BIAŁYNICKI-BIRULA FUNCTORS

1. Introduction

In this notes we study Białynicki-Birula functors. In the first section we prove some results concerning the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$, where $\mathbf{M}$ is an affine monoid k-scheme and $\mathbf{G}$ is its group of units (we assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$). These results will be used in the following sections.
We assume that k is a field. In these notes we use the following notational convention.

Remark 1.1. Since the Yoneda embedding $\mathbf{Sch}_k \hookrightarrow \widehat{\mathbf{Sch}_k}$ is full and faithful, we identify $\mathbf{Sch}_k$ with the subcategory of $\widehat{\mathbf{Sch}_k}$ consisting of representable presheaves on $\mathbf{Sch}_k$. In particular, if X is a k-scheme, then we denote by the same symbol the presheaf representable by X.

2. Relations between representations of a monoid and its group of units

In this section we study the relation between the category $\mathbf{Rep}(\mathbf{M})$ of representations of an affine monoid k-scheme $\mathbf{M}$ and the category $\mathbf{Rep}(\mathbf{G})$ of representations of its group of units $\mathbf{G}$. Let $i : k[\mathbf{M}] \to k[\mathbf{G}]$ be the morphism of k-bialgebras induced by $\mathbf{G} \hookrightarrow \mathbf{M}$. Let us first note the following elementary result.

Fact 2.1. *Assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. Then i is an injective morphism of k-algebras.*

Proof. This follows from [Görtz and Wedhorn, 2010, Proposition 9.19]. □

Fact 2.2. *The forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$ creates colimits and finite limits.*

Proof. This follows from [Monygham, 2020b, Theorem 14.3, Theorem 14.4] and the commutative triangle

$$\begin{tikzcd}
\mathbf{Rep}(\mathbf{M}) \arrow[rr] \arrow[dr] & & \mathbf{Rep}(\mathbf{G}) \arrow[dl] \\
& \mathbf{Vect}_k &
\end{tikzcd}$$

of functors. □

The theorem below characterizes representations of $\mathbf{G}$ which are contained in the image of the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$.

Theorem 2.3. *Assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. Let V be a $\mathbf{G}$-representation. Then the following are equivalent.*

- **(i)** *V is in the image of the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$.*
- **(ii)** *The coaction $d : V \to k[\mathbf{G}] \otimes_k V$ factors through $i \otimes_k 1_V : k[\mathbf{M}] \otimes_k V \hookrightarrow k[\mathbf{G}] \otimes_k V$.*

Proof. In the proof we denote by $\Delta_{\mathbf{M}}$ and $\Delta_{\mathbf{G}}$ comultiplications of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. We also denote by $\xi_{\mathbf{M}}$ and $\xi_{\mathbf{G}}$ counits of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. According to Fact 2.1 i is an injective morphism of k-algebras.
Clearly **(i)** $\Rightarrow$ **(ii)**. We prove the converse. Suppose that **(ii)** holds. Let $c : V \to k[\mathbf{M}] \otimes_k V$ be a unique morphism such that $d = (i \otimes_k 1_V) \cdot c$. It suffices to prove that c is the coaction of the bialgebra $k[\mathbf{M}]$ on V. Observe that

$$(i \otimes_k i \otimes_k 1_V) \cdot \left(1_{k[\mathbf{M}]} \otimes_k c\right) \cdot c = (i \otimes_k d) \cdot c = \left(1_{k[\mathbf{G}]} \otimes_k d\right) \cdot d = (\Delta_{\mathbf{G}} \otimes_k 1_V) \cdot d =$$

$$= (\Delta_{\mathbf{G}} \otimes_k 1_V) \cdot \big((i \otimes_k 1_V) \cdot c\big) = ((\Delta_{\mathbf{G}} \cdot i) \otimes_k 1_V) \cdot c = (i \otimes_k i \otimes_k 1_V) \cdot (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot c$$

