

DIFFERENTIABILITY

1. INTRODUCTION

In this notes we collect basic results on derivatives of functions defined on open subsets of real or complex Banach spaces.

Symbol \mathbb{K} denotes the base field which is either \mathbb{R} or \mathbb{C} .

2. PRELIMINARIES ON BOUNDED MULTILINEAR FORMS ON NORMED SPACES OVER \mathbb{K}

In this section we fix a positive integer n and consider normed spaces $\mathfrak{D}_1, \dots, \mathfrak{D}_n, \mathfrak{X}$ over \mathbb{K} .

Definition 2.1. Let $L : \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \rightarrow \mathfrak{X}$ be a \mathbb{K} -multilinear form. Suppose that there exists $C > 0$ such that

$$\|L(x_1, \dots, x_n)\| \leq C \cdot \|x_1\| \cdot \dots \cdot \|x_n\|$$

for every $x_i \in \mathfrak{D}_i$ for $i \in \{1, \dots, n\}$. Then L is *bounded*.

The following result characterizes bounded multilinear forms.

Theorem 2.2. Let $L : \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \rightarrow \mathfrak{X}$ be a \mathbb{K} -multilinear form. Then the following assertions are equivalent.

- (i) L is continuous.
- (ii) L is continuous at zero n -tuple in $\mathfrak{D}_1 \times \dots \times \mathfrak{D}_n$.
- (iii) L is bounded.

Proof. The implication (i) \Rightarrow (ii) is obvious.

Suppose that L is continuous at zero n -tuple in $\mathfrak{D}_1 \times \dots \times \mathfrak{D}_n$. Assume that there exists a sequence $\{(x_{m,1}, \dots, x_{m,n})\}_{m \in \mathbb{N}_+}$ such that $x_{m,i} \in \mathfrak{D}_i$, $\|x_{m,i}\| = 1$ for each i and

$$\|L(x_{m,1}, \dots, x_{m,n})\| \geq m$$

for each $m \in \mathbb{N}_+$. Define $y_{m,i} = \frac{1}{\sqrt[m]{m}} \cdot x_{m,i}$ for every $i \in \{1, \dots, n\}$ and every $m \in \mathbb{N}_+$. Then $\{y_{m,i}\}_{m \in \mathbb{N}_+}$ tends to zero for every $i \in \{1, \dots, n\}$ and

$$\|L(y_{m,1}, \dots, y_{m,n})\| \geq 1$$

for every $m \in \mathbb{N}_+$. This is a contradiction with the assumption that L is continuous at zero n -tuple in $\mathfrak{D}_1 \times \dots \times \mathfrak{D}_n$. Therefore, there exists $C > 0$ such that

$$\|L(x_1, \dots, x_n)\| \leq C$$

for every $x_i \in \mathfrak{D}_i$ with $\|x_i\| = 1$ for $i \in \{1, \dots, n\}$. Thus the implication (ii) \Rightarrow (iii) holds.

Assume that L is bounded. Pick $x_i \in \mathfrak{D}_i$ and $h_i \in \mathfrak{D}_i$ for $i \in \{1, \dots, n\}$. Define

$$z_0 = (x_1, \dots, x_n), \quad z_i = (x_1 + h_1, \dots, x_i + h_i, x_{i+1}, \dots, x_n)$$

for $i \in \{1, \dots, n\}$. Then

$$\begin{aligned} \|L(x_1 + h_1, \dots, x_n + h_n) - L(x_1, \dots, x_n)\| &= \left\| \sum_{i=1}^n (L(z_i) - L(z_{i-1})) \right\| \leq \\ &\leq \sum_{i=1}^n \|L(z_i) - L(z_{i-1})\| \leq \sum_{i=1}^n C \cdot \|x_1\| \cdot \dots \cdot \|x_{i-1}\| \cdot \|h_i\| \cdot \|x_{i+1}\| \cdot \dots \cdot \|x_n\| \end{aligned}$$

Thus if $(h_1, \dots, h_n) \rightarrow 0$, then $L(x_1 + h_1, \dots, x_n + h_n) - L(x_1, \dots, x_n) \rightarrow 0$. This shows that L is continuous. Hence the proof of the implication (iii) \Rightarrow (i) is completed. \square

Definition 2.3. Let $L : \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \rightarrow \mathfrak{X}$ be a bounded \mathbb{K} -multilinear form. Then we define

$$\|L\| = \sup \left\{ \|L(x_1, \dots, x_n)\| \mid \forall_{i \in \{1, \dots, n\}} x_i \in \mathfrak{D}_i \text{ and } \|x_i\| = 1 \right\}$$

and call it *the operator norm of L* .

Fact 2.4. Let $L : \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \rightarrow \mathfrak{X}$ be a bounded \mathbb{K} -multilinear form. Then

$$\|L(x_1, \dots, x_n)\| \leq \|L\| \cdot \|x_1\| \cdot \dots \cdot \|x_n\|$$

for every $(x_1, \dots, x_n) \in \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n$.

Proof. Left for the reader as an exercise. \square

Theorem 2.5. Let $L(\mathfrak{D}_1, \dots, \mathfrak{D}_n; \mathfrak{X})$ be a \mathbb{K} -vector space of bounded multilinear forms $\mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \rightarrow \mathfrak{X}$ with respect to operations defined pointwise. Suppose that \mathfrak{X} is a Banach space over \mathbb{K} . Then

$$L(\mathfrak{D}_1, \dots, \mathfrak{D}_n; \mathfrak{X}) \ni L \mapsto \|L\| \in [0, +\infty)$$

is a norm which makes $L(\mathfrak{D}_1, \dots, \mathfrak{D}_n; \mathfrak{X})$ into a Banach space over \mathbb{K} .

Proof. We left as an exercise the proof that operator norm is well defined vector space norm on $L(\mathfrak{D}_1, \dots, \mathfrak{D}_n; \mathfrak{X})$. Consider a Cauchy's sequence $\{L_m\}_{m \in \mathbb{N}}$ with respect to operator norm. Then $\{\|L_m\|\}_{m \in \mathbb{N}}$ is Cauchy's sequence and hence is convergent in \mathbb{R} . Fix $(x_1, \dots, x_n) \in \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n$. Then by Fact 2.4

$$\|(L_m - L_k)(x_1, \dots, x_n)\| \leq \|L_m - L_k\| \cdot \|x_1\| \cdot \dots \cdot \|x_n\|$$

for every $m, k \in \mathbb{N}$. This implies that $\{L_m(x_1, \dots, x_n)\}_{m \in \mathbb{N}}$ is a Cauchy's sequence in \mathfrak{X} . Since \mathfrak{X} is a Banach space over \mathbb{K} , we derive that this sequence is convergent. We define

$$L(x_1, \dots, x_n) = \lim_{m \rightarrow +\infty} L_m(x_1, \dots, x_n)$$

Note that we have

$$\|L(x_1, \dots, x_n)\| = \lim_{m \rightarrow +\infty} \|L_m(x_1, \dots, x_n)\| \leq \left(\lim_{m \rightarrow +\infty} \|L_m\| \right) \cdot \|x_1\| \cdot \dots \cdot \|x_n\|$$

