CATEGORIES OF PRESHEAVES

1. Introduction – set theoretical background

These notes deal with categories of presheaves. For some arguments and also for the underlying set-theoretic setup we use Grothendieck universes [ML98, page 22]. This implies that our arguments rely on tools that go beyond the usual Zermelo-Frankel axioms. Grothendieck universes can be defined within Zermelo-Frankel set theory as follows.

Definition 1.1. Let U be a set. We say that U is *Grothendieck universe* if the following conditions are satisfied.

(1) The set ω of finite ordinals in the sense of von Neumann is an element of U.

(2) If $y \in U$ and $x \in y$, then $y \in U$.

(3) If $x \in U$, then $\mathcal{P}(x) \in U$ and $\bigcup x \in U$.

(4) If $x \in U$, $y \subseteq U$ and $f : x \to y$ is surjective, then $y \in U$.

If U is a Grothendieck universe, then the pair $(U, \in)$ forms a model for Zermelo-Frankel theory. This implies that the existence of Grothendieck universes is independent from Zermelo-Frankel axioms. In this notes we extend the usual Zermelo-Frankel system by adding the following Tarski axiom.

Every set is an element of some Grothendieck universe.

This new formal system is called *Tarski-Grothendieck set theory*. Let U be a Grothendieck universe. We denote by $\mathbf{Set}_U$ a category whose objects are elements of U and whose morphisms are maps of sets.

Definition 1.2. Let U be a Grothendieck universe. A category $\mathcal{C}$ is *U-small* if classes of objects and morphisms of $\mathcal{C}$ are members of U.

Definition 1.3. Let U be a Grothendieck universe. A category $\mathcal{C}$ is *locally U-small* if for any pair X, Y of objects of $\mathcal{C}$ we have $\mathrm{Mor}_{\mathcal{C}}(X, Y) \in U$.

Throughout this notes we fix a Grothendieck universe U. Elements of U are called sets. We use term *class* for arbitrary sets (also these ones outside U). We denote $\mathbf{Set}_U$ by $\mathbf{Set}$. By (locally) small category we mean (locally) U-small category.

2. Creation of limits and colimits

Definition 2.1. Let $F : \mathcal{C} \to \mathcal{X}$, $D : I \to \mathcal{C}$ be functors. Suppose that $(X, \{f_i\}_{i\in I})$ is a cone in $\mathcal{X}$ for the composition $F \cdot D$. We say that a cone $(Z, \{g_i\}_{i\in I})$ in $\mathcal{C}$ for D is *a lift of* $(X, \{f_i\}_{i\in I})$ if $F(Z) = X$ and $F(g_i) = f_i$ for every $i \in I$.

Definition 2.2. Let $F : \mathcal{C} \to \mathcal{X}$, $D : I \to \mathcal{C}$ be functors. We say that F *creates limits for D* if every limiting cone for $F \cdot D$ has a unique lift to a cone for D and this lift is a limiting cone for D.

Definition 2.3. Let $F : \mathcal{C} \to \mathcal{X}$ be a functor. We say that:

(1) *F creates limits* if F creates limits for all functors $D : I \to \mathcal{C}$.

(2) *F creates small limits* if F creates limits for all functors $D : I \to \mathcal{C}$ with I being small category.

(3) *F creates finite limits* if F creates limits for all functors $D : I \to \mathcal{C}$ with I being category with finitely many objects and arrows.

Some extra material on creation of limits can be found in [ML98, V.1]. By the usual arrow inverting one defines the notion of creation of colimits.

Now we prove an important result. First we need to introduce some notation. Suppose that $\mathcal{C}$ and $\mathcal{X}$ are categories. Then we denote by $\mathbf{Fun}\,(\mathcal{C},\mathcal{X})$ the category with functors $\mathcal{C} \to \mathcal{X}$ as objects and natural transformations between them as morphisms. We also denote by $|\mathcal{C}|$ the category having the same objects as $\mathcal{C}$ but with only identities as a morphism. There exists the canonical functor $|\mathcal{C}| \to \mathcal{C}$ that induces identity map on objects. The next result describes limits and colimits in functor categories.

Theorem 2.4. *Let $\mathcal{C}$, $\mathcal{X}$ be a categories. Then the functor $\mathbf{Fun}(\mathcal{C},\mathcal{X}) \to \mathbf{Fun}(|\mathcal{C}|,\mathcal{X})$ induced by the precomposition with the functor $|\mathcal{C}| \to \mathcal{C}$ creates all limits and colimits.*

Proof. We prove that this functor creates limits. Creation of colimits can be handled similarly. Let I be a category. For every object i in I consider a functor $F_i : \mathcal{C} \to \mathcal{X}$ and for every arrow $\alpha : i \to j$ in I consider a natural transformation $F_\alpha : F_i \to F_j$. Suppose that these data gives rise to a functor $I \to \mathbf{Fun}(\mathcal{C},\mathcal{X})$. Each limiting cone over the composition of $I \to \mathbf{Fun}(\mathcal{C},\mathcal{X})$ and $\mathbf{Fun}(\mathcal{C},\mathcal{X}) \to \mathbf{Fun}(|\mathcal{C}|,\mathcal{X})$ consists of a family of objects $\{F(X)\}_{X\in\mathcal{C}}$ of $\mathcal{X}$ parametrized by objects of $\mathcal{C}$ and a family $\{f_{i,X}\}_{i\in I,\,X\in\mathcal{C}}$ of arrows in $\mathcal{X}$ parametrized by objects of $I \times \mathcal{C}$ such that the following assertion hold.

