HOMOGENEOUS MARKOV CHAINS

1. Introduction

In this notes we study discrete homogeneous Markov chains. We start in the first section with general discrete Markov chains. Main result is the theorem on existence of a Markov chain with given transition matrices and initial distribution. This theorem is basically a consequence of Daniell-Kolmogorov extension theorem proved in [Monygham, 2022]. In the second section we define homogeneous Markov chains and discuss their basic properties. Next sections are devoted to classification of states of such chains.

2. General Markov Chains

Definition 2.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{S}$ be a countable set considered as a measurable space with respect to power set σ-algebra. Suppose that $\{X_n : \Omega \to \mathcal{S}\}_{n\in\mathbb{N}}$ is a sequence of random variables such that

$$P\big(X_{n+1} = s_{n+1} \,\big|\, X_n = s_n, ..., X_0 = s_0\big) = P\big(X_{n+1} = s_{n+1} \,\big|\, X_n = s_n\big)$$

for all $n \in \mathbb{N}$ and $s_0, ..., s_{n+1} \in \mathcal{S}$ such that

$$P\big(X_n = s_n, ..., X_0 = s_0\big) > 0$$

Then $\{X_n\}_{n\in\mathbb{N}}$ is *a Markov chain with state space* $\mathcal{S}$.

Definition 2.2. Let $\mathcal{S}$ be a countable set and let $\{p_{ts}\}_{s,t\in\mathcal{S}}$ be a matrix of nonnegative reals such that

$$\sum_{t\in\mathcal{S}} p_{ts} = 1$$

for every $s \in \mathcal{S}$. Then $\{p_{ts}\}_{s,t\in\mathcal{S}}$ is *a stochastic matrix*.

Definition 2.3. Let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain with state space $\mathcal{S}$ and fix $n \in \mathbb{N}$. Suppose that $\{p_{ts}(n)\}_{s,t\in\mathcal{S}}$ is a stochastic matrix such that

$$p_{ts}(n) = P\big(X_{n+1} = t \,\big|\, X_n = s\big)$$

for every $s, t \in \mathcal{S}$ such that $P(X_n = s) > 0$. Then $\{p_{ts}(n)\}_{s,t\in\mathcal{S}}$ is called *a transition matrix of* $\{X_n\}_{n\in\mathbb{N}}$ *in n-th step*.

Definition 2.4. Let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain with state space $\mathcal{S}$. Then the distribution of X_0 is called *the initial distribution of* $\{X_n\}_{n\in\mathbb{N}}$.

Proposition 2.5. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain with state space $\mathcal{S}$. Suppose that $\{p_{ts}(n)\}_{s,t\in\mathcal{S}}$ are transition matrices and ν is the initial distribution for $\{X_n\}_{n\in\mathbb{N}}$. Then*

$$P\big(X_n = t\big) = \sum_{(s_0,s_1,...,s_{n-1})\in\mathcal{S}^n} p_{ts_{n-1}}(n-1)\cdot p_{s_{n-1}s_{n-2}}(n-2)\cdot ...\cdot p_{s_1s_0}(0)\cdot \nu(\{s_0\})$$

for every $t \in \mathcal{S}$ and $n \in \mathbb{N}$.

Proof. The proof goes by induction on n. The base case is trivial. Indeed, according to the definition of the initial distribution, we have

$$P\big(X_0 = t\big) = \nu(\{t\})$$

for every $t \in \mathcal{S}$. Suppose that the result holds for some $n \in \mathbb{N}$. Fix $t \in \mathcal{S}$. Define

$$\mathcal{S}_+ = \big\{s \in \mathcal{S} \,\big|\, P\big(X_n = s\big) > 0\big\},\ \mathcal{S}_0 = \mathcal{S} \setminus \mathcal{S}_+$$

Then

$$P\big(X_{n+1}=t\big)=\sum_{s\in\mathcal{S}}P\big(X_{n+1}=t,X_n=s\big)=\sum_{s\in\mathcal{S}_+}P\big(X_{n+1}=t,X_n=s\big)+\sum_{s\in\mathcal{S}_0}P\big(X_{n+1}=t,X_n=s\big)=$$

$$=\sum_{s\in\mathcal{S}_+}P\big(X_{n+1}=t\,\big|\,X_n=s\big)\cdot P\big(X_n=s\big)=\sum_{s\in\mathcal{S}_+}P\big(X_{n+1}=t\,\big|\,X_n=s\big)\cdot P\big(X_n=s\big)+\sum_{s\in\mathcal{S}_0}p_{ts}(n)\cdot P\big(X_n=s\big)=$$