Therefore, $L : \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \rightarrow \mathfrak{X}$ is a bounded \mathbb{K} -multilinear form. We claim that L is the limit of $\{L_m\}_{m \in \mathbb{N}}$ with respect to operator norm. For the proof fix $(x_1, \dots, x_n) \in \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n$ such that $\|x_1\| = \dots = \|x_n\| = 1$. Then

$$\|(L - L_m)(x_1, \dots, x_n)\| \leq \|L(x_1, \dots, x_n) - L_m(x_1, \dots, x_n)\| + \|L_m - L_k\|$$

Thus we have

$$\|(L - L_m)(x_1, \dots, x_n)\| \leq \limsup_{k \rightarrow +\infty} \|L_k - L_m\|$$

The left hand side does not depend on x_1, \dots, x_n and we deduce that

$$\|L - L_m\| \leq \limsup_{k \rightarrow +\infty} \|L_k - L_m\|$$

Invoking once again the assumption that $\{\|L_m\|\}_{m \in \mathbb{N}}$ is Cauchy's sequence we infer

$$\lim_{m \rightarrow +\infty} \|L - L_m\| \leq \lim_{m \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \|L_k - L_m\| = 0$$

This completes the proof. \square

Proposition 2.6. The canonical map $L(\mathfrak{D}_1, \dots, \mathfrak{D}_{n-1}; L(\mathfrak{D}_n, \mathfrak{X})) \rightarrow L(\mathfrak{D}_1, \dots, \mathfrak{D}_n; \mathfrak{X})$ which sends L in $L(\mathfrak{D}_1, \dots, \mathfrak{D}_{n-1}; L(\mathfrak{D}_n, \mathfrak{X}))$ to a \mathbb{K} -multilinear form given by formula

$$\mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \ni (x_1, \dots, x_n) \mapsto L(x_1, \dots, x_{n-1})(x_n) \in \mathfrak{X}$$

is an isometry of normed spaces.

Proof. Left for the reader as an exercise. \square

3. NOTION OF FRÉCHET DERIVATIVES

In this section we introduce derivatives and prove their basic properties. We fix Banach spaces $\mathfrak{D}, \mathfrak{X}$ over \mathbb{K} . Let U be an open subset of \mathfrak{D} and let V be an open subset of \mathfrak{X} .

Fact 3.1. Let x be a point in U and let $f : U \rightarrow V$ be a function. Suppose that there are continuous \mathbb{K} -linear maps $L_i : \mathfrak{D} \rightarrow \mathfrak{X}$ for $i = 1, 2$. If both functions

$$\{h \in \mathfrak{D} \mid h \neq 0 \text{ and } x+h \in U\} \ni h \mapsto \frac{f(x+h) - f(x) - L_i(h)}{\|h\|} \in \mathfrak{X}$$

tend to zero as $h \rightarrow 0$, then $L_1 = L_2$.

Proof. By assumption the function

$$\{h \in \mathfrak{D} \mid h \neq 0 \text{ and } x+h \in U\} \ni h \mapsto (L_1 - L_2) \left(\frac{h}{\|h\|} \right) \in \mathfrak{X}$$

tends to zero as $h \rightarrow 0$. This implies that $L_1 - L_2$ sends each vector of the unit sphere in \mathfrak{D} to zero. Thus $L_1 - L_2 = 0$ and hence $L_1 = L_2$. \square

Definition 3.2. Let x be a point in U . A function $f : U \rightarrow V$ is *differentiable at point x* if there exists a continuous \mathbb{K} -linear map $L : \mathfrak{D} \rightarrow \mathfrak{X}$ such that the function

$$\{h \in \mathfrak{D} \mid h \neq 0 \text{ and } x+h \in U\} \ni h \mapsto \frac{f(x+h) - f(x) - L(h)}{\|h\|} \in \mathfrak{X}$$

tends to zero as $h \rightarrow 0$. Moreover, the unique continuous \mathbb{K} -linear map L is *the derivative of f at x* .

Remark 3.3. Notion of differentiability defined above is named by some authors *Fréchet differentiability* after french mathematician Maurice Fréchet.

Remark 3.4. Let x be a point in U and let $f : U \rightarrow V$ be a function differentiable at point x . Then the derivative of f at x is usually denoted by $f'(x)$.

Fact 3.5. Let x be a point in U and let $f : U \rightarrow V$ be a function differentiable at x . Then f is continuous at x .

Proof. Consider the function $\phi_f(h)$ defined on the set

$$\{h \in \mathfrak{D} \mid h \neq 0 \text{ and } x+h \in U\}$$

by formula $f(x+h) - f(x) = f'(x)(h) + \phi_f(h) \cdot \|h\|$. By definition ϕ_f is continuous at zero. In order to complete the argument it suffices to note that the set $\{h \in \mathfrak{D} \mid x+h \in U\}$ contains a neighborhood of zero in \mathfrak{D} . \square

Definition 3.6. A function $f : U \rightarrow V$ is *differentiable* if it is differentiable at each point of U .

4. CHAIN RULE

Chain rule is a basic tools for calculating Fréchet derivatives of a more complex functions.

Theorem 4.1. Let $U \subseteq \mathfrak{D}$, $V \subseteq \mathfrak{X}$, $W \subseteq \mathfrak{Z}$ be open subsets of Banach spaces over \mathbb{K} and let $f : U \rightarrow V$, $g : V \rightarrow W$ be functions. Suppose that f is differentiable at some point x in U and g is differentiable at $f(x)$. Then $g \circ f$ is differentiable at x and the chain rule

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

holds.

