

1. INTRODUCTION

Throughout this notes k denote a field and \mathbf{G} denote a group scheme over k . We also fix a k -scheme X equipped with an action of \mathbf{G} determined by morphism $a : \mathbf{G} \times_k X \rightarrow X$.

2. CATEGORICAL AND GEOMETRIC QUOTIENTS

Definition 2.1. Let $q : X \rightarrow Y$ be a morphism of k -schemes such that the diagram

$$\mathbf{G} \times_k X \xrightarrow[\text{pr}_X]{a} X \xrightarrow{q} Y$$

is a cokernel in the category of k -schemes. Then $q : X \rightarrow Y$ is a *categorical quotient* of X .

Definition 2.2. Consider a cokernel

$$\mathbf{G} \times_k X \xrightarrow[\text{pr}_X]{a} X \xrightarrow{q} Y$$

in the category of locally ringed spaces over k . If Y is a scheme, then $q : X \rightarrow Y$ is a *geometric quotient* of X .

Fact 2.3. *Every geometric quotient is categorical.*

Proof. Categorical quotient is a cokernel in the category of k -schemes. On the other hand geometric quotient is a cokernel in the category of locally ringed spaces and hence it also satisfies cokernel property in its full subcategory of k -schemes. Thus every geometric quotient is categorical. \square

Corollary 2.4. *Let $q : X \rightarrow Y$ be a morphism of schemes. The following assertions are equivalent.*

(i) *The diagram*

$$\mathbf{G} \times_k X \xrightarrow[\text{pr}_X]{a} X \xrightarrow{q} Y$$

is a cokernel diagram of underlying topological spaces and the diagram

$$\mathcal{O}_Y \xrightarrow{q^\#} q_* \mathcal{O}_X \xrightarrow[q_* \text{pr}_X^\#]{q_* a^\#} q_* (\text{pr}_X)_* \mathcal{O}_{\mathbf{G} \times_k X} = q_* a_* \mathcal{O}_{\mathbf{G} \times_k X}$$

is a kernel diagram in the category of sheaves on Y .

(ii) *q is a geometric quotient of X .*

Proof. This is a consequence of [Monygham, 2019, Theorem 2.9]. \square

In the next result we give a simple example of a universal geometric quotient.

Fact 2.5. *Suppose that \mathbf{G} is an algebraic group over k . Let Y be a k -scheme and consider $\mathbf{G} \times_k Y$ with the action of \mathbf{G} induced by the regular action on the left factor. Then $\text{pr}_Y : \mathbf{G} \times_k Y \rightarrow Y$ is a universal geometric quotient.*

Let $q : X \rightarrow Y$ be a morphism of k -schemes such that $q \cdot \text{pr}_X = q \cdot a$. For a morphism $g : Y' \rightarrow Y$ of k -schemes consider the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

Then there exists a unique action $a' : \mathbf{G} \times_k X' \rightarrow X'$ of \mathbf{G} on X' such that the square above consists of \mathbf{G} -equivariant morphism (we consider Y, Y' as \mathbf{G} -schemes equipped with trivial \mathbf{G} -actions). Keeping this in mind we have the following.

Definition 2.6. A morphism $q : X \rightarrow Y$ is a *uniform categorical (geometric) quotient* of X if for every flat morphism $g : Y' \rightarrow Y$ its base change $q' : X' \rightarrow Y'$ is a categorical (geometric) quotient of X' .

Definition 2.7. A morphism $q : X \rightarrow Y$ is a *universal categorical (geometric) quotient* of X if for every morphism $g : Y' \rightarrow Y$ its base change $q' : X' \rightarrow Y'$ is a categorical (geometric) quotient of X' .

3. TYPES OF ACTIONS AND CRITERIA FOR SMOOTHNESS OF QUOTIENTS

Definition 3.1. The action of \mathbf{G} on X is *separated* if the morphism $\langle a, \text{pr}_X \rangle : \mathbf{G} \times_k X \rightarrow X \times_k X$ has closed set-theoretic image.

