1. FORMAL FUNCTORS AND REPRESENTABILITY

Example 1.1 (Formal schemes from algebraic ones)**.** Let Z be a $\mathbf{G}$-scheme and $\mathcal{I}$ be the ideal of $Z^{\mathbf{G}}$. Then $Z_n = V(\mathcal{I}^{n+1})$ is a closed $\mathbf{G}$-stable subscheme of Z for every $n \in \mathbb{N}$ and this yields to a formal $\mathbf{G}$-scheme $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$. We denote this formal $\mathbf{G}$-scheme by $\widehat{Z}$.

Now we define morphisms of formal $\mathbf{G}$-schemes.

Definition 1.2. Let $\mathcal{Z} = \{Z_n\}$ and $\mathcal{W} = \{W_n\}$ be formal $\mathbf{G}$-schemes. A *morphism* $\varphi : \mathcal{W} \to \mathcal{Z}$ *of formal* $\mathbf{G}$*-schemes* is a family of $\mathbf{G}$-equivariant morphisms $\varphi = \{\varphi_n : W_n \to Z_n\}$ such that for every $n \in \mathbb{N}$ we have a commutative square

$$\begin{tikzcd} W_{n+1} \arrow[r, "\varphi_{n+1}"] & Z_{n+1} \\ W_n \arrow[u, hook] \arrow[r, "\varphi_n"'] & Z_n \arrow[u, hook] \end{tikzcd}$$

Remark 1.3 (Morphisms of formal $\overline{\mathbf{G}}$-schemes are $\overline{\mathbf{G}}$-equivariant)**.** Let $\mathcal{W}$ and $\mathcal{Z}$ be formal $\overline{\mathbf{G}}$-schemes and consider their morphism $\varphi : \mathcal{W} \to \mathcal{Z}$ (as formal $\mathbf{G}$-schemes). Then for every $n \in \mathbb{N}$ the morphism $\varphi_n : W_n \to Z_n$ is $\overline{\mathbf{G}}$-equivariant. To see this, consider Diagram (1).

$$\begin{tikzcd} W_n \arrow[r, "r_n"'] \arrow[rr, bend left, "\varphi_n"] \arrow[dr, "\pi_{W_n}"'] & W_0 \times_{Z_0} Z_n \arrow[r, "q_n"'] \arrow[d, "p_n"] & Z_n \arrow[d, "\pi_{Z_n}"] \\ & W_0 \arrow[r, "\varphi_0"'] & Z_0 \end{tikzcd} \tag{1}$$

Since W_0 and Z_0 are equipped with trivial $\overline{\mathbf{G}}$-actions, also the pullback $W_0 \times_{Z_0} Z_n$ is a $\overline{\mathbf{G}}$-scheme and q_n is $\overline{\mathbf{G}}$-equivariant. Recall that π_{Z_n}, π_{W_n} are affine morphisms. Therefore, p_n is affine. Hence r_n is a $\mathbf{G}$-equivariant morphism between $\overline{\mathbf{G}}$-schemes separated (even affine) over W_0. Thus r_n is $\overline{\mathbf{G}}$-equivariant.

Definition 1.4. A *locally linear* $\overline{\mathbf{G}}$*-scheme* is a $\overline{\mathbf{G}}$-scheme which admits an open cover by affine $\overline{\mathbf{G}}$-stable subschemes. The category of locally linear $\overline{\mathbf{G}}$-schemes consists of those schemes and $\overline{\mathbf{G}}$-equivariant morphisms.

Let Z be a locally linear $\overline{\mathbf{G}}$-scheme. By Proposition **??**, the map $\mathcal{D}_Z \to Z$ is an isomorphism. In particular, there is a canonical morphism $\pi_Z : Z \to Z^{\mathbf{G}}$, which is the multiplication by zero. For an affine open $\overline{\mathbf{G}}$-stable cover $\{V_i\}_i$ of Z, we have $V_i = \pi_Z^{-1}(\pi_Z(V_i))$ by Proposition **??**, hence the canonical morphism $\pi_Z : Z \to Z^{\mathbf{G}}$ is affine.

Definition 1.5. Let $\mathcal{Z}$ be a formal $\overline{\mathbf{G}}$-scheme. An *algebraization* of $\mathcal{Z}$ is a $\overline{\mathbf{G}}$-scheme Z such that

(1) Z is a locally linear $\overline{\mathbf{G}}$-scheme.

(2) $\mathcal{Z}$ and $\widehat{Z}$ are isomorphic formal $\overline{\mathbf{G}}$-schemes.

By the above discussion, the morphism $\pi_Z : Z \to Z^{\mathbf{G}}$ is affine for any algebraization Z.

Theorem 1.6 (Algebraization of a formal $\overline{\mathbf{G}}$-scheme)**.** *Let $\mathcal{Z} = \{Z_n\}$ be a formal $\overline{\mathbf{G}}$-scheme. Then there exists a colimit*

$$Z = \operatorname{colim}_n Z_n$$

in the category of locally linear $\overline{\mathbf{G}}$-schemes and Z is the unique algebraization of $\mathcal{Z}$. If in addition $\mathcal{Z}$ is locally Noetherian, then π_Z is of finite type. If $\mathcal{Z}$ is locally Noetherian and Z_0 is of finite type, then also Z is of finite type.

