

## MINIMAX THEOREM AND LINEAR PROGRAMMING

### 1. SION'S MINIMAX THEOREM

**Definition 1.1.** Let  $X$  be a topological space and let  $f : X \rightarrow \mathbb{R}$  be a function. Then  $f$  is *lower-semicontinuous* if for every  $r \in \mathbb{R}$  the set

$$\{x \in X \mid f(x) \leq r\}$$

is closed. We say that  $f$  is *upper-semicontinuous* if  $-f$  is lower-semicontinuous.

**Definition 1.2.** Let  $X$  be a convex subset of a linear space over  $\mathbb{R}$  and let  $f : X \rightarrow \mathbb{R}$  be a function. Then  $f$  is *quasiconvex* if for every  $x_1, x_2 \in X$  and  $t \in [0, 1]$  we have

$$f(tx_1 + (1-t)x_2) \leq \max\{f(x_1), f(x_2)\}$$

We say that  $f$  is *quasiconcave* if  $-f$  is quasiconvex.

**Proposition 1.3.** Let  $X$  be a convex subset of a linear space over  $\mathbb{R}$ . Suppose that  $f : X \rightarrow \mathbb{R}$  is a function. Then the following are equivalent.

- (1)  $f$  is quasiconvex.
- (2) For every  $r \in \mathbb{R}$  the set  $\{x \in X \mid f(x) \leq r\}$  is convex.

*Proof.* We prove (1)  $\Rightarrow$  (2). Pick  $r \in \mathbb{R}$ ,  $x_1, x_2 \in X$  and assume that  $f(x_1), f(x_2)$  are both less or equal to  $r$ . Then

$$f(tx_1 + (1-t)x_2) \leq \max\{f(x_1), f(x_2)\} \leq r$$

for every  $t \in [0, 1]$  by (1). Thus the set  $\{x \in X \mid f(x) \leq r\}$  contains line segment joining  $x_1$  with  $x_2$  and hence it is convex. This is (2).

We prove (2)  $\Rightarrow$  (1). Pick  $x_1, x_2 \in X$  and  $t \in [0, 1]$ . Let  $r = \max\{f(x_1), f(x_2)\}$ . Then by (2) we deduce that the set  $\{x \in X \mid f(x) \leq r\}$  is convex. Hence  $f(tx_1 + (1-t)x_2) \leq r$ . This shows (1).  $\square$

**Theorem 1.4** (Sion's theorem). Let  $X$  be a convex, compact subset of a topological vector space over  $\mathbb{R}$  and let  $Y$  be a convex subset of a topological vector space over  $\mathbb{R}$ . Suppose that  $f : X \times Y \rightarrow \mathbb{R}$  is a function such that the following assertions hold.

- (1)  $f_x$  is lower-semicontinuous and quasiconvex for every  $x$  in  $X$ .
- (2)  $f_y$  is upper-semicontinuous and quasiconcave for every  $y$  in  $Y$ .

Then we have

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) = \inf_{y \in Y} \sup_{x \in X} f(x, y)$$

The proof relies on series of lemmas.

**Lemma 1.4.1.** Let  $X$  be a compact topological space and let  $f_1, f_2 : X \rightarrow \mathbb{R}$  be upper-semicontinuous functions. Suppose that

$$\{x \in X \mid f_1(x) \geq r\} \cap \{x \in X \mid f_2(x) \geq r\} = \emptyset$$

for some  $r \in \mathbb{R}$ . Then there exists  $s < r$  such that

$$\{x \in X \mid f_1(x) \geq s\} \cap \{x \in X \mid f_2(x) \geq s\} = \emptyset$$

*Proof of the lemma.* Pick an increasing sequence  $\{s_n\}_{n \in \mathbb{N}}$  of real numbers convergent to  $r$ . Define

$$F_n = \{x \in X \mid f_1(x) \geq s_n\} \cap \{x \in X \mid f_2(x) \geq s_n\}$$

for every  $n \in \mathbb{N}$ . Since  $f_1$  and  $f_2$  are upper-semicontinuous, we derive that the family of sets  $\{F_n\}_{n \in \mathbb{N}}$  consists of closed sets. Moreover, this family is nonincreasing. If  $F_n \neq \emptyset$  for all  $n \in \mathbb{N}$ , then  $\{F_n\}_{n \in \mathbb{N}}$  admits finite intersection property. Hence by compactness of  $X$  there exists  $x$  in  $X$  such that

$$\begin{aligned} x \in \bigcap_{n \in \mathbb{N}} F_n &= \{x \in X \mid \forall_{n \in \mathbb{N}} f_1(x) \geq s_n \text{ and } \forall_{n \in \mathbb{N}} f_2(x) \geq s_n\} = \\ &= \{x \in X \mid f_1(x) \geq r\} \cap \{x \in X \mid f_2(x) \geq r\} \end{aligned}$$

