QUOTIENTS OF ALGEBRAIC GROUPS

1. Introduction

Throughout this notes k denote a field and $\mathbf{G}$ denote a group scheme over k. We denote by e the identity of $\mathbf{G}$. We also fix a k-scheme X equipped with an action of $\mathbf{G}$ determined by morphism $a : \mathbf{G} \times_k X \to X$.

2. Basic properties of scheme group quotients

The following result gives scheme-theoretic criterion for topological quotient in the case of group scheme actions.

Proposition 2.1. *Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Assume that q is submersive and and the morphism $\mathbf{G} \times_k X \to X \times_Y X$ induced by a and pr_X is surjective. Then the diagram*

$$\begin{tikzcd} \mathbf{G} \times_k X \arrow[r, shift left, "a"] \arrow[r, shift right, "\mathrm{pr}_X"'] & X \arrow[r, "q"] & Y \end{tikzcd}$$

is a cokernel in the category topological spaces.

Proof. Let π_1 and π_2 be distinct projections $X \times_Y X \to X$. Pick points x_1 and x_2 in X such that $q(x_1) = q(x_2)$. Then there exists a field extension K over k such that $k(x_1) \subseteq K$ and $k(x_2) \subseteq K$. These give rise to K-points $\overline{x_1}$ and $\overline{x_2}$ of X such that their images under q is the same K-point of Y. Since we have an identification

$$\big(X \times_Y X\big)(K) = X(K) \times_{Y(K)} X(K)$$

induced by π_1 and π_2, we derive that there exists a K-point $\overline{z}$ of $X \times_Y X$ such that $\pi_1(\overline{z}) = \overline{x_1}$ and $\pi_2(\overline{z}) = \overline{x_2}$. Let z be the point of $X \times_Y X$ corresponding to $\overline{z}$. Then $\pi_1(z) = x_1$ and $\pi_2(z) = x_2$. By assumption a and pr_X induce surjection $\mathbf{G} \times_k X \twoheadrightarrow X \times_Y X$. Thus there exists a point u of $\mathbf{G} \times_k X$ such that $a(u) = x_1$ and $\mathrm{pr}_X(u) = x_2$. Thus x_1 and x_2 are identified by an equivalence relation on the underlying set of X which is determined by the pair (a, pr_X). Therefore, fibers of q are equivalence classes with respect to this relation. Since q is submersive, this implies that the diagram

$$\begin{tikzcd} \mathbf{G} \times_k X \arrow[r, shift left, "a"] \arrow[r, shift right, "\mathrm{pr}_X"'] & X \arrow[r, "q"] & Y \end{tikzcd}$$

is a cokernel in the category topological spaces. □

Now we prove a series results concerning fpqc descent. For this we fix a k-scheme Y with the trivial action of $\mathbf{G}$ and a $\mathbf{G}$-equivariant morphism $q : X \to Y$. Let $g : Y' \to Y$ be a morphism of k-schemes and consider the cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

of k-schemes. Note that X' admits a unique action a' of $\mathbf{G}$ such that the square above consists of $\mathbf{G}$-equivariant morphism (we consider g as a $\mathbf{G}$-equivariant morphism between trivial $\mathbf{G}$-schemes).

Fact 2.2. *Suppose that g is faithfully flat and quasi-compact. Assume that q' is (universally) submersive. Then q is (universally) submersive.*

Proof. It suffices to prove that submersive morphisms have descent property. This follows from the fact that g (as faithfully flat and quasi-compact morphism) and q' are submersive. Details are left for the reader. □

Fact 2.3. *Suppose that g is faithfully flat and quasi-compact. Then the canonical morphism $X' \times_{Y'} X' \to X \times_Y X$ is faithfully flat and quasi-compact and there is the cartesian square*

$$\begin{tikzcd} \mathbf{G} \times_k X' \arrow[r] \arrow[d] & \mathbf{G} \times_k X \arrow[d] \\ X' \times_{Y'} X' \arrow[r] & X \times_Y X \end{tikzcd}$$

in which the left vertical arrow is induced by $\langle a', \mathrm{pr}_{X'} \rangle : \mathbf{G} \times_k X' \to X' \times_k X'$, the right vertical arrow is induced by $\langle a, \mathrm{pr}_X \rangle : \mathbf{G} \times_k X \to X \times_k X$ and the bottom horizontal morphism is the canonical morphism.

