# TOPOLOGICAL VECTOR SPACES AND HAHN-BANACH THEOREM

## 1. Introduction

In these notes we study topological vector spaces. Among other things our goal is to prove various versions of Hahn-Banach theorem. In first sections we introduce topological vector spaces over arbitrary fields with absolute value and study their basic properties. Next we prove that all one-dimensional Hausdorff topological spaces are isomorphic. This result is used in the characterization of finite dimensional Hausdorff topological vector spaces over complete fields and in the proof of a theorem due to Riesz that all locally compact topological vector spaces are finite dimensional. Then we introduce seminormed spaces as the most important class of topological vector spaces. Characterization of finite dimensional topological vector spaces is one of the crucial ingredients of Mazur's theorem, which is the main topic of the following section. Next we introduce locally convex topological vector spaces and prove separation of convex sets for these spaces. We use Mazur's theorem and seminorms to deduce analytic version of Hahn-Banach theorem. In the final section we prove invariant version of analytic Hahn-Banach theorem.

## 2. Fields with absolute values

**Definition 2.1.** Let $\mathbb{K}$ be a field and let $|-|:\mathbb{K}\to\mathbb{R}_+\cup\{0\}$ be a function such that the following assertions hold.

**(1)** $|\alpha| = 0$ if and only if $\alpha = 0$ for every $\alpha\in\mathbb{K}$.

**(2)** $|\alpha\cdot\beta| = |\alpha|\cdot|\beta|$ for every $\alpha,\beta\in\mathbb{K}$.

**(3)** $|\alpha+\beta| \leq |\alpha|+|\beta|$ for every $\alpha,\beta\in\mathbb{K}$.

Then $\mathbb{K}$ together with $|-|$ is *a field with absolute value*.

**Example 2.2.** Let $\mathbb{K}$ be a field. Then for each $\alpha\in\mathbb{K}$ define

$$|\alpha| = \begin{cases} 0 & \text{if } \alpha = 0 \\ 1 & \text{otherwise}\end{cases}$$

Then $|-|$ is an absolute value on $\mathbb{K}$. It is *the trivial absolute value on* $\mathbb{K}$.

Throughout the notes $\mathbb{K}$ is a field with absolute value $|-|$. Note that $|-|$ induces metric

$$\mathbb{K}\times\mathbb{K}\ni(\alpha,\beta)\mapsto|\alpha-\beta|\in\mathbb{R}_+\cup\{0\}$$

In particular, $|-|$ induces topology on $\mathbb{K}$. We always consider $\mathbb{K}$ with this topology.

**Fact 2.3.** *The topology on a field $\mathbb{K}$ with absolute value is discrete if and only if $|-|$ is trivial.*

*Proof.* It suffices to prove that if topology on induced by $|-|$ is discrete, then $|-|$ is trivial. Suppose that there exists $\alpha\in\mathbb{K}$ such that $|\alpha|\notin\{0,1\}$. Then $\alpha\neq 0$ and

$$|\alpha|\cdot\left|\frac{1}{\alpha}\right| = \left|\alpha\cdot\frac{1}{\alpha}\right| = |1| = 1$$

Hence either

$$|\alpha| < 1$$

or

$$\left|\frac{1}{\alpha}\right| < 1$$

Without loss of generality we may assume that $0 < |\alpha| < 1$. Then $\{\alpha^n\}_{n\in\mathbb{N}}$ is a sequence of elements of $\mathbb{K}$ which converges to zero with respect to $|-|$. Thus the topology on $\mathbb{K}$ is not discrete. $\square$

**Definition 2.4.** The set

$$\mathbb{D} = \left\{\alpha \in \mathbb{K} \,\middle|\, |\alpha| \leq 1\right\}$$

is *the closed unit disc in* $\mathbb{K}$.

**Definition 2.5.** Suppose that every Cauchy sequence in $\mathbb{K}$ with respect to $|-|$ is convergent, then $\mathbb{K}$ is *a complete field*.

## 3. Topological vector spaces

In this section we introduce topological vector spaces and study their basic properties.

**Definition 3.1.** Let $\mathfrak{X}$ be a vector space over $\mathbb{K}$ together with a topology such that the multiplication by scalars $\cdot_{\mathfrak{X}} : \mathbb{K} \times \mathfrak{X} \to \mathfrak{X}$ and the addition $+_{\mathfrak{X}} : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ are continuous. Then $\mathfrak{X}$ is *a topological vector space over* $\mathbb{K}$.

**Fact 3.2.** *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$ and let $\mathfrak{Z}$ be its $\mathbb{K}$-subspace. Then $\mathfrak{Z}$ with subspace topology is a topological vector space over $\mathbb{K}$.*

*Proof.* Left for the reader as an exercise. $\square$

**Fact 3.3.** *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$ and let $U$ be an open neighborhood of zero in $\mathfrak{X}$. Then there exists an open neighborhood $W$ of zero in $\mathfrak{X}$ such that $W \subseteq U$ and $W = \mathbb{D} \cdot W$.*

*Proof.* Since the multiplication by scalars $\mathbb{K} \times \mathfrak{X} \to \mathfrak{X}$ is continuous, there exists an open neighborhood $V$ of zero in $\mathfrak{X}$ and a positive real number $r$ such that

$$W = \bigcup_{\alpha\in\mathbb{K},\, |\alpha|\leq r} \alpha \cdot V \subseteq U$$

Then $W$ is an open neighborhood of zero in $\mathfrak{X}$, $W \subseteq U$ and $W = \mathbb{D} \cdot W$. $\square$

**Definition 3.4.** Let $\mathfrak{X}, \mathfrak{Y}$ are topological vector spaces over $\mathbb{K}$. A map $f : \mathfrak{X} \to \mathfrak{Y}$ which is both continuous and $\mathbb{K}$-linear is *a morphism of topological vector spaces over* $\mathbb{K}$.

**Theorem 3.5.** *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$ and let $\mathfrak{U}$ be its $\mathbb{K}$-subspace. Consider the quotient map $q : \mathfrak{X} \twoheadrightarrow \mathfrak{X}/\mathfrak{U}$ in the category of vector spaces over $\mathbb{K}$ and equip $\mathfrak{X}/\mathfrak{U}$ with the quotient topology induced by $q$. Then the following assertions holds.*

**(1)** *$q$ is an open map.*

**(2)** *$\mathfrak{X}/\mathfrak{U}$ is a topological vector space over $\mathbb{K}$ and $q$ is a morphism of topological vector spaces.*

**(3)** *For every morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of topological vector spaces over $\mathbb{K}$ such that $f(\mathfrak{U}) = 0$ there exists a unique morphism $p : \mathfrak{X}/\mathfrak{U} \to \mathfrak{Y}$ of topological vector spaces over $\mathbb{K}$ which makes the triangle*

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "f"] \arrow[d, "q"'] & \mathfrak{Y} \\ \mathfrak{X}/\mathfrak{U} \arrow[ur, dotted, "p"'] & \end{tikzcd}$$

*commutative.*

**(4)** *$\mathfrak{U}$ is a closed in $\mathfrak{X}$ if and ony if $\mathfrak{X}/\mathfrak{U}$ is a Hausdorff topological space.*

For the proof we need the following result.

**Lemma 3.5.1.** *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. Then $\mathfrak{X}$ is Hausdorff if and only if zero subspace of $\mathfrak{X}$ is closed.*

*Proof of the lemma.* If $\mathfrak{X}$ is Hausdorff, then each singleton subset of $\mathfrak{X}$ is closed. Hence zero subspace of $\mathfrak{X}$ is closed.
Conversely, assume that the singleton of zero in $\mathfrak{X}$ is closed. Pick two distinct points $x_1, x_2 \in \mathfrak{X}$. There exists an open neighborhood $U$ of zero in $\mathfrak{X}$ such that $x_1 - x_2 \notin U$. Since the subtraction $\mathfrak{X} \times \mathfrak{X} \rightarrow \mathfrak{X}$ is continuous, there exists an open neighborhood $V$ of zero in $\mathfrak{X}$ such that $V - V \subseteq U$. If

$$z \in (x_1 + V) \cap (x_2 + V)$$

then $z = x_1 + z_1$ and $z = x_2 + z_2$ for some $z_1, z_2 \in V$. Hence

$$x_1 - x_2 = (z_2 - z_1) \in V - V \subseteq U$$

This is a contradiction with $x_1 - x_2 \notin U$. Thus

$$\emptyset = (x_1 + V) \cap (x_2 + V)$$

and $\mathfrak{X}$ is Hausdorff. $\square$

*Proof of the theorem.* Fix an open subset $U$ of $\mathfrak{X}$, then the set

$$q^{-1}\left(q\left(U\right)\right) = U + \mathfrak{U}$$

is open. According to the fact that $q : \mathfrak{X} \twoheadrightarrow \mathfrak{X}/\mathfrak{U}$ is a quotient topological map, we infer that $q(U)$ is open in $\mathfrak{X}/\mathfrak{U}$. Hence $q$ is an open map and the proof of **(1)** is completed.