Proof. Let L be derivative of f at x and let K be a derivative of g at $f(x)$. For h in \mathfrak{D} such that $x+h \in U$ define $\phi_f(h)$ by formula

$$f(x+h) - f(x) - L(h) = \phi_f(h) \cdot \|h\|$$

Similarly for s in \mathfrak{X} such that $f(x)+s \in V$ define $\phi_g(s)$ by formula

$$g(f(x)+s) - g(f(x)) - K(s) = \phi_g(s) \cdot \|s\|$$

Now pick nonzero h in \mathfrak{D} such that $x+h \in U$ and $f(x+h) \in V$. Then

$$\begin{aligned} \|g(f(x+h)) - g(f(x)) - K(L(h))\| &= \left\| \phi_g(f(x+h) - f(x)) \cdot \|f(x+h) - f(x)\| + K(\phi_f(h) \cdot \|h\|) \right\| \leq \\ &\leq \|\phi_g(f(x+h) - f(x))\| \cdot \|f(x+h) - f(x)\| + \|K(\phi_f(h))\| \cdot \|h\| \leq \\ &\leq \|\phi_g(f(x+h) - f(x))\| \cdot \|f(x+h) - f(x) - L(h)\| + \|\phi_g(f(x+h) - f(x))\| \cdot \|L(h)\| + \|K(\phi_f(h))\| \cdot \|h\| \leq \\ &\leq \left(\|\phi_g(f(x+h) - f(x))\| \cdot \|\phi_f(h)\| + \|\phi_g(f(x+h) - f(x))\| \cdot \|L\| + \|K\| \cdot \|\phi_f(h)\| \right) \cdot \|h\| \end{aligned}$$

According to Fact 3.5 we have $f(x+h) - f(x) \rightarrow 0$ as $h \rightarrow 0$. Hence by differentiability of f at x and g at $f(x)$ we derive that

$$\phi_g(f(x+h) - f(x)) \rightarrow 0, \phi_f(h) \rightarrow 0$$

as $h \rightarrow 0$. Since

$$\{h \in \mathfrak{D} \mid x+h \in U \text{ and } f(x+h) \in V\}$$

contains open neighborhood of zero in \mathfrak{D} , we derive that

$$\{h \in \mathfrak{D} \mid h \neq 0 \text{ and } x+h \in U\} \ni h \mapsto \frac{g(f(x+h)) - g(f(x)) - K(L(h))}{\|h\|} \in \mathfrak{Z}$$

tends to zero as $h \rightarrow 0$. This completes the proof. \square

5. MEAN VALUE INEQUALITY

The main topic of this section is extremely useful inequality, which connects derivatives with local change of a function.

Definition 5.1. Let \mathfrak{D} be an affine space over \mathbb{K} and let x_1, x_2 are points in \mathfrak{D} . The subsets

$$[x_1, x_2] = \{t \cdot x_1 + (1-t) \cdot x_2 \in \mathfrak{D} \mid t \in [0, 1]\}, (x_1, x_2) = \{t \cdot x_1 + (1-t) \cdot x_2 \in \mathfrak{D} \mid t \in (0, 1)\}$$

of \mathfrak{D} are called *the closed* and *the open interval with endpoints* x_1, x_2 , respectively.

Theorem 5.2. Let $U \subseteq \mathfrak{D}, V \subseteq \mathfrak{X}$ be open subsets of Banach spaces over \mathbb{K} and let $f : U \rightarrow V$ be a continuous function. Suppose that x_1, x_2 are points of \mathfrak{D} such that $[x_1, x_2] \subseteq U$ and f is differentiable at every point in (x_1, x_2) . Then the mean value inequality

$$\|f(x_1) - f(x_2)\| \leq \|x_1 - x_2\| \cdot \sup_{z \in (x_1, x_2)} \|f'(z)\|$$

holds.

Proof. For every $t \in [0, 1]$ we define $x(t) = t \cdot x_1 + (1-t) \cdot x_2$. Consider the continuous function $g : [0, 1] \rightarrow V$ given by formula $g(t) = f(x(t))$. Theorem 4.1 together with assumptions imply that g is differentiable over \mathbb{R} at each point of $(0, 1)$ as the composition of functions

$$[0, 1] \ni t \mapsto x(t) \in U, f : U \rightarrow V$$

Moreover, its derivative is the map

$$\mathbb{R} \ni h \mapsto h \cdot f'(x(t)) (x_1 - x_2) \in \mathfrak{X}$$

for every $t \in (0, 1)$. Thus the mean value inequality is implied by the inequality

$$\|g(1) - g(0)\| \leq \sup_{t \in (0,1)} \|g'(t)\|$$

Fix $\epsilon > 0$ and consider the set

$$S = \left\{ s \in [0, 1] \mid \|g(s) - g(0)\| \leq s \cdot \sup_{t \in (0,s)} \|g'(t)\| + s \cdot \epsilon + \epsilon \right\}$$

We shall prove that S satisfies the following assertions.

- (1) There exists $h > 0$ such that $[0, h] \subseteq S$.
- (2) For every s in $S \cap (0, 1)$ there exists $h > 0$ such that $s + h$ is contained in S .
- (3) For every increasing sequence $\{s_n\}_{n \in \mathbb{N}}$ of elements of S its limit is contained in S .

The assertion (1) holds by continuity of g at zero.

Let us prove (2). Write

$$g(s+h) - g(s) - g'(s) \cdot h = \phi_g(h) \cdot |h|$$

for $h > 0$ such that $s+h \leq 1$. Then $\phi_g(h)$ tends to zero as $h \rightarrow 0$ according to differentiability of g at $(0, 1)$. Thus

$$\begin{aligned} \|g(s+h) - g(0)\| &\leq \|g(s+h) - g(s)\| + \|g(s) - g(0)\| \leq \\ &\leq \|\phi_g(h) \cdot h + g'(s) \cdot h\| + s \cdot \sup_{t \in (0,s)} \|g'(t)\| + s \cdot \epsilon + \epsilon \leq \\ &\leq h \cdot \left(\|\phi_g(h)\| + \|g'(s)\| \right) + s \cdot \sup_{t \in (0,s)} \|g'(t)\| + s \cdot \epsilon + \epsilon \end{aligned}$$

Since $\phi_g(h)$ tends to zero as $h \rightarrow 0$, we may pick $h > 0$ such that $s+h \leq 1$ and $\|\phi_g(h)\| \leq \epsilon$. Then

$$\begin{aligned} \|g(s+h) - g(0)\| &\leq h \cdot \left(\epsilon + \|g'(s)\| \right) + s \cdot \sup_{t \in (0,s)} \|g'(t)\| + s \cdot \epsilon + \epsilon \leq \\ &\leq (s+h) \cdot \sup_{t \in (0,s]} \|g'(t)\| + \epsilon \cdot (s+h) + \epsilon \leq (s+h) \cdot \sup_{t \in (0,s+h)} \|g'(t)\| + \epsilon \cdot (s+h) + \epsilon \end{aligned}$$

and hence clearly $s+h$ is in S .

For the proof of (3) fix $\{s_n\}_{n \in \mathbb{N}}$ an increasing sequence of elements of S . Let s be its limit. For every $n \in \mathbb{N}$ we have

$$\|g(s_n) - g(0)\| \leq s_n \cdot \sup_{t \in (0,s_n)} \|g'(t)\| + s_n \cdot \epsilon + \epsilon \leq s \cdot \sup_{t \in (0,s)} \|g'(t)\| + s \cdot \epsilon + \epsilon$$

For $n \rightarrow +\infty$ we obtain

$$\|g(s) - g(0)\| \leq s \cdot \sup_{t \in (0,s)} \|g'(t)\| + s \cdot \epsilon + \epsilon$$

by continuity of g on $[0, 1]$. Thus the proof of (3) is complete.