($\star$) For every $X \in \mathcal{C}$ a pair $\left(F(X), \{f_{i,X}\}_{i\in I}\right)$ is a limiting cone for a functor $I \to \mathcal{X}$ given by $i \mapsto F_i(X)$ and $\alpha \mapsto F_\alpha(X)$ for any object i and arrow α in I.

We now show that there exists a unique lift of a pair $\left(\{F(X)\}_{X\in\mathcal{C}}, \{f_{i,X}\}_{i\in I,\,X\in\mathcal{C}}\right)$ to a cone $\left(F, \{f_i\}_{i\in I}\right)$ over the functor $I \to \mathbf{Fun}(\mathcal{C},\mathcal{X})$ described by data $\left(\{F_i\}_{i\in I}, \{F_\alpha\}_{\alpha\in\mathbf{Mor}(I)}\right)$. For this pick an arrow $f : X \to Y$. Then by **($\star$)** there exists a unique arrow $F(f) : F(X) \to F(Y)$ such that every square

$$\begin{tikzcd} F(Y) \arrow[r, "f_{i,Y}"] & F_i(Y) \\ F(X) \arrow[u, dotted, "F(f)"] \arrow[r, "f_{i,X}"'] & F_i(X) \arrow[u, "F_i(f)"'] \end{tikzcd}$$

for every $i \in I$ is commutative. Suppose that $f : X \to Y$ and $g : Y \to Z$ are arrows in $\mathcal{C}$. Then

$$f_{i,Z} \cdot F(g \cdot f) = F_i(g \cdot f) \cdot f_{i,X} = F_i(g) \cdot F_i(f) \cdot f_{i,X} = F_i(g) \cdot f_{i,Y} \cdot F(f) = f_{i,Z} \cdot F(g) \cdot F(f)$$

According to **($\star$)** we deduce that $F(g \cdot f) = F(g) \cdot F(f)$. Similarly we prove that $F(1_X) = 1_{F(X)}$. Hence there exists a unique functor $F : \mathcal{C} \to \mathcal{X}$ that extends object mapping $\{F(X)\}_{X\in\mathcal{C}}$ and such that $\{f_i : F \to F_i\}_{i\in I}$ becomes a collection of natural transformations of functors. Therefore, $\left(F, \{f_i\}_{i\in I}\right)$ is a unique lift of $\left(\{F(X)\}_{X\in\mathcal{C}}, \{f_{i,X}\}_{i\in I,\,X\in\mathcal{C}}\right)$ to a cone over the functor $I \to \mathbf{Fun}(\mathcal{C},\mathcal{X})$ described by data $\left(\{F_i\}_{i\in I}, \{F_\alpha\}_{\alpha\in\mathbf{Mor}(I)}\right)$. Now we prove that the cone $\left(F, \{f_i\}_{i\in I}\right)$ is limiting. For this assume that $\left(G, \{g_i\}_{i\in I}\right)$ is a cone over the functor $I \to \mathbf{Fun}(\mathcal{C},\mathcal{X})$ described by data $\left(\{F_i\}_{i\in I}, \{F_\alpha\}_{\alpha\in\mathbf{Mor}(I)}\right)$. By **($\star$)** we derive that for every $X \in \mathcal{C}$ there exists a unique morphism $\tau_X : G(X) \to F(X)$ such that

$$\begin{tikzcd} G(X) \arrow[rr, dotted, "\tau_X"] \arrow[dr, "g_{i,X}"'] & & F(X) \arrow[dl, "f_{i,X}"] \\ & F_i(X) & \end{tikzcd}$$

It suffices to verify that a collection $\{\tau_X\}_{X\in\mathcal{C}}$ is a natural transformation of functors $G \to F$. For this pick $f : X \to Y$. Then

$$f_{i,Y} \cdot F(f) \cdot \tau_X = F_i(f) \cdot f_{i,X} \cdot \tau_X = F_i(f) \cdot g_{i,X} = g_{i,Y} \cdot G(f) = f_{i,Y} \cdot \tau_Y \cdot G(f)$$

for every $i \in I$. According to **(⋆)** we deduce that $F(f) \cdot \tau_X = \tau_Y \cdot G(f)$. Since f is arbitrary, we derive that $\{\tau_X\}_{X\in\mathcal{C}}$ is a natural transformation of functors $G \to F$. □

Let $\mathcal{C}$, $\mathcal{X}$ be categories. For every object $X \in \mathcal{C}$ we denote by $\mathrm{ev}_X : \mathbf{Fun}(\mathcal{C}, \mathcal{X}) \to \mathcal{X}$ the functor that sends $F \in \mathbf{Fun}(\mathcal{C}, \mathcal{X})$ to $F(X)$ and $f : F \to G$ in $\mathbf{Fun}(\mathcal{C}, \mathcal{X})$ to $f_X : F(X) \to G(X)$.

