LOCALLY RINGED SPACES

1. Introduction

In this notes we study ringed and locally ringed spaces. Our main results concern existence and construction of colimits in these categories.

2. The category of locally ringed spaces

Definition 2.1. Let X be a topological space and $\mathcal{O}_X$ be a sheaf of commutative rings on X. A pair $(X, \mathcal{O}_X)$ is called *a ringed space*.

Definition 2.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. A pair $(f, f^{\#})$ consisting of a continuous map $f : X \to Y$ and a morphism $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings is called *a morphism of ringed spaces*.

Suppose that $(f, f^{\#} : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, $(g, g^{\#}) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ are morphisms of ringed spaces. Then we have the composition

$$(g, g^{\#}) \cdot (f, f^{\#}) = \big(g \cdot f, (g_* f^{\#}) \cdot g^{\#}\big) : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$$

We have the category **Rs** of ringed spaces.

Remark 2.3. The category **Rs** has all small colimits. We describe them now. We start with coproducts. Suppose that $\{(X_i, \mathcal{O}_{X_i})\}_{i \in I}$ is a family of ringed spaces. Let $\coprod_{i \in I} X_i$ be their coproduct (disjoint sum) in the category of topological spaces and for every $i \in I$ let $u_i : X_i \to \coprod_{i \in I} X_i$ be the canonical topological immersion. Next let $u_i^{\#} : \prod_{i \in I} (u_i)_* \mathcal{O}_{X_i} \to (u_i)_* \mathcal{O}_{X_i}$ be the projection on the i-th factor. Then we define a ringed space and family of morphisms of ringed spaces by

$$(X, \mathcal{O}_X) = \left(\coprod_{i \in I} X_i, \prod_{i \in I} (u_i)_* \mathcal{O}_{X_i}\right), \left\{(u_i, u_i^{\#}) : (X_i, \mathcal{O}_{X_i}) \to \left(\coprod_{i \in I} X_i, \prod_{i \in I} (u_i)_* \mathcal{O}_{X_i}\right)\right\}_{i \in I} = (X, \mathcal{O}_X)$$

This is a coproduct of $\{(X_i, \mathcal{O}_{X_i})\}_{i \in I}$ in the category of ringed spaces. Now we describe cokernels. Consider the diagram

$$(R, \mathcal{O}_R) \underset{(q, q^{\#})}{\overset{(p, p^{\#})}{\rightrightarrows}} (X, \mathcal{O}_X)$$

of ringed spaces. Let $f : X \to Y$ be a cokernel of the pair (p, q) in the category of topological spaces and consider the kernel

$$\mathcal{O}_Y \xrightarrow{f^{\#}} f_*\mathcal{O}_X \underset{f_* q^{\#}}{\overset{f_* p^{\#}}{\rightrightarrows}} f_* p_* \mathcal{O}_R = f_* q_* \mathcal{O}_R$$

in the category of sheaves of rings on Y. Then $(Y, \mathcal{O}_Y)$ together with $(f, f^{\#})$ is a cokernel of the pair $\big((p, p^{\#}), (q, q^{\#})\big)$ in the category of ringed spaces.

Definition 2.4. Let $(X, \mathcal{O}_X)$ be a ringed space such that for every x in X ring $\mathcal{O}_{X,x}$ is local. Then $(X, \mathcal{O}_X)$ is called *a locally ringed space*.

Let X be a ringed space. Suppose that U is an open subset of X and $s \in \Gamma(U, \mathcal{O}_X)$ is a section. Then we define

$$U_s = \{x \in U \,|\, s_{|x} \text{ is invertible in } \mathcal{O}_{X,x}\}$$

Fact 2.5. *Let X be a ringed space. Then the following assertions are equivalent.*

