

CONSTRUCTIBLE AND LOCALLY CONSTRUCTIBLE SETS

1. CONSTRUCTIBLE SETS

Definition 1.1. Let X be a topological space. Suppose that Z is a subset of X such that the inclusion $Z \hookrightarrow X$ is quasi-compact. Then we say that Z is *retro-compact*.

Definition 1.2. Let X be a topological space. We define *constructible subsets* of X by the following induction.

- (1) Each retro-compact open subset of X is constructible.
- (2) If E is constructible subset of X , then $X \setminus E$ is constructible.
- (3) If E_1, E_2, \dots, E_n are constructible subsets of X , then

$$\bigcup_{i=1}^n E_i$$

is constructible.

Rephrasing the definition above one can say that constructible subsets of a topological space X form an algebra of sets generated by retro-compact open subsets.

Fact 1.3. Let $f : X \rightarrow Y$ be a morphism of schemes and E be a constructible subset of Y . Then $f^{-1}(E)$ is a constructible subset of X .

Proof. We set

$$\mathcal{F} = \{E \subseteq Y \mid f^{-1}(E) \text{ is constructible}\}$$

Obviously \mathcal{F} is an algebra of subsets of Y . By the base change for quasi-compact morphisms, we derive that \mathcal{F} contains all retro-compact open subsets of Y . This implies that \mathcal{F} contains all constructible subsets of Y . \square

Now we characterize constructible subsets of affine schemes.

Proposition 1.4. Let A be a ring and E be a subset of $\text{Spec } A$. Then the following are equivalent.

- (i) E is a constructible subset of $\text{Spec } A$.
- (ii) There exists elements a_1, \dots, a_n and finitely generated ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ such that

$$E = \bigcup_{i=1}^n D(a_i) \cap V(\mathfrak{a}_i)$$

Proof of the lemma. Consider the family

$$\mathcal{F} = \left\{ \bigcup_{i=1}^n D(a_i) \cap V(\mathfrak{a}_i) \mid a_1, \dots, a_n \in A \text{ and } \mathfrak{a}_1, \dots, \mathfrak{a}_n \text{ are finitely generated ideals of } A \right\}$$

Since every retro-compact open subset of $\text{Spec } A$ is quasi-compact, it belongs to \mathcal{F} because it is a finite union of distinguished open subsets. Moreover, subsets in \mathcal{F} are closed under complements and finite unions. Therefore, \mathcal{F} contains all constructible subsets of $\text{Spec } A$. On the other hand each element of \mathcal{F} is constructible in $\text{Spec } A$. \square

Corollary 1.5. Let X be a quasi-compact and quasi-separated scheme and let E be a constructible subset of X . Then there exists an affine scheme Z together with a morphism $f : Z \rightarrow X$ of finite presentation such that $E = f(Z)$.

Proof. Since X is quasi-compact, there exists an open cover

$$X = \bigcup_{j=1}^m U_j$$

by open affines. Each $E \cap U_j$ is constructible in U_j . Write $U_j = \text{Spec } A_j$ for $1 \leq j \leq m$. Fix j . By Proposition 1.4 there exists $a_{ji} \in A$ and finitely generated ideals $\mathfrak{a}_{ji} \subseteq A_j$ for $1 \leq i \leq n_j$ such that

$$U_j \cap E = \bigcup_{i=1}^{n_j} D(a_{ji}) \cap V(\mathfrak{a}_{ji})$$

Consider a scheme $Z_j = \coprod_{i=1}^{n_j} \text{Spec } (A_j/\mathfrak{a}_{ji})_{a_{ji}}$ together with a canonical morphism $f_j : Z_j \rightarrow U_j$. Next let Z be an affine scheme $\coprod_{j=1}^m Z_j$ with a morphism $f : Z \rightarrow X$ such that $f|_{Z_j}$ is defined as f_j composed with the inclusion $U_j \hookrightarrow X$ for every $1 \leq j \leq m$. Then f is a finitely presented morphism (this uses the fact that X is quasi-separated) and $E = f(Z)$. \square

Finally we discuss constructibility for noetherian and locally noetherian topological spaces.