Since $q$ is open, we derive that $1_{\mathbb{K}} \times q$ and $q \times q$ are open. Since squares

$$\begin{tikzcd} \mathfrak{X} \times \mathfrak{X} \arrow[r, "+_{\mathfrak{X}}"] \arrow[d, "q \times q"'] & \mathfrak{X} \arrow[d, "q"] \\ \mathfrak{X}/\mathfrak{U} \times \mathfrak{X}/\mathfrak{U} \arrow[r, "+_{\mathfrak{X}/\mathfrak{U}}"'] & \mathfrak{X}/\mathfrak{U} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K} \times \mathfrak{X} \arrow[r, "\cdot_{\mathfrak{X}}"] \arrow[d, "1_{\mathbb{K}} \times q"'] & \mathfrak{X} \arrow[d, "q"] \\ \mathbb{K} \times \mathfrak{X}/\mathfrak{U} \arrow[r, "\cdot_{\mathfrak{X}/\mathfrak{U}}"'] & \mathfrak{X}/\mathfrak{U} \end{tikzcd}$$

are commutative, we deduce that the addition $+_{\mathfrak{X}/\mathfrak{U}} : \mathfrak{X}/\mathfrak{U} \times \mathfrak{X}/\mathfrak{U} \rightarrow \mathfrak{X}/\mathfrak{U}$ and the multiplication of scalars $\cdot_{\mathfrak{X}/\mathfrak{U}} : \mathbb{K} \times \mathfrak{X}/\mathfrak{U} \rightarrow \mathfrak{X}/\mathfrak{U}$ are continuous. Therefore, $\mathfrak{X}/\mathfrak{U}$ is a topological vector space over $\mathbb{K}$. It follows that $q$ is a morphism of topological vector spaces over $\mathbb{K}$ and hence **(2)** holds.

The assertion **(3)** describes the universal property which follows easily from **(2)** and the fact that $q$ is a topological quotient.

For **(4)** observe that

$$\mathfrak{U} \text{ is closed subset of } \mathfrak{X} \;\Leftrightarrow\; \text{zero subspace of } \mathfrak{X}/\mathfrak{U} \text{ is closed}$$

Thus it suffices to prove that

$$\text{zero subspace of } \mathfrak{X}/\mathfrak{U} \text{ is closed } \;\Leftrightarrow\; \mathfrak{X}/\mathfrak{U} \text{ is a Hausdorff topological space}$$

but this is a consequence of Lemma 3.5.1. $\square$

## 4. Complete topological vector spaces

We need some basic results on complete topological vector spaces. For all facts and notions related to filters on topological spaces we refer the reader to [Monygham, 2022].

**Definition 4.1.** Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. Suppose that $\mathcal{F}$ is a proper filter of subsets of $\mathfrak{X}$ such that for every open neighborhood $U$ of zero in $\mathfrak{X}$ there exists $F \in \mathcal{F}$ such that

$$F - F \subseteq U$$

Then $\mathcal{F}$ is *a Cauchy filter in* $\mathfrak{X}$.

**Definition 4.2.** A topological vector space $\mathfrak{X}$ over $\mathbb{K}$ is *complete* if every Cauchy filter in $\mathfrak{X}$ is convergent.

**Theorem 4.3.** *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$ and let $\mathfrak{Z}$ be its $\mathbb{K}$-subspace. Consider $\mathfrak{Z}$ as a topological vector space over $\mathbb{K}$ with subspace topology. Then the following assertions hold.*

**(1)** *If $\mathfrak{X}$ is complete and $\mathfrak{Z}$ is a closed in $\mathfrak{X}$, then $\mathfrak{Z}$ is complete.*

**(2)** *If $\mathfrak{Z}$ is complete and $\mathfrak{X}$ is Hausdorff, then $\mathfrak{Z}$ is closed in $\mathfrak{X}$.*

*Proof.* Consider a Cauchy filter $\mathcal{F}$ in $\mathfrak{Z}$. We define

$$\tilde{\mathcal{F}} = \left\{\tilde{F} \subseteq \mathfrak{X} \,\middle|\, \text{ there exists } F \in \mathcal{F} \text{ such that } F \subseteq \tilde{F}\right\}$$

Clearly $\tilde{\mathcal{F}}$ is a Cauchy filter in $\mathfrak{X}$. Since $\mathfrak{X}$ is complete, we derive that $\tilde{\mathcal{F}}$ is convergent to some $x$ in $\mathfrak{X}$. This together with definition of $\tilde{\mathcal{F}}$ show that for every open neighborhood $U$ of zero in $\mathfrak{X}$ there exists $F \in \mathcal{F}$ such that $F \subseteq x + U$. In particular, for every open neighborhood $U$ of zero in $\mathfrak{X}$ intersection $(x + U) \cap \mathfrak{Z}$ is nonempty. Since $\mathfrak{Z}$ is closed in $\mathfrak{X}$, it follows that $x \in \mathfrak{Z}$ and $\mathcal{F}$ is convergent to $x$. Thus $\mathfrak{Z}$ is complete.
Suppose now that $\mathfrak{Z}$ is complete. Assume that for some point $x$ in $\mathfrak{X}$ and for every open neighborhood of zero $U$ in $\mathfrak{X}$ intersection $(x + U) \cap \mathfrak{Z}$ is nonempty. Define

$$\mathcal{F} = \left\{F \subseteq \mathfrak{Z} \,\middle|\, \text{ there exists open neighborhood } U \text{ of zero in } \mathfrak{X} \text{ such that } (x + U) \cap \mathfrak{Z} \subseteq F\right\}$$

Then $\mathcal{F}$ is a Cauchy filter in $\mathfrak{Z}$. Since $\mathfrak{Z}$ is complete, $\mathcal{F}$ is convergent to some point $z$ in $\mathfrak{Z}$. By definition of $\mathcal{F}$ we have $z \in x + U$ for every open neighborhood $U$ of zero $x$. Since $\mathfrak{X}$ is Hausdorff, it follows that $z$ is identical to $x$. This proves that $\mathfrak{Z}$ is closed in $\mathfrak{X}$. ☐

**Theorem 4.4.** *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. Suppose that there exists a pseudometric $\rho : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}_+ \cup \{0\}$ which induces the topology of $\mathfrak{X}$. Then the following assertions hold.*

**(i)** *$\mathfrak{X}$ is complete.*

**(ii)** *Every Cauchy sequence with respect to $\rho$ is convergent.*

*Proof.* Assume that $\mathfrak{X}$ is complete and $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to $\rho$. Define

$$F_n = \left\{x_k \,\middle|\, k \geq n\right\}$$

for every $n \in \mathbb{N}$ and let

$$\mathcal{F} = \left\{F \subseteq \mathfrak{X} \,\middle|\, F_n \subseteq F \text{ for some } n \in \mathbb{N}\right\}$$

Since $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to pseudometric $\rho$ which induces topology on $\mathfrak{X}$, we derive that $\mathcal{F}$ is a Cauchy filter in $\mathfrak{X}$. Hence $\mathcal{F}$ is convergent to some point of $\mathfrak{X}$. This proves that $\{x_n\}_{n \in \mathbb{N}}$ is convergent to some point of $\mathfrak{X}$. Hence $\{x_n\}_{n \in \mathbb{N}}$ is convergent with respect to $\rho$. This completes the proof of **(i)** $\Rightarrow$ **(ii)**.

