

GEOMETRY OF k -FUNCTORS

1. INTRODUCTION

In this notes we present functorial approach to algebraic geometry. Our aim is to show that functorial and geometrical techniques are interrelated in a very efficient way.

2. k -FUNCTORS

Definition 2.1. The category $\mathbf{Fun}(\mathbf{Alg}_k, \mathbf{Set})$ of copresheaves on \mathbf{Alg}_k is called *the category of k -functors*.

If \mathfrak{X} and \mathfrak{Y} are k -functors, then we denote by $\mathrm{Mor}_k(\mathfrak{X}, \mathfrak{Y})$ the class of morphisms $\mathfrak{X} \rightarrow \mathfrak{Y}$ of k -functors.

Since the category of k -functors is a category of copresheaves, under assumptions specified in [Mon19a, section 5] for given k -functors $\mathfrak{X}, \mathfrak{Y}$ there exists an internal hom $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})$. Let us discuss this important notion and also related ones. For details and proofs for general case we refer to [Mon19a, section 5].

Let \mathfrak{X} and \mathfrak{Y} be A -functors for some k -algebra A . Then we denote by $\mathrm{Mor}_A(\mathfrak{X}, \mathfrak{Y})$ the class of morphisms of A -functors $\mathfrak{X} \rightarrow \mathfrak{Y}$. For every A -algebra B and a morphism $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$ of A -functors we denote by $\mathfrak{X}_B, \mathfrak{Y}_B, \sigma_B$ the restrictions $\mathfrak{X}|_{\mathbf{Alg}_B}, \mathfrak{Y}|_{\mathbf{Alg}_B}, \sigma|_{\mathbf{Alg}_B}$ of these entities to the category of B -algebras.

Fact 2.2. Let \mathfrak{X} and \mathfrak{Y} be k -functors. Assume that A is a k -algebra, B is an A -algebra, C is an B -algebra. Then the composition of maps of classes

$$\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A) \xrightarrow{\sigma \mapsto \sigma_B} \mathrm{Mor}_B(\mathfrak{X}_B, \mathfrak{Y}_B) \xrightarrow{\sigma \mapsto \sigma_C} \mathrm{Mor}_C(\mathfrak{X}_C, \mathfrak{Y}_C)$$

equals

$$\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A) \xrightarrow{\sigma \mapsto \sigma_C} \mathrm{Mor}_C(\mathfrak{X}_C, \mathfrak{Y}_C)$$

Proof. Left to the reader. □

Definition 2.3. Let \mathfrak{X} and \mathfrak{Y} be k -functors and suppose that for every k -algebra A the class $\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set. We define

$$\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(A) = \mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$$

for every k -algebra A . This is a k -functor, since for every k -algebra A and A -algebra B , we can compose a morphism $\sigma : \mathfrak{X}_A \rightarrow \mathfrak{Y}_A$ of k -functors with the forgetful functor $\mathbf{Alg}_B \rightarrow \mathbf{Alg}_A$ i.e. we have a map

$$\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(A) \ni \sigma \mapsto \sigma_B \in \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(B)$$

and these according to Fact 2.2 make $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})$ a k -functor. The k -functor $\mathcal{M}\mathrm{or}_C(\mathfrak{X}, \mathfrak{Y})$ is called a *hom k -functor of \mathfrak{X} and \mathfrak{Y}* .

We define a k -functor $\mathbf{1}$ that assigns to every k -algebra a set with one element. For every k -algebra A the restriction $\mathbf{1}_A$ is a terminal object in the category of A -functors.

Fact 2.4. Let \mathfrak{X} be a k -functor. Suppose A is a k -algebra and $x \in \mathfrak{X}(A)$. Then x determines a morphism $\mathbf{1}_A \rightarrow \mathfrak{X}_A$ that for every A -algebra B with structural morphism $f : A \rightarrow B$ sends a unique element of $\mathbf{1}_A(B)$ to $\mathfrak{X}(f)(x) \in \mathfrak{X}_A(B)$. This gives rise to a bijection

$$\mathfrak{X}(A) \cong \text{Mor}_A(\mathbf{1}_A, \mathfrak{X}_A)$$

Proof. We left to the reader as an exercise. \square

Definition 2.5. Let \mathfrak{X} be a k -functor and A be a k -algebra. The set $\mathfrak{X}(A)$ is called *the set of A -points of \mathfrak{X}* .

Now let $\mathfrak{X}, \mathfrak{Y}$ be k -functors such that for every k -algebra A the class $\text{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set. Suppose next that \mathfrak{U} is a k -functor and $\sigma : \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ is a morphism of k -functors. Fix $x \in \mathfrak{U}(A)$. We denote by $i_x : \mathbf{1}_A \rightarrow \mathfrak{U}_A$ the morphism of A -functors corresponding to x by means of Fact 2.4. Since $\mathbf{1}_A$ is terminal A -functor, a morphism $\sigma_A \cdot (\mathbf{1}_{\mathfrak{X}_A} \times i_x)$ is isomorphic to a morphism $\tau_x : \mathfrak{X}_A \rightarrow \mathfrak{Y}_A$ of A -functors. Next $x \mapsto \tau_x$ gives rise to a morphism $\tau : \mathfrak{U} \rightarrow \text{Mor}_k(\mathfrak{X}, \mathfrak{Y})$ of k -functors and hence we have a map of classes

$$\text{Mor}_k(\mathfrak{U} \times \mathfrak{X}, \mathfrak{Y}) \ni \sigma \mapsto \tau \in \text{Mor}_k(\mathfrak{U}, \text{Mor}_k(\mathfrak{X}, \mathfrak{Y}))$$

Now we have the following result [Mon19a, Theorem 5.3].

