

MONOID k -FUNCTORS AND THEIR REPRESENTATIONS

1. INTRODUCTION AND NOTATION

In these notes we study algebraic structures in the category of k -functors with special emphasis on monoid objects.

If R is a ring, then we denote by R^\times its multiplicative monoid.

2. ALGEBRAIC STRUCTURES IN THE CATEGORY OF k -FUNCTORS

In the sequel we assume that the reader is familiar with notions of a monoid, group etc. in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 2-5].

Definition 2.1. A monoid (group, abelian group, ring) k -functor is a monoid (group, abelian group, ring) object in the category of k -functors.

Example 2.2. Let \mathfrak{X} be a k -functor such that $\mathcal{M}\text{or}_k(\mathfrak{X}, \mathfrak{X})$ exists. Then $\mathcal{M}\text{or}_k(\mathfrak{X}, \mathfrak{X})$ is a monoid k -functor with respect to composition of morphisms.

Example 2.3. Basic example of a ring k -functor is a k -functor \mathfrak{K} given by

$$\mathfrak{K}(A) = k, \mathfrak{K}(f) = 1_k$$

for any k -algebra A and morphism f of k -algebras. It can be described as a constant k -functor ([ML98, page 67]) corresponding to k .

Definition 2.4. Let \mathfrak{K} be a ring k -functor. Then we denote by \mathfrak{K}^\times the k -subfunctor of \mathfrak{K} defined by

$$\mathfrak{K}^\times(A) = \mathfrak{K}(A)^\times$$

for every k -algebra A . We call \mathfrak{K}^\times the multiplicative monoid k -functor of \mathfrak{K} .

Definition 2.5. Let \mathfrak{A} be a commutative ring k -functor. An \mathfrak{A} -algebra is an \mathfrak{A} -algebra object in the category of k -functors.

3. GLOBAL REGULAR FUNCTIONS ON A k -FUNCTOR

Recall the ring k -functor \mathfrak{K} from Example 2.3. Note that a \mathfrak{K} -algebra \mathfrak{A} can be viewed as a functor $\mathfrak{A} : \mathbf{Alg}_k \rightarrow \mathbf{Alg}_k$.

Definition 3.1. The \mathfrak{K} -algebra \mathfrak{D}_k represented by the identity functor on \mathbf{Alg}_k is called the structure \mathfrak{K} -algebra.

Let $|-| : \mathbf{Alg}_k \rightarrow \mathbf{Set}$ be the forgetful k -functor. Note that $|-|$ is the underlying k -functor of \mathfrak{K} -algebra \mathfrak{D}_k . Recall that the affine line \mathbb{A}_k^1 is an affine k -scheme having k -algebra of polynomials with one variable as a k -algebra of regular functions.

Fact 3.2. Let $|-| : \mathbf{Alg}_k \rightarrow \mathbf{Set}$ be the forgetful k -functor. Then we have natural isomorphism

$$\mathfrak{P}_{\mathbb{A}_k^1} \cong |-|$$

Proof. Let B be a k -algebra. We have the following chain of identifications

$$\mathfrak{P}_{\mathbb{A}_k^1}(B) = \text{Mor}_k(\text{Spec } B, \mathbb{A}_k^1) = \text{Mor}_k(\text{Spec } B, \text{Spec } k[x]) = \text{Mor}_k(k[x], B) = |B|$$

natural in B . □

In particular, since $|-|$ carries the structure \mathfrak{K} -algebra \mathfrak{D}_k , we derive that $\mathfrak{P}_{\mathbb{A}_k^1}$ admits a structure of \mathfrak{K} -algebra isomorphic to \mathfrak{D}_k .

No we introduce regular functions on k -functors.

Definition 3.3. Let \mathfrak{X} be a k -functor and assume that $\text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$ is a set. Then $\text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$ is a k -algebra with respect to the structure induced by \mathfrak{D}_k . We call this k -algebra *the k -algebra of global regular functions on \mathfrak{X}* . Its elements are called *global regular functions on \mathfrak{X}* .

Definition 3.4. Let \mathfrak{X} be a k -functor. Suppose that A is a k -algebra, $x \in \mathfrak{X}(A)$ and $f \in \text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$. The element $f(x) \in A$ is called *the value of f on a point x* .

For given k -functor \mathfrak{X} we describe k -algebra operations on $\text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$ in terms of values of its elements on points of \mathfrak{X} . For this consider $\alpha \in k$ and $f, g \in \text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$. We have formulas

$$(f + g)(x) = f(x) + g(x), (f \cdot g)(x) = f(x) \cdot g(x), (\alpha \cdot f)(x) = \alpha \cdot f(x)$$

in which right hand side are k -algebra operations in A .

Example 3.5. Let \mathfrak{X} be a k -functor and assume that $\text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$ exists. Fix k -algebra A . Note that $\text{Mor}_A(\mathfrak{X}_A, \mathfrak{D}_A)$ is an A -algebra of global regular functions on \mathfrak{X}_A . Moreover, if B is an A -algebra, then

$$\text{Mor}_A(\mathfrak{X}_A, \mathfrak{D}_A) \ni f \mapsto f_B \in \text{Mor}_B(\mathfrak{X}_B, \mathfrak{D}_B)$$

is a morphism of A -algebras. This implies that $\text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$ admits a canonical structure of an \mathfrak{D}_k -algebra k -functor.

4. INTERNAL HOM AND PRODUCT OF k -FUNCTORS

We denote by $\mathbf{1}$ a k -functor that assigns to every k -algebra a set with one element. Then for every k -algebra A the restriction $\mathbf{1}_A$ is a terminal object in the category of A -functors.