(i) *X is a locally ringed space.*

(ii) *For every open subset U of X and every section $s \in \Gamma(U, \mathcal{O}_X)$ we have*

$$U = U_s \cup U_{(1-s)}$$

Proof. We prove **(i)** $\Rightarrow$ **(ii)**. Assume **(i)** and pick an open subset U of X together with a section $s \in \Gamma(U, \mathcal{O}_X)$. For every x in U ring $\mathcal{O}_{X,x}$ is local and hence at least one $s_{|x}, (1-s)_{|x}$ is invertible in $\mathcal{O}_{X,x}$. This implies that

$$U = U_s \cup U_{(1-s)}$$

We prove **(ii)** $\Rightarrow$ **(i)**. Assume **(ii)** and pick x in X and also $r \in \mathcal{O}_{X,x}$. Then there exists an open neighborhood U of x and an element $s \in \Gamma(U, \mathcal{O}_X)$ such that $r = s_{|x}$. Since $U = U_s \cup U_{(1-s)}$ by **(ii)**, we deduce that at least one $s_{|x}, (1-s)_{|x}$ is invertible in $\mathcal{O}_{X,x}$. This means that at least one $r, 1-r$ is invertible in $\mathcal{O}_{X,x}$. Therefore, $\mathcal{O}_{X,x}$ is local ring. □

Definition 2.6. Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be locally ringed spaces and let $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If for every x in X the induced homomorphism $f^\# : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is local, then $(f, f^\#)$ is *a morphism of locally ringed spaces*.

Note that if : $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y), (Z, \mathcal{O}_Z)$ are locally ringed spaces $(f, f^\#), (g, g^\#)$ are morphisms of locally ringed spaces, then $(g, g^\#) \cdot (f, f^\#)$ is a morphism of locally ringed spaces. Indeed, for every x in X the composition $f^\# \cdot g^\# : \mathcal{O}_{Z,g(f(x))} \rightarrow \mathcal{O}_{X,x}$ is a local morphism and this is precisely the morphism induced on stalks by $(g_* f^\#) \cdot g^\# : \mathcal{O}_Z \rightarrow g_* f_* \mathcal{O}_X$. This implies that there exists a category **Lrs** of locally ringed spaces and their morphisms. Moreover, we have the inclusion functor **Lrs** $\hookrightarrow$ **Rs** that is not full.

Fact 2.7. *Let X, Y be a locally ringed spaces and $f : X \rightarrow Y$ be a morphism of ringed spaces. Then the following are equivalent.*

(i) *f is a morphism of locally ringed spaces.*

(ii) *For every open subset V of Y and every section $s \in \Gamma(V, \mathcal{O}_Y)$ we have*

$$f^{-1}(V)_{f^\#(s)} = f^{-1}(V_s)$$

(iii) *For every open subset V of Y and every section $s \in \Gamma(V, \mathcal{O}_Y)$ we have*

$$f^{-1}(V)_{f^\#(s)} \subseteq f^{-1}(V_s)$$

Proof. We prove **(i)** $\Rightarrow$ **(ii)**. Assume **(i)**. For this note that $x \in f^{-1}(V_s)$ if and only if $f(x) \in V_s$. This holds if and only if $s_{|f(x)}$ is invertible in $\mathcal{O}_{Y,f(x)}$. Since $f^\# : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is a local morphism by **(i)**, we derive that $s_{|f(x)}$ is invertible in $\mathcal{O}_{Y,f(x)}$ if and only if $f^\#(s)_{|x} = f^\#(s_{|f(x)})$ is invertible in $\mathcal{O}_{X,x}$. This is equivalent with $x \in f^{-1}(V)_{f^\#(s)}$.
The implication **(ii)** $\Rightarrow$ **(iii)** is clear.
Now we prove that **(iii)** $\Rightarrow$ **(i)**. Assume **(iii)**. Pick x in X and consider a morphism $f^\# : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$. Suppose that $r \in \mathcal{O}_{Y,f(x)}$ and $f^\#(r) \in \mathcal{O}_{X,x}$ is invertible. Then there exists an open neighborhood V of $f(x)$ in Y and a section $s \in \Gamma(V, \mathcal{O}_Y)$ such that $s_{|f(x)} = r$. Then $f^\#(r) = f^\#(s_{|f(x)}) = f^\#(s)_{|x}$ and hence $x \in f^{-1}(V)_{f^\#(s)}$. Thus $x \in f^{-1}(V_s)$ by **(iii)** and thus $f(x) \in V_s$. Thus means that $r = s_{|f(x)} \in \mathcal{O}_{Y,f(x)}$ is invertible. □

Definition 2.8. Let $f : X \rightarrow Y$ be a morphism of ringed spaces. We say that f is *an open immersion of ringed spaces* if f is an open immersion of topological spaces (in particular $f(X)$ is an open subspace of Y) and $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ induces an isomorphism $\mathcal{O}_{f(X)} \cong (f_* \mathcal{O}_X)_{|f(X)}$.

