ALGEBRAIZATION OF FORMAL M-SCHEMES

1. Some 2-categorical limits

Consider a category $\mathcal{C}$ and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Our goal is to construct certain 2-categorical limit associated with a pair $(\mathcal{C}, T)$. Consider pairs (X, u) consisting of an object X of $\mathcal{C}$ and an isomorphism $u : T(X) \to X$ in $\mathcal{C}$. If (X, u) and (Y, w) are two such pairs, then a morphism $f : (X, u) \to (Y, u)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that the following square

$$\begin{tikzcd} T(X) \arrow[r, "u"] \arrow[d, "T(f)"'] & X \arrow[d, "f"] \\ T(Y) \arrow[r, "w"'] & Y \end{tikzcd}$$

is commutative. This data give rise to a category $\mathcal{C}(T)$. There exists a forgetful functor $\pi : \mathcal{C}(T) \to \mathcal{C}$ that sends a morphism $f : (X, u) \to (Y, w)$ to $f : X \to Y$. Moreover, there exists a natural isomorphism $\sigma : T \cdot \pi \Rightarrow \pi$ such that the component of σ on an object (X, u) of $\mathcal{C}(T)$ is u. The next result states that the data above form a certain 2-categorical limit.

Theorem 1.1. *Let $(\mathcal{C}, T)$ be a pair consiting of a category and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Suppose that $\mathcal{D}$ is a category, $P : \mathcal{D} \to \mathcal{C}$ is a functor and $\tau : T \cdot P \Rightarrow P$ is a natural isomorphisms. Then there exists a unique functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$.*

Proof. Suppose that $F : \mathcal{D} \to \mathcal{C}(T)$ is a functor such that $P = \pi \cdot F$ and $\sigma_F = \tau$. Pick an object X of $\mathcal{D}$. Then we have $\pi \cdot F(X) = P(X)$ and $\sigma_{F(X)} = \tau_X$. This implies that

$$F(X) = \left(P(X), \tau_X : T(P(X)) \to P(X)\right)$$

Next if $f : X \to Y$ is a morphism in $\mathcal{D}$, then we derive that $\pi(F(f)) = P(f)$. Hence $F(f) = P(f)$. This implies that there exists at most one functor F satisfying the properties above. Note also that formulas

$$F(X) = \left(P(X), \tau_X : T(P(X)) \to P(X)\right), F(f) = P(f)$$

for an object X in $\mathcal{D}$ and a morphism $f : X \to Y$ in $\mathcal{D}$, give rise to a functor that satisfy $P = \pi \cdot F$ and $\sigma_F = \tau$. This establishes existence and the uniqueness of F. $\square$

Assume now that the pair $(\mathcal{C}, T)$ consists of a monoidal category $\mathcal{C}$ and a monoidal endofunctor T. Then there exists a canonical monoidal structure on $\mathcal{C}(T)$. We define $(-) \otimes_{\mathcal{C}(T)} (-)$ by formula

$$(X, u) \otimes_{\mathcal{C}(T)} (Y, w) = \left(X \otimes_{\mathcal{C}} Y, (u \otimes_{\mathcal{C}} w) \cdot m_{X,Y}\right)$$

where

$$m_{X,Y} : T\left(X \otimes_{\mathcal{C}} Y\right) \to T(X) \otimes_{\mathcal{C}} T(Y)$$

is the tensor preserving isomorphism of T. We also define the unit

$$I_{\mathcal{C}(T)} = \left(I, T(I) \cong I\right)$$

where isomorphism $T(I) \cong I$ is precisely the unit preserving isomorphism of the monoidal functor T. The associativity natural isomorphism for $(-) \otimes_{\mathcal{C}(T)} (-)$ and right, left units for $I_{\mathcal{C}(T)}$ in $\mathcal{C}(T)$ are associavity natural isomorphism and right, left units for $\mathcal{C}$, respectively. The structure makes a functor $\pi : \mathcal{C}(T) \to \mathcal{C}$ strict monoidal and σ a monoidal natural isomorphism. The next result states that the data with these extra monoidal structure form a 2-categorical limit in the 2-category of monoidal categories.

Theorem 1.2. *Let $(\mathcal{C}, T)$ be a pair consiting of a monoidal category and its monoidal endofunctor $T : \mathcal{C} \to \mathcal{C}$. Suppose that $\mathcal{D}$ is a monoidal category, $P : \mathcal{D} \to \mathcal{C}$ is a monoidal functor and $\tau : T \cdot P \Rightarrow P$ is a monoidal natural isomorphisms. Then there exists a unique monoidal functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$ as monoidal functors and monoidal transformations.*

Proof. Note that F must be defined as it was described in the proof of Theorem 1.1. Namely we must have

$$F(X) = (P(X), \tau_X : T(P(X)) \to P(X)), F(f) = P(f)$$

for an object X in $\mathcal{C}$ and a morphism $f : X \to Y$ in $\mathcal{C}$.
Suppose now that F admits a structure of a monoidal functor such that $P = \pi \cdot F$ as monoidal functors. Let

$$\left\{m^F_{X,Y} : F(X \otimes_{\mathcal{D}} Y) \to F(X) \otimes_{\mathcal{C}(T)} F(Y)\right\}_{X,Y \in \mathcal{C}}, \phi^F : F(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$$

be the data forming that structure. Since π is a strict monoidal functor and $P = \pi \cdot F$ as monoidal functors, we derive that for any objects X, Y of $\mathcal{C}$

$$\pi(m^F_{X,Y}) : P(X \otimes_{\mathcal{D}} Y) \to P(X) \otimes_{\mathcal{C}} P(Y)$$

is the tensor preserving isomorphism $m^P_{X,Y} : P(X \otimes_{\mathcal{D}} Y) \to P(X) \otimes_{\mathcal{C}} P(Y)$ of the monoidal functor P. By the same argument

$$\pi(\phi_F) : P(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$$

is the unit preserving isomorphism $\phi^P : P(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$ of P. Thus we deduce that for any objects X, Y of $\mathcal{C}$ we have $m^F_{X,Y} = m^P_{X,Y}$ and $\phi^F = \phi^P$. This implies that there exists at most one monoidal functor F such that $P = \pi \cdot F$ as monoidal functors.
On the other hand define $m^F_{X,Y} = m^P_{X,Y}$ for objects X, Y in $\mathcal{C}$ and $\phi^F = \phi^P$. We check now that F equipped with these data is a monoidal functor. Fix objects X, Y in $\mathcal{C}$. The square

$$\begin{tikzcd}
T\left(P\left(X \otimes_{\mathcal{D}} Y\right)\right) \arrow[r, "\tau_{X \otimes_{\mathcal{C}} Y}"] \arrow[d, "T\left(m^P_{X,Y}\right)"'] & P\left(X \otimes_{\mathcal{C}} Y\right) \arrow[d, "m^P_{X,Y}"] \\
T\left(P(X) \otimes_{\mathcal{C}} P(Y)\right) \arrow[r, "(\tau_X \otimes_{\mathcal{C}} \tau_Y) \cdot m^T_{P(X),P(Y)}"'] & P(X) \otimes_{\mathcal{C}} P(Y)
\end{tikzcd}$$

is commutative due to the fact that $\tau : T \cdot P \Rightarrow P$ is a monoidal natural isomorphisms. This implies that $m^F_{X,Y}$ is a morphism in $\mathcal{C}(T)$. It follows that $m^F_{X,Y}$ is a natural isomorphism and due to the definition of associativity in $\mathcal{C}(T)$, we derive its compatibility with $m^F_{X,Y}$. Similarly, since the square

$$\begin{tikzcd}
T\left(P\left(I_{\mathcal{D}}\right)\right) \arrow[r, "\tau_{I_{\mathcal{D}}}"] \arrow[d, "T\left(\phi^P\right)"'] & P\left(I_{\mathcal{D}}\right) \arrow[d, "\phi^P"] \\
T\left(I_{\mathcal{C}}\right) \arrow[r, "\phi^T"'] & I_{\mathcal{C}}
\end{tikzcd}$$

is commutative, we deduce that ϕ^F is a morphism in $\mathcal{C}(T)$. By definition of left and right unit in $\mathcal{C}(T)$, we derive their compatibility with ϕ^F. This finishes the verification of the fact that F with $\{m^F_{X,Y}\}_{X,Y \in \mathcal{C}}$ and ϕ^F is a monoidal functor. Definitions of $\{m^F_{X,Y}\}_{X,Y \in \mathcal{C}}$ and ϕ^F show that the identities $P = \pi \cdot F$ holds on the level of monoidal structures. Since the 2-forgetful functor from

2-category of monoidal categories into 2-category of categories is faithful on 2-cells, the identity $\sigma_F = \tau$ of natural isomorphisms is also the identity of monoidal natural isomorphisms. □

Theorem 1.3. *Let $(\mathcal{C}, T)$ be a pair consiting of a category and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Assume that T preserves colomits. Then the following assertions hold.*

(1) *$\pi : \mathcal{C}(T) \to \mathcal{C}$ creates colimits.*

(2) *Suppose that $\mathcal{D}$ is a category, $P : \mathcal{D} \to \mathcal{C}$ a functor preserving small colimits and $\tau : T \cdot P \Rightarrow P$ a natural isomorphisms. Then the unique functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$ preserves small colimits.*

Proof. Let I be a small category and $D : I \to \mathcal{C}(T)$ be a diagram such that the composition $\pi \cdot D : I \to \mathcal{C}$ admits a colimit given by cocone $(X, \{g_i\}_{i \in I})$. Since T preserves colimits, we derive that $(T(X), \{T(u_i)\}_{i \in I})$ is a colimit of $T \cdot \pi \cdot D : I \to \mathcal{C}$. Now $\sigma_D : T \cdot \pi \cdot D \to \pi \cdot D$ is a natural isomorphism. Hence there exists a unique arrow $u : T(X) \to X$ such that $u \cdot T(g_i) = g_i \cdot \sigma_{D(i)}$ for $i \in I$. Clearly u is an isomorphism and hence (X, u) is an object of $\mathcal{C}(T)$. Moreover, the family $\{g_i\}_{i \in I}$ together with (X, u) is a colimiting cocone over D. This proves **(1)**. Now **(2)** is a consequence of **(1)**. □

Now we apply the results above to certain more general diagrams of categories.