Now we spell out the main idea of the proof: the $\overline{\mathbf{G}}$-scheme Z required in Theorem 1.6 is equal to $\operatorname{Spec}_{Z_0}\mathcal{A}$, where $\mathcal{A}$ is the limit of $\mathcal{A}_n$ *in the category of* $\overline{\mathbf{G}}$*-algebras*; in other words each isotypic component of $\mathcal{A}$ is the limit of isotypic components of $\mathcal{A}_n$. Our first goal is to prove a stabilization result. We denote by $\operatorname{Irr}(\mathbf{G})$ the set of isomorphism types of irreducible $\mathbf{G}$-representations and by $\operatorname{Irr}(\overline{\mathbf{G}}) \subset \operatorname{Irr}(\mathbf{G})$ the subset of $\overline{\mathbf{G}}$-representations. For $\lambda \in \operatorname{Irr}(\mathbf{G})$ and a quasi-coherent $\overline{\mathbf{G}}$-module $\mathcal{C}$ on Z_0 we denote by $\mathcal{C}[\lambda] \subset \mathcal{C}$ the $\overline{\mathbf{G}}$-submodule such that $H^0(U, \mathcal{C}[\lambda]) \subset H^0(U, \mathcal{C})$ is the union of all $\mathbf{G}$-subrepresentations of $H^0(U, \mathcal{C})$ isomorphic to λ (i.e., the isotypic component of λ).

Lemma 1.6.1 (stabilization on an isotypic component)**.** *Let $\lambda \in \operatorname{Irr}(\overline{\mathbf{G}})$. Then there exists a number $n_\lambda \in \mathbb{N}$ such that the following holds. Let $\mathcal{Z} = \{Z_n\}$ be a formal $\overline{\mathbf{G}}$-scheme and $\{\mathcal{A}_{n+1} \twoheadrightarrow \mathcal{A}_n\}$ be the associated sequence of quasi-coherent $\overline{\mathbf{G}}$-algebras. Then for every $n > n_\lambda$ the surjection*

$$\mathcal{A}_n[\lambda] \twoheadrightarrow \mathcal{A}_{n-1}[\lambda]$$

is an isomorphism. If $\lambda_0 \in \operatorname{Irr}(\overline{\mathbf{G}})$ is the trivial representation, then we may take $n_{\lambda_0} = 0$.

Proof of Lemma 1.6.1. The claims are preserved under field extension, so we may assume our field is algebraically closed (hence perfect) so we may use the Kempf's torus. Fix a grading on $k[\overline{\mathbf{G}}]$ induced by a Kempf's torus for k as in Corollary **??**. Denote by $A_\lambda \subseteq \mathbb{N}$ the set of weights which appear in $k[\mathbf{G}]_\lambda$. Since $\dim_k k[\mathbf{G}]_\lambda$ is finite by Proposition **??**, the set A_λ is finite. Put

$$n_\lambda = \sup A_\lambda.$$

Fix $n > n_\lambda$ and let $\mathcal{I}_n = \ker(\mathcal{A}_n \to \mathcal{A}_0)$. Then we have a decomposition with respect to the chosen torus

$$\mathcal{A}_n = \bigoplus_{i \geq 0} (\mathcal{A}_n)[i],$$

By Corollary **??**, we have $\mathcal{I}_n = \oplus_{i\geq 1}(\mathcal{A}_n)[i]$. Since $n > n_\lambda$ we have

$$\mathcal{I}_n^n \subset \bigoplus_{i \geq n} (\mathcal{A}_n)[i] \subseteq \bigoplus_{i \notin A_\lambda} (\mathcal{A}_n)[i]$$

Hence, $\mathcal{I}_n^n[\lambda] = 0$. But $\mathcal{I}_n^n[\lambda] = \ker(\mathcal{A}_n[\lambda] \to \mathcal{A}_{n-1}[\lambda])$, thus $\mathcal{A}_n[\lambda] \to \mathcal{A}_{n-1}[\lambda]$ is an isomorphism. Finally note that $A_{\lambda_0} = \{0\}$. This implies that $n_{\lambda_0} = 0$. ☐

Proof of Theorem 1.6. Let $\mathcal{A}_n$ be the quasi-coherent $\overline{\mathbf{G}}$-algebras as in (**??**). For $\lambda \in \operatorname{Irr}(\overline{\mathbf{G}})$ we define $\mathcal{A}[\lambda] := \mathcal{A}_n[\lambda]$, where $n \geq n_\lambda$ as in Lemma 1.6.1.

$$\mathcal{A} = \bigoplus_{\lambda \in \operatorname{Irr}(\overline{\mathbf{G}})} \mathcal{A}[\lambda] = \bigoplus_{\lambda \in \operatorname{Irr}(\overline{\mathbf{G}})} \mathcal{A}_{n_\lambda}[\lambda].$$

