

# RADON-NIKODYM THEOREM, HAHN-JORDAN DECOMPOSITION AND LEBESGUE DECOMPOSITION

## 1. INTRODUCTION

This notes are devoted to more advanced notions in measure theory. Tools presented here are indispensable in probability theory and statistics. We refer to [\[Monygham, 2018\]](#) for extensive introduction to measure theory.

## 2. SIGNED AND COMPLEX MEASURES

In this section we define extension of the usual notion of measure. Denote by  $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$  the topological space obtained as a two-point compactification of the line  $\mathbb{R}$ . Addition is partially defined operation on  $\overline{\mathbb{R}}$  given by the following rules

$$(+\infty) + r = +\infty = r + (+\infty), (-\infty) + r = -\infty = r + (-\infty)$$

for every  $r \in \mathbb{R}$

**Definition 2.1.** Let  $(X, \Sigma)$  be a measurable space. A *signed measure* on  $(X, \Sigma)$  is a function  $\nu : \Sigma \rightarrow \overline{\mathbb{R}}$  such that  $\nu(\emptyset) = 0$  and

$$\nu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \nu(A_n)$$

for every family  $\{A_n\}_{n \in \mathbb{N}}$  of pairwise disjoint subsets of  $\Sigma$ . We also say that  $\nu$  is a *real measure* on  $(X, \Sigma)$  if it is signed measure and takes values in  $\mathbb{R}$ .

**Definition 2.2.** Let  $(X, \Sigma)$  be a measurable space. A *complex measure* is a function  $\nu : \Sigma \rightarrow \mathbb{C}$  such that  $\nu(\emptyset) = 0$  and

$$\nu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \nu(A_n)$$

for every family  $\{A_n\}_{n \in \mathbb{N}}$  of pairwise disjoint subsets of  $\Sigma$ .

**Definition 2.3.** Let  $(X, \Sigma)$  be a measurable space and let  $\mu$  be a measure on  $(X, \Sigma)$ . Assume that  $\nu$  is either complex or signed measure on  $(X, \Sigma)$ . Suppose that for every set  $A$  in  $\Sigma$  we have

$$\mu(A) = 0 \Rightarrow \nu(A) = 0$$

Then we write  $\nu \ll \mu$  and say that  $\nu$  is *absolutely continuous with respect to  $\mu$* .

**Definition 2.4.** Let  $(X, \Sigma)$  be a measurable space and let  $\mu, \nu$  be two measures (either complex or signed) on  $(X, \Sigma)$ . Suppose that there exists a set  $S \in \Sigma$  such that

$$\mu(A \cap S) = 0, \nu(A \setminus S) = 0$$

for every  $A \in \Sigma$ . Then we write  $\nu \perp \mu$  and say that  $\nu$  is *singular with respect to  $\mu$* .

## 3. HAHN-JORDAN DECOMPOSITION

**Theorem 3.1** (Hahn-Jordan decomposition). *Let  $(X, \Sigma)$  be a measurable space and  $\nu : \Sigma \rightarrow \overline{\mathbb{R}}$  be a signed measure. Then there exists the unique pair of measures  $\nu_+, \nu_- : \Sigma \rightarrow [0, +\infty]$  such that*

$$\nu = \nu_+ - \nu_-$$

and  $\nu_+ \perp \nu_-$ .

For the proof we need the following notion.

**Definition 3.2.** Let  $(X, \Sigma, \nu)$  be a space with signed measure. A set  $A \in \Sigma$  is *positive* if for every subset  $B$  of  $A$  such that  $B \in \Sigma$  we have inequality  $\nu(B) \geq 0$ .

**Lemma 3.2.1.** Let  $B \in \Sigma$  be a set such that  $\nu(B) \in \mathbb{R}$  and  $\nu(B) > 0$ . Then there exists a positive set  $C \subseteq B$  such that  $\nu(B) \leq \nu(C)$ .

*Proof of the lemma.* First note that all sets  $A \in \Sigma$  contained in  $B$  have finite measure (we left the proof as an exercise for the reader). For every subset  $A \in \Sigma$  contained in  $B$  we define

$$\delta(A) = \max \left\{ \frac{1}{2} \inf \{ \nu(D) \mid D \text{ is a subset of } A \text{ in } \Sigma \}, -1 \right\}$$

Note that  $\delta(A) \leq 0$ . Now we define a sequence  $\{D_n\}_{n \in \mathbb{N}}$  of disjoint subsets of  $B$  and members of  $\Sigma$ . This is done recursively as follows. If  $D_0, \dots, D_n$  are defined, then we pick  $D_{n+1}$  as a subset of  $B \setminus (D_0 \cup \dots \cup D_n)$  in  $\Sigma$  such that

$$\nu(D_{n+1}) \leq \delta(B \setminus (D_0 \cup \dots \cup D_n))$$

Let

$$C = B \setminus \bigcup_{n \in \mathbb{N}} D_n$$

be a subset of  $B$ . Clearly  $C \in \Sigma$  and for every  $n \in \mathbb{N}$  we have

$$\delta(C) \geq \delta(B \setminus (D_0 \cup \dots \cup D_n))$$

Thus

$$\nu(C) = \nu(B) - \sum_{n \in \mathbb{N}} \nu(D_n) \geq \nu(B) - \sum_{n \in \mathbb{N}} \delta(B \setminus (D_0 \cup \dots \cup D_n)) = \nu(B) - \sum_{n \in \mathbb{N}} \delta(C)$$