Theorem 2.9. *The inclusion functor* **Lrs** $\hookrightarrow$ **Rs** *creates all small colimits.*

Proof. Suppose that $\left\{(X_i, \mathcal{O}_{X_i})\right\}_{i\in I}$ is a family of locally ringed spaces. Then using notation of Remark 2.3 we have a coproduct in the category of ringed spaces

$$(X, \mathcal{O}_X) = \left(\coprod_{i\in I} X_i, \prod_{i\in I}(u_i)_*\mathcal{O}_{X_i}\right), \left\{(u_i, u_i^\#) : (X_i, \mathcal{O}_{X_i}) \to \left(\coprod_{i\in I} X_i, \prod_{i\in I}(u_i)_*\mathcal{O}_{X_i}\right)\right\}_{i\in I} = (X, \mathcal{O}_X)$$

Note that for every $i \in I$ morphism $(u_i, u_i^\#)$ is an open immersion of ringed spaces. Thus $(X, \mathcal{O}_X)$ is a locally ringed space and for every $i \in I$ morphism $(u_i, u_i^\#)$ is a morphism of locally ringed spaces. This shows that the inclusion functor **Lrs** $\hookrightarrow$ **Rs** creates coproducts.
Consider now the diagram

$$(R, \mathcal{O}_R) \overset{(p,p^\#)}{\underset{(q,q^\#)}{\rightrightarrows}} (X, \mathcal{O}_X)$$

of locally ringed spaces and their morphisms. Let

$$(R, \mathcal{O}_R) \overset{(p,p^\#)}{\underset{(q,q^\#)}{\rightrightarrows}} (X, \mathcal{O}_X) \xrightarrow{(f,f^\#)} (Y, \mathcal{O}_Y)$$

be the cokernel in the category of ringed spaces described in Remark 2.3. In order to show that the inclusion functor **Lrs** $\hookrightarrow$ **Rs** creates cokernels it suffices to check that $(Y, \mathcal{O}_Y)$ is a locally ringed space and $(f, f^\#)$ is a morphism of locally ringed spaces. For this pick an open subset V of Y and $s \in \Gamma(V, \mathcal{O}_Y)$. Then $p^\#(f^\#(s)) = q^\#(f^\#(s))$ by definition of $\mathcal{O}_Y$ and also $p^{-1}(f^{-1}(V)) = q^{-1}(f^{-1}(V))$ by definition of f. Hence

$$p^{-1}\left(f^{-1}(V)_{f^\#(s)}\right) = p^{-1}\left(f^{-1}(V)\right)_{p^\#(f^\#(s))} = q^{-1}\left(f^{-1}(V)\right)_{q^\#(f^\#(s))} = q^{-1}\left(f^{-1}(V)_{f^\#(s)}\right)$$

by Fact 2.7. Since f is a topological cokernel of (p, q), we derive that there exists an open subset W of V such that

$$f^{-1}(W) = f^{-1}(V)_{f^\#(s)}$$

Now $f^\#(s_{|W}) = f^\#(s)_{|f^{-1}(W)} = f^\#(s)_{|f^{-1}(V)_{f^\#(s)}}$ is an invertible element of $\Gamma(W, f_*\mathcal{O}_X)$. Denote its inverse by t. We have

$$f_*p^\#(f^\#(s)) \cdot f_*p^\#(t) = f_*p^\#(f^\#(s) \cdot t) = 1 = f_*q^\#(f^\#(s) \cdot t) = f_*q^\#(f^\#(s)) \cdot f_*q^\#(t)$$

and hence

$$f_*p^\#(t) = f_*q^\#(t)$$

This implies by definition of $\mathcal{O}_Y$ that there exists $r \in \Gamma(W, \mathcal{O}_Y)$ such that $f^\#(r) = t$. Now

