

FILTERS IN TOPOLOGY

1. INTRODUCTION

In these short notes we study filters of subsets with their applications to topological spaces. Filters were introduced in [Cartan, 1937] as an effective tool in studying general topological spaces. Here we recapitulate Cartan's approach. Our main goal is to give a concise proof of Tychonoff's theorem on compact spaces.

2. FILTERS

Definition 2.1. Let X be a set and let \mathcal{F} be a nonempty family of subsets of X . Assume that the following assertions hold.

- (1) \mathcal{F} is closed under finite intersections.
- (2) If F_1 and F_2 are subsets of X such that $F_1 \in \mathcal{F}$ and $F_1 \subseteq F_2$, then $F_2 \in \mathcal{F}$.

Then \mathcal{F} is a *filter of subsets of X* .

Definition 2.2. Let X be a set and let \mathcal{F} be a filter of subsets of X . If $\emptyset \notin \mathcal{F}$, then \mathcal{F} is a *proper filter*.

Filters are functorial as it is displayed in the following notion.

Definition 2.3. Let \mathcal{F} be a filter of subsets of a set X and let $f : X \rightarrow Y$ be a map. Then a filter

$$f(\mathcal{F}) = \{G \subseteq Y \mid \text{there exists } F \in \mathcal{F} \text{ such that } f(F) \subseteq G\}$$

of subsets of Y is the *image of \mathcal{F} under f* .

Let us note the following results.

Fact 2.4. Let \mathcal{F} be a filter of subsets of a set X and let $f : X \rightarrow Y$ be a map. If \mathcal{F} is a proper filter, then $f(\mathcal{F})$ is a proper filter.

Proof. Left for the reader as an exercise. □

Now we introduce the notion of ultrafilter and prove by invoking axiom of choice that they exist.

Definition 2.5. Let \mathcal{F} be a proper filter of subsets of a set X such that for every proper filter $\tilde{\mathcal{F}}$ if $\mathcal{F} \subseteq \tilde{\mathcal{F}}$, then $\mathcal{F} = \tilde{\mathcal{F}}$. Then \mathcal{F} is an *ultrafilter of subsets of X* .

Proposition 2.6. Let X be a set and let \mathcal{F} be a proper filter of subsets of X . Then there exists an ultrafilter $\tilde{\mathcal{F}}$ of subsets of X such that $\mathcal{F} \subseteq \tilde{\mathcal{F}}$.

Proof. Consider the family

$$F = \{\mathcal{G} \mid \mathcal{G} \text{ is a proper filter of subsets of } X \text{ and } \mathcal{F} \subseteq \mathcal{G}\}$$

Note that F is nonempty, because $\mathcal{F} \in F$. The inclusion introduces partial order on F and if $L \subseteq F$ is a linearly ordered subset, then

$$\bigcup L$$

is a proper filter. Hence each chain in (F, \subseteq) admits an upper bound. By Zorn's lemma implies that (F, \subseteq) has a maximal element $\tilde{\mathcal{F}}$. Clearly $\tilde{\mathcal{F}}$ is an ultrafilter of subsets of X which contains \mathcal{F} . □

3. FILTERS AND CONVERGENCE IN TOPOLOGICAL SPACES

The following notion play an important role.

Definition 3.1. Let (X, τ) be a topological space and let \mathcal{F} be a proper filter of subsets of X . Consider a point x in X . Suppose that for every open neighborhood U of x we have $U \in \mathcal{F}$. Then filter \mathcal{F} *converges to x with respect to τ* .

Proposition 3.2. Let $(X, \tau), (Y, \theta)$ be topological spaces and let $f : X \rightarrow Y$ be a map. Then the following assertions are equivalent.

- (i) f is a continuous map $(X, \tau) \rightarrow (Y, \theta)$.
- (ii) If \mathcal{F} is a proper filter of subsets of X convergent to some point x with respect to τ , then $f(\mathcal{F})$ converges to $f(x)$ with respect to θ .

Proof. Suppose that f is a continuous map $(X, \tau) \rightarrow (Y, \theta)$. Fix a proper filter \mathcal{F} of subsets of X convergent to x with respect to τ . Fix an open neighborhood V of $f(x)$ with respect to θ . By continuity of f we have $f^{-1}(V) \in \tau$. Thus $f^{-1}(V)$ is an open neighborhood of x with respect to τ . Hence $f^{-1}(V) \in \mathcal{F}$ and we infer that $V \in f(\mathcal{F})$. Since V is arbitrary open neighborhood of $f(x)$ with respect to θ , we derive that $f(\mathcal{F})$ converges to $f(x)$ with respect to θ . This proves the implication (i) \Rightarrow (ii).

Suppose now that (ii) holds. Fix a point x in X and consider an open neighborhood V of $f(x)$ with respect to θ . Define

$$\mathcal{F} = \{F \subseteq X \mid U \setminus f^{-1}(V) \subseteq F \text{ for some open neighborhood } U \text{ of } x \text{ with respect to } \tau\}$$

Then \mathcal{F} is a filter of subsets of X . Note that

$$Y \setminus V = f(X \setminus f^{-1}(V)) \in f(\mathcal{F})$$

This implies that $V \notin f(\mathcal{F})$. If \mathcal{F} is a proper filter, then it converges to x with respect τ and thus $f(\mathcal{F})$ converges to $f(x)$ with respect to θ . Since $V \notin f(\mathcal{F})$, the filter $f(\mathcal{F})$ cannot converge to $f(x)$ with respect to θ . Therefore, \mathcal{F} is not a proper filter. This means that there exists an open neighborhood U of x with respect to τ such that $U \subseteq f^{-1}(V)$. This proves that f is continuous at x as a map $(X, \tau) \rightarrow (Y, \theta)$. Since $x \in X$ is arbitrary, we derive the implication (ii) \Rightarrow (i). \square

REFERENCES

[Cartan, 1937] Cartan, H. (1937). Théorie des filtres. *CR Acad. Sci. Paris*, 205:595–598.