Using these three assertions we complete the proof. Note first that by (1) the set S contains some elements of $(0, 1)$ and by (3) it contains its least upper bound. According to (2) the least upper bound of S cannot be contained in $(0, 1)$. Thus the least upper bound of S is 1. This proves that

$$\|g(1) - g(0)\| \leq \sup_{t \in (0,1)} \|g'(t)\| + 2 \cdot \epsilon$$

for every $\epsilon > 0$ and thus

$$\|g(1) - g(0)\| \leq \sup_{t \in (0,1)} \|g'(t)\|$$

The proof is complete. □

Corollary 5.3. *Let $U \subseteq \mathfrak{D}, V \subseteq \mathfrak{X}$ be open subsets of Banach spaces over \mathbb{K} and let $f : U \rightarrow V$ be a differentiable function. If U is connected and derivative of f at each point of U is the zero map, then f is constant.*

Proof. Theorem 5.2 shows that for every open convex set $W \subseteq U$ the restriction $f|_W$ is constant. Let y be some element of $f(U)$. It follows that the set $f^{-1}(y)$ is open. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of elements of $f^{-1}(y)$ which is convergent to some point x in U . Pick an open and convex neighborhood W of x . Then for sufficiently large $n \in \mathbb{N}$ we have $x_n \in W$ and thus $x \in f^{-1}(y)$. Therefore, $f^{-1}(y)$ is closed. Hence $f^{-1}(y)$ is a clopen nonempty subset of a connected set U . This shows that $U = f^{-1}(y)$. \square

6. CONVERGENCE OF SEQUENCES OF DIFFERENTIABLE FUNCTIONS

In this section we prove important result concerning convergence of differentiable functions.

Definition 6.1. Let X be a topological space and let Y be a metric space. Let $\{f_n : X \rightarrow Y\}_{n \in \mathbb{N}}$ be a sequence of functions and $f : X \rightarrow Y$ be a function. Suppose that for every point x in X there exists an open neighborhood W of x in X such that the sequence $\{f_n|_W\}_{n \in \mathbb{N}}$ converges uniformly to $f|_W$. Then the sequence $\{f_n\}_{n \in \mathbb{N}}$ is *locally uniformly convergent* to f .

Theorem 6.2. Let $U \subseteq \mathfrak{D}$ be open subset of a Banach space \mathfrak{D} over \mathbb{K} , let \mathfrak{X} be a Banach space over \mathbb{K} and let $\{f_n : U \rightarrow \mathfrak{X}\}_{n \in \mathbb{N}}$ be a sequence of functions. Assume that the following assertions hold.

- (1) U is connected.
- (2) There exists u in U such that the sequence $\{f_n(u)\}_{n \in \mathbb{N}}$ is convergent to some element of \mathfrak{X} .
- (3) f_n is differentiable for every $n \in \mathbb{N}$.
- (4) The sequence of maps

$$\{U \ni x \mapsto f'_n(x) \in L(\mathfrak{D}, \mathfrak{X})\}_{n \in \mathbb{N}}$$

is locally uniformly convergent to a continuous map $g : U \rightarrow L(\mathfrak{D}, \mathfrak{X})$.

Then the sequence $\{f_n\}_{n \in \mathbb{N}}$ converges locally uniformly to a differentiable function $f : U \rightarrow \mathfrak{X}$ and $f'(x) = g(x)$ for every $x \in U$.

Proof. Suppose that z is a point of U such that $\{f_n(z)\}_{n \in \mathbb{N}}$ is convergent. Let W be a bounded, open and convex neighborhood of z in \mathfrak{D} and assume that

$$\{W \ni x \mapsto f'_n(x) \in L(\mathfrak{D}, \mathfrak{X})\}_{n \in \mathbb{N}}$$

converges uniformly. Let x be a point of W . Then

$$\begin{aligned} \|f_n(x) - f_m(x)\| &\leq \| (f_n(x) - f_m(x)) - (f_n(z) - f_m(z)) \| + \|f_n(z) - f_m(z)\| \\ &\leq \|x - z\| \cdot \sup_{y \in (x, z)} \|f'_n(y) - f'_m(y)\| + \|f_n(z) - f_m(z)\| \end{aligned}$$

Hence

$$\sup_{x \in W} \|f_n(x) - f_m(x)\| \leq \text{diam}(W) \cdot \sup_{x \in W} \|f'_n(x) - f'_m(x)\| + \|f_n(z) - f_m(z)\|$$

Since $\{f_n(z)\}_{n \in \mathbb{N}}$ is convergent, \mathfrak{X} is complete and

$$\{W \ni x \mapsto f'_n(x) \in L(\mathfrak{D}, \mathfrak{X})\}_{n \in \mathbb{N}}$$

converges uniformly, we derive that $\{f_n|_W\}_{n \in \mathbb{N}}$ converges uniformly. This proves that the sequence $\{f_n|_W\}_{n \in \mathbb{N}}$ is uniformly convergent for every bounded, open and convex subset W of U such that the sequence

$$\{W \ni x \mapsto f'_n(x) \in L(\mathfrak{D}, \mathfrak{X})\}_{n \in \mathbb{N}}$$

converges uniformly and there exists $z \in W$ such that $\{f_n(z)\}_{n \in \mathbb{N}}$ is convergent.

We define \mathcal{W} as the largest open subset of U such that $\{f_n|_{\mathcal{W}}\}_{n \in \mathbb{N}}$ converges locally uniformly. Note that $u \in \mathcal{W}$ according to the first part of the proof. Suppose that $\{z_n\}_{n \in \mathbb{N}}$ is a sequence of

elements of \mathcal{W} convergent to some point z in U . Pick a bounded, open and convex neighborhood W of z such that

$$\{W \ni x \mapsto f'_n(x) \in L(\mathfrak{D}, \mathfrak{X})\}_{n \in \mathbb{N}}$$

converges uniformly. Then for sufficiently large $n \in \mathbb{N}$ we have $z_n \in W$. Thus $\{f_n|_W\}_{n \in \mathbb{N}}$ is uniformly convergent by the first part of the proof and hence z is in \mathcal{W} . This implies that \mathcal{W} is a closed subset of U . Hence it is a clopen and nonempty subset of U . Since U is connected, we have $U = \mathcal{W}$ and $\{f_n\}_{n \in \mathbb{N}}$ is locally uniformly convergent to some function $f : U \rightarrow \mathfrak{X}$.

