MINIMAX THEOREM AND LINEAR PROGRAMMING

1. Sion's minimax theorem

For reader's convenience we recall few notions.

Definition 1.1. Let X be a topological space and let $f : X \to \mathbb{R}$ be a function. Then f is *lower-semicontinuous* if for every $r \in \mathbb{R}$ the set

$$\{x \in X \,|\, f(x) \leq r\}$$

is closed. We say that f is *upper-semicontinuous* if $-f$ is lower-semicontinuous.

Definition 1.2. Let X be a convex subset of a linear space over $\mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function. Then f is *convex* if for every $x_1, x_2 \in X$ and $t \in [0,1]$ we have

$$f\left(tx_1 + (1-t)x_2\right) \leq tf(x_1) + (1-t)f(x_2)$$

We say that f is *concave* if $-f$ is convex.

Definition 1.3. Let X be a convex subset of a linear space over $\mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function. Then f is *quasiconvex* if for every $x_1, x_2 \in X$ and $t \in [0,1]$ we have

$$f\left(tx_1 + (1-t)x_2\right) \leq \max\left\{f(x_1), f(x_2)\right\}$$

We say that f is *quasiconcave* if $-f$ is quasiconvex.

Fact 1.4. *Every convex function is quasiconvex.*

Proof. We left the proof to the reader. □

Proposition 1.5. *Let X be a convex subset of a linear space over $\mathbb{R}$. Suppose that $f : X \to \mathbb{R}$ is a function. Then the following are equivalent.*

(1) *f is quasiconvex.*

(2) *For every $r \in \mathbb{R}$ the set $\{x \in X \,|\, f(x) \leq r\}$ is convex.*

Proof. We prove **(1)** $\Rightarrow$ **(2)**. Pick $r \in \mathbb{R}$, $x_1, x_2 \in X$ and assume that $f(x_1), f(x_2)$ are both less or equal to r. Then

$$f\left(tx_1 + (1-t)x_2\right) \leq \max\left\{f(x_1), f(x_2)\right\} \leq r$$

for every $t \in [0,1]$ by **(1)**. Thus the set $\{x \in X \,|\, f(x) \leq r\}$ contains line segment joining x_1 with x_2 and hence it is convex. This is **(2)**.
We prove **(2)** $\Rightarrow$ **(1)**. Pick $x_1, x_2 \in X$ and $t \in [0,1]$. Let $r = \max\left\{f(x_1), f(x_2)\right\}$. Then by **(2)** we deduce that the set $\{x \in X \,|\, f(x) \leq r\}$ is convex. Hence $f\left(tx_1 + (1-t)x_2\right) \leq r$. This shows **(1)**. □

Theorem 1.6 (Sion's theorem)**.** *Let X be a convex, compact subset of a topological vector space over $\mathbb{R}$ and let Y be a convex subset of a topological vector space over $\mathbb{R}$. Suppose that $f : X \times Y \to \mathbb{R}$ is a function such that the following assertions hold.*

(1) *f_x is upper-semicontinuous and quasiconcave for every x in X.*

(2) *f_y is lower-semicontinuous and quasiconvex for every y in Y.*

Then we have

$$\inf_{x \in X} \sup_{y \in Y} f(x,y) = \sup_{y \in Y} \inf_{x \in X} f(x,y)$$

For every $y \in Y$ and $r \in \mathbb{R}$ we denote

$$L_{f,r,y} = \{x \in X \,|\, f(x,y) \leq r\}$$

Note that for each $r \in \mathbb{R}$ the family $\{L_{f,r,y}\}_{y \in Y}$ consists of convex, compact subsets of X.

Lemma 1.6.1. *Let X be a convex, compact subset of a topological vector space over $\mathbb{R}$ and let Y be a convex subset of a topological vector space over $\mathbb{R}$. Suppose that $f : X \times Y \to \mathbb{R}$ is a function such that the following assertions hold.*

(1) *f_x is upper-semicontinuous and quasiconcave for every x in X.*

(2) *f_y is lower-semicontinuous and quasiconvex for every y in Y.*

For every $y \in Y$ and $r \in \mathbb{R}$ we denote

$$L_{f,r,y} = \{x \in X \,|\, f(x,y) \leq r\}$$

If a family $\{L_{f,r,y}\}_{y \in Y}$ consists of nonempty sets, then it admits finite intersection property.