Theorem 2.6. Let $\mathfrak{X}, \mathfrak{Y}$ be k -functors. Assume that for every k -algebra A the class $\text{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set. Then the map

$$\text{Mor}_k(\mathfrak{U} \times \mathfrak{X}, \mathfrak{Y}) \rightarrow \text{Mor}_k(\mathfrak{U}, \text{Mor}_k(\mathfrak{X}, \mathfrak{Y}))$$

described above is a bijection natural in \mathfrak{U} .

Now we start a discussion of a more geometric flavour. For every k -algebra A we denote by k_A the k -functor given by

$$k_A(B) = \text{Hom}_k(A, B), \quad k_A(g) = \text{Hom}_k(1_A, f)$$

for every k -algebra B and for every morphism $g : B \rightarrow C$ of k -algebras. Note that if $f : A \rightarrow B$ is a morphism of k -algebras, then there exists a morphism of k -functors $k_f : k_B \rightarrow k_A$ given by formula

$$k_f(C) = \text{Hom}_k(f, 1_C)$$

where C is a k -algebra. These are general definitions that make sense in any category of copresheaves c.f. [Mon19a, section 7].

Definition 2.7. Let \mathfrak{X} be a k -functor. We say that \mathfrak{X} is *corepresentable* if \mathfrak{X} is isomorphic to k_A for some k -algebra A .

Definition 2.8. Let $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of k -functors. Assume that for every k -algebra A and every morphism $\tau : k_A \rightarrow \mathfrak{X}$ of k -functors there exist an ideal \mathfrak{a} in A and morphism $\tau' : k_{A/\mathfrak{a}} \rightarrow \mathfrak{X}$ such that the square

$$\begin{array}{ccc} k_{A/\mathfrak{a}} & \xrightarrow{\tau'} & \mathfrak{X} \\ k_q \downarrow & & \downarrow \sigma \\ k_A & \xrightarrow{\tau} & \mathfrak{X} \end{array}$$

is cartesian, where $q : A \rightarrow A/\mathfrak{a}$ is the quotient morphism of k -algebras. Then σ is a *closed immersion of k -functors*.

Consider a k -algebra A and an ideal \mathfrak{a} in A . We define a k -subfunctor $D(\mathfrak{a})$ of k_A by

$$D(\mathfrak{a})(B) = \{f : A \rightarrow B \mid 1 \in f(\mathfrak{a})B\}$$

In other words $f : A \rightarrow B$ is a B -point of $D(\mathfrak{a})$ if and only if we have cartesian diagram in the category of k -schemes

$$\begin{array}{ccc}
 \emptyset & \longrightarrow & \operatorname{Spec} A/\mathfrak{a} \\
 \downarrow & & \downarrow \operatorname{Spec} q \\
 \operatorname{Spec} B & \xrightarrow{\operatorname{Spec} f} & \operatorname{Spec} A
 \end{array}$$

where $q : A \rightarrow A/\mathfrak{a}$ is the quotient morphism.

Definition 2.9. Let $\sigma : \mathfrak{X} \rightarrow \mathfrak{X}$ be a morphism of k -functors. Assume that for every k -algebra A and every morphism $\tau : k_A \rightarrow \mathfrak{X}$ of k -functors there exist an ideal \mathfrak{a} in A and a morphism $\tau' : D(\mathfrak{a}) \rightarrow \mathfrak{X}$ of k -functors such that the square

$$\begin{array}{ccc}
 D(\mathfrak{a}) & \xrightarrow{\tau'} & \mathfrak{X} \\
 \downarrow & & \downarrow \sigma \\
 k_A & \xrightarrow{\tau} & \mathfrak{X}
 \end{array}$$

is cartesian. Then σ is an open immersion of k -functors.

Fact 2.10. The class of open (closed) immersions of k -functors is closed under base change.

Proof. Left to the reader. □

3. ZARISKI LOCAL k -FUNCTORS AND ZARISKI SHEAVES

In this part we use notion of a Grothendieck topology on a category. For this notion we refer the reader to [Mon19b].

Definition 3.1. Let $\{f_i : X_i \rightarrow X\}_{i \in I}$ be a family of morphisms of k -schemes. We say that $\{f_i\}_{i \in I}$ is a Zariski covering of X if the following conditions are satisfied.

- (1) For every $i \in I$ morphism f_i is an open immersion of schemes.
- (2) Morphism $\coprod_{i \in I} X_i \rightarrow X$ induced by $\{f_i\}_{i \in I}$ is surjective.

The collection of all Zariski coverings on \mathbf{Sch}_k is a Grothendieck pretopology.

Definition 3.2. We call the Grothendieck topology generated by pretopology of Zariski coverings the Zariski topology on \mathbf{Sch}_k . A presheaf on \mathbf{Sch}_k that is a sheaf with respect to Zariski topology on \mathbf{Sch}_k is called a Zariski sheaf.

Let \mathfrak{X} be a presheaf on the category of k -schemes. Recall that by [Mon19b, Theorem 3.5] \mathfrak{X} is a Zariski sheaf if and only if for every k -scheme X and for every Zariski covering $\{f_i : X_i \rightarrow X\}$ of X the diagram

$$\mathfrak{X}(X) \xrightarrow{\langle \mathfrak{X}(f_i) \rangle_{i \in I}} \prod_{i \in I} \mathfrak{X}(X_i) \xrightleftharpoons[\langle \mathfrak{X}(f''_{ij}) \cdot pr_j \rangle_{(i,j)}]{\langle \mathfrak{X}(f'_{ij}) \cdot pr_i \rangle_{(i,j)}} \prod_{(i,j) \in I \times I} \mathfrak{X}(X_i \times_X X_j)$$

is a kernel of a pair of arrows, where for every $(i, j) \in I \times I$ morphisms f'_{ij} and f''_{ji} form a cartesian square

$$\begin{array}{ccc}
X_i \times_X X_j & \xrightarrow{f''_{ij}} & X_j \\
f'_{ij} \downarrow & & \downarrow f_j \\
X_i & \xrightarrow{f_i} & X
\end{array}$$

Now we repeat this definitions for k -algebras and k -functors.

