an abelian variety over k, let **G** be a separated group scheme over k and let $f : \mathbf{A} \to \mathbf{G}$ be a morphism of schemes over k. Suppose that the diagram

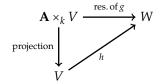


is commutative. Then f is a morphism of groups schemes over k.

Proof. We define a morphism $g : \mathbf{A} \times_k \mathbf{A} \to \mathbf{G}$ given by

$$(x_1, x_2) \mapsto f(x_1) \cdot f(x_2) \cdot f(x_1 \cdot x_2)^{-1}$$

where A is a k-algebra and x_1, x_2 are A-points of \mathbf{A} . It suffices to show that g factors through Spec $k(e_{\mathbf{G}})$. For this we may change base to an algebraic closure of k by faitfully flat descent. So we may assume that the field k is algebraically closed and \mathbf{A} is connected. Then the projection onto second factor $\pi: \mathbf{A} \times_k \mathbf{A} \to \mathbf{A}$ is proper and $k = \Gamma(\mathbf{A}, \mathcal{O}_{\mathbf{A}})$ implies that $\pi^{\#}$ is an isomorphism of sheaves on \mathbf{A} . Moreover, note that $\pi^{-1}(e_{\mathbf{A}}) \subseteq g^{-1}(e_{\mathbf{G}})$. Indeed, this follows from the assumption that $f(e_{\mathbf{A}}) = e_{\mathbf{G}}$. By Theorem 3.1 we deduce that there exist an affine neighborhood V of $e_{\mathbf{A}}$, an affine neighborhood V of $e_{\mathbf{G}}$ and a morphism V such that V such that V is V and the diagram



is commutative. Hence for every k-point v of V we have the restiction $g_{|\mathbf{A}\times_k \operatorname{Spec} k(v)}$ factors through $\operatorname{Spec} k(h(v))$. Since $g(v,e_{\mathbf{A}})=e_{\mathbf{G}}$, we derive that $h(v)=e_{\mathbf{G}}$ and thus $g_{|\mathbf{A}\times_k \operatorname{Spec} k(v)}$ factors through $\operatorname{Spec} k(e_{\mathbf{G}})$. This holds for any k-point of V. Therefore, $g_{|\mathbf{A}\times_k V}$ factors through $\operatorname{Spec} k(e_{\mathbf{G}})$. Consider the kernel $i:Z \to \mathbf{A}\times_k \mathbf{A}$ of a pair consisting of g and a morphism $\mathbf{A}\times_k \mathbf{A} \to \mathbf{G}$ that factorizes through $\operatorname{Spec} k(e_{\mathbf{G}})$. Since \mathbf{G} is separated, we derive that i is a closed immersion. Moreover, i dominates $\mathbf{A}\times_k V$. Since $\mathbf{A}\times_k V$ is schematically dense open subset of $\mathbf{A}\times_k \mathbf{A}$ (because $\mathbf{A}\times_k \mathbf{A}$ is integral), we derive that i is an isomorphism and hence g factors through $\operatorname{Spec} k(e_{\mathbf{G}})$. \square

Corollary 3.4. Let A be an abelian variety over k. Then A is a commutative group scheme over k.

Proof. Consider the morphism $(-)^{-1} : \mathbf{A} \to \mathbf{A}$. By Theorem 3.3 we derive $(-)^{-1}$ is a morphism of group schemes over k. Hence \mathbf{A} is a commutative group scheme.

4. Transporters

Definition 4.1. Let \mathfrak{G} be a monoid k-functor and let $\alpha : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X}$ be an action of \mathfrak{G} on a k-functor \mathfrak{X} . Suppose that $\mathfrak{Y}_1, \mathfrak{Y}_2$ are k-subfunctors of \mathfrak{X} . For every k-algebra A we define

$$\operatorname{Transp}_{\mathfrak{G}}(\mathfrak{Y}_{1},\mathfrak{Y}_{2})(A) = \left\{ g \in \mathfrak{G}(A) \, \middle| \, \alpha_{g}(\mathfrak{Y}_{1}(A)) \subseteq \mathfrak{Y}_{2}(A) \right\}$$

where as usual α_g is a slice of α along g. Then Transp_{\mathfrak{G}} $(\mathfrak{Y}_1,\mathfrak{Y}_2)$ is a k-subfunctor of \mathfrak{G} . It is called the transporter of \mathfrak{Y}_1 into \mathfrak{Y}_2 with respect to α .

REFERENCES

[Mon19] Monygham. Geometry of k-functors. github repository: "Monygham/Pedo-mellon-a-minno", 2019.

[Mon20] Monygham. Monoid k-functors and their representations. github repository: "Monygham/Pedo-mellon-a-minno", 2020.