Definition 1.4. A diagram

$$\dots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \dots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

of categories and functors is called *a telescope of categories*.

Definition 1.5. Let

$$\dots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \dots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

be a telescope of monoidal categories and monoidal cocontinuous functors. Then *a 2-categorical limit of the telescope* consists of a monoidal category $\mathcal{C}$, a family of monoidal cocontinuous functors $\{\pi_n : \mathcal{C} \to \mathcal{C}_n\}_{n \in \mathbb{N}}$ and a family of monoidal natural isomorphisms $\{\sigma_n : F_{n+1} \cdot \pi_{n+1} \Rightarrow \pi_n\}_{n \in \mathbb{N}}$ such that the following universal property holds. For any monoidal category $\mathcal{D}$, family $\{P_n : \mathcal{D} \to \mathcal{C}_n\}_{n \in \mathbb{N}}$ of cocontinuous monoidal functors and a family $\{\tau_n : F_n P_{n+1} \Rightarrow P_n\}_{n \in \mathbb{N}}$ of monoidal natural isomorphisms there exists a unique monoidal cocontinuous functor $F : \mathcal{D} \to \mathcal{C}$ satisfying $P_n = \pi_n \cdot F$ and $(\sigma_n)_F = \tau_n$ for every $n \in \mathbb{N}$.

Corollary 1.6. *Let*

$$\dots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \dots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

be a telescope of monoidal categories and monoidal cocontinuous functors. Then its 2-limit exists.

Proof. We decompose the task of constructing its 2-limit as follows. First note that one may form a product $\mathcal{C} = \prod_{n \in \mathbb{N}} \mathcal{C}_n$. Next the functors $\{F_n\}_{n \in \mathbb{N}}$ induce an endofunctor $T = \prod_{n \in \mathbb{N}} F_n \times t$, where $\mathbf{1}$ is the terminal category (it has single object and single identity arrow) and $t : \mathcal{C}_0 \to \mathbf{1}$ is the unique functor. Consider the category $\mathcal{C}(T)$. We define $\{\pi_n : \mathcal{C}(T) \to \mathcal{C}_n\}_{n \in \mathbb{N}}$ to be a family of functors given by coordinates of $\pi : \mathcal{C}(T) \to \mathcal{C}$ and $\{\sigma_n : F_n \cdot \pi_{n+1} \Rightarrow \pi_n\}_{n \in \mathbb{N}}$ to be a family of natural isomorphisms given by coordinates of $\sigma : \pi \cdot T \Rightarrow \pi$. Now this data form a 2-limit of the telescope by compilation of Theorem 1.2 and Theorem 1.3. □

2. FORMAL **G**-SCHEMES

This section is devoted to introducing some notions from formal geometry that are central in this notes. We fix a group scheme $\mathbf{G}$ over k.

Definition 2.1. *A formal* $\mathbf{G}$*-scheme* consists of a sequence $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ of $\mathbf{G}$-schemes together with $\mathbf{G}$-equivariant closed immersions

$$Z_0 \hookrightarrow Z_1 \hookrightarrow ... \hookrightarrow Z_n \hookrightarrow Z_{n+1} \hookrightarrow ...$$

satisfying the following assertions.

(1) We have $Z_0 = Z_n^{\mathbf{G}}$ scheme-theoretically for every $n \in \mathbb{N}$.

(2) Let $\mathcal{I}_n$ be an ideal of $\mathcal{O}_{Z_n}$ defining Z_0. Then for every $m \leq n$ the subscheme $Z_m \subset Z_n$ is defined by $\mathcal{I}_n^{m+1}$.

Example 2.2. Let Z be a $\mathbf{G}$-scheme. Consider a quasi-coherent ideal $\mathcal{I}$ of fixed point subscheme $Z^{\mathbf{G}}$ of Z. Then for every $n \in \mathbb{N}$ ideal $\mathcal{I}^n$ is $\mathbf{G}$-equivariant and hence

$$V(\mathcal{I}) \hookrightarrow V(\mathcal{I}^2) \hookrightarrow ... \hookrightarrow V(\mathcal{I}^n) \hookrightarrow ...$$

is a formal $\mathbf{G}$-scheme. We denote it by $\widehat{Z}$.

Definition 2.3. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ and $\mathcal{W} = \{W_n\}_{n\in\mathbb{N}}$ are formal $\mathbf{G}$-schemes. Then *a morphism* $f : \mathcal{Z} \to \mathcal{W}$ *of formal* $\mathbf{G}$*-schemes* consists of a family of $\mathbf{G}$-equivariant morphisms $f = \{f_n : Z_n \to W_n\}_{n\in\mathbb{N}}$ such that the diagram

$$\begin{tikzcd}
Z_0 \arrow[r, hook] \arrow[d, "f_0"] & Z_1 \arrow[r, hook] \arrow[d, "f_1"] & ... \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[d, "f_n"] & Z_{n+1} \arrow[r, hook] \arrow[d, "f_{n+1}"] & ... \\
W_0 \arrow[r, hook] & W_1 \arrow[r, hook] & ... \arrow[r, hook] & W_n \arrow[r, hook] & W_{n+1} \arrow[r, hook] & ...
\end{tikzcd}$$

is commutative.

Definition 2.4. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{G}$-scheme. Then there we have the corresponding telescope of monoidal categories

$$... \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_{n+1}) \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_n) \longrightarrow ... \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_2) \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_1) \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_0)$$

and cocontinuous monoidal functors given by restricting $\mathbf{G}$-equivariant quasi-coherent sheaves to closed $\mathbf{G}$-subschemes. Then we define *a category* $\mathfrak{Qcoh}(\mathcal{Z})$ *of quasi-coherent sheaves on* $\mathcal{Z}$ as a monoidal category which is a 2-limit of the telescope above. This category is defined uniquely up to a monoidal equivalence.

Let Z be a $\mathbf{G}$-scheme and let $\mathcal{I}$ be a quasi-coherent ideal of $Z^{\mathbf{G}}$. We have a commutative diagram

$$\begin{tikzcd}
V(\mathcal{I}) \arrow[r, hook] \arrow[drrr, hook, bend right] & V(\mathcal{I}^2) \arrow[r, hook] \arrow[drr, hook] & ... \arrow[r, hook] & V(\mathcal{I}^n) \arrow[r, hook] \arrow[d, hook] & ... \\
 & & & Z &
\end{tikzcd}$$

in the category of $\mathbf{G}$-schemes. Thus restriction functors $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}_{\mathbf{G}}(V(\mathcal{I}^n))$ for $n \in \mathbb{N}$ induce a unique cocontinuous monoidal functor $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}(\widehat{Z})$.

Definition 2.5. Let Z be a $\mathbf{G}$-scheme. Then a unique cocontinuous monoidal functor $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}(\widehat{Z})$ is called *the comparison functor*.

Definition 2.6. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{G}$-scheme. A $\mathbf{G}$-scheme Z is called *an algebraization of* $\mathcal{Z}$ if the following two conditions are satisfied.

(1) $\mathcal{Z}$ is isomorphic to $\widehat{Z}$ in the category of formal $\mathbf{G}$-schemes.

(2) The comparison functor $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}\left(\widehat{Z}\right)$ is an equivalence of monoidal categories.

3. Diagonalisable monoid k-schemes

Consider an abstract commutative monoid Γ. Consider the monoid k-algebra $k[\Gamma]$. Recall that $k[\Gamma]$ as a free k-vector space over k and its elements can be uniquely written as

$$\sum_{\gamma\in\Gamma} k_\gamma \cdot \gamma$$

where almost all k_γ are zero for $\gamma \in \Gamma$. Next the k-algebra $k[\Gamma]$ admits a structure of a commutative bialgebra with a comultiplication given by

$$k[\Gamma] \ni \sum_{\gamma\in\Gamma} k_\gamma \cdot \gamma \to \sum_{\gamma\in\Gamma} k_\gamma \cdot (\gamma \otimes \gamma) \in k[\Gamma] \otimes_k k[\Gamma]$$

and a counit

$$k[\Gamma] \ni \sum_{\gamma\in\Gamma} k_\gamma \cdot \gamma \mapsto \sum_{\gamma\in\Gamma} k_\gamma \in k$$

This makes $\operatorname{Spec} k[\Gamma]$ into a monoid k-scheme. We denote this monoid k-scheme by $\mathbf{D}_\Gamma$. For an alternative description note that we have identifications

$$\mathfrak{P}_{\mathbf{D}_\Gamma}(A) \cong \operatorname{Mor}_k\left(k[\Gamma], A\right) \cong \mathbf{Mon}\left(\Gamma, A^\times\right)$$

natural in k-algebra A, where the right hand side denotes the set of morphisms of monoids from Γ to the multiplicative monoid $A^\times$ of A. The k-functor

$$\mathbf{Alg}_k \ni A \mapsto \mathbf{Mon}\left(\Gamma, A^\times\right) \in \mathbf{Set}$$

is a monoid k-functor with respect to multiplication of monoid homomorphisms in $\mathbf{Mon}\left(\Gamma, A^\times\right)$ for every k-algebra A. Hence the identification above makes the functor of points $\mathfrak{P}_{\mathbf{D}_\Gamma}$ into the monoid k-functor and induces precisely the bialgebra structure on $k[\Gamma]$ described above.
Note that if $g : \Gamma_1 \to \Gamma_2$ is a morphism of commutative monoids, then $k[g] : k[\Gamma_1] \to k[\Gamma_2]$ is a morphism of bialgebras (with respect to the structure described above). We denote $\operatorname{Spec} k[g]$ by $\mathbf{D}_g$.

Definition 3.1. Let $\mathbf{M}$ be a monoid k-scheme. We say that $\mathbf{M}$ is *diagonalisable* if there exists an abstract commutative monoid Γ such that $\mathbf{M}$ is visomorphic to $\mathbf{D}_\Gamma$ as a monoid k-scheme.

Now we prove the following important result.