Definition 3.3. Let $\{f_i : A \rightarrow A_i\}_{i \in I}$ be a family of morphisms of k -algebras. We say that $\{f_i\}_{i \in I}$ is a *Zariski covering* of A if the following conditions are satisfied.

- (1) For every $i \in I$ morphism $\text{Spec } f_i$ is an open immersion of schemes.
- (2) Morphism $\coprod_{i \in I} \text{Spec } A_i \rightarrow \text{Spec } A$ induced by $\{\text{Spec } f_i\}_{i \in I}$ is surjective.

The collection of all Zariski coverings on \mathbf{Alg}_k induces on its opposite category \mathbf{Aff}_k of affine k -schemes a Grothendieck pretopology.

Definition 3.4. We call the Grothendieck topology generated by pretopology of Zariski coverings the *Zariski topology* on \mathbf{Aff}_k . A k -functor that is a sheaf with respect to Zariski topology on \mathbf{Aff}_k is called a *Zariski local k -functor*.

Let \mathfrak{X} be a k -functor. Again by [Mon19b, Theorem 3.5] \mathfrak{X} is a Zariski local k -functor if and only if for every k -algebra A and for every Zariski covering $\{f_i : A \rightarrow A_i\}$ of A the diagram

$$\mathfrak{X}(A) \xrightarrow{\langle \mathfrak{X}(f_i) \rangle_{i \in I}} \prod_{i \in I} \mathfrak{X}(A_i) \xrightleftharpoons[\langle \mathfrak{X}(f''_{ij}) \cdot \text{pr}_j \rangle_{(i,j)}]{\langle \mathfrak{X}(f'_{ij}) \cdot \text{pr}_i \rangle_{(i,j)}} \prod_{(i,j) \in I \times I} \mathfrak{X}(A_i \otimes_A A_j)$$

is a kernel of a pair of arrows, where for every $(i, j) \in I \times I$ morphisms f'_{ij} and f''_{ji} form a cocartesian square

$$\begin{array}{ccc}
A & \xrightarrow{f_j} & A_j \\
f_i \downarrow & & \downarrow f'_{ji} \\
A_i & \xrightarrow{f'_{ij}} & A_i \otimes_A A_j
\end{array}$$

Now we state the main result of this section.

Theorem 3.5. *Let*

$$\widehat{\mathbf{Sch}}_k \longrightarrow \text{the category of } k\text{-functors}$$

be the restriction of presheaves on \mathbf{Sch}_k to copresheaves on \mathbf{Alg}_k (k -functors) induced by the contravariant functor $\text{Spec} : \mathbf{Alg}_k \rightarrow \mathbf{Sch}_k$. Then it induces an equivalence of categories between Zariski sheaves on \mathbf{Sch}_k and Zariski local k -functors.

Proof. Note that \mathbf{Aff}_k with Zariski topology is a dense subsite ([Mon19b, definition 4.4]) of \mathbf{Sch}_k with Zariski topology. Hence the result is a special case of a more general theorem [Mon19b, Theorem 4.6]. \square

4. TRANSPORTERS

Definition 4.1. Let X be a k -scheme. Suppose that there exists an open affine cover $X = \bigcup_{i \in I} X_i$ such that k -algebra $\Gamma(X_i, \mathcal{O}_{X_i})$ is free as a k -module. Then we say that X is a *locally free k -scheme*.

Next theorem is the main result of this section.

Theorem 4.2. *Let $j : \mathfrak{X}' \rightarrow \mathfrak{X}$ be a closed immersion of k -functors and X be a locally free k -scheme. Suppose that classes $\text{Mor}_A(X_A, \mathfrak{X}_A)$ are sets for every k -algebra A . Then classes $\text{Mor}_A(X_A, \mathfrak{X}'_A)$ are sets for every k -algebra A and the morphism*

$$\text{Mor}_k(1_X, j) : \text{Mor}_k(X, \mathfrak{X}') \rightarrow \text{Mor}_k(X, \mathfrak{X})$$

is a closed immersion of k -functors.

It is useful to isolate crucial steps in the argument. For this we proceed by proving some lemmas.

Lemma 4.2.1. *Suppose that A is a commutative ring. Let $j : \mathfrak{X}' \rightarrow \mathfrak{X}$ be a closed immersion of A -functors and X be an affine A -scheme such that $\Gamma(X, \mathcal{O}_X)$ is a free A -module. Assume that $\tau : X \rightarrow \mathfrak{X}$ is a morphism of A -functors. Then there exists an ideal $\mathfrak{a} \subseteq A$ such that for every A -algebra B the restriction τ_B factors through j_B if and only if the structure morphism $f : A \rightarrow B$ of B satisfies $\mathfrak{a} \subseteq \ker(f)$.*

Proof of the lemma. Consider a cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & \mathfrak{X}' \\ j' \downarrow & & \downarrow j \\ X & \xrightarrow{\tau} & \mathfrak{X} \end{array}$$