$$=\sum_{s\in\mathcal{S}_+}p_{ts}(n)\cdot P\big(X_n=s\big)+\sum_{s\in\mathcal{S}_0}p_{ts}(n)\cdot P\big(X_n=s\big)=\sum_{s\in\mathcal{S}}p_{ts}(n)\cdot P\big(X_n=s\big)$$

and by inductive assumption we have

$$\sum_{s\in\mathcal{S}}p_{ts}(n)\cdot P\big(X_n=s\big)=\sum_{s\in\mathcal{S}}p_{ts}(n)\cdot\left(\sum_{(s_0,\dots s_{n-1})\in\mathcal{S}^n}p_{ss_{n-1}}(n-1)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})\right)=$$

$$=\sum_{s\in\mathcal{S},(s_0,\dots s_{n-1})\in\mathcal{S}^n}p_{ts}(n)\cdot p_{ss_{n-1}}(n-1)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})=$$

$$=\sum_{(s_0,\dots s_{n-1},s_n)\in\mathcal{S}^n}p_{ts_n}(n)\cdot p_{s_ns_{n-1}}(n-1)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})$$

and hence

$$P\big(X_{n+1}=t\big)=\sum_{(s_0,\dots s_{n-1},s_n)\in\mathcal{S}^n}p_{ts_n}(n)\cdot p_{s_ns_{n-1}}(n-1)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})$$

□

Proposition 2.5 shows that Markov chains are determined by initial distributions and transition matrices. The following result establishes its converse.

Theorem 2.6. *Let $\mathcal{S}$ be a countable set considered as a measurable space with respect to its power set σ-algebra. Suppose that ν is a probability distribution on $\mathcal{S}$ and $\{p_{ts}(n)\}_{s,t\in\mathcal{S}}$ for $n\in\mathbb{N}$ is a sequence of stochastic matrices. Then there exists a probability space $(\Omega,\mathcal{F},P)$ and a Markov chain $\{X_n:\Omega\to\mathcal{S}\}_{n\in\mathbb{N}}$ such that ν is the initial distribution of $\{X_n\}_{n\in\mathbb{N}}$ and $\{p_{ts}(n)\}_{s,t\in\mathcal{S}}$ for $n\in\mathbb{N}$ are its transition matrices.*

Proof. Suppose that $[n]=\{0,1,\dots,n\}$ for $n\in\mathbb{N}$. Then $\mathcal{S}^{[n]}=\underbrace{\mathcal{S}\times\ldots\times\mathcal{S}}_{n\text{ times}}$ together with its power algebra of subsets is a measurable space. We define a probability measure μ_n in $\mathcal{S}^{[n]}$ by formula

$$\mu_n\left(\{(s_0,s_1,\dots,s_n)\}\right)=p_{s_ns_{n-1}}(n-1)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})$$

In order to verify that μ_n is well defined note that

$$\sum_{(s_0,s_1,\dots,s_n)\in\mathcal{S}^{[n]}}\mu_n\left(\{(s_0,s_1,\dots,s_n)\}\right)=\sum_{(s_0,s_1,\dots,s_n)\in\mathcal{S}^{[n]}}p_{s_ns_{n-1}}(n-1)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})=$$

$$=\sum_{s_0\in\mathcal{S}}\sum_{s_1\in\mathcal{S}}\cdots\sum_{s_{n-2}\in\mathcal{S}}\sum_{s_{n-1}\in\mathcal{S}}\sum_{s_n\in\mathcal{S}}p_{s_ns_{n-1}}(n-1)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})=$$

$$=\sum_{s_0\in\mathcal{S}}\sum_{s_1\in\mathcal{S}}\cdots\sum_{s_{n-2}\in\mathcal{S}}\sum_{s_{n-1}\in\mathcal{S}}\left(\sum_{s_n\in\mathcal{S}}p_{s_ns_{n-1}}(n-1)\right)\cdot p_{s_{n-1}s_{n-2}}(n-2)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})=$$

$$=\sum_{s_0\in\mathcal{S}}\sum_{s_1\in\mathcal{S}}\cdots\sum_{s_{n-2}\in\mathcal{S}}\sum_{s_{n-1}\in\mathcal{S}}p_{s_{n-1}s_{n-2}}(n-2)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})$$

