

INTRODUCTION TO MEASURE THEORY

1. FAMILIES OF SETS

In this section we study various families of sets that are important in the development of measure theory.

Definition 1.1. Let X be a set and $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of X . We define the following types of families.

- (1) \mathcal{F} is an *algebra* if it contains X and is closed under finite unions, intersections and complements.
- (2) \mathcal{F} is a σ -*algebra* if it is an algebra and is closed under countable unions.
- (3) \mathcal{F} is a *monotone family* if it is closed under unions of countable non-decreasing sequences and under intersections of countable non-increasing sequences.
- (4) \mathcal{F} is a π -*system* if it is closed under finite intersections.
- (5) \mathcal{F} is a λ -*system* if it contains X and is closed under complements and countable disjoint unions.

Fact 1.2. Let X be a set and $\{\mathcal{F}_i\}_{i \in I}$ be a class of families subsets of X . Suppose that \mathcal{F}_i is an algebra (σ -algebra, monotone family, π -system, λ -system) for every $i \in I$. Then the intersection $\bigcap_{i \in I} \mathcal{F}_i$ is an algebra (σ -algebra, monotone family, π -system, λ -system).

Proof. Left as an exercise. □

Definition 1.3. Let \mathcal{F} be a family of subsets of X . We denote by $\sigma(\mathcal{F})$, $\lambda(\mathcal{F})$ and $\mathcal{M}(\mathcal{F})$ intersections of all σ -algebras, λ -systems and monotone families containing \mathcal{F} , respectively. We call them σ -*algebra*, λ -*system* and *monotone family generated by \mathcal{F}* , respectively.

Theorem 1.4 (Dynkin's π - λ lemma). Let X be a set and \mathcal{P} be a π -system of its subsets. Then $\lambda(\mathcal{P}) = \sigma(\mathcal{P})$.

For the proof we need the following result.

Lemma 1.4.1. Let \mathcal{L} be a λ -system. Then for every $A \in \mathcal{L}$ family

$$\mathcal{L}_A = \{B \in \mathcal{L} \mid A \cap B \in \mathcal{L}\}$$

is a λ -system.

Proof of the lemma. Since $A \in \mathcal{L}$, we have $X \in \mathcal{L}_A$. Suppose now that $B \in \mathcal{L}_A$. Then $A \cap B \in \mathcal{L}$. Since $X \setminus A \in \mathcal{L}$, we derive that also $(A \cap B) \cup (X \setminus A) \in \mathcal{L}$ and hence

$$A \cap (X \setminus B) = X \setminus ((A \cap B) \cup (X \setminus A)) \in \mathcal{L}$$

Thus $X \setminus B \in \mathcal{L}_A$. Finally note that \mathcal{L}_A is closed under countable disjoint unions. □

Proof of the theorem. Fix $A \in \mathcal{P}$. Define \mathcal{L}_A as in Lemma 1.4.1 with $\mathcal{L} = \lambda(\mathcal{P})$. Then \mathcal{L}_A is a λ -system. Moreover, \mathcal{L}_A contains \mathcal{P} . Hence $\mathcal{L}_A = \lambda(\mathcal{P})$. This shows that $\lambda(\mathcal{P})$ is closed under intersections with members of \mathcal{P} . Now fix $A \in \lambda(\mathcal{P})$ and define \mathcal{L}_A as in Lemma 1.4.1 with $\mathcal{L} = \lambda(\mathcal{P})$. Then $\mathcal{P} \subseteq \mathcal{L}_A$ and \mathcal{L}_A is a λ -system. Thus $\mathcal{L}_A = \lambda(\mathcal{P})$. This proves that $\lambda(\mathcal{P})$ is a π -system. A π -system that is simultaneously a λ -system is a σ -algebra. Thus $\sigma(\mathcal{P}) \subseteq \lambda(\mathcal{P})$. Since it is clear that $\lambda(\mathcal{P}) \subseteq \sigma(\mathcal{P})$, we derive that $\lambda(\mathcal{P}) = \sigma(\mathcal{P})$. □

Theorem 1.5 (Halmos's lemma on monotone classes). *Let X be a set and \mathcal{A} be an algebra of its subsets. Then $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$.*

For the proof we need the following easy results. Their proofs are left to the reader.

Lemma 1.5.1. *Let \mathcal{M} be a monotone family. Then for every $A \in \mathcal{M}$ family*

$$\mathcal{M}_A = \{B \in \mathcal{M} \mid A \cap B \in \mathcal{M}\}$$

is monotone.