Clearly $\mathcal{A}[\lambda_0] = \mathcal{A}_0 = \mathcal{O}_{Z_0}$ canonically (where λ_0 is the trivial representation), hence $\mathcal{A}$ is an $\mathcal{O}_{Z_0}$-module. Actually $\mathcal{A} = \lim_n \mathcal{A}_n$ in the category of quasi-coherent $\overline{\mathbf{G}}$-modules on Z_0. We construct the algebra structure on $\mathcal{A}$. For this pick $\eta_1, \eta_2 \in \operatorname{Irr}(\overline{\mathbf{G}})$. Fix the finite set $\{\lambda_1, \ldots, \lambda_s\} \subseteq \operatorname{Irr}(\overline{\mathbf{G}})$ of representations which appear in $k[\overline{\mathbf{G}}]_{\eta_1} \otimes_k k[\overline{\mathbf{G}}]_{\eta_2}$. Then, for every $n \in \mathbb{N}$, we have the multiplication

$$\mathcal{A}_n[\eta_1] \otimes_k \mathcal{A}_n[\eta_2] \to \mathcal{A}_n[\eta_1] \cdot \mathcal{A}_n[\eta_2] \subseteq \bigoplus_{i=1}^{s} \mathcal{A}_n[\lambda_i]$$

and by Lemma 1.6.1 these morphisms can be identified for $n \geq \sup\{n_{\eta_1}, n_{\eta_2}, n_{\lambda_1}, \ldots, n_{\lambda_s}\}$. We define

$$\mathcal{A}[\eta_1] \otimes_k \mathcal{A}[\eta_2] \to \bigoplus_{i=1}^{s} \mathcal{A}[\lambda_i] \subseteq \mathcal{A}$$

as a morphism induced by the multiplication morphism for any $n \geq \sup\{n_{\eta_1}, n_{\eta_2}, n_{\lambda_1}, \dots, n_{\lambda_s}\}$. This gives an $\mathcal{O}_{Z_0}$-algebra structure on $\mathcal{A}$, so $\mathcal{A}$ is in fact the limit of $\mathcal{A}_n$ is the category of $\overline{\mathbf{G}}$-algebras. Note that from the description of $\mathcal{A}$ it follows that for every $n \in \mathbb{N}$ we have a surjective morphism $p_n : \mathcal{A} \twoheadrightarrow \mathcal{A}_n$ of $\overline{\mathbf{G}}$-algebras. We denote its kernel by $\mathcal{J}_n$ and we put $\mathcal{J} := \mathcal{J}_0$. The natural injection $\mathcal{O}_{Z_0} = \mathcal{A}_0 \to \mathcal{A}$ is a section of p_0, so that we have

$$\mathcal{J} = \bigoplus_{\lambda \in \mathrm{Irr}(\overline{\mathbf{G}}) \smallsetminus \{\lambda_0\}} \mathcal{A}[\lambda].$$

We also denote by $\mathcal{I}_n$ the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$ for $n \in \mathbb{N}$. Then $\mathcal{I}_n = \mathcal{J}/\mathcal{J}_n$. Fix $m \in \mathbb{N}$ and consider $n \in \mathbb{N}$ such that $n \geq m$. Since $\mathcal{Z}$ is a formal $\overline{\mathbf{G}}$-scheme, the sheaf $\mathcal{I}_n^{m+1}$ is the kernel of the morphism $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_m$. Thus

$$\mathcal{J}_m/\mathcal{J}_n = \mathcal{I}_n^{m+1} = (\mathcal{J}^{m+1} + \mathcal{J}_n)/\mathcal{J}_n.$$

Both $\mathcal{J}_m$ and $\mathcal{J}^{m+1}$ are $\mathrm{Irr}(\overline{\mathbf{G}})$-graded and for given $\lambda \in \mathrm{Irr}(\overline{\mathbf{G}})$ and $n \gg 0$ the isotypic component $\mathcal{J}_n[\lambda]$ is zero by Lemma 1.6.1. Hence $\mathcal{J}_m = \mathcal{J}^{m+1}$ for every $m \in \mathbb{N}$. We define

$$Z = \mathrm{Spec}_{Z_0}(\mathcal{A})$$

and we denote by $\pi : Z \to Z_0$ the structural morphism. The scheme Z inherits a $\overline{\mathbf{G}}$-action from $\mathcal{A}$. For every $n \in \mathbb{N}$ the zero-set of $\mathcal{J}^{n+1} \subseteq \mathcal{A}$ is a $\overline{\mathbf{G}}$-scheme isomorphic to Z_n. Hence $\mathcal{Z}$ is isomorphic to $\widehat{Z}$. Thus Z is an algebraization of $\mathcal{Z}$. Since $\mathcal{A} = \lim \mathcal{A}_n$, we have $Z = \mathrm{colim}\, Z_n$ in the category of locally linear $\overline{\mathbf{G}}$-schemes.

It remains to prove uniqueness of algebraization. Let $Z' = \mathrm{Spec}_{Z_0}\mathcal{A}'$ be an algebraization of $\mathcal{Z} = \{Z_n\}$. Then $Z_n \hookrightarrow Z'$, so by the universal property of colimit, we obtain a $\overline{\mathbf{G}}$-morphism $Z \to Z'$, corresponding to $\mathcal{A}' \to \mathcal{A}$. It induces epimorphisms $\mathcal{A}' \twoheadrightarrow \mathcal{A}_n$ for all n. For each $\lambda \in \mathrm{Irr}(\overline{\mathbf{G}})$, the composition

$$\mathcal{A}'[\lambda] \to \mathcal{A}[\lambda] \simeq \mathcal{A}_{n_\lambda}[\lambda]$$

is an epimorphism, hence $\mathcal{A}' \to \mathcal{A}$ is an epimorphism. The kernel of $\mathcal{A}' \to \mathcal{A}$ is equal to

$$\bigcap_n \ker(\mathcal{A}' \to \mathcal{A}_n) = \bigcap_n \ker(\mathcal{A}' \to \mathcal{A}_0)^n.$$

To prove that this kernel is zero, we may enlarge the field to an algebraically closed field, so the result follows from Corollary **??**.