Suppose that every Cauchy sequence with respect to $\rho$ is convergent in $\mathfrak{X}$. Consider a Cauchy filter $\mathcal{F}$ in $\mathfrak{X}$. Since topology of $\mathfrak{X}$ is pseudometrizable, we derive that there exists a countable basis $\{U_n\}_{n\in\mathbb{N}}$ of open neighborhoods of zero in $\mathfrak{X}$. There exists a decreasing sequence $\{F_n\}$ of elements of $\mathcal{F}$ such that

$$F_n - F_n \subseteq U_n$$

for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let $x_n \in F_n$. Then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\rho$. Hence it is convergent to some point $x$ in $\mathfrak{X}$. Pick an open neighborhood $U$ of zero in $\mathfrak{X}$. Consider open neighborhood $W$ of zero in $\mathfrak{X}$ such that $W + W \subseteq U$. For sufficiently large $n \in \mathbb{N}$ we have

$$F_n - F_n \subseteq W,\ x_n - x \in W$$

If $z \in F_n$, then

$$x - z = (x - x_n) + (x_n - z) \in W + (F_n - F_n) \subseteq W + W \subseteq U$$

Hence $F_n \subseteq x + U$. This proves that $\mathcal{F}$ is convergent to $x$. The implication **(ii)** $\Rightarrow$ **(i)** holds. □

**Theorem 4.5.** *Let $\{\mathfrak{X}_i\}_{i\in I}$ be a family of topological vector space over $\mathbb{K}$. Then the following assertions are equivalent.*

- **(i)** *$\mathfrak{X}_i$ is complete for every $i \in I$.*
- **(ii)** *$\prod_{i\in I} \mathfrak{X}_i$ is complete topological vector space over $\mathbb{K}$.*

*Proof.* We denote $\prod_{i\in I} \mathfrak{X}_i$ by $\mathfrak{X}$ and let $pr_i : \mathfrak{X} \to \mathfrak{X}_i$ be canonical projection on $i$-th axis.

Assume that $\mathfrak{X}_i$ is complete for every $i \in I$. Suppose that $\mathcal{F}$ is a Cauchy filter in $\mathfrak{X}$. Then $pr_i(\mathcal{F})$ is a Cauchy filter in $\mathfrak{X}_i$ for each $i$. Since $\mathfrak{X}_i$ is complete, we derive that $pr_i(\mathcal{F})$ is convergent to some point $x_i$ in $\mathfrak{X}_i$. Define $x \in \mathfrak{X}$ by condition $pr_i(x) = x_i$ for each $i \in I$. Then $\mathcal{F}$ is convergent to $x$. Thus $\mathfrak{X}$ is a complete topological vector space over $\mathbb{K}$.

Suppose now that $\mathfrak{X}$ is complete. Fix $i_0$ in $I$ and consider a Cauchy filter $\mathcal{F}$ in $\mathfrak{X}_{i_0}$. Define

$$\tilde{F} = \{\underbrace{F}_{i_0} \times \underbrace{\{0\}}_{i\neq i_0} \subseteq \mathfrak{X} \,|\, F \in \mathcal{F}\}$$

Then $\tilde{\mathcal{F}}$ is a Cauchy filter in $\mathfrak{X}$. Hence $\tilde{\mathcal{F}}$ is convergent to some point $x$ in $\mathfrak{X}$. Then $\mathcal{F} = pr_{i_0}(\tilde{\mathcal{F}})$ is convergent to $pr_{i_0}(x)$. Thus $\mathfrak{X}_{i_0}$ is complete. Since $i_0$ is arbitrary, we derive that $\mathfrak{X}_i$ is complete for every $i \in I$. □

**Corollary 4.6.** *Let $\mathbb{K}$ be a complete field. Topological vector space $\mathbb{K}^n$ over $\mathbb{K}$ is complete for each $n \in \mathbb{N}$.*

*Proof.* This is a direct consequence of Theorems 4.4 and 4.5. □

## 5. Finite dimensional topological vector spaces

**Fact 5.1.** *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. Suppose that $f : \mathbb{K}^n \to \mathfrak{X}$ is a $\mathbb{K}$-linear map for some $n \in \mathbb{N}$. Then $f$ is continuous.*

*Proof.* Let $\{e_1, ..., e_n\}$ be the canonical basis of $\mathbb{K}^n$. For every $i$ let $pr_i : \mathbb{K}^n \to \mathbb{K}$ be the projection onto $i$-th axis and let $m_i : \mathbb{K} \to \mathfrak{X}$ be the composition of the multiplication of scalars $\mathbb{K} \times \mathfrak{X} \to \mathfrak{X}$ with the continuous embedding $\mathbb{K} \ni \alpha \mapsto (\alpha, f(e_i)) \in \mathbb{K} \times \mathfrak{X}$. Since $\mathrm{pr}_i$ and $m_i$ are continuous for each $i$, we derive that their compositions $m_i \cdot pr_i$ are also continuous. According to the fact that the addition $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ is continuous, we infer that the sum

$$\sum_{i=1}^{n} m_i \cdot pr_i$$

is continuous. This sum is equal to $f$. Thus $f$ is continuous. □

**Theorem 5.2.** *Let $\mathfrak{X}$ be a one-dimensional topological vector space over $\mathbb{K}$. Then the following assertions hold.*

**(1)** *If $\mathfrak{X}$ is Hausdorff and the absolute value on $\mathbb{K}$ is nontrivial, then every $\mathbb{K}$-linear isomorphism $\mathfrak{X} \to \mathbb{K}$ is a homeomorphism.*

**(2)** *If $\mathfrak{X}$ is not Hausdorff, then the topology on $\mathfrak{X}$ is indiscrete.*

*Proof.* Assume that $\mathfrak{X}$ is Hausdorff. Let $f : \mathfrak{X} \to \mathbb{K}$ be a $\mathbb{K}$-linear isomorphism. The topology on $\mathbb{K}$ is not discrete by Fact 2.3. Thus for each positive real number $r$ there exists nonzero $\gamma \in \mathbb{K}$ such that $|\gamma| < r$. Consider $x_\gamma$ in $\mathfrak{X}$ such that $f(x_\gamma) = \gamma$. It is unique element of $\mathfrak{X}$. Since $\mathfrak{X}$ is Hausdorff, by Fact 3.3 there exists an open neighborhood $W$ of zero in $\mathfrak{X}$ such that $\mathbb{D} \cdot W = W$ and $x_\gamma \notin W$. Then $\mathbb{D} \cdot f(W) = f(W)$ and $\gamma \notin f(W)$. This proves that $f(W)$ is a subset of

$$\{\alpha \in \mathbb{K} \mid |\alpha| < r\}$$

Therefore, $f$ is continuous at zero and hence $f$ is continuous. On the other hand map $f^{-1} : \mathbb{K} \to \mathfrak{X}$ is continuous by Fact 5.1. This means that $f$ is a homeomorphism.

Suppose now that $\mathfrak{X}$ is not Hausdorff. Theorem 3.5 implies that zero subspace is not closed in $\mathfrak{X}$. Since in every topological vector space closure of a subspace is a subspace, we derive that $\mathfrak{X}$ is the closure of its zero subspace. This shows that $\mathfrak{X}$ is indiscrete. □

**Example 5.3.** Let $\mathbb{K}$ be field of real numbers with trivial absolute value and let $\mathbb{R}$ be the set of all real numbers with the natural topology. Then $\mathbb{R}$ is one-dimensional topological vector space over $\mathbb{K}$, which is not isomorphic to $\mathbb{K}$.

**Corollary 5.4.** *Suppose that absolute value on $\mathbb{K}$ is nontrivial. Let $f : \mathfrak{X} \to \mathbb{K}$ be a $\mathbb{K}$-linear map between topological vector spaces over $\mathbb{K}$. Then the following are equivalent.*

**(i)** *$f$ is continuous.*

**(ii)** $\ker(f)$ *is a closed subspace of $\mathfrak{X}$.*

*Proof.* Follows immediately from Theorems 3.5 and 5.2. □

**Theorem 5.5.** *Let $\mathbb{K}$ be a complete field with nontrivial absolute value and let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. If $\mathfrak{X}$ is Hausdorff and of dimension $n$ over $\mathbb{K}$ for some $n \in \mathbb{N}$, then $\mathfrak{X}$ is isomorphic with $\mathbb{K}^n$.*

*Proof.* The proof goes on induction by $n \in \mathbb{N}$. For $n = 0$ it is clear. Suppose that the result holds for $n \in \mathbb{N}$. Assume that $\mathfrak{X}$ is a Hausdorff topological vector space over $\mathbb{K}$ of dimension $n+1$. By induction each $n$-dimensional subspace of $\mathfrak{X}$ is isomorphic to $\mathbb{K}^n$ and hence by Corollary 4.6 it is complete. Thus Theorem 4.3 asserts that all $n$-dimensional subspaces are closed in $\mathfrak{X}$. Corollary 5.4 implies that each $\mathbb{K}$-linear map $f : \mathfrak{X} \to \mathbb{K}$ is continuous. Therefore, every $\mathbb{K}$-linear map $\Phi : \mathfrak{X} \to \mathbb{K}^{n+1}$ is continuous. Next $\Phi^{-1}$ is continuous according to Fact 5.1. Therefore, $\mathfrak{X}$ is isomorphic to $\mathbb{K}^{n+1}$ as a topological vector space over $\mathbb{K}$. The proof is completed. □

**Example 5.6.** The subspace

$$\mathbb{Q} + \sqrt{2} \cdot \mathbb{Q} \subseteq \mathbb{R}$$

is a two-dimensional Hausdorff topological vector space over $\mathbb{Q}$. Note that each of its one-dimensional subspaces is dense. Hence

$$\mathbb{Q} + \sqrt{2} \cdot \mathbb{Q} \ncong \mathbb{Q} \times \mathbb{Q}$$

as topological vector spaces over $\mathbb{Q}$.