Lemma 1.5.2. *Let \mathcal{M} be a monotone family. Then a family*

$$\mathcal{M}^c = \{A \in \mathcal{M} \mid X \setminus A \in \mathcal{M}\}$$

is monotone.

Proof of the theorem. Fix $A \in \mathcal{A}$. Define \mathcal{M}_A as in Lemma 1.5.1 with $\mathcal{M} = \mathcal{M}(\mathcal{A})$. Then \mathcal{M}_A is a monotone family. Moreover, \mathcal{M}_A contains \mathcal{A} . Hence $\mathcal{M}_A = \mathcal{M}(\mathcal{A})$. This shows that $\mathcal{M}(\mathcal{A})$ is closed under intersections with members of \mathcal{A} . Now fix $A \in \mathcal{M}(\mathcal{A})$ and define \mathcal{M}_A as in Lemma 1.5.1 with $\mathcal{M} = \mathcal{M}(\mathcal{A})$. Then $\mathcal{A} \subseteq \mathcal{M}_A$ and \mathcal{M}_A is a monotone family. Thus $\mathcal{M}_A = \mathcal{M}(\mathcal{A})$. This proves that $\mathcal{M}(\mathcal{A})$ is closed under finite intersections. According to Lemma 1.5.2 we derive that $\mathcal{M}(\mathcal{A})^c$ is a monotone family and contains \mathcal{A} . Hence $\mathcal{M}(\mathcal{A})^c = \mathcal{M}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$ is closed under complements. Therefore, $\mathcal{M}(\mathcal{A})$ is a σ -algebra. Thus $\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$. Since it is clear that $\mathcal{M}(\mathcal{A}) \subseteq \sigma(\mathcal{A})$, we derive that $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$. \square

2. MEASURABLE SPACES AND MEASURES

Definition 2.1. A pair (X, Σ) consisting of a set X together with a σ -algebra Σ of its subsets is called a *measurable space*.

Definition 2.2. Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces. A function $f : X_1 \rightarrow X_2$ is called a *measurable map* if $f^{-1}(A) \in \Sigma_1$ for every $A \in \Sigma_2$.

Measurable spaces and their morphisms form a category.

Definition 2.3. Let X be a set and Σ be an algebra of its subsets. A function $\mu : \Sigma \rightarrow [0, +\infty]$ such that $\mu(\emptyset) = 0$ and

$$\mu\left(\bigcup_{n=0}^m A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

for every family $\{A_n\}_{0 \leq n \leq m}$ of pairwise disjoint subsets in Σ is called an *additive function*. If in addition μ satisfies

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

for every family $\{A_n\}_{n \in \mathbb{N}}$ of pairwise disjoint subsets in Σ such that $\bigcup_{n \in \mathbb{N}} A_n \in \Sigma$, then μ is called a σ -*additive function*. Moreover, if $\mu : \Sigma \rightarrow [0, +\infty]$ is a σ -additive function and Σ is a σ -algebra, then μ is called a *measure*.

Definition 2.4. A tuple (X, Σ, μ) consisting of a measurable space (X, Σ) and a measure $\mu : \Sigma \rightarrow [0, +\infty]$ is called a *space with measure*.

Definition 2.5. Let (X, Σ, μ) be a space with measure. We say that it is *finite* if $\mu(X)$ is finite. We say that it is σ -*finite* if there exists a sequence $\{X_n\}_{n \in \mathbb{N}}$ of subsets of Σ such that $\mu(X_n)$ is finite for every $n \in \mathbb{N}$ and $X = \bigcup_{n \in \mathbb{N}} X_n$.

Theorem 2.6. Let (X, Σ) be a measurable space and $\mu_1, \mu_2 : \Sigma \rightarrow [0, +\infty]$ be measures such that $\mu_1(X) = \mu_2(X)$ is finite. Suppose that \mathcal{P} is a π -system of subsets of X such that $\Sigma = \sigma(\mathcal{P})$ and $\mu_1(A) = \mu_2(A)$ for every $A \in \mathcal{P}$. Then $\mu_1 = \mu_2$.

Proof. Define $\mathcal{F} = \{A \in \Sigma \mid \mu_1(A) = \mu_2(A)\}$. Straightforward verification shows that \mathcal{F} is a λ -system. By assumption $\mathcal{P} \subseteq \mathcal{F}$. Therefore, $\lambda(\mathcal{P}) \subseteq \mathcal{F}$. By Theorem 1.5 we deduce that $\Sigma = \sigma(\mathcal{P}) = \lambda(\mathcal{P}) \subseteq \mathcal{F} \subseteq \Sigma$. Hence $\mathcal{F} = \Sigma$. \square

Definition 2.7. Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be spaces with measures. A function $f : X_1 \rightarrow X_2$ is called a *morphism of spaces with measures* if f is a morphism of measurable spaces and for every $A \in \Sigma_2$ we have equality $\mu_2(A) = \mu_1(f^{-1}(A))$.