Fact 1.6. *Let X be a locally noetherian topological space. Then the algebra of constructible sets of X is generated by open subsets of X .*

Proof. Every open subset of a locally noetherian topological space is retro-compact. \square

Proposition 1.7. *Let X be a noetherian topological space. Suppose that E is a subset of X such that for every irreducible closed subset F of X either $E \cap F$ contains open nonempty subset of F or $E \cap F = \emptyset$. Then E is constructible.*

Proof. Note that by Fact 1.6 every closed subset of X is constructible. Assume that E is not constructible. We set

$$\mathcal{F} = \{F \subseteq X \mid F \text{ is closed subset of } X \text{ and } E \cap F \text{ is not constructible in } X\}$$

First note that $X \in \mathcal{F}$. Since X is noetherian, there exists the minimal (with respect to inclusion) subset F in \mathcal{F} . If F is not irreducible, then $F = F' \cup F''$ for some nonempty closed proper subsets F', F'' of F . Since F is minimal in \mathcal{F} , we deduce that both $E \cap F'$ and $E \cap F''$ are constructible and hence $E \cap F = (E \cap F') \cup (E \cap F'')$ is constructible. This is a contradiction. Hence F must be irreducible. Since $E \cap F$ is not constructible, it is nonempty. Hence there exists nonempty subset $U \subseteq E \cap F$ open in F . According to $F \setminus U \subset F$ we infer that $E \cap (F \setminus U)$ is constructible. Thus

$$E \cap F = U \cup (E \cap (F \setminus U))$$

is constructible as a union of constructible sets. This is a contradiction. Therefore, E is constructible. \square

2. NOETHER NORMALIZATION LEMMA

In this section we prove important theorem on the structure of commutative and finitely generated k -algebras.

Theorem 2.1 (Noether normalization lemma). *Let k be a field and A be a finitely generated k -algebra. Then there exist elements z_1, \dots, z_n in A algebraically independent over k such that*

$$k[z_1, \dots, z_n] \subseteq A$$

is a finite extension of rings.

Proof. Let \mathcal{A} be a family of finitely generated k -subalgebras of A such that for every $B \in \mathcal{A}$ extension $B \subseteq A$ is finite. Clearly $A \in \mathcal{A}$ so \mathcal{A} is nonempty. Now suppose that $n \in \mathbb{N}$ is a minimal number of k -algebra generators of any element in \mathcal{A} . Then there exist $z_1, \dots, z_n \in A$ such that $k[z_1, \dots, z_n] \subseteq A$ is finite. We show now that z_1, \dots, z_n are algebraically independent over k . Let $k[x_1, \dots, x_n]$ be a polynomial k -algebra and assume that there exists nonzero $f \in k[x_1, \dots, x_n]$ such that $f(z_1, \dots, z_n) = 0$. Write

$$f(x_1, \dots, x_n) = \sum_{(d_1, \dots, d_n) \in F} a_{d_1, \dots, d_n} \cdot x_1^{d_1} \cdot \dots \cdot x_n^{d_n}$$

where $F \subseteq \mathbb{N}^n$ is a finite subset and $a_{d_1, \dots, d_n} \in k$ are nonzero. Since f is nonzero, we derive that F is nonempty. Define

$$m = 1 + \max_{(d_1, \dots, d_n) \in F} \max_{1 \leq i \leq n} d_i$$

Next define $g \in k[z_2, \dots, z_n][x]$ by formula

$$g(x) = f(x, z_2 - z_1^m + x^m, z_3 - z_1^{m^2} + x^{m^2}, \dots, z_n - z_1^{m^{n-1}} + x^{m^{n-1}})$$

Now we prove that g is a monic polynomial of variable x . Let \leq be the lexicographical order on \mathbb{N}^n that is

$$(d_1, \dots, d_n) \leq (e_1, \dots, e_n) \text{ if } d_i \leq e_i \text{ for } i = \max \{j \mid 1 \leq j \leq n \text{ and } d_j \neq e_j\}$$