Theorem 3.2. Let $q : X \rightarrow Y$ be a geometric quotient of X . Assume that q is universally submersive. Then the following assertions are equivalent.

- (i) The action of \mathbf{G} on X is separated.
- (ii) Y is separated.

Proof. We have a cartesian square

$$\begin{array}{ccc} X \times_Y X & \hookrightarrow & X \times_k X \\ \downarrow & & \downarrow q \times_k q \\ Y & \xrightarrow{\Delta_Y} & Y \times_k Y \end{array}$$

It follows that $X \times_Y X \hookrightarrow X \times_k X$ is a locally closed immersion. Since q is a geometric quotient, we derive that $\langle a, \text{pr}_X \rangle$ factors as a surjective morphism $\mathbf{G} \times_k X \twoheadrightarrow X \times_Y X$ followed by the immersion $X \times_Y X \hookrightarrow X \times_k X$. Thus the action of \mathbf{G} on X is separated if and only if $X \times_Y X$ is a closed subscheme of $X \times_k X$. Since q is universally submersive, we derive that $q \times_k q$ is submersive. As the square above is cartesian we derive that $\Delta_Y(Y) \subseteq Y \times_k Y$ is closed if and only if $X \times_Y X \subseteq X \times_k X$ is closed. Therefore, Y is separated if and only if the action of \mathbf{G} on X is separated. \square

Definition 3.3. The action of \mathbf{G} on X is *free* if the morphism $\langle a, \text{pr}_X \rangle : \mathbf{G} \times_k X \rightarrow X \times_k X$ is a closed immersion.

Definition 3.4. Let x be a k -point of X . Suppose that the orbit morphism $\mathbf{G} \rightarrow X$ of x given by the composition

$$\mathbf{G} = \mathbf{G} \times_k \text{Spec } k \xrightarrow{\text{induced by } x} \mathbf{G} \times_k X \longrightarrow X$$

is a closed immersion. Then the action of \mathbf{G} on X has a *closed free orbit* at x .

Fact 3.5. *If the action of \mathbf{G} on X is free, then every k -point of X has a closed free orbit.*

The following is important result concerning smoothness of geometric quotients.

Theorem 3.6. *Let k be an algebraically closed field and let \mathbf{G} be a smooth group k -scheme. Suppose that $q : X \rightarrow Y$ is a geometric quotient locally of finite type and assume that Y is the spectrum of a complete local noetherian k -algebra such that the residue field of the closed point of Y is k . Then the following assertions hold.*

- (1) *Suppose that x is a k -point of X which has a closed free orbit. Then there exists a \mathbf{G} -equivariant, étale and surjective morphism $f : \mathbf{G} \times_k Y \rightarrow X$ such that the triangle*

$$\begin{array}{ccc} \mathbf{G} \times_k Y & \xrightarrow{f} & X \\ \text{pr}_Y \searrow & & \swarrow q \\ & Y & \end{array}$$

is commutative.

- (2) *If the action of \mathbf{G} on X is free, then f is an isomorphism.*

The proof relies on two algebraic lemmas. The first describe free actions in context of complete rings and the second is a version of Hensel's lemma.

Lemma 3.6.1. *Let (R, \mathfrak{m}, k) be a complete local noetherian k -algebra and let $\sigma : R \rightarrow R[[x_1, \dots, x_n]]$ be a local morphism into a ring of formal power series over R . Assume that the composition*

$$R \xrightarrow{\sigma} R[[x_1, \dots, x_n]] \xrightarrow{f \mapsto f \bmod (x_1, \dots, x_n)} R$$

is the identity and the composition

$$R \xrightarrow{\sigma} R[[x_1, \dots, x_n]] \xrightarrow{f \mapsto f \bmod \mathfrak{m}} (R/\mathfrak{m})[[x_1, \dots, x_n]] = k[[x_1, \dots, x_n]]$$

is surjective. Consider elements y_1, \dots, y_n of R such that $\sigma(y_i) \bmod \mathfrak{m} = x_i$ for $i = 1, \dots, n$. Then the composition

$$R \xrightarrow{\sigma} R[[x_1, \dots, x_n]] \xrightarrow{f \mapsto f \bmod (y_1, \dots, y_n)} (R/(y_1, \dots, y_n))[[x_1, \dots, x_n]]$$

is an isomorphism.