Since $i \otimes_k i \otimes_k 1_V$ is a monomorphism, we deduce that $\left(1_{k[\mathbf{M}]} \otimes_k c\right) \cdot c = (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot c$. Moreover, we have

$$(\xi_{\mathbf{G}} \otimes_k 1_V) \cdot d = (\xi_{\mathbf{G}} \otimes_k 1_V) \cdot ((i \otimes_k 1_V) \cdot c) = (\xi_{\mathbf{M}} \otimes_k 1_V) \cdot c$$

and hence $(\xi_{\mathbf{M}} \otimes_k 1_V) \cdot c$ is the canonical isomorphism $V \cong k \otimes_k V$. Thus c is the coaction of $k[\mathbf{M}]$ and $d = (i \otimes_k 1_V) \cdot c$. Therefore, V is in the image of $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$. □

Theorem 2.4. *Assume that* $\mathbf{G}$ *is open and schematically dense in* $\mathbf{M}$. *Then* $\mathbf{Rep}(\mathbf{M})$ *is a full subcategory of* $\mathbf{Rep}(\mathbf{G})$ *closed under subobjects and quotients.*

Proof. In the proof we denote by $\Delta_{\mathbf{M}}$ and $\Delta_{\mathbf{G}}$ comultiplications of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. We also denote by $\xi_{\mathbf{M}}$ and $\xi_{\mathbf{G}}$ counits of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. According to Fact 2.1 i is an injective morphism of k-algebras.
We first prove that $\mathbf{Rep}(\mathbf{M})$ is a full subcategory of $\mathbf{Rep}(\mathbf{G})$. For this consider $\mathbf{M}$-representations V, W and a their morphism $f : V \to W$ as $\mathbf{G}$-representations. Let c_V and c_W be coactions of $k[\mathbf{M}]$ on V and W, respectively. Our goal is to prove that f is a morphism of $\mathbf{M}$-representations. Consider the diagram

$$\begin{tikzcd}
k[\mathbf{G}] \otimes_k V \arrow[r, "1_{k[\mathbf{G}]} \otimes_k f"] & k[\mathbf{G}] \otimes_k W \\
k[\mathbf{M}] \otimes_k V \arrow[u, "i \otimes_k 1_V"] \arrow[r, dotted, "1_{k[\mathbf{M}]} \otimes_k f"] & k[\mathbf{M}] \otimes_k W \arrow[u, "i \otimes_k 1_W"'] \\
V \arrow[u, "c_V"] \arrow[r, "f"'] & W \arrow[u, "c_W"']
\end{tikzcd}$$

in which the outer square is commutative. Our goal is to prove that the bottom square is commutative. We have

$$(i \otimes_k 1_W) \cdot c_W \cdot f = \left(1_{k[\mathbf{G}]} \otimes_k f\right) \cdot (i \otimes_k 1_V) \cdot c_V = (i \otimes_k 1_W) \cdot \left(1_{k[\mathbf{M}]} \otimes_k f\right) \cdot c_V$$

Since $i \otimes_k 1_W$ is a monomorphism, we deduce that $c_W \cdot f = \left(1_{k[\mathbf{M}]} \otimes_k f\right) \cdot c_V$. Hence f is a morphism of $\mathbf{M}$-representations.
Next we prove that $\mathbf{Rep}(\mathbf{M})$ is a subcategory of $\mathbf{Rep}(\mathbf{G})$ that is closed under subquotients. Consider an $\mathbf{M}$-representation V and its quotient $\mathbf{G}$-representations $q : V \twoheadrightarrow W$. We show that W is a quotient $\mathbf{M}$-representation of V. Let c_V be the coaction of $\mathbf{M}$ on V and let d_W be the coaction of $\mathbf{G}$ on W. We have a commutative diagram

$$\begin{tikzcd}
k[\mathbf{G}]\otimes_k V \arrow[r, two heads, "1_{k[\mathbf{G}]}\otimes_k q"] & k[\mathbf{G}]\otimes_k W \\
k[\mathbf{M}]\otimes_k V \arrow[u, "i\otimes_k 1_V"] & \\
V \arrow[u, "c_V"] \arrow[r, two heads, "q"'] & W \arrow[uu, "d_W"']
\end{tikzcd}$$