Fact 4.1. Let \mathfrak{X} be a k -functor. Suppose A is a k -algebra and $x \in \mathfrak{X}(A)$. Then x determines a morphism $\mathbf{1}_A \rightarrow \mathfrak{X}_A$ that for every A -algebra B with structural morphism $f : A \rightarrow B$ sends a unique element of $\mathbf{1}_A(B)$ to $\mathfrak{X}(f)(x) \in \mathfrak{X}_A(B)$. This gives rise to a bijection

$$\mathfrak{X}(A) \cong \text{Mor}_A(\mathbf{1}_A, \mathfrak{X}_A)$$

Proof. Left to the reader as an exercise. □

The discussion below is partially an application of the main result in [Mon19a, section 6]. For reader's convenience we make our presentation self-contained.

Definition 4.2. Let $\mathfrak{U}, \mathfrak{X}, \mathfrak{Y}$ be k -functors and let $\sigma : \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of k -functors. Fix $z \in \mathfrak{U}(A)$ for some k -algebra A . We denote by $i_z : \mathbf{1}_A \rightarrow \mathfrak{U}_A$ the morphism of A -functors corresponding to z by Fact 4.1. Since $\mathbf{1}_A$ is terminal A -functor, a morphism $\sigma_A \cdot (i_z \times \mathbf{1}_{\mathfrak{X}_A})$ is isomorphic to a morphism $\sigma_z : \mathfrak{X}_A \rightarrow \mathfrak{Y}_A$ of A -functors. We call σ_z *the slice of σ over z* .

Definition 4.3. Let $\mathfrak{X}, \mathfrak{Y}$ be k -functors. Let \mathfrak{J} be a k -functor such that $\mathfrak{J}(A)$ is a subset of a class $\text{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ for every k -algebra A . Assume that for every morphism $f : A \rightarrow B$ of k -algebras and every $\sigma \in \mathfrak{J}(A)$ we have

$$\mathfrak{J}(f)(\sigma) = \sigma_B$$

where $\sigma_B \in \text{Mor}_B(\mathfrak{X}_B, \mathfrak{Y}_B)$ is the restriction of σ along f . Then we call \mathfrak{J} a *k -subfunctor of internal hom of \mathfrak{X} and \mathfrak{Y}* .

Definition 4.4. Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{U}$ be k -functors and let $\sigma : \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of k -functors. Suppose that \mathfrak{J} is a k -subfunctor of internal hom of \mathfrak{X} and \mathfrak{Y} . Assume that $\sigma_z : \mathfrak{X}_A \rightarrow \mathfrak{Y}_A$ is contained in $\mathfrak{J}(A)$ for every k -algebra A and $z \in \mathfrak{U}(A)$. Then we call σ a *family of \mathfrak{J} -morphisms parametrized by \mathfrak{U}* .

Let \mathfrak{J} be a k -subfunctor of internal hom of \mathfrak{X} and \mathfrak{Y} . Assume that $\sigma : \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ is a \mathfrak{J} -family of morphism parametrized by \mathfrak{U} . Then the family of maps

$$\mathfrak{U}(A) \ni z \mapsto \sigma_z \in \mathfrak{J}(A)$$

gives rise to a morphism $\tau : \mathfrak{U} \rightarrow \mathfrak{J}$ of k -functors. Indeed, for a morphism $f : A \rightarrow B$ of k -algebras and $z \in \mathfrak{U}(A)$ we have

$$\sigma_B \cdot (i_{\mathfrak{U}(f)(z)} \times 1_{\mathfrak{X}_B}) = (\sigma_A \cdot (i_z \times 1_{\mathfrak{X}_A}))_B$$

and hence $\sigma_{\mathfrak{U}(f)(z)} = (\sigma_z)_B$. This gives rise to a map Φ of classes

$$\left\{ \text{families } \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y} \text{ of } \mathfrak{J}\text{-morphisms parametrized by } \mathfrak{U} \right\} \ni \sigma \mapsto \tau \in \text{Mor}_k(\mathfrak{U}, \mathfrak{J})$$

Consider next a morphism $\tau : \mathfrak{U} \rightarrow \mathfrak{J}$ of k -functors and define $\sigma : \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ by formula $\sigma^A(z, x) = (\tau^A(z))^A(x)$ for every k -algebra A and points $z \in \mathfrak{U}(A)$, $x \in \mathfrak{X}(A)$. Let $f : A \rightarrow B$ be a morphism of k -algebras. Then

$$\begin{aligned} \sigma^B(\mathfrak{U}(f)(z), \mathfrak{X}(f)(x)) &= (\tau^B(\mathfrak{U}(f)(z)))^B(\mathfrak{X}(f)(x)) = \left((\tau^A(z))_B \right)^B(\mathfrak{X}(f)(x)) = \\ &= (\tau^A(z))^B(\mathfrak{X}(f)(x)) = \mathfrak{Y}(f) \left((\tau^A(z))^A(x) \right) = \mathfrak{Y}(f) (\sigma^A(z, x)) \end{aligned}$$

Thus $\sigma : \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ is a morphism of k -functors. For every k -algebra A and $z \in \mathfrak{U}(A)$ we have $\sigma_z = \tau^A(z)$. Indeed, let $f : A \rightarrow B$ be a morphism of k -algebras and x be an element in $\mathfrak{X}(B)$ then we have

$$(\sigma_z)^B(x) = \sigma^B(\mathfrak{U}(f)(z), x) = (\tau^B(\mathfrak{U}(f)(z)))^B(x) = \left((\tau^A(z))_B \right)^B(x) = (\tau^A(z))^B(x)$$

Hence σ is a family of \mathfrak{J} -morphisms parametrized by \mathfrak{U} . This gives rise to a map Ψ of classes

$$\text{Mor}_k(\mathfrak{U}, \mathfrak{J}) \ni \tau \mapsto \sigma \in \left\{ \text{families } \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y} \text{ of } \mathfrak{J}\text{-morphisms parametrized by } \mathfrak{U} \right\}$$

Now we have the following result, which is an instance [Mon19a, Theorem 6.3]. To make presentation self-contained we give a complete proof.