Since j is a closed immersion of A -functors, we derive by Fact 2.10 that j' is a closed immersion. By assumption X is affine. Hence X' is a functor of points of some A -scheme and $j' : X' \rightarrow X$ is (induced by) a closed immersion of A -schemes. Next let B be an A -algebra with the structure morphism $f : A \rightarrow B$. Then τ_B factors through j_B if and only if the projection $\text{Spec } B \times_{\text{Spec } A} X \rightarrow X$ induced by f factors through X' . Let $A[X]$ be the A -algebra of global regular functions on X and let \mathfrak{J} be an ideal in $A[X]$ such that $A[X]/\mathfrak{J} = A[X']$ is the A -algebra of global regular functions of X' . With this notation we derive that the projection $\text{Spec } B \times_{\text{Spec } A} X \rightarrow X$ induced by f factors through X' if and only if the morphism $A[X] \rightarrow B \otimes_A A[X]$ induced by f sends every element of \mathfrak{J} to zero. Since $A[X]$ is a free A -module, we write $A[X] = A^{\oplus I}$ for some index set I . Then the morphism $A[X] \rightarrow B \otimes_A A[X]$ induced by f is just $f^{\oplus I} : A^{\oplus I} \rightarrow B^{\oplus I}$. We have $f^{\oplus I}(\mathfrak{J}) = 0$ if and only if $(pr_i^B \cdot f^{\oplus I})(\mathfrak{J}) = 0$ for every $i \in I$, where $pr_i^B : B^{\oplus I} \rightarrow B$ is the projection on i -th component. Pick $i \in I$ and consider the commutative diagram

$$\begin{array}{ccc} A^{\oplus I} & \xrightarrow{f^{\oplus I}} & B^{\oplus I} \\ pr_i^A \downarrow & & \downarrow pr_i^B \\ A & \xrightarrow{f} & B \end{array}$$

In the diagram pr_i^A is the projection on i -th component. Diagram implies that $(pr_i^B \cdot f^{\oplus I})(\mathfrak{J}) = 0$ for every $i \in I$ if and only if $(f \cdot pr_i^A)(\mathfrak{J}) = 0$ for every $i \in I$. This is equivalent with the condition that $f(\mathfrak{a}) = 0$ for ideal \mathfrak{a} in A generated by $\sum_{i \in I} pr_i^A(\mathfrak{J})$. Thus the lemma is proved. \square

Lemma 4.2.2. *Suppose that A is a commutative ring. Let $j : \mathfrak{X}' \rightarrow \mathfrak{X}$ be a closed immersion of A -functors and X be an A -scheme with open cover*

$$X = \bigcup_{i \in I} X_i$$

Assume that $\tau : X \rightarrow \mathfrak{X}$ is a morphism of A -functors. Fix an A -algebra B . Then τ_B factors through j_B if and only if $(\tau|_{X_i})_B$ factors through j_B for every $i \in I$.

Proof of the lemma. If τ_B factors through j_B , then also $(\tau_{|X_i})_B$ factors through j_B for every $i \in I$. It suffices to prove the converse. So suppose that $(\tau_{|X_i})_B$ factors through j_B for every $i \in I$. Since j is a closed immersion of A -functors and X is an A -scheme, there exists a cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & \mathfrak{X}' \\ j' \downarrow & & \downarrow j \\ X & \xrightarrow{\tau} & \mathfrak{X} \end{array}$$

where $j' : X' \rightarrow X$ is (induced by) a closed immersion of A -schemes (this follows from Fact 2.10 and Fact ??). For each $i \in I$ let $j'_i : j'^{-1}(X_i) \rightarrow X_i$ be the restriction of j' . We have the induced cartesian square

$$\begin{array}{ccc} j'^{-1}(X_i) & \longrightarrow & \mathfrak{X}' \\ j'_i \downarrow & & \downarrow j \\ X_i & \xrightarrow{\tau_{|X_i}} & \mathfrak{X} \end{array}$$

Now $(\tau_{|X_i})_B$ factors through j_B . Together with Fact ?? this shows that $(j'_i)_B$ is an isomorphism of B -schemes. This holds for every $i \in I$. Hence j'_B is an isomorphism of B -schemes (again by application of Fact ??). Therefore, τ_B factors through j_B . \square

Proof of the theorem. Let A be a k -algebra. The restriction functor $(-)|_{\mathbf{Alg}_A} = (-)_A$ preserves all closed immersions. Thus j_A is a closed immersion of A -functors and hence we derive that $j_A : \mathfrak{X}'_A \rightarrow \mathfrak{X}_A$ is a monomorphism of A -functors. Thus we have an injective map of classes

$$\mathrm{Mor}_A(1_{X_A}, j_A) : \mathrm{Mor}_A(X_A, \mathfrak{X}'_A) \hookrightarrow \mathrm{Mor}_A(X_A, \mathfrak{X}_A)$$

Hence if $\mathrm{Mor}_A(X_A, \mathfrak{X}_A)$ is a set, then $\mathrm{Mor}_A(X_A, \mathfrak{X}'_A)$ is a set. All these facts imply that both internal homs

$$\mathcal{M}\mathrm{or}_k(X, \mathfrak{X}'), \mathcal{M}\mathrm{or}_k(X, \mathfrak{X})$$

exist and morphism $\mathcal{M}\mathrm{or}_k(1_X, j)$ of k -functors is a monomorphism. Our task is to prove that it is a closed immersion. For this consider a k -algebra A and a morphism $\sigma : k_A \rightarrow \mathcal{M}\mathrm{or}_k(X, \mathfrak{X})$ of k -functors that sends 1_A to some morphism $\tau : X_A \rightarrow \mathfrak{X}_A$ of A -functors. Consider a cartesian square

$$\begin{array}{ccc} \mathfrak{U} & \longrightarrow & \mathcal{M}\mathrm{or}_k(X, \mathfrak{X}') \\ \downarrow & & \downarrow \mathcal{M}\mathrm{or}_k(1_X, j) \\ k_A & \xrightarrow{\sigma} & \mathcal{M}\mathrm{or}_k(X, \mathfrak{X}) \end{array}$$