Now we are ready to present some consequences of results obtained in this section.

**Corollary 5.7.** *Let $\mathbb{K}$ be a complete field with nontrivial absolute value and let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. Fix a number $n \in \mathbb{N}$. Then every morphism $f : \mathfrak{X} \to \mathbb{K}^n$ of topological vector spaces over $\mathbb{K}$ is open.*

*Proof.* Note that $\ker(f)$ is closed in $\mathfrak{X}$. Hence the quotient map $\mathfrak{X}/\ker(f)$ is Hausdorff by Theorem 3.5. By Theorem 5.5 we derive that $\mathfrak{X}/\ker(f) \cong \mathbb{K}^n$ as topological vector spaces over $\mathbb{K}$. Hence $f$ is the quotient map $q : \mathfrak{X} \to \mathfrak{X}/\ker(f)$ followed by an isomorphism $\mathfrak{X}/\ker(f) \cong \mathbb{K}^n$ of topological vector spaces over $\mathbb{K}$. Theorem 3.5 implies that $q$ is open. Therefore, $f$ is open. □

**Theorem 5.8** (Riesz). *Let $\mathbb{K}$ be a locally compact field with nontrivial absolute value and let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. Then $\mathfrak{X}$ is locally compact if and only if $\mathfrak{X}$ is finite dimensional and Hausdorff.*

*Proof.* First note that each locally compact field is complete.

Suppose that $\mathfrak{X}$ is locally compact. Then there exists a compact subset $K$ of $\mathfrak{X}$ which contains an open neighborhood of zero. By compactness of $K$ and fact that it contains open neighborhood of zero in $\mathfrak{X}$ we deduce that there exist $s \in \mathbb{N}_+$ and $z_1, .., z_s \in \mathfrak{X}$ such that

$$K \subseteq \bigcup_{i=1}^{s} \left( z_i + \frac{1}{2} \cdot K \right)$$

Let $\mathfrak{Z}$ be a $\mathbb{K}$-linear subspace of $\mathfrak{X}$ spanned by $z_1, ..., z_s$. By easy induction we have

$$K \subseteq \mathfrak{Z} + \frac{1}{2^n} \cdot K$$

for every $n \in \mathbb{N}_+$. Fix now an open neighborhood $U$ of zero in $\mathfrak{X}$. Pick $n \in \mathbb{N}_+$ such that

$$\frac{1}{2^n} \cdot K \subseteq -U$$

The number $n$ exists due to compactness of $K$. Then

$$K \subseteq \mathfrak{Z} + \frac{1}{2^n} \cdot K \subseteq \mathfrak{Z} - U$$

and hence for every $x \in K$ there exists $z \in \mathfrak{Z}$ such that $z \in x + U$. This proves that $K \subseteq \mathbf{cl}(\mathfrak{Z})$. Since $\mathfrak{Z}$ is finite dimensional and Hausdorff, Theorem 5.5 implies that $\mathfrak{Z}$ is isomorphic to $\mathbb{K}^n$ for some $n \in \mathbb{N}$. Corollary 4.6 shows that $\mathfrak{Z}$ is a complete topological vector space over $\mathbb{K}$. Next by Theorem 4.3 we deduce that $\mathfrak{Z}$ is closed in $\mathfrak{X}$. Hence $K \subseteq \mathfrak{Z}$. Since $K$ contains open neighborhood of zero in $\mathfrak{X}$, we derive that $\mathfrak{Z} = \mathfrak{X}$.

On the other hand if $\mathfrak{X}$ is finite dimensional and Hausdorff, then Theorem 5.5 shows that $\mathfrak{X}$ is isomorphic to $\mathbb{K}^n$ for some $n \in \mathbb{N}$. Since $\mathbb{K}$ is locally compact, we derive that $\mathfrak{X}$ is locally compact. □

## 6. MAZUR'S THEOREM

In this section assume that $\mathbb{K}$ is either real numbers field $\mathbb{R}$ of complex numbers field $\mathbb{C}$ with usual absolute values.

**Theorem 6.1** (Mazur). *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$ and let $U$ be an open and convex subset of $\mathfrak{X}$. Suppose that $\mathfrak{U}$ is a $\mathbb{K}$-subspace of $\mathfrak{X}$ such that $\mathfrak{U}$ does not intersect with $U$. Then there exists a $\mathbb{K}$-linear continuous map $f : \mathfrak{X} \to \mathbb{K}$ such that $\mathfrak{U} \subseteq \ker(f)$ and $0 \notin f(U)$.*

For the proof we need the following result.

**Lemma 6.1.1.** *Let $\mathfrak{X}$ be a two-dimensional Hausdorff topological vector space over $\mathbb{R}$ and let $U$ be an open and convex subset which does not contain zero of $\mathfrak{X}$. Then there exists one-dimensional subspace $L$ of $\mathfrak{X}$ which does not intersect $U$.*

*Proof of the lemma.* Theorem 5.5 implies that we may assume that $\mathfrak{X}$ is $\mathbb{R}^2$. Consider

$$S^1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$$

and a retraction $r : \mathbb{R}^2 \setminus \{0\} \to S^1$ given by formula

$$r(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

Note that $r$ is an open map. Thus $\tilde{U} = r(U)$ is an open subset of $S^1$. Let $i : S^1 \to S^1$ be a homeomorphism given by formula $i(x,y) = (-x,-y)$. Since $U$ is convex and does not contain zero, sets $i(\tilde{U})$ and $\tilde{U}$ have empty intersection. According to the fact that $S^1$ is connected, we deduce that $i(\tilde{U}) \cup \tilde{U}$ is a proper subset of $S^1$. This is the case if and only if there exists $(x,y) \in S^1$ such that $(x,y) \notin \tilde{U}$ and $(-x,-y) \notin \tilde{U}$. Then one-dimensional subspace $\mathbb{R} \cdot (x,y)$ of $\mathfrak{X}$ does not intersect $U$. □

*Proof of the theorem.* Assume first that $\mathbb{K}$ is $\mathbb{R}$. By Zorn's lemma there exists maximal $\mathbb{R}$-subspace $\mathfrak{Z}$ such that $\mathfrak{U} \subseteq \mathfrak{Z}$ and $\mathfrak{Z}$ does not intersect $U$. Since $U$ is open, we derive that $\mathbf{cl}(\mathfrak{Z})$ does not intersect $U$. This shows that $\mathfrak{Z}$ is a closed subspace of $\mathfrak{X}$. Now consider the quotient map $q : \mathfrak{X} \twoheadrightarrow \mathfrak{X}/\mathfrak{Z}$. By Theorem 3.5 space $\mathfrak{X}/\mathfrak{Z}$ is Hausdorff and $q(U)$ is an open set. Moreover, $q(U)$ does not intersect zero and is convex. Suppose that there exists two-dimensional $\mathbb{R}$-subspace $\mathfrak{Y}$ of $\mathfrak{X}/\mathfrak{Z}$. Applying Lemma 6.1.1 to $\mathfrak{Y}$ and $\mathfrak{Y} \cap q(U)$ we deduce that there exists a one-dimensional $\mathbb{R}$-subspace $L$ of $\mathfrak{X}/\mathfrak{Z}$ such that $L$ does not intersect $q(U)$. Then $q^{-1}(L)$ is $\mathbb{R}$-subspace of $\mathfrak{X}$ strictly containing $\mathfrak{Z}$ which does not intersect $U$. This is contradiction with maximality of $\mathfrak{Z}$. Thus $\mathfrak{X}/\mathfrak{Z}$ contains no two-dimensional subspaces and hence it is one-dimensional. According to Theorem 5.5 we have isomorphism $\phi : \mathfrak{X}/\mathfrak{Z} \to \mathbb{R}$ of topological vector spaces over $\mathbb{R}$. The composition $f = \phi \cdot q$ satisfies the assertion of the theorem and this completes the proof for $\mathbb{R}$.