Since  $\nu(C) \in \mathbb{R}$ , we derive that  $\delta(C) = 0$ . This implies that  $C$  is a positive set and  $\nu(C) \geq \nu(B)$ .  $\square$

*Proof of the theorem.* Assume that for every  $A \in \Sigma$  we have  $\nu(A) \in \mathbb{R} \cup \{-\infty\}$ . Now consider

$$\alpha = \sup \{ \nu(C) \mid C \text{ is positive} \}$$

We can find a nondecreasing sequence  $\{\alpha_n\}_{n \in \mathbb{N}}$  of nonnegative real numbers that converges to  $\alpha$  and such that for every  $n \in \mathbb{N}$  there exists a positive set  $C_n$  with  $\nu(C_n) = \alpha_n$ . Now pick  $P = \bigcup_{n \in \mathbb{N}} C_n$ . Obviously  $P$  is positive and  $\nu(P) = \alpha$ . In particular,  $\alpha \in \mathbb{R}$ . Assume that there exists  $B \in \Sigma$  such that  $B \subseteq X \setminus P$  and  $\nu(B) > 0$ . According to Lemma 3.2.1 we deduce that there exists a positive set  $C$  inside  $B$  such that  $\nu(B) \leq \nu(C)$ . Then we get

$$\alpha = \nu(P) < \nu(P) + \nu(C) = \nu(P \cup C)$$

and  $P \cup C$  is positive. This contradicts the definition of  $\alpha$ . Hence for every  $B \subseteq X \setminus P$  such that  $B \in \Sigma$  we have  $\nu(B) \leq 0$ . Thus measures

$$\nu_+(A) = \nu(A \cap P), \quad \nu_-(A) = -\nu(A \setminus P)$$

defined for  $A \in \Sigma$  satisfy the assertion of the theorem. This finishes the proof of the Hahn-Jordan decomposition under the assumption that  $\nu(A) \in \mathbb{R} \cup \{-\infty\}$  for all  $A \in \Sigma$ .

If  $\nu(A) \in \mathbb{R} \cup \{+\infty\}$  for every  $A \in \Sigma$ , then we apply the result above for  $-\nu$ . Finally the case  $\nu(A_1) = -\infty$  and  $\nu(A_2) = +\infty$  for some  $A_1, A_2 \in \Sigma$  yields to the contradiction. Hence Hahn-Jordan decomposition is proved.  $\square$

**Corollary 3.3.** Let  $(X, \Sigma)$  be a measurable space and  $\nu : \Sigma \rightarrow \overline{\mathbb{R}}$  be a signed measure. Then either  $\nu_+$  or  $\nu_-$  is finite.

*Proof.* According to Theorem 3.1 we have  $\nu = \nu_+ - \nu_-$  and both  $\nu_+, \nu_-$  are measures such that  $\nu_+ \perp \nu_-$ . We cannot have  $\nu_+(X) = \nu_-(X) = +\infty$ , because then  $\nu(X)$  would be undefined. This implies that either  $\nu_+(X) \in \mathbb{R}$  or  $\nu_-(X) \in \mathbb{R}$ .  $\square$

## 4. LEBESGUE DECOMPOSITION

**Definition 4.1.** Let  $(X, \Sigma)$  be a measurable space and  $\mu : \Sigma \rightarrow \overline{\mathbb{R}}$  be a signed measure. We say that  $\mu$  is  $\sigma$ -finite if there exists a decomposition

$$X = \bigcup_{n \in \mathbb{N}} X_n$$

onto pairwise disjoint elements of  $\Sigma$  such that  $\mu(X_n) \in \mathbb{R}$  for every  $n \in \mathbb{N}$ .

**Theorem 4.2** (Lebesgue decomposition). *Let  $(X, \Sigma)$  be a measurable space and let  $\mu$  be a measure on  $(X, \Sigma)$ . Suppose that  $\nu$  is either a signed and  $\sigma$ -finite measure or a complex measure on  $(X, \Sigma)$ . Then there exists a unique decomposition*

$$\nu = \nu_s + \nu_a$$

of measure  $\nu$  such that  $\nu_s \perp \mu$  and  $\nu_a \ll \mu$ .