Corollary 2.5. *Let $\mathcal{C}$, $\mathcal{X}$ and I be categories and let $D : I \to \mathbf{Fun}(\mathcal{C}, \mathcal{X})$ be a functor. Suppose that for every $X \in \mathcal{C}$ the functor $\mathrm{ev}_X \cdot D : I \to \mathcal{X}$ admits a limit (colimit). Then D admits a limit (colimit). Moreover, suppose that $(F, \{f_i\}_{i\in I})$ is a cone (cocone) over D. Then the following are equivalent.*

(i) *$(F, \{f_i\}_{i\in I})$ is a limiting cone (colimiting cocone) over D.*

(ii) *$(F, \{f_i\}_{i\in I})$ is a cone (cocone) over D and for every $X \in \mathcal{C}$ the pair $(F(X), \{f_{i,X}\}_{i\in I})$ is a limiting cone (colimiting cocone) over $\mathrm{ev}_X \cdot D$.*

Proof. The assumption that for every $X \in \mathcal{C}$ the functor $\mathrm{ev}_X \cdot D : I \to \mathcal{X}$ admits a limit (colimit) implies that the composition of D with the functor $\mathbf{Fun}(\mathcal{C}, \mathcal{X}) \to \mathbf{Fun}(|\mathcal{C}|, \mathcal{X})$ induced by the canonical functor $|\mathcal{C}| \to \mathcal{C}$ admits a limit (colimit). Now by Theorem 2.4 we derive that the functor $\mathbf{Fun}(\mathcal{C}, \mathcal{X}) \to \mathbf{Fun}(|\mathcal{C}|, \mathcal{X})$ creates limits and colimits. Hence D admits a limit (colimit). More precisely there exists a limiting cone (colimiting cocone) $(F, \{f_i\}_{i\in I})$ over D such that for every $X \in \mathcal{C}$ the pair $(F(X), \{f_{i,X}\}_{i\in I})$ is a limiting cone (colimiting cocone) over $\mathrm{ev}_X \cdot D$. Since any two limiting cones (colimiting cocones) over given functor are isomorphic, we deduce that **(i)** ⇒ **(ii)**. On the other hand if $(F, \{f_i\}_{i\in I})$ is a cone (cocone) over D and for every $X \in \mathcal{C}$ the pair $(F(X), \{f_{i,X}\}_{i\in I})$ is a limiting cone (colimiting cocone) over $\mathrm{ev}_X \cdot D$, then, according to the fact that $\mathbf{Fun}(\mathcal{C}, \mathcal{X}) \to \mathbf{Fun}(|\mathcal{C}|, \mathcal{X})$ creates limits and colimits, we derive that $(F, \{f_i\}_{i\in I})$ is a limiting cone (colimiting cocone) over D. Thus **(ii)** ⇒ **(i)** holds. □

3. Presheaves

Definition 3.1. Let $\mathcal{C}$ be a locally small category. We denote by $\widehat{\mathcal{C}}$ the category $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ and we call it *the category of presheaves on* $\mathcal{C}$.

Definition 3.2. Let $\mathcal{C}$ be a locally small category. For every object $X \in \mathcal{C}$ we define $h_X = \mathrm{Mor}_{\mathcal{C}}(-, X)$. We call it *the presheaf represented by* X. Next for every morphism $f : X \to Y$ in $\mathcal{C}$ we define a natural transformation $h_f : h_X \to h_Y$ given by formula

$$\mathrm{Mor}_{\mathcal{C}}(Z, X) \ni g \mapsto f \cdot g \in \mathrm{Mor}_{\mathcal{C}}(Z, Y)$$

This defines a functor $h : \mathcal{C} \to \widehat{\mathcal{C}}$ called *the Yoneda embedding of* $\mathcal{C}$.

Theorem 3.3 (Yoneda lemma)**.** *Let $\mathcal{C}$ be a locally small category. For every object $X \in \mathcal{C}$ and a presheaf $F \in \widehat{\mathcal{C}}$ map*

$$\mathrm{Mor}_{\widehat{\mathcal{C}}}(h_X, F) \to F(X)$$

given by formula $p \mapsto p(1_X)$ is a bijection natural in both X and F.