Proof of the lemma. For convenience let ϕ denote the morphism given by the rule $r \mapsto \sigma(r) \bmod (y_1, \dots, y_n)$. Also denote $R/(y_1, \dots, y_n)$ by S . According to assumptions we have

$$\sigma(y_i) = x_i + y_i + \sum_{j=1}^n x_j \cdot \mathfrak{m}[[x_1, \dots, x_n]]$$

for each i . Thus $\phi(y_i) = \sum_{j=1}^n f_{ij} \cdot x_j$ where $f_{ij} \in S$ are elements such that the matrix $[f_{ij}]_{1 \leq i, j \leq n}$ is invertible in S . Hence

$$S[[x_1, \dots, x_n]] = S[[\phi(y_1), \dots, \phi(y_n)]]$$

and ϕ composed with $S[[\phi(y_1), \dots, \phi(y_n)]] \rightarrow S$ is the quotient morphism $R \rightarrow S$. From this observations we derive that ϕ is surjective. It remains to prove that it is injective. Consider z in R such that $\phi(z) = 0$. Suppose that $z \in (y_1, \dots, y_n)^m$ for some $m \in \mathbb{N}$. Write

$$z = \sum_{\alpha \in \Lambda} c_\alpha \cdot y_1^{\alpha_1} \dots y_n^{\alpha_n}$$

for some $c_\alpha \in R$ where $\Lambda = \{\alpha \in \mathbb{N}^n \mid \alpha_1 + \dots + \alpha_n = m\}$. Hence

$$0 = \phi(z) = \sum_{\alpha \in \Lambda} \phi(c_\alpha) \cdot \phi(y_1)^{\alpha_1} \dots \phi(y_n)^{\alpha_n}$$

Thus $\phi(c_\alpha) \in (\phi(y_1), \dots, \phi(y_n))$ for every $\alpha \in \Lambda$. Since ϕ composed with $S[[\phi(y_1), \dots, \phi(y_n)]] \rightarrow S$ is the quotient morphism $R \twoheadrightarrow S$, we derive that

$$c_\alpha \bmod (y_1, \dots, y_n) = \phi(c_\alpha) \bmod (\phi(y_1), \dots, \phi(y_n)) = 0$$

for every $\alpha \in \Lambda$. Thus $c_\alpha \in (y_1, \dots, y_n)$ for every $\alpha \in \Lambda$, which implies that $z \in (y_1, \dots, y_n)^{m+1}$. Thus we proved that

$$\phi(z) = 0 \text{ and } z \in (y_1, \dots, y_n)^m \Rightarrow z \in (y_1, \dots, y_n)^{m+1}$$

By \mathfrak{m} -adic completeness of R this implies that $\phi(z) = 0$ if and only if $z = 0$. Hence ϕ is also injective. \square

Lemma 3.6.2. *Let (R, \mathfrak{m}) be a complete local noetherian ring and let $R \rightarrow S$ be a local morphism into a local noetherian ring. Suppose that there exists a finitely generated R -submodule N of S such that*

$$S = N + \mathfrak{m}S$$

Then $S = N$.

Proof of the lemma. Pick s in S . Since $S = N + \mathfrak{m}S$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $x_n \in \mathfrak{m}^n N$ and

$$s - \sum_{i \leq n} x_i \in \mathfrak{m}^{n+1} S$$

According to the assumption that (R, \mathfrak{m}) is complete with respect to \mathfrak{m} -adic topology and N is finitely generated over R , we deduce that N is complete with respect to \mathfrak{m} -adic topology. Hence there exists a unique element x in N such that

$$x = \sum_{n \in \mathbb{N}} x_n$$

where above series is convergent with respect to \mathfrak{m} -adic topology. Note also that