Theorem 4.5. *Let $\mathfrak{X}, \mathfrak{Y}$ be k -functors and let \mathfrak{J} be a k -subfunctor of internal hom of \mathfrak{X} and \mathfrak{Y} . Then maps*

$$\Phi : \left\{ \text{families } \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y} \text{ of } \mathfrak{J}\text{-morphisms parametrized by } \mathfrak{U} \right\} \rightarrow \text{Mor}_k(\mathfrak{U}, \mathfrak{J})$$

and

$$\Psi : \text{Mor}_k(\mathfrak{U}, \mathfrak{J}) \rightarrow \left\{ \text{families } \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y} \text{ of } \mathfrak{J}\text{-morphisms parametrized by } \mathfrak{U} \right\}$$

are mutually inverse bijections.

Proof. Pick a morphism $\tau : \mathfrak{U} \rightarrow \mathfrak{J}$ of k -functors. Let A be a k -algebra and $z \in \mathfrak{U}(A)$. In the discussion preceding the statement we showed that $\Psi(\tau)_z = \tau^A(z)$. Thus

$$(\Phi(\Psi(\tau)))^A(z) = \Psi(\tau)_z = \tau^A(z)$$

and hence $\Phi \cdot \Psi$ is the identity.

Pick a family of \mathfrak{J} -morphism $\sigma : \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ parametrized by \mathfrak{U} . Let A be a k -algebra and $z \in \mathfrak{U}(A)$, $x \in \mathfrak{X}(A)$ be points. Then

$$(\Psi(\Phi(\sigma)))^A(z, x) = (\Phi(\sigma)^A(z))^A(x) = \sigma_z^A(x) = \sigma^A(z, x)$$

Thus $\Psi \cdot \Phi$ is the identity map. □

Now we formulate some consequences of Theorem 4.5.

Corollary 4.6. Let $\mathfrak{X}, \mathfrak{Y}$ be k -functors. Assume that for every k -algebra A the class $\text{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set. Then there is a bijection

$$\text{Mor}_k(\mathfrak{U} \times \mathfrak{X}, \mathfrak{Y}) \rightarrow \text{Mor}_k(\mathfrak{U}, \text{Mor}_k(\mathfrak{X}, \mathfrak{Y}))$$

of classes.

Definition 4.7. Let $\mathfrak{X}, \mathfrak{Y}$ be k -functors. If $\text{Iso}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set for every k -algebra A , then we define a k -subfunctor $\text{Iso}_k(\mathfrak{X}, \mathfrak{Y})$ of $\text{Mor}_k(\mathfrak{X}, \mathfrak{Y})$ by

$$\text{Iso}_k(\mathfrak{X}, \mathfrak{Y})(A) = \text{Iso}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$$

for every k -algebra A . We call $\text{Iso}_k(\mathfrak{X}, \mathfrak{Y})$ the k -functor of isomorphism.

Definition 4.8. Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{U}$ be k -functors and let $\sigma : \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of k -functors. Assume that $\sigma_z : \mathfrak{X}_A \rightarrow \mathfrak{Y}_A$ is an isomorphism of A -functors for every k -algebra A . Then we call σ a family of isomorphisms parametrized by \mathfrak{U} .

Corollary 4.9. Let $\mathfrak{U}, \mathfrak{X}, \mathfrak{Y}$ be k -functors and suppose that for every k -algebra A the class $\text{Iso}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set. The following map

$$\left\{ \text{families } \mathfrak{U} \times \mathfrak{X} \rightarrow \mathfrak{Y} \text{ of isomorphism parametrized by } \mathfrak{U} \right\} \rightarrow \text{Mor}_k(\mathfrak{U}, \text{Iso}_k(\mathfrak{X}, \mathfrak{Y}))$$

is a bijection of classes.

5. ACTIONS OF MONOID k -FUNCTORS

In this section we assume that the reader is familiar with notion of an action of a monoid object in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 5].

Let \mathfrak{G} be a monoid k -functor and \mathfrak{X} be a k -functor together with an action $\alpha : \mathfrak{G} \times \mathfrak{X} \rightarrow \mathfrak{X}$. Next assume that k -functor $\text{Mor}_k(\mathfrak{X}, \mathfrak{X})$ exists. By Example 2.2 it is a monoid k -functor. We define a morphism $\rho : \mathfrak{G} \rightarrow \text{Mor}_k(\mathfrak{X}, \mathfrak{X})$ of k -functors by formula $\rho(x) = \alpha_x$. Note that by discussion preceding Theorem 4.5, we deduce that ρ is a well defined morphism of k -functors. We show now that ρ is a morphism of monoids. For this pick k -algebra A and $x, y \in \mathfrak{G}(A)$. Since α is an action, we deduce that $\alpha_{x \cdot y} = \alpha_x \cdot \alpha_y$ and hence also

$$\rho(x \cdot y) = \alpha_{x \cdot y} = \alpha_x \cdot \alpha_y = \rho(x) \cdot \rho(y)$$

Therefore, ρ is a morphism of monoid k -functors. This shows how to construct a morphism of monoid k -functors ρ from an action α of \mathfrak{G} .