*Proof.* Suppose first that  $\nu$  is a finite measure. Consider

$$\alpha = \sup_{A \in \Sigma, \mu(A)=0} \nu(A)$$

Since  $\nu$  is finite, we derive that  $\alpha \in \mathbb{R}$ . Consider a sequence  $\{A_n\}_{n \in \mathbb{N}}$  such that  $A_n \in \Sigma$ ,  $\mu(A_n) = 0$  for every  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow +\infty} \nu(A_n) = \alpha$ . Define  $S = \bigcup_{n \in \mathbb{N}} A_n$ . Then  $\mu(S) = 0$  and  $\nu(S) = \alpha$ . Moreover, if  $A \in \Sigma$  and  $A \cap S = \emptyset$ , then  $\mu(A) = 0$  implies  $\nu(A) = 0$ . Now we define  $\nu_s(A) = \nu(A \cap S)$  and  $\nu_a(A) = \nu(A \setminus S)$  for every  $A \in \Sigma$ . Then  $\nu = \nu_s + \nu_a$  and  $\nu_s \perp \mu$ ,  $\nu_a \ll \mu$ .

Now assume that  $\nu$  is  $\sigma$ -finite measure. There exists a decomposition

$$X = \bigcup_{n \in \mathbb{N}} X_n$$

onto pairwise disjoint elements of  $\Sigma$  such that  $\nu(X_n) \in \mathbb{R}$  for every  $n \in \mathbb{N}$ . We define  $\nu_n(A) = \nu(A \cap X_n)$  for each  $n \in \mathbb{N}$  and  $A \in \Sigma$ . Then  $\nu_n$  is a finite measure. By the case above we find  $\nu_n = \nu_{ns} + \nu_{na}$  and  $\nu_{ns} \perp \mu$ ,  $\nu_{na} \ll \mu$  for some measures on  $\Sigma$ . Now we define

$$\nu_s = \sum_{n \in \mathbb{N}} \nu_{ns}, \quad \nu_a = \sum_{n \in \mathbb{N}} \nu_{na}$$

Then  $\nu = \nu_s + \nu_a$  and  $\nu_s \perp \mu$ ,  $\nu_a \ll \mu$ .

Now consider the case when  $\nu$  is  $\sigma$ -finite and signed measure. According to Theorem 3.1 we write  $\nu = \nu_+ - \nu_-$  for measures  $\nu_+, \nu_-$  such that  $\nu_+ \perp \nu_-$ . Then  $\nu_+, \nu_-$  are  $\sigma$ -finite measures. According to previous case we can write  $\nu_+ = \nu_{+s} + \nu_{+a}$ ,  $\nu_- = \nu_{-s} + \nu_{-a}$  for measures such that  $\nu_{+s} \perp \mu$ ,  $\nu_{-s} \perp \mu$ ,  $\nu_{+a} \ll \mu$ ,  $\nu_{-a} \ll \mu$ . Then  $\nu_s = \nu_{+s} - \nu_{-s}$ ,  $\nu_a = \nu_{+a} - \nu_{-a}$  are signed measures and  $\nu_s \perp \mu$ ,  $\nu_a \ll \mu$ .

Finally assume that  $\nu$  is complex. Then  $\nu = \nu^r + i \cdot \nu^i$ , where  $\nu^r$  and  $\nu^i$  are finite, signed measures. From the case above we have decompositions

$$\nu^r = \nu_s^r + \nu_a^r, \quad \nu^i = \nu_s^i + \nu_a^i$$

and  $\nu_s^r \perp \mu$ ,  $\nu_s^i \perp \mu$ ,  $\nu_a^r \ll \mu$ ,  $\nu_a^i \ll \mu$ . Then complex measures

$$\nu_s = \nu_s^r + i \cdot \nu_s^i, \quad \nu_a = \nu_a^r + i \cdot \nu_a^i$$

satisfy  $\nu_s \perp \mu$ ,  $\nu_a \ll \mu$ . □

## 5. RADON-NIKODYM THEOREM

In this section we prove famous result of Radon and Nikodym.

**Theorem 5.1** (Radon-Nikodym). *Let  $(X, \Sigma)$  be a measurable space and let  $\mu$  be a  $\sigma$ -finite measure on  $(X, \Sigma)$ . Suppose that  $\nu \ll \mu$  for  $\nu$  that is either complex measure or  $\sigma$ -finite, signed measure. Then there exists a measurable function  $f : X \rightarrow \mathbb{C}$  such that*

$$\nu(A) = \int_A f d\mu$$

for every  $A \in \Sigma$ .

*Proof for finite measures  $\mu, \nu$ .* Fix  $n \in \mathbb{N}$  and  $k \in \mathbb{N}$ . According to Theorem 3.1 there exists a set  $P_{n,k} \in \Sigma$  such that

$$\left(\nu - \frac{k}{2^n} \cdot \mu\right)(A \cap P_{n,k}) \geq 0, \left(\nu - \frac{k}{2^n} \cdot \mu\right)(A \setminus P_{n,k}) \leq 0$$

for every  $A \in \Sigma$ . We may assume that  $P_{n,0} = X$ ,  $P_{n,k+1} \subseteq P_{n,k}$  and  $P_{n,k} = P_{n+1,2k}$  for every  $n, k \in \mathbb{N}$ . Since  $\nu \ll \mu$  and  $\nu$  is finite, we derive that

$$\mu\left(\bigcap_{k \in \mathbb{N}} P_{n,k}\right) = \nu\left(\bigcap_{k \in \mathbb{N}} P_{n,k}\right) = 0$$

and may assume that this set is empty for every  $n \in \mathbb{N}$ . Pick  $n \in \mathbb{N}$ . We define a function  $f_n : X \rightarrow \mathbb{C}$  by formula

$$f_n(x) = \sum_{k \in \mathbb{N}} \frac{k}{2^n} \cdot \chi_{P_{n,k} \setminus P_{n,k+1}}(x)$$