Spaces with measures and their morphisms form a category.

3. OUTER MEASURES AND CARATHÉODORY'S CONSTRUCTION

Definition 3.1. Let X be a set and $\mu^* : \mathcal{P}(X) \rightarrow [0, +\infty]$ be a function. Suppose that $\mu^*(\emptyset) = 0$, $\mu^*(A) \leq \mu^*(B)$ for every subset A of a set B contained in X and

$$\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)$$

for every family $\{A_n\}_{n \in \mathbb{N}}$ of subsets of X . Then we say that μ^* is an *outer measure* on X .

Theorem 3.2 (Carathéodory's construction). *Let X be a set and $\mu^* : \mathcal{P}(X) \rightarrow [0, +\infty]$ be an outer measure on X . We define a family of sets $\Sigma_{\mu^*} \subseteq \mathcal{P}(X)$ by condition*

$$A \in \Sigma_{\mu^*} \Leftrightarrow \forall E \subseteq X \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$$

Then the following assertions hold.

- (1) Σ_{μ^*} is an σ -algebra of subsets of X .
- (2) For every family $\{A_n\}_{n \in \mathbb{N}}$ of pairwise disjoint subsets of Σ_{μ^*} and every subset E of X we have

$$\mu^*\left(E \cap \bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu^*(E \cap A_n)$$

*In particular, $\mu^*_{|\Sigma_{\mu^*}}$ is a measure.*

- (3) Every subset A of X such that $\mu^*(A) = 0$ is contained in Σ_{μ^*} . In particular, $\mu^*_{|\Sigma_{\mu^*}}$ is complete.

The proof is encapsulated in two lemmas.

Lemma 3.2.1. Σ_{μ^*} is an algebra of sets.

Proof of the lemma. Clearly $\emptyset \in \Sigma_{\mu^*}$ and $A \in \Sigma_{\mu^*} \Leftrightarrow X \setminus A \in \Sigma_{\mu^*}$. It suffices to prove that Σ_{μ^*} is closed under unions. For a subset B of X we denote $X \setminus B$ by B^c . Now assume that $A_1, A_2 \in \Sigma_{\mu^*}$ and pick a subset E of X . Then

$$\mu^*(E) = \mu^*(E \cap A_1) + \mu^*(E \cap A_1^c) = \mu^*(E \cap A_1) + \mu^*(E \cap A_1^c \cap A_2) + \mu^*(E \cap A_1^c \cap A_2^c)$$

Since we have equalities

$$E \cap A_1 = (E \cap (A_1 \cup A_2)) \cap A_1, \quad E \cap A_1^c \cap A_2 = (E \cap (A_1 \cup A_2)) \cap A_1^c$$

we derive that $\mu^*(E \cap (A_1 \cup A_2)) = \mu^*(E \cap A_1) + \mu^*(E \cap A_1^c \cap A_2)$. Similarly we have equality

$$E \cap A_1^c \cap A_2^c = E \cap (A_1 \cup A_2)^c$$

and hence $\mu^*(E \cap A_1^c \cap A_2^c) = \mu^*(E \cap (A_1 \cup A_2)^c)$. Therefore, we have

$$\mu^*(E) = \mu^*(E \cap (A_1 \cup A_2)) + \mu^*(E \cap (A_1 \cup A_2)^c)$$

Thus we proved that $A_1 \cup A_2 \in \Sigma_{\mu^*}$. Therefore, Σ_{μ^*} is a family of subsets of X closed under finite unions, complements and containing \emptyset . Thus Σ_{μ^*} is an algebra of sets. \square

Lemma 3.2.2. Let $\{A_n\}_{n \in \mathbb{N}}$ be a family of pairwise disjoint subsets of Σ_{μ^*} . Then $\bigcup_{n \in \mathbb{N}} A_n \in \Sigma_{\mu^*}$ and for every subset E of X there is an equality

$$\mu^* \left(E \cap \bigcup_{n \in \mathbb{N}} A_n \right) = \sum_{n \in \mathbb{N}} \mu^*(E \cap A_n)$$