Theorem 5.1. Let \mathfrak{G} be a monoid k -functor and let \mathfrak{X} be a k -functor such that $\text{Mor}_k(\mathfrak{X}, \mathfrak{X})$ exists. Suppose that

$$\left\{ \text{actions of } \mathfrak{G} \text{ on } \mathfrak{X} \right\} \longrightarrow \left\{ \text{Morphisms } \rho : \mathfrak{G} \rightarrow \text{Mor}_k(\mathfrak{X}, \mathfrak{X}) \text{ of monoid } k\text{-functors} \right\}$$

is a map of classes described above. Then it is bijection.

Proof. Our goal is to construct the inverse of the map. Substitute $\mathfrak{J} = \text{Mor}_k(\mathfrak{X}, \mathfrak{X})$ in Theorem 4.5. Consider maps

$$\Phi : \left\{ \text{families } \mathfrak{G} \times \mathfrak{X} \rightarrow \mathfrak{X} \text{ of morphisms} \right\} \rightarrow \text{Mor}_k(\mathfrak{G}, \text{Mor}_k(\mathfrak{X}, \mathfrak{X}))$$

and

$$\Psi : \text{Mor}_k(\mathfrak{G}, \text{Mor}_k(\mathfrak{X}, \mathfrak{X})) \rightarrow \left\{ \text{families } \mathfrak{G} \times \mathfrak{X} \rightarrow \mathfrak{X} \text{ of morphisms} \right\}$$

in that Theorem. Then the map in the statement above is the restriction of Φ to \mathfrak{G} -actions on \mathfrak{X} on the right and morphisms $\mathfrak{G} \rightarrow \text{Mor}_k(\mathfrak{X}, \mathfrak{X})$ of monoid k -functors on the left. Since by Theorem 4.5 maps Φ and Ψ are mutually inverse, it suffices to check that Ψ sends a morphism $\rho : \mathfrak{G} \rightarrow$

$\text{Mor}_k(\mathfrak{X}, \mathfrak{X})$ of monoids to an action of \mathfrak{G} on \mathfrak{X} . For this denote $\Psi(\rho)$ by α . Consider k -algebra A and A -points $x, y \in \mathfrak{G}(A)$, $z \in \mathfrak{X}(A)$. Then

$$\alpha(y, \alpha(x, z)) = \rho(y)(\rho(x)(z)) = (\rho(y) \cdot \rho(x))(z) = \rho(x \cdot y)(z) = \alpha(x \cdot y, z)$$

Therefore, α is an action of \mathfrak{G} on \mathfrak{X} . □

Proposition 5.2. *Let \mathfrak{G} be a monoid k -functor and let $\mathfrak{X}_1, \mathfrak{X}_2$ be k -functors such that $\text{Mor}_k(\mathfrak{X}_1, \mathfrak{X}_1), \text{Mor}_k(\mathfrak{X}_2, \mathfrak{X}_2)$ exist. Suppose that $\alpha_1 : \mathfrak{G} \times \mathfrak{X}_1 \rightarrow \mathfrak{X}_1$, $\alpha_2 : \mathfrak{G} \times \mathfrak{X}_2 \rightarrow \mathfrak{X}_2$ are actions of \mathfrak{G} , respectively. Suppose that $\sigma : \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$ is a morphism of k -functors. Then the following assertions are equivalent.*

(i) *The square*

$$\begin{array}{ccc} \mathfrak{G} \times \mathfrak{X}_1 & \xrightarrow{1_{\mathfrak{G}} \times \sigma} & \mathfrak{G} \times \mathfrak{X}_2 \\ \alpha_1 \downarrow & & \downarrow \alpha_2 \\ \mathfrak{X}_1 & \xrightarrow{\sigma} & \mathfrak{X}_2 \end{array}$$

is commutative.

(ii) *For every k -algebra A and $x \in \mathfrak{G}(A)$ we have*

$$\sigma_A \cdot \rho_1(x) = \rho_2(x) \cdot \sigma_A$$

where $\rho_1 : \mathfrak{G} \rightarrow \text{Mor}_k(\mathfrak{X}_1, \mathfrak{X}_1)$ and $\rho_2 : \mathfrak{G} \rightarrow \text{Mor}_k(\mathfrak{X}_2, \mathfrak{X}_2)$ are morphism of monoid k -functors corresponding to α_1 and α_2 , respectively.

Proof. Conditions expressed in (i) and (ii) are directly translatable to each other by virtue of the bijection in Theorem 5.1. □

Definition 5.3. Let \mathfrak{G} be a monoid k -functor and let $(\mathfrak{X}_1, \alpha_1), (\mathfrak{X}_2, \alpha_2)$ be k -functors with actions of \mathfrak{G} . Suppose that $\sigma : \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$ is a morphism k -functors such that the square

$$\begin{array}{ccc} \mathfrak{G} \times \mathfrak{X}_1 & \xrightarrow{1_{\mathfrak{G}} \times \sigma} & \mathfrak{G} \times \mathfrak{X}_2 \\ \alpha_1 \downarrow & & \downarrow \alpha_2 \\ \mathfrak{X}_1 & \xrightarrow{\sigma} & \mathfrak{X}_2 \end{array}$$

is commutative. Then σ is called an \mathfrak{G} -equivariant morphism.

6. MODULES OVER RING k -FUNCTORS

Definition 6.1. Let \mathfrak{R} be a ring k -functor. Suppose that \mathfrak{M} is an abelian group k -functor and there exists a morphism $\mathfrak{R} \times \mathfrak{M} \rightarrow \mathfrak{M}$ of k -functors that for each k -algebra A makes $\mathfrak{M}(A)$ into an $\mathfrak{R}(A)$ -module. Then we say that \mathfrak{M} is a *module k -functor over \mathfrak{R}* .