Assume that each scheme Z_n is locally Noetherian over k. Then $\mathcal{I}_n$ is a coherent $\mathcal{A}_n$-module, thus $\mathcal{I}_n^i/\mathcal{I}^{i+1}$ is a coherent $\mathcal{A}_0$-module for all i. The series

$$0 = \mathcal{I}_n^{n+1} \subset \mathcal{I}^n \subset \dots \subset \mathcal{I} \subset \mathcal{A}_n$$

has coherent subquotients, hence $\mathcal{A}_n$ is a coherent $\mathcal{O}_{Z_n}$-algebra. Thus $\mathcal{A}[\lambda]$ is a coherent $\mathcal{O}_{Z_0}$-module for every $\lambda \in \mathrm{Irr}(\overline{\mathbf{G}})$. The claim that π is of finite type is local on $Z^{\mathbf{G}}$, hence we may assume that $Z^{\mathbf{G}}$ is quasi-compact. The sheaf $\mathcal{J}/\mathcal{J}^2 \subseteq \mathcal{A}_1$ is coherent so there exists a finite set $\lambda_1, \dots, \lambda_r \in \mathrm{Irr}(\overline{\mathbf{G}}) \smallsetminus \{\lambda_0\}$ such that the morphism

$$\bigoplus_{i=1}^r \mathcal{A}[\lambda_i] \to \mathcal{J}/\mathcal{J}^2$$

induced by $\mathcal{A} \twoheadrightarrow \mathcal{A}_2$ is surjective. Let $\mathcal{B} \subset \mathcal{A}$ be the quasi-coherent $\mathcal{O}_{Z_0}$-subalgebra generated by the coherent subsheaf $\mathcal{M} := \bigoplus_{i=1}^r \mathcal{A}[\lambda_i] \subseteq \mathcal{A}$. Let $\overline{k}$ be an algebraic closure of k and let $\mathcal{A}' = \mathcal{A} \otimes \overline{k}$. Fix a Kempf's torus over $\overline{k}$ and the associated grading $\mathcal{A}' = \bigoplus_{i \geq 0} \mathcal{A}'[i]$ as in Corollary **??**. Then $\mathcal{J} = \bigoplus_{i \geq 1} \mathcal{A}'[i]$ is a graded ideal and $\mathcal{J}/\mathcal{J}^2$ is generated by the graded (coherent) subsheaf $\mathcal{M}' = \bigoplus_{i=1}^r \mathcal{A}'[\lambda_i]$. By graded Nakayama's lemma, the ideal $\mathcal{J}$ itself is generated by (the elements of) $\mathcal{M}'$. Then by induction on the degree, $\mathcal{A}'$ is generated by $\mathcal{M}'$ as an algebra. In other words, $\mathcal{A}' = \mathcal{B} \otimes \overline{k}$. Thus also $\mathcal{A} = \mathcal{B}$ and so $\mathcal{A}$ is of finite type over $\mathcal{O}_{Z_0}$. $\square$

With the proof of Theorem 1.6 in hand, we can easily algebraize also equivariant mappings between formal schemes.

Proposition 1.7 (Algebraization of morphisms of formal $\overline{\mathbf{G}}$-schemes)**.** *Let $\mathcal{W} = \{W_n\}$ and $\mathcal{Z} = \{Z_n\}$ be formal $\overline{\mathbf{G}}$-schemes. Let W and Z be algebraizations of $\mathcal{W}$ and $\mathcal{Z}$ respectively (see Theorem 1.6). Then every $\overline{\mathbf{G}}$-morphism $\widehat{\varphi}\colon \mathcal{W} \to \mathcal{Z}$ is the formalization of a unique $\overline{\mathbf{G}}$-equivariant morphism $\varphi\colon W \to Z$.*

Proof. The map $\widehat{\varphi}$ induces maps $W_n \to Z_n \hookrightarrow Z$. By Theorem 1.6, the scheme W is a colimit of W_n in the category of locally linear $\overline{\mathbf{G}}$-schemes. By the universal property of the colimit, we obtain a unique $\overline{\mathbf{G}}$-equivariant morphism $W \to Z$. □

It turns out that for each $n \in \mathbb{N}$ the functor P_n admits a right adjoint. We construct this right adjoint now. Let X be an object of $\mathcal{C}_n$. For every $m \in \mathbb{N}$ we define

$$X_m = \begin{cases} G_{m-1}...G_{n+1}G_n(X) & \text{if } m > n \\ X & \text{if } m = n \\ F_m...F_{n-2}F_{n-1}(X) & \text{if } m < n \end{cases}$$

and

$$u_m = \begin{cases} \xi_{G_{m-1}...G_{n+1}G_n(X)} & \text{if } m \geq n \\ 1_{F_m...F_{n-2}F_{n-1}(X)} & \text{if } m < n \end{cases}$$

where $\xi_{G_{m-1}...G_{n+1}G_n(X)} : F_mG_mG_{m-1}...G_{n+1}G_n(X) \to G_{m-1}...G_{n+1}G_n(X)$ is a counit of the adjoint functors F_m and G_m, which is an isomorphism as G_m is full and faithful. We define $Q_n(X) = (\{X_n\}_{n\in\mathbb{N}}, \{u_n\}_{n\in\mathbb{N}})$.