Since $\mathcal{M}\mathrm{or}_k(1_X, j)$ is a monomorphism, we may consider \mathfrak{U} as a k -subfunctor of k_A . For every k -algebra B subset $\mathfrak{U}(B) \subseteq \mathrm{Mor}_k(A, B) = k_A(B)$ consists of A -algebras B with structure morphisms $f : A \rightarrow B$ such that τ_B factors through $j_B : \mathfrak{X}'_B \rightarrow \mathfrak{X}_B$. Since X is a locally free k -scheme, we deduce that X_A is (a functor of points of) a locally free A -scheme. Pick an open affine cover $X_A = \bigcup_{i \in I} X_i$ such that $\Gamma(X_i, \mathcal{O}_X)$ is a free A -module. Now Lemma 4.2.2 implies that τ_B factors through j_B if and only if $(\tau_{|X_i})_B$ factors through j_B for every $i \in I$. Next by Lemma 4.2.1 we deduce that $(\tau_{|X_i})_B$ factors through j_B for given $i \in I$ if and only if $f(\mathfrak{a}_i) = 0$ for some ideal $\mathfrak{a}_i \subseteq A$ independent of f .

Thus \mathcal{U} consists of all morphisms $f : A \rightarrow B$ of k -algebras such that $f(\mathfrak{a}) = 0$ where $\mathfrak{a} = \sum_{i \in I} \mathfrak{a}_i$. Therefore, $\mathcal{U} \hookrightarrow k_A$ is isomorphic with $k_{A/\mathfrak{a}} \hookrightarrow k_A$ and hence $\mathcal{M}or_k(1_X, j)$ is a closed immersion of k -functors. \square

The Theorem 4.2 is a simple yet powerful result. Before giving any interesting applications we state its immediate consequence.

5. SCHEMES AS k -FUNCTORS

Let X be a k -scheme. We define a k -functor \mathfrak{P}_X by formula

$$\mathfrak{P}_X(A) = \mathcal{M}or_k(\mathrm{Spec} A, X)$$

That is \mathfrak{P}_X is the restriction of the presheaf on \mathbf{Sch}_k represented by X to the category \mathbf{Alg}_k along the functor $\mathrm{Spec} : \mathbf{Alg}_k \rightarrow \mathbf{Sch}_k$. Next if $f : X \rightarrow Y$ is a morphism of k -schemes, then \mathfrak{P}_f is the restriction of a morphism of presheaves on \mathbf{Sch}_k represented by f to the category of k -algebras along $\mathrm{Spec} : \mathbf{Alg}_k \rightarrow \mathbf{Sch}_k$. Thus we have a functor

$$\mathbf{Sch}_k \xrightarrow{\mathfrak{P}} \text{the category of } k\text{-functors}$$

Definition 5.1. Let X be a k -scheme. Then \mathfrak{P}_X is called *the k -functor of points of X* .

Fact 5.2. *Functor*

$$\mathbf{Sch}_k \xrightarrow{\mathfrak{P}} \text{the category of } k\text{-functors}$$

is full, faithful and its image consists of Zariski local k -functors.

Proof. Note that the presheaf h_X on \mathbf{Sch}_k represented by X is a Zariski sheaf. Indeed, this just rephrase standard fact that morphism of schemes can be glued in Zariski topology. Next according to Theorem 3.5 the functor $\mathrm{Spec} : \mathbf{Alg}_k \rightarrow \mathbf{Sch}_k$ induces an equivalence of the category of Zariski sheaves and local Zariski k -functors. Thus \mathfrak{P}_X is a local Zariski k -functor and it is full and faithful. \square

Definition 5.3. Let F, G be presheaves on \mathbf{Sch}_k and let $f : F \rightarrow G$ be their morphism. Suppose that $x \in G(X)$ for some k -scheme X . To every x of this type one can associate the cartesian square of presheaves

$$\begin{array}{ccc} h_X \times_G F & \longrightarrow & F \\ \pi_x \downarrow & & \downarrow f \\ h_X & \longrightarrow & G \end{array}$$

in which bottom vertical morphism $h_X \rightarrow G$ is canonically identified with x . We say that f is:

- (1) *an open immersion* if for every k -scheme X and $x \in G(X)$ morphism π_x is isomorphic to the image under Yoneda embedding of some open immersion of k -schemes.
- (2) *a closed immersion* if for every k -scheme X and $x \in G(X)$ morphism π_x is isomorphic to the image under Yoneda embedding of some closed immersion of k -schemes.

Proposition 5.4. *Let $f : F \rightarrow G$ be a morphism of presheaves on \mathbf{Sch}_k . Suppose that f is either open or closed immersion. Then f is a monomorphism of presheaves.*

Proof. Fix an element $y \in G(X)$. Consider a cartesian square

$$\begin{array}{ccc}
h_X \times_G F & \longrightarrow & F \\
\pi_y \downarrow & & \downarrow f \\
h_X & \longrightarrow & G
\end{array}$$

in which y determines a morphism $h_X \rightarrow G$. Morphism f is either open or closed immersion. Hence there exists a monomorphism $j : Y \rightarrow X$ of k -schemes such that π_y is isomorphic with h_j . Yoneda embedding preserves monomorphisms. Thus h_j is a monomorphism of presheaves. This implies that π_y is a monomorphism of presheaves for every k -scheme X and $y \in G(X)$. In particular, there exists at most one element $x \in F(X)$ such that $f(x) = y$. Since $y \in G(X)$ is arbitrary, we deduce that f is a monomorphism of presheaves. \square

Definition 5.5. Let F be a presheaf on \mathbf{Sch}_k and $\{f_i : F_i \rightarrow F\}_{i \in I}$ be a family of open immersions. Then for every k -scheme X and $x \in F(X)$ we have a family of open immersions $\{f_{i,x} : U_{i,x} \rightarrow X\}_{i \in I}$ defined by cartesian squares

$$\begin{array}{ccc}
h_{U_{i,x}} & \longrightarrow & F_i \\
h_{f_{i,x}} \downarrow & & \downarrow f_i \\
h_X & \longrightarrow & F
\end{array}$$

in which bottom vertical morphism $h_X \rightarrow F$ is canonically identified with x . We say that $\{f_i\}_{i \in I}$ is an open cover of F if for every k -scheme X and $x \in F(X)$ we have

$$X = \bigcup_{i \in I} f_{i,x}(U_{i,x})$$

Theorem 5.6. Let F be a presheaf on \mathbf{Sch}_k . Then the following are equivalent.