Definition 6.2. Let \mathfrak{R} be an ring k -functor and let $\mathfrak{M}_1, \mathfrak{M}_2$ be module k -functors over \mathfrak{R} . Suppose that $\sigma : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ is a morphism of abelian group k -functors such that the diagram

$$\begin{array}{ccc} \mathfrak{R} \times \mathfrak{M}_1 & \xrightarrow{1_{\mathfrak{R}} \times \sigma} & \mathfrak{R} \times \mathfrak{M}_2 \\ \alpha_1 \downarrow & & \downarrow \alpha_2 \\ \mathfrak{M}_1 & \xrightarrow{\sigma} & \mathfrak{M}_2 \end{array}$$

is commutative, where $\alpha_i : \mathfrak{R} \times \mathfrak{M}_i \rightarrow \mathfrak{M}_i$ define \mathfrak{R} -module structure on \mathfrak{M}_i for $i = 1, 2$. Then σ is a morphism of modules over \mathfrak{R} .

Let \mathfrak{M}_1 and \mathfrak{M}_2 be module k -functors over \mathfrak{R} . We denote by

$$\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{M}_1, \mathfrak{M}_2)$$

the class of all morphisms of modules $\mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ over \mathfrak{R} . We denote the category of \mathfrak{R} -modules by $\mathbf{Mod}(\mathfrak{R})$.

Definition 6.3. Let \mathfrak{M}_1 and \mathfrak{M}_2 be module k -functors over \mathfrak{R} . Assume that $\mathrm{Hom}_{\mathfrak{R}_A}((\mathfrak{M}_1)_A, (\mathfrak{M}_2)_A)$ is a set for every k -algebra A . Then we define a k -subfunctor $\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{M}_1, \mathfrak{M}_2)$ of internal hom of \mathfrak{M}_1 and \mathfrak{M}_2 by formula

$$\mathbf{Alg}_k \ni A \mapsto \mathrm{Hom}_{\mathfrak{R}_A}((\mathfrak{M}_1)_A, (\mathfrak{M}_2)_A) \in \mathbf{Set}$$

We call $\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{M}_1, \mathfrak{M}_2)$ a k -functor of module morphisms of \mathfrak{M}_1 and \mathfrak{M}_2 .

If \mathfrak{M} is a module k -functor over some ring k -functor \mathfrak{R} , then we denote (if it exists) $\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{M}, \mathfrak{M})$ by $\mathrm{End}_{\mathfrak{R}}(\mathfrak{M})$.

Example 6.4. Let \mathfrak{M} be a module over a ring k -functor \mathfrak{R} . Assume that $\mathrm{End}_{\mathfrak{R}}(\mathfrak{M})$ exists. Then $\mathrm{End}_{\mathfrak{R}}(\mathfrak{M})$ is a ring k -functor with respect to composition of morphisms of modules as the multiplication and the usual addition of module morphisms. Moreover, if \mathfrak{A} is a commutative ring k -functor, then $\mathrm{End}_{\mathfrak{A}}(\mathfrak{M})$ (if exists) admits additional structure of a \mathfrak{A} -algebra k -functor induced via a unique morphism $\mathfrak{A} \rightarrow \mathrm{End}_{\mathfrak{R}}(\mathfrak{M})$ of ring k -functors that sends $1 \mapsto 1_{\mathfrak{M}}$.

Let \mathfrak{A} be a commutative ring k -functor and let \mathfrak{R} be a \mathfrak{A} -algebra k -functor. This means that there exists a morphism $\mathfrak{A} \rightarrow \mathfrak{R}$ of ring k -functors and for every k -algebra A induced morphism $\mathfrak{A}(A) \rightarrow \mathfrak{R}(A)$ sends $\mathfrak{A}(A)$ to the center of a ring $\mathfrak{R}(A)$. Fix a module \mathfrak{M} over \mathfrak{A} . Next assume that k -functor $\mathrm{End}_{\mathfrak{A}}(\mathfrak{M})$ exists. By Example 6.4 it is a ring k -functor.

Definition 6.5. In the setting above suppose that $\alpha : \mathfrak{R} \times \mathfrak{M} \rightarrow \mathfrak{M}$ is a morphism of k -functors. Suppose that α makes \mathfrak{M} into \mathfrak{R} -module and moreover, for every k -algebra A and for every point $x \in \mathfrak{R}(A)$ morphism α_x is a morphism of \mathfrak{A}_A -modules. Then α is called a \mathfrak{A} -linear \mathfrak{R} -action on \mathfrak{M} .

We continue the discussion. We assume that we are given an \mathfrak{A} -linear \mathfrak{R} -action $\alpha : \mathfrak{R} \times \mathfrak{M} \rightarrow \mathfrak{M}$ on \mathfrak{M} . We define a morphism $\rho : \mathfrak{R} \rightarrow \mathrm{End}_{\mathfrak{A}}(\mathfrak{M})$ of k -functors by formula $\rho(x) = \alpha_x$. As in Section 5 we can prove that ρ is a morphism of ring k -functors. Now we have the following result.

Theorem 6.6. Let \mathfrak{R} be an algebra k -functor over commutative ring \mathfrak{A} k -functor and let \mathfrak{M} be a \mathfrak{A} -module such that $\mathrm{End}_{\mathfrak{A}}(\mathfrak{M})$ exists. Suppose that

$$\left\{ \mathfrak{A} \text{ linear actions of } \mathfrak{R} \text{ on } \mathfrak{M} \right\} \longrightarrow \left\{ \text{Morphisms } \rho : \mathfrak{R} \rightarrow \mathrm{End}_{\mathfrak{A}}(\mathfrak{M}) \text{ of ring } k\text{-functors} \right\}$$

is a map of classes described above. Then it is bijection.