Proposition 1.8. *Let $Q_n : \mathcal{C}_n \to \mathcal{C}(\mathbb{T})$ be a that sends X*

2. Thick subcategories

Definition 2.1. Let $\mathcal{C}$ be an abelian category and let $\mathcal{S}$ be its full subcategory. Suppose that for every exact sequence in $\mathcal{C}$

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

we have $X \in \mathcal{S}$ if and only if $X', X'' \in \mathcal{S}$. Then $\mathcal{S}$ is called *a thick subcategory of $\mathcal{C}$*.

Definition 2.2. A category $\mathcal{C}$ is called *well-powered* if the class of subobjects of X is a set for every object X in $\mathcal{C}$.

Proposition 2.3. *Let $\mathcal{C}$ be an* **Ab**3*-category and let $\mathcal{S}$ be a thick subcategory. Assume that $\mathcal{S}$ is closed under small direct sums. For every object X in $\mathcal{C}$ there exists a unique subobject $S(X)$ such that for every morphism $f : Y \to X$ in $\mathcal{C}$ with Y in $\mathcal{S}$ we have $f(Y) \subseteq S(X)$.*

Proof. One can prove the result invoking general adjoint functor theorems [Mac Lane, 1998, Chapter V, Sections 5 and 6]. For self-containment we present the complete proof below.
Fix an object X of $\mathcal{C}$. Since $\mathcal{C}$ is well-powered, the class $\{Y_i\}_{i\in I}$ of subobjects of X that belong to $\mathcal{S}$ is a set. Since $\mathcal{S}$ is closed under small direct sums we derive that $\sum_{i\in I} Y_i \subseteq X$ is in $\mathcal{S}$. Indeed, this is the image of the canonical morphism

$$\bigoplus_{i\in I} Y_i \longrightarrow X$$

and since $\mathcal{S}$ is a thick subcategory closed under small direct sums, we deduce that this image is an object of $\mathcal{S}$. Thus $S(X) = \sum_{i\in I} Y_i$ is the largest subobject of X contained in $\mathcal{S}$. This implies the statement. □

Fact 2.4. *Let $\mathcal{C}$ be an* $\mathbf{Ab}3$*-category and let $\mathcal{S}$ be a thick subcategory. Assume that $\mathcal{S}$ is closed under small direct sums. For every X in $\mathcal{C}$ let $S(X)$ be the largest subobject of X contained in $\mathcal{S}$. Then $S : \mathcal{C} \to \mathcal{S}$ is a left exact functor.*

Proof. Left to the reader. □

3. Existence of the algebraization

Definition 3.1. Let $\mathbf{M}$ be a affine monoid k-scheme. Let $\mathcal{K} : \mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbb{A}^1_k)$ be an exact functor such that the triangle

$$\begin{tikzcd} \mathbf{Rep}(\mathbf{M}) \arrow[rr, "\mathcal{K}"] \arrow[dr, "|-|"'] & & \mathbf{Rep}(\mathbb{A}^1_k) \arrow[dl, "|-|"] \\ & \mathbf{Vect}_k & \end{tikzcd}$$

is commutative. Then we say that $\mathcal{K}$ is *a Kempf functor for* $\mathbf{M}$.

4. Formal $\mathbf{M}$-schemes

Let $\mathbf{M}$ be a affine monoid k-scheme.

Definition 4.1. Let X be a $\mathbf{M}$-scheme. We say that X is *a locally linear* $\mathbf{M}$*-scheme* if there exists an open cover of X consisting of affine and $\mathbf{M}$-stable subchemes of X.

Definition 4.2. *A formal* $\mathbf{M}$*-scheme* consists of a sequence $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ of $\mathbf{M}$-schemes together with $\mathbf{M}$-equivariant closed immersions

$$Z_0 \hookrightarrow Z_1 \hookrightarrow \ldots \hookrightarrow Z_n \hookrightarrow Z_{n+1} \hookrightarrow \ldots$$

satisfying the following assertions.

(1) $\mathbf{M}$-scheme Z_0 is locally linear.

(2) Let $\mathcal{I}_n$ be an ideal of $\mathcal{O}_{Z_n}$ defining Z_0. Then for every $m \leq n$ the subscheme $Z_m \subset Z_n$ is defined by $\mathcal{I}_n^{m+1}$.