$$x - \sum_{i \leq n} x_i \in \mathfrak{m}^{n+1} N$$

for every $n \in \mathbb{N}$. Thus we have

$$s - x = \left(s - \sum_{i \leq n} x_i \right) - \left(x - \sum_{i \leq n} x_i \right) \in \mathfrak{m}^{n+1} S + \mathfrak{m}^{n+1} N = \mathfrak{m}^{n+1} S$$

for every $n \in \mathbb{N}$. Hence

$$x - s \in \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n S$$

Since $R \rightarrow S$ is local morphism and S is a local ring, we deduce that $\mathfrak{m}S$ is contained in the maximal ideal of S . By assumptions S is noetherian. Therefore, S is separated with respect to \mathfrak{m} -adic topology. This implies that

$$0 = \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n S$$

Thus $s - x = 0$ and we infer that s is an element of N . This completes the proof that $S = N$. \square

In what follows we shall denote by $\mathbf{G}x$ the closed subscheme determined by the orbit morphism $\mathbf{G} \rightarrow X$ of a k -point x of X which has a closed free orbit. For readers convenience we include the following lemmas, which have topological content.

Lemma 3.6.3. *Let $q : X \rightarrow Y$ be a geometric quotient and assume that Y is the spectrum of a local k -algebra such that the residue field of the closed point o of Y is k . Let x be a k -point of X with free closed orbit, then $q^{-1}(o) = \mathbf{G}x$ as closed subschemes of X .*

Proof of the lemma. Morphism q induces the morphism of residue fields $k(q(x)) \hookrightarrow k(x) = k$ over k . This implies that $k(q(x)) = k$ and hence $q(x)$ is a k -point of Y . Note that o is the unique k -point of Y . Thus $q(x) = o$. Clearly $q^{-1}(o)$ is a closed \mathbf{G} -stable subscheme of X (it is the preimage of o under \mathbf{G} -equivariant q), that contains x . Since $\mathbf{G}x$ is the smallest closed \mathbf{G} -stable subscheme of X containing x , we deduce that $\mathbf{G}x \subseteq q^{-1}(o)$ scheme-theoretically. Consider the pair of arrows

$$\mathbf{G} \times_k X \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{\text{pr}_X} \end{array} X$$

Passing to functors of points we obtain that $a^{-1}(\mathbf{G}x) = \text{pr}_X^{-1}(\mathbf{G}x)$. Since q is the cokernel of the pair (a, pr_X) in the category of topological spaces, we deduce that there exists a closed subset Z of Y such that $q^{-1}(Z) = \mathbf{G}x$. Clearly $o \in Z$ and hence $q^{-1}(o) \subseteq \mathbf{G}x$ set-theoretically. On the other hand above we proved that $\mathbf{G}x \subseteq q^{-1}(o)$ scheme-theoretically. This can only happen if $q^{-1}(o) = \mathbf{G}x$ as closed subschemes of X . \square

Lemma 3.6.4. *Let $q : X \rightarrow Y$ be a geometric quotient and assume that Y is the spectrum of a local k -algebra such that the residue field of the closed point o of Y is k . Let U be an open \mathbf{G} -stable subset of X which contain a k -point. Then $U = X$.*

Proof of the lemma. Consider the pair of arrows

$$\mathbf{G} \times_k X \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{\text{pr}_X} \end{array} X$$

Since U is \mathbf{G} -stable open subset of X , we derive that $\text{pr}_X^{-1}(U) = a^{-1}(U)$. Next by definition q is the cokernel of the above pair in the category of topological spaces. Hence there exists an open subset V of Y such that $U = q^{-1}(V)$. Since U contains a k -point of X , we deduce as in Lemma 3.6.3 that $o \in V$. Thus $V = Y$ and finally $U = q^{-1}(V) = X$. \square