and hence $d_W(W) \subseteq k[\mathbf{M}]\otimes_k W$. Thus Theorem 2.3 implies that W is a representation of $\mathbf{M}$ and q is a morphism of $\mathbf{M}$-representations. This shows that $\mathbf{Rep}(\mathbf{M})$ is a subcategory of $\mathbf{Rep}(\mathbf{G})$ closed under quotients. Next let $j : U \hookrightarrow V$ be a $\mathbf{G}$-subrepresentation of a $\mathbf{M}$-representation V. By what we proved above the cokernel $q : V \twoheadrightarrow W$ of j in $\mathbf{Rep}(\mathbf{G})$ is contained in $\mathbf{Rep}(\mathbf{M})$. Since both $\mathbf{Rep}(\mathbf{M})$ and $\mathbf{Rep}(\mathbf{G})$ are abelian and the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$ is exact, we derive that the kernel of q in $\mathbf{Rep}(\mathbf{M})$ coincides with its kernel in $\mathbf{Rep}(\mathbf{G})$. Thus U is a $\mathbf{M}$-representation and $j : U \hookrightarrow V$ is a morphism of $\mathbf{M}$-representations. Hence $\mathbf{Rep}(\mathbf{M})$ is the category of $\mathbf{Rep}(\mathbf{G})$ closed under subobjects. $\square$

Theorem 2.5. *Assume that* $\mathbf{G}$ *is open and schematically dense in* $\mathbf{M}$. *Let* V *be a* $\mathbf{G}$*-representation of* $\mathbf{G}$. *There exists an* $\mathbf{M}$*-representation* W *and a surjective morpism* $q : V \twoheadrightarrow W$ *of* $\mathbf{G}$*-representations such that for every* $\mathbf{M}$*-representation* U *and a morphism* $f : V \to U$ *of* $\mathbf{G}$*-representations there exists a unique morphism* $\tilde{f} : W \to U$ *of* $\mathbf{M}$*-representations making the triangle*

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, two heads, "q"'] & U \\
W \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

commutative.

Proof. Assume first that V is finite dimensional. Let $\mathcal{K}$ be a set of $\mathbf{G}$-subrepresentations of V that consists of all $K \subseteq V$ such that V/K carries a structure of $\mathbf{M}$-representation. Clearly $\mathcal{K} = \varnothing$ because $\{0\} \in \mathcal{K}$. Since V is finite dimensional, there exists a finite subset $\{K_1, ..., K_n\} \subseteq \mathcal{K}$ such that

$$\bigcap_{i=1}^{n} K_i = \bigcap_{K\in\mathcal{K}} K$$

Then a morphism

$$V/\left(\bigcap_{K\in\mathcal{K}} K\right) \ni v \mapsto \left(v \bmod K_i\right)_{1\leq i\leq n} \in \bigoplus_{i=1}^{n} V/K_i$$

is a monomorphism and hence by Theorem 2.4 the quotient $W = V/\left(\bigcap_{K\in\mathcal{K}} K\right)$ is an $\mathbf{M}$-representation. Let $q : V \twoheadrightarrow W$ be the canonical epimorphism. Consider now a morphism $f : V \to U$ of $\mathbf{G}$-representations, where U is an $\mathbf{M}$-representation. Then $\mathrm{im}(f)$ is a $\mathbf{G}$-subrepresentation of U and by Theorem 2.4 we derive that $\mathrm{im}(f)$ is an $\mathbf{M}$-representation. This implies that $\ker(f)$ is in $\mathcal{K}$. Hence f factors through q. Thus there exists a unique morphism $\tilde{f} : W \to U$ of $\mathbf{G}$-representations such that $\tilde{f}\cdot q = f$. This completes the proof in case when V is finite dimensional.
Now consider the general V. Let $\mathcal{F}$ be the set of all finite dimensional $\mathbf{G}$-representations of V. According to [Monygham, 2020b, Corollary 15.2] we deduce that $V = \mathrm{colim}_{F\in\mathcal{F}} F$. By the case considered above we deduce that for every F in $\mathcal{F}$ there exists a universal morphism $q_F : F \to W_F$ of $\mathbf{G}$-representations into an $\mathbf{M}$-representation W_F. Note that if $F_1 \subseteq F_2$ are two elements of $\mathcal{F}$, then