- (i) $F \cong h_X$ for some k -scheme X .
- (ii) F is a Zariski sheaf and there exists an open cover $\{v_i : h_{V_i} \rightarrow F\}_{i \in I}$ such that $\{V_i\}_{i \in I}$ are affine k -schemes.
- (iii) F is a Zariski sheaf and there exists an open cover $\{v_i : h_{V_i} \rightarrow F\}_{i \in I}$ such that $\{V_i\}_{i \in I}$ are k -schemes.

Proof. We prove (i) \Rightarrow (ii). Since $F \cong h_X$ and properties in (ii) are stable under isomorphism, we deduce that we can replace F by h_X . So it suffices to show that h_X satisfies (ii). By definition every k -scheme X admits an open cover $\{v_i : V_i \rightarrow X\}_{i \in I}$ by affine k -schemes. Since Yoneda embedding $h : \mathbf{Sch}_k \rightarrow \widehat{\mathbf{Sch}}_k$ preserves fiber-products, we derive that $\{h_{v_i}\}_{i \in I}$ is an open cover in the category of presheaves. Thus h_X admits an open cover by presheaves representable by affine k -schemes. Next suppose that $\{f_i : U_i \rightarrow U\}_{i \in I}$ is a Zariski covering of a k -scheme U and $\{g_i : U_i \rightarrow X\}_{i \in I}$ is a family of morphisms of k -schemes such that $g_i|_{U_i \times_U U_j} = g_j|_{U_i \times_U U_j}$ for every pair $(i, j) \in I \times I$. Then we can glue $\{g_i\}_{i \in I}$ into a unique morphism of k -schemes $g : U \rightarrow X$ such that $g \cdot f_i = g_i$ for every $i \in I$. This shows that h_X is a Zariski sheaf.

The implication (ii) \Rightarrow (iii) is a consequence of the fact that every affine k -scheme is a k -scheme.

Assume now that (iii) holds. Fix elements $i, j \in I$ and consider a cartesian square

$$\begin{array}{ccc}
 h_{V_i} \times_F h_{V_j} & \xrightarrow{v'_j} & h_{V_j} \\
 v'_i \downarrow & & \downarrow v_j \\
 h_{V_i} & \xrightarrow{v_i} & F
 \end{array}$$

in the category $\widehat{\mathbf{Sch}}_k$. Since v_i is an open immersion, we derive that there exists an open subscheme $V_{ij} \subseteq V_i$ and an isomorphism $p_{ij} : h_{V_{ij}} \rightarrow h_{V_i} \times_F h_{V_j}$ such that the triangle

$$\begin{array}{ccc}
 h_{V_{ij}} & \xrightarrow{p_{ij}} & h_{V_i} \times_F h_{V_j} \\
 & \searrow & \swarrow v'_i \\
 & h_{V_i} &
 \end{array}$$

is commutative. Similarly since v_j is an open immersion, we derive that there exists an open subscheme $V_{ji} \subseteq V_j$ and an isomorphism $p_{ji} : h_{V_{ji}} \rightarrow h_{V_i} \times_F h_{V_j}$ such that the triangle

$$\begin{array}{ccc}
 h_{V_{ji}} & \xrightarrow{p_{ji}} & h_{V_i} \times_F h_{V_j} \\
 & \searrow & \swarrow v'_j \\
 & h_{V_j} &
 \end{array}$$

is commutative. Now we define an isomorphism of k -schemes $\phi_{ij} : V_{ij} \rightarrow V_{ji}$ by requirement $h_{\phi_{ij}} = p_{ji}^{-1} \cdot p_{ij}$. Then the data consisting of families $\{V_i\}_{i \in I}$, $\{V_{ij}\}_{(i,j) \in I \times I}$ and $\{\phi_{ij}\}_{(i,j) \in I \times I}$ satisfy the following assertions.

- (1) $V_{ij} \subseteq V_i$ is an open subscheme for every $i \in I$ and $j \in J$.
- (2) $V_{ii} = V_i$ and $\phi_{ii} = 1_{V_i}$ for every $i \in I$.
- (3) $\phi_{ij} : V_{ij} \rightarrow V_{ji}$ is an isomorphism of k -schemes for every $(i, j) \in I \times I$.
- (4) For every pair $(i, j) \in I \times I$ and $k \in I$ isomorphism ϕ_{ij} restricts to an isomorphism

$$\phi'_{ij,k} : V_{ij} \cap V_{ik} \rightarrow V_{ji} \cap V_{jk}$$

of k -schemes.