$$1 = f^\#(s) \cdot t = f^\#(s \cdot r)$$

and since $f^\#$ is injective, we derive that r is an inverse of s in $\Gamma(W, \mathcal{O}_Y)$. Thus s is an invertible element of $\Gamma(W, \mathcal{O}_Y)$. Hence $W \subseteq V_s$. Since $f^{-1}(W) = f^{-1}(V)_{f^\#(s)}$ and f is surjective, we derive that $W = V_s$ and

$$f^{-1}(V_s) = f^{-1}(V)_{f^\#(s)}$$

This holds for every section of $\mathcal{O}_Y$ on V. This property together with Fact 2.5 applied to a locally ringed space X yield

$$f^{-1}\left(V_s \cup V_{(1-s)}\right) = f^{-1}(V_s) \cup f^{-1}(V_{(1-s)}) = f^{-1}(V)_{f^\#(s)} \cup f^{-1}(V)_{1-f^\#(s)} = f^{-1}(V)$$

Again since f is surjective, we deduce that $V = V_s \cup V_{(1-s)}$. By Fact 2.5 we deduce that Y is a locally ringed space. Next by Fact 2.7 and equality

$$f^{-1}(V_s) = f^{-1}(V)_{f^\#(s)}$$

we deduce that $(f, f^\#)$ is a morphism of locally ringed spaces. $\square$

Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be ringed spaces and $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be their morphism. From now by the usual abuse of notation we say that X, Y are ringed spaces and $f : X \to Y$ is their morphism.
Suppose now that X is a locally ringed space. For every x in X we denote by $\mathfrak{m}_x$ the unique maximal ideal of $\mathcal{O}_{X,x}$ and we denote by $k(x)$ the field $\mathcal{O}_{X,x}/\mathfrak{m}_x$. Next if U is an open neighborhood of x and $s \in \Gamma(U, \mathcal{O}_X)$ is a section, then we define $s(x) \in k(x)$ as an element

$$s_{|x} \bmod \mathfrak{m}_x \in k(x)$$

In particular, for every open subset U of X and a section $s \in \Gamma(U, \mathcal{O}_X)$ we have

$$U_s = \{x \in U \,|\, s(x) \neq 0\}$$

Definition 2.10. Consider a pair of morphism of locally ringed spaces

$$\begin{tikzcd} R \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & X \end{tikzcd}$$

and let U be an open subset of X. If $p^{-1}(U) = q^{-1}(U)$, then U is *a saturated subset for a pair* (p, q).

Corollary 2.11. *Consider a cokernel*

$$\begin{tikzcd} R \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

in the category of locally ringed spaces. Suppose that U is an open subset of X saturated with respect to (p, q). Then $f(U)$ is open in Y and the diagram

$$\begin{tikzcd} p^{-1}(U) = q^{-1}(U) \arrow[r, shift left, "p_U"] \arrow[r, shift right, "q_U"'] & U \arrow[r, "f_U"] & f(U) \end{tikzcd}$$

is a cokernel of (p_U, q_U), where $f_U : U \to f(U)$ is induced by f and p_U, q_U are induced by p, q, respectively.

Proof. The proof follows from Theorem 2.9 and the construction of cokernels in **Rs** descibed by Remark 2.3. □

3. RECOLLEMENT OF RINGED SPACES

Definition 3.1. Let X, R be objects and let $p, q : R \to X$ be morphism in some category $\mathcal{C}$. Suppose that for every object Y of $\mathcal{C}$ we have injective map

$$\mathrm{Mor}_{\mathcal{C}}(Y, R) \xhookrightarrow{\langle \mathrm{Mor}_{\mathcal{C}}(1_Y, p_1), \mathrm{Mor}_{\mathcal{C}}(1_Y, p_2) \rangle} \mathrm{Mor}_{\mathcal{C}}(Y, X) \times \mathrm{Mor}_{\mathcal{C}}(Y, X)$$

that exhibits $\mathrm{Mor}_{\mathcal{C}}(Y, R)$ as an equivalence relation in $\mathrm{Mor}_{\mathcal{C}}(Y, X) \times \mathrm{Mor}_{\mathcal{C}}(Y, X)$. Then (R, p, q) is called *an equivalence relation in $\mathcal{C}$*.

