LEBESGUE SPACES AND THEIR DUALS

1. Introduction

In these notes

2. Dual spaces to L^p for $p \in (1, +\infty)$

Let (X, Σ, μ) be a space with measure and let p be a real in $(1, +\infty)$. Define $q \in (1, +\infty)$ to be the unique number which satisfies

$$\frac{1}{p} + \frac{1}{q} = 1$$

Assume that $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$ with usual absolute value. We start by proving the following result.

Proposition 2.1. *Let g be a function in $L^q(\mu, \mathbb{K})$. Then*

$$\|g\|_q = \sup\left\{ \left|\int_X g \cdot f \, d\mu\right| \in \mathbb{R}_+ \cup \{0\} \,\middle|\, f \in L^p(\mu, \mathbb{K}) \text{ such that } \|f\|_p = 1 \right\}$$

Proof. According to Hölder inequality

$$\left|\int_X g \cdot f \, d\mu\right| \leq \int_X |g| \cdot |f| \, d\mu \leq \|g\|_q \cdot \|f\|_p$$

Thus for $f \in L^p(\mu, \mathbb{K})$ such that $\|f\|_p = 1$ we have

$$\left|\int_X g \cdot f \, d\mu\right| \leq \|g\|_q$$

Therefore, it suffices to prove that

$$\|g\|_q \leq \sup\left\{ \left|\int_X g \cdot f \, d\mu\right| \in \mathbb{R}_+ \cup \{0\} \,\middle|\, f \in L^p(\mu, \mathbb{K}) \text{ such that } \|f\|_p = 1 \right\}$$

under the assumption that $\|g\|_q \neq 0$. Define

$$f(x) = \begin{cases} \|g\|_q^{1-q} \cdot \frac{|g(x)|^q}{g(x)} & \text{if } g(x) \neq 0 \\ 0 & \end{cases}$$

Then $f \in L^p(\mu, \mathbb{K})$ and even more precisely we have

$$\|f\|_p = \left(\int_X \|g\|_q^{(1-q)\cdot p} \cdot |g|^{(q-1)\cdot p} \, d\mu\right)^{\frac{1}{p}} = \left(\int_X \|g\|_q^{-q} \cdot |g|^q \, d\mu\right)^{\frac{1}{p}} = \left(\|g\|_q^{-q} \cdot \int_X \cdot |g|^q \, d\mu\right)^{\frac{1}{p}} = 1$$

Note that

$$\left|\int_X g \cdot f \, d\mu\right| = \int_X \|g\|_q^{(1-q)} \cdot |g|^q \, d\mu = \|g\|_q^{(1-q)} \cdot \int_X |g|^q \, d\mu = \|g\|_q^{(1-q)} \cdot \|g\|_q^{(q)} = \|g\|_q$$

and this completes the proof. □

The following theorem is the main result of this section.

Theorem 2.2. *Let $\Lambda : L^p(\mu, \mathbb{K}) \to \mathbb{K}$ be a continuous $\mathbb{K}$-linear map. Then there exists $g \in L^q(\mu, \mathbb{K})$ such that*

$$\Lambda(f) = \int_X g \cdot f \, d\mu$$

for every $f \in L^p(\mu, \mathbb{K})$. Moreover, g is uniquely defined up to a set of measure μ equal to zero.

We start by the following observation.

Lemma 2.2.1. *Let $\Lambda : L^p(\mu, \mathbb{R}) \to \mathbb{R}$ be a continuous $\mathbb{R}$-linear map. For each set $S \in \Sigma$ we define*

$$\Lambda_S(f) = \Lambda\left(\mathbb{1}_S \cdot f\right)$$

for every $f \in L^p(\mu, \mathbb{R})$. Then the following assertions hold.

(1) *$\Lambda_S : L^p(\mu, \mathbb{R}) \to \mathbb{R}$ is a continuous $\mathbb{R}$-linear map.*

(2) *Now if $S \subseteq T$ are two sets in Σ, then*

$$\|\Lambda_S\| \leq \|\Lambda_T\| \leq \|\Lambda\|$$

(3) *There exists a σ-finite subset S in Σ such that $\|\Lambda_S\| = \|\Lambda\|$.*