Proof of the lemma. We prove that $\bigcup_{n \in \mathbb{N}} A_n \in \Sigma_{\mu^*}$. For this observe that we have

$$\begin{aligned} \mu^*(E) &\leq \mu^* \left(E \cap \bigcup_{n \in \mathbb{N}} A_n \right) + \mu^* \left(E \setminus \bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n \in \mathbb{N}} \mu^*(E \cap A_n) + \mu^* \left(E \setminus \bigcup_{n \in \mathbb{N}} A_n \right) = \\ &= \lim_{m \rightarrow +\infty} \left(\mu^* \left(E \cap \bigcup_{n=0}^m A_n \right) + \mu^* \left(E \setminus \bigcup_{n=0}^m A_n \right) \right) \leq \lim_{m \rightarrow +\infty} \left(\mu^* \left(E \cap \bigcup_{n=0}^m A_n \right) + \mu^* \left(E \setminus \bigcup_{n=0}^m A_n \right) \right) = \mu^*(E) \end{aligned}$$

and the last equality holds, since $\bigcup_{n=0}^m A_n \in \Sigma_{\mu^*}$ by Lemma 3.2.1. This implies that we have equalities everywhere above. Hence

$$\mu^* \left(E \cap \bigcup_{n \in \mathbb{N}} A_n \right) = \sum_{n \in \mathbb{N}} \mu^*(E \cap A_n)$$

and $\bigcup_{n \in \mathbb{N}} A_n \in \Sigma_{\mu^*}$. □

Proof of the theorem. Lemma 3.2.1 and Lemma 3.2.2 imply that Σ_{μ^*} is a σ -algebra and statement (2) holds. It suffices to verify that statement (3) holds. For this pick a subset A of X such that $\mu^*(A) = 0$. Then for every subset E of X we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \setminus A) = \mu^*(E \setminus A) \leq \mu^*(E)$$

Hence $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$ and thus $A \in \Sigma_{\mu^*}$. □

Next result is a general tool of constructing measures.

Theorem 3.3 (Carathéodory extension). Let X be a set and Σ be some algebra of its subsets. Suppose that $\mu : \Sigma \rightarrow [0, +\infty]$ is a σ -additive function. Now for every subset A in X we define

$$\mu^*(A) = \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_n) \mid A_n \in \Sigma \text{ for every } n \in \mathbb{N} \text{ and } A \subseteq \bigcup_{n \in \mathbb{N}} A_n \right\}$$

Then $\mu^* : \mathcal{P}(X) \rightarrow [0, +\infty]$ is an outer measure, $\Sigma \subseteq \Sigma_{\mu^*}$ and $\mu^*_{|\Sigma} = \mu$. Moreover, if $\mu(X)$ is finite, then $\mu^*_{|\sigma(\Sigma)}$ is a unique extension of μ to a measure on $\sigma(\Sigma)$.

Proof. Standard verification shows that μ^* is an outer measure. Note that

$$\mu^*(A) = \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_n) \mid \{A_n\}_{n \in \mathbb{N}} \text{ is a family of pairwise disjoint subsets of } \Sigma \text{ and } A \subseteq \bigcup_{n \in \mathbb{N}} A_n \right\}$$

for every subset A of X . Let A be element of Σ and let E be an arbitrary subset of X . Fix $\epsilon > 0$. By the remark above there exists a family $\{A_n\}_{n \in \mathbb{N}}$ of pairwise disjoint elements of Σ such that

$$E \subseteq \bigcup_{n \in \mathbb{N}} A_n, \quad \sum_{n \in \mathbb{N}} \mu(A_n) \leq \mu^*(E) + \epsilon$$

By definition of μ^* we have $\mu^*(E \cap A) \leq \sum_{n \in \mathbb{N}} \mu(A_n \cap A)$, $\mu^*(E \setminus A) \leq \sum_{n \in \mathbb{N}} \mu(A_n \setminus A)$ and hence

$$\begin{aligned} \mu^*(E) &\leq \mu^*(E \cap A) + \mu^*(E \setminus A) \leq \sum_{n \in \mathbb{N}} \mu(A_n \cap A) + \sum_{n \in \mathbb{N}} \mu(A_n \setminus A) = \\ &= \sum_{n \in \mathbb{N}} (\mu(A_n \cap A) + \mu(A_n \setminus A)) = \sum_{n \in \mathbb{N}} \mu(A_n) \leq \mu^*(E) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we derive that $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$ and hence $A \in \Sigma_{\mu^*}$. Thus $\Sigma \subseteq \Sigma_{\mu^*}$. Once again fix $A \in \Sigma$. Then for every family $\{A_n\}_{n \in \mathbb{N}}$ of pairwise disjoint elements of Σ such that $A \subseteq \bigcup_{n \in \mathbb{N}} A_n$ we have $\mu(A) = \sum_{n \in \mathbb{N}} \mu(A_n \cap A) \leq \sum_{n \in \mathbb{N}} \mu(A_n)$ and thus $\mu(A) \leq \mu^*(A)$. Obviously $\mu^*(A) \leq \mu(A)$. Therefore, for every $A \in \Sigma$ we have $\mu(A) = \mu^*(A)$. Together with