$$\begin{tikzcd} F_1 \arrow[r, two heads, "q_{F_1}"] \arrow[d, hook] & W_{F_1} \arrow[d] \\ F_2 \arrow[r, two heads, "q_{F_2}"'] & W_{F_2} \end{tikzcd}$$

Thus $\{W_F\}_{F \in \mathcal{F}}$ together with morphisms $W_{F_1} \to W_{F_2}$ for $F_1 \subseteq F_2$ in $\mathcal{F}$ form a diagram parametrized by the poset $\mathcal{F}$. The category $\mathbf{Rep}(\mathbf{M})$ has small colimits ([Monygham, 2020b, Corollary 14.5]) and we define $W = \mathrm{colim}_{F \in \mathcal{F}} W_F$. This is also a colimit of this diagram in the category $\mathbf{Rep}(\mathbf{G})$ by Fact 2.2. We also define $q = \mathrm{colim}_{F \in \mathcal{F}} q_F : V = \mathrm{colim}_{F \in \mathcal{F}} F \to W$. Since a colimit of a family of epimorphisms is an epimorphism, we derive that q is an epimorphism of $\mathbf{G}$-representations. Suppose now that $f : V \to U$ is a morphism of $\mathbf{G}$-representations and U is an $\mathbf{M}$-representation. Then $f_{|F}$ uniquely factors through q_F for every F in $\mathcal{F}$. Hence by universal property of colimits we derive that f factors through q in a unique way. This completes the proof. □

3. Białynicki-Birula functors

In this section we fix an affine group k-scheme $\mathbf{G}$. Let $\mathbf{M}$ be an affine monoid k-scheme with zero $\mathbf{o}$ such that $\mathbf{G}$ is its group of units. Note that if Y is a k-scheme, then $\mathbf{M} \times_k Y$ admits canonical action of $\mathbf{M}$ and

Definition 3.1. Let X be a k-scheme equipped with an action of $\mathbf{G}$. For every k-scheme Y we define

$$\mathcal{D}_X(Y) = \left\{\gamma : \mathbf{M} \times_k Y \to X \,\middle|\, \gamma \text{ is } \mathbf{G}\text{-equivariant}\right\}$$

This gives gives rise to a subfunctor $\mathcal{D}_X$ of $\mathrm{Mor}_k(\mathbf{M} \times_k (-), X) : \mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Set}$. We call it *the Białynicki-Birula functor of* X.

Remark 3.2. Let X be a k-scheme equipped with an action of $\mathbf{G}$. Then there are canonical morphism of functors

$$\begin{tikzcd} \mathcal{D}_X \arrow[r, "i_X"] \arrow[d, "r_X"] & X \\ X^{\mathbf{G}} \arrow[u, bend left, "s_X"] & \end{tikzcd}$$

which we define now. First let us explain that in the diagram X stands for the presheaf representable by the k-scheme X (Remark 1.1) and $X^{\mathbf{G}}$ denotes the functor of fixed points of X ([Monygham, 2020a, Definition 7.1]). Now fix k-scheme Y and $\gamma \in \mathcal{D}_X(Y)$, then we define

$$i_X(\gamma) = \gamma_{|\{e\} \times_k X} = \gamma \cdot \langle e, 1_X \rangle,\; r_X(\gamma) = \gamma_{|\{o\} \times_k X} = \gamma \cdot \langle o, 1_X \rangle$$

where $e : \mathrm{Spec}\, k \to \mathbf{M}$ is the unit of $\mathbf{M}$ and $\mathbf{o} : \mathrm{Spec}\, k \to \mathbf{M}$ is the zero. Next if $f : Y \to X$ is a morphism in $X^{\mathbf{G}}(Y)$, then we define

$$s_X(f) = f \cdot pr_Y$$

where $pr_Y : \mathbf{M} \times_k Y \to Y$ is the projection. Finally note that $r_X \cdot s_X = 1_{X^{\mathbf{G}}}$.