Proof. Fix $p : h_X \to F$ for some $X \in \mathcal{C}$ and $F \in \widehat{\mathcal{C}}$. Denote $x = p(1_X)$. Next let $f : Y \to X$ be a morphism in $\mathcal{C}$. Since p is natural transformation, we derive that the diagram

$$\begin{tikzcd} h_X(Y) \arrow[r, "p_Y"] & F(Y) \\ h_X(X) \arrow[u, "h_X(f)"] \arrow[r, "p_X"'] & F(X) \arrow[u, "F(f)"'] \end{tikzcd}$$

is commutative. Thus $p_Y(f) = p_Y(h_X(f)(1_X)) = F(f)(x)$. This shows that for every object $Y \in \mathcal{C}$ and every morphism $f : Y \to X$ we have $p_Y(f) = F(f)(x)$. Hence p is uniquely determined by x. This proves that the map described in the statement is injective. Now we prove that it is surjective. For this fix an element $x \in F(X)$ and define $p : h_X \to F$ by formula $p_Y(f) = F(f)(x)$ for every morphism $f : Y \to X$ in $\mathcal{C}$. Consider morphisms $g : Z \to Y$ and $f : Y \to X$ in $\mathcal{C}$ and note that

$$F(g)\,(p_Y(f)) = F(g) \cdot F(f)\,(x) = F(f \cdot g)(x) = p_Z(f \cdot g) = p_Z\,(h_X(g)(f))$$

Thus p is a morphism of presheaves and $p(1_X) = x$.
It remains to prove that the map in the statement is natural with respect to X and F. This is left to the reader as an exercise. □

Corollary 3.4. *Let $\mathcal{C}$ be a locally small category. The functor $h : \mathcal{C} \to \widehat{\mathcal{C}}$ is full and faithful.*

Proof. Fully faithfulness follows from Theorem 3.3. □

Now we investigate small limits and colimits in presheaf categories. For this fix a locally small category $\mathcal{C}$ and $X \in \mathcal{C}$. We denote by $\mathrm{ev}_X : \widehat{\mathcal{C}} \to \mathbf{Set}$ the functor that sends a presheaf F to $F(X)$ and a morphism $f : F \to G$ to f_X.

Corollary 3.5. *Fix a locally small category $\mathcal{C}$. Let I be a category and let $D : I \to \widehat{\mathcal{C}}$ be a functor. If I is a small category, then D admits a limit (colimit). Moreover, for a cone (cocone) $\left(F, \{f_i\}_{i \in I}\right)$ over D the following assertions are equivalent.*

- **(i)** *$\left(F, \{f_i\}_{i \in I}\right)$ is a limiting cone (colimiting cocone) over D.*
- **(ii)** *$\left(F, \{f_i\}_{i \in I}\right)$ is a cone (cocone) over D and for every $X \in \mathcal{C}$ the pair $\left(F(X), \{f_{i,X}\}_{i \in I}\right)$ is a limiting cone (colimiting cocone) over $\mathrm{ev}_X \cdot D$.*

Proof. By [ML98, V.1, Theorem 1 and Exercise 8] we know that the category **Set** admits both small limits and small colimits. Now it suffices to use Corollary 2.5. □

Finally we add one notational remark. Let $\mathcal{C}$ be a locally small category and F, G be presheaves on $\mathcal{C}$. Then we denote by $\mathrm{Mor}_{\mathcal{C}}(F, G)$ the class of morphisms of presheaves with domain F and codomain G.

4. Classes of generators

Definition 4.1. Let $\mathcal{C}$ be a category. A class $\mathcal{K}$ of objects of $\mathcal{C}$ is called *a class of generators for $\mathcal{C}$* if for any pair of distinct and parallel arrows

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

there exists $Z \in \mathcal{K}$ and a morphism $h : Z \to X$ such that $f \cdot h \neq g \cdot h$.

Now we introduce special case of the notion of the class of generators of category. For this we need one more definition.

Definition 4.2. Let $\mathcal{C}$ be a category and X be an object of $\mathcal{C}$. *An object of $\mathcal{C}$ over X* is a morphism $f : Y \to X$ in $\mathcal{C}$. If $f_1 : Y_1 \to X$, $f_2 : Y_2 \to X$ are objects of $\mathcal{C}$ over X, then *a morphism over X* between these objects consists of a morphism $f : Y_1 \to Y_2$ in $\mathcal{C}$ such that the following triangle

$$\begin{tikzcd} Y_1 \arrow[rr, "f"] \arrow[dr, "f_1"'] & & Y_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

is commutative. This defines *the category of objects of $\mathcal{C}$ over X*.

For every object X of a category $\mathcal{C}$ we denote by $\mathcal{C}/X$ the category of objects over X. Next suppose that X is an object of $\mathcal{C}$ and $\mathcal{K}$ is a subclass of the class of objects of $\mathcal{C}$. We denote by $\mathcal{K}/X$ the full subcategory of $\mathcal{C}/X$ that consists of morphisms $f : K \to X$ such that K is in $\mathcal{K}$. For every such class we denote by π_X the canonical functor $\mathcal{K}/X \to \mathcal{K}$ that sends every arrow $f : K \to X$ in $\mathcal{K}/X$ to K. In the case of considerations in which multiple distinct classes are involved we specify more precise notation. Next suppose that $f : X \to Y$ is a morphism in a category $\mathcal{C}$. Then the composition with f induces a functor $\mathcal{C}/X \to \mathcal{C}/Y$. We denote this functor by $\mathcal{C}/f$. Now if $\mathcal{K}$ is a class of objects of $\mathcal{C}$, then we denote by $\mathcal{K}/f$ the functor $\mathcal{K}/X \to \mathcal{K}/Y$ induced by $\mathcal{C}/f$.