Proof. Note that squares

$$\begin{tikzcd} X' \times_{Y'} X' \arrow[r] \arrow[d] & X' \times_Y X' \arrow[d] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X' \times_Y X' \arrow[r] \arrow[d] & X \times_Y X \arrow[d] \\ X' \times_k X' \arrow[r, "g' \times_k g'"'] & X \times_k X \end{tikzcd}$$

are cartesian. Since both g and $g' \times_k g'$ are faithfully flat and quasi-compact, we derive that both morphisms $X' \times_{Y'} X' \to X' \times_Y X'$ and $X' \times_Y X' \to X \times_Y X$ are faithfully flat and quasi-compact. Then their composition i.e. the canonical morphism $X' \times_{Y'} X' \to X \times_Y X$ is faithfully flat and quasi-compact. □

Finally we need the following notion

Definition 2.4. Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Consider a pair

$$q_*\mathcal{O}_X \overset{q_*a^\#}{\underset{q_*\mathrm{pr}_X^\#}{\rightrightarrows}} q_*\left(\mathrm{pr}_X\right)_* \mathcal{O}_{\mathbf{G}\times_k X} = q_* a_* \mathcal{O}_{\mathbf{G}\times_k X}$$

of morphisms of sheaves of rings on Y. Suppose that $q^\# : \mathcal{O}_Y \to q_*\mathcal{O}_X$ is a kernel of this pair. Then *$\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q.*

Proposition 2.5. *Suppose that g is faitfully flat and quasi-compact. Assume that q' is quasi-compact, semiseparated and $\mathcal{O}_{Y'}$ is the sheaf of $\mathbf{G}$-invariants for q'. Then $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q.*

Proof. We denote by a' the action of $\mathbf{G}$ on X'. First note that q is semiseparated and quasi-compact morphism as these classes of morphisms admit descent along quasi-compact and faithfully flat morphisms. Since q is quasi-compact, semiseparated and g is flat, we derive that for every quasi-coherent sheaf $\mathcal{F}$ on X the canonical morphism $q'_*g'^*\mathcal{F} \to g^*q_*\mathcal{F}$ is an isomorphism. Thus the diagram

$$\mathcal{O}_{Y'} \xrightarrow{q^\#} q'_*\mathcal{O}_{X'} \overset{q'_*a'^\#}{\underset{q'_*\mathrm{pr}^\#_{X'}}{\rightrightarrows}} q'_*\left(\mathrm{pr}_{X'}\right)_*\mathcal{O}_{\mathbf{G}\times_k X'} = q'_*a'_*\mathcal{O}_{\mathbf{G}\times_k X'}$$

is isomorphic to the diagram

$$g^*\mathcal{O}_Y \xrightarrow{g^*q^\#} g^*\left(q_*\mathcal{O}_X\right) \overset{g^*q_*a^\#}{\underset{g^*q_*\mathrm{pr}^\#_X}{\rightrightarrows}} g^*\left(q_*\left(\mathrm{pr}_X\right)_*\mathcal{O}_{\mathbf{G}\times_k X}\right) = g^*\left(q_*a_*\mathcal{O}_{\mathbf{G}\times_k X}\right)$$

Since $\mathcal{O}_{Y'}$ is the sheaf of $\mathbf{G}$-invariants for q', the first diagram is a kernel diagram. Hence the second is a kernel diagram. According to the fact that g is faithfully flat we deduce that the diagram

$$\mathcal{O}_Y \xrightarrow{q^\#} q_*\mathcal{O}_X \overset{q_*a^\#}{\underset{q_*\mathrm{pr}^\#_X}{\rightrightarrows}} q_*\left(\mathrm{pr}_X\right)_*\mathcal{O}_{\mathbf{G}\times_k X} = q_*a_*\mathcal{O}_{\mathbf{G}\times_k X}$$

is also a kernel diagram. Thus $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q. □

3. Categorical and geometric quotients

Definition 3.1. Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Suppose that the following conditions hold.

(1) q is submersive.