- (5) For every triple $(i, j, k) \in I \times I \times I$ we have

$$\phi'_{ik,j} = \phi'_{jk,i} \cdot \phi'_{ij,k}$$

Thus by [GD71, Chapitre 0, 4.1.7] family $\{V_i\}_{i \in I}$ can be considered as an open cover of a ringed k -space X in such a way that for any elements $i, j \in I$ the square

$$\begin{array}{ccc}
 h_{V_i \cap V_j} & \hookrightarrow & h_{V_j} \\
 \downarrow & & \downarrow v_j \\
 h_{V_i} & \xrightarrow{v_i} & F
 \end{array}$$

is cartesian (the intersection $V_i \cap V_j$ in the diagram is taken inside X). Since X admits an open cover by a k -schemes, it is itself a k -scheme. Next we construct a morphism $f : h_X \rightarrow F$. For this note that for each $i \in I$ morphism v_i gives rise to an element $x_i \in F(V_i)$. Since $v_i|_{h_{V_i \cap V_j}} = v_j|_{h_{V_i \cap V_j}}$ for any two $i, j \in I$, we deduce that $x_i|_{V_i \cap V_j} = x_j|_{V_i \cap V_j}$. Next we apply the fact that F is a Zariski sheaf to construct an element $x \in F(X)$ such that $x|_{V_i} = x_i$ for every $i \in I$. Now x determines a morphism $f : h_X \rightarrow F$ such that the following square

$$\begin{array}{ccc} & h_{V_i} & \\ \swarrow & & \searrow v_i \\ h_X & \xrightarrow{g} & F \end{array}$$

Now let Y be a k -scheme and pick $y \in F(Y)$. Suppose that $g : h_Y \rightarrow F$ is a morphism corresponding to y . Pick $i \in I$. Since $v_i : h_{V_i} \rightarrow F$ is an open immersion, there exists open subscheme $W_i \subseteq Y$ that fits in a cartesian square

$$\begin{array}{ccc} h_{W_i} & \xrightarrow{g_i} & h_{V_i} \\ \downarrow & & \downarrow v_i \\ h_Y & \xrightarrow{g} & F \end{array}$$

By Yoneda lemma g_i corresponds to $k_i \in h_{V_i}(W_i)$. By definition $k_i : W_i \rightarrow V_i$ is a morphism of k -schemes. Next for $i \in I$ and $j \in I$ we have \square

6. EQUIVALENCE RELATIONS AND QUOTIENTS

In this section we introduce internal binary relations in categories. The first part of the section are categorical reformulations and generalizations of standard notions. We encourage the reader to interpret everything that we introduce in the category of sets.

Definition 6.1. Let \mathcal{C} be a category with products and let X be an object of \mathcal{C} . Suppose that $p_1, p_2 : R \rightarrow X$ are morphisms such that $\langle p_1, p_2 \rangle : R \rightarrow X \times X$ is a monomorphism. Then a triple (R, p_1, p_2) is called a *binary relation on X* .

If \mathcal{C} is a category with products and (R, p_1, p_2) is a binary relation on some object X of \mathcal{C} , then for every object Y of \mathcal{C} we have the inclusion

$$\text{Mor}_{\mathcal{C}}(Y, R) \xhookrightarrow{\text{Mor}_{\mathcal{C}}(1_Y, \langle p_1, p_2 \rangle)} \text{Mor}_{\mathcal{C}}(Y, X \times X) = \text{Mor}_{\mathcal{C}}(Y, X) \times \text{Mor}_{\mathcal{C}}(Y, X)$$

and thus we can interpret it as a binary relation on the class $\text{Mor}_{\mathcal{C}}(Y, X)$.

Suppose now that \mathcal{C} is a category with fiber products and (R, p_1, p_2) is a binary relation on some object X of \mathcal{C} . Consider a cartesian square

$$\begin{array}{ccc} T & \xrightarrow{q_2} & R \\ q_1 \downarrow & & \downarrow p_1 \\ R & \xrightarrow{p_2} & X \end{array}$$

Denote by r morphism $p_2 \cdot q_1 = p_1 \cdot q_2$. Let $\pi_1, \pi_3 : X \times X \times X \rightarrow$ be projections on first and third factor, respectively. The morphism $\langle p_1 \cdot q_1, r, p_2 \cdot q_2 \rangle : T \rightarrow X \times X \times X$ composed with the morphism $\langle \pi_1, \pi_3 \rangle : X \times X \times X \rightarrow X \times X$ factors through $\langle p_1, p_2 \rangle : R \hookrightarrow X \times X$ if there exists $t : T \rightarrow R$ such that the square

$$\begin{array}{ccc} T & \xrightarrow{\langle p_1 \cdot q_1, r, p_2 \cdot q_2 \rangle} & X \times X \times X \\ \downarrow t & & \downarrow \langle \pi_1, \pi_3 \rangle \\ R & \xrightarrow{\langle p_1, p_2 \rangle} & X \times X \end{array}$$

is commutative. Note that t is unique. Also from the diagram it follows that $t : T \rightarrow R$ can be described as the unique morphism such that $p_1 \cdot t = p_1 \cdot q_1$ and $p_2 \cdot t = p_2 \cdot q_2$.

Definition 6.2. Let \mathcal{C} be a category with fiber products and let (R, p_1, p_2) be a binary relation on some object X of \mathcal{C} .

- (1) (R, p_1, p_2) is *reflexive* if the diagonal $\delta : X \rightarrow X \times X$ factors through $\langle p_1, p_2 \rangle$.
- (2) (R, p_1, p_2) is *symmetric* if $\langle p_1, p_2 \rangle = s \cdot \langle p_1, p_2 \rangle$, there $s : X \times X \rightarrow X \times X$ is the cartesian symmetry.
- (3) Consider the cartesian square

$$\begin{array}{ccc} T & \xrightarrow{q_2} & R \\ q_1 \downarrow & & \downarrow p_1 \\ R & \xrightarrow{p_2} & X \end{array}$$

(R, p_1, p_2) is *transitive* if there exists $t : T \rightarrow R$ such that $p_1 \cdot t = p_1 \cdot q_1$ and $p_2 \cdot t = p_2 \cdot q_2$.

- (4) Finally, we say that (R, p_1, p_2) is an *equivalence relation* if it is reflexive, symmetric and transitive.

Fact 6.3. Let \mathcal{C} be a category with fiber products and let (R, p_1, p_2) be a binary relation on some object X of \mathcal{C} . Then the following assertions are equivalent.