Definition 4.3. Let $\mathcal{C}$ be a category and $\mathcal{K}$ be a class of objects of $\mathcal{C}$. Suppose that for every object X of $\mathcal{C}$ a pair

$$\left(X, \{f\}_{f \in \mathcal{K}/X}\right)$$

is a colimiting cocone of a functor given as the composition of $\pi_X : \mathcal{K}/X \to \mathcal{K}$ with the inclusion functor $\mathcal{K} \hookrightarrow \mathcal{C}$. Then we call $\mathcal{K}$ *a dense class of generators for $\mathcal{C}$*.

Let $\mathcal{C}$ be a locally small category and $\mathcal{K}$ be a class of objects of $\mathcal{C}$. We also denote by $\mathcal{K}$ the corresponding full subcategory of $\mathcal{C}$. We define a functor $\Gamma_{\mathcal{K}} : \mathcal{C} \to \widehat{\mathcal{K}}$ as the composition of the Yoneda embedding $\mathcal{C} \to \widehat{\mathcal{C}}$ with the restriction functor $\widehat{\mathcal{C}} \to \widehat{\mathcal{K}}$.

Theorem 4.4. *Let $\mathcal{C}$ be a locally small category and $\mathcal{K}$ be a class of objects of $\mathcal{C}$. Then the following are equivalent.*

(i) *$\mathcal{K}$ is a (dense) class of generators for $\mathcal{C}$.*

(ii) *The functor*

$$\Gamma_{\mathcal{K}} : \mathcal{C} \to \widehat{\mathcal{K}}$$

is (full and) faithful.

Proof. First we need to introduce some notation. For every object X of $\mathcal{C}$ we denote by $F_X : \mathcal{K}/X \to \mathcal{C}$ the functor obtained as the compositon of $\pi_X : \mathcal{K}/X \to \mathcal{K}$ with the inclusion functor $\mathcal{K} \hookrightarrow \mathcal{C}$. We also denote by Γ_X the value of Γ on X and for every object Y of $\mathcal{C}$ we denote by $\mathrm{Cocone}_Y(F_X)$ the class of cocones with Y as the vertex over the functor F_X. Finally if $g : X \to Y$ is a morphism of $\mathcal{C}$, then we denote by Γ_g a natural morphism $\Gamma_X \to \Gamma_Y$ induced by g.
Suppose now that X and Y are objects of $\mathcal{C}$. Let $\sigma : \Gamma_X \to \Gamma_Y$ be a natural transformation. Then one can show that $\{\sigma(f)\}_{f \in \mathcal{K}/X}$ is a cocone of F_X with vertex in Y and moreover, the map

$$\mathrm{Mor}_{\mathcal{K}}\left(\Gamma_X, \Gamma_Y\right) \ni \sigma \mapsto \{\sigma(f)\}_{f \in \mathcal{K}/X} \in \mathrm{Cocone}_Y(F_X)$$

is bijective. We have a commutative triangle

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{K}}(\Gamma_X, \Gamma_Y) \arrow[rr, "\sigma \mapsto \{\sigma(f)\}_{f \in \mathcal{K}/X}"] & & \mathrm{Cocone}_Y(F_X) \\
& \mathrm{Mor}_{\mathcal{C}}(X,Y) \arrow[ul, "g \mapsto \Gamma_g"] \arrow[ur, "g \mapsto \{g \cdot f\}_{f \in \mathcal{K}/X}"'] &
\end{tikzcd}$$

From this we derive that Γ is (full and) faithful if and only if

$$\mathrm{Mor}_{\mathcal{C}}(X,Y) \ni g \mapsto \{g \cdot f\}_{f \in \mathcal{K}/X} \in \mathrm{Cocone}_Y(F_X)$$

is (bijective) injective for any pair X, Y of objects in $\mathcal{C}$. This map is (bijective) injective for any pair X, Y of objects in $\mathcal{C}$ if and only if $\mathcal{K}$ is a class of (dense) generators for $\mathcal{C}$. This proves theorem. $\square$

Corollary 4.5. *Let $\mathcal{C}$ be a locally small category. Then the class of representable presheaves $\{h_X\}_{X \in \mathcal{C}}$ is a dense class of generators for $\widehat{\mathcal{C}}$.*