Proof of the lemma. Assertions **(1)** and **(2)** are left for the reader as an exercises.
We prove **(3)**. Suppose that $f \in L^p(\mu, \mathbb{R})$ satisfies $\|f\|_p \leq 1$. Then there exists a nondecreasing sequence $\{S_n\}_{n \in \mathbb{N}}$ of sets in Σ such that $\mu(S_n)$ is finite for every $n \in \mathbb{N}$ and $\{\mathbb{1}_{S_n} \cdot f\}_{n \in \mathbb{N}}$ converges to f in $L^p(\mu, \mathbb{R})$. Hence

$$\Lambda(f) = \lim_{n \to +\infty} \Lambda_{S_n}(f)$$

It follows that

$$\|\Lambda\| = \sup\left\{ \|\Lambda_S\| \,\middle|\, S \in \Sigma \text{ such that } \mu(S) \text{ is finite } \right\}$$

Hence there exists a nondecreasing sequence $\{S_n\}_{n \in \mathbb{N}}$ of sets in Σ such that $\mu(S_n)$ is finite for every $n \in \mathbb{N}$ and

$$\|\Lambda\| = \lim_{n \to +\infty} \|\Lambda_{S_n}\|$$

Then the union

$$S = \bigcup_{n \in \mathbb{N}} S_n$$

is in Σ is σ-finite and satisfies $\|\Lambda\| = \|\Lambda_S\|$. $\square$

We prove the theorem by gradually considering more general cases.

Proof for finite μ and $\mathbb{K} = \mathbb{R}$. Assume that μ is finite measure and $\mathbb{K}$ is equal to $\mathbb{R}$. We have finite signed measure

$$\Sigma \ni A \mapsto \Lambda\left(\mathbb{1}_A\right) \in \mathbb{R}$$

According to Radon-Nikodym there exists $g \in L^1(\mu, \mathbb{R})$ such that

$$\Lambda\left(\mathbb{1}_A\right) = \int_X g \cdot \mathbb{1}_A \, d\mu$$

for every A in Σ. It follows that

$$\Lambda(f) = \int_X g \cdot f \, d\mu$$

for every $f \in L^\infty(\mu, \mathbb{R})$. For each $n \in \mathbb{N}_+$ define $A_n = \left\{x \in X \,\middle|\, |g(x)| \leq n\right\}$ and consider a measurable and bounded function $f_n : X \to \mathbb{R}$ given by formula

$$f_n(x) = \begin{cases} \mathbb{1}_{A_n}(x) \cdot \frac{|g(x)|^q}{g(x)} & \text{if } g(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\int_X \mathbb{1}_{A_n} \cdot |g|^q \, d\mu = \int_X g \cdot f_n \, d\mu = \Lambda\left(f_n\right) \leq \|\Lambda\| \cdot \|f_n\|_p =$$

$$= \|\Lambda\| \cdot \|\mathbb{1}_{A_n} \cdot |g|^{q-1}\|_p = \|\Lambda\| \cdot \left(\int_X \mathbb{1}_{A_n} \cdot \left(|g|^{q-1}\right)^p d\mu\right)^{\frac{1}{p}} = \|\Lambda\| \cdot \left(\int_X \mathbb{1}_{A_n} \cdot |g|^q \, d\mu\right)^{\frac{1}{p}}$$

and thus

$$\left(\int_X \mathbb{1}_{A_n} \cdot |g|^q \, d\mu\right)^{\frac{1}{q}} \leq \|\Lambda\|$$

By monotone convergence we have

$$\|g\|_q = \lim_{n\to+\infty} \left(\int_X \mathbb{1}_{A_n} \cdot |g|^q \, d\mu\right)^{\frac{1}{q}} \leq \|\Lambda\|$$

Hence $g \in L^q(\mu, \mathbb{R})$. It follows that

$$L^p(\mu, \mathbb{R}) \ni f \mapsto \int_X g \cdot f \, d\mu \in \mathbb{R}$$

is continuous $\mathbb{R}$-linear map, which coincides with Λ on the space of μ-simple functions. Since μ-simple functions are dense in $L^p(\mu, \mathbb{R})$, we derive that

$$\Lambda(f) = \int_X g \cdot f \, d\mu$$

for every $f \in L^p(\mu, \mathbb{R})$. This completes the proof. □

Proof for σ-finite μ and $\mathbb{K} = \mathbb{R}$. Assume that μ is σ-finite measure and $\mathbb{K}$ is equal to $\mathbb{R}$. Since μ is σ-finite, there exist a nondecreasing sequence $\{X_n\}_{n\in\mathbb{N}}$ of sets in Σ such that

$$X = \bigcup_{n\in\mathbb{N}} X_n$$

and $\mu(X_n)$ is finite for $n \in \mathbb{N}$. According to the case considered above and Lemma 2.2.1 for each $n \in \mathbb{N}$ there exists $g_n \in L^q(\mu, \mathbb{R})$ such that