- (i) For every object Y of \mathcal{C} the binary relation

$$\text{Mor}_{\mathcal{C}}(Y, R) \xrightarrow{\text{Mor}_{\mathcal{C}}(1_Y, \langle p_1, p_2 \rangle)} \text{Mor}_{\mathcal{C}}(Y, X \times X) = \text{Mor}_{\mathcal{C}}(Y, X) \times \text{Mor}_{\mathcal{C}}(Y, X)$$

is reflexive (symmetric, transitive).

- (ii) (R, p_1, p_2) is reflexive (symmetric, transitive).

Proof. We may pass to larger Grothendieck universe V such that \mathcal{C} is locally V -small. Then one can use the standard Yoneda lemma in order to derive translation from (i) and (ii). \square

Proposition 6.4. Let \mathcal{C} be a category with fiber products and let $f : X \rightarrow Y$ be a morphism \mathcal{C} . Consider a cartesian square

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{p_2} & X \\ p_1 \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

Then $(X \times_Y X, p_1, p_2)$ is an equivalence relation on X .

Proof. For the proof pick an object of Z and by Fact 6.3 it suffices to prove that

$$\text{Mor}_{\mathcal{C}}(Z, X \times_Y X) \xrightarrow{\text{Mor}_{\mathcal{C}}(1_Z, \langle p_1, p_2 \rangle)} \text{Mor}_{\mathcal{C}}(Z, X \times X) = \text{Mor}_{\mathcal{C}}(Z, X) \times \text{Mor}_{\mathcal{C}}(Z, X)$$

is an equivalence relation. Note that we have fiber product of classes

$$\begin{array}{ccc} \text{Mor}_{\mathcal{C}}(Z, X \times_Y X) & \xrightarrow{\text{Mor}_{\mathcal{C}}(1_Z, p_2)} & \text{Mor}_{\mathcal{C}}(Z, X) \\ \text{Mor}_{\mathcal{C}}(1_Z, p_1) \downarrow & & \downarrow \text{Mor}_{\mathcal{C}}(1_Z, f) \\ \text{Mor}_{\mathcal{C}}(Z, X) & \xrightarrow{\text{Mor}_{\mathcal{C}}(1_Z, f)} & \text{Mor}_{\mathcal{C}}(Z, Y) \end{array}$$

and hence $\text{Mor}_{\mathcal{C}}(Z, X \times_Y X)$ contains these pairs (g, h) of morphisms in $\text{Mor}_{\mathcal{C}}(Z, X)$ such that $f \cdot g = f \cdot h$. This is clearly an equivalence relation. \square

Definition 6.5. Let \mathcal{C} be a category with fiber products and let $f : X \rightarrow Y$ be a morphism in \mathcal{C} . Consider a cartesian square

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{p_2} & X \\ p_1 \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

Then the equivalence relation $(X \times_Y X, p_1, p_2)$ is called a *kernel pair* of f .

Definition 6.6. Let \mathcal{C} be a category with fiber products and let (R, p_1, p_2) be an equivalence relation on some object X of \mathcal{C} . The morphism $f : X \rightarrow Y$ is a *quotient* of (R, p_1, p_2) if the fork

$$R \xrightarrow[p_2]{p_1} X \xrightarrow{f} Y$$

is a cokernel of (p_1, p_2) . An equivalence relation that admits a quotient is called *effective*.

Proposition 6.7. Let \mathcal{C} be a category and let $g_1, g_2 : Z \rightarrow X$ be morphisms in \mathcal{C} . Suppose that

$$Z \xrightarrow[g_2]{g_1} X \xrightarrow{f} Y$$

is a cokernel of (g_1, g_2) . Then f is also a cokernel of its kernel pair.

Proof. Let $(X \times_Y X, p_1, p_2)$ be a kernel pair of f . Since $f \cdot g_1 = f \cdot g_2$, we derive that there exists a unique morphism $u : Z \rightarrow X \times_Y X$ such that $g_1 = p_1 \cdot u$ and $g_2 = p_2 \cdot u$. Hence we have a commutative diagram

$$\begin{array}{ccccc}
 Z & \xrightleftharpoons[g_2]{g_1} & X & \xrightarrow{f} & Y \\
 \downarrow u & & \downarrow = & & \\
 X \times_Y X & \xrightleftharpoons[p_2]{p_1} & X & &
 \end{array}$$

Suppose now that $h : X \rightarrow T$ is a morphism such that $h \cdot p_1 = h \cdot p_2$. Then

$$h \cdot g_1 = h \cdot p_1 \cdot u = h \cdot p_1 \cdot u = h \cdot g_2$$

and hence there exists a unique $k : Y \rightarrow T$ such that $k \cdot f = h$. This finishes the proof. \square

Proposition 6.8. *Let \mathcal{C} be a category with fiber products and let (R, p_1, p_2) be an equivalence relation on some object X of \mathcal{C} . If (R, p_1, p_2) is effective, then it is a kernel pair of its quotient.*

Proof. Let $f : X \rightarrow Y$ be a quotient of (R, p_1, p_2) . \square

Theorem 6.9. *Let (R, p_1, p_2) be an equivalence relation in \mathbf{Sch}_k . Suppose that $p_1, p_2 : R \rightarrow X$ are open immersions locally on R . Then there exists a k -scheme Y and a morphism $f : X \rightarrow Y$ such that the fork*

$$R \xrightleftharpoons[p_2]{p_1} X \xrightarrow{f} Y$$

is a cokernel of the pair p_1, p_2 . Moreover, f is open immersion locally on X .

Proof. Let $f : X \rightarrow Y$ be a quotient of X by R in the category of topological spaces. We check now that f is an open immersion locally on X . \square

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