Proof. We want to apply Theorem 4.4 to $\widehat{\mathcal{C}}$. Our issue is that in general $\widehat{\mathcal{C}}$ is not a locally small category. To fix this we must be specific and work with Grothendieck universes [ML98, page 22]. We assume (c.f. Section 1) that our base Grothendieck universe is U. Then $\mathbf{Set} = \mathbf{Set}_U$ is the category of U-small sets and $\mathcal{C}$ is a locally U-small category. Next $\widehat{\mathcal{C}} = \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}_U)$ is a presheaf category. Now we fix another universe V that contains U and such that $\mathcal{C}$ is V-small. We denote by $\mathbf{Set}_V$ the category of V-small sets. We can apply Theorem 4.4 to a locally V-small category $\widehat{\mathcal{C}}$. Consider the composition of the Yoneda embedding $\widehat{\mathcal{C}} \to \mathbf{Fun}\left(\left(\widehat{\mathcal{C}}\right)^{\mathrm{op}}, \mathbf{Set}_V\right)$ with the restriction $\mathbf{Fun}\left(\left(\widehat{\mathcal{C}}\right)^{\mathrm{op}}, \mathbf{Set}_V\right) \to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}_V)$ induced by the usual Yoneda embedding $h : \mathcal{C} \to \widehat{\mathcal{C}}$. The composition is isomorphic with the functor $\widehat{\mathcal{C}} = \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}_U) \to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}_V)$ induced by the inclusion $\mathbf{Set}_U \hookrightarrow \mathbf{Set}_V$. Hence it is full and faithful. Now (replacing our base universe U by V) we can apply Theorem 4.4 to a locally V-small category $\widehat{\mathcal{C}}$ and derive the statement. $\square$

5. Internal hom

We start by making few remarks. Let $\mathcal{C}$ be a locally small category and let X be an object of $\mathcal{C}$. Recall that $\pi_X : \mathcal{C}/X \to \mathcal{C}$ is a functor that sends morphism $f : Y \to X$ to Y. For every presheaf F on $\mathcal{C}$ we denote by $F_{|X}$ the functor

$$F \cdot (\pi_X)^{\mathrm{op}} : (\mathcal{C}/X)^{\mathrm{op}} \to \mathbf{Set}$$

The map $F \mapsto F_{|X}$ extends to a functor $\widehat{\mathcal{C}} \to \widehat{\mathcal{C}/X}$. Let $\mathbf{1}_{|X}$ denote a presheaf on $\mathcal{C}/X$ that assigns to every object over X a set with one element. According to Corollary 3.5 we derive that $\mathbf{1}_{|X}$ is a terminal object in $\widehat{\mathcal{C}/X}$.

Fact 5.1. *Let $\mathcal{C}$ be a category and let F be a presheaf on $\mathcal{C}$. Suppose that $x \in F(X)$ for some X in $\mathcal{C}$. Then x determines a morphism $\mathbf{1}_{|X} \to F_{|X}$ that for every object f in $\mathcal{C}/X$ sends a unique element of $\mathbf{1}_{|X}(f)$ to $F(f)(x) \in F_{|X}(f)$. This gives rise to a bijection*

$$F(X) \cong \mathrm{Mor}_{\mathcal{C}/X}\left(\mathbf{1}_{|X}, F_{|X}\right)$$

Proof. We left to the reader as an exercise. $\square$

Let $\mathcal{C}$ be a locally small category. If $f : X \to Y$ is a morphism in $\mathcal{C}$, then we have a functor $\widehat{\mathcal{C}/Y} \to \widehat{\mathcal{C}/X}$ induced by the precomposition with $(\mathcal{C}/f)^{\mathrm{op}}$.

Definition 5.2. Let $\mathcal{C}$ be a locally small category and let F, G be presheaves on $\mathcal{C}$. Assume that for every object X in $\mathcal{C}$ the class $\mathrm{Mor}_{\mathcal{C}/X}\left(F_{|X}, G_{|X}\right)$ is a set. We define

$$\mathcal{M}\mathrm{or}_{\mathcal{C}}\left(F,G\right)\left(X\right) = \mathrm{Mor}_{\mathcal{C}/X}(F_{|X}, G_{|X})$$

for every X in $\mathcal{C}$. This is a presheaf on $\mathcal{C}$, since for every morphism $f : X \to Y$, we can compose a morphism $\sigma : F_{|Y} \to G_{|Y}$ of presheaves with $(\mathcal{C}/f)^{\mathrm{op}}$ i.e. we have a map

$$\mathcal{M}\mathrm{or}_{\mathcal{C}}(F,G)(Y) \ni \sigma \mapsto \sigma_{(\mathcal{C}/f)^{\mathrm{op}}} \in \mathcal{M}\mathrm{or}_{\mathcal{C}}(F,G)(X)$$

and these make $\mathcal{M}\mathrm{or}_{\mathcal{C}}(F,G)$ a functor. The presheaf $\mathcal{M}\mathrm{or}_{\mathcal{C}}(F,G)$ is called *an internal hom of F and G*.

Let F, G and H be presheaves on a locally small category $\mathcal{C}$ and assume that $\mathcal{M}\mathrm{or}_{\mathcal{C}}(F,G)$ exists. Fix a morphism of presheaves $\sigma : H \times F \to G$. Pick an object X in $\mathcal{C}$ and $x \in H(X)$. Let $i_x : \mathbf{1}_{|X} \to H_{|X}$ be a morphism determined by $x \in H(X)$ as in Fact 5.1. Then $\sigma_{|X} \cdot \left(i_x \times 1_{F_{|X}}\right)$ yields a morphism $\tau_x : F_{|X} \to G_{|X}$. Suppose now that $f : Y \to X$ is a morphism in $\mathcal{C}$. We have