Clearly  $f_n$  is a measurable function with real, nonnegative values. If  $m, n \in \mathbb{N}$  and  $n \leq m$ , then we have

$$f_n(x) \leq f_m(x) \leq f_n(x) + \frac{1}{2^n}$$

Thus  $\{f_n\}_{n \in \mathbb{N}}$  is a nondecreasing sequence of measurable functions convergent uniformly to a measurable function  $f : X \rightarrow \mathbb{C}$ . Moreover, for every  $A \in \Sigma$  and  $n \in \mathbb{N}$  we have

$$\begin{aligned} \nu(A) - \frac{1}{2^n} \mu(A) &= \sum_{k \in \mathbb{N}} \nu(A \cap (P_{n,k} \setminus P_{n,k+1})) - \frac{1}{2^n} \mu(A) \leq \\ &\leq \sum_{k \in \mathbb{N}} \frac{k+1}{2^n} \mu(A \cap (P_{n,k} \setminus P_{n,k+1})) - \frac{1}{2^n} \sum_{k \in \mathbb{N}} \mu(A \cap (P_{n,k} \setminus P_{n,k+1})) \leq \\ &\leq \sum_{k \in \mathbb{N}} \frac{k}{2^n} \mu(A \cap (P_{n,k} \setminus P_{n,k+1})) \leq \sum_{k \in \mathbb{N}} \nu(A \cap (P_{n,k} \setminus P_{n,k+1})) = \nu(A) \end{aligned}$$

and since

$$\int_A f_n d\mu = \sum_{k \in \mathbb{N}} \frac{k}{2^n} \mu(A \cap (P_{n,k} \setminus P_{n,k+1}))$$

we derive that

$$\nu(A) - \frac{1}{2^n} \mu(A) \leq \int_A f_n d\mu \leq \nu(A)$$

This inequality together with monotone convergence theorem imply that

$$\nu(A) = \lim_{n \rightarrow +\infty} \int_A f_n d\mu = \int_A f d\mu$$

This finishes the proof for finite measures  $\nu, \mu$ . □

*Reduction to finite case.* Assume now that  $\nu$  and  $\mu$  are  $\sigma$ -finite measures on  $(X, \Sigma)$ . Then there exists a decomposition

$$X = \bigcup_{n \in \mathbb{N}} X_n$$

onto disjoint subsets in  $\Sigma$  such that  $\nu(X_n) \in \mathbb{R}$  and  $\mu(X_n) \in \mathbb{R}$  for every  $n \in \mathbb{N}$ . For every  $n \in \mathbb{N}$  we define  $\nu_n(A) = \nu(A \cap X_n)$  and  $\mu_n(A) = \mu(A \cap X_n)$  for  $A \in \Sigma$ . Since  $\nu \ll \mu$ , we derive that  $\nu_n \ll \mu_n$  for every  $n \in \mathbb{N}$ . Measures  $\{\nu_n\}_{n \in \mathbb{N}}$  and  $\{\mu_n\}_{n \in \mathbb{N}}$  are finite. By finite case of the theorem we deduce that for each  $n \in \mathbb{N}$  there exists a measurable function  $f_n : X \rightarrow \mathbb{C}$  such that

$$\nu_n(A) = \int_A f_n d\mu_n$$

for every  $A \in \Sigma$ . Note that  $f_n$  admits real, nonnegative values  $\mu$ -almost everywhere and can be set equal to zero outside  $X_n$ . Thus

$$\nu_n(A) = \int_A f_n d\mu_n = \int_A f_n d\mu$$

for every  $A \in \Sigma$ . Therefore, we deduce that

$$\nu(A) = \sum_{n \in \mathbb{N}} \nu(A \cap X_n) = \sum_{n \in \mathbb{N}} \nu_n(A) = \sum_{n \in \mathbb{N}} \int_A f_n d\mu = \int_A \left( \sum_{n \in \mathbb{N}} f_n \right) d\mu$$

by monotone convergence theorem.