Next assume that $\mathbb{K}$ is $\mathbb{C}$. Since $\mathfrak{X}$ is a topological vector space over $\mathbb{C}$, it is also topological vector space over $\mathbb{R}$. Hence there exists an $\mathbb{R}$-linear continuous map $\tilde{f} : \mathfrak{X} \to \mathbb{R}$ such that $\mathfrak{U} \subseteq \ker(\tilde{f})$ and $0 \notin \tilde{f}(U)$. Consider $f : \mathfrak{X} \to \mathbb{C}$ given by formula

$$f(x) = \tilde{f}(x) - \sqrt{-1} \cdot \tilde{f}\left(\sqrt{-1} \cdot x\right)$$

for $x$ in $\mathfrak{X}$. Then $f$ is a $\mathbb{C}$-linear continuous map such that $\mathfrak{U} \subseteq \ker(f)$ and $0 \notin f(U)$. □

The result above is often called geometric Hahn-Banach theorem.

## 7. Locally convex spaces and separation theorem

In this section assume that $\mathbb{K}$ is either real numbers field $\mathbb{R}$ of complex numbers field $\mathbb{C}$ with usual absolute values.

**Definition 7.1.** Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. Suppose that every open neighborhood of zero in $\mathfrak{X}$ contains an open and convex neighborhood of zero. Then $\mathfrak{X}$ is *a locally convex space over* $\mathbb{K}$.

**Theorem 7.2.** *Let $\mathfrak{X}$ be a locally convex space over $\mathbb{R}$. Suppose that $K$ and $C$ are disjoint, nonempty, convex subsets of $\mathfrak{X}$ such that $K$ is quasi-compact and $C$ is closed in $\mathfrak{X}$. Then there exists a continuous $\mathbb{R}$-linear map $f : \mathfrak{X} \to \mathbb{R}$ and a point $x \in \mathfrak{X}$ such that*

$$f(K - x) \subseteq \mathbb{R}_-,\ f(C - x) \subseteq \mathbb{R}_+$$

*Proof.* For each $x \in K$ there exists open neighborhood $W_x$ of zero in $\mathfrak{X}$ such that

$$(x + W_x + W_x) \cap C = \emptyset$$

Since $K$ is quasi-compact, there are $x_1, ..., x_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} (x_i + W_{x_i})$$

Define

$$W = \bigcap_{i=1}^{n} W_{x_i}$$

Then $W$ is an open neighborhood of zero in $\mathfrak{X}$ such that $(K+W) \cap C = \emptyset$. Fix now an open and convex neighborhood of zero in $\mathfrak{X}$ such that $V \subseteq W$. Such set exists according to the fact that $\mathfrak{X}$ is locally convex. Note that

$$(K+V) \cap C = \emptyset$$

It follows that subset

$$U = (K+V) - C$$

of $\mathfrak{X}$ is open, convex and does not contain zero. Invoking Theorem 6.1 we get a continuous $\mathbb{R}$-linear map $f : \mathfrak{X} \to \mathbb{R}$ such that $0 \notin f(U)$. Corollary 5.7 implies that $f$ is an open map. It follows that $f(K+V), f(C)$ are disjoint intervals in $\mathbb{R}$. Since $f(K)$ is a compact interval contained in an open interval $f(K+V)$, we deduce that there exists $x \in \mathfrak{X}$ such that $f(x)$ is strictly between $f(K)$ and $f(C)$. Clearly it is also strictly between $f(K)$ and $f(C)$. Thus $f(K-x)$ and $f(C-x)$ are strictly separated by zero in $\mathbb{R}$. Without loss of generality we may assume that $f(K-x) \subseteq \mathbb{R}_-$ and $f(C-x) \subseteq \mathbb{R}_+$. □

## 8. Analytic Hahn-Banach theorem

**Definition 8.1.** Let $\mathfrak{X}$ be a vector space over $\mathbb{R}$ and let $p : \mathfrak{X} \to \mathbb{R}$ be a map. Suppose that

$$p(x_1 + x_2) \leq p(x_1) + p(x_2)$$

for all $x_1, x_2 \in \mathfrak{X}$ and

$$p(r \cdot x) = r \cdot p(x)$$

for each $x \in \mathfrak{X}$ and each $r \in \mathbb{R}_+$. Then $p$ is *a sublinear map*.

**Theorem 8.2** (Hahn-Banach)**.** *Let $\mathfrak{X}$ be a vector space over $\mathbb{R}$ and let $p : \mathfrak{X} \to \mathbb{R}$ be a sublinear map. Suppose that $\mathfrak{U}$ is an $\mathbb{R}$-subspace of $\mathfrak{X}$ and $f : \mathfrak{U} \to \mathbb{R}$ is an $\mathbb{R}$-linear map such that $f(x) \leq p(x)$ for every $x$ in $\mathfrak{U}$. Then there exists an $\mathbb{R}$-linear map $\tilde{f} : \mathfrak{X} \to \mathbb{R}$ such that $\tilde{f} \leq p$ and $\tilde{f}_{|\mathfrak{U}} = f$.*

We need the following result, which shows that each sublinear map give rise to a seminorm.

**Lemma 8.2.1.** *Let $\mathfrak{X}$ be a vector space over $\mathbb{R}$ and let $p : \mathfrak{X} \to \mathbb{R}$ be a sublinear map. Consider $q : \mathfrak{X} \to \mathbb{R}$ given by formula*

$$q(x) = \max\{p(x), p(-x)\}$$

*for $x \in \mathfrak{X}$. Then $q$ is a seminorm on $\mathfrak{X}$ and $p$ is continuous with respect to $q$.*

*Proof of the lemma.* Note that $q$ is a sublinear map. Since

$$0 \leq p(x) + p(-x)$$

for $x \in \mathfrak{X}$, we derive that the image of $q$ is $\mathbb{R}_+ \cup \{0\}$. Moreover, $q(x) = q(-x)$ for each $x$ in $\mathfrak{X}$. Therefore, $q$ is a seminorm on $\mathfrak{X}$. Observe that

$$|p(x_1) - p(x_2)| \leq q(x_1 - x_2)$$

and hence $p$ is continuous with respect to the topology induced by $q$ on $\mathfrak{X}$. □

*Proof of the theorem.* By Lemma 8.2.1 we may assume that $\mathfrak{X}$ is a topological vector space over $\mathbb{R}$ and $p$ is a continuous map on $\mathfrak{X}$. Define

$$U = \big\{(x,r) \in \mathfrak{X} \times \mathbb{R}\,\big|\, p(x) < r\big\},\ \mathfrak{Z} = \big\{(x, f(x)) \in \mathfrak{X} \times \mathbb{R}\,\big|\, x \in \mathfrak{U}\big\}$$

It follows that $U$ is a convex open subset of $\mathfrak{X} \times \mathbb{R}$ and $\mathfrak{Z}$ is an $\mathbb{R}$-subspace of $\mathfrak{X} \times \mathbb{R}$ such that $U \cap \mathfrak{Z} = \emptyset$. By Theorem 6.1 there exists a codimension one $\mathbb{R}$-linear subspace $\mathfrak{M}$ of $\mathfrak{X} \times \mathbb{R}$ such that $\mathfrak{Z} \subseteq \mathfrak{M}$ and $U \cap \mathfrak{M} = \emptyset$. Let $\pi$ be the projection $\mathfrak{X} \times \mathbb{R} \to \mathfrak{X}$. It follows from the properties of $\mathfrak{M}$, that $\pi_{|\mathfrak{M}} : \mathfrak{W} \to \mathfrak{X}$ is an $\mathbb{R}$-linear isomorphism. Hence there exists an $\mathbb{R}$-linear map $\tilde{f} : \mathfrak{X} \to \mathbb{R}$ such that

$$\mathfrak{M} = \big\{(x, \tilde{f}(x)) \in \mathfrak{X} \times \mathbb{R}\,\big|\, x \in \mathfrak{X}\big\}$$

Since $\mathfrak{Z} \subseteq \mathfrak{M}$, we deduce that $\tilde{f}_{|\mathfrak{U}} = f$. According to $U \cap \mathfrak{W} = \emptyset$, we have $\tilde{f} \leq p$. This completes the proof. □

## 9. Invariant Hahn-Banach theorem

In this section we prove invariant version of Hahn-Banach theorem. This theorem appears implicitly in [Banach, 1979, Chapitre II, §3] and is related to the previous Banach's manuscript [Banach, 1923] in which the author shows the existence of translation invariant finitely additive extension of Lebesgue measure on real line.