(2) The morphism $\mathbf{G}\times_k X \to X\times_Y X$ induced by $\langle a, \mathrm{pr}_x\rangle : \mathbf{G}\times_k X \to X\times_k X$ is surjective.

(3) $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariant for q.

Then q is *a geometric quotient of* X.

Corollary 3.2. *Let q be a geometric quotient of X. Then the diagram*

$$\mathbf{G}\times_k X \overset{a}{\underset{\mathrm{pr}_X}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel in the category of ringed spaces.

Proof. Due to the fact that $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q it suffices to prove that

$$\mathbf{G}\times_k X \overset{a}{\underset{\mathrm{pr}_X}{\rightrightarrows}} X \xrightarrow{q} Y$$

is the cokernel in the category of topological spaces. This follows from Proposition 2.1. □

Definition 3.3. Let $q : X \to Y$ be a morphism of k-schemes such that the diagram

$$\mathbf{G}\times_k X \overset{a}{\underset{\mathrm{pr}_X}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel in the category of k-schemes. Then $q : X \to Y$ is *a categorical quotient of* X.

Fact 3.4. *Every geometric quotient is categorical.*

Proof. Categorical quotient is a cokernel in the category of k-schemes. On the other hand geometric quotient is a cokernel in the category of locally ringed spaces and hence it also satisfies cokernel property in its full subcategory of k-schemes. Thus every geometric quotient is categorical. □

Let $q : X \to Y$ be a morphism of k-schemes such that $q \cdot \mathrm{pr}_X = q \cdot a$. For a morphism $g : Y' \to Y$ of k-schemes consider the cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Then there exists a unique action $a' : \mathbf{G} \times_k X' \to X'$ of $\mathbf{G}$ on X' such that the square above consists of $\mathbf{G}$-equivariant morphism (we consider Y, Y' as $\mathbf{G}$-schemes equipped with trivial $\mathbf{G}$-actions). Keeping this in mind we have the following.

Definition 3.5. A morphism $q : X \to Y$ is *a uniform categorical (geometric) quotient of* X if for every flat morphism $g : Y' \to Y$ its base change $q' : X' \to Y'$ is a categorical (geometric) quotient of X'.

Definition 3.6. A morphism $q : X \to Y$ is *a universal categorical (geometric) quotient of* X if for every morphism $g : Y' \to Y$ its base change $q' : X' \to Y'$ is a categorical (geometric) quotient of X'.

In the next result we give a simple example of a universal geometric quotient.

Proposition 3.7. *Suppose that* $\mathbf{G}$ *is a quasi-compact group scheme over* k*. Let* Y *be a* k*-scheme and consider* $\mathbf{G} \times_k Y$ *with the action of* $\mathbf{G}$ *induced by the regular action on the left factor. Then* $\mathrm{pr}_Y : \mathbf{G} \times_k Y \to Y$ *is a universal geometric quotient.*

Proof. Clearly pr_Y is univerally submersive (it is even universally open). Let $\mu : \mathbf{G} \times_k \mathbf{G} \to \mathbf{G}$ be the multiplication morphism and let $\pi_{23} : \mathbf{G} \times_k \mathbf{G} \times Y \to \mathbf{G} \times_k Y$ be the projection on the last two factors. Then the morphism

$$\mathbf{G} \times_k \mathbf{G} \times_k Y \to (\mathbf{G} \times_k Y) \times_Y (\mathbf{G} \times_k Y) = \mathbf{G} \times_k \mathbf{G} \times_k Y$$

induced by $\langle \mu \times_k 1_Y, \pi_{23} \rangle : \mathbf{G} \times_k \mathbf{G} \times_k Y \to (\mathbf{G} \times_k Y) \times_k (\mathbf{G} \times_k Y)$ is an isomorphism. We show that $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for pr_Y. For this pick an affine open subset V of Y. It suffices to check that the diagram

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{\mathrm{pr}_Y^\#} \Gamma\left(\mathbf{G} \times_k V, \mathcal{O}_{\mathbf{G} \times_k Y}\right) \underset{\pi_{23}^\#}{\overset{(\mu \times_k 1_Y)^\#}{\rightrightarrows}} \Gamma\left(\mathbf{G} \times_k \mathbf{G} \times_k V, \mathcal{O}_{\mathbf{G} \times_k \mathbf{G} \times_k Y}\right)$$

is a kernel. Since $\mathbf{G}$ is quasi-compact and separated (every group k-scheme is separated), we derive that the diagram above is isomorphic with