$$\Lambda_{X_n}(f) = \int_X g_n \cdot f \, d\mu$$

for every $f \in L^p(\mu, \mathbb{R})$. We may also assume that $g_{n|X\setminus X_n} = 0$ and $g_{n+1|X_n} = g_{n|X_n}$ for every $n \in \mathbb{N}$. Let g be a pointwise limit of a sequence $\{g_n\}_{n\in\mathbb{N}}$. Then g is a measurable real valued function on X. Moreover, we have $g_n = \mathbb{1}_{X_n} \cdot g$ for each $n \in \mathbb{N}$. By Proposition 2.1, Lemma 2.2.1 and monotone convergence we have

$$\|g\|_q = \lim_{n\to+\infty} \|g_n\|_q = \lim_{n\to+\infty} \|\Lambda_{X_n}\| \leq \|\Lambda\|$$

This implies that $g \in L^q(\mu, \mathbb{R})$. Fix $f \in L^p(\mu, \mathbb{R})$. Then sequence $\{\mathbb{1}_{X_n} \cdot f\}_{n\in\mathbb{N}}$ converges to f in $L^p(\mu, \mathbb{R})$ and hence

$$\Lambda(f) = \lim_{n\to+\infty} \Lambda(\mathbb{1}_{X_n} \cdot f) = \lim_{n\to+\infty} \Lambda_{X_n}(f)$$

On the other hand by dominated convergence theorem

$$\int_X g \cdot f \, d\mu = \lim_{n\to+\infty} \int_X g_n \cdot f \, d\mu = \lim_{n\to+\infty} \Lambda_{X_n}(f)$$

This completes the proof. □

Proof for $\mathbb{K} = \mathbb{R}$. According to Lemma 2.2.1 there exists a σ-finite set S in Σ such that $\|\Lambda_S\| = \|\Lambda\|$. According to previous case there exists $g \in L^q(\mu, \mathbb{R})$ such that

$$\Lambda(f) = \int_X g \cdot f \, d\mu$$

for every $f \in L^p(\mu, \mathbb{R})$. We may also assume that $g_{|X\setminus S} = 0$. Suppose now that T is a σ-finite set in Σ such that $S \subseteq T$. Then there exists $g_T \in L^q(\mu, \mathbb{R})$ such that

$$\Lambda_T(f) = \int_X g_T \cdot f \, d\mu$$

for every $f \in L^p(\mu, \mathbb{R})$. We may assume that $\mathbb{1}_S \cdot g_T = g$. Proposition 2.1 implies that

$$\|\Lambda\| = \|\Lambda_S\| = \|g\|_q \leq \|h\|_q \leq \|\Lambda_{S\cup T}\| \leq \|\Lambda\|$$

Thus $\|g\|_q = \|g_T\|_q$ and this proves that we may assume that $g_T = g$. Fix now $f \in L^p(\mu, \mathbb{R})$ and consider

$$T = \{x \in X \,|\, f(x) \neq 0\} \cup S$$

Then T is σ-finite set in Σ and $S \subseteq T$. Hence

$$\Lambda(f) = \Lambda_T(f) = \int_X g_T \cdot f \, d\mu = \int_X g \cdot f \, d\mu$$

Since $f \in L^p(\mu, \mathbb{R})$ is arbitrary, the proof is completed. □

Proof for $\mathbb{K} = \mathbb{C}$. According to already proved case there exist $g_r, g_i \in L^q(\mu, \mathbb{R})$ such that

$$\operatorname{Re}\Lambda(f) = \int_X g_r \cdot f \, d\mu, \ \operatorname{Im}\Lambda(f) = \int_X g_i \cdot f \, d\mu$$

for every $f \in L^p(\mu, \mathbb{R})$. Then $g = g_r + i \cdot g_r$ is a function in $L^q(\mu, \mathbb{C})$ and

$$\Lambda(f) = \int_X g \cdot f \, d\mu$$

for every $f \in L^p(\mu, \mathbb{C})$. □

3. Dual to L^1

In this section we fix a space with measure (X, Σ, μ). Assume that $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$ with usual absolute value. We start by proving the following result.