Since $F \subseteq \mathbb{N}^n$ is finite, there exists (M_1, \dots, M_n) in F that is the greatest with respect to lexicographical order \leq restricted to F . This implies that

$$d_1 + d_2 \cdot m + d_3 \cdot m^2 + \dots + d_n \cdot m^{n-1} < M_1 + M_2 \cdot m + M_3 \cdot m^2 + \dots + M_n \cdot m^{n-1}$$

for every $(d_1, \dots, d_n) \in F$. This fact and a precise investigation of how coefficients of powers of x in g are calculated show that g is monic. Note also that $g(z_1) = f(z_1, z_2, \dots, z_n) = 0$. This implies that z_1 is integral over $k[z_2, \dots, z_n]$ and hence $k[z_2, \dots, z_n] \subseteq A$ is a finite extension of rings. This proves that $k[z_2, \dots, z_n] \in \mathcal{A}$ and contradicts the definition of n . Therefore, such f does not exist and this proves that z_1, \dots, z_n are algebraically independent over k . \square

3. LOCALLY CONSTRUCTIBLE SETS AND CHEVALLEY'S THEOREM

Definition 3.1. Let X be a topological space. A subset E of X is called *locally constructible* in X if for every point x in X there exists an open neighbourhood U of x in X such that $E \cap U$ is constructible in U .

Theorem 3.2. Let X be a scheme and E be a subset of X . Then the following are equivalent.

- (i) E is locally constructible.
- (ii) $E \cap U$ is constructible in U for every open quasi-compact and quasi-separated subset U of X .
- (iii) $E \cap U$ is constructible in U for every affine open subset U of X .

The proof is based on the following result.

Lemma 3.2.1. Let U be a quasi-separated scheme and W be its open affine subset. Then every constructible subset E of W is constructible in U .

Proof of the lemma. For every $f \in \Gamma(W, \mathcal{O}_U)$ nonvanishing set W_f of f in W is affine. Since U is quasi-separated, we derive that W_f is retro-compact in U and hence constructible. Suppose now that $\mathcal{I} \subseteq \Gamma(W, \mathcal{O}_U)$ is an ideal generated by $f_1, \dots, f_n \in \Gamma(W, \mathcal{O}_U)$ and $V(\mathcal{I}) \subseteq W$ is a vanishing set of this ideal in W . Then

$$V(\mathcal{I}) = \left(U \setminus \bigcup_{i=1}^n W_{f_i} \right) \setminus (U \setminus W)$$

Since U, W_{f_i} for $1 \leq i \leq n$ and W are constructible in U , we derive that $V(\mathcal{I})$ is constructible in U . Since constructible sets of U form an algebra of sets, the assertion follows from Proposition

1.4. \square

Proof of the theorem. Suppose that E is a locally constructible subset of X and U is an open quasi-compact and quasi-separated subset of X . There exists a finite open cover $U = \bigcup_{j=1}^m W_j$ such that each W_j is affine and $E \cap W_j$ is constructible in W_j . According to Lemma 3.2.1 we deduce that each $E \cap W_j$ is constructible in U . Hence

$$E \cap U = \bigcup_{j=1}^m (E \cap W_j)$$

is constructible in U . This proves that (i) \Rightarrow (ii).

Implications (ii) \Rightarrow (iii) and (iii) \Rightarrow (i) follow from definition. \square

Theorem 3.3 (Chevalley's theorem on images). *Let $f : X \rightarrow Y$ be a morphism of schemes of finite presentation and E be a locally constructible subset of X . Then $f(E)$ is locally constructible in Y .*

We start by a sequence of reductions. Since the question is local on Y , one can assume that Y is affine. Then X is quasi-compact and quasi-separated. By Theorem 3.2 we deduce that E is constructible on X . Next by Corollary 1.5 we may assume that $E = X$ and X is affine. Now since f is of finite presentation, there exists a cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\quad} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{\quad g \quad} & Y' \end{array}$$

with Y' the spectrum of a finitely generated \mathbb{Z} -algebra, f' is of finite type and affine X' . We have

$$f(X) = g^{-1}(f'(X'))$$

Since a preimage of a constructible subset is constructible by Fact 1.3, it suffices to prove that $f'(X')$ is constructible. Hence we may assume that the base is noetherian. Thus our goal is to prove that $f(X)$ is constructible in Y under assumptions that Y is a noetherian affine scheme and f is of finite type. For the proof of this statement we need the following interesting application of Theorem 2.1