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{f \mapsto 1 \otimes f} \Gamma(\mathbf{G}, \mathcal{O}_{\mathbf{G}}) \otimes_k \Gamma(V, \mathcal{O}_Y) \underset{\chi \otimes f \mapsto 1 \otimes \chi \otimes f}{\overset{\chi \otimes f \mapsto \mu^\#(\chi) \otimes f}{\rightrightarrows}} \Gamma(\mathbf{G}, \mathcal{O}_{\mathbf{G}}) \otimes_k \Gamma(\mathbf{G}, \mathcal{O}_{\mathbf{G}}) \otimes_k \Gamma(V, \mathcal{O}_Y)$$

□

4. Geometric quotients of separated actions

Definition 4.1. The action of $\mathbf{G}$ on X is *separated* if the morphism $\langle a, \mathrm{pr}_X\rangle : \mathbf{G} \times_k X \to X \times_k X$ has closed set-theoretic image.

Theorem 4.2. *Let $q : X \to Y$ be a geometric quotient of X. Assume that q is universally submersive. Then the following assertions are equivalent.*

- **(i)** *The action of $\mathbf{G}$ on X is separated.*
- **(ii)** *Y is separated.*

Proof. We have a cartesian square

$$\begin{tikzcd} X \times_Y X \arrow[r, hook] \arrow[d] & X \times_k X \arrow[d, "q \times_k q"] \\ Y \arrow[r, hook, "\Delta_Y"'] & Y \times_k Y \end{tikzcd}$$

It follows that $X \times_Y X \hookrightarrow X \times_k X$ is a locally closed immersion. Since q is a geometric quotient, we derive that $\langle a, \mathrm{pr}_X\rangle$ factors as a surjective morphism $\mathbf{G} \times_k X \twoheadrightarrow X \times_Y X$ followed by the immersion $X \times_Y X \hookrightarrow X \times_k X$. Thus the action of $\mathbf{G}$ on X is separated if and only if $X \times_Y X$ is a closed subscheme of $X \times_k X$. Since q is universally submersive, we derive that $q \times_k q$ is submersive. As the square above is cartesian we derive that $\Delta_Y(Y) \subseteq Y \times_k Y$ is closed if and only if $X \times_Y X \subseteq X \times_k X$ is closed. Therefore, Y is separated if and only if the action of $\mathbf{G}$ on X is separated. $\square$

5. Geometric quotients of free actions and principal bundles

Definition 5.1. The action of $\mathbf{G}$ on X is *free* if the morphism $\langle a, \mathrm{pr}_X\rangle : \mathbf{G} \times_k X \to X \times_k X$ is a closed immersion.

Definition 5.2. Let x be a k-point of X. Suppose that *the orbit morphism* $\mathbf{G} \to X$ *of* x given by the composition

$$\mathbf{G} = \mathbf{G} \times_k \operatorname{Spec} k \xhookrightarrow{\text{induced by } x} \mathbf{G} \times_k X \longrightarrow X$$

is a closed immersion. Then the action of $\mathbf{G}$ on X has *a closed free orbit at x*.

Fact 5.3. *If the action of $\mathbf{G}$ on X is free, then every k-point of X has a closed free orbit.*

The following result states that over special type of local complete noetherian k-algebras geometric quotients of free actions correspond to trivial $\mathbf{G}$-bundles.

Theorem 5.4. *Suppose that k is an algebraically closed field and $\mathbf{G}$ is a smooth algebraic group over k. Let $q : X \to Y$ be a geometric quotient locally of finite type and let Y be the spectrum of a complete local noetherian k-algebra such that the residue field of the closed point of Y is k. Then the following assertions hold.*

- **(1)** *If x is a k-point of X which has a closed free orbit, then there exists a $\mathbf{G}$-equivariant, étale and surjective morphism $f : \mathbf{G} \times_k Y \to X$ such that the triangle*

$$\begin{tikzcd} \mathbf{G} \times_k Y \arrow[rr, "f"] \arrow[dr, "\mathrm{pr}_Y"'] & & X \arrow[dl, "q"] \\ & Y & \end{tikzcd}$$

is commutative and the morphism

$$Y = \operatorname{Spec} k \times_k Y \xhookrightarrow{e \times_k 1_Y} \mathbf{G} \times_k Y \xrightarrow{f} X$$

is a section of q.