Fix x in U and let W be an open neighborhood of zero in \mathfrak{D} such that $[x, x+h] \subseteq U$ for every $h \in W$. We apply Theorem 5.2 to a function $k_n : W \rightarrow \mathfrak{X}$ given by formula

$$k_n(h) = f_n(x+h) - f'_n(x)(h)$$

with derivative $k'_n(h) = f'_n(x+h) - f'_n(x)$ for all $h \in W$. We deduce that

$$\begin{aligned} \|f_n(x+h) - f_n(x) - f'_n(x)(h)\| &= \|k_n(h) - k_n(0)\| \leq \|h\| \cdot \sup_{z \in (0,h)} \|k'_n(z)\| = \\ &= \|h\| \cdot \sup_{z \in (0,h)} \|f'_n(x+z) - f'_n(x)\| = \|h\| \cdot \sup_{z \in (x, x+h)} \|f'_n(z) - f'_n(x)\| \end{aligned}$$

For $n \rightarrow +\infty$ we obtain that

$$\|f(x+h) - f(x) - g(x)(h)\| \leq \|h\| \cdot \sup_{z \in (x, x+h)} \|g(z) - g(x)\|$$

Since g is continuous, we derive that

$$\lim_{h \rightarrow 0} \sup_{z \in (x, x+h)} \|g(z) - g(x)\| = 0$$

and thus $f'(x) = g(x)$. □

7. PARTIAL DERIVATIVES

We fix Banach spaces $\mathfrak{D}_1, \dots, \mathfrak{D}_n, \mathfrak{X}$ over \mathbb{K} for some positive integer n . For each $i \in \{1, \dots, n\}$ let $U_i \subseteq \mathfrak{D}_i$ be an open subset and let V be an open subset of \mathfrak{X} .

Definition 7.1. Consider a function $f : U_1 \times \dots \times U_n \rightarrow V$ and a point $x = (x^1, \dots, x^n) \in U_1 \times \dots \times U_n$. Fix i in $\{1, \dots, n\}$. Suppose that the restriction

$$f|_{\{(x^1, \dots, x^{i-1})\} \times U_i \times \{(x^{i+1}, \dots, x^n)\}} : U_i \rightarrow V$$

is differentiable at x^i . Then its derivative is *the partial derivative of f at x along i -th axis*.

Remark 7.2. In the situation of the definition above we usually denote the partial derivative of f at x along i -th axis by the symbol

$$\frac{\partial f}{\partial x_i}(x)$$

Note that $\frac{\partial f}{\partial x_i}(x) : \mathfrak{D}_i \rightarrow \mathfrak{X}$ is a continuous \mathbb{K} -linear map.

Proposition 7.3. Let $f : U_1 \times \dots \times U_n \rightarrow V$ be a function differentiable at some point $x \in U_1 \times \dots \times U_n$. Fix $i \in \{1, \dots, n\}$. Then

$$\frac{\partial f}{\partial x_i}(x) : \mathfrak{D}_i \rightarrow \mathfrak{X}$$

exists and is the composition of the canonical inclusion $\mathfrak{D}_i \hookrightarrow \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n$ with $f'(x) : \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \rightarrow \mathfrak{X}$.

Proof. Suppose that ϕ_f is a function given by formula

$$\{s \in \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \mid s \neq 0 \text{ and } x+s \in U\} \ni h \mapsto \frac{f(x+s) - f(x) - f'(x)(s)}{\|s\|} \in \mathfrak{X}$$

Denote the restriction $f|_{\{(x^1, \dots, x^{i-1})\} \times U_i \times \{(x^{i+1}, \dots, x^n)\}}$ by f_i and denote the inclusion $\mathfrak{D}_i \hookrightarrow \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n$ by j_i . Recall that for every $h \in \mathfrak{D}_i$ we have $\|h\| = \|j_i(h)\|$. Pick $h \in \mathfrak{D}_i$ such that $h \neq 0$ and $x_i + h \in U_i$. We have

$$\frac{f_i(x_i + h) - f_i(x_i) - (f'(x) \cdot j_i)(h)}{\|h\|} = \frac{f(x + j_i(h)) - f(x) - f'(x)(j_i(h))}{\|j_i(h)\|} = \phi_f(j_i(h))$$

Since by definition of $f'(x)$ the function $\phi_f(s)$ tends to zero as $s \rightarrow 0$, we derive that $\phi_f(j_i(h))$ tends to zero as $h \rightarrow 0$. Thus the partial derivative of f at x along i -th axis exists and is given by formula $f'(x) \cdot j_i$. \square

It is reasonable to ask for the converse of Proposition 7.3. The next theorem gives useful answer to this question under some additional assumptions.

Theorem 7.4. *Let $f : U_1 \times \dots \times U_n \rightarrow V$ be a function and let x be a point in $U_1 \times \dots \times U_n$. Suppose that the following two assertions hold.*

- (1) $\frac{\partial f}{\partial x_i}(u)$ exist for each $i \in \{1, \dots, n\}$ and every point $u \in U_1 \times \dots \times U_n$.
- (2) For each $i \in \{1, \dots, n\}$ the map

$$U_1 \times \dots \times U_n \ni u \mapsto \frac{\partial f}{\partial x_i}(u) \in L(\mathfrak{D}_i, \mathfrak{X})$$

is continuous at x .

Then f is differentiable at x and

$$f'(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) \cdot \text{pr}_i$$

where $\text{pr}_i : \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \rightarrow \mathfrak{D}_i$ is the projection onto i -th axis.

Proof. Pick $h \in \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n$ such that $x + h \in U_1 \times \dots \times U_n$. Write $x = (x^1, \dots, x^n)$ and $h = (h^1, \dots, h^n)$. Let $z_0 = x$ and

$$z_i = (x^n, \dots, x^{i+1}, x^i + h^i, \dots, x^1 + h^1)$$

for each $i \in \{1, \dots, n\}$. Then

$$\begin{aligned} f(x + h) - f(x) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)(h^i) &= \\ &= \sum_{i=1}^n \left(f(z_i) - f(z_{i-1}) - \frac{\partial f}{\partial x_i}(z_{i-1})(h^i) \right) + \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}(z_{i-1})(h^i) - \frac{\partial f}{\partial x_i}(x)(h^i) \right) \end{aligned}$$

For each $i \in \{1, \dots, n\}$ define $\phi_i(h^i)$ by formula

$$f(z_i) - f(z_{i-1}) - \frac{\partial f}{\partial x_i}(z_{i-1})(h^i) = \phi_i(h^i) \cdot \|h^i\|$$

Then we have

$$\begin{aligned} &\left\| f(x + h) - f(x) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)(h^i) \right\| \leq \\ &\leq \|h\| \cdot \sum_{i=1}^n \left(\|\phi_i(h^i)\| + \left\| \frac{\partial f}{\partial x_i}(z_{i-1}) - \frac{\partial f}{\partial x_i}(x) \right\| \right) \end{aligned}$$

