FLATNESS

1. Introduction

In this notes we assume that rings are unital but possibly noncommutative. For homological algebra (exact sequences, the notion of projective module, properties of Tor etc.) the reader should consult [Weibel, 1995].

2. Filtered colimits in the category of modules

Definition 2.1. Let I be a category. Suppose that the following conditions are satisfied.

(1) For any objects $i, j \in I$ there exists an object $k \in I$ and a diagram

$$\begin{tikzcd} & k & \\ i \arrow[ur] & & j \arrow[ul] \end{tikzcd}$$

(2) For any pair of parallel morphisms in I

$$i \rightrightarrows j$$

there exist an object $k \in I$ and a morphism $j \to k$ such that, the following diagram is commutative

$$i \rightrightarrows j \longrightarrow k$$

Then we say that I is *a filtered category*.

Let R be a ring.

Proposition 2.2. *Let I be a small filtered category. Then the functor sending I-indexed diagram of left R-modules to its colimit is exact.*

Proof. Suppose that

$$\left\{0 \longrightarrow K_i \xrightarrow{r_i} M_i \xrightarrow{p_i} N_i \longrightarrow 0\right\}_{i \in I}$$

is an I-indexed family of short exact sequences. Consider a complex

$$0 \longrightarrow K \xrightarrow{r} M \xrightarrow{p} N \longrightarrow 0$$

where $K = \mathrm{colim}_{i\in I} K_i$, $M = \mathrm{colim}_{i\in I} M_i$, $N = \mathrm{colim}_{i\in I} N_i$, $r = \mathrm{colim}_{i\in I} r_i$ and $p = \mathrm{colim}_{i\in I} p_i$. Clearly the complex is right exact. It suffices to prove that r is a monomorphism. For $i \in I$ denote by

$v_i : K_i \to K$, $u_i : M_i \to M$ structural morphisms. Pick $k \in K$ such that $r(k) = 0$. Since I is filtered, we have

$$K = \sum_{i \in I} v_i(K_i),\ M = \sum_{i \in I} u_i(M_i)$$

Thus there exists $i_0 \in I$ and $k_{i_0} \in K_{i_0}$ such that $v_{i_0}(k_{i_0}) = k$. We have $u_{i_0}\left(r_{i_0}(k_{i_0})\right) = r(k) = 0$. Again using the fact that I is filtered, we deduce that there exist $i_1 \in I$ and a morphism $\alpha : i_0 \to i_1$ such that $u_\alpha\left(r_{i_0}(k_{i_0})\right) = 0$, where $u_\alpha : M_{i_0} \to M_{i_1}$ is a morphism in the I-indexed diagram $\{M_i\}_{i \in I}$. Now let $k_{i_1} = v_\alpha(k_{i_0})$, where $v_\alpha : K_{i_0} \to K_{i_1}$ is a morphism in the I-indexed diagram $\{K_i\}_{i \in I}$. Then $v_{i_1}(k_{i_1}) = k$ and $r_{i_1}(k_{i_1}) = 0$. Since r_{i_1} is a monomorphism, we derive that $k_{i_1} = 0$ and hence $k = v_{i_1}(k_{i_1}) = 0$. Thus r is a monomorphism. □

Corollary 2.3. *Let M be a right R-module. Then for every $i \in \mathbb{N}$ functor $\mathrm{Tor}_i^R(M, -)$ defined on the category of left R-modules and with values in the category of abelian groups preserves filtered colimits.*

Proof. Let I be a small filtered category and $\{N_i\}_{i \in I}$ be an I-indexed diagram of left R-modules. Fix a projective resolution $P_\bullet \to M$ of M. Since tensor product commutes with colimits, we have

$$\mathrm{colim}_{i \in I}\left(P_\bullet \otimes_R N_i\right) = P_\bullet \otimes_R \mathrm{colim}_{i \in I} N_i$$

in the category of complexes of abelian groups. Since exact functors preserve kernels, cokernels and images, we derive by Proposition 2.2 that for every $n \in \mathbb{N}$ there is an identification

$$\mathrm{Tor}_n^R\left(M, \mathrm{colim}_{i \in I} N_i\right) = H_n\left(P_\bullet \otimes_R \mathrm{colim}_{i \in I} N_i\right) = H_n\left(\mathrm{colim}_{i \in I}\left(P_\bullet \otimes_R N_i\right)\right) =$$

$$= \mathrm{colim}_{i \in I} H_n\left(P_\bullet \otimes_R N_i\right) = \mathrm{colim}_{i \in I} \mathrm{Tor}_n^R\left(M, N_i\right)$$

of cocones. □

3. Homological characterizations of flatness

Let R be a ring with unit.