(2) *If the action of* $\mathbf{G}$ *on* X *is free, then* f *is an isomorphism.*

The proof relies on two algebraic lemmas. The first describe free actions in context of complete rings and the second is a version of Hensel's lemma.

Lemma 5.4.1. *Let* $(R, \mathfrak{m}, k)$ *be a complete local noetherian* k*-algebra and let* $\sigma : R \to R[[x_1, ..., x_n]]$ *be a local morphism into a ring of formal power series over* R. *Assume that the composition*

$$R \xrightarrow{\sigma} R[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod (x_1, ..., x_n)} R$$

is the identity and the composition

$$R \xrightarrow{\sigma} R[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod \mathfrak{m}} (R/\mathfrak{m})\,[[x_1, ..., x_n]] = k[[x_1, ..., x_n]]$$

is surjective. Consider elements $y_1, ..., y_n$ *of* R *such that* $\sigma(y_i) \bmod \mathfrak{m} = x_i$ *for* $i = 1, ..., n$. *Then the composition*

$$R \xrightarrow{\sigma} R[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod (y_1, ..., y_n)} (R/(y_1, ..., y_n))\,[[x_1, ..., x_n]]$$

is an isomorphism.

Proof of the lemma. For convienience let ϕ denote the morphism given by the rule $r \mapsto \sigma(r) \bmod (y_1, ..., y_n)$. Also denote $R/(y_1, ..., y_n)$ by S. According to assumptions we have

$$\sigma(y_i) = x_i + y_i + \sum_{j=1}^{n} x_j \cdot \mathfrak{m}[[x_1, ..., x_n]]$$

for each i. Thus $\phi(y_i) = \sum_{j=1}^{n} f_{ij} \cdot x_j$ where $f_{ij} \in S$ are elements such that the matrix $\left[f_{ij}\right]_{1 \leq i,j \leq n}$ is invertible in S. Hence

$$S[[x_1, ..., x_n]] = S[[\phi(y_1), ..., \phi(y_n)]]$$

and ϕ composed with $S[[\phi(y_1), ..., \phi(y_n)]] \twoheadrightarrow S$ is the quotient morphism $R \twoheadrightarrow S$. From this observations we derive that ϕ is surjective. It remains to prove that it is injective. Consider z in R such that $\phi(z) = 0$. Suppose that $z \in (y_1, ..., y_n)^m$ for some $m \in \mathbb{N}$. Write

$$z = \sum_{\alpha \in \Lambda} c_\alpha \cdot y_1^{\alpha_1} ... y_n^{\alpha_n}$$

for some $c_\alpha \in R$ where $\Lambda = \{\alpha \in \mathbb{N}^n \,|\, \alpha_1 + ... + \alpha_n = m\}$. Hence

$$0 = \phi(z) = \sum_{\alpha \in \Lambda} \phi(c_\alpha) \cdot \phi(y_1)^{\alpha_1} ... \phi(y_n)^{\alpha_n}$$

Thus $\phi(c_\alpha) \in (\phi(y_1),...,\phi(y_n))$ for every $\alpha \in \Lambda$. Since ϕ composed with $S[[\phi(y_1),...,\phi(y_n)]] \twoheadrightarrow S$ is the quotient morphism $R \twoheadrightarrow S$, we derive that

$$c_\alpha \bmod (y_1,...,y_n) = \phi(c_\alpha) \bmod (\phi(y_1),...,\phi(y_n)) = 0$$

for every $\alpha \in \Lambda$. Thus $c_\alpha \in (y_1,...,y_n)$ for every $\alpha \in \Lambda$, which implies that $z \in (y_1,...,y_n)^{m+1}$. Thus we proved that

$$\phi(z) = 0 \text{ and } z \in (y_1,...,y_n)^m \Rightarrow z \in (y_1,...,y_n)^{m+1}$$

By $\mathfrak{m}$-adic completeness of R this implies that $\phi(z) = 0$ if and only if $z = 0$. Hence ϕ is also injective. □

Lemma 5.4.2. *Let $(R,\mathfrak{m})$ be a complete local noetherian ring and let $R \to S$ be a local morphism into a local noetherian ring. Suppose that there exists a finitely generated R-submodule N of S such that*

$$S = N + \mathfrak{m}S$$

Then $S = N$.