By definition of partial derivative along i -th axis at z_{i-1} we derive that $\phi_i(h^i) \rightarrow 0$ as h^i tends to zero. If $h \rightarrow 0$, then by continuity of partial derivatives we have

$$\frac{\partial f}{\partial x_i}(z_{i-1}) - \frac{\partial f}{\partial x_i}(x) \rightarrow 0$$

for every i . These results imply that

$$\sum_{i=1}^n \left(\|\phi_i(h^i)\| + \left\| \frac{\partial f}{\partial x_i}(z_{i-1}) - \frac{\partial f}{\partial x_i}(x) \right\| \right) \rightarrow 0$$

for $h \rightarrow 0$. This completes the proof. \square

8. HIGHER ORDER DERIVATIVES

We introduce higher order Fréchet derivatives. We fix Banach spaces $\mathfrak{D}, \mathfrak{X}$ over \mathbb{K} . Let U be an open subset of \mathfrak{D} and let V be an open subset of \mathfrak{X} .

Definition 8.1. Let $f : U \rightarrow V$ be a function. For each natural number m we define m -th derivative $f^{(m)}$ of f by recursive formula

$$f^{(0)} = f, f^{(m)} = \left(f^{(m-1)} \right)' \text{ for } m > 1$$

If $f^{(m)}$ exists for some natural number m , then f is m -times differentiable on U .

Note that the definition above gives m -th derivative as a function defined on the whole domain.

Remark 8.2. Let $f : U \rightarrow V$ be a m -times differentiable function on U . Then $f^{(m)}$ can be identified with a function

$$f^{(m)} : U \rightarrow L(\underbrace{\mathfrak{D}, \dots, \mathfrak{D}}_{m \text{ times}}; \mathfrak{X})$$

Indeed, the original codomain of $f^{(m)}$ is $L(\mathfrak{D}, L(\mathfrak{D}, \dots, L(\mathfrak{D}, \mathfrak{X}) \dots))$ and according to Proposition 2.6 we have canonical isometry

$$\underbrace{L(\mathfrak{D}, L(\mathfrak{D}, \dots, L(\mathfrak{D}, \mathfrak{X}) \dots))}_{m \text{ times } \mathfrak{D} \text{ symbol}} = L(\underbrace{\mathfrak{D}, \dots, \mathfrak{D}}_{m \text{ times}}; \mathfrak{X})$$

Thus we can regard $f^{(m)}$ as a function on U taking values in $L(\underbrace{\mathfrak{D}, \dots, \mathfrak{D}}_{m \text{ times}}; \mathfrak{X})$.

Now we introduce the notion of the higher derivative defined locally for a point in the domain.

Definition 8.3. Let $f : U \rightarrow V$ be a function on U and let x be a point of U . Let m be a positive integer. Suppose that f is $(m-1)$ -times differentiable on some open neighborhood of x in U . Then m -th derivative of f at x is the derivative of $f^{(m-1)}$ at x . If it exists, then f is m -times differentiable at x .

Remark 8.4. Let x be a point in U and let $f : U \rightarrow V$ be a function. Let m be a positive integer. Assume that f is m -times differentiable at x . Then the m -th derivative of f at x is usually denoted by $f^{(m)}(x)$. Similarly to Remark 8.2 we identify $f^{(m)}(x)$ with an element in $L(\underbrace{\mathfrak{D}, \dots, \mathfrak{D}}_{m \text{ times}}; \mathfrak{X})$.

Theorem 8.5. Let $f : U \rightarrow V$ be a function on U and let x be a point of U . Suppose that f is m -times differentiable at x for some integer m greater or equal 2. Then

$$f^{(m)}(x) \in L(\underbrace{\mathfrak{D}, \dots, \mathfrak{D}}_{m \text{ times}}; \mathfrak{X})$$

is a symmetric \mathbb{K} -multilinear form.

Proof for the second derivative. Pick $h, s \in \mathfrak{D}$. Assume that r is a positive real number greater than norms of h and s . Let I be an open interval in \mathbb{R} containing zero and such that

$$I \subseteq \left\{ t \in \mathbb{R} \mid \forall \xi, \eta \in \mathfrak{D} \left(\|\xi\| < r \text{ and } \|\eta\| < r \right) \Rightarrow x + t \cdot \xi + t \cdot \eta \in U \right\}$$

Consider the expression

$$F(t) = f(x + t \cdot h + t \cdot s) - f(x + t \cdot s) - f(x + t \cdot h) + f(x) - t^2 \cdot f''(x)(s, h)$$

defined for $t \in I$. For fixed $t \in I$ define a function

$$g(\xi) = f(x + t \cdot \xi + t \cdot s) - f(x + t \cdot \xi) - t^2 \cdot f''(x)(s, \xi)$$

Then g is defined for each $\xi \in \mathfrak{D}$ with $\|\xi\| < r$. It is differentiable function and we have formula

$$g'(\xi) = t \cdot f'(x + t \cdot \xi + t \cdot s) - t \cdot f'(x + t \cdot \xi) - t^2 \cdot f''(x)(s)$$

which follows from Theorem 4.1. Thus by Theorem 5.2

$$\begin{aligned} \|F(t)\| &= \|g(h) - g(0)\| \leq \|h\| \cdot \sup_{\xi \in (0, h)} \|g'(\xi)\| = \\ &= t \cdot \|h\| \cdot \sup_{\xi \in (0, h)} \|f'(x + t \cdot \xi + t \cdot s) - f'(x + t \cdot \xi) - t \cdot f''(x)(s)\| \end{aligned}$$

We write

$$f'(x + t \cdot \xi + t \cdot s) = f'(x) + f''(x)(t \cdot \xi + t \cdot s) + \phi(t \cdot \xi + t \cdot s) \cdot t \cdot \|\xi + s\|$$

and

$$f'(x + t \cdot \xi) = f'(x) + f''(x)(t \cdot \xi) + \phi(t \cdot \xi) \cdot t \cdot \|\xi\|$$

Therefore, we have

$$\begin{aligned} \|F(t)\| &\leq t \cdot \|h\| \cdot \sup_{\xi \in (0, h)} \|f'(x + t \cdot \xi + t \cdot s) - f'(x + t \cdot \xi) - t \cdot f''(x)(s)\| = \\ &= t \cdot \|h\| \cdot \sup_{\xi \in (0, h)} \|\phi(t \cdot \xi + t \cdot s) \cdot t \cdot \|\xi + s\| - \phi(t \cdot \xi) \cdot t \cdot \|\xi\|\| = \\ &= t^2 \cdot \|h\| \cdot \sup_{\xi \in (0, h)} \|\phi(t \cdot \xi + t \cdot s) \cdot \|\xi + s\| - \phi(t \cdot \xi) \cdot \|\xi\|\| \end{aligned}$$