Assume now that both  $\nu$  is  $\sigma$ -finite, signed measure. In this situation we may write  $\nu = \nu_+ - \nu_-$  for measures  $\nu_+, \nu_-$  such that  $\nu_+ \perp \nu_-$ . Then  $\nu_+ \ll \mu$  and  $\nu_- \ll \mu$ . There exists a set  $P \in \Sigma$  such that  $\nu_-(P) = \nu_+(X \setminus P) = 0$ . By the case considered previously there exist measurable functions  $f_+ : X \rightarrow \mathbb{C}, f_- : X \rightarrow \mathbb{C}$  such that

$$\nu_+(A) = \int_A f_+ d\mu, \nu_-(A) = \int_A f_- d\mu$$

for every  $A \in \Sigma$ . Moreover, we may assume that  $f_+$  is equal to zero outside  $P$  and  $f_-$  is equal to zero outside  $X \setminus P$ . From this we have

$$\nu(A) = \nu_+(A) + \nu_-(A) = \int_A f_+ d\mu + \int_A f_- d\mu = \int_A (f_+ - f_-) d\mu$$

for every  $A \in \Sigma$ .

Suppose that  $\nu$  is complex measure. Write  $\nu = \nu_r - i \cdot \nu_i$ . Then both  $\nu_r, \nu_i$  are finite, signed measures. Moreover, we have  $\nu_r \ll \mu, \nu_i \ll \mu$ . There exist measurable functions  $f_r : X \rightarrow \mathbb{C}$  and  $f_i : X \rightarrow \mathbb{C}$  that are real valued and satisfy

$$\nu_r(A) = \int_A f_r d\mu, \nu_i(A) = \int_A f_i d\mu$$

for every  $A \in \Sigma$ . Thus

$$\nu(A) = \nu_r(A) + i \cdot \nu_i(A) = \int_A f_r d\mu + i \cdot \int_A f_i d\mu = \int_A (f_r + i \cdot f_i) d\mu$$

for every  $A \in \Sigma$ . □

## 6. BANACH SPACE OF COMPLEX MEASURES

**Proposition 6.1.** *Let  $\mu$  be a complex measure on a measurable space  $(X, \Sigma)$ . For every  $A \in \Sigma$  we define*

$$|\mu|(A) = \sup \left\{ \sum_{n \in \mathbb{N}} |\mu(A_n)| \mid A = \bigcup_{n \in \mathbb{N}} A_n \text{ is a partition of } A \text{ onto subsets in } \Sigma \right\}$$

*Then  $|\mu|$  is a finite measure on  $(X, \Sigma)$ .*

*Proof.* Let  $\mu = \mu^r + i \cdot \mu^i$  be decomposition onto real and imaginary part. Then  $\mu^r, \mu^i$  are finite, signed measures. By Theorem 3.1 we derive that there exist decompositions  $\mu^r = \mu_+^r - \mu_-^r, \mu^i = \mu_+^i - \mu_-^i$  such that  $\mu_+^r, \mu_-^r, \mu_+^i, \mu_-^i$  are finite measures and  $\mu_+^r \perp \mu_-^r, \mu_+^i \perp \mu_-^i$ . Then for every partition

$$A = \bigcup_{n \in \mathbb{N}} A_n$$

of  $A \in \Sigma$  onto sets in  $\Sigma$  we have

$$\begin{aligned} \sum_{n \in \mathbb{N}} |\mu(A_n)| &= \sum_{n \in \mathbb{N}} \sqrt{(\mu^r(A_n))^2 + (\mu^i(A_n))^2} \leq \sum_{n \in \mathbb{N}} (|\mu^r(A_n)| + |\mu^i(A_n)|) \leq \\ &\leq \sum_{n \in \mathbb{N}} (\mu_+^r(A_n) + \mu_-^r(A_n) + \mu_+^i(A_n) + \mu_-^i(A_n)) = \mu_+^r(A) + \mu_-^r(A) + \mu_+^i(A) + \mu_-^i(A) \end{aligned}$$

Right hand side of the inequality does not depend on the partition and hence

$$|\mu|(A) \leq \mu_+^r(A) + \mu_-^r(A) + \mu_+^i(A) + \mu_-^i(A)$$

This implies that  $|\mu|(A) \in \mathbb{R}$  for every  $A \in \Sigma$ . Note also that  $|\mu|(\emptyset) = 0$ . Suppose now that  $A \in \Sigma$  and we have partitions

$$A = \bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} C_n, A_n = \bigcup_{m \in \mathbb{N}} A_{n,m} \text{ for every } n \in \mathbb{N}$$

onto subsets in  $\Sigma$ . Then

$$\sum_{n \in \mathbb{N}} |\mu(C_n)| \leq \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} |\mu(A_n \cap C_m)| \leq \sum_{n \in \mathbb{N}} |\mu|(A_n)$$

and

$$\sum_{n \in \mathbb{N}} \left( \sum_{m \in \mathbb{N}} |\mu(A_{n,m})| \right) \leq |\mu|(A)$$

These inequalities imply that

$$|\mu|(A) \leq \sum_{n \in \mathbb{N}} |\mu|(A_n) \leq |\mu|(A)$$

Therefore,  $|\mu|$  is a finite measure. □

**Definition 6.2.** Let  $\mu$  be a complex measure on  $(X, \Sigma)$ . Then we define

$$\|\mu\| = |\mu|(X)$$

and call it *the total variation of  $\mu$* .

**Theorem 6.3.** Let  $(X, \Sigma)$  be a measurable space and  $\mathcal{M}(X, \Sigma)$  be a space of all complex measures on  $(X, \Sigma)$ . Then the following assertions hold.