Lemma 3.3.1. *Let A be a domain and $f : A \rightarrow B$ be an injective morphism of finite type. Then there exists nonzero $s \in A$ such that the image of $\text{Spec } f : \text{Spec } B \rightarrow \text{Spec } A$ contains the distinguished set $D(s)$ of $\text{Spec } A$.*

Proof of the lemma. Let $S = A \setminus \{0\}$. Then $K = S^{-1}A$ is a field of fractions of A and $S^{-1}B$ is a finitely generated K -algebra. By Theorem 2.1 we derive that there exists $\frac{b_1}{s_1}, \dots, \frac{b_n}{s_n} \in S^{-1}B$ algebraically independent over K such that

$$K \left[\frac{b_1}{s_1}, \dots, \frac{b_n}{s_n} \right] \subseteq S^{-1}B$$

is a finite extension of rings. Here $b_1, \dots, b_n \in B$ and $s_1, \dots, s_n \in S$. It follows that

$$K[b_1, \dots, b_n] \subseteq S^{-1}B$$

is a finite extension of rings and b_1, \dots, b_n are algebraically independent over K . There exists a finite set c_1, \dots, c_m that generates B as an $A[b_1, \dots, b_n]$ -algebra and all these elements are integral over $K[b_1, \dots, b_n]$. This implies that for every $1 \leq i \leq m$ there exists a monic polynomial $f_i \in K[b_1, \dots, b_n][x]$ such that $f_i(c_i) = 0$. Now there are finitely many coefficients of each f_i and each of them is some algebraic expression in b_1, \dots, b_n having coefficients in $K = S^{-1}A$. This implies that there exists nonzero $s \in A$ such that f_i is a monic polynomial in $A_s[b_1, \dots, b_n][x]$ for every $1 \leq i \leq m$. Hence the extension

$$A_s[b_1, \dots, b_n] \subseteq B_s$$

is finite. We also know that b_1, \dots, b_n are algebraically independent over K . Thus $A_s \subseteq B_s$ can be decomposed as a polynomial extension followed by a finite extension

$$A_s \subseteq A_s[b_1, \dots, b_n] \subseteq B_s$$

Both polynomial extension and finite extension induce surjective morphism on prime spectra. Thus the morphism $\text{Spec } B_s \rightarrow \text{Spec } A_s$ induced by $\text{Spec } f$ is surjective. Hence $D(s) \subseteq \text{Spec } A$ is in the image of $\text{Spec } f$. \square

Proof of the theorem. Let $f : X \rightarrow Y$ be a finite type morphism with Y affine and noetherian. As we explained above it suffices to prove that $f(X)$ is constructible. Suppose that F is an irreducible closed subset of Y . We consider it as a subscheme of Y with integral structure. By Lemma 3.3.1 we deduce that either the image of a morphism $f^{-1}(F) \rightarrow F$ induced by f contains nonempty open subset of F or this image is empty. Thus for every irreducible closed subset F of Y either $f(X) \cap F$ contains nonempty open subset of F or $f(X) \cap F = \emptyset$. By Proposition 1.7 we derive that $f(X)$ is constructible in Y . \square

Corollary 3.4 (Characterization of locally constructible sets on qcqs schemes). *Let X be a quasi-compact and quasi-separated scheme. Then the following are equivalent.*

- (i) E is locally constructible.
- (ii) E is constructible.
- (iii) There exists an affine scheme Z and a morphism $f : Z \rightarrow X$ of finite presentation such that $E = f(Z)$.

Proof. The implication (i) \Rightarrow (ii) follows from Theorem 3.2. The assertion (ii) \Rightarrow (iii) is a consequence of Corollary 1.5 and (iii) \Rightarrow (i) follows from Theorem 3.3. \square

Next result is simple but worth noted.

Fact 3.5. *Let $f : X \rightarrow Y$ be a morphism of schemes and E be a locally constructible subset of Y . Then $f^{-1}(E)$ is a locally constructible subset of X .*

Proof. This is an immediate consequence of Fact 1.3 and the definition of locally constructible sets. \square

REFERENCES