Definition 3.1. Let M be a right R-module. We say that M is *flat* if the functor $M \otimes_R (-)$ defined on the category of left R-modules and with values in the category of abelian groups is exact.

Proposition 3.2. *Let I be a filtered category and $\{M_i\}_{i \in I}$ be an I-indexed diagram of flat right R-modules. Then $\mathrm{colim}_{i \in I} M_i$ is a flat right R-module.*

Proof. Proposition 2.2 implies that filtered colimits of short exact sequences of abelian groups are short exact seqeunces. Thus filtered colimits of flat right R-modules are flat. □

Proposition 3.3 (Homological criterion for flatness)**.** *Let M be a right R-module. Then the following are equivalent.*

- **(i)** *For every finitely generated left ideal $I \subseteq R$ morphism $M \otimes_R I \to M$ induced by the inclusion of I in R is a monomorphism.*
- **(ii)** $\mathrm{Tor}_1^R(M, R/I) = 0$ *for every finitely generated left ideal $I \subseteq R$.*
- **(iii)** *M is flat.*
- **(iv)** $\mathrm{Tor}_i^R(M, N) = 0$ *for every left R-module N and $i > 0$.*

Proof. The implication **(i)** ⇒ **(ii)** is straightforward.

Suppose that **(ii)** holds. Then for every left ideal $I \subseteq R$ we can write $I = \mathrm{colim}_{\lambda \in \Lambda} I_\lambda$, where $\{I_\lambda\}_{\lambda \in \Lambda}$ is a filtered set of all finitely generated left ideals of R contained in I. This induces a presentation of R/I as a filtered colimit of the system $\{R/I_\lambda\}_{\lambda \in \Lambda}$ and thus by Corollary 2.3 we have

$$\mathrm{Tor}_1^R(M, R/I) = \mathrm{colim}_{\lambda \in \Lambda} \mathrm{Tor}_1^R(M, R/I_\lambda) = 0$$

Now suppose that N is a finitely generated left module over R. Then we can decompose N such that it fits in an exact sequence

$$0 \longrightarrow K \longrightarrow N \xrightarrow{q} R/I \longrightarrow 0$$

Now we have $\mathrm{Tor}_1^R(M, K) = 0$ implies that $\mathrm{Tor}_1^R(M, N) = 0$. Therefore, using induction on the minimal number of generators of finitely generated left R-module we may prove that $\mathrm{Tor}_1^R(M, N) = 0$ for every finitely generated left R-module. Since every left R-module is a filtered colimit of its finitely generated left R-submodules, we derive by Corollary 2.3 that $\mathrm{Tor}_1^R(M, N) = 0$ for every left R-module N. Using first terms of the long exact sequence for Tor associated with $M \otimes_R (-)$ we deduce **(iii)**.
Now if M is flat, then tensoring with M is exact. This means that tensor product of a free resolution of a left R-module N with M has trivial higher homologies. Thus $\mathrm{Tor}_i^R(M, N) = 0$ for $i > 0$. This proves **(iii)** $\Rightarrow$ **(iv)**.
Finally **(iv)** $\Rightarrow$ **(i)** is obvious. □

4. Flatness in terms of equations

Let R be a ring with unit.

Proposition 4.1. *Let M be a right R-module and N be a left R-module. Suppose that $\{y_i\}_{i\in I}$ is a set of generators for N and $\{x_i\}_{i\in I}$ is a set of elements of M. Suppose that all x_i for $i \in I$ except of finitely many are zero. Assume that*

$$\sum_{i\in I} x_i \otimes y_i = 0$$

in tensor product $M \otimes_R N$. Then there exist $n \in \mathbb{N}$, $\{a_{ik}\}_{i\in I, 1\le k\le n}$ in R and $\{z_k\}_{1\le k\le n}$ in M such that

$$x_i = \sum_{k=1}^{n} z_k a_{ki}$$

for every $i \in I$ and

$$\sum_{i\in I} a_{ki} y_i = 0$$

for every $1 \le k \le n$.