Theorem 3.2. *Suppose that k is commutative ring such that* $\operatorname{Spec} k$ *is connected (i.e. k has no nontrivial idempotents). Consider the functor*

$$\begin{tikzcd}
\Gamma_1 \arrow[d, "g"'] & & \mathbf{D}_{\Gamma_1} \\
{} \arrow[r, mapsto, phantom, ""] & {} & \\
\Gamma_2 & & \mathbf{D}_{\Gamma_2} \arrow[u, "\mathbf{D}_g"']
\end{tikzcd}
\qquad
\left(\Gamma_1 \xrightarrow{g} \Gamma_2\right) \longmapsto \left(\mathbf{D}_{\Gamma_2} \xrightarrow{\mathbf{D}_g} \mathbf{D}_{\Gamma_1}\right)$$

defined on the category of commutative monoids and with values in the category of monoid schemes over k. This functor preserves finite products and induces an equivalence of categories between abstract commutative monoids and diagonalisable monoid schemes over k.

Proof. Suppose that Γ_1,Γ_2 are commutative monoids and $f : k[\Gamma_1] \to k[\Gamma_2]$ is a morphism of bialgebras over k. Let Δ_1,ξ_1 and Δ_2,ξ_2 be comultiplications and counits for $k[\Gamma_1], k[\Gamma_2]$, respectively. Fix $\gamma \in \Gamma_1$ and suppose that $f(\gamma) = \sum_{\gamma'\in\Gamma_2} k_{\gamma'}\cdot\gamma'$. The fact that f is a morphism of bialgebras over k implies that

$$\Delta_2\left(f\left(\gamma\right)\right) = \left(f\otimes_k f\right)\left(\Delta_1(\gamma)\right) = \left(f\otimes_k f\right)\left(\gamma\otimes_k\gamma\right) = f(\gamma)\otimes_k f(\gamma)$$

Substituting $\sum_{\gamma'\in\Gamma_2} k_{\gamma'}\cdot\gamma'$ for $f(\gamma)$ we deduce that

$$\sum_{\gamma'\in\Gamma_2} k_{\gamma'}\cdot\left(\gamma'\otimes\gamma'\right) = \sum_{\gamma'\in\Gamma_2}\sum_{\gamma''\in\Gamma_2} k_{\gamma'}\cdot k_{\gamma''}\cdot\left(\gamma'\otimes\gamma''\right)$$

Thus we derive that

$$k_{\gamma'}\cdot k_{\gamma''} = \begin{cases} 0 & \text{if } \gamma'\neq\gamma'' \\ k_{\gamma'} & \text{if } \gamma' = \gamma''\end{cases}$$

Since there are no nontrivial idempotents in k, this implies that $k_{\gamma'} = 0,1$ for each $\gamma'\in\Gamma_2$. Again by the fact that f is a morphism of k-bialgebras, we derive that

$$\xi_1(\gamma) = \xi_2\left(f(\gamma)\right)$$

Substituting $\sum_{\gamma'\in\Gamma_2} k_{\gamma'}\cdot\gamma'$ for $f(\gamma)$ yields that

$$\sum_{\gamma'\in\Gamma_2} k_{\gamma'} = 1$$

Combining this with previously established fact that $k_{\gamma'} = 0,1$ for each $\gamma'\in\Gamma_2$ we deduce that there exists precisely one $\gamma'\in\Gamma_2$ such that $f(\gamma) = \gamma'$. This proves that $f(\Gamma_1)\subseteq\Gamma_2$. Since f preserves multiplication and unit, we deduce that $f = k[g]$ for some homomorphism of abstract monoids $g : \Gamma_1\to\Gamma_2$. Thus the functor described in the statement is full.
It is also clearly faithful. Indeed, for two distinct morphisms of monoids $g_1, g_2 : \Gamma_1\to\Gamma_2$ we have $k[g_1]\neq k[g_2]$ and hence $\operatorname{Spec} k[g_1]\neq\operatorname{Spec} k[g_2]$.
By definition of diagonalisable monoid the image of the functor is an essential subcategory of the category of diagonalisable k-schemes.
Finally, consider commutative monoids Γ_1,Γ_2 and note that isomorphism

$$k[\Gamma_1\times\Gamma_2]\ni\sum_{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2} k_{(\gamma_1,\gamma_2)}\cdot(\gamma_1,\gamma_2)\mapsto\sum_{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2} k_{(\gamma_1,\gamma_2)}\cdot\gamma_1\otimes\gamma_2\in k[\Gamma_1]\otimes_k k[\Gamma_2]$$

is a morphism of k-bialgebras. This implies that the functor described in the statement preserves binary products. The functor preserves terminal objects, since k is a monoid k-algebra for trivial (zero) commutative monoid. □

4. Representations of diagonalisable monoid k-schemes

Definition 4.1. Let Γ be a commutative monoid and let $\mathbf{D}_\Gamma$ be the corresponding monoid k-scheme. Suppose that V is a representation of $\mathbf{D}_\Gamma$ with respect to a morphism of monoid k-functors given by

$$\mathfrak{P}_{\mathbf{D}_\Gamma}(A) = \mathbf{Mod}\left(\Gamma, A^\times\right)\ni f\mapsto f(\gamma)\cdot(-)\in\mathcal{L}_V(A)$$

where γ is a fixed element of Γ. Then V is called *a representation of* $\mathbf{D}_\Gamma$ *of weight* γ.

Fact 4.2. *Let Γ be a commutative monoid and let γ be its element. Suppose that V is a representation of $\mathbf{D}_\Gamma$ of weight γ. Then V can be equivalently described as a comodule over $k[\Gamma]$ with respect to the following coaction*

$$V_\gamma\ni v\mapsto\gamma\otimes v\in k[\Gamma]\otimes_k V_\gamma$$

Proof. Denote by $\rho : \mathfrak{P}_{\mathbf{D}_\Gamma}\to\mathcal{L}_V$ the morphism of monoid k-functors that makes a V into a representation of $\mathbf{D}_\Gamma$. Then $\rho\left(1_{\mathbf{D}_\Gamma}\right)$ is a morphism of $k[\Gamma]$-modules

$$k[\Gamma]\otimes_k V\ni 1\otimes v\mapsto\gamma\otimes v\in k[\Gamma]\otimes_k V$$

We obtain the coaction of $k[\Gamma]$ on V corresponding to ρ by transforming morphism $\rho\left(1_{\mathbf{D}_\Gamma}\right)$ via the canonical isomorphism

$$\mathrm{Hom}_{k[\Gamma]}\left(k[\Gamma]\otimes_k V, k[\Gamma]\otimes_k V\right)\cong \mathrm{Hom}_k\left(V, k[\Gamma]\otimes_k V\right)$$

Thus this coaction is given by formula

$$V\ni v\mapsto \gamma\otimes v\ni k[\Gamma]\otimes_k V$$

$\square$

Fact 4.3. *Let Γ be a commutative monoid and let $\mathbf{D}_\Gamma$ be the corresponding monoid k-scheme. Suppose that V_1, V_2 are representations of $\mathbf{D}_\Gamma$ and assume that V_1, V_2 have weights γ_1, γ_2 with $\gamma_1\neq\gamma_2$. Then*

$$\mathrm{Hom}_{\mathbf{D}_\Gamma}\left(V_1, V_2\right) = 0$$

Proof. This follows from Fact 4.2. $\square$

Let Γ be a commutative monoid and let $\mathbf{D}_\Gamma$ be the corresponding monoid k-scheme. For every representation V of $\mathbf{D}_\Gamma$ and fixed γ in Γ define

$$V[\gamma] = \left\{v\in V \,\middle|\, d(v) = \gamma\otimes v\right\}$$

where $d : V\to k[\Gamma]\otimes_k V$ is the coaction. Then $V[\gamma]$ is a subrepresentation of V. Note that according to Fact 4.2 $V[\gamma]$ is a subrepresentation of V of weight γ.

Proposition 4.4. *Let Γ be a commutative monoid and let $\mathbf{D}_\Gamma$ be the corresponding monoid k-scheme. For every representation V of $\mathbf{D}_\Gamma$ we have a direct sum*

$$V = \bigoplus_{\gamma\in\Gamma} V[\gamma]$$

Proof. Let Δ, ξ be the comultiplication and the counit of $k[\Gamma]$, respectively. Let $d : V\to k[\Gamma]\otimes_k V$ be a coaction. Fix $v\in V$. Then we have a unique decomposition $d(v) = \sum_{\gamma\in\Gamma}\gamma\otimes v_\gamma$. Then

$$\sum_{\gamma\in\Gamma}\gamma\otimes\gamma\otimes v_\gamma = \left(\Delta\otimes_k 1_V\right)\left(d(v)\right) = \left(1_{k[\Gamma]}\otimes_k d\right)\left(d(v)\right) = \sum_{\gamma\in\Gamma}\gamma\otimes d(v_\gamma)$$

This implies that $d(v_\gamma) = \gamma\otimes v_\gamma$ and hence $v_\gamma\in V[\gamma]$. On the other hand we have

$$v = \xi\left(d(v)\right) = \sum_{\gamma\in\Gamma} v_\gamma$$

Thus

$$v\in\sum_{\gamma\in\Gamma} V[\gamma]$$

Hence

$$V = \sum_{\gamma\in\Gamma} V[\gamma]$$

Moreover, suppose that $\sum_{\gamma\in\Gamma} v_\gamma = \sum_{\gamma\in\Gamma} v'_\gamma$ for some $v_\gamma, v'_\gamma\in V[\gamma]$. Then

$$\sum_{\gamma\in\Gamma}\gamma\otimes v_\gamma = d\left(\sum_{\gamma\in\Gamma} v_\gamma\right) = d\left(\sum_{\gamma\in\Gamma} v'_\gamma\right) = \sum_{\gamma\in\Gamma}\gamma\otimes v'_\gamma$$

and hence $v_\gamma = v'_\gamma$ for each $\gamma\in\Gamma$. This proves the direct decomposition of V as we claimed. $\square$

Corollary 4.5. *Let k be a field. Suppose that Γ is a commutative monoid and let $\mathbf{D}_\Gamma$ be the corresponding monoid k-scheme. Then the category* **Rep** $\left(\mathbf{D}_\Gamma\right)$ *is semisimple. Moreover, each irreducible representation of $\mathbf{D}_\Gamma$ is isomorphic to one-dimensional representation of weight γ for a unique $\gamma\in\Gamma$.*

Proof. This is a consequence of Fact 4.3 and Proposition 4.4. $\square$

5. Diagonalisable group k-schemes

Let Γ be an abstract commutative group. Then in addition to k-bialgebra structure the k-algebra $k[\Gamma]$ admits an antipode map

$$k[\Gamma] \ni \sum_{\gamma\in\Gamma} k_\gamma \cdot \gamma \mapsto \sum_{\gamma\in\Gamma} k_\gamma \cdot \gamma^{-1} \in k[\Gamma]$$

That makes $k[\Gamma]$ into a commutative Hopf k-algebra. Thus $\mathbf{D}_\Gamma$ is a group k-scheme in this case. The forgetful functor $|-| : \mathbf{Ab} \to \mathbf{CMon}$ sending commutative (abelian) group to its underlying commutative monoid admits left adjoint $(-)_{\mathbf{Grp}} : \mathbf{CMon} \to \mathbf{Ab}$. Hence for every commutative monoid Γ there exists a universal commutative group $\Gamma_{\mathbf{Grp}}$ generated by Γ. This is used in the following result.