**Definition 9.1.** Let $\mathfrak{X}$ be a vector space over $\mathbb{R}$ and let $\mathcal{G}$ be a semigroup of $\mathbb{R}$-linear endomorphisms of $\mathfrak{X}$. An $\mathbb{R}$-linear subspace $\mathfrak{U}$ is $\mathfrak{G}$-*invariant* if $g\,(\mathfrak{U}) \subseteq \mathfrak{U}$ for every $g \in \mathcal{G}$.

**Example 9.2.** Let $\mathfrak{X}$ be a vector space over $\mathbb{R}$ and let $\mathcal{G}$ be a semigroup of $\mathbb{R}$-linear endomorphisms of $\mathfrak{X}$. Then by obvious reasons $\mathfrak{X}$ is $\mathcal{G}$-invariant subspace.

**Definition 9.3.** Let $\mathfrak{X}$ be a vector space over $\mathbb{R}$, let $\mathcal{G}$ be a semigroup of $\mathbb{R}$-linear endomorphisms of $\mathfrak{X}$ and let $\mathfrak{U}$ be a $\mathcal{G}$-invariant subspace of $\mathfrak{X}$. An $\mathbb{R}$-linear map $f : \mathfrak{U} \to \mathbb{R}$ is $\mathcal{G}$-*invariant* if

$$f\,(g(x)) = f(x)$$

for every $x \in \mathfrak{U}$ and $g \in \mathcal{G}$.

Now we are ready to state the result.

**Theorem 9.4.** *Let $\mathfrak{X}$ be a vector space over $\mathbb{R}$ and let $\mathcal{G}$ be a commutative semigroup of $\mathbb{R}$-linear endomorphisms of $\mathfrak{X}$. Suppose that $p : \mathfrak{X} \to \mathbb{R}$ is a sublinear map such that*

$$p\,(g(x)) \leq p(x)$$

*for every $x \in \mathfrak{X}$ and $g \in \mathcal{G}$. If $\mathfrak{U}$ is a $\mathcal{G}$-invariant subspace of $\mathfrak{X}$ and $f : \mathfrak{U} \to \mathbb{R}$ is an $\mathbb{R}$-linear and $\mathcal{G}$-invariant map such that $f(x) \leq p(x)$ for all $x \in \mathfrak{U}$, then there exists an $\mathbb{R}$-linear and $\mathcal{G}$-invariant map $\tilde{f} : \mathfrak{X} \to \mathbb{R}$ such that $\tilde{f} \leq p$ and $\tilde{f}_{|\mathfrak{U}} = f$.*

The proof presented below is an adaptation of the original Banach's proof from [Banach, 1979, Chapitre II, §3].

*Proof.* For each $x \in \mathfrak{X}$ we define

$$q(x) = \inf\left\{\frac{1}{n}\cdot p\left(\sum_{i=1}^{n} g_i(x)\right)\,\middle|\, \text{for some } n \in \mathbb{N}_+ \text{ and } g_1, ..., g_n \in \mathcal{G}\right\}$$

Fix $n \in \mathbb{N}_+$ and $g_1, ..., g_n \in \mathcal{G}$. Then

$$\frac{1}{n}\cdot p\left(\sum_{i=1}^{n} g_i(x)\right) \geq -\frac{1}{n}\cdot p\left(\sum_{i=1}^{n} -g_i(x)\right) \geq -\frac{1}{n}\cdot\sum_{i=1}^{n} p\,(g_i(-x)) \geq -\frac{1}{n}\cdot\sum_{i=1}^{n} p\,(-x) = -p(-x)$$

and thus $q(x) \in \mathbb{R}$ for every $x \in \mathfrak{X}$. Clearly $q(r \cdot x) = r \cdot q(x)$ for every $x \in \mathfrak{X}$ and $r \in \mathbb{R}_+$. Suppose now that $x_1, x_2 \in \mathfrak{X}$ and $\epsilon > 0$. Then there exist $n, m \in \mathbb{N}_+$ and $g_1, ...g_n, h_1, ..., h_m \in \mathcal{G}$ such that

$$\frac{1}{n} \cdot p\left(\sum_{i=1}^{n} g_i(x_1)\right) \leq q(x_1) + \epsilon,\ \frac{1}{m} \cdot p\left(\sum_{i=1}^{m} h_i(x_2)\right) \leq q(x_2) + \epsilon$$

Then

$$q(x_1 + x_2) \leq \frac{1}{n \cdot m} \cdot p\left(\sum_{i=1}^{n}\sum_{j=1}^{m} (g_i \cdot h_j)(x_1 + x_2)\right) \leq$$
$$\leq \frac{1}{n \cdot m} \cdot p\left(\sum_{i=1}^{n}\sum_{j=1}^{m} (g_i \cdot h_j)(x_1)\right) + \frac{1}{n \cdot m} \cdot p\left(\sum_{i=1}^{n}\sum_{j=1}^{m} (g_i \cdot h_j)(x_2)\right) \leq$$
$$\leq \frac{1}{n \cdot m} \cdot p\left(\sum_{j=1}^{m} h_j\left(\sum_{i=1}^{n} g_i(x_1)\right)\right) + \frac{1}{n \cdot m} \cdot p\left(\sum_{i=1}^{n} g_i\left(\sum_{j=1}^{m} h_j(x_2)\right)\right) \leq$$
$$\leq \frac{1}{n \cdot m} \cdot \sum_{j=1}^{m} p\left(h_j\left(\sum_{i=1}^{n} g_i(x_1)\right)\right) + \frac{1}{n \cdot m} \cdot \sum_{i=1}^{n} p\left(g_i\left(\sum_{j=1}^{m} h_j(x_2)\right)\right) \leq$$
$$\leq \frac{1}{n \cdot m} \cdot \sum_{j=1}^{m} p\left(\sum_{i=1}^{n} g_i(x_1)\right) + \frac{1}{n \cdot m} \cdot \sum_{i=1}^{n} p\left(\sum_{j=1}^{m} h_j(x_2)\right) \leq$$
$$\leq \frac{1}{n} \cdot p\left(\sum_{i=1}^{n} g_i(x_1)\right) + \frac{1}{m} \cdot p\left(\sum_{j=1}^{m} h_j(x_2)\right) \leq q(x_1) + q(x_2) + 2 \cdot \epsilon$$

Note that in order to prove the inequality above we used the fact that $\mathcal{G}$ is commutative. We deduced that

$$q(x_1 + x_2) \leq q(x_1) + q(x_2) + 2 \cdot \epsilon$$

for every $x_1, x_2 \in \mathfrak{X}$ and $\epsilon > 0$. This proves that $q : \mathfrak{X} \to \mathbb{R}$ is a sublinear map. We claim that

$$q(x - g(x)) = q(g(x) - x) = 0$$

for all $x \in \mathfrak{X}$ and for every $g \in \mathcal{G}$. For this note that

$$q(x - g(x)) \leq \frac{1}{n} \cdot p\left(\sum_{i=1}^{n} g^i(x - g(x))\right) = \frac{1}{n} \cdot p\left(g(x) - g^{n+1}(x)\right) \leq$$
$$\leq \frac{1}{n}\left(p(g(x)) + p\left(-g^{n+1}(x)\right)\right) \leq \frac{1}{n}(p(x) + p(-x))$$

for every $n \in \mathbb{N}_+$. Hence

$$q(x - g(x)) \leq 0$$

Similar argument shows that

$$q(g(x) - x) \leq 0$$

for all $x \in \mathfrak{X}$. Since $q$ is sublinear map, we derive

$$0 \leq q(x - g(x)) + q(g(x) - x)$$

and the claim is proved. Observe that

$$f(x) = \frac{1}{n} \cdot \sum_{i=1}^{n} f(g_i(x)) = \frac{1}{n} \cdot f\left(\sum_{i=1}^{n} g_i(x)\right) \leq \frac{1}{n} \cdot p\left(\sum_{i=1}^{n} g_i(x)\right)$$

for every $x \in \mathfrak{U}$, $n \in \mathbb{N}_+$ and $g_1, ..., g_n \in \mathcal{G}$. Hence $f(x) \leq q(x)$ for all $x \in \mathfrak{U}$. Now by Theorem 8.2 there exists an $\mathbb{R}$-linear map $\tilde{f} : \mathfrak{X} \to \mathbb{R}$ such that $\tilde{f} \leq q$ and $\tilde{f}_{|\mathfrak{U}} = f$. Fix $x \in \mathfrak{X}$ and $g \in \mathcal{G}$. Since

$$\tilde{f}(x - g(x)) \leq q(x - g(x)) = 0,\ \tilde{f}(g(x) - x) \leq q(g(x) - x) = 0$$

we derive that

$$\tilde{f}(x) = \tilde{f}(g(x))$$

This shows that $\tilde{f}$ is $\mathcal{G}$-invariant. Note that $q \leq p$ and thus $\tilde{f} \leq p$. Hence $\tilde{f}$ satsifies the assertion. □