Proof of the lemma. For any such X, Y and f pick $r \in \mathbb{R}$ such that $L_{f,r,y}$ is nonempty for every y in Y. Suppose that $y_1, ..., y_m$ are points in Y. We want to show that

$$\bigcap_{i=1}^{m} L_{f,r,y_i} \neq \emptyset$$

Consider $X' = L_{f,r,y_m}$. This is a convex, compact and nonempty subset of a topological vector space over $\mathbb{R}$. Next let $f' : X' \times Y \to \mathbb{R}$ be the restriction of f. Then

$$L_{f',r,y} = X' \cap L_{f,r,y} = L_{f,r,y_m} \cap L_{f,r,y}$$

and it suffices to prove that

$$\bigcap_{i=1}^{m-1} L_{f',r,y_i} \neq \emptyset$$

Hence the proof goes on induction on m provided that we prove first that $L_{f,r,y_1} \cap L_{f,r,y_2} \neq \emptyset$ for any $y_1, y_2 \in Y$ and every X, Y, f and $r \in \mathbb{R}$ such that the family $\{L_{f,r,y}\}_{y \in Y}$ consists of nonempty sets. Assume by contradiction that $L_{f,r,y_1} \cap L_{f,r,y_2} = \emptyset$. Suppose that $y \in [y_1, y_2]$ and $x \in L_{f,r,y}$. Since f_x is quasiconcave, we derive that

$$\min\{f(x,y_1), f(x,y_2)\} \leq f(x,y) \leq r$$

and hence $x \in L_{f,r,y_1} \cup L_{f,r,y_2}$. This implies that for every $y \in [y_1, y_2]$ we have $L_{f,r,y} \subseteq L_{,fr,y_1} \cup L_{f,r,y_2}$. Since sets L_{f,r,y_1}, L_{f,r,y_2} are disjoint, we deduce that either $L_{f,r,y} \subseteq L_{f,r,y_1}$ or $L_{f,r,y} \subseteq L_{f,r,y_2}$ for every $y \in Y$. Now we define

$$F_1 = \{y \in [y_1, y_2] \,|\, L_{f,r,y} \subseteq L_{f,r,y_1}\},\ F_2 = \{y \in [y_1, y_2] \,|\, L_{f,r,y} \subseteq L_{f,r,y_2}\}$$

We proved that $F_1 \cup F_2 = [y_1, y_2]$ and $F_1 \cap F_2 = \emptyset$. Next fix $i = 1, 2$ and suppose that $\{z_n\}_{n \in \mathbb{N}}$ is a sequence of points in F_1 convergent to some point $z \in [y_1, y_2]$. Pick $x \in L_{f,r,z}$. Since f_x is upper-semicontinuous, we have

$$\limsup_{n \to +\infty} f(x, z_n) \leq f(x,z) \leq r$$

Thus there exists $n \in \mathbb{N}$ such that $f(x, z_n) \leq r$. This implies that $x \in L_{f,r,y_i}$ and hence $L_{f,r,z}$ intersects with L_{f,r,y_i}. Therefore, we deduce that $L_{f,r,z} \subseteq L_{f,r,y_i}$. This implies that $z \in F_i$. Hence F_1, F_2 are closed subsets of $[y_1, y_2]$, which are disjoint and with union equal to $[y_1, y_2]$. This is contradiction with the fact that $[y_1, y_2]$ is connected. This proves that $L_{f,r,y_1} \cap L_{f,r,y_2} \neq \emptyset$. □

Proof of the theorem. We always have

$$\sup_{y \in Y} \inf_{x \in X} f(x,y) \leq \inf_{x \in X} \sup_{y \in Y} f(x,y)$$

Next pick $r \in \mathbb{R}$ such that

$$\sup_{y \in Y} \inf_{x \in X} f(x,y) < r$$

Then $\{L_{f,r,y}\}_{y \in Y}$ consists of nonempty and compact subsets of X. By Lemma 1.6.1 we deduce that $\{L_{f,r,y}\}_{y \in Y}$ admits finite intersection property. Hence there exists x in X such that

$$x \in \bigcap_{y \in Y} L_{f,r,y}$$

Therefore, $f(x,y) \leq r$ for every $y \in Y$. This implies that

$$\inf_{x \in X} \sup_{y \in Y} f(x,y) \leq \sup_{y \in Y} f(x,y) \leq r$$

This shows that

$$\inf_{x \in X} \sup_{y \in Y} f(x,y) \leq \sup_{y \in Y} \inf_{x \in X} f(x,y)$$

□

2. Strong duality theorem