Proposition 5.1. *Let Γ be a commutative monoid. Then the canonical morphism $\Gamma \to \Gamma_{\mathbf{Grp}}$ induces a monomorphism of monoid k-schemes*

$$\mathbf{D}_{\Gamma_{\mathbf{Grp}}} \hookrightarrow \mathbf{D}_\Gamma$$

that identifies $\mathbf{D}_{\Gamma_{\mathbf{Grp}}}$ with $(\mathbf{D}_\Gamma)^*$.

Proof. For every k-algebra we have an isomorphism of groups

$$\mathbf{Mon}\left(\Gamma, A^\times\right)^* \cong \mathbf{Mon}\left(\Gamma, A^*\right) \cong \mathbf{Mon}\left(\Gamma_{\mathbf{Grp}}, A^*\right) \cong \mathbf{Mon}\left(\Gamma_{\mathbf{Grp}}, A^\times\right)$$

natural in A. Note that this natural isomorphisms identifies $\mathfrak{P}^*_{\mathbf{D}_\Gamma}$ with $\mathfrak{P}_{\mathbf{D}_{\Gamma_{\mathbf{Grp}}}}$ by morphism induced by the unit $\Gamma \to \Gamma_{\mathbf{Grp}}$ of the adjunction $|-| \dashv (-)_{\mathbf{Grp}}$. □

Corollary 5.2. *Let $\mathbf{G}$ be a group k-scheme. Suppose that G is isomorphic to $\mathbf{D}_\Gamma$ as a monoid k-scheme for some commutative monoid Γ. Then Γ is a group.*

Proof. Suppose that $\mathbf{G} \cong \mathbf{D}_\Gamma$ as a monoid k-schemes. We derive that $\mathbf{D}_\Gamma$ is a group k-scheme. Hence $\mathbf{D}_{\Gamma_{\mathbf{Grp}}} \hookrightarrow \mathbf{D}_\Gamma$ is an isomorphism of monoid k-schemes. This implies that $\Gamma = \Gamma_{\mathbf{Grp}}$ and thus Γ is an abstract group. □

Definition 5.3. Let $\mathbf{G}$ be a group k-scheme. We say that $\mathbf{G}$ is *diagonalisable group k-scheme* if it is diagonalisable as a monoid scheme over k.

Example 5.4. Let $\mathbb{Z}$ be a commutative group of additive integers. We denote by $\mathbb{G}_m$ the monoid k-scheme $\mathbf{D}_\mathbb{Z}$. Note that $\mathbb{G}_m$ represents the group k-functor

$$\mathbf{Alg}_k \ni A \mapsto A^* \in \mathbf{Ab}$$

We call $\mathbb{G}_m$ *the multiplicative group over k.*

Definition 5.5. Let $\mathfrak{G}$ be a monoid k-functor. Then the morphisms $\mathfrak{G} \to \mathfrak{P}_{\mathbb{G}_m}$ of monoid k-functors are called *characters of* $\mathfrak{G}$. They form a group $\mathcal{X}(\mathfrak{G})$ called *the group of characters of* $\mathfrak{G}$.

Corollary 5.6. *Suppose that k is commutative ring such that* $\operatorname{Spec} k$ *is connected (i.e. k has no nontrivial idempotents). Functors*

$$\begin{array}{ccccccc}
\Gamma_1 & & \mathbf{D}_{\Gamma_1} & \quad & \mathbf{G}_1 & & \mathcal{X}(\mathbf{G}_1) \\
{\scriptstyle g}\big\downarrow & \longmapsto & \big\uparrow{\scriptstyle \mathbf{D}_g} & & {\scriptstyle f}\big\downarrow & \longmapsto & \big\uparrow{\scriptstyle \mathcal{X}(f)} \\
\Gamma_2 & & \mathbf{D}_{\Gamma_2} & & \mathbf{G}_2 & & \mathcal{X}(\mathbf{G}_2)
\end{array}$$

induce an equivalence between categories of abstract commutative groups and diagonalisable group schemes over k..

Proof. This is a consequence of Theorem 3.2. □

6. Preliminaries

LS TODO: Większość z wyników, które tutaj są, powinna być w teoretycznym wstępie. Idea jest taka, by tutaj w zasadzie tylko przygotować notację do dowodu głównego twierdzenia.

6.1. Results on linear representations.

Proposition 6.1. *Let $\mathbf{M}$ be an affine monoid k-scheme and let V be a representation of $\mathbf{M}$. Then for every k-algebra A the natural morphism of A-modules*

$$V^{\mathbf{M}} \otimes_k A \to (A \otimes_k V)^{\mathbf{M}_A}$$

is an isomorphism.

Proof. Note that we have a left exact sequence of k-vector spaces defining invariants

$$0 \longrightarrow V^{\mathbf{M}} \longrightarrow V \xrightarrow{\Delta - p} \Gamma(\mathbf{M}, \mathcal{O}_{\mathbf{M}}) \otimes_k V$$

where $\Delta : V \to \Gamma(\mathbf{M}, \mathcal{O}_{\mathbf{M}}) \otimes_k V$ is the coaction and $p : V \to \Gamma(\mathbf{M}, \mathcal{O}_{\mathbf{M}}) \otimes_k V$ is the trivial coaction defined by formula $p(v) = 1 \otimes v$ for every v in V. Now tensoring the sequence with k-algebra A yields a left exact sequence

$$0 \longrightarrow V^{\mathbf{M}} \otimes_k A \longrightarrow A \otimes_k V \xrightarrow{\Delta_A - p_A} \Gamma(\mathbf{M}_A, \mathcal{O}_{\mathbf{M}_A}) \otimes_A (A \otimes_k V)$$

where Δ_A is the coaction on $A \otimes_k V$ induced by Δ and p_A is the trivial coaction on $A \otimes_k V$. This shows that $V^{\mathbf{M}} \otimes_k A \to (A \otimes_k V)^{\mathbf{M}_A}$ is an isomorphism. □

Proposition 6.2. *Let $\mathbf{G}$ be an affine group k-scheme and let V, W be representations of $\mathbf{G}$. If V is finite dimensional, then for every k-algebra A the canonical morphism*

$$A \otimes_k \mathrm{Hom}_{\mathbf{G}}(V, W) \longrightarrow \mathrm{Hom}_{\mathbf{G}_A}(A \otimes_k V, A \otimes_k W)$$

is an isomorphism of A-modules.

Proof. Fix a k-algebra A. Since V is finite dimensional, for every k-algebra B there exists an isomorphism $B \otimes_k \mathrm{Hom}_k(V, W) \to \mathrm{Hom}_B(B \otimes_k V, B \otimes_k W)$ of B-modules natural in B. This implies that $\mathrm{Hom}_k(V, W)$ is a representation of $\mathbf{G}$ via the action given by formula

$$(g \cdot f)(v) = g \cdot f(g^{-1} \cdot v)$$

where $f \in \mathrm{Hom}_B(B \otimes_k V, B \otimes_k W)$, $v \in B \otimes_k V$ and $g \in \mathfrak{P}_{\mathbf{G}}(B)$. Similarly $\mathrm{Hom}_A(A \otimes_k V, A \otimes_k W)$ is a representation of $\mathbf{G}_K$ and the canonical isomorphism $A \otimes_k \mathrm{Hom}_k(V, W) \to \mathrm{Hom}_A(A \otimes_k V, A \otimes_k W)$ of A-modules is $\mathbf{G}_A$-equivariant. Now we apply Proposition 6.1 to derive a chain of isomorphisms

$$\mathrm{Hom}_A(A \otimes_k V, A \otimes_k W)^{\mathbf{G}_A} \cong (A \otimes_k \mathrm{Hom}_k(V, W))^{\mathbf{G}_A} \cong A \otimes_k \mathrm{Hom}_k(V, W)^{\mathbf{G}}$$

of A-modules. Since we have identifications

$$\mathrm{Hom}_{\mathbf{G}_A}(A \otimes_k V, A \otimes_k W) \cong \mathrm{Hom}_A(A \otimes_k V, A \otimes_k W)^{\mathbf{G}_A}, \ \mathrm{Hom}_{\mathbf{G}}(V, W) \cong \mathrm{Hom}_k(V, W)^{\mathbf{G}}$$

we deduce the statement. □

Proposition 6.3. *Let $\mathbf{G}$ be an affine group scheme over k and let V, W be $\mathbf{G}$-representation such that $\mathrm{Hom}_{\mathbf{G}}(U, W) = 0$ for every finite dimensional $\mathbf{G}$-subrepresentation of V. Then for every k-algebra A we have*

$$\mathrm{Hom}_{\mathbf{G}_A}(A \otimes_k V, A \otimes_k W) = 0$$

Proof. Let $\mathcal{F}$ be a set of all finite dimensional $\mathbf{G}$-subrepresentations of V. Since V is a $\mathbf{G}$-representation and $\mathbf{G}$ is an affine group k-scheme, we have

$$V = \mathrm{colim}_{U\in\mathcal{F}}\, U$$

Fix k-algebra A then we have identifications of A-modules

$$\mathrm{Hom}_{\mathbf{G}_A}\left(A\otimes_k V, A\otimes_k W\right) = \mathrm{Hom}_{\mathbf{G}_A}\left(A\otimes_k \mathrm{colim}_{U\in\mathcal{F}} U, A\otimes_k W\right) =$$
$$= \mathrm{Hom}_{\mathbf{G}_A}\left(\mathrm{colim}_{U\in\mathcal{F}} A\otimes_k U, A\otimes_k W\right) = \lim_{U\in\mathcal{F}} \mathrm{Hom}_{\mathbf{G}_A}\left(A\otimes_k U, A\otimes_k W\right) =$$
$$= \lim_{U\in\mathcal{F}}\Big(A\otimes_k \mathrm{Hom}_{\mathbf{G}}\left(U, W\right)\Big) = 0$$

where we apply Proposition 6.2. □

Corollary 6.4. *Let $\mathbf{G}$ be an affine group scheme over k and let $\mathfrak{G}$ be a monoid k-functor. Denote by Λ the set of isomorphism classes of irreducible $\mathbf{G}$-representations. Suppose that V is a representation of both $\mathbf{G}$ and $\mathfrak{G}$ and assume that their actions on V commute. Assume that V is completely reducible as a $\mathbf{G}$-representation and consider the decomposition*

$$V = \bigoplus_{\lambda\in\Lambda} V[\lambda]$$

onto isotypic components with respect to the action of $\mathbf{G}$. Then for every λ in Λ the subspace $V[\lambda]$ is a $\mathfrak{G}$-subrepresentation of V.