## 10. Seminormed and normed spaces

**Definition 10.1.** Let $\mathfrak{X}$ be a vector space over $\mathbb{K}$. Let $\|-\| : \mathfrak{X} \to \mathbb{R}_+ \cup \{0\}$ be a function such that the following assertions hold.

**(1)** $\|0\| = 0$

**(2)** $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$ for every $\alpha \in \mathbb{K}$ and $x \in \mathfrak{X}$.

**(3)** $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ for every $x_1, x_2 \in \mathfrak{X}$.

Then $\|-\|$ is *a seminorm on* $\mathfrak{X}$.

**Definition 10.2.** Let $\mathfrak{X}$ be a vector space over $\mathbb{K}$ and let $\|-\|$ be a seminorm on $\mathfrak{X}$. Suppose that $\|x\| = 0$ if and only if $x = 0$ for every $x \in \mathfrak{X}$. Then $\|-\|$ is *a norm on* $\mathfrak{X}$.

**Definition 10.3.** A vector space $\mathfrak{X}$ over $\mathbb{K}$ together with a seminorm $\|-\|$ on $\mathfrak{X}$ is *a seminormed space over* $\mathbb{K}$. If $\|-\|$ is a norm, then this structure is *a normed space over* $\mathbb{K}$.

**Fact 10.4.** *Let $\mathfrak{X}$ be a seminormed space over $\mathbb{K}$ with respect to $\|-\|$. Consider the function $\rho : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}_+ \cup \{0\}$ given by formula*

$$\rho(x_1, x_2) = \|x_1 - x_2\|$$

*for every $x_1, x_2 \in \mathfrak{X}$. Then $\rho$ is a pseudometric and the topology induced by $\rho$ makes $\mathfrak{X}$ into a topological vector space over $\mathbb{K}$.*

*Moreover, $\rho$ is a metric if and only if $\|-\|$ is a norm.*

*Proof.* Suppose that $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are sequences in $\mathfrak{X}$ which converge in $\rho$ to $x, y \in \mathfrak{X}$, respectively. Then by subadditivity of $\|-\|$ we obtain

$$\rho(x + y, x_n + y_n) \leq \rho(x, x_n) + \rho(y, y_n)$$

for every $n \in \mathbb{N}$. It follows that $\{x_n + y_n\}_{n\in\mathbb{N}}$ is convergent to $x + y \in \mathfrak{X}$ with respect to $\rho$. This proves that $+_{\mathfrak{X}} : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ is continuous with respect to topology induced by $\rho$.

Next suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{X}$ convergent to $x \in \mathfrak{X}$ with respect to $\rho$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{K}$ convergent to $\alpha \in \mathbb{K}$ with respect to $|-|$. Then by positive homogeneity we obtain

$$\rho(\alpha \cdot x, \alpha_n \cdot x_n) = |\alpha| \cdot \rho(x, x_n) + |\alpha - \alpha_n| \cdot \|x_n\|$$

for every $n \in \mathbb{N}$. It follows that $\{\alpha_n \cdot x_n\}_{n\in\mathbb{N}}$ is convergent in $\mathfrak{X}$ to $\alpha \cdot x$ with respect to $\rho$. This proves that $\cdot_{\mathfrak{X}} : \mathbb{K} \times \mathfrak{X} \to \mathfrak{X}$ is continuous.

The last assertion is left for the reader as an exercise. □

**Remark 10.5.** Let $\mathfrak{X}$ be a seminormed space over $\mathbb{K}$ with respect to $\|-\|$. Then $\mathfrak{X}$ is implicitly considered a topological vector space over $\mathbb{K}$ and a pseudometric space with respect to the topology and the pseudometric induced by $\|-\|$.

Often we consider several seminormed spaces at the same time. In these situations by abuse of notation we denote their seminorms by the same symbol – this should not cause confusion, since we can identify a seminorm by the type of the argument to which it is applied.

**Definition 10.6.** For $n \in \mathbb{N}_+$ let $\mathfrak{X}_1, ..., \mathfrak{X}_n$ be seminormed spaces over $\mathbb{K}$. Let $\mathfrak{X}$ be a seminormed space over $\mathbb{K}$ and let $L : \mathfrak{X}_1 \times ... \times \mathfrak{X}_n \to \mathfrak{X}$ be a $\mathbb{K}$-multilinear form. Suppose that there exists $c \in \mathbb{R}_+$ such that

$$\|L(\mathbf{x})\| \leq c \cdot \|\pi_1(\mathbf{x})\| \cdot ... \cdot \|\pi_n(\mathbf{x})\|$$

for every $\mathbf{x} \in \mathfrak{X}_1 \times ... \times \mathfrak{X}_n$. Then $L$ is *a bounded* $\mathbb{K}$*-multilinear form*.

**Theorem 10.7.** *For $n \in \mathbb{N}_+$ let $\mathfrak{X}_1, ..., \mathfrak{X}_n$ be seminormed spaces over $\mathbb{K}$. Let $\mathfrak{X}$ be a seminormed space over $\mathbb{K}$ and let $L : \mathfrak{X}_1 \times ... \times \mathfrak{X}_n \to \mathfrak{X}$ be a $\mathbb{K}$-multilinear form. Then the following assertions are equivalent.*

**(i)** *$L$ is continuous.*

**(ii)** *$L$ is continuous at zero $n$-tuple in $\mathfrak{X}_1 \times ... \times \mathfrak{X}_n$.*

**(iii)** *$L$ is bounded.*

*Proof.* Clearly **(i)** $\Rightarrow$ **(ii)**.

Suppose that $L$ is continuous at zero $n$-tuple in $\mathfrak{X}_1 \times ... \times \mathfrak{X}_n$. Assume that $L$ is not bounded. Then there exists a sequence $\{\mathbf{x}_m\}_{m \in \mathbb{N}}$ of elements in $\mathfrak{X}_1 \times ... \times \mathfrak{X}_n$ such that $\|\pi_i(\mathbf{x}_m)\| = 1$ for each $i$ and

$$\|L(\mathbf{x}_m)\| > m + 1$$

We define $\mathbf{z}_m$ by formula

$$\pi_i(\mathbf{z}_m) = \frac{1}{\sqrt[n]{m+1}} \cdot \pi_i(\mathbf{x}_m)$$

for every $1 \leq i \leq n$ and $m \in \mathbb{N}$. Then $\{\mathbf{z}_m\}_{m \in \mathbb{N}}$ converges to zero tuple in $\mathfrak{X}_1 \times ... \times \mathfrak{X}_n$. It follows that

$$\lim_{m \to +\infty} \left\| \frac{1}{m+1} \cdot L(\mathbf{x}_m) \right\| = \lim_{m \to +\infty} \|L(\mathbf{z}_m)\| = 0$$

On the other hand

$$\left\| \frac{1}{m+1} \cdot L(\mathbf{x}_m) \right\| > 1$$

for every $m \in \mathbb{N}$. This is contradiction. The implication **(ii)** $\Rightarrow$ **(iii)** is proved.

The implication **(iii)** $\Rightarrow$ **(i)** is immediate. □

**Definition 10.8.** For $n \in \mathbb{N}_+$ let $\mathfrak{X}_1, ..., \mathfrak{X}_n$ be seminormed spaces over $\mathbb{K}$. Let $\mathfrak{X}$ be a seminormed space over $\mathbb{K}$ and let $L : \mathfrak{X}_1 \times ... \times \mathfrak{X}_n \to \mathfrak{X}$ be a bounded $\mathbb{K}$-multilinear form. Then

$$\|L\| = \sup \left\{ \|L(\mathbf{x})\| \,\middle|\, \mathbf{x} \in \mathfrak{X}_1 \times ... \times \mathfrak{X}_n \text{ and } \|\pi_i(\mathbf{x})\| = 1 \text{ for all } i \right\}$$

is *the operator seminorm of $L$*.