(1)  $\mathcal{M}(X, \Sigma)$  is a  $\mathbb{C}$ -linear space.

(2) The mapping

$$\mathcal{M}(X, \Sigma) \ni \mu \mapsto \|\mu\| \in [0, +\infty)$$

is a norm.

(3) Suppose that  $\{\mu_n\}_{n \in \mathbb{N}}$  is a sequence of complex measures on  $(X, \Sigma)$  that is a Cauchy sequence with respect to total variation. Then there exists a complex measure  $\mu$  such that

$$\lim_{n \rightarrow +\infty} \mu_n = \mu$$

Moreover, for every  $A \in \Sigma$  we have

$$\lim_{n \rightarrow +\infty} \mu_n(A) = \mu(A)$$

*Proof.* We left (1) and (2) for the reader as an exercise. Fix  $A \in \Sigma$ . Then

$$|\mu_n(A) - \mu_m(A)| \leq |\mu_n - \mu_m|(A) \leq \|\mu_n - \mu_m\|$$

for every  $n, m \in \mathbb{N}$ . Since  $\{\mu_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence with respect to total variation, we derive that there exists the limit  $\mu(A)$  of  $\{\mu_n(A)\}_{n \in \mathbb{N}}$ . Suppose that

$$A = \bigcup_{k \in \mathbb{N}} A_k$$

for  $A \in \Sigma$  and  $A_k \in \Sigma$  for  $k \in \mathbb{N}$ . Assume that sets  $\{A_k\}_{k \in \mathbb{N}}$  are disjoint. Pick  $N \in \mathbb{N}$ . Then

$$\begin{aligned} \sum_{k=0}^N |\mu_n(A_k) - \mu(A_k)| &= \lim_{m \rightarrow +\infty} \sum_{k=0}^N |\mu_n(A_k) - \mu_m(A_k)| \leq \\ &\leq \limsup_{m \rightarrow +\infty} \sum_{k \in \mathbb{N}} |\mu_n(A_k) - \mu_m(A_k)| \leq \limsup_{m \rightarrow +\infty} |\mu_n - \mu_m|(A) = \limsup_{m \rightarrow +\infty} \|\mu_n - \mu_m\| \end{aligned}$$

This implies that

$$\sum_{k \in \mathbb{N}} |\mu_n(A_k) - \mu(A_k)| \leq \limsup_{m \rightarrow +\infty} \|\mu_n - \mu_m\|$$

regardless of set  $A$  and partition  $\{A_k\}_{k \in \mathbb{N}}$ . Thus we deduce that there exists a sequence  $\{a_n\}_{n \in \mathbb{N}}$  of real numbers, convergent to zero such that

$$\sum_{k \in \mathbb{N}} |\mu_n(A_k) - \mu(A_k)| \leq a_n$$

for every  $n \in \mathbb{N}$ ,  $A \in \Sigma$  and partition  $\{A_k\}_{k \in \mathbb{N}}$  as above. Therefore, for fixed  $N \in \mathbb{N}$  we have

$$\begin{aligned} \left| \mu(A) - \sum_{k=0}^N \mu(A_k) \right| &\leq |\mu(A) - \mu_n(A)| + \left| \mu_n(A) - \sum_{k=0}^N \mu_n(A_k) \right| + \sum_{k=0}^N |\mu_n(A_k) - \mu(A_k)| \leq \\ &\leq |\mu(A) - \mu_n(A)| + \left| \mu_n(A) - \sum_{k=0}^N \mu_n(A_k) \right| + \sum_{k \in \mathbb{N}} |\mu_n(A_k) - \mu(A_k)| \leq 2a_n + \left| \mu_n(A) - \sum_{k=0}^N \mu_n(A_k) \right| \end{aligned}$$

Hence we derive that

$$\mu(A) = \sum_{k \in \mathbb{N}} \mu(A_k)$$

thus  $\mu$  is a complex measure and according to

$$\sum_{k \in \mathbb{N}} |\mu_n(A_k) - \mu(A_k)| \leq a_n$$

for every  $n \in \mathbb{N}$  we deduce that

$$\lim_{n \rightarrow +\infty} |\mu_n - \mu|(A) = 0$$

for every  $A \in \Sigma$ . Hence also  $\lim_{n \rightarrow +\infty} \|\mu_n - \mu\| = 0$ . This finishes the proof of (3).  $\square$

## 7. APPLICATIONS OF RADON-NIKODYM THEOREM

**Proposition 7.1.** *Let  $\mu$  be a measure on a measurable space  $(X, \Sigma)$  and  $f : X \rightarrow \mathbb{C}$  be a measurable function taking real, nonnegative values. We define*

$$\nu(A) = \int_A f d\mu$$

*for every  $A \in \Sigma$ . Then  $\nu$  is a measure on  $(X, \Sigma)$  and the equality*

$$\int_X g d\nu = \int_X g \cdot f d\mu$$

*holds for every measurable function  $g : X \rightarrow \mathbb{C}$  that is either  $\nu$ -integrable or takes real, nonnegative values.*