Proof. Consider a free left R-module F on a set I and a morphism $\phi : F \to N$ given by $\phi(e_i) = y_i$, where e_i is a free generator corresponding to $i \in I$. Let $K = \ker(\phi)$. Applying $M \otimes_R (-)$ we derive that $M \otimes_R K$ maps onto the kernel of $1_M \otimes_R \phi$. Next by assumptions $(1_M \otimes_R \phi)(\sum_{i\in I} x_i \otimes e_i) = \sum_{i\in I} x_i \otimes y_i = 0$. Thus $\sum_{i\in I} x_i \otimes e_i$ is equal to $\sum_{k=1}^{n} z_k \otimes f_k$ for $z_k \in M$, $f_k \in K$ and $n \in \mathbb{N}$. We can write $f_k = \sum_{i\in I} a_{ki} e_i$. Then we have

$$\sum_{k=1}^{n} z_k \otimes f_k = \sum_{k=1}^{n} z_k \otimes \left(\sum_{i\in I} a_{ki} e_i\right) = \sum_{k=1}^{n}\sum_{i\in I} (z_k \otimes a_{ki} e_i) = \sum_{i\in I}\sum_{k=1}^{n} (z_k a_{ki} \otimes e_i) = \sum_{i\in I}\left(\sum_{k=1}^{n} z_k a_{ki}\right) \otimes e_i$$

We deduce that $x_i = \sum_{k=1}^{n} z_k a_{ki}$ and $0 = \sum_{i\in I} a_{ki} y_i$ for every $i \in I$ and $1 \le k \le n$. □

Theorem 4.2 (Equational criteria for flatness)**.** *Let M be a right R-module. Then the following are equivalent.*

(i) *M is flat.*

(ii) *For every set of elements $\{x_i\}_{i=1,\dots,n}$ in M and a relation*

$$\sum_{i=1}^{n} x_i a_i = 0$$

where $a_i \in R$ there exist elements $z_k \in M$ and $r_{ki} \in R$ for $1 \le k \le l$ such that

$$x_i = \sum_{k=1}^{l} z_k r_{ki},\ \sum_{i=1}^{n} r_{ki} a_i = 0$$

for every $1 \leq i \leq n$ and $1 \leq k \leq l$.

(iii) *For every finitely presented right R-module N, every morphism $\phi : N \to M$ and every finitely generated R-submodule $K \subseteq \ker(\phi)$ there exists a factorization*

$$\begin{tikzcd} & G \arrow[dr, "\theta"] & \\ N \arrow[ur, "\psi"] \arrow[rr, "\phi"'] & & M \end{tikzcd}$$

where G is a finitely generated free right R-module and $K \subseteq \ker(\psi)$.

(iv) *For every set of elements $\{x_i\}_{i=1,...,n}$ in M and a finite set of relations*

$$\sum_{i=1}^{n} x_i a_{ij} = 0$$

where $1 \leq j \leq m$ and $a_{ij} \in R$ there exist elements $z_k \in M$ and $r_{ki} \in R$ for $1 \leq k \leq l$ such that

$$x_i = \sum_{k=1}^{l} z_k r_{ki},\ 0 = \sum_{i=1}^{n} r_{ki} a_{ij}$$

for every $1 \leq i \leq n$, $1 \leq j \leq m$ and $1 \leq k \leq l$.

Proof. Suppose that M is flat. We show then that **(ii)** holds. We have relation

$$\sum_{i=1}^{n} x_i a_i = 0$$

Let $I = \sum_{1 \leq i \leq n} R a_i \subseteq R$ be a left ideal. Since M is flat, the canonical morphism $M \otimes_R I \to M$ is a monomorphism. It sends $\sum_{i=1}^{n} x_i \otimes a_i$ to $\sum_{i=1}^{n} x_i a_i = 0$. It follows that

$$\sum_{i=1}^{n} x_i \otimes a_i = 0$$

in $M \otimes_R I$. Thus by Proposition 4.1 there exist $\{r_{ki}\}_{1 \leq i \leq n, 1 \leq k \leq l}$ in R and $\{z_k\}_{1 \leq k \leq l}$ in M such that

$$x_i = \sum_{k=1}^{l} z_k r_{ki},\ 0 = \sum_{i=1}^{n} r_{ki} a_i$$

for every $1 \leq i \leq n$ and $1 \leq k \leq l$.
Now we prove that **(ii)** $\Rightarrow$ **(iii)**. Suppose first that N is a finitely generated and free right R-module, $\phi : N \to M$ is a morphism and $K \subseteq \ker(\phi)$ is finitely generated. Note that our result easily follows from **(ii)**, if $K \subseteq \ker(\phi)$ is generated by a single element. Now easy induction on the number of generators for $K \subseteq \ker(\phi)$ yields the assertion **(iii)** in the case of finitely generated free right R-module N.
Suppose now that N is a finitely presented right R-module, $\phi : N \to M$ is a morphism and $K \subseteq \ker(\phi)$ is a finitely generated submodule. Take an epimorphism $f : F \to N$ where F is a finitely generated free left R-module. Let $\phi' = \phi f$ and pick a factorization