Repeating this simplification $(n-1)$-times more we obtain

$$\sum_{(s_0,s_1,\dots,s_n)\in\mathcal{S}^{[n]}}\mu_n\left(\{(s_0,s_1,\dots,s_n)\}\right)=\sum_{s_0\in\mathcal{S}}\nu(\{s_0\})=1$$

This proves that μ_n is well defined. Suppose next that $n_1\le n_2$. Then we have a projection $\pi_{n_2,n_1}:\mathcal{S}^{[n_2]}\to\mathcal{S}^{[n_1]}$ and

$$\left(\pi_{n_2,n_1}\right)_*\mu_{n_2}\left(\{(s_0,s_1,\dots,s_{n_1})\}\right)=\mu_{n_2}\left(\pi^{-1}_{n_2,n_1}\left((s_0,s_1,\dots,s_{n_1})\right)\right)=$$

$$= \sum_{s_{n_1+1}\in\mathcal{S}} \cdots \sum_{s_{n_2}\in\mathcal{S}} \mu_{n_2}\left(\{(s_0,s_1,...,s_{n_1},s_{n_1+1},...,s_{n_2})\}\right) =$$

$$= \sum_{s_{n_1+1}\in\mathcal{S}} \cdots \sum_{s_{n_2}\in\mathcal{S}} p_{s_{n_2}s_{n_2-1}}(n_2-1)\cdot ... \cdot p_{s_{n_1+1}s_{n_1}}(n_1)\cdot p_{s_{n_1}s_{n_1-1}}(n_1-1)\cdot ... \cdot p_{s_1s_0}(0)\cdot \nu(\{s_0\}) =$$

$$= \left(\sum_{s_{n_1+1}\in\mathcal{S}} \cdots \sum_{s_{n_2}\in\mathcal{S}} p_{s_{n_2}s_{n_2-1}}(n_2-1)\cdot ... \cdot p_{s_{n_1+1}s_{n_1}}(n_1)\right)\cdot p_{s_{n_1}s_{n_1-1}}(n_1-1)\cdot ... \cdot p_{s_1s_0}(0)\cdot \nu(\{s_0\}) =$$

$$= p_{s_{n_1}s_{n_1-1}}(n_1-1)\cdot ... \cdot p_{s_1s_0}(0)\cdot \nu(\{s_0\}) = \mu_{n_1}\left(\{(s_0,s_1,...,s_{n_1})\}\right)$$

This proves that $\left(\pi_{n_2,n_1}\right)_*\mu_{n_2} = \mu_{n_1}$ for every pair of natural numbers $n_1 \leq n_2$. Now suppose that F is a finite subset of $\mathbb{N}$ and pick $n \in \mathbb{N}$ such that $F \subseteq [n]$. Next suppose that $\pi_{n,F} : \mathcal{S}^{[n]} \to \mathcal{S}^F$ is the projection on axes parametrized by elements of F. We define

$$\mu_F = \left(\pi_{n,F}\right)_*\mu_n$$

Then μ_F is a probability measure and, since $\left(\pi_{n_2,n_1}\right)_*\mu_{n_2} = \mu_{n_1}$ for every pair of natural numbers $n_1 \leq n_2$, we derive that μ_F does not depend on particular choice of $n \in \mathbb{N}$ such that $F \subseteq [n]$. It follows that if $F_1 \subseteq F_2$ are arbitrary finite subsets of $\mathbb{N}$ and $\pi_{F_2,F_1} : \mathcal{S}^{F_2} \to \mathcal{S}^{F_1}$ is the projection, then

$$\left(\pi_{F_2,F_1}\right)_*\mu_{F_2} = \mu_{F_1}$$

Moreover, we have $\mu_n = \mu_{[n]}$ for every $n \in \mathbb{N}$. Note also that each measure μ_F is inner regular with respect to discrete topology on $\mathcal{S}^F$. Therefore, by [Monygham, 2022, Theorem 2.2] there exists a unique probability measure μ defined on $\mathcal{S}^{\mathbb{N}}$ with σ-algebra $\mathcal{P}(\mathcal{S})^{\otimes\mathbb{N}}$ such that

$$\left(\pi_F\right)_*\mu = \mu_F$$

for every finite subset F of $\mathbb{N}$. We set $\Omega = \mathcal{S}^{\mathbb{N}}, \mathcal{F} = \mathcal{P}(\mathcal{S})^{\otimes\mathbb{N}}$ and $P = \mu$. Then for each $n \in \mathbb{N}$ we define $X_n : \Omega \to \mathcal{S}$ as the projection onto n-th axis. We describe now joint distribution of $(X_0,...,X_n)$ for some $n \in \mathbb{N}$. For this fix $s_0,...,s_n \in \mathcal{S}$ and note that