Proposition 3.1. *Let g be a function in $L^1(\mu, \mathbb{K})$ and let $\Lambda : L^1(\mu, \mathbb{K}) \to \mathbb{K}$ be a continuous map. Assume that*

$$\Lambda(f) = \int_X g \cdot f d\mu$$

for every $f \in L^1(\mu, \mathbb{K})$ and

$$L^1(\mu, \mathbb{R}) \ni f \mapsto \int_X g \cdot f \, d\mu \in \mathbb{K}$$

is continuous. Then

$$\|g\|_\infty = \sup\left\{ \left|\int_X g \cdot f \, d\mu\right| \in \mathbb{R}_+ \cup \{0\} \,\middle|\, f \in L^1(\mu, \mathbb{R}) \text{ such that } \|f\|_1 = 1 \right\}$$

Proof. Suppose that

$$h(x) = \begin{cases} \frac{|g(x)|}{g(x)} & \text{if } g(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then h is a bounded and measurable function. For $r \in \mathbb{R}_+$ we define

$$A_r = \{x \in X \,|\, |g(x)| \geq r\}$$

If $\mu(A_r) > 0$ for some $r \in \mathbb{R}_+$, then we also define

$$m_r = \int_X \mathbb{1}_{A_r} \cdot |g| \, d\mu, \ f_r = h \cdot \frac{1}{m_r} \cdot \mathbb{1}_{A_r} \cdot |g|$$

Then $f_r : X \to \mathbb{K}$ is μ-integrable and $\|f_r\|_1 = 1$. Let L be a norm of a continuous $\mathbb{K}$-linear map

$$L^1(\mu, \mathbb{K}) \ni f \mapsto \int_X g \cdot f \, d\mu \in \mathbb{K}$$

Thus if $r \in \mathbb{R}_+$ satisfies $\mu(A_r) > 0$, then

$$\left|\int_X g \cdot f_r \, d\mu\right| \leq L$$

On the other hand

$$\left|\int_X g \cdot f_r \, d\mu\right| = \left|\int_X (g \cdot h) \cdot \frac{1}{m_r} \cdot \mathbb{1}_{A_r} \cdot |g| \, d\mu\right| =$$

$$= \int_X (\mathbb{1}_{A_r} \cdot |g|) \cdot \frac{1}{m_r} \cdot \mathbb{1}_{A_r} \cdot |g| \, d\mu \geq r \cdot \int_X \frac{1}{m_r} \cdot \mathbb{1}_{A_r} \cdot |g| \, d\mu = r$$

This implies that $r \leq L$. We derive that g is essentially bounded and $\|g\|_\infty \leq L$. □

The following theorem is the main result of this section.

Theorem 3.2. *Let $\Lambda : L^1(\mu, \mathbb{K}) \to \mathbb{K}$ be a continuous $\mathbb{K}$-linear map. Then there exists $g \in L^\infty(\mu, \mathbb{K})$ such that*

$$\Lambda(f) = \int_X g \cdot f \, d\mu$$

for every $f \in L^1(\mu, \mathbb{K})$ and $\|g\|_\infty = \|\Lambda\|$. Moreover, g is uniquely defined up to a set of measure μ equal to zero.

We prove the theorem by gradually considering more general cases.

Proof for finite μ and $\mathbb{K} = \mathbb{R}$. Assume that μ is finite measure and $\mathbb{K}$ is equal to $\mathbb{R}$. We have finite signed measure

$$\Sigma \ni A \mapsto \Lambda(\mathbb{1}_A) \in \mathbb{R}$$

According to Radon-Nikodym there exists $g \in L^1(\mu, \mathbb{R})$ such that

$$\Lambda(\mathbb{1}_A) = \int_X g \cdot \mathbb{1}_A \, d\mu$$

for every A in Σ. It follows that

$$\Lambda(f) = \int_X g \cdot f \, d\mu$$

for every $f \in L^\infty(\mu, \mathbb{R})$. For every $r \in \mathbb{R}_+$ define

$$A_r = \{x \in X \,|\, |g(x)| \geq r\},\ B_r = \{x \in X \,|\, |g(x)| \leq r\}$$

Fix $r \in \mathbb{R}_+$ such that $\mu(A_r) > 0$. We define

$$f(x) = \begin{cases} \frac{1}{\mu(A_r)} \cdot \frac{|g(x)|}{g(x)} & \text{if } x \in A_r \\ 0 & \text{otherwise} \end{cases}$$

Then $\|f\|_1 = 1$ and $f \in L^\infty(\mu, \mathbb{R})$. Note that

$$r = \int_X r \cdot |f| \, d\mu \leq \int_X |g| \cdot |f| \, d\mu = \int_X g \cdot f \, d\mu = \Lambda(f) \leq \|\Lambda\|$$