Proof of the theorem. Denote by o the closed point of Y and by e the unit of \mathbf{G} . We first prove (1). Assume that x is a k -point of X which has a closed free orbit. Consider the surjective morphism $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{\mathbf{G},e}$ induced by the orbit morphism $\mathbf{G} \hookrightarrow X$ of x . Since \mathbf{G} is smooth over k , the ring $\mathcal{O}_{\mathbf{G},e}$ is regular. Pick a system of parameters x_1, \dots, x_n of $\mathcal{O}_{\mathbf{G},e}$ and let y_1, \dots, y_n be elements of $\mathcal{O}_{X,x}$ such that y_i is sent to x_i by the morphism $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{\mathbf{G},e}$ for $1 \leq i \leq n$. Define S to be the quotient ring $\mathcal{O}_{X,x}/(y_1, \dots, y_n)$. Let $f : \mathbf{G} \times_k \text{Spec } S \rightarrow X$ be the unique \mathbf{G} -equivariant morphism induced by the surjection $\mathcal{O}_{X,x} \twoheadrightarrow S$. The morphism q induces the morphism $q^\# : \mathcal{O}_{Y,o} \rightarrow \mathcal{O}_{X,x}$ and hence the morphism $\mathcal{O}_{Y,o} \rightarrow S$. By Lemma 3.6.3 we have

$$S/\mathfrak{m}_o S = k$$

where \mathfrak{m}_o is the maximal ideal of $\mathcal{O}_{Y,o}$. According to Lemma 3.6.2 we derive that $\mathcal{O}_{Y,o} \rightarrow S$ is surjective. Let $f : \mathbf{G} \times_k \text{Spec } S \rightarrow X$ be the morphism induced by the surjection $\mathcal{O}_{X,x} \twoheadrightarrow S$. We have a commutative square

$$\begin{array}{ccc} \mathbf{G} \times_k \text{Spec } S & \xrightarrow{f} & X \\ \text{pr}_{\text{Spec } S} \downarrow & & \downarrow q \\ \text{Spec } S & \xrightarrow{j} & Y \end{array}$$

where j is a closed immersion induced by $\mathcal{O}_{Y,0} \twoheadrightarrow S$. According to assumptions q is locally of finite type. Moreover, \mathbf{G} is locally algebraic group over k and hence $\mathrm{pr}_{\mathrm{Spec} S}$ is locally of finite type. These two facts together with the fact that $\mathrm{Spec} S \hookrightarrow Y$ is a closed immersion of noetherian schemes (and thus is of finite type) imply that f is locally of finite type. Then by Lemma 3.6.1 we deduce that f induces an isomorphism

$$\widehat{\mathcal{O}_{X,x}} \longrightarrow \hat{S}[[x_1, \dots, x_n]] = \hat{S} \hat{\otimes}_k \widehat{\mathcal{O}_{\mathbf{G},e}}$$

of completions. Since f is locally of finite type, it follows that f is étale at a k -point of $\mathbf{G} \times_k \mathrm{Spec} S$ determined by the unique k -point of $\mathrm{Spec} S$ and $e \in \mathbf{G}$. Let U be an étale locus of f . It contains a k -point and hence it is nonempty. Moreover, U is open (it is étale locus). Since f is \mathbf{G} -equivariant, we derive that U is \mathbf{G} -stable. Similarly $f(U)$ is open \mathbf{G} -stable subset of X and $x \in f(U)$. Thus by Lemma 3.6.4 we deduce that

$$U = \mathbf{G} \times_k \mathrm{Spec} S, f(U) = X$$

Therefore, f is étale and surjective. Now we pullback $q : X \rightarrow Y$ along the closed immersion $\mathrm{Spec} S \hookrightarrow Y$. We obtain a cartesian square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{j}} & X \\ \tilde{q} \downarrow & & \downarrow q \\ \mathrm{Spec} S & \xrightarrow{j} & Y \end{array}$$

Then f factors as a morphism $\mathbf{G} \times_k \mathrm{Spec} S \rightarrow \tilde{X}$ followed by a closed immersion \tilde{j} . Since f is étale and surjective, we deduce that \tilde{j} is étale and surjective. This implies that \tilde{j} is an isomorphism of schemes. \square

REFERENCES

[Monygham, 2019] Monygham (2019). Locally ringed spaces. *github repository: "Monygham/Pedo-mellon-a-minno"*.