Proof. Part of the structure V as the $\mathfrak{G}$-representation is the morphism $\rho : \mathfrak{G} \to \mathcal{L}_V$ of k-monoids. Fix k-algebra A and $g \in \mathfrak{G}(A)$. Since actions of $\mathbf{G}$ and $\mathfrak{G}$ on V commute, morphism $\rho(g) : A\otimes_k V \to A\otimes_k V$ of A-modules is a morphism of $\mathbf{G}_A$-representation. According to Proposition 6.3 we derive that

$$\mathrm{Hom}_{\mathbf{G}_A}\left(A\otimes_k V[\lambda_1], A\otimes_k V[\lambda_2]\right) = 0$$

for distinct $\lambda_1, \lambda_2 \in \Lambda$. Thus

$$\rho(g)\left(A\otimes_k V[\lambda]\right) \subseteq A\otimes_k V[\lambda]$$

for every λ in Λ. This holds for every k-algebra A and $g \in \mathfrak{G}(A)$. Hence $V[\lambda]$ is $\mathfrak{G}$-subrepresentation of V. □

6.2. **Locally linear schemes.**

Definition 6.5. Let $\mathbf{M}$ be a monoid k-scheme and let X be a $\mathbf{M}$-scheme. Suppose that each point of X admits an open affine $\mathbf{M}$-stable neighborhood. Then we say that X is *a locally linear* $\mathbf{M}$*-scheme*.

Proposition 6.6. *Let $\mathbf{M}$ be an affine monoid k-scheme and let X be a $\mathbf{M}$-scheme. Suppose that there exists a quasi-coherent $\mathbf{M}$-equivariant ideal $\mathcal{I}$ on X with nilpotent sections. Consider an open subset U of X. Then the following are equivalent.*

(1) *U is $\mathbf{M}$-stable.*

(2) *$U\cap V(\mathcal{I})$ is $\mathbf{M}$-stable.*

Proof. Let $\alpha : \mathbf{M}\times X \to X$ be the action of $\mathbf{M}$ on X. Fix open subset U of X. If U is $\mathbf{M}$-stable, then $U\cap V(\mathcal{I})$ is $\mathbf{M}$-stable. So suppose that $U\cap V(\mathcal{I})$ is $\mathbf{M}$-stable. Since $\mathcal{I}$ has nilpotent sections and $\mathbf{M}$ is affine, we derive that closed immersions $U\cap V(\mathcal{I}) \hookrightarrow U$ and $\mathbf{M}\times(U\cap V(\mathcal{I})) \hookrightarrow \mathbf{M}\times U$ induce homeomorphisms on topological spaces. Consider the commutative diagram

$$\begin{tikzcd}
\mathbf{M}\times U \arrow[r, "\alpha_{|U\cap V(\mathcal{I})}"] & X \\
\mathbf{M}\times(U\cap V(\mathcal{I})) \arrow[u, hook] \arrow[r] & U\cap V(\mathcal{I}) \arrow[u, hook]
\end{tikzcd}$$

where the bottom horizontal arrow is the induced action on $U \cap V(\mathcal{I})$ and vertical morphisms are homeomorphisms. The commutativity of the diagram implies that $\alpha(\mathbf{M} \times U)$ is contained set-theoretically in U. Since U is open in X, we derive that morphism of schemes $\alpha_{|\mathbf{M}\times U}$ factors through U. Hence U is $\mathbf{M}$-stable. □

Corollary 6.7. *Let $\mathbf{M}$ be an affine monoid k-scheme and let X be a $\mathbf{M}$-scheme. Suppose that there exists a quasi-coherent $\mathbf{M}$-equivariant ideal $\mathcal{I}$ on X such that $\mathcal{I}^n = 0$ for $n \in \mathbb{N}$. Consider an open subset U of X. Then the following are equivalent.*

(1) *U is $\mathbf{M}$-stable and affine.*

(2) *$U \cap V(\mathcal{I})$ is $\mathbf{M}$-stable and affine.*

Proof. Since $\mathcal{I}^n = 0$, we derive that U is affine if and only if $U \cap V(\mathcal{I})$ is affine. Combining this with Proposition 6.6, we deduce the result. □

Corollary 6.8. *Let $\mathbf{M}$ be an affine monoid k-scheme and let X be a $\mathbf{M}$-scheme. Suppose that there exists a quasi-coherent $\mathbf{M}$-equivariant ideal $\mathcal{I}$ on X such that $\mathcal{I}^n = 0$ for $n \in \mathbb{N}$. Then X is locally linear $\mathbf{M}$-scheme if and only if $V(\mathcal{I})$ is locally linear $\mathbf{M}$-scheme.*

Proof. This is a consequence of Corollary 6.7. □

6.3. Affine monoid schemes with zero.

Proposition 6.9. *Let $\mathbf{M}$ be an affine monoid k-scheme with zero and let X be a locally linear $\mathbf{M}$-scheme. Then there exists an affine $\mathbf{M}$-equivariant morphism*

$$X \xrightarrow{r} X^{\mathbf{M}}$$

such that $r_{|X^{\mathbf{M}}} = 1_{X^{\mathbf{M}}}$.

Proof. Consider the action $\alpha : \mathbf{M} \times X \to X$ of $\mathbf{M}$ on X. Since X is locally linear and $\mathbf{M}$ is affine, we derive that α is an affine morphism of k-schemes. Now if $\mathbf{o}$ is a zero of $\mathbf{M}$, then we define a morphism

$$X \xrightarrow{\cong} \mathbf{o} \times X \hookrightarrow \mathbf{M} \times X \xrightarrow{\alpha} X$$

The morphism above is affine (as a composition of affine morphisms) and induces multiplication by $\mathbf{o}$ on functors of points $\mathbf{o} \cdot (-) : \mathfrak{P}_X \to \mathfrak{P}_X$. Now $\mathbf{o} \cdot (-) : \mathfrak{P}_X \to \mathfrak{P}_X$ factors as an $f P_{\mathbf{M}}$-equivariant epimorphism $\mathfrak{P}_X \twoheadrightarrow \mathfrak{P}_{X^{\mathbf{M}}}$ composed with a closed immersion $\mathfrak{P}_{X^{\mathbf{M}}} \hookrightarrow \mathfrak{P}_X$. The $\mathfrak{P}_{\mathbf{M}}$-equivariant epimorphism $\mathfrak{P}_X \to \mathfrak{P}_{X^{\mathbf{M}}}$ corresponds to a $\mathbf{M}$-equivariant morphism $r : X \to X^{\mathbf{M}}$ of k-schemes such that $r_{|X^{\mathbf{M}}} = 1_{X^{\mathbf{M}}}$. Moreover, the composition of r with a closed immersion $X^{\mathbf{M}} \hookrightarrow X$ is an affine morphism. Thus r is affine. □

6.4. M-equivariant quasi-coherent sheaves.

LS TODO: Tu trzeba zdefiniować i następnie opisać przypadek schematu z trywialnym działaniem, bo on jest najważniejszy

6.5. Kempf monoids.

Definition 6.10. Let $\mathbf{M}$ be a monoid k-scheme. Suppose that the following conditions hold.

(1) $\mathbf{M}$ is affine, geometrically connected and geometrically normal.

(2) There exists zero $\mathbf{o}$ in $\mathbf{M}$.

(3) There exists a torus T over k contained in the center of $\mathbf{M}$ such that the closure $\mathbf{cl}(T)$ of T in $\mathbf{M}$ contains $\mathbf{o}$.

LS TODO: Tutaj trzeba zdefiniować monoidy Kempfa. Najpierw trzeba porządnie spisać

LS TODO: (or rather Jelisiejew :D)

Then **M** is called *Kempf monoid*.

Let **M** be a Kempf monoid and let **G** be its group of units. If V is a representation of **G** and λ is a class in Λ, then we denote by $V[\lambda] \subseteq V$ the sum of all irreducible T-subpresentations of V of isomorphism type λ. Since T is a central subgroup of **G**, we derive by Proposition **??** that $V[\lambda]$ is a **G**-representation of V.
Suppose that Z is a k-scheme with trivial action of **M**. If $\mathcal{F}$ is a quasi-coherent sheaf on Z equipped with **G**-action, then we denote by $\mathcal{F}[\lambda]$ a sheaf given by

$$U \mapsto \mathcal{F}(U)[\lambda]$$

for every open affine subset U of Z. Then $\mathcal{F}[\lambda] \subseteq \mathcal{F}$ is a **G**-quasi-coherent subsheaf of $\mathcal{F}$.

6.6. Formal M-schemes.

Definition 6.11. Let **M** be a monoid k-scheme having **G** as the group of units. *A formal* **M**-*scheme* is a formal **G**-scheme $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$ scheme Z_n is **M**-scheme and the sequence of closed immersions

$$Z_0 \hookrightarrow Z_1 \hookrightarrow \dots \hookrightarrow Z_n \hookrightarrow Z_{n+1} \hookrightarrow \dots$$

consists of **M**-equivariant morphisms.