$$\begin{tikzcd} & G \arrow[dr, "\theta"] & \\ F \arrow[ur, "g"] \arrow[rr, "\phi'"'] & & M \end{tikzcd}$$

where G is a finitely generated free right R-module and $f^{-1}(K) \subseteq \ker(g)$. Such a factorization exists according to the fact that $f^{-1}(K)$ is a finitely generated submodule of $\ker(\phi')$. Since $\ker(f) \subseteq f^{-1}(K)$, we deduce that g factorizes through f. This proves the implication.
Assume that **(iii)** holds. Suppose that $\{x_i\}_{i=1,...,n}$ are in M and that we have a finite set of relations

$$\sum_{i=1}^{n} x_i a_{ij} = 0$$

where $1 \leq j \leq m$ and $a_{ij} \in R$. Let F be a right free R-module of rank n with basis $e_1,...,e_n$. Define a morphism $\phi : F \to M$ by $\phi(e_i) = x_i$ for $1 \leq i \leq n$. Then

$$K = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} e_i a_{ij} \right) R \subseteq \ker(\phi)$$

is finitely generated. Hence by **(iii)** there exist a finitely generated free right R-module G and morphisms $\psi : F \to G$, $\theta : G \to M$ such that $\phi = \theta \cdot \psi$ and $K \subseteq \ker(\psi)$. Next if $f_1,...,f_l$ is a basis of G, then we pick $z_k = \theta(f_k)$ for $1 \leq k \leq l$. There exist $r_{ki} \in R$ for $1 \leq k \leq l$ and $1 \leq i \leq n$ such that $\psi(e_i) = \sum_{k=1}^{l} f_k r_{ki}$ for $1 \leq i \leq n$. Now straightforward verification shows that $z_k \in M$ and $r_{ki} \in R$ for $1 \leq k \leq l$ and $1 \leq i \leq n$ satisfy **(iv)**.
Now assume that **(iv)** holds. Let I be a finitely generated left ideal in R. Suppose that a_i for $1 \leq i \leq n$ are generators of I. We are going to prove that the canonical morphism $M \otimes_R I \to M$ is a monomorphism. This implies **(i)** due to Proposition 3.3. Assume that there exist x_i for $1 \leq i \leq n$ in M such that $\sum_{i=1}^{n} x_i \otimes a_i \in M \otimes_R I$ is in the kernel of $M \otimes_R I \to M$. This means that $\sum_{i=1}^{n} x_i a_i = 0$ in M. According to **(iv)** there exist $z_k \in M$ and $r_{ki} \in R$ for $1 \leq k \leq l$ and $1 \leq i \leq n$ such that

$$x_i = \sum_{k=1}^{l} z_k r_{ki},\ 0 = \sum_{i=1}^{n} r_{ki} a_i$$

Thus

$$\sum_{i=1}^{n} x_i \otimes a_i = \sum_{i=1}^{n} \left(\sum_{k=1}^{l} z_k r_{ki} \right) \otimes a_i = \sum_{i=1}^{n} \sum_{k=1}^{l} (z_k r_{ki} \otimes a_i) = \sum_{k=1}^{l} \sum_{i=1}^{n} (z_k \otimes r_{ki} a_i) = \sum_{k=1}^{l} z_k \otimes \left(\sum_{i=1}^{n} r_{ki} a_i \right) = 0$$

Hence the kernel of the morphism $M \otimes_R I \to M$ is trivial. □

5. Categorical characterizations of flatness

Let R be a ring with unit.

Theorem 5.1 (Lazard's theorem). *A right R-module M is flat if and only if it is a colimit of a filtered diagram of finitely generated free right R-modules.*