$$\left(\sigma_{|X} \cdot \left(i_x \times 1_{F_{|X}}\right)\right)_{(\mathcal{C}/f)^{\mathrm{op}}} = \left(\sigma_{|X}\right)_{(\mathcal{C}/f)^{\mathrm{op}}} \cdot \left((i_x)_{(\mathcal{C}/f)^{\mathrm{op}}} \times \left(1_{F_{|X}}\right)_{(\mathcal{C}/f)^{\mathrm{op}}}\right) = \sigma_{|Y} \cdot \left(i_{F(f)(x)} \times 1_{F_{|Y}}\right)$$

because $(i_x)_{(\mathcal{C}/f)^{\mathrm{op}}} = i_{F(f)(x)}$. This implies that $(\tau_x)_{(\mathcal{C}/f)^{\mathrm{op}}} = \tau_{F(f)(x)}$. Hence $\tau : H \to \mathcal{M}\mathrm{or}_{\mathcal{C}}\left(F,G\right)$ given by

$$H(X) \ni x \mapsto \tau_x \in \mathrm{Mor}_{\mathcal{C}/X}\left(F_{|X}, G_{|X}\right)$$

is a morphism of presheaves. This defines a map of classes

$$\mathrm{Mor}_{\mathcal{C}}\left(H \times F, G\right) \in \sigma \mapsto \tau \in \mathrm{Mor}_{\mathcal{C}}\left(H, \mathcal{M}\mathrm{or}_{\mathcal{C}}\left(F,G\right)\right)$$

Theorem 5.3. *Let $\mathcal{C}$ be a locally small category and F, G be presheaves on $\mathcal{C}$. Assume that for every object X in $\mathcal{C}$ the class $\mathrm{Mor}_{\mathcal{C}/X}\left(F_{|X}, G_{|X}\right)$ is a set. Then the map*

$$\mathrm{Mor}_{\mathcal{C}}\left(H \times F, G\right) \to \mathrm{Mor}_{\mathcal{C}}\left(H, \mathcal{M}\mathrm{or}_{\mathcal{C}}\left(F,G\right)\right)$$

described above is a bijection natural in H.

Proof. The fact that the map in the statement is natural in H is left to the reader as an exercise. Pick an object X in $\mathcal{C}$. We verify now that the map

$$\mathrm{Mor}_{\mathcal{C}}\left(h_X \times F, G\right) \to \mathrm{Mor}_{\mathcal{C}}\left(h_X, \mathcal{M}\mathrm{or}_{\mathcal{C}}\left(F,G\right)\right)$$

is a bijection. Pick a morphism $\sigma : h_X \times F \to G$ of presheaves and suppose that $\tau : h_X \to \mathcal{M}\mathrm{or}_{\mathcal{C}}(F,G)$ is its value under the discussed map. According to Yoneda lemma (Theorem 3.3) τ is uniquely determined by its value on 1_X. We denote this value by ρ. Thus it suffices to prove that

$$\mathrm{Mor}_{\mathcal{C}}\left(h_X \times F, G\right) \ni \sigma \mapsto \rho \in \mathrm{Mor}_{\mathcal{C}/X}\left(F_{|X}, G_{|X}\right)$$

is bijective. We retrieve ρ by means of procedure described before the statement of this theorem. Firstly 1_X according to Fact 5.1 determines a morphism $i : \mathbf{1}_{|X} \to (h_X)_{|X}$. Now $\rho \in \mathrm{Mor}_{\mathcal{C}/X}\left(F_{|X}, G_{|X}\right)$ is isomorphic with $\sigma_{|X} \cdot \left(i \times 1_{F_{|X}}\right)$. Hence for every $f : Y \to X$ and $y \in F(Y)$ we have

$$\rho_f(y) = \sigma_Y(f, y)$$

This implies that σ and ρ are mutually determined and thus

$$\mathrm{Mor}_{\mathcal{C}}\left(h_X \times F, G\right) \to \mathrm{Mor}_{\mathcal{C}}\left(h_X, \mathcal{M}\mathrm{or}_{\mathcal{C}}\left(F,G\right)\right)$$

is a bijection.
Now we prove the general case. We know that the map

$$\mathrm{Mor}_{\mathcal{C}}\left(H \times F, G\right) \to \mathrm{Mor}_{\mathcal{C}}\left(H, \mathcal{M}\mathrm{or}_{\mathcal{C}}\left(F,G\right)\right)$$

is natural in H and is bijective when H is a representable presheaf. Now the following statements hold.

(1) Every presheaf is canonically the colimit of representable presheaves by Corollary 4.5.

(2) The functor $(-) \times F : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ preserves colimits (this follows from cartesian closedness of **Set** [ML98, page 98] and Corollary 3.5).

(3) Suppose that V is a Grothendieck universe that contains the base universe U and such that $\widehat{\mathcal{C}}$ is V-locally small. Then the functor

$$\mathrm{Mor}_{\mathcal{C}}(-, \mathcal{M}or_{\mathcal{C}}(F,G)) : \widehat{\mathcal{C}} \to \mathbf{Set}_V$$

preserves colimits [ML98, V.4, Theorem 1].