It follows that $\|g\|_\infty \leq \Lambda$. Therefore, $g \in L^\infty(\mu, \mathbb{R})$. It follows that

$$L^1(\mu, \mathbb{R}) \ni f \mapsto \int_X g \cdot f \, d\mu \in \mathbb{R}$$

is continuous $\mathbb{R}$-linear map, which coincides with Λ on the space of μ-simple functions. Since μ-simple functions are dense in $L^1(\mu, \mathbb{R})$, we derive that

$$\Lambda(f) = \int_X g \cdot f \, d\mu$$

for every $f \in L^1(\mu, \mathbb{R})$. It remains to prove that $\|\Lambda\| \leq \|g\|_\infty$. For this pick $f \in L^1(\mu, \mathbb{R})$ such that $\|f\|_1 = 1$. Then

$$|\Lambda(f)| = \left|\int_X g \cdot f \, d\mu\right| \leq \int_X |g| \cdot |f| \, d\mu \leq \|g\|_\infty \cdot \int_X |f| \, d\mu = \|g\|_\infty$$

and hence $\|\Lambda\| \leq \|g\|_\infty$. This completes the proof. □

Proof for $\mathbb{K} = \mathbb{R}$. For each set S in Σ such that $\mu(S) \in \mathbb{R}_+ \cup \{0\}$ we define

$$\Lambda_S(f) = \Lambda\left(\mathbb{1}_S \cdot f\right)$$

for every $f \in L^1(\mu, \mathbb{R})$. Then $\Lambda_S : L^1(\mu, \mathbb{R}) \to \mathbb{R}$ is a continuous $\mathbb{R}$-linear map. Now if $S \subseteq T$ are two sets in Σ such that $\mu(S), \mu(T) \in \mathbb{R}_+ \cup \{0\}$, then

$$\|\Lambda_S\| \leq \|\Lambda_T\| \leq \|\Lambda\|$$

Suppose now that f is a function in $L^1(\mu, \mathbb{R})$ such that $\|f\|_1 \leq 1$. Then there exists a nondecreasing sequence $\{S_n\}_{n \in \mathbb{N}}$ of sets in Σ such that $\mu(S_n)$ is finite for every $n \in \mathbb{N}$ and $\{\mathbb{1}_{S_n} \cdot f\}_{n \in \mathbb{N}}$ converges to f in $L^1(\mu, \mathbb{R})$. Hence

$$\Lambda(f) = \lim_{n \to +\infty} \Lambda_{S_n}(f)$$

It follows that

$$\|\Lambda\| = \sup\left\{\|\Lambda_S\| \,\middle|\, S \in \Sigma \text{ such that } \mu(S) \text{ is finite }\right\}$$

It follows that there exists a nondecreasing sequence $\{S_n\}_{n \in \mathbb{N}}$ of sets in Σ such that $\mu(S_n)$ is finite for every $n \in \mathbb{N}$ and

$$\|\Lambda\| = \lim_{n \to +\infty} \|\Lambda_{S_n}\|$$

According to already proved case there exists $g_n \in L^\infty(\mu, \mathbb{R})$ such that $g_n = 0$ outside S_n and

$$\Lambda_{S_n}(f) = \int_X g_n \cdot f \, d\mu$$

for every $f \in L^1(\mu, \mathbb{R})$. Moreover, $\|g_n\|_\infty = \|\Lambda_{S_n}\|$ for each $n \in \mathbb{N}$. We may assume that $g_{n+1}(x) = g_n(x)$ for $x \in S_n$. Define $g : X \to \mathbb{R}$ as a pointwise limit of $\{g_n\}_{n \in \mathbb{N}}$. According to Proposition 2.1 we have

$$\|g_n\|_q = \|\Lambda_{S_n}\|$$

for every $n \in \mathbb{N}$. Taking limits of both sides and using monotone convergence we obtain $\|g\|_q = \|\Lambda\|$. In particular, we have $g \in L^q(\mu, \mathbb{R})$. Fix $f \in L^p(\mu, \mathbb{R})$. Then $g \cdot f \in L^1(\mu, \mathbb{R})$ which follows from Hölder inequality. By dominated convergence theorem

$$\Lambda(f) = \lim_{n \to +\infty} \Lambda_{S_n}(f) = \lim_{n \to +\infty} \int_X g_n \cdot f \, d\mu = \lim_{n \to +\infty} \int_X g \cdot \mathbb{1}_{S_n} \cdot f \, d\mu = \int_X g \cdot f \, d\mu$$

Thus

$$\Lambda(f) = \int_X g \cdot f \, d\mu$$

for every $f \in L^p(\mu, \mathbb{R})$. $\square$