$\Sigma \subseteq \Sigma_{\mu^*}$ this implies that $\mu^*_{|\sigma(\Sigma)}$ is a measure that extends μ . Now we prove the uniqueness of extension under the assumption that $\mu(X)$ is finite. This follows immediately from Theorem 2.6. \square

4. OUTER METRIC MEASURES

Definition 4.1. Let X be a topological space. The σ -algebra \mathcal{B}_X generated by all open sets of X is called the σ -algebra of Borel subsets of X .

Definition 4.2. Let (X, d) be a metric space and $\mu^* : \mathcal{P}(X) \rightarrow [0, +\infty]$ be an outer measure. We say that μ^* is a *metric outer measure* if

$$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$$

for any two subsets E_1, E_2 of X with $\text{dist}(E_1, E_2) = \inf_{x_1 \in E_1, x_2 \in E_2} d(x_1, x_2) > 0$.

Theorem 4.3 (Carathéodory). Let (X, d) be a metric space and $\mu^* : \mathcal{P}(X) \rightarrow [0, +\infty]$ be an outer metric measure on X . Then the σ -algebra \mathcal{B}_X of Borel subsets of X is contained in Σ_{μ^*} .

Proof. Let U be an open subset of X . Define $F = X \setminus U$ and $U_n = \{x \in X \mid \text{dist}(x, F) > \frac{1}{2^n}\}$ for $n \in \mathbb{N}$. Then $\{U_n\}_{n \in \mathbb{N}}$ form an ascending family of open sets and $U = \bigcup_{n \in \mathbb{N}} U_n$. Fix now a subset E of X such that $\mu^*(E) \in \mathbb{R}$. We define $E_n = E \cap U_n$ for every $n \in \mathbb{N}$. Since μ^* is an outer metric measure, we derive that

$$\mu^*\left(\bigcup_{n=0}^m E_{2n+1} \setminus E_{2n}\right) = \sum_{n=0}^m \mu^*(E_{2n+1} \setminus E_{2n}), \quad \mu^*\left(\bigcup_{n=1}^m E_{2n} \setminus E_{2n-1}\right) = \sum_{n=1}^m \mu^*(E_{2n} \setminus E_{2n-1})$$

for every positive integer m . Thus we derive

$$\sum_{n \in \mathbb{N}} \mu^*(E_{2n+1} \setminus E_{2n}) \leq \mu^*(E) \in \mathbb{R}, \quad \sum_{n \in \mathbb{N}} \mu^*(E_{2n} \setminus E_{2n-1}) \leq \mu^*(E) \in \mathbb{R}$$

Hence we have $\sum_{n \in \mathbb{N}} \mu^*(E_{n+1} \setminus E_n) \leq 2 \cdot \mu^*(E) \in \mathbb{R}$. Using the fact that μ^* is an outer measure, we derive that

$$\mu(E_m) \leq \mu^*(E \cap U) \leq \mu^*(E_m) + \sum_{n \geq m} \mu^*(E_{n+1} \setminus E_n)$$

for every $m \in \mathbb{N}$. Hence these inequalities yield $\lim_{m \rightarrow +\infty} \mu^*(E_m) = \mu^*(E \cap U)$. Now we have $\mu^*(E_m) + \mu^*(E \setminus U) \leq \mu^*(E) \leq \mu^*(E \cap U) + \mu^*(E \setminus U)$ for every $m \in \mathbb{N}$. The first inequality holds due to the fact that μ^* is an outer metric measure. We derive that $\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$. Note that if $\mu^*(E) = +\infty$, then inequality $\mu^*(E) \leq \mu^*(E \cap U) + \mu^*(E \setminus U)$ must be equality. Hence for every subset E of X we have $\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$. This implies that $U \in \Sigma_{\mu^*}$. Since U is an arbitrary open subset of X , we deduce that $\mathcal{B}_X \subseteq \Sigma_{\mu^*}$. \square