Since f' is differentiable at x , we derive that

$$\lim_{t \rightarrow 0} \phi(t \cdot \xi + t \cdot s) = \lim_{t \rightarrow 0} \phi(t \cdot \xi) = 0$$

and hence

$$\lim_{t \rightarrow 0} \frac{F(t)}{t^2} = 0$$

This implies that

$$\lim_{t \rightarrow 0} \frac{f(x + t \cdot h + t \cdot s) - f(x + t \cdot s) - f(x + t \cdot h) + f(x)}{t^2} = f''(x)(s, h)$$

Since the left hand side is symmetric with respect to s and h , we deduce that it also converges to $f''(x)(h, s)$ as $t \rightarrow 0$. Thus $f''(x)(s, h) = f''(x)(h, s)$. According to the fact that h and s are arbitrary we infer that $f''(x)$ is a symmetric \mathbb{K} -bilinear form. \square

Proof of the general case. We proved the theorem for $m = 2$. Suppose that it holds for some $m \geq 2$. We prove it for $m + 1$. For this assume that f is $(m + 1)$ -times differentiable at x . By shrinking domain of f we may assume that f is m -times differentiable function on U . Pick elements $h_1, h_2, h_3, \dots, h_{m+1} \in \mathfrak{D}$ and fix a permutation σ of the set $\{2, 3, \dots, m + 1\}$. Consider the composition of $f^{(m)} : U \rightarrow L(\underbrace{\mathfrak{D}, \dots, \mathfrak{D}}_{m \text{ times}}; \mathfrak{X})$ with the map $\text{ev}_{h_2, h_3, \dots, h_{m+1}} : L(\underbrace{\mathfrak{D}, \dots, \mathfrak{D}}_{m \text{ times}}; \mathfrak{X}) \rightarrow \mathfrak{X}$ given by formula

$L \mapsto L(h_2, h_3, \dots, h_{m+1})$. According to Theorem 4.1 we derive that the derivative of this composition at x is a \mathbb{K} -linear map

$$f^{(m+1)}(x)(-, h_2, h_3, \dots, h_{m+1}) : \mathfrak{D} \rightarrow \mathfrak{X}$$

Similarly the derivative at x of the composition of $f^{(m)} : U \rightarrow L(\underbrace{\mathfrak{D}, \dots, \mathfrak{D}}_{m \text{ times}}; \mathfrak{X})$ with the map $\text{ev}_{h_{\sigma(2)}, h_{\sigma(3)}, \dots, h_{\sigma(m+1)}} : L(\underbrace{\mathfrak{D}, \dots, \mathfrak{D}}_{m \text{ times}}; \mathfrak{X}) \rightarrow \mathfrak{X}$ given by formula $L \mapsto L(h_{\sigma(2)}, h_{\sigma(3)}, \dots, h_{\sigma(m+1)})$ is a \mathbb{K} -linear map

$$f^{(m+1)}(x)(-, h_{\sigma(2)}, h_{\sigma(3)}, \dots, h_{\sigma(m+1)}) : \mathfrak{D} \rightarrow \mathfrak{X}$$

Since we have $\text{ev}_{h_2, h_3, \dots, h_{m+1}} \cdot f^{(m)} = \text{ev}_{h_{\sigma(2)}, h_{\sigma(3)}, \dots, h_{\sigma(m+1)}} \cdot f^{(m)}$ (by inductive assumption), we deduce that $f^{(m+1)}(x)(-, h_2, h_3, \dots, h_{m+1}) = f^{(m+1)}(x)(-, h_{\sigma(2)}, h_{\sigma(3)}, \dots, h_{\sigma(m+1)})$. In particular, we derive that

$$f^{(m+1)}(x)(h_1, h_2, h_3, \dots, h_{m+1}) = f^{(m+1)}(x)(h_1, h_{\sigma(2)}, h_{\sigma(3)}, \dots, h_{\sigma(m+1)})$$

for every permutation σ of the set $\{2, \dots, m, m+1\}$. Next observe that

$$f^{(m+1)}(x) = \left(f^{(m-1)}\right)''(x)$$

and hence

$$\begin{aligned} f^{(m+1)}(x)(h_1, h_2, h_3, \dots, h_{m+1}) &= \left(f^{(m-1)}\right)''(x)(h_1, h_2)(h_3, \dots, h_{m+1}) = \\ &= \left(f^{(m-1)}\right)''(x)(h_2, h_1)(h_3, \dots, h_{m+1}) = f^{(m+1)}(x)(h_2, h_1, h_3, \dots, h_{m+1}) \end{aligned}$$

by the symmetry of the second derivative. Let us summarize these results in slightly different form. For every elements $h_1, h_2, h_3, \dots, h_{m+1} \in \mathfrak{D}$ we have

$$f^{(m+1)}(x)(h_1, h_2, h_3, \dots, h_{m+1}) = f^{(m+1)}(x)(h_{\sigma(1)}, h_{\sigma(2)}, h_{\sigma(3)}, \dots, h_{\sigma(m+1)})$$

for each permutation σ of $\{1, \dots, m+1\}$ such that $\sigma(1) = 1$ and

$$f^{(m+1)}(x)(h_1, h_2, h_3, \dots, h_{m+1}) = f^{(m+1)}(x)(h_2, h_1, h_3, \dots, h_{m+1})$$

This implies that

$$f^{(m+1)}(x)(h_1, h_2, h_3, \dots, h_{m+1}) = f^{(m+1)}(x)(h_{\sigma(1)}, h_{\sigma(2)}, h_{\sigma(3)}, \dots, h_{\sigma(m+1)})$$

for every permutation σ of $\{1, \dots, m+1\}$ and every elements $h_1, h_2, h_3, \dots, h_{m+1} \in \mathfrak{D}$. This completes the proof that $f^{(m+1)}(x)$ is a symmetric \mathbb{K} -multilinear form. \square

9. TAYLOR FORMULAS

In this section we fix Banach spaces $\mathfrak{D}, \mathfrak{X}$ over \mathbb{K} . Let U be an open subset of \mathfrak{D} and let V be an open subset of \mathfrak{X} .

Remark 9.1. Let x be a point in U and let $f : U \rightarrow V$ be a function. Let m be a positive integer. Assume that f is m -times differentiable at x . For every h in \mathfrak{D} we denote

$$f^{(m)}(x)(\underbrace{h, \dots, h}_{m \text{ times}})$$

by $f^{(m)}(x) \cdot h^m$.