Proof. If M is a filtered colimit of finitely generated flat right R-modules, then Proposition 3.2 implies that M is flat.
Assume now that M is flat. Consider a set of symbols $E = \{e_m \mid m \in M\}$. For every finite subset $\alpha \subseteq E$ let F_α be a right free R-module generated by symbols in α. Next for every such α let $q_\alpha : F_\alpha \to M$ be a morphism defined by formula $q_\alpha(e_m) = m$ for $e_m \in \alpha$.
Next we define a small diagram category I. Objects of I are finite subsets $\alpha \subseteq E$. Morphisms $f : \alpha \to \beta$ for any two finite subsets α, $\beta \subseteq E$ are morphisms of right R-modules $f : F_\alpha \to F_\beta$ such that $q_\beta \cdot f = q_\alpha$. The composition of morphisms in I is given by the usual composition of morphisms of right R-modules.
We will now show that I is a filtered category. Pick α_1, $\alpha_2 \in I$. Let $\alpha = \alpha_1 \cup \alpha_2$. Then α is well defined object of I. Moreover, canonical inclusions $\alpha_1 \subseteq \alpha$, $\alpha_2 \subseteq \alpha$ give rise to morphisms $f_1 : F_{\alpha_1} \to F_\alpha$ and $f_2 : F_{\alpha_2} \to F_\alpha$ in the category of right R-modules and hence give rise to morphisms $f_1 : \alpha_1 \to \alpha$ and $f_2 : \alpha_2 \to \alpha$ in I. This verifies the first axiom of filtered category for I. Now if $f, g : \alpha \to \beta$ are two morphisms in I, then

$$q_\beta \cdot (f - g) = q_\alpha - q_\alpha = 0$$

in the category of right R-modules. Hence $(f-g)(F_\alpha)$ is a finitely generated right R-submodule of F_β contained in the kernel of q_β. Using Theorem 4.2 we derive that there exists some finite subset $\gamma \subseteq E$ and a morphism $h : F_\beta \to F_\gamma$ such that $h \cdot (f-g) = 0$ and $q_\gamma \cdot h = q_\beta$. This implies that $h : \beta \to \gamma$ is a morphism in I and $h \cdot f = h \cdot g$. Hence I verifies the second axiom for filtered category.

Now we define a diagram of finitely generated free right R-modules indexed by I. We send each object α of I to right R-module F_α and we send $f : \alpha \to \beta$ in I to $f : F_\alpha \to F_\beta$ in the category of right R-modules. It is clear that it is well defined I-indexed diagram.

Finally it suffices to verify that $q_\alpha : F_\alpha \to M$ for $\alpha \in I$ admit the universal property of colimit for the I-indexed diagram defined above. For this let N be some right R-module and $r_\alpha : F_\alpha \to N$ for $\alpha \in I$ be morphisms such that $r_\beta \cdot f = r_\alpha$ for every $f : \alpha \to \beta$ in I. Now we define a function $s : M \to N$ by formula

$$s(m) = r_\alpha(e_m)$$

for any $m \in M$ and any $\alpha \in I$ such that $e_m \in \alpha$. It is easy to verify that the function s is well defined. Moreover, it is a unique function that satisfies $s \cdot q_\alpha = r_\alpha$.

We will show now that s is a morphism of right R-modules. Pick $x \in R$ and $m \in M$. Consider $\alpha \in I$ such that $e_m, e_{mx} \in \alpha$. Since $q_\alpha(e_m x - e_{mx}) = mx - mx = 0$ and M is flat, by Theorem 4.2 there exist $\beta \in I$ and a morphism $f : \alpha \to \beta$ in I such that $f(e_m x - e_{mx}) = 0$. Hence we deduce that

$$s(m)x - s(mx) = r_\alpha(e_m)x - r_\alpha(e_{mx}) = r_\alpha(e_m x - e_{mx}) = r_\beta\left(f\left(e_m x - e_{mx}\right)\right) = 0$$

Similar argument shows that for $m_1, m_2 \in M$ the relation $s(m_1 + m_2) - (s(m_1) + s(m_2)) = 0$ is satisfied.

Now according to the fact that $s : M \to N$ is a unique morphism of cocones in the category of right R-modules, we deduce that

$$M = \mathrm{colim}_{\alpha \in I} F_\alpha$$

□

Corollary 5.2. *Let M be a right R-module of finite presentation. Then M is flat if and only if it is projective.*

Proof. Using Theorem 5.1 we derive that $M = \mathrm{colim}_{\alpha \in I} F_\alpha$, where I is a filtered category and $\{F_\alpha\}_{\alpha \in I}$ is I-indexed diagram of finitely generated right free R-modules. Next we have that

$$\mathrm{Hom}_R(M, M) = \mathrm{colim}_{\alpha \in I} \mathrm{Hom}_R(M, F_\alpha)$$

by finite presentation of M. Thus there exists an $\alpha \in I$ and a morphism $f : M \to F_\alpha$ such that $q_\alpha \cdot f = 1_M$ for the structural morphism $q_\alpha : F_\alpha \to M$. This means that q_α is a retraction. Hence M is a direct summand of a right free R-module P_β. Thus it is projective. □

References

[Weibel, 1995] Weibel, C. (1995). *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press.