Definition 4.3. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ and $\mathcal{W} = \{W_n\}_{n\in\mathbb{N}}$ are formal $\mathbf{M}$-schemes. Then *a morphism* $f : \mathcal{Z} \to \mathcal{W}$ *of formal* $\mathbf{M}$*-schemes* consists of a family of $\mathbf{M}$-equivariant morphisms $f = \{f_n : Z_n \to W_n\}_{n\in\mathbb{N}}$ such that the diagram

$$\begin{tikzcd} Z_0 \arrow[r, hook] \arrow[d, "f_0"'] & Z_1 \arrow[r, hook] \arrow[d, "f_1"'] & \ldots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[d, "f_n"'] & Z_{n+1} \arrow[r, hook] \arrow[d, "f_{n+1}"'] & \ldots \\ W_0 \arrow[r, hook] & W_1 \arrow[r, hook] & \ldots \arrow[r, hook] & W_n \arrow[r, hook] & W_{n+1} \arrow[r, hook] & \ldots \end{tikzcd}$$

is commutative.

Definition 4.4. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{M}$-scheme. *A quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{Z}$* consists of a family $\left(\{\mathcal{F}_n\}_{n\in\mathbb{N}}, \{\phi_{n,m}\}_{n,m\in\mathbb{N},m\leq n}\right)$ such that the following are satisfied.

(1) $\mathcal{F}_n$ is a quasi-coherent sheaf on Z_n with $\mathbf{M}$-linearization.

(2) $\phi_{n,m} : \mathcal{F}_{n|Z_m} \to \mathcal{F}_m$ is an isomorphism of quasi-coherent sheaves with $\mathbf{M}$-linearizations for any pair $n, m \in \mathbb{N}$ such that $m \leq n$.

(3) The composition

$$\phi_{m,l} \cdot \phi_{n,m|Z_l} : \left(\mathcal{F}_{n|Z_m}\right)_{|Z_l} \to \mathcal{F}_l$$

and the morphism

$$\phi_{n,l} : \mathcal{F}_{n|Z_l} \to \mathcal{F}_l$$

are canonically isomorphic for any $n, m, l \in \mathbb{N}$ such that $l \leq m \leq n$.

Definition 4.5. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{M}$-scheme. Suppose that $\mathcal{F} = \left(\{\mathcal{F}_n\}_{n\in\mathbb{N}}, \{\phi_{n,m}\}_{n,m\in\mathbb{N},m\leq n}\right)$ and $\mathcal{G} = \left(\{\mathcal{G}_n\}_{n\in\mathbb{N}}, \{\psi_{n,m}\}_{n,m\in\mathbb{N},m\leq n}\right)$ are quasi-coherent sheaves on $\mathcal{Z}$. *A morphism $\theta : \mathcal{F} \to \mathcal{G}$ of quasi-coherent sheaves on $\mathcal{Z}$* consists of a family $\{\theta_n : \mathcal{F}_n \to \mathcal{G}_n\}_{n\in\mathbb{N}}$ of morphisms of quasi-coherent sheaves with $\mathbf{M}$-linearizations such that squares

$$\begin{tikzcd} \mathcal{F}_{n|Z_m} \arrow[r, "\phi_{n,m}"] \arrow[d, "\theta_{n|Z_m}"'] & \mathcal{F}_m \arrow[d, "\theta_m"] \\ \mathcal{G}_{n|Z_m} \arrow[r, "\psi_{n,m}"'] & \mathcal{F}_m \end{tikzcd}$$

are commutative for any $n, m \in \mathbb{N}$ and $m \leq n$.

If $\mathcal{Z}$ is a formal $\mathbf{M}$-scheme, then we denote by $\mathfrak{Qcoh}(\mathcal{Z})$ its category of quasi-coherent sheaves.

Definition 4.6. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{M}$-scheme. A pair $(Z, \mathcal{I})$ consisting of a $\mathbf{M}$-scheme Z together with a quasi-coherent ideal $\mathcal{I}$ equipped with $\mathbf{M}$-linearization is called *an algebraization of $\mathcal{Z}$* if the following two conditions are satisfied.

(1) $\mathcal{Z}$ is isomorphic to $\widehat{Z}_{\mathcal{I}} = \{V(\mathcal{I}^n)\}_{n\in\mathbb{N}}$ in the category of formal $\mathbf{M}$-schemes.

(2) The canonical functor $\mathfrak{Qcoh}_{\mathbf{M}}(Z) \to \mathfrak{Qcoh}\left(\widehat{Z}_{\mathcal{I}}\right)$ is an equivalence of categories.

5. Telescopes of categories and their 2-limits

Definition 5.1. A diagram

$$\ldots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \ldots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

of categories and functors is called *a telescope of categories*.

We fix a telescope

$$\ldots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \ldots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

of categories. Our goal is to construct its 2-categorical limit. Consider pairs $\mathcal{X} = (\{X_n\}_{n\in\mathbb{N}}, \{u_n\}_{n\in\mathbb{N}})$ such that the following assertions hold.

(1) X_n is an object of $\mathcal{C}_n$ for every $n \in \mathbb{N}$.

(2) $u_n : F_n(X_{n+1}) \to X_n$ is an isomorphism in $\mathcal{C}_n$ for every $n \in \mathbb{N}$.