Proof of the lemma. Pick s in S. Since $S = N + \mathfrak{m}S$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in \mathfrak{m}^n N$ and

$$s - \sum_{i\le n} x_i \in \mathfrak{m}^{n+1}S$$

According to the assumption that $(R,\mathfrak{m})$ is complete with respect to $\mathfrak{m}$-adic topology and N is finitely generated over R, we deduce that N is complete with respect to $\mathfrak{m}$-adic topology. Hence there exists a unique element x in N such that

$$x = \sum_{n\in\mathbb{N}} x_n$$

where above series is convergent with respect to $\mathfrak{m}$-adic topology. Note also that

$$x - \sum_{i\le n} x_i \in \mathfrak{m}^{n+1}N$$

for every $n \in \mathbb{N}$. Thus we have

$$s - x = \left(s - \sum_{i\le n} x_i\right) - \left(x - \sum_{i\le n} x_i\right) \in \mathfrak{m}^{n+1}S + \mathfrak{m}^{n+1}N = \mathfrak{m}^{n+1}S$$

for every $n \in \mathbb{N}$. Hence

$$x - s \in \bigcap_{n\in\mathbb{N}} \mathfrak{m}^n S$$

Since $R \to S$ is local morphism and S is a local ring, we deduce that $\mathfrak{m}S$ is contained in the maximal ideal of S. By assumptions S is noetherian. Therefore, S is separated with respect to $\mathfrak{m}$-adic topology. This implies that

$$0 = \bigcap_{n\in\mathbb{N}} \mathfrak{m}^n S$$

Thus $s - x = 0$ and we infer that s is an element of N. This completes the proof that $S = N$. □

In what follows we shall denote by $\mathbf{G}x$ the closed subscheme determined by the orbit morphism $\mathbf{G} \to X$ of a k-point x of X which has a closed free orbit. For readers convienience we include the following lemmas, which have topological content.

Lemma 5.4.3. *Let $q : X \to Y$ be a geometric quotient and assume that Y is the spectrum of a local k-algebra such that the residue field of the closed point o of Y is k. Let x be a k-point of X with free closed orbit, then $q^{-1}(o) = \mathbf{G}x$ as closed subschemes of X.*

Proof of the lemma. Morphism q induces the morphism of residue fields $k(q(x)) \hookrightarrow k(x) = k$ over k. This implies that $k(q(x)) = k$ and hence $q(x)$ is a k-point of Y. Note that o is the unique k-point of Y. Thus $q(x) = o$. Clearly $q^{-1}(o)$ is a closed $\mathbf{G}$-stable subscheme of X (it is the preimage of o under $\mathbf{G}$-equivariant q), that contains x. Since $\mathbf{G}x$ is the smallest closed $\mathbf{G}$-stable subscheme of X containing x, we deduce that $\mathbf{G}x \subseteq q^{-1}(o)$ scheme-theoretically. Consider the pair of arrows

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X$$

Passing to functors of points we obtain that $a^{-1}(\mathbf{G}x) = \mathrm{pr}_X(\mathbf{G}.x)$. Since q is the cokernel of the pair (a, pr_X) in the category of topological spaces, we deduce that there exists a closed subset Z of Y such that $q^{-1}(Z) = \mathbf{G}x$. Clearly $o \in Z$ and hence $q^{-1}(o) \subseteq \mathbf{G}x$ set-theoretically. On the other hand above we proved that $\mathbf{G}x \subseteq q^{-1}(o)$ scheme-theoretically. This can only happen if $q^{-1}(o) = \mathbf{G}x$ as closed subschemes of X. □