This is contradiction. Thus  $F_{n_0} = \emptyset$  for some  $n_0 \in \mathbb{N}$  and hence for  $s = s_{n_0}$  we have

$$\{x \in X \mid f_1(x) \geq s\} \cap \{x \in X \mid f_2(x) \geq s\} = \emptyset$$

Thus the assertion is proved.  $\square$

**Lemma 1.4.2.** *Let  $X$  be a convex, compact subset of a topological vector space over  $\mathbb{R}$  and let  $Y$  be a convex subset of a topological vector space over  $\mathbb{R}$ . Suppose that  $f : X \times Y \rightarrow \mathbb{R}$  is a function such that the following assertions hold.*

(1)  $f_x$  is lower-semicontinuous and quasiconvex for every  $x$  in  $X$ .

(2)  $f_y$  is upper-semicontinuous and quasiconcave for every  $y$  in  $Y$ .

For every  $y \in Y$  and  $r \in \mathbb{R}$  we denote

$$L_{r,y} = \{x \in X \mid f_y(x) \geq r\}$$

If every set in the family  $\{L_{r,y}\}_{y \in Y}$  is nonempty, then any two sets in this family have nonempty intersection.

*Proof of the lemma.* Suppose conversely that  $L_{r,y_1} \cap L_{r,y_2} = \emptyset$  for some  $y_1, y_2 \in Y$ . Then by Lemma 1.4.1 ( $f_y$  are upper-semicontinuous for all  $y$  so it can be applied) we derive that there exists  $s < r$  such that  $L_{s,y_1} \cap L_{s,y_2} = \emptyset$ . Note that  $L_{r,y} \subseteq L_{s,y}$  for all  $y \in Y$  and hence  $\{L_{s,y}\}_{y \in Y}$  consists of nonempty sets. Let  $[y_1, y_2]$  be an interval joining  $y_1$  and  $y_2$  in  $Y$ . Pick now  $y \in [y_1, y_2]$ . If  $x \in L_{s,y}$ , then

$$s \geq f_y(x) = f_x(y) \leq \max\{f_x(y_1), f_x(y_2)\} = \max\{f_{y_1}(x), f_{y_2}(x)\}$$

by the fact that  $f_x$  is quasiconvex. This implies that  $x \in L_{s,y_1} \cup L_{s,y_2}$ . Thus  $L_{s,y} \subseteq L_{s,y_1} \cup L_{s,y_2}$ . Since  $f_y$  is quasiconcave, we derive that  $L_{s,y}$  is convex and hence connected. Sets  $L_{s,y_1}, L_{s,y_2}$  are closed ( $f_y$  is upper-semicontinuous for each  $y$ ) and disjoint. Therefore, we have either  $L_{s,y} \subseteq L_{s,y_1}$  or  $L_{s,y} \subseteq L_{s,y_2}$  for every  $y \in [y_1, y_2]$ . Now we define

$$F_i = \{y \in [y_1, y_2] \mid L_{s,y} \subseteq L_{s,y_i}\}$$

for  $i = 1, 2$ . Then we have  $[y_1, y_2] = F_1 \cup F_2$  and  $F_1 \cap F_2 = \emptyset$ . Next since  $L_{r,y} \subseteq L_{s,y}$  for every  $y \in Y$ , we deduce that

$$F_i = \{y \in [y_1, y_2] \mid L_{r,y} \subseteq L_{s,y_i}\}$$

for  $i = 1, 2$ . Fix now  $i = 1, 2$ . Consider a sequence  $\{z_n\}_{n \in \mathbb{N}}$  of elements in  $F_i$  convergent to some  $z \in [y_1, y_2]$ . Pick  $x \in L_{r,z}$ . Since  $f_x$  is lower-semicontinuous, we derive that

$$\liminf_{n \rightarrow +\infty} f_{z_n}(x) = \liminf_{n \rightarrow +\infty} f_x(z_n) \geq f_x(z) = f_z(x) \geq r$$