Let **M** be an affine monoid k-scheme with zero and let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal **M**-scheme. Suppose that **M** is a monoid with zero. Then by Proposition 6.9, we derive that $\mathcal{Z}$ is a part of the commutative diagram

$$\begin{tikzcd}
Z_0 \arrow[r, hook] \arrow[drr, bend right, two heads, "r_0 = 1_{Z_0}"'] & Z_1 \arrow[r, hook] \arrow[dr, two heads, "r_1"] & \dots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[dl, bend left, two heads, "r_n"] & \dots \\
 & & Z_0 & &
\end{tikzcd}$$

in which vertical morphisms $r_n : Z_n \twoheadrightarrow Z_0$ are affine morphisms such that $r_{n|Z_0} = 1_{Z_0}$. This implies that Z_n is affine over Z_0 for each $n \in \mathbb{N}$ and hence we write $\mathrm{Spec}_{Z_0}\mathcal{A}_n$ for $n \in \mathbb{N}$, where $\mathcal{A}_n$ is a quasi-coherent Z_0-algebra equipped with the action of **M**. Moreover, we have $\mathcal{A}_0 = \mathcal{O}_{Z_0}$. The diagram above induces the following sequence of epimorphisms

$$\dots \twoheadrightarrow \mathcal{A}_{n+1} \twoheadrightarrow \mathcal{A}_n \twoheadrightarrow \dots \twoheadrightarrow \mathcal{A}_1 \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$$

of quasi-coherent $\mathcal{O}_{Z_0}$-algebras with **M**-action. Denote by **G** the group of units of **M**. If **G** is schematically dense in **M** (for instance if **M** is integral), then we have $Z_0 = Z_n^{\mathbf{G}} = Z_n^{\mathbf{M}}$ and hence Z_0 admits trivial **M**-action. This alternative description of formal **M**-schemes will be used in the proof of the main theorem.

7. Formal functors and representability - OLD

Theorem 7.1 (Algebraization of a formal $\overline{\mathbf{G}}$-scheme)**.** *Let $\mathcal{Z} = \{Z_n\}$ be a formal $\overline{\mathbf{G}}$-scheme. Then there exists a colimit*

$$Z = \mathrm{colim}_n Z_n$$

in the category of locally linear $\overline{\mathbf{G}}$-schemes and Z is the unique algebraization of $\mathcal{Z}$. If in addition $\mathcal{Z}$ is locally Noetherian, then π_Z is of finite type. If $\mathcal{Z}$ is locally Noetherian and Z_0 is of finite type, then also Z is of finite type.

Now we spell out the main idea of the proof: the $\overline{\mathbf{G}}$-scheme Z required in Theorem 7.1 is equal to $\operatorname{Spec}_{Z_0}\mathcal{A}$, where $\mathcal{A}$ is the limit of $\mathcal{A}_n$ *in the category of* $\overline{\mathbf{G}}$*-algebras*; in other words each isotypic component of $\mathcal{A}$ is the limit of isotypic components of $\mathcal{A}_n$. Our first goal is to prove a stabilization result. We denote by $\operatorname{Irr}(\mathbf{G})$ the set of isomorphism types of irreducible $\mathbf{G}$-representations and by $\operatorname{Irr}(\overline{\mathbf{G}}) \subset \operatorname{Irr}(\mathbf{G})$ the subset of $\overline{\mathbf{G}}$-representations. For $\lambda \in \operatorname{Irr}(\mathbf{G})$ and a quasi-coherent $\overline{\mathbf{G}}$-module $\mathcal{C}$ on Z_0 we denote by $\mathcal{C}[\lambda] \subset \mathcal{C}$ the $\overline{\mathbf{G}}$-submodule such that $H^0(U, \mathcal{C}[\lambda]) \subset H^0(U, \mathcal{C})$ is the union of all $\mathbf{G}$-subrepresentations of $H^0(U, \mathcal{C})$ isomorphic to λ (i.e., the isotypic component of λ).

Lemma 7.1.1 (stabilization on an isotypic component)**.** *Let $\lambda \in \operatorname{Irr}(\overline{\mathbf{G}})$. Then there exists a number $n_\lambda \in \mathbb{N}$ such that the following holds. Let $\mathcal{Z} = \{Z_n\}$ be a formal $\overline{\mathbf{G}}$-scheme and $\{\mathcal{A}_{n+1} \twoheadrightarrow \mathcal{A}_n\}$ be the associated sequence of quasi-coherent $\overline{\mathbf{G}}$-algebras. Then for every $n > n_\lambda$ the surjection*

$$\mathcal{A}_n[\lambda] \twoheadrightarrow \mathcal{A}_{n-1}[\lambda]$$

is an isomorphism. If $\lambda_0 \in \operatorname{Irr}(\overline{\mathbf{G}})$ is the trivial representation, then we may take $n_{\lambda_0} = 0$.

Proof of Lemma 7.1.1. The claims are preserved under field extension, so we may assume our field is algebraically closed (hence perfect) so we may use the Kempf's torus. Fix a grading on $k[\overline{\mathbf{G}}]$ induced by a Kempf's torus for k as in Corollary **??**. Denote by $A_\lambda \subseteq \mathbb{N}$ the set of weights which appear in $k[\mathbf{G}]_\lambda$. Since $\dim_k k[\mathbf{G}]_\lambda$ is finite by Proposition **??**, the set A_λ is finite. Put

$$n_\lambda = \sup A_\lambda.$$

Fix $n > n_\lambda$ and let $\mathcal{I}_n = \ker(\mathcal{A}_n \to \mathcal{A}_0)$. Then we have a decomposition with respect to the chosen torus

$$\mathcal{A}_n = \bigoplus_{i \geq 0} (\mathcal{A}_n)[i],$$

By Corollary **??**, we have $\mathcal{I}_n = \oplus_{i \geq 1}(\mathcal{A}_n)[i]$. Since $n > n_\lambda$ we have

$$\mathcal{I}_n^n \subset \bigoplus_{i \geq n} (\mathcal{A}_n)[i] \subseteq \bigoplus_{i \notin A_\lambda} (\mathcal{A}_n)[i]$$

Hence, $\mathcal{I}_n^n[\lambda] = 0$. But $\mathcal{I}_n^n[\lambda] = \ker(\mathcal{A}_n[\lambda] \to \mathcal{A}_{n-1}[\lambda])$, thus $\mathcal{A}_n[\lambda] \to \mathcal{A}_{n-1}[\lambda]$ is an isomorphism. Finally note that $A_{\lambda_0} = \{0\}$. This implies that $n_{\lambda_0} = 0$. □

Proof of Theorem 7.1. Let $\mathcal{A}_n$ be the quasi-coherent $\overline{\mathbf{G}}$-algebras as in (**??**). For $\lambda \in \operatorname{Irr}(\overline{\mathbf{G}})$ we define $\mathcal{A}[\lambda] := \mathcal{A}_n[\lambda]$, where $n \geq n_\lambda$ as in Lemma 7.1.1.

$$\mathcal{A} = \bigoplus_{\lambda \in \operatorname{Irr}(\overline{\mathbf{G}})} \mathcal{A}[\lambda] = \bigoplus_{\lambda \in \operatorname{Irr}(\overline{\mathbf{G}})} \mathcal{A}_{n_\lambda}[\lambda].$$

Clearly $\mathcal{A}[\lambda_0] = \mathcal{A}_0 = \mathcal{O}_{Z_0}$ canonically (where λ_0 is the trivial representation), hence $\mathcal{A}$ is an $\mathcal{O}_{Z_0}$-module. Actually $\mathcal{A} = \lim_n \mathcal{A}_n$ in the category of quasi-coherent $\overline{\mathbf{G}}$-modules on Z_0. We construct the algebra structure on $\mathcal{A}$. For this pick $\eta_1, \eta_2 \in \operatorname{Irr}(\overline{\mathbf{G}})$. Fix the finite set $\{\lambda_1, \ldots, \lambda_s\} \subseteq \operatorname{Irr}(\overline{\mathbf{G}})$ of representations which appear in $k[\overline{\mathbf{G}}]_{\eta_1} \otimes_k k[\overline{\mathbf{G}}]_{\eta_2}$. Then, for every $n \in \mathbb{N}$, we have the multiplication

$$\mathcal{A}_n[\eta_1] \otimes_k \mathcal{A}_n[\eta_2] \to \mathcal{A}_n[\eta_1] \cdot \mathcal{A}_n[\eta_2] \subseteq \bigoplus_{i=1}^{s} \mathcal{A}_n[\lambda_i]$$

and by Lemma 7.1.1 these morphisms can be identified for $n \geq \sup\{n_{\eta_1}, n_{\eta_2}, n_{\lambda_1}, ..., n_{\lambda_s}\}$. We define

$$\mathcal{A}[\eta_1] \otimes_k \mathcal{A}[\eta_2] \to \bigoplus_{i=1}^{s} \mathcal{A}[\lambda_i] \subseteq \mathcal{A}$$

as a morphism induced by the multiplication morphism for any $n \geq \sup\{n_{\eta_1}, n_{\eta_2}, n_{\lambda_1}, \ldots, n_{\lambda_s}\}$. This gives an $\mathcal{O}_{Z_0}$-algebra structure on $\mathcal{A}$, so $\mathcal{A}$ is in fact the limit of $\mathcal{A}_n$ is the category of $\overline{\mathbf{G}}$-algebras. Note that from the description of $\mathcal{A}$ it follows that for every $n \in \mathbb{N}$ we have a surjective morphism $p_n : \mathcal{A} \twoheadrightarrow \mathcal{A}_n$ of $\overline{\mathbf{G}}$-algebras. We denote its kernel by $\mathcal{J}_n$ and we put $\mathcal{J} := \mathcal{J}_0$. The natural injection $\mathcal{O}_{Z_0} = \mathcal{A}_0 \to \mathcal{A}$ is a section of p_0, so that we have

$$\mathcal{J} = \bigoplus_{\lambda \in \mathrm{Irr}(\overline{\mathbf{G}}) \smallsetminus \{\lambda_0\}} \mathcal{A}[\lambda].$$

We also denote by $\mathcal{I}_n$ the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$ for $n \in \mathbb{N}$. Then $\mathcal{I}_n = \mathcal{J}/\mathcal{J}_n$. Fix $m \in \mathbb{N}$ and consider $n \in \mathbb{N}$ such that $n \geq m$. Since $\mathcal{Z}$ is a formal $\overline{\mathbf{G}}$-scheme, the sheaf $\mathcal{I}_n^{m+1}$ is the kernel of the morphism $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_m$. Thus

$$\mathcal{J}_m/\mathcal{J}_n = \mathcal{I}_n^{m+1} = (\mathcal{J}^{m+1} + \mathcal{J}_n)/\mathcal{J}_n.$$

Both $\mathcal{J}_m$ and $\mathcal{J}^{m+1}$ are $\mathrm{Irr}(\overline{\mathbf{G}})$-graded and for given $\lambda \in \mathrm{Irr}(\overline{\mathbf{G}})$ and $n \gg 0$ the isotypic component $\mathcal{J}_n[\lambda]$ is zero by Lemma 7.1.1. Hence $\mathcal{J}_m = \mathcal{J}^{m+1}$ for every $m \in \mathbb{N}$. We define

$$Z = \mathrm{Spec}_{Z_0}(\mathcal{A})$$

and we denote by $\pi : Z \to Z_0$ the structural morphism. The scheme Z inherits a $\overline{\mathbf{G}}$-action from $\mathcal{A}$. For every $n \in \mathbb{N}$ the zero-set of $\mathcal{J}^{n+1} \subseteq \mathcal{A}$ is a $\overline{\mathbf{G}}$-scheme isomorphic to Z_n. Hence $\mathcal{Z}$ is isomorphic to $\widehat{Z}$. Thus Z is an algebraization of $\mathcal{Z}$. Since $\mathcal{A} = \lim \mathcal{A}_n$, we have $Z = \mathrm{colim}\, Z_n$ in the category of locally linear $\overline{\mathbf{G}}$-schemes.