Remark 3.3. Let X be a k-scheme equipped with an action of $\mathbf{G}$. Then $\mathbf{M}$ (actually the presheaf of monoids represented by $\mathbf{M}$) acts on $\mathcal{D}_X$. Indeed, fix k-scheme Y, $\gamma \in \mathcal{D}_X(Y)$ and $m : Y \to \mathbf{M}$. Then we define the product

$$m\gamma = \gamma \cdot \langle m, 1_Y \rangle$$

and this determines an action of $\mathbf{M}$ on $\mathcal{D}_X$. Moreover, with respect to this action i_X is $\mathbf{G}$-equivariant and r_X, s_X are $\mathbf{M}$-equivariant ($X^{\mathbf{G}}$ is equipped with trivial action of $\mathbf{M}$).

Remark 3.4. Let X, Y be k-schemes equipped with actions of $\mathbf{G}$ and let $f : X \to Y$ be a $\mathbf{G}$-equivariant morphism, then there exists a morphism of functors $\mathcal{D}_f : \mathcal{D}_X \to \mathcal{D}_Y$ given by

$$\mathcal{D}_f(\gamma) = f \cdot \gamma$$

for every element γ of the functor $\mathcal{D}_X$.

Let X be a k-scheme equipped with an action of $\mathbf{G}$. It is useful to discuss subfunctors of $\mathcal{D}_X$ defined by closed $\mathbf{G}$-stable subschemes of X.

Theorem 3.5. *Let X be a k-scheme equipped with an action of the group $\mathbf{G}$. Suppose that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. If $j : Z \hookrightarrow X$ is a closed $\mathbf{G}$-stable subscheme of X, then the square*

$$\begin{tikzcd} \mathcal{D}_Z \arrow[r, "\mathcal{D}_j"] \arrow[d, "i_Z"'] & \mathcal{D}_X \arrow[d, "i_X"] \\ Z \arrow[r, "j"'] & X \end{tikzcd}$$

is cartesian in the category of presheaves on $\mathbf{Sch}_k$.

Proof. The fact that the square is commutative follows by examination of definitions in Remarks 3.2 and 3.4. Pick k-scheme Y, $f : Y \to Z$ and $\gamma \in \mathcal{D}_X(Y)$ such that $j \cdot f = i_X(\gamma)$. This is depicted in the diagram

$$\begin{tikzcd} & \gamma \arrow[d, mapsto, "i_X"] \\ f \arrow[r, mapsto, "j"'] & j \cdot f = \gamma_{|\{e\} \times_k X} \end{tikzcd}$$

Our goal is to show that there exists a unique $\mathbf{G}$-equivariant morphism $\eta : \mathbf{M} \times_k Y \to U$ such that $\mathcal{D}_j(\eta) = \gamma$ and $i_Z(\eta) = f$. This is depicted by the diagram

$$\begin{tikzcd} \eta \arrow[r, mapsto, "\mathcal{D}_j"] \arrow[d, mapsto, "r_U"'] & \gamma = j \cdot \eta \\ f = \eta_{|\{e\} \times_k X} & \end{tikzcd}$$