If $\mathcal{X} = (\{X_n\}_{n\in\mathbb{N}}, \{u_n\}_{n\in\mathbb{N}})$ and $\mathcal{Y} = (\{Y_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}})$ are two such pairs, then a morphism $f : \mathcal{X} \to \mathcal{Y}$ consists of a family $\{f_n : X_n \to Y_n\}_{n\in\mathbb{N}}$ of morphisms such that squares

$$\begin{tikzcd} F_n(X_{n+1}) \arrow[r, "u_n"] \arrow[d, "F_n(f_{n+1})"'] & X_n \arrow[d, "f_n"] \\ F_n(Y_{n+1}) \arrow[r, "w_n"'] & Y_n \end{tikzcd}$$

are commutative for $n \in \mathbb{N}$. This data gives rise to a category $\lim_{n\in\mathbb{N}} \mathcal{C}_n$. Next for every $n \in \mathbb{N}$ we define a functor $\pi_n : \lim_{n\in\mathbb{N}} \mathcal{C}_n \to \mathcal{C}_n$ that sends a morphism $f : \mathcal{X} \to \mathcal{Y}$ to $f_n : X_n \to Y_n$. Finally we define a natural isomorphism

$$\begin{tikzcd} & \lim_{n\in\mathbb{N}} \mathcal{C}_n \arrow[dl, "\pi_{n+1}"'] \arrow[dr, "\pi_n"] \arrow[d, phantom, "\Longrightarrow \sigma_n"] & \\ \mathcal{C}_{n+1} \arrow[rr, "F_n"'] & & \mathcal{C}_n \end{tikzcd}$$

by setting its component on $\mathcal{X} = (\{X_n\}_{n\in\mathbb{N}}, \{u_n\}_{n\in\mathbb{N}})$ to be $u_n : F_n(X_{n+1}) \to X_n$. Since $F_n\pi_{n+1}(\mathcal{X}) = F_n(X_{n+1})$ and $\pi_n(\mathcal{X}) = X_n$ this makes sense. The next result states that the data above form a 2-categorical limit over the telescope.

Theorem 5.2. *Let*

$$\ldots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \ldots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

be a telescope a categories. Suppose that $\mathcal{C}$ is a category, $\{P_n : \mathcal{C} \to \mathcal{C}_n\}_{n\in\mathbb{N}}$ is a family of functors and $\{\tau_n : F_nP_{n+1} \Rightarrow P_n\}_{n\in\mathbb{N}}$ is a family of natural isomorphisms. Then there exists a unique functor $F : \mathcal{C} \to \lim_{n\in\mathbb{N}} \mathcal{C}_n$ such that $P_n = \pi_nF$ and $(\sigma_n)_F = \tau_n$ for every $n \in \mathbb{N}$.

Proof. Suppose that $F : \mathcal{C} \to \lim_{n\in\mathbb{N}} \mathcal{C}_n$ is a functor such that $P_n = \pi_nF$ and $(\sigma_n)_F = \tau_n$ for every $n \in \mathbb{N}$. Pick an object X of $\mathcal{C}$. Then we have $\pi_nF(X) = P_n(X)$ and $(\sigma_n)_{F(X)} = (\tau_n)_X$. This implies that

$$F(X) = \big(\{P_n(X)\}_{n\in\mathbb{N}}, \{(\tau_n)_X : F_nP_{n+1}(X) \to P_n(X)\}_{n\in\mathbb{N}}\big)$$

Next if $f : X \to Y$ is a morphism in $\mathcal{C}$, then we derive that $\pi_nF(f) = P_n(f)$ for $n \in \mathbb{N}$. Hence $F(f) = \{P_n(f)\}_{n\in\mathbb{N}}$. This implies that the functor F can be completely recovered from the fact that $P_n = \pi_nF$ and $(\sigma_n)_F = \tau_n$ for every $n \in \mathbb{N}$. Note also that formulas

$$F(X) = \big(\{P_n(X)\}_{n\in\mathbb{N}}, \{(\tau_n)_X : F_nP_{n+1}(X) \to P_n(X)\}_{n\in\mathbb{N}}\big),\ F(f) = \{P_n(f)\}_{n\in\mathbb{N}}$$

for an object X in $\mathcal{C}$ and a morphism $f : X \to Y$ in $\mathcal{C}$, give rise to a functor that satisfy $P_n = \pi_nF$ and $(\sigma_n)_F = \tau_n$ for every $n \in \mathbb{N}$. This establishes existence and the uniqueness of F. □

Assume now that the telescope consists of monoidal categories and that for each $n \in \mathbb{N}$ functor F_n is monoidal. Then there exists a canonical monoidal structure on $\lim_{n\in\mathbb{N}} \mathcal{C}_n$. We define $(-) \otimes_{\lim_{n\in\mathbb{N}} \mathcal{C}_n} (-)$ by formula

$$\mathcal{X} \otimes_{\lim_{n\in\mathbb{N}} \mathcal{C}_n} \mathcal{Y} = \big(\{X_n \otimes_{\mathcal{C}_n} Y_n\}_{n\in\mathbb{N}}, \big\{(u_n \otimes_{\mathcal{C}_n} w_n) \cdot m_{X_{n+1},Y_{n+1}}\big\}_{n\in\mathbb{N}}\big)$$

where

$$m_{X_{n+1},Y_{n+1}} : F_n\left(X_{n+1} \otimes_{\mathcal{C}_{n+1}} Y_{n+1}\right) \to F_n(X_{n+1}) \otimes_{\mathcal{C}_n} F_n(Y_{n+1})$$

