QUOTIENTS OF ALGEBRAIC GROUPS

1. Introduction

Throughout this notes k denote a field and $\mathbf{G}$ denote a group scheme over k. We denote by e the identity of $\mathbf{G}$. We also fix a k-scheme X equipped with an action of $\mathbf{G}$ determined by morphism $a : \mathbf{G} \times_k X \to X$.

2. Basic properties of scheme group quotients

The following result gives scheme-theoretic criterion for topological quotient in the case of group scheme actions.

Proposition 2.1. *Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Assume that q is submersive and and the morphism $\mathbf{G} \times_k X \to X \times_Y X$ induced by a and pr_X is surjective. Then the diagram*

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel in the category topological spaces.

Proof. Let π_1 and π_2 be distinct projections $X \times_Y X \to X$. Pick points x_1 and x_2 in X such that $q(x_1) = q(x_2)$. Then there exists a field extension K over k such that $k(x_1) \subseteq K$ and $k(x_2) \subseteq K$. These give rise to K-points $\overline{x_1}$ and $\overline{x_2}$ of X such that their images under q is the same K-point of Y. Since we have an identification

$$\big(X \times_Y X\big)(K) = X(K) \times_{Y(K)} X(K)$$

induced by π_1 and π_2, we derive that there exists a K-point $\overline{z}$ of $X \times_Y X$ such that $\pi_1(\overline{z}) = \overline{x_1}$ and $\pi_2(\overline{z}) = \overline{x_2}$. Let z be the point of $X \times_Y X$ corresponding to $\overline{z}$. Then $\pi_1(z) = x_1$ and $\pi_2(z) = x_2$. By assumption a and pr_X induce surjection $\mathbf{G} \times_k X \twoheadrightarrow X \times_Y X$. Thus there exists a point u of $\mathbf{G} \times_k X$ such that $a(u) = x_1$ and $\mathrm{pr}_X(u) = x_2$. Thus x_1 and x_2 are identified by an equivalence relation on the underlying set of X which is determined by the pair (a, pr_X). Therefore, fibers of q are equivalence classes with respect to this relation. Since q is submersive, this implies that the diagram

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel in the category topological spaces. $\square$

Now we prove a series results concerning fpqc descent. For this we fix a k-scheme Y with the trivial action of $\mathbf{G}$ and a $\mathbf{G}$-equivariant morphism $q : X \to Y$. Let $g : Y' \to Y$ be a morphism of k-schemes and consider the cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

of k-schemes. Note that X' admits a unique action a' of $\mathbf{G}$ such that the square above consists of $\mathbf{G}$-equivariant morphism (we consider g as a $\mathbf{G}$-equivariant morphism between trivial $\mathbf{G}$-schemes).

Fact 2.2. *Suppose that g is faithfully flat and quasi-compact. Assume that q' is (universally) submersive. Then q is (universally) submersive.*

Proof. It suffices to prove that submersive morphisms have descent property. This follows from the fact that g (as faithfully flat and quasi-compact morphism) and q' are submersive. Details are left for the reader. $\square$

Fact 2.3. *Suppose that g is faithfully flat and quasi-compact. Then the canonical morphism $X' \times_{Y'} X' \to X \times_Y X$ is faithfully flat and quasi-compact and there is the cartesian square*

$$\begin{tikzcd} \mathbf{G} \times_k X' \arrow[r] \arrow[d] & \mathbf{G} \times_k X \arrow[d] \\ X' \times_{Y'} X' \arrow[r] & X \times_Y X \end{tikzcd}$$

in which the left vertical arrow is induced by $\langle a', \mathrm{pr}_{X'} \rangle : \mathbf{G} \times_k X' \to X' \times_k X'$, the right vertical arrow is induced by $\langle a, \mathrm{pr}_X \rangle : \mathbf{G} \times_k X \to X \times_k X$ and the bottom horizontal morphism is the canonical morphism.

Proof. Note that squares

$$\begin{tikzcd} X' \times_{Y'} X' \arrow[r] \arrow[d] & X' \times_Y X' \arrow[d] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X' \times_Y X' \arrow[r] \arrow[d] & X \times_Y X \arrow[d] \\ X' \times_k X' \arrow[r, "g' \times_k g'"'] & X \times_k X \end{tikzcd}$$

are cartesian. Since both g and $g' \times_k g'$ are faithfully flat and quasi-compact, we derive that both morphisms $X' \times_{Y'} X' \to X' \times_Y X'$ and $X' \times_Y X' \to X \times_Y X$ are faithfully flat and quasi-compact. Then their composition i.e. the canonical morphism $X' \times_{Y'} X' \to X \times_Y X$ is faithfully flat and quasi-compact. $\square$

Finally we need the following notion

Definition 2.4. Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Consider a pair

$$q_* \mathcal{O}_X \underset{q_* \mathrm{pr}_X^\#}{\overset{q_* a^\#}{\rightrightarrows}} q_* \left(\mathrm{pr}_X\right)_* \mathcal{O}_{\mathbf{G} \times_k X} = q_* a_* \mathcal{O}_{\mathbf{G} \times_k X}$$

of morphisms of sheaves of rings on Y. Suppose that $q^\# : \mathcal{O}_Y \to q_* \mathcal{O}_X$ is a kernel of this pair. Then *$\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q.*

Proposition 2.5. *Suppose that g is faitfully flat and quasi-compact. Assume that q' is quasi-compact, semiseparated and $\mathcal{O}_{Y'}$ is the sheaf of $\mathbf{G}$-invariants for q'. Then $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q.*

Proof. We denote by a' the action of $\mathbf{G}$ on X'. First note that q is semiseparated and quasi-compact morphism as these classes of morphisms admit descent along quasi-compact and faithfully flat morphisms. Since q is quasi-compact, semiseparated and g is flat, we derive that for every quasi-coherent sheaf $\mathcal{F}$ on X the canonical morphism $q'_*g'^*\mathcal{F} \to g^*q_*\mathcal{F}$ is an isomorphism. Thus the diagram

$$\mathcal{O}_{Y'} \xrightarrow{q^\#} q'_*\mathcal{O}_{X'} \underset{q'_*\mathrm{pr}_{X'}^\#}{\overset{q'_*a'^\#}{\rightrightarrows}} q'_*\left(\mathrm{pr}_{X'}\right)_*\mathcal{O}_{\mathbf{G}\times_k X'} = q'_*a'_*\mathcal{O}_{\mathbf{G}\times_k X'}$$

is isomorphic to the diagram

$$g^*\mathcal{O}_Y \xrightarrow{g^*q^\#} g^*\left(q_*\mathcal{O}_X\right) \underset{g^*q_*\mathrm{pr}_X^\#}{\overset{g^*q_*a^\#}{\rightrightarrows}} g^*\left(q_*\left(\mathrm{pr}_X\right)_*\mathcal{O}_{\mathbf{G}\times_k X}\right) = g^*\left(q_*a_*\mathcal{O}_{\mathbf{G}\times_k X}\right)$$

Since $\mathcal{O}_{Y'}$ is the sheaf of $\mathbf{G}$-invariants for q', the first diagram is a kernel diagram. Hence the second is a kernel diagram. According to the fact that g is faithfully flat we deduce that the diagram

$$\mathcal{O}_Y \xrightarrow{q^\#} q_*\mathcal{O}_X \underset{q_*\mathrm{pr}_X^\#}{\overset{q_*a^\#}{\rightrightarrows}} q_*\left(\mathrm{pr}_X\right)_*\mathcal{O}_{\mathbf{G}\times_k X} = q_*a_*\mathcal{O}_{\mathbf{G}\times_k X}$$

is also a kernel diagram. Thus $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q. □

3. Categorical and geometric quotients

Definition 3.1. Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Suppose that the following conditions hold.

(1) q is submersive.

(2) The morphism $\mathbf{G} \times_k X \to X \times_Y X$ induced by $\langle a, \mathrm{pr}_x\rangle : \mathbf{G} \times_k X \to X \times_k X$ is surjective.

(3) $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariant for q.

Then q is *a geometric quotient of* X.

Corollary 3.2. *Let q be a geometric quotient of X. Then the diagram*

$$\mathbf{G}\times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel in the category of ringed spaces.

Proof. Due to the fact that $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q it suffices to prove that

$$\mathbf{G}\times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

is the cokernel in the category of topological spaces. This follows from Proposition 2.1. □

Definition 3.3. Let $q : X \to Y$ be a morphism of k-schemes such that the diagram

$$\mathbf{G}\times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel in the category of k-schemes. Then $q : X \to Y$ is *a categorical quotient of* X.

Fact 3.4. *Every geometric quotient is categorical.*

Proof. Categorical quotient is a cokernel in the category of k-schemes. On the other hand geometric quotient is a cokernel in the category of locally ringed spaces and hence it also satisfies cokernel property in its full subcategory of k-schemes. Thus every geometric quotient is categorical. □

Let $q : X \to Y$ be a morphism of k-schemes such that $q \cdot \mathrm{pr}_X = q \cdot a$. For a morphism $g : Y' \to Y$ of k-schemes consider the cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Then there exists a unique action $a' : \mathbf{G} \times_k X' \to X'$ of $\mathbf{G}$ on X' such that the square above consists of $\mathbf{G}$-equivariant morphism (we consider Y, Y' as $\mathbf{G}$-schemes equipped with trivial $\mathbf{G}$-actions). Keeping this in mind we have the following.

Definition 3.5. A morphism $q : X \to Y$ is *a uniform categorical (geometric) quotient of* X if for every flat morphism $g : Y' \to Y$ its base change $q' : X' \to Y'$ is a categorical (geometric) quotient of X'.

Definition 3.6. A morphism $q : X \to Y$ is *a universal categorical (geometric) quotient of* X if for every morphism $g : Y' \to Y$ its base change $q' : X' \to Y'$ is a categorical (geometric) quotient of X'.

Corollary 3.7. *Let $g : Y' \to Y$ be a faithfully flat and quasi-compact morphism. Suppose that q' is a geometric quotient, then q is a geometric quotient.*

Proof. This follows from Facts 2.2, 2.3 and Proposition 2.5. □

In the next result we give a simple example of a universal geometric quotient.

Proposition 3.8. *Suppose that $\mathbf{G}$ is a quasi-compact group scheme over k. Let Y be a k-scheme and consider $\mathbf{G} \times_k Y$ with the action of $\mathbf{G}$ induced by the regular action on the left factor. Then $\mathrm{pr}_Y : \mathbf{G} \times_k Y \to Y$ is a universal geometric quotient.*

Proof. Clearly pr_Y is univerally submersive (it is even universally open). Let $\mu : \mathbf{G} \times_k \mathbf{G} \to \mathbf{G}$ be the multiplication morphism and let $\pi_{23} : \mathbf{G} \times_k \mathbf{G} \times Y \to \mathbf{G} \times_k Y$ be the projection on the last two factors. Then the morphism

$$\mathbf{G} \times_k \mathbf{G} \times_k Y \to (\mathbf{G} \times_k Y) \times_Y (\mathbf{G} \times_k Y) = \mathbf{G} \times_k \mathbf{G} \times_k Y$$

induced by $\langle \mu \times_k 1_Y, \pi_{23} \rangle : \mathbf{G} \times_k \mathbf{G} \times_k Y \to (\mathbf{G} \times_k Y) \times_k (\mathbf{G} \times_k Y)$ is an isomorphism. We show that $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for pr_Y. For this pick an affine open subset V of Y. It suffices to check that the diagram

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{\mathrm{pr}_Y^{\#}} \Gamma\left(\mathbf{G} \times_k V, \mathcal{O}_{\mathbf{G} \times_k Y}\right) \underset{\pi_{23}^{\#}}{\overset{(\mu \times_k 1_Y)^{\#}}{\rightrightarrows}} \Gamma\left(\mathbf{G} \times_k \mathbf{G} \times_k V, \mathcal{O}_{\mathbf{G} \times_k \mathbf{G} \times_k Y}\right)$$

is a kernel. Since $\mathbf{G}$ is quasi-compact and separated (every group k-scheme is separated), we derive that the diagram above is isomorphic with

$$\Gamma(V,\mathcal{O}_Y) \xrightarrow{f\mapsto 1\otimes f} \Gamma(\mathbf{G},\mathcal{O}_{\mathbf{G}})\otimes_k \Gamma(V,\mathcal{O}_Y) \underset{\chi\otimes f\mapsto 1\otimes\chi\otimes f}{\overset{\chi\otimes f\mapsto \mu^{\#}(\chi)\otimes f}{\rightrightarrows}} \Gamma(\mathbf{G},\mathcal{O}_{\mathbf{G}})\otimes_k \Gamma(\mathbf{G},\mathcal{O}_{\mathbf{G}})\otimes_k \Gamma(V,\mathcal{O}_Y)$$

Thus the first diagram is the kernel diagram if $f\mapsto 1\otimes f$ induces an isomorphism of $\Gamma(V,\mathcal{O}_Y)$ with subspace of $\Gamma(\mathbf{G},\mathcal{O}_{\mathbf{G}})\otimes_k \Gamma(V,\mathcal{O}_Y)$ given by formula

$$\left\{\chi\in\Gamma(\mathbf{G},\mathcal{O}_{\mathbf{G}})\,\middle|\,\mu^{\#}(\chi)=1\otimes\chi\right\}\otimes_k\Gamma(V,\mathcal{O}_Y)$$

Hence it suffices to prove that

$$\left\{\chi\in\Gamma(\mathbf{G},\mathcal{O}_{\mathbf{G}})\,\middle|\,\mu^{\#}(\chi)=1\otimes\chi\right\} = \text{constant functions on } \mathbf{G}$$

For this pick a k-algebra A and let $g:\operatorname{Spec} A\to\mathbf{G}$ be an A-point. Next let $e:\operatorname{Spec} A\to\mathbf{G}$ be an A-point of $\mathbf{G}$ which corresponds to the identity element of $\mathbf{G}$. Suppose that a regular function χ in $\mathbf{G}$ satisfies $\mu^{\#}(\chi)=1\otimes\chi$. Then

$$g^{\#}(\chi)=\langle g,e\rangle^{\#}\mu^{\#}(\chi)=\langle g,e\rangle^{\#}(1\otimes\chi)=e^{\#}(\chi)$$

Recall that e is given by the composition of the structural morphism $\operatorname{Spec} A\to\operatorname{Spec} k$ and the k-point $\operatorname{Spec} k\to\mathbf{G}$ determined by the identity of $\mathbf{G}$. Thus $g^{\#}(\chi)$ is an element of k. Since this follows for every $g:\operatorname{Spec} A\to\mathbf{G}$, we derive that χ is a constant function. This completes the proof of our claim that

$$\Gamma(V,\mathcal{O}_Y) \xrightarrow{\mathrm{pr}_Y^{\#}} \Gamma\left(\mathbf{G}\times_k V,\mathcal{O}_{\mathbf{G}\times_k Y}\right) \underset{\pi_{23}^{\#}}{\overset{(\mu\times_k 1_Y)^{\#}}{\rightrightarrows}} \Gamma\left(\mathbf{G}\times_k\mathbf{G}\times_k V,\mathcal{O}_{\mathbf{G}\times_k\mathbf{G}\times_k Y}\right)$$

is the kernel diagram and hence $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for pr_Y. Therefore, we proved that pr_Y is a geometric quotient of $\mathbf{G}\times_k Y$. Consider any morphism $Y'\to Y$. Then base change of pr_Y along this morphism is $\mathrm{pr}_{Y'}$. We conclude that pr_Y is a universal geometric quotient for $\mathbf{G}\times_k Y$. □

4. Geometric quotients of separated actions

Definition 4.1. The action of $\mathbf{G}$ on X is *separated* if the morphism $\langle a,\mathrm{pr}_X\rangle:\mathbf{G}\times_k X\to X\times_k X$ has closed set-theoretic image.

Theorem 4.2. *Let $q:X\to Y$ be a geometric quotient of X. Assume that q is universally submersive. Then the following assertions are equivalent.*

- **(i)** *The action of $\mathbf{G}$ on X is separated.*
- **(ii)** *Y is separated.*

Proof. We have a cartesian square

$$\begin{tikzcd} X\times_Y X \arrow[r, hook] \arrow[d] & X\times_k X \arrow[d, "q\times_k q"] \\ Y \arrow[r, hook, "\Delta_Y"'] & Y\times_k Y \end{tikzcd}$$

It follows that $X\times_Y X\hookrightarrow X\times_k X$ is a locally closed immersion. Since q is a geometric quotient, we derive that $\langle a,\mathrm{pr}_X\rangle$ factors as a surjective morphism $\mathbf{G}\times_k X\twoheadrightarrow X\times_Y X$ followed by the immersion $X\times_Y X\hookrightarrow X\times_k X$. Thus the action of $\mathbf{G}$ on X is separated if and only if $X\times_Y X$ is a closed subscheme of $X\times_k X$. Since q is universally submersive, we derive that $q\times_k q$ is submersive. As the square above is cartesian we derive that $\Delta_Y(Y) \subseteq Y \times_k Y$ is closed if and only if $X \times_Y X \subseteq X \times_k X$ is closed. Therefore, Y is separated if and only if the action of $\mathbf{G}$ on X is separated. $\square$

5. Geometric quotients of free actions and principal bundles

Definition 5.1. The action of $\mathbf{G}$ on X is *free* if the morphism $\langle a, \mathrm{pr}_X \rangle : \mathbf{G} \times_k X \to X \times_k X$ is a closed immersion.

Definition 5.2. Let x be a k-point of X. Suppose that *the orbit morphism* $\mathbf{G} \to X$ *of* x given by the composition

$$\mathbf{G} = \mathbf{G} \times_k \operatorname{Spec} k \xhookrightarrow{\text{induced by } x} \mathbf{G} \times_k X \longrightarrow X$$

is a closed immersion. Then the action of $\mathbf{G}$ on X has *a closed free orbit at* x.

Fact 5.3. *If the action of* $\mathbf{G}$ *on* X *is free, then every* k*-point of* X *has a closed free orbit.*

The following result states that over special type of local complete noetherian k-algebras geometric quotients of free actions correspond to trivial $\mathbf{G}$-bundles.

Theorem 5.4. *Suppose that* k *is an algebraically closed field and* $\mathbf{G}$ *is a smooth algebraic group over* k*. Let* $q : X \to Y$ *be a geometric quotient locally of finite type and let* Y *be the spectrum of a complete local noetherian* k*-algebra such that the residue field of the closed point of* Y *is* k*. Then the following assertions hold.*

(1) *If* x *is a* k*-point of* X *which has a closed free orbit, then there exists a* $\mathbf{G}$*-equivariant, étale and surjective morphism* $f : \mathbf{G} \times_k Y \to X$ *such that the triangle*

$$\begin{tikzcd} \mathbf{G} \times_k Y \arrow[rr, "f"] \arrow[dr, "\mathrm{pr}_Y"'] & & X \arrow[dl, "q"] \\ & Y & \end{tikzcd}$$

is commutative and the morphism

$$Y = \operatorname{Spec} k \times_k Y \xhookrightarrow{e \times_k 1_Y} \mathbf{G} \times_k Y \xrightarrow{\ f\ } X$$

is a section of q*.*

(2) *If the action of* $\mathbf{G}$ *on* X *is free, then* f *is an isomorphism.*

The proof relies on two algebraic lemmas. The first describe free actions in context of complete rings (it is [Mumford et al., 1994, lemma on page 18]) and the second is a version of Hensel's lemma.

Lemma 5.4.1. *Let* $(R, \mathfrak{m}, k)$ *be a complete local noetherian* k*-algebra and let* $\sigma : R \to R[[x_1, ..., x_n]]$ *be a local morphism into a ring of formal power series over* R*. Assume that the composition*

$$R \xrightarrow{\ \sigma\ } R[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod (x_1, ..., x_n)} R$$

is the identity and the composition

$$R \xrightarrow{\ \sigma\ } R[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod \mathfrak{m}} (R/\mathfrak{m})\,[[x_1, ..., x_n]] = k[[x_1, ..., x_n]]$$

is surjective. Consider elements $y_1, ..., y_n$ of R such that $\sigma(y_i) \bmod \mathfrak{m} = x_i$ for $i = 1, ..., n$. Then the composition

$$R \xrightarrow{\sigma} R[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod (y_1, ..., y_n)} (R/(y_1, ..., y_n))\,[[x_1, ..., x_n]]$$

is an isomorphism.

Proof of the lemma. For convienience let ϕ denote the morphism given by the rule $r \mapsto \sigma(r) \bmod (y_1, ..., y_n)$. Also denote $R/(y_1, ..., y_n)$ by S. According to assumptions we have

$$\sigma(y_i) = x_i + y_i + \sum_{j=1}^{n} x_j \cdot \mathfrak{m}[[x_1, ..., x_n]]$$

for each i. Thus $\phi(y_i) = \sum_{j=1}^{n} f_{ij} \cdot x_j$ where $f_{ij} \in S$ are elements such that the matrix $\left[f_{ij}\right]_{1 \le i,j \le n}$ is invertible in S. Hence

$$S[[x_1, ..., x_n]] = S[[\phi(y_1), ..., \phi(y_n)]]$$

and ϕ composed with $S[[\phi(y_1), ..., \phi(y_n)]] \twoheadrightarrow S$ is the quotient morphism $R \twoheadrightarrow S$. From this observations we derive that ϕ is surjective. It remains to prove that it is injective. Consider z in R such that $\phi(z) = 0$. Suppose that $z \in (y_1, ..., y_n)^m$ for some $m \in \mathbb{N}$. Write

$$z = \sum_{\alpha \in \Lambda} c_\alpha \cdot y_1^{\alpha_1} ... y_n^{\alpha_n}$$

for some $c_\alpha \in R$ where $\Lambda = \{\alpha \in \mathbb{N}^n \,|\, \alpha_1 + ... + \alpha_n = m\}$. Hence

$$0 = \phi(z) = \sum_{\alpha \in \Lambda} \phi(c_\alpha) \cdot \phi(y_1)^{\alpha_1} ... \phi(y_n)^{\alpha_n}$$

Thus $\phi(c_\alpha) \in (\phi(y_1), ..., \phi(y_n))$ for every $\alpha \in \Lambda$. Since ϕ composed with $S[[\phi(y_1), ..., \phi(y_n)]] \twoheadrightarrow S$ is the quotient morphism $R \twoheadrightarrow S$, we derive that

$$c_\alpha \bmod (y_1, ..., y_n) = \phi(c_\alpha) \bmod (\phi(y_1), ..., \phi(y_n)) = 0$$

for every $\alpha \in \Lambda$. Thus $c_\alpha \in (y_1, ..., y_n)$ for every $\alpha \in \Lambda$, which implies that $z \in (y_1, ..., y_n)^{m+1}$. Thus we proved that

$$\phi(z) = 0 \text{ and } z \in (y_1, ..., y_n)^m \Rightarrow z \in (y_1, ..., y_n)^{m+1}$$

By $\mathfrak{m}$-adic completeness of R this implies that $\phi(z) = 0$ if and only if $z = 0$. Hence ϕ is also injective. □

Lemma 5.4.2. *Let $(R, \mathfrak{m})$ be a complete local noetherian ring and let $R \to S$ be a local morphism into a local noetherian ring. Suppose that there exists a finitely generated R-submodule N of S such that*

$$S = N + \mathfrak{m}S$$

Then $S = N$.

Proof of the lemma. Pick s in S. Since $S = N + \mathfrak{m}S$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $x_n \in \mathfrak{m}^n N$ and

$$s - \sum_{i \le n} x_i \in \mathfrak{m}^{n+1} S$$

According to the assumption that $(R, \mathfrak{m})$ is complete with respect to $\mathfrak{m}$-adic topology and N is finitely generated over R, we deduce that N is complete with respect to $\mathfrak{m}$-adic topology. Hence there exists a unique element x in N such that

$$x = \sum_{n \in \mathbb{N}} x_n$$

where above series is convergent with respect to $\mathfrak{m}$-adic topology. Note also that

$$x - \sum_{i \le n} x_i \in \mathfrak{m}^{n+1} N$$

for every $n \in \mathbb{N}$. Thus we have

$$s - x = \left(s - \sum_{i \leq n} x_i\right) - \left(x - \sum_{i \leq n} x_i\right) \in \mathfrak{m}^{n+1}S + \mathfrak{m}^{n+1}N = \mathfrak{m}^{n+1}S$$

for every $n \in \mathbb{N}$. Hence

$$x - s \in \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n S$$

Since $R \to S$ is local morphism and S is a local ring, we deduce that $\mathfrak{m}S$ is contained in the maximal ideal of S. By assumptions S is noetherian. Therefore, S is separated with respect to $\mathfrak{m}$-adic topology. This implies that

$$0 = \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n S$$

Thus $s - x = 0$ and we infer that s is an element of N. This completes the proof that $S = N$. □

In what follows we shall denote by $\mathbf{G}x$ the closed subscheme determined by the orbit morphism $\mathbf{G} \to X$ of a k-point x of X which has a closed free orbit. For readers convienience we include the following lemmas, which have topological content.

Lemma 5.4.3. *Let $q : X \to Y$ be a geometric quotient and assume that Y is the spectrum of a local k-algebra such that the residue field of the closed point o of Y is k. Let x be a k-point of X with free closed orbit, then $q^{-1}(o) = \mathbf{G}x$ as closed subschemes of X.*

Proof of the lemma. Morphism q induces the morphism of residue fields $k(q(x)) \hookrightarrow k(x) = k$ over k. This implies that $k(q(x)) = k$ and hence $q(x)$ is a k-point of Y. Note that o is the unique k-point of Y. Thus $q(x) = o$. Clearly $q^{-1}(o)$ is a closed $\mathbf{G}$-stable subscheme of X (it is the preimage of o under $\mathbf{G}$-equivariant q), that contains x. Since $\mathbf{G}x$ is the smallest closed $\mathbf{G}$-stable subscheme of X containing x, we deduce that $\mathbf{G}x \subseteq q^{-1}(o)$ scheme-theoretically. Consider the pair of arrows

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X$$

Passing to functors of points we obtain that $a^{-1}(\mathbf{G}x) = \mathrm{pr}_X(\mathbf{G}.x)$. Since q is the cokernel of the pair (a, pr_X) in the category of topological spaces, we deduce that there exists a closed subset Z of Y such that $q^{-1}(Z) = \mathbf{G}x$. Clearly $o \in Z$ and hence $q^{-1}(o) \subseteq \mathbf{G}x$ set-theoretically. On the other hand above we proved that $\mathbf{G}x \subseteq q^{-1}(o)$ scheme-theoretically. This can only happen if $q^{-1}(o) = \mathbf{G}x$ as closed subschemes of X. □

Lemma 5.4.4. *Let $q : X \to Y$ be a geometric quotient and assume that Y is the spectrum of a local k-algebra such that the residue field of the closed point o of Y is k. Let U be an open $\mathbf{G}$-stable subset of X which contain a k-point. Then $U = X$.*

Proof of the lemma. Consider the pair of arrows

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X$$

Since U is $\mathbf{G}$-stable open subset of X, we derive that $\mathrm{pr}_X^{-1}(U) = a^{-1}(U)$. Next by definition q is the cokernel of the above pair in the category of topological spaces. Hence there exists an open subset V of Y such that $U = q^{-1}(V)$. Since U contains a k-point of X, we deduce as in Lemma 5.4.3 that $o \in V$. Thus $V = Y$ and finally $U = q^{-1}(V) = X$. □

Proof of the theorem. We first prove **(1)**. Denote by o the closed point of Y. Assume that x is a k-point of X which has a closed free orbit. Consider the surjective morphism $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{\mathbf{G},e}$ induced by the orbit morphism $\mathbf{G} \hookrightarrow X$ of x. Since $\mathbf{G}$ is smooth over k, the ring $\mathcal{O}_{\mathbf{G},e}$ is regular. Pick a system of parameters $x_1,...,x_n$ of $\mathcal{O}_{\mathbf{G},e}$ and let $y_1,...,y_n$ be elements of $\mathcal{O}_{X,x}$ such that y_i is send to x_i by the morphism $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{\mathbf{G},e}$ for $1 \leq i \leq n$. Define S to be the quotient ring $\mathcal{O}_{X,x}/(y_1,...,y_n)$. The morphism q induces the morphism $q^{\#} : \mathcal{O}_{Y,o} \to \mathcal{O}_{X,x}$ and hence the morphism $\mathcal{O}_{Y,o} \to S$. By Lemma 5.4.3 we have

$$S/\mathfrak{m}_o S = k$$

where $\mathfrak{m}_o$ is the maximal ideal of $\mathcal{O}_{Y,o}$. According to Lemma 5.4.2 we derive that $\mathcal{O}_{Y,o} \to S$ is surjective. Let $f : \mathbf{G} \times_k \operatorname{Spec} S \to X$ be the unique $\mathbf{G}$-equivariant morphism induced by the surjection $\mathcal{O}_{X,x} \twoheadrightarrow S$. We have a commutative square

$$\begin{tikzcd} \mathbf{G} \times_k \operatorname{Spec} S \arrow[r, "f"] \arrow[d, "\mathrm{pr}_{\operatorname{Spec} S}"'] & X \arrow[d, "q"] \\ \operatorname{Spec} S \arrow[r, hook, "j"'] & Y \end{tikzcd}$$

where j is a closed immersion induced by $\mathcal{O}_{Y,o} \twoheadrightarrow S$. According to assumptions q is locally of finite type. Moreover, $\mathbf{G}$ is an algebraic group over k and hence $\mathrm{pr}_{\operatorname{Spec} S}$ is locally of finite type. These two assertions together with the fact that $\operatorname{Spec} S \hookrightarrow Y$ is a closed immersion of noetherian schemes (and thus is of finite type) imply that f is locally of finite type. Then by Lemma 5.4.1 we deduce that f induces an isomorphism

$$\widehat{\mathcal{O}_{X,x}} \longrightarrow \hat{S}[[x_1,...,x_n]] = \hat{S}\hat{\otimes}_k\widehat{\mathcal{O}_{\mathbf{G},e}}$$

of complete local rings. Since f is locally of finite type, it follows that f is étale at a k-point of $\mathbf{G} \times_k \operatorname{Spec} S$ determined by the unique k-point of $\operatorname{Spec} S$ and $e \in \mathbf{G}$. Let U be an étale locus of f. It contains a k-point and hence it is nonempty. Moreover, U is open (it is étale locus) subset of X. Since f is $\mathbf{G}$-equivariant, we derive that U is $\mathbf{G}$-stable. Similarly $f(U)$ is open $\mathbf{G}$-stable subset of X and $x \in f(U)$. Thus by Lemma 5.4.4 we deduce that

$$U = \mathbf{G} \times_k \operatorname{Spec} S,\, f(U) = X$$

Therefore, f is étale and surjective. Now we pullback $q : X \to Y$ along the closed immersion $\operatorname{Spec} S \hookrightarrow Y$. We obtain a cartesian square

$$\begin{tikzcd} \tilde{X} \arrow[r, hook, "\tilde{j}"] \arrow[d, "\tilde{q}"'] & X \arrow[d, "q"] \\ \operatorname{Spec} S \arrow[r, hook, "j"'] & Y \end{tikzcd}$$

Then f factors as a morphism $\mathbf{G} \times_k \operatorname{Spec} S \to \tilde{X}$ followed by a closed immersion $\tilde{j}$. Since f is étale and surjective, we deduce that $\tilde{j}$ is étale and surjective. This implies that $\tilde{j}$ is an isomorphism of schemes. By definition of geometric quotient we derive that the diagram

$$\mathcal{O}_Y \xrightarrow{q^{\#}} q_*\mathcal{O}_X \overset{q_*a^{\#}}{\underset{q_*\mathrm{pr}_X^{\#}}{\rightrightarrows}} q_*\left(\mathrm{pr}_X\right)_* \mathcal{O}_{\mathbf{G}\times_k X} = q_*a_*\mathcal{O}_{\mathbf{G}\times_k X}$$

is the kernel in the category of sheaves on Y. Hence $q^{\#} : \mathcal{O}_Y \to q_*\mathcal{O}_X$ is a monomorphism of sheaves. On the other hand we have

$$q^{\#} = j_* q_* \left(\tilde{j}^{-1}\right)^{\#} \cdot j_* \tilde{q}^{\#} \cdot j^{\#}$$

and thus $j^{\#}$ is a monomorphism. Since j is a closed immersion, we infer that j is an isomorphism. Therefore, we can identify $\operatorname{Spec} S$ with Y. Then f is a morphism which makes the triangle

$$\begin{tikzcd} \mathbf{G} \times_k Y \arrow[rr, "f"] \arrow[dr, "\mathrm{pr}_Y"'] & & X \arrow[dl, "q"] \\ & Y & \end{tikzcd}$$

commutative. This completes the proof of **(1)**.
For the proof of **(2)** consider the section $s : Y \hookrightarrow X$ described in **(1)**. Then f fits into a cartesian square

$$\begin{tikzcd} \mathbf{G} \times_k Y \arrow[r, "f"] \arrow[d, "1_{\mathbf{G}} \times_Y s"'] & X \times_Y Y = X \arrow[d, "1_X \times_Y s"] \\ \mathbf{G} \times_k X \arrow[r, hook, "\phi"'] & X \times_Y X \end{tikzcd}$$

where ϕ is a closed immersion induced by the closed immersion $\langle a, \mathrm{pr}_X \rangle : \mathbf{G} \times_k X \hookrightarrow X \times_k X$ (the action of $\mathbf{G}$ on X is free). Thus f is a closed immersion. By **(1)** it is étale and surjective. Therefore, f is an isomorphism. □

Remark 5.5. We expect that Theorem 5.4 holds for prime spectra of strictly henselian rings.

Now we introduce sufficient condition for smoothness of geometric quotient in case of locally algebraic k-schemes.

Corollary 5.6. *Suppose that $\mathbf{G}$ is a smooth algebraic group over k. Let $q : X \to Y$ be a morphism of finite type between k-schemes locally of finite type. Assume that q is a uniform geometric quotient of X and x is a k-point of X with closed free orbit. Then q is smooth at x.*

Proof. Since smoothness descent along faithfully flat morphisms, we may assume that k is algebraically closed. Let $y = q(x)$. Then y is a k-point of Y. Now $1_{\operatorname{Spec} \widehat{\mathcal{O}_{Y,y}}} \times_k q$ is a geometric quotient and $\widehat{\mathcal{O}_{Y,y}}$ is a complete local noetherian k-algebra with k as a residue field. Moreover, x is a k-point of $\operatorname{Spec} \widehat{\mathcal{O}_{Y,y}} \times_k X$ with a closed free orbit. By Theorem 5.4 we deduce that $1_{\operatorname{Spec} \widehat{\mathcal{O}_{Y,y}}} \times_k q$ is smooth. From descent for smoothness we infer that q is smooth at x. □

Definition 5.7. Let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism into a k-scheme Y equipped with the trivial $\mathbf{G}$-action. Suppose that q is faithfully flat, quasi-compact morphism and the square

$$\begin{tikzcd} \mathbf{G} \times_k X \arrow[r, "a"] \arrow[d, "\mathrm{pr}_X"'] & X \arrow[d, "q"] \\ X \arrow[r, "q"'] & Y \end{tikzcd}$$

is cartesian. Then q is *a principal* $\mathbf{G}$*-bundle*.

Now we use Theorem 5.4 to describe principal $\mathbf{G}$-bundles in the category of locally algebraic k-schemes.

Theorem 5.8. *Suppose that* $\mathbf{G}$ *is a smooth algebraic group over* k*. Let* $q : X \to Y$ *be a morphism of finite type between* k*-schemes locally of finite type. Then the following assertions are equivalent.*

- **(i)** *q is a universal geometric quotient and the action of* $\mathbf{G}$ *on* X *is free.*
- **(ii)** *q is a uniform geometric quotient and the action of* $\mathbf{G}$ *on* X *is free.*
- **(iii)** *q is a principal* $\mathbf{G}$*-bundle.*

Proof. Clearly **(i)** ⇒ **(ii)**. Suppose that **(ii)** holds. Let $\overline{k}$ be an algebraic closure of k. Then $1_{\mathrm{Spec}\,\overline{k}} \times_k q$ is a uniform quotient and the action of $\mathrm{Spec}\,\overline{k} \times_k \mathbf{G}$ on $\mathrm{Spec}\,\overline{k} \times_k X$ induced by the action of $\mathbf{G}$ on X is free. Moreover, if $1_{\mathrm{Spec}\,\overline{k}} \times_k q$ is a principal $\mathrm{Spec}\,\overline{k} \times_k \mathbf{G}$-bundle, then q is a $\mathbf{G}$-bundle. This follows from the observation that property of being a principal bundle descents along faithfuly flat and quasi-compact base extensions. Thus we may assume that k is algebraically closed. Next we pick a k-point y of Y and consider base change $1_{\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}}} \times_Y q$. This is a geometric quotient (because morphism $\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}} \to Y$ is flat) and a morphism of finite type. Moreover, the action of $\mathbf{G}$ on $\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}} \times_Y X$ is free. Since $\widehat{\mathcal{O}_{Y,y}}$ is a complete noetherian k-algebra with residue field k, we derive by Theorem 5.4 that $\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}} \times_Y q$ is isomorphic as a $\mathbf{G}$-equivariant morphism with $\mathrm{pr}_{\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}}}$. By faithfuly flat descent for flat morphism we deduce that q is flat at every point in the fiber $q^{-1}\left(\mathrm{Spec}\,\mathcal{O}_{Y,y}\right)$. Since y is an arbitrary k-point, it follows that q is flat at every point of X which specializes to a k-point. Every point of X is a generization of a k-point (X is locally of finite type and k is algebraically closed). Thus q is flat. It is also surjective (as it is a geometric quotient) and quasi-compact (it is of finite type). Therefore, it is faithfully flat and quasi-compact morphism. In order to obtain **(iii)** it remains to prove that the morphism $\Phi : \mathbf{G} \times_k X \to X \times_Y X$ induced by a and pr_X is an isomorphism. Note that it is a closed immersion (the action of $\mathbf{G}$ on X is closed). Moreover, $1_{\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}}} \times_Y \Phi$ is an isomorphism due to the fact that $1_{\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}}} \times_Y q$ is isomorphic as a $\mathbf{G}$-equivariant morphism with $\mathrm{pr}_{\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}}}$. By faithfully flat descent we infer that $1_{\mathrm{Spec}\,\mathcal{O}_{Y,y}} \times_Y \Phi$ is an isomorphism. This holds for every k-point y in Y. Thus Φ induces an isomorphism $\mathcal{O}_{X\times_Y X,\Phi(z)} \to \mathcal{O}_{\mathbf{G}\times_k X,z}$ for every k-point z of $X \times_Y X$. Hence a closed immersion Φ is an isomorphism. This completes the proof of **(ii)** ⇒ **(iii)**.
Assume now that **(iii)** holds. Then the square

$$\begin{array}{ccc} \mathbf{G} \times_k X & \xrightarrow{a} & X \\ {\scriptstyle \mathrm{pr}_X}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[q]{} & Y \end{array}$$

is cartesian and q is faithfully flat and quasi-compact. By Proposition 3.8 morphism pr_X is a universal geometric quotient. According to Corollary 3.7 we derive that q is universal geometric quotient. Moreover, the cartesian square above shows that the morphism $\mathbf{G} \times_k X \to X \times_Y X$ induced by a and pr_X is an isomorphism. Thus the action of $\mathbf{G}$ on X is free. This is **(i)**. Hence **(iii)** ⇒ **(i)** holds. □

6. Nagata's theorem

We start by proving the following result which give yet another characterization of linearly reductive groups.

Theorem 6.1. *Let $\mathbf{G}$ be a smooth affine algebraic group over k. Then the following assertions are equivalent.*

- **(i)** *$\mathbf{G}$ is linearly reductive.*
- **(ii)** *For every finitely dimensional linear representation V of $\mathbf{G}$ and for every nonzero $\mathbf{G}$-invariant element v in V there exists a $\mathbf{G}$-invariant linear function $f : V \to k$ such that $f(v) \neq 0$.*

We need the following easy result.

Lemma 6.1.1. *Let $\mathbf{G}$ be an algebraic group over k which satisfies **(ii)**. Suppose that V is a finitely dimensional representation of $\mathbf{G}$. Then the map*

$$\mathrm{Hom}_k(V,k)^{\mathbf{G}} \ni f \mapsto f_{|V^{\mathbf{G}}} \in \mathrm{Hom}_k\left(V^{\mathbf{G}},k\right)$$

is an isomorphism of vector spaces over k.

Proof of the lemma. The image of the map in the statement is a k-vector subspace $W \subseteq \mathrm{Hom}_k\left(V^{\mathbf{G}},k\right)$ such that for every nonzero element v in $V^{\mathbf{G}}$ there exists f in W such that $f(v) \neq 0$ (this is a consequence of **(ii)**). It follows that W cannot be proper subspace of $\mathrm{Hom}_k\left(V^{\mathbf{G}},k\right)$. Hence the map in the statement is an epimorphism. Now fix a nonzero $\mathbf{G}$-invariant linear function $f : V \to k$. By **(ii)** there exists a $\mathbf{G}$-invariant linear function $w : \mathrm{Hom}_k(V,k) \to k$ such that $w(f) = 0$. Note that the canonical isomorphism

$$V \cong \mathrm{Hom}_k\left(\mathrm{Hom}_k(V,k),k\right)$$

of k-vector spaces is a morphism of representations of $\mathbf{G}$. Thus w is defined in terms of evaluation in some $\mathbf{G}$-invariant vector v in V. Therefore, $f(v) \neq 0$ and hence $f_{|V^{\mathbf{G}}} \neq 0$. Thus the map described in the statement is also a monomorphism. □

Proof of the theorem. Suppose that **(i)** holds. Consider a $\mathbf{G}$-invariant nonzero vector v in a finitely dimensional representation V of $\mathbf{G}$. Then $k \cdot v \subseteq V$ is a $\mathbf{G}$-subrepresentation. Since $\mathbf{G}$ is linearly reductive, there exists a morphism of $\mathbf{G}$-representations which is a left inverse of $k \cdot v \hookrightarrow V$. This morphism can be identified with a $\mathbf{G}$-invariant linear function $f : V \to k$ such that $f(v) \neq 0$. Hence **(i)** $\Rightarrow$ **(ii)**.
Now suppose that **(ii)** holds. Pick an epimorphism $\theta : V \twoheadrightarrow W$ of finitely dimensional representations V of $\mathbf{G}$. Assume that there exists a nonzero $\mathbf{G}$-invariant vector w in W such that $w \notin \theta\left(V^{\mathbf{G}}\right)$. By Lemma 6.1.1 there exists f in $\mathrm{Hom}_k(W,k)^{\mathbf{G}}$ such that $f_{|\theta\left(V^{\mathbf{G}}\right)} = 0$ and $f(w) \neq 0$. Then $f \cdot \theta$ is a nonzero element of $\mathrm{Hom}(V,k)^{\mathbf{G}}$ such that $(f \cdot \theta)_{|V^{\mathbf{G}}} = 0$. This is impossible according to Lemma 6.1.1. Hence $\theta^{\mathbf{G}} : V^{\mathbf{G}} \to W^{\mathbf{G}}$ is an epimorphism. Now assume that $\theta : V \twoheadrightarrow W$ is an epimorphism of arbitrary linear representations of $\mathbf{G}$. Since $\mathbf{G}$ is affine, every linear representation of $\mathbf{G}$ is rational (i.e. it is a sum of its finitely dimensional subrepresentations). This together with the finitely dimensional case considered above imply that $\theta^{\mathbf{G}} : V^{\mathbf{G}} \to W^{\mathbf{G}}$ is an epimorphism. Thus the functor $(-)^{\mathbf{G}} : \mathbf{Rep}(\mathbf{G}) \to \mathbf{Vect}_k$ is exact. □

The result above motivates the following notion.

Definition 6.2. Let $\mathbf{G}$ be a smooth affine algebraic group. Suppose that for every finitely dimensional representation V of $\mathbf{G}$ and for every nonzero $\mathbf{G}$-invariant vector v of V there exists a homogenous $\mathbf{G}$-invariant polynomial $f : V \to k$ such that $f(v) \neq 0$. Then $\mathbf{G}$ is *geometrically reductive*.

We state here the following celebrated result.

Theorem 6.3. *If* $\mathbf{G}$ *is reductive, then it is geometrically reductive.*

The result above is due to Haboush and its proof can be found in [Haboush, 1975].
The following theorem shows that geometric reductivity admits up to an integral extension the same property as linear reductivity (see also Remark 6.5 below).

Theorem 6.4. *Suppose that* $\mathbf{G}$ *is geometrically reductive. Let* A *be a* k*-algebra such that* $\operatorname{Spec} A$ *admits an action of* $\mathbf{G}$ *and let* $\mathfrak{a}$ *be a* $\mathbf{G}$*-stable ideal of* A*. We consider* $A^{\mathbf{G}}/A^{\mathbf{G}} \cap \mathfrak{a}$ *as a* k*-subalgebra of* $(A/\mathfrak{a})^{\mathbf{G}}$ *by means of the canonical inclusion* $A^{\mathbf{G}}/A^{\mathbf{G}} \cap \mathfrak{a} \hookrightarrow A/\mathfrak{a}$*. For every element* $x \in (A/\mathfrak{a})^{\mathbf{G}}$ *there exists positive integer* r *such that* $x^r \in A^{\mathbf{G}}/A^{\mathbf{G}} \cap \mathfrak{a}$*.*

Proof. Let $d : A \to k[\mathbf{G}] \otimes_k A$ be the coaction of $\mathbf{G}$ on A. Pick an element $x_0 \in A$ which maps to x modulo $\mathfrak{a}$. Consider finitely dimensional vector subspace $V \subseteq A$ over k such that V is a $\mathbf{G}$-subrepresentation of A and $x_0 \in V$. Since x is x_0 modulo $\mathfrak{a}$, we derive that $c(x_0) - 1 \otimes x_0$ is in ideal of $k[\mathbf{G}] \otimes_k A$ generated by $k[\mathbf{G}] \otimes_k \mathfrak{a}$. Thus $W = k \cdot x_0 + V \cap \mathfrak{a} \subseteq A$ is finitely dimensional $\mathbf{G}$-subrepresentation of A. Let $\lambda : W \to k$ be a k-linear form such that $\lambda(x_0) = 1$ and $\lambda_{|V \cap \mathfrak{a}} = 0$. Since $\mathbf{G}$ is geometrically reductive there exists $f \in \operatorname{Sym}_r(W)^{\mathbf{G}}$ such that $f(\lambda) = 1$. Since the canonical morphism $\operatorname{Sym}_r(W) \to A$ is a morphism of representations of $\mathbf{G}$, we deduce that f is mapped under this morphism to some $\mathbf{G}$-invariant element y in A. Note that f is sum of an r-th symmetric power of x_0 and some element of $\operatorname{Sym}_r(V \cap \mathfrak{a})$. Thus $y \bmod \mathfrak{a} = x^r$. Hence $x^r \in A^{\mathbf{G}}/A^{\mathbf{G}} \cap \mathfrak{a}$. □

Remark 6.5. Let $\mathbf{G}$ be an algebraic group $\mathbf{G}$ which acts on $\operatorname{Spec} A$ for some k-algebra A and let $\mathfrak{a}$ be a $\mathbf{G}$-stable ideal of A. Then the sequence

$$0 \longrightarrow \mathfrak{a}^{\mathbf{G}} \longrightarrow A^{\mathbf{G}} \longrightarrow (A/\mathfrak{a})^{\mathbf{G}}$$

is left exact and it induces a monomorphism $A^{\mathbf{G}}/\mathfrak{a} \cap A^{\mathbf{G}} = A^{\mathbf{G}}/\mathfrak{a}^{\mathbf{G}} \hookrightarrow (A/\mathfrak{a})^{\mathbf{G}}$. If $\mathbf{G}$ is linearly reductive, then the sequence is exact and this monomorphism is an isomorphism. Theorem 6.4 states that if $\mathbf{G}$ is geometrically reductive, then the monomorphism $A^{\mathbf{G}}/\mathfrak{a} \cap A^{\mathbf{G}} \hookrightarrow (A/\mathfrak{a})^{\mathbf{G}}$ is integral.

Now we are going to formulate the main result of this section.

Theorem 6.6. *Suppose that* $\mathbf{G}$ *is geometrically reductive. Let* A *be a finitely generated* k*-algebra such that* $\operatorname{Spec} A$ *admits an action of* $\mathbf{G}$*. Then* $A^{\mathbf{G}}$ *is finitely generated* k*-algebra.*

The theorem above was first proved by Nagata and here we follow Nagata's original proof. In the argument we denote the coaction of $k[\mathbf{G}]$ on A by $d : A \to k[\mathbf{G}] \otimes_k A$. The proof relies on a series of partial results.

Lemma 6.6.1. *Let* $A \hookrightarrow B$ *be an integral morphism of* k*-algebras and suppose that* B *is finitely generated over* k*. Then* A *is finitely generated.*

Proof of the lemma. Suppose that $b_1, ..., b_r$ are generators of B as a k-algebra. For every $1 \leq i \leq r$ we have a polynomial relation

$$b_i^{n_i} + a_{i,n_i-1} b_i^{n_i-1} + ... + a_{i,1} b_i + a_{i,0} = 0$$

where $n_i > 0$ and $a_{i,j} \in A$ for $0 \leq j \leq n_i - 1$. Suppose that $\tilde{A}$ is a k-subalgebra of A generated by $a_{i,j}$ for $1 \leq i \leq r$ and $0 \leq j \leq n_i - 1$. Then B is finite over $\tilde{A}$. Since $\tilde{A} \subseteq A \subseteq B$ and $\tilde{A}$ is noetherian, we derive that A is finite over $\tilde{A}$. Hence A is finitely generated over k. □

Lemma 6.6.2. *Suppose that $\mathbf{G}$ is geometrically reductive. Let A be a k-algebra such that* $\operatorname{Spec} A$ *admits an action of $\mathbf{G}$. Assume that A contains $\mathbf{G}$-invariant zero divisor and that for every proper $\mathbf{G}$-stable ideal $\mathfrak{a}$ of A the k-algebra $(A/\mathfrak{a})^{\mathbf{G}}$ is finitely generated over k. Then $A^{\mathbf{G}}$ is finitely generated over k.*

Proof of the lemma. Let f be a $\mathbf{G}$-invariant zero divisor of A. By assumption both k-algebras $(A/fA)^{\mathbf{G}}$ and $(A/\operatorname{ann}(f))^{\mathbf{G}}$ are finitely generated over k. Now by combination of Lemma 6.6.1 and Theorem 6.4 we obtain that $A^{\mathbf{G}}/A^{\mathbf{G}} \cap fA$ and $A^{\mathbf{G}}/A^{\mathbf{G}} \cap \operatorname{ann}(f)$ are finitely generated over k. Let B be a finitely generated k-subalgebra of $A^{\mathbf{G}}$ which maps surjectively onto $A^{\mathbf{G}}/A^{\mathbf{G}} \cap fA$ and $A^{\mathbf{G}}/A^{\mathbf{G}} \cap \operatorname{ann}(f)$. Let $u_1, ..., u_n$ be elements in A such that the image of $B \cdot u_1 + ... + B \cdot u_n \subseteq A$ modulo $\operatorname{ann}(f)$ contains a finite B-module $(A/\operatorname{ann}(f))^{\mathbf{G}}$. Fix $a \in A^{\mathbf{G}}$. Since B maps surjectively onto $A^{\mathbf{G}}/A^{\mathbf{G}} \cap fA$, there exist $b \in B$ and $c \in A$ such that $a - b = fc$. Note that $fc \in A^{\mathbf{G}}$ and thus

$$(1 \otimes f)\,(d(c) - 1 \otimes c) = 0$$

This implies that c is send to $(A/\operatorname{ann}(f))^{\mathbf{G}}$ modulo $\operatorname{ann}(f)$. Then $c \in B \cdot u_1 + ... + B \cdot u_n$. Hence $a - b \in B \cdot fu_1 + ... + B \cdot fu_n$. Therefore, $a \in B[fu_1, ..., fu_n]$. This completes the proof that $A^{\mathbf{G}}$ is finitely generated over k. □

Lemma 6.6.3. *Suppose that $\mathbf{G}$ is geometrically reductive. Let $A = \oplus_{n \in \mathbb{N}} A_n$ be a $\mathbb{N}$-graded k-algebra such that A admits an action of $\mathbf{G}$. Assume that A_n is a $\mathbf{G}$-subrepresentation of A for every $n \in \mathbb{N}$ and that for every proper $\mathbf{G}$-stable homogenous ideal $\mathfrak{a}$ of A the k-algebra $(A/\mathfrak{a})^{\mathbf{G}}$ is finitely generated over k. If A contains $\mathbf{G}$-invariant zero divisor, then $A^{\mathbf{G}}$ is finitely generated over k.*

Proof of the lemma. Let f be a $\mathbf{G}$-invariant zero divisor of A. We may pick f such that it is homogenous. Then both ideals fA and $\operatorname{ann}(f)$ are homogenous, $\mathbf{G}$-stable and proper in A. Now we proceed as in the proof of Lemma 6.6.2. □

Proof of the theorem. We first prove the theorem in case of $\mathbb{N}$-graded k-algebras and then reduce the general case to this graded case.
Assume that $A = \bigoplus_{n \in \mathbb{N}} A_n$ is $\mathbb{N}$-graded in such a way that $A_0 = k$ and A_n is a $\mathbf{G}$-subrepresentation of A for every $n \in \mathbb{N}$. Since A is finitely generated over k and by virtue of noetherian induction, we assume that $(A/\mathfrak{a})^{\mathbf{G}}$ is finitely generated over k for every homogenous $\mathbf{G}$-stable proper ideal $\mathfrak{a}$ of A. If there are $\mathbf{G}$-invariant zero divisors of A, then by Lemma 6.6.3 we deduce that $A^{\mathbf{G}}$ is finitely generated over k. So we may assume that $A^{\mathbf{G}}$ contains no zero divisors of A. Pick a nonzero homogenous element $f \in A^{\mathbf{G}}$ of positive degree. If there are no such elements, then $A^{\mathbf{G}} = A_0 = k$ and the result holds. So we may assume that such an element exists. Note that it is noninvertible. Consider $x \in A$ such that $fx \in A^{\mathbf{G}}$. Then

$$0 = d(fx) - 1 \otimes fx = d(f) \cdot d(x) - (1 \otimes f) \cdot (1 \otimes x) = (1 \otimes f)\,(d(x) - 1 \otimes x)$$

Since f is not a zero divisor in A, we derive that $1 \otimes f$ is not a zero divisor in $k[\mathbf{G}] \otimes_k A$. Thus $d(x) = 1 \otimes x$ and $x \in A^{\mathbf{G}}$. This shows that $fA \cap A^{\mathbf{G}} = fA^{\mathbf{G}}$. By Theorem 6.4 $(A/fA)^{\mathbf{G}}$ is integral over $A^{\mathbf{G}}/fA \cap A^{\mathbf{G}} = A^{\mathbf{G}}/fA^{\mathbf{G}}$. Note that $(A/fA)^{\mathbf{G}}$ is finitely generated over k by inductive assumption. According to Lemma 6.6.1 we obtain that $A^{\mathbf{G}}/fA^{\mathbf{G}}$ is finitely generated over k. Clearly

$$A^{\mathbf{G}} = \bigoplus_{n \in \mathbb{N}} A_n^{\mathbf{G}}$$

and hence $A^{\mathbf{G}}/fA^{\mathbf{G}}$ inherits $\mathbb{N}$-grading from A. The ideal generated by elements of positive degree $\left(A^{\mathbf{G}}/fA^{\mathbf{G}}\right)_+$ is finitely generated (as is every ideal in noetherian ring). Hence also

$$\left(A^{\mathbf{G}}\right)_+ = \bigoplus_{n \in \mathbb{N}_+} A_n^{\mathbf{G}}$$

is finitely generated (generating set consists of lifts of generators of $\left(A^{\mathbf{G}}/fA^{\mathbf{G}}\right)_+$ and f). This implies that $A^{\mathbf{G}}$ is finitely generated over $A_0^{\mathbf{G}} = k$.

Now assume that A is an arbitrary finitely generated k-algebra. By noetherian induction we may assume that $(A/\mathfrak{a})^{\mathbf{G}}$ is finitely generated over k for every proper $\mathbf{G}$-stable ideal $\mathfrak{a}$ of A. Pick a finitely dimensional $\mathbf{G}$-subrepresentation V of A which contains some finite set of generators of A as a k-algebra. Define $S = \mathrm{Sym}(V)$ and $S_n = \mathrm{Sym}_n(V)$ for every $n \in \mathbb{N}$. Then S is $\mathbb{N}$-graded, $S_0 = k$ and $\mathbf{G}$ acts on $\mathrm{Spec}\, S$ in such a way that S_n is a $\mathbf{G}$-subrepresentation of S for every S. By the case considered above $S^{\mathbf{G}}$ is finitely generated over k. The canonical (induced by $V \hookrightarrow A$) surjective morphism $S \twoheadrightarrow A$ of k-algebras is also a morphism of representations of $\mathbf{G}$. Let I be its kernel. Then I is a $\mathbf{G}$-stable ideal of S. By Theorem 6.4 we derive that $A^{\mathbf{G}} = (S/I)^{\mathbf{G}}$ is integral over its finitely generated k-subalgebra $S^{\mathbf{G}}/I \cap S^{\mathbf{G}}$. Moreover, by Lemma 6.6.2 we may assume that $A^{\mathbf{G}}$ does not contain zero divisors of A. In particular, it is an integral domain. Hence $S^{\mathbf{G}}/I \cap S^{\mathbf{G}}$ is a domain. Let B be the integral closure of $S^{\mathbf{G}}/I \cap S^{\mathbf{G}}$ in the field L of fractions of $A^{\mathbf{G}}$. Since B is integral over $A^{\mathbf{G}}$, Lemma 6.6.1 shows that it suffices to prove that B is finitely generated over k. This will follow, if we can show that B is a finite $S^{\mathbf{G}}/I \cap S^{\mathbf{G}}$-module. Since fields are Nagata rings, we may reduce this question to proving that L is a finite extension of the field K of fractions of $S^{\mathbf{G}}/I \cap S^{\mathbf{G}}$. Since $K \subseteq L$ is algebraic (due to the fact that $S^{\mathbf{G}}/I \cap S^{\mathbf{G}} \hookrightarrow A^{\mathbf{G}}$ is integral), it suffices to show that L is finitely generated field over K. For this pick a set S of nonzero divisors of A. Note that S is a multiplicative subset of A. Fix a maximal ideal $\mathfrak{m} \subseteq S^{-1}A$. Since nonzero elements of $\mathfrak{m} \cap A^{\mathbf{G}}$ are zero divisors of A, we derive that $\mathfrak{m} \cap A^{\mathbf{G}} = 0$. Thus L is a subfield of $S^{-1}A/\mathfrak{m}$. The inclusion $A \hookrightarrow S^{-1}A$ induces an isomorphism between the fraction field of $A/\mathfrak{m} \cap A$ and the field $S^{-1}A/\mathfrak{m}$. By our assumption A is finitely generated over k. Thus the fraction field of $A/\mathfrak{m} \cap A$ is finitely generated over k as field. It follows that L is a field finitely generated over k. This implies that L is a field finitely generated over K. Therefore, $A^{\mathbf{G}}$ is finitely generated k-algebra. □

7. Good categorical quotients

References

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