$$P\big(X_n = s_n,...,X_0 = s_0\big) = \mu\left(\{(s_0,s_1,...,s_n)\}\times\mathcal{S}^{\mathbb{N}\setminus[n]}\right) = \left(\pi_{[n]}\right)_*\mu\left(\{(s_0,s_1,...,s_n)\}\right) =$$

$$= \mu_n\left(\{(s_0,...,s_n)\}\right) = p_{s_ns_{n-1}}(n-1)\cdot ... \cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})$$

In addition for $n \in \mathbb{N}$ and $t,s \in \mathcal{S}$ we have formulas It follows that for $n \in \mathbb{N}$ and $t \in \mathcal{S}$ we have

$$P(X_n = t) = \mu\left(\mathcal{S}^{[n-1]}\times\{t\}\times\mathcal{S}^{\mathbb{N}\setminus[n]}\right) = \left(\pi_{[n]}\right)_*\mu\left(\mathcal{S}^{[n-1]}\times\{t\}\right) =$$

$$= \mu_n\left(\mathcal{S}^{[n-1]}\times\{t\}\right) = \sum_{(s_0,s_1,...,s_{n-1})\in\mathcal{S}^{[n-1]}} p_{ts_{n-1}}(n-1)\cdot ... \cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})$$

and

$$P(X_{n+1} = t, X_n = s) = \mu\left(\mathcal{S}^{[n-1]}\times\{(s,t)\}\times\mathcal{S}^{\mathbb{N}\setminus[n]}\right) = \left(\pi_{[n]}\right)_*\mu\left(\mathcal{S}^{[n-1]}\times\{(s,t)\}\right) =$$

$$= \mu_n\left(\mathcal{S}^{[n-1]}\times\{(s,t)\}\right) = \sum_{(s_0,s_1,...,s_{n-1})\in\mathcal{S}^{[n-1]}} p_{ts}(n)\cdot p_{ss_{n-1}}(n-1)\cdot ... \cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})$$

We claim that $\{X_n\}_{n\in\mathbb{N}}$ is a Markov chain with initial distribution ν and transition matrices $\{p_{ts}(n)\}_{s,t\in\mathcal{S}}$ for $n \in \mathbb{N}$. In order to prove the claim we use the description of joint distributions $(X_1,...,X_n)$ and distribution of X_n for every $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $s_0,...,s_n,s_{n+1} \in \mathcal{S}$ such that

$$P\big(X_n = s_n,...,X_0 = s_0\big) > 0$$

we have

$$P\left(X_{n+1} = s_{n+1}\,|\,X_n = s_n,...,X_0 = s_0\right) = \frac{P\big(X_{n+1} = s_{n+1}, X_n = s_n,...,X_0 = s_0\big)}{P\big(X_n = s_n,...,X_0 = s_0\big)} =$$

$$= \frac{p_{s_{n+1}s_n}(n)\cdot p_{s_ns_{n-1}}(n-1)\cdot ... \cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})}{p_{s_ns_{n-1}}(n-1)\cdot ... \cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})} = p_{s_{n+1}s_n}(n)$$

On the other hand we have

$$P\left(X_{n+1}=s_{n+1}\,|\,X_n=s_n\right)=\frac{P\big(X_{n+1}=s_{n+1},X_n=s_n\big)}{P\big(X_n=s_n\big)}=$$

$$=\frac{\sum_{(s_0,s_1,\ldots,s_{n-1})\in\mathcal{S}^{[n-1]}} p_{s_{n+1}s_n}(n)\cdot p_{s_ns_{n-1}}(n-1)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})}{\sum_{(s_0,s_1,\ldots,s_{n-1})\in\mathcal{S}^{[n-1]}} p_{s_ns_{n-1}}(n-1)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})}=$$

$$=\frac{p_{s_{n+1}s_n}(n)\cdot\Big(\sum_{(s_0,s_1,\ldots,s_{n-1})\in\mathcal{S}^{[n-1]}} p_{s_ns_{n-1}}(n-1)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})\Big)}{\sum_{(s_0,s_1,\ldots,s_{n-1})\in\mathcal{S}^{[n-1]}} p_{s_ns_{n-1}}(n-1)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})}=p_{s_{n+1}s_n}(n)$$

Therefore, we have

$$P\left(X_{n+1}=s_{n+1}\,|\,X_n=s_n,\ldots,X_0=s_0\right)=P\left(X_{n+1}=s_{n+1}\,|\,X_n=s_n\right)$$

Thus $\{X_n\}_{n\in\mathbb{N}}$ is a Markov chain. Moreover, if $t,s\in\mathcal{S}$ and $n\in\mathbb{N}$ are such that

$$P(X_n=s)>0$$

then we have

$$P\left(X_{n+1}=t\,|\,X_n=s\right)=\frac{P\big(X_{n+1}=t,X_n=s\big)}{P\big(X_n=s\big)}=$$

$$=\frac{\sum_{(s_0,s_1,\ldots,s_{n-1})\in\mathcal{S}^{[n-1]}} p_{ts}(n)\cdot p_{ss_{n-1}}(n-1)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})}{\sum_{(s_0,s_1,\ldots,s_{n-1})\in\mathcal{S}^{[n-1]}} p_{ss_{n-1}}(n-1)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})}=$$

$$=\frac{p_{ts}(n)\cdot\Big(\sum_{(s_0,s_1,\ldots,s_{n-1})\in\mathcal{S}^{[n-1]}} p_{ss_{n-1}}(n-1)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})\Big)}{\sum_{(s_0,s_1,\ldots,s_{n-1})\in\mathcal{S}^{[n-1]}} p_{ss_{n-1}}(n-1)\cdot\ldots\cdot p_{s_1s_0}(0)\cdot\nu(\{s_0\})}=p_{ts}(n)$$

and we have

$$P(X_0=s)=\nu(\{s\})$$

This completes the proof of the claim. Hence the theorem is proved. □

3. Homogeneous Markov chains and their state spaces

So far we discussed general Markov chains. However, the following special case is very important.

Definition 3.1. Let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain with state space $\mathcal{S}$. Suppose that there exists a stochastic matrix $\{p_{ts}\}_{s,t\in\mathcal{S}}$ such that

$$p_{ts}=P\big(X_{n+1}=t\,\big|\,X_n=s\big)$$

for every $s,t\in\mathcal{S}$ such that $P(X_n=s)>0$. Then $\{X_n\}_{n\in\mathbb{N}}$ is *a homogeneous Markov chain*.

Definition 3.2. Let $\{X_n\}_{n\in\mathbb{N}}$ be a homogeneous Markov chain with state space $\mathcal{S}$ and transition matrix P. A state t is *accessible* from state s for $s,t\in\mathcal{S}$ if there exists $n\in\mathbb{N}_+$ such that

$$\left(P^n\right)_{ts}>0$$

If t is accessible from s, then we write $s\to t$.

Fact 3.3. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a homogeneous Markov chain with state space $\mathcal{S}$. If $s_1\to s_2$ and $s_2\to s_3$ for some $s_1,s_2,s_3\in\mathcal{S}$, then also $s_1\to s_3$.*

Proof. Let P be a transition matrix of $\{X_n\}_{n\in\mathbb{N}}$. By assumptions

$$\left(P^{n_2}\right)_{s_3s_2}>0,\ \left(P^{n_1}\right)_{s_2s_1}>0,$$

for some $n_1,n_2\in\mathbb{N}_+$. Hence

$$\left(P^{n_1+n_2}\right)_{s_3s_1}\geq\left(P^{n_2}\right)_{s_3s_2}\cdot\left(P^{n_1}\right)_{s_2s_1}>0$$

Thus $s_1\to s_3$. □

Definition 3.4. Let $\{X_n\}_{n\in\mathbb{N}}$ be a homogeneous Markov chain with state space $\mathcal{S}$ and transition matrix P. Consider a subset C of $\mathcal{S}$. Suppose that for every $t \in \mathcal{S}$ if $s \to t$ for $s \in C$, then $t \in C$. Then C is *a closed set of states*.

Let $\{X_n\}_{n\in\mathbb{N}}$ be a homogeneous Markov chain with state space $\mathcal{S}$ and transition matrix P. Consider states s, t in $\mathcal{S}$ and define

$$F_{ts} = \sum_{n=1}^{+\infty} (P^n)_{ts}$$

Definition 3.5.

References

[Monygham, 2022] Monygham (2022). Daniell-Kolmogorov extension theorem. *github repository: "Monygham/Pedo-mellon-a-minno"*.