In order to achieve this it suffices to prove that γ factors through j. First note that the assumption $\gamma_{|\{e\} \times_k Y} = j \cdot f$ implies that

$$\gamma_{|\mathbf{G} \times_k Y} = j \cdot f \cdot pr_Y$$

where $pr_Y : \mathbf{G} \times_k Y \to Y$ is the projection. This implies that $\gamma_{|\mathbf{G} \times_k}$ factors through j. Consider scheme-theoretic preimage $\gamma^{-1}(Z)$. Then $\gamma^{-1}(Z)$ is a closed $\mathbf{G}$-stable (as an inverse image of a $\mathbf{G}$-stable closed subscheme under the $\mathbf{G}$-equivariant morphism) subscheme of $\mathbf{M} \times_k Y$, which contains $\mathbf{G} \times_k Y$. Since $\mathbf{G}$ is open, schematically dense in $\mathbf{M}$ and k is a field, we derive that $\mathbf{G} \times_k Y$ is open and schematically dense in $\mathbf{M} \times_k Y$. Thus $\gamma^{-1}(Z) = \mathbf{M} \times_k Y$ and hence γ factors through j. $\square$

In order to prove interesting result in the spirit of Theorem 3.5 which concerns open $\mathbf{G}$-stable subschemes, we need to assume that $\mathbf{M}$ is a Kempf monoid.

Theorem 3.6. *Let X be a k-scheme equipped with an action of the group $\mathbf{G}$ of units of a Kempf monoid $\mathbf{M}$. If $j : U \hookrightarrow X$ is an open $\mathbf{G}$-stable subscheme of X, then the square*

$$\begin{tikzcd} \mathcal{D}_U \arrow[r, "\mathcal{D}_j"] \arrow[d, "r_U"'] & \mathcal{D}_X \arrow[d, "r_X"] \\ U^{\mathbf{G}} \arrow[r, "j^{\mathbf{G}}"'] & X^{\mathbf{G}} \end{tikzcd}$$

is cartesian in the category of presheaves on $\mathbf{Sch}_k$.

As we shall see this result follows from the following.

Lemma 3.6.1. *Let K be an algebraicaly closed field over k. Suppose that*

$$\mathbf{M}_K = \operatorname{Spec} K \times_k \mathbf{M}, \mathbf{G}_K = \operatorname{Spec} K \times_k \mathbf{G}$$

and let $\mathbf{o}_K$ be the unique K-point of $\mathbf{M}_K$ lying over $\mathbf{o}$. Let V be an open $\mathbf{G}_K$-stable subscheme of $\mathbf{M}_K$ such that $\mathbf{o}_K \in V$. Then $V = \mathbf{M}_K$.

Proof of the lemma. Since $\mathbf{M}$ is a Kempf monoid, there exists a closed embedding of monoids $v : \mathbb{A}^1_K \hookrightarrow \mathbf{M}_K$ preserving zeros such that $v_{|\mathbb{G}_{m,K}} \subseteq \mathbf{G}_K$. Fix a point $p \in \mathbf{M}_K$ and let $u : \operatorname{Spec} k(p) \to \mathbf{M}_K$ be the associated morphism of K-schemes. Consider the composition

$$\begin{tikzcd} \mathbb{A}^1_{k(p)} = \mathbb{A}^1_K \times_K \operatorname{Spec} k(p) \arrow[r, hook, "v \times_K u"] \arrow[rr, bend left, "h"] & \mathbf{M}_K \times_K \mathbf{M}_K \arrow[r, hook] & \mathbf{M}_K \end{tikzcd}$$

where the second morphism is the multiplication. Clearly h is $\mathbf{G}_{m,k(p)}$-equivariant. Hence $h^{-1}(V)$ is an open $\mathbf{G}_{m,k(p)}$-stable subscheme of $\mathbb{A}^1_{k(p)}$ containing zero of this monoid $k(p)$-scheme (because $\mathbf{o}_K \in V$ by assumption). Since the only open $\mathbf{G}_{m,k(p)}$-stable subscheme of $\mathbb{A}^1_{k(p)}$ containing zero is $\mathbb{A}^1_{k(p)}$, we derive that $h^{-1}(V) = \mathbb{A}^1_{k(p)}$. Thus $p \in V$. Since p is arbitrary point of $\mathbf{M}_K$, we derive that $V = \mathbf{M}_K$. □