Therefore, we derive that the map in the question is bijective for every presheaf H. □

6. Subpresheaves of internal hom

Let $\mathcal{C}$ be a locally small category and let F, G be a presheaves on $\mathcal{C}$. The requirement that $\mathrm{Mor}_{\mathcal{C}/X}\left(F_{|X}, G_{|X}\right)$ is a set for every object X in $\mathcal{C}$ is a serious limitation of Theorem 5.3. In this section we explain a useful result which addresses this issue.

Definition 6.1. Let $\mathcal{C}$ be a locally small category and let F, G, J be presheaves on $\mathcal{C}$. Suppose that for every object X in $\mathcal{C}$ there exists an inclusion of classes $J(X) \subseteq \mathrm{Mor}_{\mathcal{C}/X}\left(F_{|X}, G_{|X}\right)$ such that the square of maps (horizontal arrows in the square are inclusions) of classes

$$\begin{tikzcd} J(Y) \arrow[r, hook] \arrow[d, "J(f)"'] & \mathrm{Mor}_{\mathcal{C}/Y}\left(F_{|Y}, G_{|Y}\right) \arrow[d, "\sigma \mapsto \sigma_{(\mathcal{C}/f)^{\mathrm{op}}}"] \\ J(X) \arrow[r, hook] & \mathrm{Mor}_{\mathcal{C}/X}\left(F_{|X}, G_{|X}\right) \end{tikzcd}$$

is commutative for every morphism $f : X \to Y$ in $\mathcal{C}$. Then we say that J is *a subpresheaf of internal hom of F and G*.

Let $\mathcal{C}$ be a locally small category, F, G, H be presheaves on $\mathcal{C}$. Fix a morphism $\sigma : H \times F \to G$ of presheaves. Recall form the previous section that for every object X in $\mathcal{C}$ and x in $H(X)$ we denote by $i_x : \mathbf{1}_{|X} \to H_{|X}$ a unique morphism determined by x (Fact 5.1). Next we denote by $\tau_x : F_{|X} \to G_{|X}$ a unique morphism isomorphic with $\sigma_{|X} \cdot \left(i_x \times 1_{F_{|X}}\right)$.

Definition 6.2. Let $\mathcal{C}$ be a locally small category, F, G, H be presheaves on $\mathcal{C}$ and assume that J is a subpresheaf of internal hom of F and G. Then a morphism of presheaves $\sigma : H \times F \to G$ is called *a family of J-morphisms parametrized by H* if for every object X in $\mathcal{C}$ and every x in $H(X)$ we have $\tau_x \in J(X)$.

We continue discussion started before the definition. Let us now assume that $\sigma : H \times F \to G$ is a family of J-morphisms parametrized by H for some subpresheaf J of internal hom of F and G. Then $\tau : H \to J$ given by

$$H(X) \ni x \mapsto \tau_x \in J(X)$$

is a morphism of presheaves. The proof is identical to the proof of the analogous statement preceding Theorem 5.3. This gives rise to a map of classes

$$\left\{\text{families of } J\text{-morphisms parametrized by } H\right\} \ni \sigma \mapsto \tau \in \mathrm{Mor}_{\mathcal{C}}(H, J)$$

Theorem 6.3. *Let $\mathcal{C}$ be a locally small category and F, G be presheaves on $\mathcal{C}$. Assume that J is a subpresheaf of internal hom of F and G. Then the map*

$$\left\{\textit{families of } J\textit{-morphisms parametrized by } H\right\} \to \mathrm{Mor}_{\mathcal{C}}(H, J)$$

described above is a bijection natural in H.

Proof. We enlarge our base universe U to a Grothendieck universe V such that $\mathcal{C}$ is V-small. Then $\mathcal{M}\mathrm{or}_{\mathcal{C}}(F,G) \in \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}_V)$ and J is a legitimate subobject of $\mathcal{M}\mathrm{or}_{\mathcal{C}}(F,G)$ in $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}_V)$. For every $H \in \widehat{\mathcal{C}} = \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}_U) \subseteq \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}_V)$ we have a bijection

$$\mathrm{Mor}_{\mathcal{C}}(H \times F, G) \to \mathrm{Mor}_{\mathcal{C}}(H, \mathcal{M}\mathrm{or}_{\mathcal{C}}(F,G))$$

natural in H. This follows according to Theorem 5.3 applied to the enlarged category of presheaves $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}_V)$. Finally this bijection induces a bijection

$$\left\{\text{families of } J\text{-morphisms parametrized by } H\right\} \to \mathrm{Mor}_{\mathcal{C}}(H, J)$$

on its subclasses, which is natural in H and is given by the rule described in the discussion preceding the statement of the theorem. □

7. Remarks on copresheaf categories

Definition 7.1. Let $\mathcal{C}$ be a locally small category. The category $\mathbf{Fun}(\mathcal{C}, \mathbf{Set})$ is called *the category of copresheaves on* $\mathcal{C}$.

All results stated above for categories of presheaves hold for categories of copresheaves by virtue of the identification

$$\mathbf{Fun}(\mathcal{C}, \mathbf{Set}) = \mathbf{Fun}\left((\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}, \mathbf{Set}\right)$$

References

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.