Proof. The proof is similar to the proof of Theorem 5.1. □

7. MONOID ALGEBRA $\mathfrak{D}_k[\mathfrak{G}]$ AND ITS MODULES

Definition 7.1. Let \mathfrak{G} be a monoid k -functor. Then we construct an \mathfrak{D}_k -algebra $\mathfrak{D}_k[\mathfrak{G}]$ as follows. For every k -algebra A we define

$$\mathfrak{D}_k[\mathfrak{G}](A) = A[\mathfrak{G}(A)]$$

where the right hand side is monoid A -algebra for the abstract monoid $\mathfrak{G}(A)$. The structure of monoid k -functor on \mathfrak{G} and \mathfrak{R} -algebra \mathfrak{D}_k makes $\mathfrak{D}_k[\mathfrak{G}]$ into a ring k -functor. Moreover, we have a morphism $\mathfrak{D}_k \rightarrow \mathfrak{D}_k[\mathfrak{G}]$ which for every k -algebra A is given by the canonical inclusion

$$A \hookrightarrow A[\mathfrak{G}(A)]$$

Thus $\mathfrak{D}_k[\mathfrak{G}]$ is \mathfrak{D}_k -algebra. We call $\mathfrak{D}_k[\mathfrak{G}]$ a monoid \mathfrak{D}_k -algebra over \mathfrak{G} .

Fact 7.2. Let \mathfrak{G} be a monoid k -functor and let \mathfrak{R} be an \mathfrak{D}_k -algebra k -functor. Then every morphism

$$\sigma : \mathfrak{G} \rightarrow \mathfrak{R}^\times$$

of monoid k -functors admits a unique extension

$$\tilde{\sigma} : \mathfrak{D}_k[\mathfrak{G}] \rightarrow \mathfrak{R}$$

to a morphism of \mathfrak{D}_k -algebras.

Proof. This follows from the analogical universal property of algebras over abstract monoids. \square

Definition 7.3. Let \mathfrak{G} be a monoid k -functor and let \mathfrak{M} be a module over \mathfrak{D}_k . Suppose that $\alpha : \mathfrak{G} \times \mathfrak{M} \rightarrow \mathfrak{M}$ is an action of \mathfrak{G} such that for any k -algebra A and point $x \in \mathfrak{G}(A)$ morphism $\alpha_x : \mathfrak{M}_A \rightarrow \mathfrak{M}_A$ is a morphism of \mathfrak{D}_A -modules. Then α is called a *linear \mathfrak{G} -action on \mathfrak{M}* .

Suppose now that \mathfrak{G} is a monoid k -functor and \mathfrak{M} is a module \mathfrak{D}_k . Note that every linear \mathfrak{G} -action $\alpha : \mathfrak{G} \times \mathfrak{M} \rightarrow \mathfrak{M}$ extends uniquely to a \mathfrak{D}_k -linear action $\mathfrak{D}_k[\mathfrak{G}] \times \mathfrak{M} \rightarrow \mathfrak{M}$ of monoid \mathfrak{D}_k -algebra. This gives a bijection

$$\left\{ \text{Linear actions of } \mathfrak{G} \text{ on } \mathfrak{M} \right\} \longrightarrow \left\{ \mathfrak{D}_k\text{-linear actions } \mathfrak{D}_k[\mathfrak{G}] \times \mathfrak{M} \rightarrow \mathfrak{M} \right\}$$

Next assume that k -functor $\mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M})$ exists. By Example 6.4 it is an \mathfrak{D}_k -algebra k -functor. Next by Theorem 6.6 we have a bijection

$$\left\{ \mathfrak{D}_k\text{-linear actions of } \mathfrak{D}_k[\mathfrak{G}] \times \mathfrak{M} \rightarrow \mathfrak{M} \right\} \longrightarrow \left\{ \text{Morphisms } \mathfrak{D}_k[\mathfrak{G}] \rightarrow \mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M}) \text{ of } \mathfrak{D}_k\text{-algebras} \right\}$$

Finally Fact 7.2 implies that we have a bijection

$$\left\{ \text{Morphisms } \mathfrak{D}_k[\mathfrak{G}] \rightarrow \mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M}) \text{ of } \mathfrak{D}_k\text{-algebras} \right\} \longrightarrow \left\{ \text{Morphisms } \mathfrak{G} \rightarrow \mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M}) \text{ of monoids} \right\}$$

This chain of bijections sends a linear action $\alpha : \mathfrak{G} \times \mathfrak{M} \rightarrow \mathfrak{M}$ of \mathfrak{G} to a morphism $\rho : \mathfrak{G} \rightarrow \mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M})$ of monoid k -functors given by $\rho(x) = \alpha_x$ for every $x \in \mathfrak{G}(A)$ and every k -algebra A . We proved the following result.

Proposition 7.4. Let \mathfrak{G} be a monoid k -functor and let \mathfrak{M} be a \mathfrak{D}_k -module such that $\mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M})$ exists. Then the following classes are in canonical bijections described above.

- (1) Linear actions of \mathfrak{G} on \mathfrak{M} .
- (2) \mathfrak{D}_k -linear actions $\mathfrak{D}_k[\mathfrak{G}] \times \mathfrak{M} \rightarrow \mathfrak{M}$. These are precisely $\mathfrak{D}_k[\mathfrak{G}]$ -modules.
- (3) Morphisms $\mathfrak{D}_k[\mathfrak{G}] \rightarrow \mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M})$ of \mathfrak{D}_k -algebras.
- (4) Morphisms $\mathfrak{G} \rightarrow \mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M})$ of monoids.

Moreover, the bijection between class (1) and (2) does not require the existence of $\mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M})$.