Proof of the theorem. The fact that the square is commutative follows by examination of definitions in Remarks 3.2 and 3.4. Pick k-scheme Y, $f \in U^{\mathbf{G}}$ and $\gamma \in \mathcal{D}_X(Y)$ such that $j^{\mathbf{G}}(f) = r_X(\gamma)$. This is depicted in the diagram

$$\begin{tikzcd} & \gamma \arrow[d, mapsto, "r_X"] \\ f \arrow[r, mapsto, "j^{\mathbf{G}}"'] & j \cdot f = \gamma_{|\{\mathbf{o}\} \times_k X} \end{tikzcd}$$

Our goal is to show that there exists a unique $\mathbf{G}$-equivariant morphism $\eta : \mathbf{M} \times_k Y \to U$ such that $\mathcal{D}_j(\eta) = \gamma$ and $r_U(\eta) = f$. This is depicted by the diagram

$$\begin{tikzcd} \eta \arrow[r, mapsto, "\mathcal{D}_j"] \arrow[d, mapsto, "r_U"'] & \gamma = j \cdot \eta \\ f = \eta_{|\{\mathbf{o}\}\times_k X} & \end{tikzcd}$$

In order to achieve this it suffices to prove that γ factors through j. Consider $W = \gamma^{-1}(U)$. Note that W is an open $\mathbf{G}$-stable (as an inverse image of a $\mathbf{G}$-stable open subscheme under the $\mathbf{G}$-equivariant morphism) subscheme of $\mathbf{M} \times_k Y$, which contains $\{\mathbf{o}\} \times_k Y$. Lemma 3.6.1 asserts that for every geometric point $\overline{y}$ of Y we have $W_{\overline{y}} = \mathbf{M}_{k(\overline{y})}$, where $W_{\overline{y}}$ is the fiber over $\overline{y}$ of the projection $\mathbf{M} \times_k Y \to Y$ restricted to W. Since W is open subscheme of $\mathbf{M} \times_k Y$, this implies that $W = \mathbf{M} \times_k Y$ and hence γ factors through j. □

As we shall see below both Theorems are extremely useful properties of Białynicki-Birula functors. Now we introduce a formal version of this functor.

Definition 3.7. Let $\mathbf{M}$ be an affine monoid k-scheme with zero $\mathbf{o}$ and let $\mathbf{G}$ be its group of units. For every $n \in \mathbb{N}$ let $\mathbf{M}_n \hookrightarrow \mathbf{M}$ be an n-th infinitesimal neighborhood of $\mathbf{o}$ in $\mathbf{M}$. Let X be a k-scheme equipped with an action of $\mathbf{G}$. For every k-scheme Y we define

$$\widehat{\mathcal{D}}_X(Y) = \left\{\{\gamma_n : \mathbf{M}_n \times_k Y \to X\}_{n\in\mathbb{N}} \,\middle|\, \forall_{n\in\mathbb{N}}\, \gamma_n \text{ is } \mathbf{G}\text{-equivariant and } \gamma_{n+1|\mathbf{M}_n\times_k Y} = \gamma_n\right\}$$

This gives gives rise to a functor $\widehat{\mathcal{D}}_X$. We call it *the formal Białynicki-Birula functor of* X.

Remark 3.8. Let $\mathbf{M}$ be an affine monoid k-scheme with zero $\mathbf{o}$ and let $\mathbf{G}$ be its group of units. Let X be a k-scheme equipped with an action of $\mathbf{G}$. Then there exists a canonical morphism of functors $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ given by

$$\gamma \mapsto \{\gamma_{|\mathbf{M}_n\times_k Y}\}_{n\in\mathbb{N}}$$

for every $\gamma \in \mathcal{D}_X(Y)$ and every k-scheme Y.

4. Representability of Białynicki-Birula functor for locally linear schemes

We first investigate open covers of Białynicki-Birula functor

Remark 4.1. Since $\mathbf{G}$ is geometrically connected and locally algebraic it follows by [Monygham, 2020a, Theorem 7.2] that for every k-scheme X equipped with an action of $\mathbf{G}$ there exists closed subscheme $X^{\mathbf{G}}$ of X representing

References

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