*Proof.* Suppose that  $A = \bigcup_{n \in \mathbb{N}} A_n$  for  $A \in \Sigma$  and  $A_n \in \Sigma$  for every  $n \in \mathbb{N}$ . Assume also that  $\{A_n\}_{n \in \mathbb{N}}$  are pairwise disjoint. Then by monotone convergence theorem

$$\nu(A) = \int_A f d\mu = \int_X \chi_A \cdot f d\mu = \int_X \left( \sum_{n \in \mathbb{N}} \chi_{A_n} \cdot f \right) d\mu = \sum_{n \in \mathbb{N}} \int_X \chi_{A_n} \cdot f d\mu = \sum_{n \in \mathbb{N}} \int_{A_n} f d\mu = \sum_{n \in \mathbb{N}} \nu(A_n)$$

Moreover, we have  $\nu(\emptyset) = 0$ . Thus  $\nu$  is a measure on  $(X, \Sigma)$ .

For the second part of the statement note that the family of measurable functions  $g : X \rightarrow \mathbb{C}$  satisfying equality

$$\int_X g d\nu = \int_X g \cdot f d\mu$$

contains  $\{\chi_A\}_{A \in \Sigma}$ , is closed under  $\mathbb{R}_{\geq 0}$ -linear combinations of measurable functions taking nonnegative values, if it contains nondecreasing sequence  $\{g_n : X \rightarrow \mathbb{C}\}_{n \in \mathbb{N}}$  of measurable functions taking only nonnegative values, then it also contains its limit. Thus this family contains all measurable functions  $g : X \rightarrow \mathbb{C}$  taking nonnegative values. Since every real valued,  $\nu$ -integrable function  $g : X \rightarrow \mathbb{C}$  is a difference of a two  $\nu$ -integrable functions taking nonnegative values, we deduce that this family contains all real,  $\nu$ -integrable functions. Finally, if it contains two  $\nu$ -integrable, real valued functions, then it contains its  $\mathbb{C}$ -linear combination. Thus it contains all  $\nu$ -integrable functions.  $\square$

**Theorem 7.2.** *Let  $\mu$  be a complex measure on a measurable space  $(X, \Sigma)$ . There exists a measurable function  $f : X \rightarrow \mathbb{C}$  such that*

$$\mu(A) = \int_A f d|\mu|$$

*for every  $A \in \Sigma$  and  $|f(x)| = 1$  for every  $x$  in  $X$ .*

For the proof we need the following result.

**Lemma 7.2.1.** Let  $\mu$  be a measure on  $(X, \Sigma)$ . Suppose that  $f : X \rightarrow \mathbb{C}$  is a measurable function and  $F$  is a closed subset of  $\mathbb{C}$ . Assume that for every  $A \in \Sigma$  such that  $\mu(A) > 0$ , we have

$$\frac{1}{\mu(A)} \int_A f d\mu \in F$$

Then  $\mu(X \setminus f^{-1}(F)) = 0$ .

*Proof of the lemma.* Let  $D$  be a closed disc in  $\mathbb{C}$  such that  $D \cap F = \emptyset$ . If  $\mu(f^{-1}(D)) > 0$ , then

$$\frac{1}{\mu(f^{-1}(D))} \int_{f^{-1}(D)} f d\mu \in D$$

by convexity of  $D$ . This implies that for every closed disc in  $\mathbb{C}$  disjoint from  $F$  we have  $\mu(f^{-1}(D)) = 0$ . Since  $\mathbb{C} \setminus F$  can be covered by such discs, we derive that  $\mu(X \setminus f^{-1}(F)) = 0$ .  $\square$

*Proof of the theorem.* Consider Radon-Nikodym derivative  $f : X \rightarrow \mathbb{C}$  of  $\mu$  with respect to  $|\mu|$ . It exists according to Theorem 5.1. Since

$$\frac{1}{\mu(A)} \left| \int_A f d|\mu| \right| \leq \frac{1}{\mu(A)} \int_A |f| d|\mu| = \frac{|\mu|(A)}{\mu(A)} \leq 1$$

for every  $A \in \Sigma$  such that  $\mu(A) > 0$ , we derive by Lemma 7.2.1 that  $f(x) \in D$  almost everywhere with respect to measure  $|\mu|$ , where  $D$  is a closed unit disc in  $\mathbb{C}$ . Changing values of  $f$  on a set of measure  $|\mu|$  equal to zero, we may assume that  $f(x) \in D$  for every  $x$  in  $X$ .