Therefore, the inequality  $s < r$  implies that there exists  $n \in \mathbb{N}$  such that  $f_{z_n}(x) \geq s$ . Hence  $x \in L_{s,z_n}$ . The point  $z_n$  is contained in  $F_i$  and hence  $x \in L_{s,z_n} \subseteq L_{s,y_i}$ . Thus  $\emptyset = L_{s,y_i} \cap L_{r,z} \subseteq L_{s,y_i} \cap L_{s,z}$ . Thus  $L_{s,z} \subseteq L_{s,y_i}$  and  $z \in F_i$ . We deduce that set  $F_i$  is closed in  $[y_1, y_2]$ . Note that we obtain

$$[y_1, y_2] = F_1 \cup F_2, \emptyset = F_1 \cap F_2$$

for some closed subsets  $F_1, F_2$  of  $[y_1, y_2]$ . This is a contradiction with connectedness.  $\square$

Let  $\mathcal{F}$  be a class of all functions  $f : X \times Y \rightarrow \mathbb{R}$  such that the following assertions hold.

- (1)  $X$  is a convex, compact subset of a topological vector space over  $\mathbb{R}$  and  $Y$  is a convex subset of a topological vector space over  $\mathbb{R}$ .
- (2)  $f_x$  is lower-semicontinuous and quasiconvex for every  $x$  in  $X$ .
- (3)  $f_y$  is upper-semicontinuous and quasiconcave for every  $y$  in  $Y$ .

For every  $f : X \times Y \rightarrow \mathbb{R}$  and  $r \in \mathbb{R}$  we denote by  $\mathcal{L}_{f,r}$  the family of sets

$$L_{f,r,y} = \{x \in X \mid f_y(x) \geq r\}$$

parametrized by  $y \in Y$ .

**Lemma 1.4.3.** *For every  $f \in \mathcal{F}$  the following assertion holds. If for some  $r \in \mathbb{R}$  the family  $\mathcal{L}_{f,r}$  consists of nonempty sets, then it has finite intersection property.*

*Proof of the lemma.* Suppose that  $y_1, \dots, y_m$  are points in  $Y$ . We want to show that

$$\bigcap_{i=1}^m L_{f,r,y_i} \neq \emptyset$$

The case  $m = 1$  holds by assumption so we may assume that  $m \geq 2$ . Consider  $X' = L_{f,r,y_m}$ . This is a convex, compact and nonempty subset of a topological vector space over  $\mathbb{R}$ . Next let  $f' : X' \times Y \rightarrow \mathbb{R}$  be the restriction of  $f$ . Then  $f' \in \mathcal{F}$ . In addition

$$L_{f',r,y} = X' \cap L_{f,r,y} = L_{f,r,y_m} \cap L_{f,r,y}$$

and by Lemma 1.4.2 this is nonempty set. So it suffices to prove that

$$\bigcap_{i=1}^{m-1} L_{f',r,y_i} \neq \emptyset$$

Hence the proof goes on induction on  $m$ . □

*Proof of the theorem.* We always have

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) \leq \inf_{y \in Y} \sup_{x \in X} f(x, y)$$

Next pick  $r \in \mathbb{R}$  such that

$$r < \inf_{y \in Y} \sup_{x \in X} f(x, y)$$

Then  $\mathcal{L}_{f,r}$  consists of nonempty and compact subsets of  $X$ . By Lemma 1.4.3 we deduce that  $\mathcal{L}_{f,r}$  admits finite intersection property. Hence there exists  $x$  in  $X$  such that

$$r \leq f(x, y)$$

for every  $y \in Y$ . This implies that

$$r \leq \inf_{y \in Y} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} f(x, y)$$

This shows that

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) \leq \inf_{y \in Y} \sup_{x \in X} f(x, y)$$

□

## 2. CONVEX AND CONCAVE FUNCTIONS

**Definition 2.1.** Let  $X$  be a convex subset of a linear space over  $\mathbb{R}$  and let  $f : X \rightarrow \mathbb{R}$  be a function. Then  $f$  is *convex* if for every  $x_1, x_2 \in X$  and  $t \in [0, 1]$  we have

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

We say that  $f$  is *concave* if  $-f$  is convex.

**Fact 2.2.** Every convex function is quasiconvex.

*Proof.* We left the proof to the reader. □