Lemma 5.4.4. *Let $q : X \to Y$ be a geometric quotient and assume that Y is the spectrum of a local k-algebra such that the residue field of the closed point o of Y is k. Let U be an open $\mathbf{G}$-stable subset of X which contain a k-point. Then $U = X$.*

Proof of the lemma. Consider the pair of arrows

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X$$

Since U is $\mathbf{G}$-stable open subset of X, we derive that $\mathrm{pr}_X^{-1}(U) = a^{-1}(U)$. Next by definition q is the cokernel of the above pair in the category of topological spaces. Hence there exists an open subset V of Y such that $U = q^{-1}(V)$. Since U contains a k-point of X, we deduce as in Lemma 5.4.3 that $o \in V$. Thus $V = Y$ and finally $U = q^{-1}(V) = X$. □

Proof of the theorem. We first prove **(1)**. Denote by o the closed point of Y. Assume that x is a k-point of X which has a closed free orbit. Consider the surjective morphism $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{\mathbf{G},e}$ induced by the orbit morphism $\mathbf{G} \hookrightarrow X$ of x. Since $\mathbf{G}$ is smooth over k, the ring $\mathcal{O}_{\mathbf{G},e}$ is regular. Pick a system of parameters $x_1, ..., x_n$ of $\mathcal{O}_{\mathbf{G},e}$ and let $y_1, ..., y_n$ be elements of $\mathcal{O}_{X,x}$ such that y_i is send to x_i by the morphism $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{\mathbf{G},e}$ for $1 \leq i \leq n$. Define S to be the quotient ring $\mathcal{O}_{X,x}/(y_1, ..., y_n)$. The morphism q induces the morphism $q^{\#} : \mathcal{O}_{Y,o} \to \mathcal{O}_{X,x}$ and hence the morphism $\mathcal{O}_{Y,o} \to S$. By Lemma 5.4.3 we have

$$S/\mathfrak{m}_o S = k$$

where $\mathfrak{m}_o$ is the maximal ideal of $\mathcal{O}_{Y,o}$. According to Lemma 5.4.2 we derive that $\mathcal{O}_{Y,o} \to S$ is surjective. Let $f : \mathbf{G} \times_k \operatorname{Spec} S \to X$ be the unique $\mathbf{G}$-equivariant morphism induced by the surjection $\mathcal{O}_{X,x} \twoheadrightarrow S$. We have a commutative square

$$\begin{tikzcd}
\mathbf{G} \times_k \operatorname{Spec} S \arrow[r, "f"] \arrow[d, "\mathrm{pr}_{\operatorname{Spec} S}"'] & X \arrow[d, "q"] \\
\operatorname{Spec} S \arrow[r, hook, "j"'] & Y
\end{tikzcd}$$

where j is a closed immersion induced by $\mathcal{O}_{Y,o} \twoheadrightarrow S$. According to assumptions q is locally of finite type. Moreover, $\mathbf{G}$ is an algebraic group over k and hence $\mathrm{pr}_{\operatorname{Spec} S}$ is locally of finite type. These two assertions together with the fact that $\operatorname{Spec} S \hookrightarrow Y$ is a closed immersion of noetherian schemes (and thus is of finite type) imply that f is locally of finite type. Then by Lemma 5.4.1 we deduce that f induces an isomorphism

$$\widehat{\mathcal{O}_{X,x}} \longrightarrow \hat{S}[[x_1, \dots, x_n]] = \hat{S}\hat{\otimes}_k \widehat{\mathcal{O}_{\mathbf{G},e}}$$

of complete local rings. Since f is locally of finite type, it follows that f is étale at a k-point of $\mathbf{G} \times_k \operatorname{Spec} S$ determined by the unique k-point of $\operatorname{Spec} S$ and $e \in \mathbf{G}$. Let U be an étale locus of f. It contains a k-point and hence it is nonempty. Moreover, U is open (it is étale locus) subset of X. Since f is $\mathbf{G}$-equivariant, we derive that U is $\mathbf{G}$-stable. Similarly $f(U)$ is open $\mathbf{G}$-stable subset of X and $x \in f(U)$. Thus by Lemma 5.4.4 we deduce that

$$U = \mathbf{G} \times_k \operatorname{Spec} S,\ f(U) = X$$

Therefore, f is étale and surjective. Now we pullback $q : X \to Y$ along the closed immersion $\operatorname{Spec} S \hookrightarrow Y$. We obtain a cartesian square

$$\begin{tikzcd} \tilde{X} \arrow[r, hook, "\tilde{j}"] \arrow[d, "\tilde{q}"'] & X \arrow[d, "q"] \\ \operatorname{Spec} S \arrow[r, hook, "j"'] & Y \end{tikzcd}$$