It remains to prove uniqueness of algebraization. Let $Z' = \mathrm{Spec}_{Z_0}\mathcal{A}'$ be an algebraization of $\mathcal{Z} = \{Z_n\}$. Then $Z_n \hookrightarrow Z'$, so by the universal property of colimit, we obtain a $\overline{\mathbf{G}}$-morphism $Z \to Z'$, corresponding to $\mathcal{A}' \to \mathcal{A}$. It induces epimorphisms $\mathcal{A}' \twoheadrightarrow \mathcal{A}_n$ for all n. For each $\lambda \in \mathrm{Irr}(\overline{\mathbf{G}})$, the composition

$$\mathcal{A}'[\lambda] \to \mathcal{A}[\lambda] \simeq \mathcal{A}_{n_\lambda}[\lambda]$$

is an epimorphism, hence $\mathcal{A}' \to \mathcal{A}$ is an epimorphism. The kernel of $\mathcal{A}' \to \mathcal{A}$ is equal to

$$\bigcap_n \ker(\mathcal{A}' \to \mathcal{A}_n) = \bigcap_n \ker(\mathcal{A}' \to \mathcal{A}_0)^n.$$

To prove that this kernel is zero, we may enlarge the field to an algebraically closed field, so the result follows from Corollary **??**.

Assume that each scheme Z_n is locally Noetherian over k. Then $\mathcal{I}_n$ is a coherent $\mathcal{A}_n$-module, thus $\mathcal{I}_n^i/\mathcal{I}^{i+1}$ is a coherent $\mathcal{A}_0$-module for all i. The series

$$0 = \mathcal{I}_n^{n+1} \subset \mathcal{I}^n \subset \ldots \subset \mathcal{I} \subset \mathcal{A}_n$$

has coherent subquotients, hence $\mathcal{A}_n$ is a coherent $\mathcal{O}_{Z_n}$-algebra. Thus $\mathcal{A}[\lambda]$ is a coherent $\mathcal{O}_{Z_0}$-module for every $\lambda \in \mathrm{Irr}(\overline{\mathbf{G}})$. The claim that π is of finite type is local on $Z^{\mathbf{G}}$, hence we may assume that $Z^{\mathbf{G}}$ is quasi-compact. The sheaf $\mathcal{J}/\mathcal{J}^2 \subseteq \mathcal{A}_1$ is coherent so there exists a finite set $\lambda_1, \ldots, \lambda_r \in \mathrm{Irr}(\overline{\mathbf{G}}) \smallsetminus \{\lambda_0\}$ such that the morphism

$$\bigoplus_{i=1}^{r} \mathcal{A}[\lambda_i] \to \mathcal{J}/\mathcal{J}^2$$

induced by $\mathcal{A} \twoheadrightarrow \mathcal{A}_2$ is surjective. Let $\mathcal{B} \subset \mathcal{A}$ be the quasi-coherent $\mathcal{O}_{Z_0}$-subalgebra generated by the coherent subsheaf $\mathcal{M} := \bigoplus_{i=1}^r \mathcal{A}[\lambda_i] \subseteq \mathcal{A}$. Let $\overline{k}$ be an algebraic closure of k and let $\mathcal{A}' = \mathcal{A} \otimes \overline{k}$. Fix a Kempf's torus over $\overline{k}$ and the associated grading $\mathcal{A}' = \bigoplus_{i \geq 0} \mathcal{A}'[i]$ as in Corollary **??**. Then $\mathcal{J} = \bigoplus_{i \geq 1} \mathcal{A}'[i]$ is a graded ideal and $\mathcal{J}/\mathcal{J}^2$ is generated by the graded (coherent) subsheaf $\mathcal{M}' = \bigoplus_{i=1}^r \mathcal{A}'[\lambda_i]$. By graded Nakayama's lemma, the ideal $\mathcal{J}$ itself is generated by (the elements of) $\mathcal{M}'$. Then by induction on the degree, $\mathcal{A}'$ is generated by $\mathcal{M}'$ as an algebra. In other words, $\mathcal{A}' = \mathcal{B} \otimes \overline{k}$. Thus also $\mathcal{A} = \mathcal{B}$ and so $\mathcal{A}$ is of finite type over $\mathcal{O}_{Z_0}$. □

7.1. **Linear algebraic monoids.**

Proposition 7.2. *Let* $\mathbf{M}$ *be a monoid k-scheme. Then the k-functor of units* $\mathfrak{P}^*_{\mathbf{M}}$ *of* $\mathfrak{P}_{\mathbf{M}}$ *is representable by a group k-scheme* $\mathbf{M}^*$. *Moreover, if* $\mathbf{M}$ *is affine and of finite type over k, then* $\mathbf{M}^*$ *is an open subscheme of* $\mathbf{M}$.

Proof. Note that $\mathfrak{P}^*_{\mathbf{M}}$ fits into a cartesian square

$$\begin{tikzcd} \mathfrak{P}^*_{\mathbf{M}} \arrow[r] \arrow[d] & \mathbf{1} \arrow[d, "\mathfrak{P}_e"] \\ \mathfrak{P}_{\mathbf{M}} \times \mathfrak{P}_{\mathbf{M}} \arrow[r, "\mathfrak{P}_m"'] & \mathfrak{P}_{\mathbf{M}} \end{tikzcd}$$

where $m : \mathbf{M} \times \mathbf{M} \to \mathbf{M}$ is the multiplication and $e : \operatorname{Spec} k \to \mathbf{M}$ is the unit. Since the functor

$$\widehat{\mathbf{Sch}_k} \longrightarrow \textit{the category of k-functors}$$

preserves fiber products, we derive that $\mathfrak{P}^*_{\mathbf{M}}$ is isomorphic to $\mathfrak{P}_{\mathbf{M}^*}$, where $\mathbf{M}^*$ is a k-scheme defined by the cartesian diagram

$$\begin{tikzcd} \mathbf{M}^* \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d, "e"] \\ \mathbf{M} \times \mathbf{M} \arrow[r, "m"'] & \mathbf{M} \end{tikzcd}$$

Since $\mathfrak{P}_{\mathbf{M}^*} \cong \mathfrak{P}^*_{\mathbf{M}}$, we deduce that $\mathbf{M}^*$ admits a structure of a group k-scheme.
Now suppose that $\mathbf{M}$ is affine monoid k-scheme of finite type over k. Then there exist a finite dimensional vector space V over k and a closed immersion $i : \mathbf{M} \to L(V)$ of monoid k-schemes. □

LS TODO: Skończyć dowód.

Definition 7.3. Let $\mathbf{M}$ be an affine monoid k-scheme. Suppose that the group $\mathbf{G}$ of units of $\mathbf{M}$ is an algebraic group over k and that the open immersion $\mathbf{G} \hookrightarrow \mathbf{M}$ is schematically dense. Then $\mathbf{M}$ is *a linear algebraic monoid over k*.

Definition 7.4. Let $\mathbf{M}$ be a linear algebraic monoid over k. Suppose that the group $\mathbf{G}$ of units of $\mathbf{M}$ is (linearly) reductive. Then $\mathbf{M}$ is *a (linearly) reductive monoid over k*.

8. Toruses and toric monoid k-schemes

Definition 8.1. Let T be an affine algebraic group over k. Suppose that there exists $n \in \mathbb{N}$ such that for every algebraically closed extension K of k there exists an isomorphism

$$T_K \cong \operatorname{Spec} K \times \underbrace{\mathbb{G}_m \times \mathbb{G}_m \times \ldots \times \mathbb{G}_m}_{n \text{ times}}$$

of group schemes over K. Then T is called *a torus over k*.

Example 8.2. If $T \cong \underbrace{\mathbb{G}_m \times \mathbb{G}_m \times \ldots \times \mathbb{G}_m}_{n \text{ times}}$, then T is a torus. We call toruses T of this form *split toruses*.