Theorem 9.2 (Taylor's theorem). *Let x be a point in U and let $f : U \rightarrow V$ be a function. Let m be a positive integer. Assume that f is m -times differentiable at x . Consider the function $\phi : \{h \in \mathfrak{D} \mid h \neq 0 \text{ and } x+h \in U\} \rightarrow \mathfrak{X}$ defined by formula*

$$f(x+h) - \sum_{i=0}^m \frac{1}{i!} \cdot f^{(i)}(x) \cdot h^i = \phi(h) \cdot \|h\|^m$$

Then $\phi(h) \rightarrow 0$ as $h \rightarrow 0$.

Proof. The proof goes by induction. The case $m = 0$ follows from the definition of Fréchet derivative. Suppose that the result holds for some $m \in \mathbb{N}$ and assume that f is $(m+1)$ -times differentiable at x . Without loss of generality we may assume that f is m -times differentiable on U . Consider the function $g : \{h \in \mathfrak{D} \mid h \neq 0 \text{ and } x+h \in U\} \rightarrow \mathfrak{X}$ given by formula

$$g(h) = f(x+h) - \sum_{i=0}^{m+1} \frac{1}{i!} \cdot f^{(i)}(x) \cdot h^i$$

There exists open subset W of \mathfrak{D} such that $W \subseteq \{h \in \mathfrak{D} \mid h \neq 0 \text{ and } x+h \in U\}$ and $W \cup \{0\}$ is an open neighborhood of zero in \mathfrak{D} . According to Theorem 8.5 we have

$$g'(h) = f'(x+h) - \sum_{i=1}^{m+1} \frac{1}{(i-1)!} \cdot f^{(i)}(x) \left(\underbrace{h, \dots, h}_{i-1 \text{ times}}, - \right) = f'(x+h) - \sum_{i=0}^m \frac{1}{i!} \cdot (f')^{(i)}(x) \cdot h^i$$

for every $h \in W$. Suppose that $\psi : W \rightarrow \mathfrak{X}$ is a function given by formula

$$f'(x+h) - \sum_{i=0}^m \frac{1}{i!} \cdot (f')^{(i)}(x) \cdot h^i = \psi(h) \cdot \|h\|^m$$

Pick $h \in W$. By Theorem 5.2

$$\begin{aligned} \|h\|^{m+1} \cdot \|\phi(h)\| &= \left\| f(x+h) - \sum_{i=0}^{m+1} \frac{1}{i!} \cdot f^{(i)}(x) \cdot h^i \right\| = \|g(h) - g(0)\| \leq \\ &\leq \|h\| \cdot \sup_{s \in (0,h)} \|g'(s)\| = \|h\| \cdot \sup_{s \in (0,h)} \left(\|s\|^m \cdot \|\psi(s)\| \right) = \|h\|^{m+1} \cdot \sup_{s \in (0,h)} \|\psi(s)\| \end{aligned}$$

and thus $\|\phi(h)\| \leq \sup_{s \in (0,h)} \|\psi(s)\|$. Induction hypothesis applied to $f' : U \rightarrow L(\mathfrak{D}, \mathfrak{X})$ and point x takes form

$$\lim_{h \rightarrow 0} \psi(h) = 0$$

Therefore, we also have $\sup_{s \in (0,h)} \|\psi(s)\| \rightarrow 0$ as $h \rightarrow 0$. Hence by inequality $\|\phi(h)\| \leq \sup_{s \in (0,h)} \|\psi(s)\|$ which holds for every $h \in W$ we infer $\phi(h) \rightarrow 0$ for $h \rightarrow 0$. This completes the proof. \square

Theorem 9.3. Let $f : U \rightarrow V$ be m -times differentiable function for some $m \in \mathbb{N}$. Suppose that x is a point in U and h is an element of \mathfrak{D} such that $[x, x+h] \subseteq U$ and f is $(m+1)$ -times differentiable at every point of $(x, x+h)$. Then

$$\left\| f(x+h) - \sum_{i=0}^m \frac{1}{i!} \cdot f^{(i)}(x) \cdot h^i \right\| \leq \frac{\|h\|^{m+1}}{(m+1)!} \cdot \sup_{\xi \in (x, x+h)} \|f^{(m+1)}(\xi)\|$$

Proof. The proof goes by induction. The case $m = 0$ follows from Theorem 5.2. Suppose that the result holds for some $m \in \mathbb{N}$ and assume that f is $(m+1)$ -times differentiable on U and $(m+2)$ -times differentiable at every point of $(x, x+h)$, where $x \in U$ and $h \in \mathfrak{D}$ are such that $[x, x+h] \subseteq U$. Consider the function $g : [0, 1] \rightarrow \mathfrak{X}$ given by formula

$$g(t) = f(x+t \cdot h) - \sum_{i=0}^{m+1} \frac{1}{i!} \cdot f^{(i)}(x) \cdot (t \cdot h)^i$$

According to Theorem 8.5 we have

$$g'(t) = f'(x+t \cdot h)(h) - \sum_{i=1}^{m+1} \frac{1}{(i-1)!} \cdot f^{(i)}(x) \left(\underbrace{t \cdot h, \dots, t \cdot h}_{i-1 \text{ times}}, h \right) =$$

$$= \left(f'(x + t \cdot h) - \sum_{i=0}^m \frac{1}{i!} \cdot (f')^{(i)}(x) \cdot (t \cdot h)^i \right)(h)$$

for every $t \in (0, 1)$. Induction hypothesis applied to $f' : U \rightarrow L(\mathfrak{D}, \mathfrak{X})$ and point x implies

$$\left\| f'(x + t \cdot h) - \sum_{i=0}^m \frac{1}{i!} \cdot (f')^{(i)}(x) \cdot (t \cdot h)^i \right\| \leq \frac{t^{m+1} \cdot \|h\|^{m+1}}{(m+1)!} \cdot \sup_{\xi \in (x, x+h)} \|(f')^{(m+1)}(\xi)\|$$

Now Theorem 5.2 implies that

$$\begin{aligned} & \left\| f(x+h) - \sum_{i=0}^{m+1} \frac{1}{i!} \cdot f^{(i)}(x) \cdot h^i \right\| = \|g(1) - g(0)\| \leq \\ & \leq \sup_{t \in (0,1)} \|g'(t)\| \leq \|h\| \cdot \sup_{t \in (0,1)} \left\| f'(x + t \cdot h) - \sum_{i=0}^m \frac{1}{i!} \cdot (f')^{(i)}(x) \cdot (t \cdot h)^i \right\| \leq \\ & \leq \sup_{t \in (0,1)} \frac{t^{m+1} \cdot \|h\|^{m+2}}{(m+1)!} \cdot \sup_{\xi \in (x, x+h)} \|(f')^{(m+1)}(\xi)\| = \frac{t^{m+1} \cdot \|h\|^{m+2}}{(m+1)!} \cdot \sup_{\xi \in (x, x+h)} \|f^{(m+2)}(\xi)\| \end{aligned}$$

□