Then f factors as a morphism $\mathbf{G} \times_k \operatorname{Spec} S \to \tilde{X}$ followed by a closed immersion $\tilde{j}$. Since f is étale and surjective, we deduce that $\tilde{j}$ is étale and surjective. This implies that $\tilde{j}$ is an isomorphism of schemes. By definition of geometric quotient we derive that the diagram

$$\mathcal{O}_Y \xrightarrow{q^{\#}} q_*\mathcal{O}_X \underset{q_*\mathrm{pr}_X^{\#}}{\overset{q_*a^{\#}}{\rightrightarrows}} q_*\left(\mathrm{pr}_X\right)_* \mathcal{O}_{\mathbf{G}\times_k X} = q_*a_*\mathcal{O}_{\mathbf{G}\times_k X}$$

is the kernel in the category of sheaves on Y. Hence $q^{\#} : \mathcal{O}_Y \to q_*\mathcal{O}_X$ is a monomorphism of sheaves. On the other hand we have

$$q^{\#} = j_*q_*\left(\tilde{j}^{-1}\right)^{\#} \cdot j_*\tilde{q}^{\#} \cdot j^{\#}$$

and thus $j^{\#}$ is a monomorphism. Since j is a closed immersion, we infer that j is an isomorphism. Therefore, we can identify $\operatorname{Spec} S$ with Y. Then f is a morphism which makes the triangle

$$\begin{tikzcd} \mathbf{G}\times_k Y \arrow[rr, "f"] \arrow[dr, "\mathrm{pr}_Y"'] & & X \arrow[dl, "q"] \\ & Y & \end{tikzcd}$$

commutative. This completes the proof of **(1)**.

For the proof of **(2)** consider the section $s : Y \hookrightarrow X$ described in **(1)**. Then f fits into a cartesian square

$$\begin{tikzcd}
\mathbf{G} \times_k Y \arrow[r, "f"] \arrow[d, "1_{\mathbf{G}} \times_Y s"'] & X \times_Y Y = X \arrow[d, "1_X \times_Y s"] \\
\mathbf{G} \times_k X \arrow[r, hook, "\phi"'] & X \times_Y X
\end{tikzcd}$$

where ϕ is a closed immersion induced by the closed immersion $\langle a, \mathrm{pr}_X \rangle : \mathbf{G} \times_k X \hookrightarrow X \times_k X$ (the action of $\mathbf{G}$ on X is free). Thus f is a closed immersion. By **(1)** it is étale and surjective. Therefore, f is an isomorphism. □

Definition 5.5. Let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism into a k-scheme Y equipped with the trivial $\mathbf{G}$-action. Suppose that q is faithfully flat, quasi-compact morphism and the square

$$\begin{tikzcd}
\mathbf{G} \times_k X \arrow[r, "a"] \arrow[d, "\mathrm{pr}_X"'] & X \arrow[d, "q"] \\
X \arrow[r, "q"'] & Y
\end{tikzcd}$$

is cartesian. Then q is *a principal* $\mathbf{G}$*-bundle*.

Now we use Theorem 5.4 to describe principal $\mathbf{G}$-bundles in the category of locally algebraic k-schemes.

Theorem 5.6. *Suppose that* $\mathbf{G}$ *is a smooth algebraic group over* k*. Let* $q : X \to Y$ *be a morphism locally of finite type between* k*-schemes locally of finite type. Then the following assertions are equivalent.*

(i) *q is a uniform geometric quotient and the action of* $\mathbf{G}$ *on* X *is free.*

(ii) *q is a principal* $\mathbf{G}$*-bundle.*

Proof. Suppose that **(i)** holds. Clearly q is locally of finite type. Since q is uniform quotient, we may consider base □

References