**Theorem 10.9.** *For $n \in \mathbb{N}_+$ let $\mathfrak{X}_1, ..., \mathfrak{X}_n$ be seminormed spaces over $\mathbb{K}$ and let $\mathfrak{X}$ be a seminormed space over $\mathbb{K}$. Let $L(\mathfrak{X}_1, ..., \mathfrak{X}_n; \mathfrak{X})$ be a $\mathbb{K}$-vector space of bounded $\mathbb{K}$-multilinear forms $\mathfrak{X}_1 \times ... \times \mathfrak{X}_n \to \mathfrak{X}$ with respect to operations defined pointwise. Then*

$$L(\mathfrak{X}_1, ..., \mathfrak{X}_n; \mathfrak{X}) \ni L \mapsto \|L\| \in \mathbb{R}_+ \cup \{0\}$$

*is a seminorm.*

*If $\mathfrak{X}$ is a normed space over $\mathbb{K}$, then $L(\mathfrak{X}_1, ..., \mathfrak{X}_n; \mathfrak{X})$ is normed space over $\mathbb{K}$.*

*If $\mathfrak{X}$ is complete, then $L(\mathfrak{X}_1, ..., \mathfrak{X}_n; \mathfrak{X})$ is complete.*

*Proof.* Clearly the operator seminorm is zero for zero $\mathbb{K}$-multilinear map. Suppose that $L_1, L_2 \in L(\mathfrak{X}_1, ..., \mathfrak{X}_n; \mathfrak{X})$. Fix $\mathbf{x} \in \mathfrak{X}_1 \times ... \times \mathfrak{X}_n$ such that $\|\pi_i(\mathbf{x})\| = 1$ for each $i$. Then

$$\|(L_1 + L_2)(\mathbf{x})\| \leq \|L_1(\mathbf{x})\| + \|L_2(\mathbf{x})\| \leq \|L_1\| + \|L_2\|$$

and, since $\mathbf{x}$ is arbitrary, we derive that

$$\|L_1 + L_2\| \leq \|L_1\| + \|L_2\|$$

Next suppose that $L \in L(\mathfrak{X}_1, ..., \mathfrak{X}_n; \mathfrak{X})$ and $\alpha \in \mathbb{K}$. Again fix $\mathbf{x} \in \mathfrak{X}_1 \times ... \times \mathfrak{X}_n$ such that $\|\pi_i(\mathbf{x})\| = 1$ for each $i$. Then
$$\|(\alpha \cdot L)(\mathbf{x})\| = |\alpha| \cdot \|L(\mathbf{x})\| \leq |\alpha| \cdot \|L\|$$
and, since $\mathbf{x}$ is arbitrary, we derive that
$$\|\alpha \cdot L\| \leq |\alpha| \cdot \|L\|$$
This completes the proof that $L(\mathfrak{X}_1, ..., \mathfrak{X}_n; \mathfrak{X})$ is a seminormed space over $\mathbb{K}$.

Suppose that $\mathfrak{X}$ is normed and $L \in L(\mathfrak{X}_1, ..., \mathfrak{X}_n; \mathfrak{X})$ satisfies $\|L\| = 0$. It follows that $L(\mathbf{x}) = 0$ for every $\mathbf{x} \in \mathfrak{X}_1 \times ... \times \mathfrak{X}_n$ such that $\|\pi_i(\mathbf{x})\| = 1$ for each $i$. This implies that $L = 0$ and hence $L(\mathfrak{X}_1, ..., \mathfrak{X}_n; \mathfrak{X})$ is normed space over $\mathbb{K}$.

Consider a Cauchy sequence $\{L_m\}_{m\in\mathbb{N}}$ with respect to operator seminorm. Fix $\mathbf{x} \in \mathfrak{X}_1 \times ... \times \mathfrak{X}_n$. Then
$$\|(L_m - L_k)(\mathbf{x})\| \leq \|L_m - L_k\| \cdot \|\pi_1(\mathbf{x})\| \cdot ... \cdot \|\pi_n(\mathbf{x})\|$$
for every $m, k \in \mathbb{N}$. This implies that $\{L_m(\mathbf{x})\}_{m\in\mathbb{N}}$ is a Cauchy's sequence in $\mathfrak{X}$. Since $\mathfrak{X}$ is complete, we derive by Theorem 4.4 that this sequence is convergent. We define
$$L(\mathbf{x}) = \lim_{m\to+\infty} L_m(\mathbf{x})$$
for every $\mathbf{x} \in \mathfrak{X}_1 \times ... \times \mathfrak{X}_n$. We also have
$$\big|\|L_m\| - \|L_k\|\big| \leq \|L_m - L_k\|$$
for every $m, k \in \mathbb{N}$. Thus $\{\|L_m\|\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and hence it is convergent in $\mathbb{R}$. Note that we have
$$\|L(\mathbf{x})\| = \lim_{m\to+\infty}\|L_m(\mathbf{x})\| \leq \left(\lim_{m\to+\infty}\|L_m\|\right) \cdot \|\pi_1(\mathbf{x})\| \cdot ... \cdot \|\pi_n(\mathbf{x})\|$$
Therefore, $L : \mathfrak{X}_1 \times ... \times \mathfrak{X}_n \to \mathfrak{X}$ is a bounded $\mathbb{K}$-multilinear form. We claim that $L$ is the limit of $\{L_m\}_{m\in\mathbb{N}}$ with respect to the operator seminorm. For the proof fix $\mathbf{x} \in \mathfrak{X}_1 \times ... \times \mathfrak{X}_n$ such that $\|\pi_1(\mathbf{x})\| = ... = \|\pi_n(\mathbf{x})\| = 1$. Then
$$\|(L - L_m)(\mathbf{x})\| \leq \|L(\mathbf{x}) - L_k(\mathbf{x})\| + \|L_k - L_m\|$$
Thus we have
$$\|(L - L_m)(\mathbf{x})\| \leq \limsup_{k\to+\infty}\|L_k - L_m\|$$
Since the left hand side does not depend on $\mathbf{x}$, we deduce that
$$\|L - L_m\| \leq \limsup_{k\to+\infty}\|L_k - L_m\|$$
Invoking once again the assumption that $\{L_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence we infer
$$\lim_{m\to+\infty}\|L - L_m\| \leq \lim_{m\to+\infty}\limsup_{k\to+\infty}\|L_k - L_m\| = 0$$
This completes the proof. □

**Definition 10.10.** Let $\mathfrak{X}$ be a normed space over $\mathbb{K}$ with respect to a norm $\|-\|$. If $\mathfrak{X}$ is a complete topological vector space over $\mathbb{K}$, then $\mathfrak{X}$ is *a Banach space over* $\mathbb{K}$.

**Theorem 10.11.** *Let $\mathfrak{X}$ be a seminormed space over $\mathbb{K}$ with respect to a seminorm $\|-\|$. Let $\mathfrak{U}$ be a $\mathbb{K}$-linear subspace. Consider the quotient map $q : \mathfrak{X} \twoheadrightarrow \mathfrak{X}/\mathfrak{U}$ in the category of vector spaces over $\mathbb{K}$ and define the map $\|-\|_{\mathfrak{X}/\mathfrak{U}} : \mathfrak{X}/\mathfrak{U} \to \mathbb{R}_+ \cup \{0\}$ by formula*
$$\|x + \mathfrak{U}\|_{\mathfrak{X}/\mathfrak{U}} = \inf_{u\in\mathfrak{U}}\|x + u\|$$
*Then the following assertions hold.*

**(1)** *$\mathfrak{X}/\mathfrak{U}$ is a seminormed space over $\mathbb{K}$ with respect to $\|-\|_{\mathfrak{X}/\mathfrak{U}}$.*

**(2)** *$q : \mathfrak{X} \twoheadrightarrow \mathfrak{X}/\mathfrak{U}$ is an open map of topological spaces.*

**(3)** *$\mathfrak{U}$ is closed in $\mathfrak{X}$ if and only if $\|-\|_{fX/\mathfrak{U}}$ is a norm.*

*Proof.* □

## References

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