Suppose next that  $0 < \alpha < 1$  and denote  $A_\alpha = f^{-1}(\{z \in \mathbb{C} \mid |f(z)| \leq \alpha\})$ . Let

$$A_\alpha = \bigcup_{n \in \mathbb{N}} A_n$$

be a decomposition on disjoint subsets in  $\Sigma$ . Then

$$\sum_{n \in \mathbb{N}} |\mu(A_n)| = \sum_{n \in \mathbb{N}} \left| \int_{A_n} f d|\mu| \right| \leq \sum_{n \in \mathbb{N}} \int_{A_n} |f| d|\mu| \leq \alpha \cdot \sum_{n \in \mathbb{N}} |\mu|(A_n) = \alpha \cdot |\mu|(A_\alpha)$$

Hence

$$|\mu|(A_\alpha) \leq \alpha \cdot |\mu|(A_\alpha)$$

Therefore,  $|\mu|(A_\alpha) = 0$ . Since  $\alpha$  is arbitrary number in  $(0, 1)$ , we deduce that

$$|\mu|\left(\left\{z \in \mathbb{C} \mid |f(z)| < 1\right\}\right) = 0$$

Thus changing values of  $f$  on a set of measure  $|\mu|$  equal to zero, we may assume that  $|f(x)| = 1$  for every  $x$  in  $X$ .  $\square$

**Corollary 7.3.** Let  $\mu$  be a measure on a measurable space  $(X, \Sigma)$  and  $f : X \rightarrow \mathbb{C}$  be a  $\mu$ -integrable function. Define

$$\nu(A) = \int_A f d\mu$$

for every  $A \in \Sigma$ . Then  $\nu$  is a complex measure on  $(X, \Sigma)$  and

$$|\nu|(A) = \int_A |f| d\mu$$

for every  $A \in \Sigma$ .

*Proof.* Clearly  $\nu(A) \in \mathbb{C}$  for every  $A \in \Sigma$ . Suppose that  $A = \bigcup_{n \in \mathbb{N}} A_n$  for  $A \in \Sigma$  and  $A_n \in \Sigma$  for every  $n \in \mathbb{N}$ . Assume also that  $\{A_n\}_{n \in \mathbb{N}}$  are pairwise disjoint. Then by dominated convergence theorem

$$\nu(A) = \int_A f d\mu = \int_X \chi_A \cdot f d\mu = \int_X \left( \sum_{n \in \mathbb{N}} \chi_{A_n} \cdot f \right) d\mu = \sum_{n \in \mathbb{N}} \int_X \chi_{A_n} \cdot f d\mu = \sum_{n \in \mathbb{N}} \int_{A_n} f d\mu = \sum_{n \in \mathbb{N}} \nu(A_n)$$



Moreover, we have  $\nu(\emptyset) = 0$ . Thus  $\nu$  is a complex measure on  $(X, \Sigma)$ . Since  $f$  is  $\mu$ -integrable, there exists a  $\sigma$ -finite subset  $\Omega \in \Sigma$  such that  $|f(x)| = 0$  for  $x \notin \Omega$ . We define  $\tilde{\mu}(A) = \mu(A \cap \Omega)$  for every  $A \in \Sigma$ . Clearly

$$\nu(A) = \int_A f d\mu = \int_A f d\tilde{\mu}$$

for every  $A \in \Sigma$ . Hence we have  $|\nu| \ll \tilde{\mu}$  by definition of  $\nu$  and  $|\nu|$ . Note that  $\tilde{\mu}$  is a  $\sigma$ -finite measure. By Theorem 5.1 there exists a measurable function  $g : X \rightarrow \mathbb{C}$  equal to zero outside  $\Omega$  such that

$$|\nu|(A) = \int_A g d\tilde{\mu} = \int_A g d\mu$$

for every  $A \in \Sigma$ . We may assume that  $g$  takes only nonnegative values. By Theorem 7.2 there exists a function  $h : X \rightarrow \mathbb{C}$  such that

$$\nu(A) = \int_A h d|\nu|$$

for every  $A \in \Sigma$  and  $|h(x)| = 1$  for all  $x$  in  $X$ . By Proposition 7.1 we deduce that

$$\int_A f d\mu = \nu(A) = \int_A h d|\nu| = \int_A h \cdot g d\mu$$

for every  $A \in \Sigma$ . Therefore,  $f = h \cdot g$  almost everywhere with respect to  $\mu$ . Thus

$$g(x) = |h(x)| \cdot g(x) = |f(x)|$$

almost everywhere with respect to  $\mu$ . □

**Corollary 7.4.** *Let  $(X, \Sigma)$  be a measurable space and  $\mu$  be a measure on  $\Sigma$ . Then there exists an isometrical embedding*

$$L^1(X, \mu) \ni f \mapsto \left( \Sigma \ni A \mapsto \int_A f d\mu \in \mathbb{C} \right) \in \mathcal{M}(X, \Sigma)$$

*of Banach spaces. If in addition  $\mu$  is  $\sigma$ -finite, then it is surjective map onto the subspace of  $\mathcal{M}(X, \Sigma)$  consisting of complex measures which are absolutely continuous with respect to  $\mu$ .*

*Proof.* The first assertion follows from Corollary 7.3 and Theorem 6.3. The second is a recapitulation of Theorem 5.1. □

## REFERENCES

[Monygham, 2018] Monygham (2018). Introduction to measure theory. *github repository: "Monygham/Pedo-mellon-a-minno"*.