is the tensor preserving isomorphism of F_n. We also define the unit

$$I_{\lim_{n\in\mathbb{N}} \mathcal{C}_n} = \big(\{I_{\mathcal{C}_n}\}_{n\in\mathbb{N}}, \{F_n(I_{\mathcal{C}_{n+1}}) \cong I_{\mathcal{C}_n}\}_{n\in\mathbb{N}}\big)$$

where isomorphisms $F_n(I_{\mathcal{C}_{n+1}}) \cong I_{\mathcal{C}_n}$ are precisely the unit preserving isomorphisms of monoidal functors F_n for every $n \in \mathbb{N}$. The associativity natural isomorphism for $(-) \otimes_{\lim_{n\in\mathbb{N}} \mathcal{C}_n} (-)$ and right, left units for $I_{\lim_{n\in\mathbb{N}}\mathcal{C}_n}$ in $\lim_{n\in\mathbb{N}}\mathcal{C}_n$ are defined as tuples of the corresponding natural isomorphisms of $\mathcal{C}_n$ for $n \in \mathbb{N}$. With respect to this monoidal structure functors $\{\pi_n\}_{n\in\mathbb{N}}$ are **strict monoidal functors** and $\{\sigma_n\}_{n\in\mathbb{N}}$ are monoidal natural isomorphisms. The next result states that the data with these extra monoidal structure form a 2-categorical limit over the telescope in the 2-category of monoidal categories.

Theorem 5.3. *Let*

$$\ldots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \ldots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

be a telescope of a monoidal categories and monoidal functors. Suppose that $\mathcal{C}$ is a monoidal category, $\{P_n : \mathcal{C} \to \mathcal{C}_n\}_{n\in\mathbb{N}}$ is a family of monoidal functors and $\{\tau_n : F_n P_{n+1} \Rightarrow P_n\}_{n\in\mathbb{N}}$ is a family of natural monoidal isomorphisms. Then there exists a unique monoidal functor $F : \mathcal{C} \to \lim_{n\in\mathbb{N}}\mathcal{C}_n$ such that $P_n = \pi_n F$ and $(\sigma_n)_F = \tau_n$ for every $n \in \mathbb{N}$ as monoidal functors and monoidal transformations, respectively.

Proof. Note that F must be defined precisely as it was detected in the proof of Theorem 5.2. Namely we must have

$$F(X) = \left(\{P_n(X)\}_{n\in\mathbb{N}}, \{(\tau_n)_X : F_n P_{n+1}(X) \to P_n(X)\}_{n\in\mathbb{N}}\right),\ F(f) = \{P_n(f)\}_{n\in\mathbb{N}}$$

for an object X in $\mathcal{C}$ and a morphism $f : X \to Y$ in $\mathcal{C}$. Suppose also that F admits a structure of a monoidal functor such that $P_n = \pi_n F$ as monoidal functors for every $n \in \mathbb{N}$. Let

$$\left\{m^F_{X,Y} : F(X \otimes_{\mathcal{C}} Y) \to F(X) \otimes_{\lim_{n\in\mathbb{N}}\mathcal{C}_n} F(Y)\right\}_{X,Y\in\mathcal{C}},\ \phi^F : F(I_{\mathcal{C}}) \to I_{\lim_{n\in\mathbb{N}}}$$

be the data forming that structure. Since $\{\pi_n\}_{n\in\mathbb{N}}$ are strict monoidal functors and $P_n = \pi_n F$ as monoidal functors for every $n \in \mathbb{N}$, we derive that for any objects X, Y of $\mathcal{C}$

$$\pi_n(m^F_{X,Y}) : P_n(X \otimes_{\mathcal{C}} Y) \to P_n(X) \otimes_{\mathcal{C}_n} P_n(Y)$$

is the tensor preserving isomorphism $m^{P_n}_{X,Y} : P_n(X \otimes_{\mathcal{C}} Y) \to P_n(X) \otimes_{\mathcal{C}_n} P_n(Y)$ of the monoidal functor P_n for every $n \in \mathbb{N}$. By the same argument

$$\pi_n(\phi_F) : P_n(I_{\mathcal{C}}) \to I_{\mathcal{C}_n}$$

is the unit preserving isomorphism $\phi^{P_n} : P_n(I_{\mathcal{C}}) \to I_{\mathcal{C}_n}$ of P_n. Thus we deduce that for any objects X, Y of $\mathcal{C}$ we have $m^F_{X,Y} = \{m^{P_n}_{X,Y}\}_{n\in\mathbb{N}}$ and $\phi^F = \{\phi^{P_n}\}_{n\in\mathbb{N}}$. This implies the uniqueness of a monoidal structure on F such that $P_n = \pi_n F$ as monoidal functors for every $n \in \mathbb{N}$. On the other hand define

$$m^F_{X,Y} = \{m^{P_n}_{X,Y}\}_{n\in\mathbb{N}}$$

for objects X, Y in $\mathcal{C}$ and

$$\phi^F = \{\phi^{P_n}\}_{n\in\mathbb{N}}$$

Then

□

References

[Mac Lane, 1998] Mac Lane, S. (1998). *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition.