BIAŁYNICKI-BIRULA FUNCTORS

1. Introduction

In this notes we study Białynicki-Birula functors. In the first section we prove some results concerning the forgetful functor $Rep(M) \to Rep(G)$, where M is an affine monoid k-scheme and G is its group of units (we assume that G is open and schematically dense in M). These results will be used in following sections.

We assume that *k* is a field.

2. RELATIONS BETWEEN REPRESENTATIONS OF A MONOID AND ITS GROUP OF UNITS

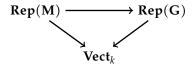
In this section we study the relation between the category $\mathbf{Rep}(\mathbf{M})$ of representations of an affine monoid k-scheme \mathbf{M} and the category $\mathbf{Rep}(\mathbf{G})$ of representations of its group of units \mathbf{G} . Let $i:k[\mathbf{M}] \to k[\mathbf{G}]$ be the morphism of k-bialgebras induced by $\mathbf{G} \hookrightarrow \mathbf{M}$. Let us first note the following elementary result.

Fact 2.1. Let M be an affine monoid k-scheme and let G be its group of units. Assume that G is open and schematically dense in M. Then i is an injective morphism of k-algebras.

Proof. This follows from [Görtz and Wedhorn, 2010, Proposition 9.19]. □

Fact 2.2. Let M be an affine monoid k-scheme and let G be its group of units. Then the forgetful functor $Rep(M) \to Rep(G)$ creates colimits and finite limits.

Proof. This follows from [Monygham, 2020, Theorem 14.3, Theorem 14.4] and the commutative triangle



of functors. \Box

The theorem below characterizes representations of G which are contained in the image of the forgetful functor $Rep(M) \rightarrow Rep(G)$.

Theorem 2.3. Let M be an affine monoid k-scheme and let G be its group of units. Assume that G is open and schematically dense in M. Let V be a G-representation. Then the following are equivalent.

- (i) V is in the image of the forgetful functor $Rep(M) \rightarrow Rep(G)$.
- (ii) The coaction $d: V \to k[\mathbf{G}] \otimes_k V$ factors through $i \otimes_k 1_V : k[\mathbf{M}] \otimes_k V \hookrightarrow k[\mathbf{G}] \otimes_k V$.

Proof. In the proof we denote by $\Delta_{\mathbf{M}}$ and $\Delta_{\mathbf{G}}$ comultiplications of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. We also denote by $\xi_{\mathbf{M}}$ and $\xi_{\mathbf{G}}$ counits of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. According to Fact 2.1 i is an injective morphism of k-algebras.

Clearly (i) \Rightarrow (ii). We prove the converse. Suppose that (ii) holds. Let $c: V \to k[\mathbf{M}] \otimes_k V$ be

a unique morphism such that $d = (i \otimes_k 1_V) \cdot c$. It suffices to prove that c is the coaction of the bialgebra $k[\mathbf{M}]$ on V. Observe that

$$(i \otimes_k i \otimes_k 1_V) \cdot (1_{k[\mathbf{M}]} \otimes_k c) \cdot c = (i \otimes_k d) \cdot c = (1_{k[\mathbf{G}]} \otimes_k d) \cdot d = (\Delta_{\mathbf{G}} \otimes_k 1_V) \cdot d = (\Delta_{\mathbf{$$

$$= (\Delta_{\mathbf{G}} \otimes_k 1_V) \cdot ((i \otimes_k 1_V) \cdot c) = ((\Delta_{\mathbf{G}} \cdot i) \otimes_k 1_V) \cdot c = (i \otimes_k i \otimes_k 1_V) \cdot (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot c$$

Since $i \otimes_k i \otimes_k 1_V$ is a monomorphism, we deduce that $(1_{k[\mathbf{M}]} \otimes_k c) \cdot c = (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot c$. Moreover, we have

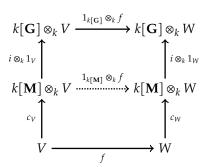
$$(\xi_{\mathbf{G}} \otimes_k 1_V) \cdot d = (\xi_{\mathbf{G}} \otimes_k 1_V) \cdot ((i \otimes_k 1_V) \cdot c) = (\xi_{\mathbf{M}} \otimes_k 1_V) \cdot c$$

and hence $(\xi_{\mathbf{M}} \otimes_k 1_V) \cdot c$ is the canonical isomorphism $V \cong k \otimes_k V$. Thus c is the coaction of $k[\mathbf{M}]$ and $d = (i \otimes_k 1_V) \cdot c$. Therefore, V is in the image of $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$.

Theorem 2.4. Let M be an affine monoid k-scheme and let G be its group of units. Assume that G is open and schematically dense in M. Then Rep(M) is a full subcategory of Rep(G) closed under subobjects and quotients.

Proof. In the proof we denote by $\Delta_{\mathbf{M}}$ and $\Delta_{\mathbf{G}}$ comultiplications of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. We also denote by $\xi_{\mathbf{M}}$ and $\xi_{\mathbf{G}}$ counits of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. According to Fact 2.1 i is an injective morphism of k-algebras.

We first prove that $\mathbf{Rep}(\mathbf{M})$ is a full subcategory of $\mathbf{Rep}(\mathbf{G})$. For this consider \mathbf{M} -representations V,W and a their morphism $f:V\to W$ as \mathbf{G} -representations. Let c_V and c_W be coactions of $k[\mathbf{M}]$ on V and W, respectively. Our goal is to prove that f is a morphism of \mathbf{M} -representations. Consider the diagram

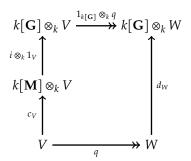


in which the outer square is commutative. Our goal is to prove that the bottom square is commutative. We have

$$(i \otimes_k 1_W) \cdot c_W \cdot f = (1_{k[\mathbf{G}]} \otimes_k f) \cdot (i \otimes_k 1_V) \cdot c_V = (i \otimes_k 1_W) \cdot (1_{k[\mathbf{M}]} \otimes_k f) \cdot c_V$$

Since $i \otimes_k 1_W$ is a monomorphism, we deduce that $c_W \cdot f = (1_{k[\mathbf{M}]} \otimes_k f) \cdot c_V$. Hence f is a morphism of \mathbf{M} -representations.

Next we prove that $\mathbf{Rep}(\mathbf{M})$ is a subcategory of $\mathbf{Rep}(\mathbf{G})$ that is closed under subquotients. Consider an \mathbf{M} -representation V and its quotient \mathbf{G} -representations $q:V \twoheadrightarrow W$. We show that W is a quotient \mathbf{M} -representation of V. Let c_V be the coaction of \mathbf{M} on V and let d_W be the coaction of \mathbf{G} on W. We have a commutative diagram



and hence $d_W(W) \subseteq k[\mathbf{M}] \otimes_k W$. Thus Theorem 2.3 implies that W is a representation of \mathbf{M} and q is a morphism of \mathbf{M} -representations. This shows that $\mathbf{Rep}(\mathbf{M})$ is a subcategory of $\mathbf{Rep}(\mathbf{G})$ closed under quotients. Next let $j: U \hookrightarrow V$ be a \mathbf{G} -subrepresentation of a \mathbf{M} -representation V. By what we proved above the cokernel $q: V \twoheadrightarrow W$ of j in $\mathbf{Rep}(\mathbf{G})$ is contained in $\mathbf{Rep}(\mathbf{M})$. Since both $\mathbf{Rep}(\mathbf{M})$ and $\mathbf{Rep}(\mathbf{G})$ are abelian and the forgetful functor $\mathbf{Rep}(\mathbf{M}) \rightarrow \mathbf{Rep}(\mathbf{G})$ is exact, we derive that the kernel of q in $\mathbf{Rep}(\mathbf{M})$ coincides with its kernel in $\mathbf{Rep}(\mathbf{G})$. Thus U is a \mathbf{M} -representation and $j: U \hookrightarrow V$ is a morphism of \mathbf{M} -representations. Hence $\mathbf{Rep}(\mathbf{M})$ is the category of $\mathbf{Rep}(\mathbf{G})$ closed under subobjects.

Theorem 2.5. Let M be an affine monoid k-scheme and let G be its group of units. Assume that G is open and schematically dense in M. Let V be a G-representation of G. There exists an G-representation G and a surjective morpism G: G-representations such that for every G-representation G and a morphism G: G-representations there exists a unique morphism G: G-representations making the triangle



commutative.

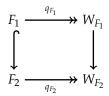
Proof. Assume first that V is finite dimensional. Let $\mathcal K$ be a set of **G**-subrepresentations of V that consists of all $K \subseteq V$ such that V/K carries a structure of **M**-representation. Clearly $\mathcal K = \emptyset$ because $\{0\} \in \mathcal K$. Since V is finite dimensional, there exists a finite subset $\{K_1, ..., K_n\} \subseteq \mathcal K$ such that

$$\bigcap_{i=1}^{n} K_i = \bigcap_{K \in \mathcal{K}} K$$

Then a morphism

$$V/\left(\bigcap_{K\in\mathcal{K}}K\right)\ni v\mapsto \left(v\bmod K_i\right)_{1\leq i\leq n}\in\bigoplus_{i=1}^nV/K_i$$

is a monomorphism and hence by Theorem 2.4 the quotient $W = V/(\bigcap_{K \in \mathcal{K}} K)$ is an M-representation. Let $q:V \twoheadrightarrow W$ be the canonical epimorphism. Consider now a morphism $f:V \to U$ of G-representations, where U is an M-representation. Then $\operatorname{im}(f)$ is a G-subrepresentation of U and by Theorem 2.4 we derive that $\operatorname{im}(f)$ is an M-representation. This implies that $\ker(f)$ is in \mathcal{K} . Hence f factors through g. Thus there exists a unique morphism $f:W \to U$ of G-representations such that $f \cdot g = f$. This completes the proof in case when V is finite dimensional. Now consider the general V. Let \mathcal{F} be the set of all finite dimensional G-representations of V. According to [Monygham, 2020, Corollary 15.2] we deduce that $V = \operatorname{colim}_{F \in \mathcal{F}} F$. By the case considered above we deduce that for every F in \mathcal{F} there exists a universal morphism $g_F: F \to W_F$ of G-representations into an M-representation W_F . Note that if $F_1 \subseteq F_2$ are two elements of \mathcal{F} , then



Thus $\{W_F\}_{F\in\mathcal{F}}$ together with morphisms $W_{F_1} \to W_{F_2}$ for $F_1 \subseteq F_2$ in \mathcal{F} form a diagram parametrized by the poset \mathcal{F} . The category $\mathbf{Rep}(\mathbf{M})$ has small colimits ([Monygham, 2020, Corollary 14.5]) and we define $W = \mathrm{colim}_{F\in\mathcal{F}}W_F$. This is also a colimit of this diagram in the category $\mathbf{Rep}(\mathbf{G})$ by Fact 2.2. We also define $q = \mathrm{colim}_{F\in\mathcal{F}}q_F : V = \mathrm{colim}_{F\in\mathcal{F}}F \to W$. Since a colimit of a family of epimorphisms is an epimorphism, we derive that q is an epimorphism of \mathbf{G} -representations. Suppose now that $f: V \to U$ is a morphism of \mathbf{G} -representations and U is an \mathbf{M} -representation. Then $f_{|F}$ uniquely factors through q_F for every F in \mathcal{F} . Hence by universal property of colimits we derive that f factors through q in a unique way. This completes the proof.

3. Białynicki-Birula functors and its representability for locally linear schemes ${\tt References}$

[Görtz and Wedhorn, 2010] Görtz, U. and Wedhorn, T. (2010). Algebraic Geometry: Part I: Schemes. With Examples and Exercises. Advanced Lectures in Mathematics.

[Monygham, 2020] Monygham (2020). Monoid k-functors and their representations. *github repository: "Monygham/Pedomellon-a-minno"*.