Now in a similar manner we can describe morphisms.

Proposition 7.5. Let \mathfrak{G} be a monoid k -functor and let $\mathfrak{M}_1, \mathfrak{M}_2$ be k -functors of \mathfrak{D}_k -modules such that $\mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M}_1), \mathcal{E}nd_{\mathfrak{D}_k}(\mathfrak{M}_2)$ exist. Suppose that $\alpha_1 : \mathfrak{G} \times \mathfrak{M}_1 \rightarrow \mathfrak{M}_1, \alpha_2 : \mathfrak{G} \times \mathfrak{M}_2 \rightarrow \mathfrak{M}_2$ are linear actions of \mathfrak{G} , respectively. Suppose that $\sigma : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ is a morphism of modules over \mathfrak{D}_k . Then the following assertions are equivalent.

- (i) The square

$$\begin{array}{ccc}
\mathfrak{G} \times \mathfrak{M}_1 & \xrightarrow{1_{\mathfrak{G}} \times \sigma} & \mathfrak{G} \times \mathfrak{M}_2 \\
\alpha_1 \downarrow & & \downarrow \alpha_2 \\
\mathfrak{M}_1 & \xrightarrow{\sigma} & \mathfrak{M}_2
\end{array}$$

is commutative.

(ii) The square

$$\begin{array}{ccc}
\mathfrak{D}_k[\mathfrak{G}] \times \mathfrak{M}_1 & \xrightarrow{1_{\mathfrak{D}_k[\mathfrak{G}]} \times \sigma} & \mathfrak{D}_k[\mathfrak{G}] \times \mathfrak{M}_2 \\
\tilde{\alpha}_1 \downarrow & & \downarrow \tilde{\alpha}_2 \\
\mathfrak{M}_1 & \xrightarrow{\sigma} & \mathfrak{M}_2
\end{array}$$

is commutative, where $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are \mathfrak{D}_k -linear actions of $\mathfrak{D}_k[\mathfrak{G}]$ corresponding to α_1 and α_2 , respectively. This states that σ is a morphism of $\mathfrak{D}_k[\mathfrak{G}]$ -modules.

(iii) For every k -algebra A and $x \in \mathfrak{G}(A)$ we have

$$\sigma_A \cdot \tilde{\rho}_1(x) = \tilde{\rho}_2(x) \cdot \sigma_A$$

where $\tilde{\rho}_1 : \mathfrak{D}_k[\mathfrak{G}] \rightarrow \text{End}_{\mathfrak{D}_k}(\mathfrak{M}_1)$ and $\tilde{\rho}_2 : \mathfrak{D}_k[\mathfrak{G}] \rightarrow \text{End}_{\mathfrak{D}_k}(\mathfrak{M}_2)$ are morphism of \mathfrak{D}_k -algebras corresponding to $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, respectively.

(iv) For every k -algebra A and $x \in \mathfrak{G}(A)$ we have

$$\sigma_A \cdot \rho_1(x) = \rho_2(x) \cdot \sigma_A$$

where $\rho_1 : \mathfrak{G} \rightarrow \text{End}_{\mathfrak{D}_k}(\mathfrak{M}_1)$ and $\rho_2 : \mathfrak{G} \rightarrow \text{End}_{\mathfrak{D}_k}(\mathfrak{M}_2)$ are restrictions of $\tilde{\rho}_1$ and $\tilde{\rho}_2$, respectively.

The equivalence of (1) and (2) does not require the existence of $\text{End}_{\mathfrak{D}_k}(\mathfrak{M}_1)$ and $\text{End}_{\mathfrak{D}_k}(\mathfrak{M}_2)$.

Proof. Conditions expressed in (i)-(iv) are directly translatable to each other by virtue of bijections in Proposition 9.5. \square

Let \mathfrak{G} be a monoid k -functor. We denote by $\mathbf{Mod}(\mathfrak{D}_k[\mathfrak{G}])$ the category of $\mathfrak{D}_k[\mathfrak{G}]$ -modules.

8. EXAMPLE OF \mathfrak{G} -ACTION: REGULAR FUNCTIONS k -FUNCTOR

First we need the following notion.

Definition 8.1. Let $(-)^{\text{op}} : \mathbf{Mon} \rightarrow \mathbf{Mon}$ be the functor of opposite monoid and let \mathfrak{G} be a monoid k -functor. Then the composition $\mathfrak{G}^{\text{op}} = (-)^{\text{op}} \cdot \mathfrak{G}$ is called the *opposite monoid k -functor* of \mathfrak{G} .

Let \mathfrak{G} be a monoid k -functor. In this section we discuss important example of a $\mathfrak{D}_k[\mathfrak{G}]$ -module. Fix a k -functor \mathfrak{X} for which $\text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$ exists. Recall that by Example 3.5 $\text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$ is \mathfrak{D}_k -algebra k -functor. Let $\alpha : \mathfrak{G} \times \mathfrak{X} \rightarrow \mathfrak{X}$ be an action of \mathfrak{G} on \mathfrak{X} . For every k -algebra A we have a map of sets

$$\text{Mor}_A(\mathfrak{X}_A, (\mathfrak{D}_k)_A) \ni f \mapsto f \cdot \alpha_x \in \text{Mor}_A(\mathfrak{X}_A, (\mathfrak{D}_k)_A)$$

where $x \in \mathfrak{G}(A)$. From this description it follows that the map $f \mapsto f \cdot \alpha_x$ is a morphism of A -algebras. Moreover, note that if $y \in \mathfrak{G}(A)$ is some other A -point, then $(f \cdot \alpha_x) \cdot \alpha_y = f \cdot \alpha_{x \cdot y}$, where $x \cdot y \in \mathfrak{G}(A)$ is a product of x and y . Thus the opposite monoid $\mathfrak{G}^{\text{op}}(A)$ acts on the A -algebra $\text{Mor}_A(\mathfrak{X}_A, (\mathfrak{D}_k)_A)$ by morphism of A -algebras. Next for every A -algebra B and every point $y \in \mathfrak{X}(B)$ we have

$$(f \cdot \alpha_x)(y) = f(\alpha_x(y))$$

This proves the following result.