The next theorem is a categorical reformulation of the *recollement* technique [GD71, Chapitre 0, 4.1.7].

Theorem 3.2. *Let $X = \coprod_{i \in I} X_i$, $R = \coprod_{i,j \in I} R_{ij}$ be disjoint sums of ringed spaces and let $p, q : R \to X$ be morphisms of ringed spaces such that the following assertions hold.*

(1) *For any $i, j \in I$ morphism $p_{|R_{ij}}$ induces an open immersion $R_{ij} \hookrightarrow X_i$ and morphism $q_{|R_{ij}}$ induces an open immersion $R_{ij} \hookrightarrow X_j$.*

(2) *For every $i \in I$ morphisms $p_{|R_{ii}}$ and $q_{|R_{ii}}$ are equal and induce an isomorphisms $R_{ii} \to X_i$.*

(3) *Triple (R, p, q) is an equivalence relation on X in the category of ringed spaces over k.*

Let

$$R \overset{p}{\underset{q}{\rightrightarrows}} X \xrightarrow{f} Y$$

be a cokernel of a pair (p,q) *in the category of ringed spaces. Then* f *induces an isomorphism of ringed spaces* $X_i \cong f(X_i)$ *for every* $i \in I$.

Lemma 3.2.1. *Consider the assumptions as above. Suppose that in addition there exists* $i \in I$ *such that morphism* p *induces an isomorphism* $R_{ji} \cong X_j$ *for every* $j \in I$. *Then* $Y = f(X_i)$ *and* f *induces an isomorphism of ringed spaces* $X_i \cong Y$.

Proof of the lemma. For every $j \in I$ let $p_{ji} : R_{ji} \to X_j$ be an isomorphism induced by p and let $q_{ji} : R_{ji} \to X_i$ be an open immersion induced by q. For every $j \in I$ we define $g_j : X_j \to X_i$ by formula $q_{ji} \cdot p_{ji}^{-1}$. Next we define $g : X \to X_i$ such that $g_{|X_j} = g_j$. We have $g \cdot p = g \cdot q$. Suppose now that Z is a ringed space and $h : X \to Z$ is a morphism of ringed spaces such that $h \cdot p = h \cdot q$. We denote $h_{|X_j}$ by h_j for every $j \in I$. Then $h_j \cdot p_{ji} = h_i \cdot q_{ji}$ and hence $h_j = h_i \cdot (q_{ij} \cdot p_{ij}^{-1}) = h_i \cdot g_j$ for every $j \in I$. Thus we have $h = h_i \cdot g$ and h_i is a unique morphism of ringed spaces with this property. Therefore, if such $i \in I$ exists, then

$$R \overset{p}{\underset{q}{\rightrightarrows}} X \xrightarrow{g} X_i$$

is a cokernel in the category of ringed spaces. Moreover, substituting f for h in our discussion. We deduce that $f_i = f_{|X_i} : X_i \to Y$ is a unique morphism of ringed spaces such that $f_i \cdot g = f$. Since both $g : X \to X_i$ and $f : X \to Y$ are cokernels of the same pair (p,q), we derive that f_i is an isomorphism. Thus f induces an isomorphism of ringed spaces $X_i \cong Y$. □

Proof of the theorem. We first prove that f is topologically open map. For this consider an open subset U of X. Then

$$f^{-1}(f(U)) = p(q^{-1}(U))$$

and since p is an open continuous map, we derive that $f^{-1}(f(U))$ is open. By Remark 2.3 f is quotient map of topologically spaces and hence $f(U)$ is open in Y.

Next fix $i \in I$ and note that f induces a homeomorphism $X_i \cong f(X_i)$. In order to show that f induces an isomorphism of ringed spaces $X_i \cong f(X_i)$ by Proposition 2.11 we may restrict diagram

$$R \overset{p}{\underset{q}{\rightrightarrows}} X \xrightarrow{f} Y$$

to the open subset $f^{-1}(f(x_i))$ saturated with respect to p, q. Now our claim follows by Lemma 3.2.1. □

References

[GD71] Alexander Grothendieck and Jean Dieudonné. *Éléments de géométrie algébrique I*, volume 166 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, 1971.