Proposition 8.2. *Let \mathfrak{X} be a k -functor and let $\alpha : \mathfrak{G} \times \mathfrak{X} \rightarrow \mathfrak{X}$ be an action of a monoid k -functor \mathfrak{G} . Suppose that $\text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$ exists. Then \mathfrak{G}^{op} acts canonically on \mathfrak{D}_k -algebra k -functor $\text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$ by morphisms of \mathfrak{D}_k -algebras.*

Let us note one important consequence of this result.

Corollary 8.3. *Let \mathfrak{G} be a monoid k -functor. The action of $\mathfrak{G} \times \mathfrak{G}^{\text{op}}$ on \mathfrak{G} induces the action of $\mathfrak{G} \times \mathfrak{G}^{\text{op}}$ on \mathfrak{D}_k -algebra k -functor $\text{Mor}_k(\mathfrak{X}, \mathfrak{D}_k)$ by morphisms of \mathfrak{D}_k -algebras.*

9. LINEAR REPRESENTATIONS OF A MONOID k -FUNCTORS

We start the discussion with some results that relates categories $\mathbf{Mod}(k)$ and $\mathbf{Mod}(\mathfrak{D}_k)$.

Example 9.1. Let V be a k -module. We define a k -functor V_a . We set

$$V_a(A) = A \otimes_k V, \quad V_a(f) = f \otimes_k 1_V$$

for every k -algebra A and every morphism $f : A \rightarrow B$ of k -algebras. Note that V_a is \mathfrak{D}_k -module. Suppose that $\phi : V \rightarrow W$ is a morphism of k -modules, then we define $\phi_a : V_a \rightarrow W_a$ by formula

$$\phi_a^A = 1_A \otimes_k \sigma$$

for every k -algebra. Then ϕ_a is a morphism of \mathfrak{D}_k -modules.

Proposition 9.2. *The functor $(-)_a : \mathbf{Mod}(k) \rightarrow \mathbf{Mod}(\mathfrak{D}_k)$ is full and faithful.*

Proof. Fix k -modules V, W . Then

$$\text{Hom}_{\mathfrak{D}_k}(V_a, W_a) \ni \sigma \mapsto \sigma^k \in \text{Hom}_k(V, W)$$

and

$$\text{Hom}_k(V, W) \ni \phi \mapsto \phi_a \in \text{Hom}_{\mathfrak{D}_k}(V_a, W_a)$$

are mutually inverse bijections. Hence the functor is full and faithful. \square

Example 9.3. Let V be a k -module. We define a k -functor \mathcal{L}_V . We set

$$\mathcal{L}_V(A) = \text{Hom}_A(A \otimes_k V, A \otimes_k V)$$

for every k -algebra A . Next for every morphism $f : A \rightarrow B$ of k -algebras and every morphism $\phi : A \otimes_k V \rightarrow A \otimes_k V$ of A -modules we define $\mathcal{L}_V(f)(\phi)$ as a unique morphism of B -modules such that the diagram

$$\begin{array}{ccc} A \otimes_k V & \xrightarrow{\phi} & A \otimes_k V \\ f \otimes_k 1_V \downarrow & & \downarrow f \otimes_k 1_V \\ B \otimes_k V & \xrightarrow{\mathcal{L}_V(\phi)} & B \otimes_k V \end{array}$$

is commutative. Note also that $\mathcal{L}_V(A)$ is an A -algebra. Hence \mathcal{L}_V is a monoid $k\mathfrak{D}_k$ -algebra.

Remark 9.4. Let V be a k -module. Proposition 9.2 implies that there are bijective maps that make the square

$$\begin{array}{ccc} \mathcal{L}_V(A) & \xrightarrow{\cong} & \text{End}_{\mathfrak{D}_A}((V_a)_A, (V_a)_A) \\ \mathcal{L}_V(f) \downarrow & & \downarrow \sigma \mapsto \sigma_B \\ \mathcal{L}_V(B) & \xrightarrow{\cong} & \text{End}_{\mathfrak{D}_B}((V_a)_B, (V_a)_B) \end{array}$$

commutative for every morphism $f : A \rightarrow B$ of k -algebras. This induces an identification $\mathcal{L}_V = \text{End}_{\mathfrak{D}_k}(V_a)$ of \mathfrak{D}_k -algebras.

Fix monoid k -functor \mathfrak{G} . Now we have the following consequence of our discussion concerning modules over $\mathfrak{D}_k[\mathfrak{G}]$.

Corollary 9.5. *Let \mathfrak{G} be a monoid k -functor and let V be a k -module. Then the following classes are in canonical bijections.*

- (1) *Linear actions of \mathfrak{G} on V_a .*
- (2) *\mathfrak{D}_k -linear actions $\mathfrak{D}_k[\mathfrak{G}] \times V_a \rightarrow V_a$. These are precisely $\mathfrak{D}_k[\mathfrak{G}]$ -modules.*
- (3) *Morphisms $\mathfrak{D}_k[\mathfrak{G}] \rightarrow \mathcal{L}_V$ of \mathfrak{D}_k -algebras.*
- (4) *Morphisms $\mathfrak{G} \rightarrow \mathcal{L}_V$ of monoids.*

Proof. This follows from Proposition 9.5. □

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