Example 8.3. Define

$$\mathbf{S}^1 = \operatorname{Spec} k[x,y]/(x^2+y^2-1)$$

a scheme over k and let $\mathfrak{P}_{\mathbf{S}^1}$ be its functor of points. Then for every k-algebra A we have

$$\mathfrak{P}_{\mathbf{S}^1}(A) = \left\{(u,v) \in A \times A \,\middle|\, u^2+v^2 = 1\right\}$$

There is also a morphism $\mathfrak{P}_{\mathbf{S}^1} \times \mathfrak{P}_{\mathbf{S}^1} \to \mathfrak{P}_{\mathbf{S}^1}$ of k-functors given by

$$\mathfrak{P}_{\mathbf{S}^1}(A) \times \mathfrak{P}_{\mathbf{S}^1}(A) \to \mathfrak{P}_{\mathbf{S}^1} \ni ((u_1,v_1),(u_2,v_2)) \mapsto (u_1u_2 - v_1v_2, u_1v_2 + u_2v_1) \in \mathfrak{P}_{\mathbf{S}^1}(A)$$

for every k-algebra A. This makes $\mathfrak{P}_{\mathbf{S}^1}$ into a group k-functor. Thus $\mathbf{S}^1$ with the group structure described above is an affine algebraic group over k. We call it *the circle group over k*.
Now suppose that $\operatorname{char}(k) = 2$ and K is an algebraically closed extension of k. Consider an element $i \in K$ such that $i^2 = -1$. For every K-algebra A we have a map

$$\mathfrak{P}_{\mathbf{S}^1}(A) \ni (u,v) \mapsto u + iv \in A^*$$

First note that this map is bijective. Indeed, its inverse is given by

$$A^* \ni a \mapsto \left(\frac{1}{2}(a+a^{-1}), \frac{1}{2i}(a-a^{-1})\right) \in \mathfrak{P}_{\mathbf{S}^1}(A)$$

Moreover, the map $\mathfrak{P}_{\mathbf{S}^1}(A) \to A^*$ is a homomorphism of abstract groups. Thus $\mathfrak{P}_{\mathbf{S}^1}$ resricted to the category $\mathbf{Alg}_K$ of K-algebras is isomorphic with $\mathfrak{P}_{\operatorname{Spec} K \times \mathbb{G}_m}$ as a group k-functor. Hence

$$\mathbf{S}^1_K \cong \operatorname{Spec} K \times \mathbb{G}_m$$

as algebraic group schemes over K. Hence $\mathbf{S}^1$ is a torus over k.
Now assume that $k = \mathbb{R}$. Then abstract groups

$$\mathfrak{P}_{\mathbf{S}^1}(\mathbb{R}) = \left\{z \in \mathbb{C} \,\middle|\, |z| = 1\right\} \subseteq \mathbb{C}^*,\ \mathbb{R}^*$$

are not isomorphic. Indeed, the left hand side group has infinite torsion subgroup and the right hand side group has torsion subgroup equal to $\{-1,1\}$. This implies that over $\mathbb{R}$ algebraic groups $\mathbf{S}^1$ and $\mathbb{G}_m$ are not isomorphic. Hence $\mathbf{S}^1$ is not a split torus over $\mathbb{R}$.

Corollary 8.4. *Let T be a torus over k. Then T is a linearly reductive algebraic group.*

Definition 8.5. Let T be a torus over k and let $\mathbf{M}$ be a linearly reductive monoid having T as the group of units. Then $\mathbf{M}$ is *a toric monoid over k*

9. ALGEBRAIZATION OF FORMAL **M**-SCHEMES

Now we prove the main result.

Theorem 9.1. *Let $\mathbf{M}$ be a Kempf monoid with unit group $\mathbf{G}$ and let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{M}$-scheme. Then there exists an algebraization Z of $\mathcal{Z}$. Moreover, the following assertions hold.*

(1) *Z is $\mathbf{M}$-scheme.*

(2) *The canonical morphism $\pi : Z \to Z_0$ is an affine morphism.*

Moreover, if $\mathcal{Z}$ is locally noetherian, then π is of finite type.

Let $\mathbf{G}$ be the group of units of $\mathbf{M}$. According to the fact that $\mathbf{M}$ is integral, we derive that $\mathbf{G}$ is schematically dense in $\mathbf{M}$ and hence Z_0 admits trivial $\mathbf{M}$-action. Since $\mathbf{M}$ has zero, formal $\mathbf{M}$-scheme $\mathcal{Z}$ corresponds to a sequence

$$\ldots \twoheadrightarrow \mathcal{A}_{n+1} \twoheadrightarrow \mathcal{A}_n \twoheadrightarrow \ldots \twoheadrightarrow \mathcal{A}_1 \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$$

of **M**-quasi-coherent $\mathcal{O}_{Z_0}$-algebras such that $Z_n = \operatorname{Spec}_{Z_0}\mathcal{A}_n$ for every $n \in \mathbb{N}$. Next since **M** is a Kempf monoid, there exists a closed subgroup T of the center $Z(\mathbf{G})$ such that T is a torus and the scheme-theoretic closure $\overline{T}$ of T in **M** contains the zero **o** of **M**. Then $\overline{T}$ is toric monoid over k with group of units T and with zero. Let $\mathbf{Irr}(\overline{T}), \mathbf{Irr}(T)$ be sets of isomorphism classes of irreducible representations of $\overline{T}$ and T, respectively. Suppose that λ_0 is the class of trivial irreducible representation of $\overline{T}$. The proof of the theorem is based on the following results.

Lemma 9.1.1. *Let $\{V_\lambda\}_{\lambda \in \mathbf{Irr}(\overline{T})}$ be a set of irreducible representations of $\overline{T}$ such that V_λ is in class λ. Denote by λ_0 the isomorphism class of trivial one-dimensional representations of $\overline{T}$. Then for every $\lambda \in \mathbf{Irr}(\overline{T})$ there exists $n_\lambda \in \mathbb{N}$ with the following property. For each $n > n_\lambda$ and any $\lambda_1, ..., \lambda_n \in \mathbf{Irr}(\overline{T}) \setminus \{\lambda_0\}$ the representation*

$$\bigotimes_{i=1}^{n} V_{\lambda_i}$$

for $1 \leq i \leq n$ has trivial isotypic component of type λ. Moreover, we may pick $n_{\lambda_0} = 0$.

Proof of the lemma. Denote by T the group of units of $\overline{T}$. By assumption T is a torus over k. Let K be an algebraically closed extension of k. Then $\overline{T}_K = \overline{T} \times_{\operatorname{Spec} k} \operatorname{Spec} K$ is an affine toric variety over $T_K = T \times_{\operatorname{Spec} k} \operatorname{Spec} K$. Since

$$T_K = \operatorname{Spec} K \times \underbrace{\mathbb{G}_m \times \mathbb{G}_m \times ... \times \mathbb{G}_m}_{N\text{ times}} = \operatorname{Spec} K[\mathbb{Z}^N]$$

we derive that

$$\overline{T}_K = \operatorname{Spec} K[S]$$

for some abstract submonoid S of $\mathbb{Z}^N$. Moreover, the open immersion $T_K \hookrightarrow \overline{T}_K$ is induced by the inclusion $S \hookrightarrow \mathbb{Z}^N$. Since $\overline{T}$ admits a zero, we derive that

$$\mathfrak{m} = \bigoplus_{s \in S \setminus \{0\}} K \cdot s \subseteq K[S]$$

is an ideal in $K[S]$. This implies that $S \setminus \{0\}$ is closed under addition. Next since $\overline{T}$ is of finite type over k, we derive that $S \setminus \{0\}$ is a finitely generated semigroup. By there exists $f : \mathbb{Z}^N \to \mathbb{Z}$ such that $f_{|S \setminus \{0\}} > 0$. Now we fix $\lambda \in \mathbf{Irr}(\overline{T})$. Then there exists a finite subset A of S and $n_s \in \mathbb{N}$ for each $s \in A$ such that we have decomposition

LS TODO: Książka Coxa

$$K \otimes_k V_\lambda = \bigoplus_{s \in A} (Ks)^{\oplus n_s}$$

onto irreducible representations of $\overline{T}_K$. Let $n_\lambda = \sup_{s \in A} f(s)$. Pick $n > n_\lambda$ and $\lambda_1, ..., \lambda_n \in \mathbf{Irr}(\overline{T}) \setminus \{\lambda_0\}$. Then representation $K \otimes_k \bigotimes_{i=1}^n V_{\lambda_i}$ is a direct sum of representations

$$K(s_1 + ... + s_n) = \bigotimes_{i=1}^{n} Ks_i$$

for some $s_1, ..., s_n \in S \setminus \{0\}$. Since

$$f(s_1 + ... + s_n) = f(s_1) + ... + f(s_n) \geq n > n_\lambda = \sup_{s \in A} f(s)$$

Thus for every $s \in A$ we have $s \neq s_1 + ... + s_n$. Thus V_λ cannot be a direct summand of $\bigotimes_{i=1}^n V_{\lambda_i}$. Also note that $K \otimes_k V_{\lambda_0}$ is one-dimensional trivial representation of $\overline{T}_K$. Hence $n_{\lambda_0} = 0$. □

Lemma 9.1.2. *Fix λ in $\mathbf{Irr}(\overline{T})$. Then there exists a number $n_\lambda \in \mathbb{N}$ such that the following holds. For every $n > n_\lambda$ the surjection*

$$\mathcal{A}_{n+1}[\lambda] \twoheadrightarrow \mathcal{A}_n[\lambda]$$

is an isomorphism. If λ_0 is the isomorphism type of trivial representation of $\mathbf{G}$, then $n_{\lambda_0} = 0$.

Proof of the lemma. Let $\mathcal{I}_n$ be a quasi-coherent ideal of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$. Since $\mathcal{Z}$ is a formal **M**-scheme, the kernel of $\mathcal{A}_{n+1} \twoheadrightarrow \mathcal{A}_n$ is $\mathcal{I}_{n+1}^n$. Note also that the image of the composition

$$\underbrace{\mathcal{I}_{n+1} \otimes_k \mathcal{I}_{n+1} \otimes_k \dots \otimes_k \mathcal{I}_{n+1}}_{n \text{ times}} \longrightarrow\!\!\!\!\!\to \underbrace{\mathcal{I}_{n+1} \otimes_{\mathcal{O}_{Z_0}} \mathcal{I}_{n+1} \otimes_{\mathcal{O}_{Z_0}} \dots \otimes_{\mathcal{O}_{Z_0}} \mathcal{I}_{n+1}}_{n \text{ times}} \longrightarrow \mathcal{A}_{n+1}$$

is $\mathcal{I}_{n+1}^n$. Pick $n_\lambda \in \mathbb{N}$ as in Lemma 9.1.1 (note that $n_{\lambda_0} = 0$). If $n > n_\lambda$, then by Lemma 9.1.1 we derive that

$$\left(\underbrace{\mathcal{I}_{n+1} \otimes_k \mathcal{I}_{n+1} \otimes_k \dots \otimes_k \mathcal{I}_{n+1}}_{n \text{ times}}\right)[\lambda] = 0$$

Since the composition above is a morphism of sheaves with $\overline{T}$-linearization, we derive that $\mathcal{I}_{n+1}^n[\lambda] = 0$ for $n > n_\lambda$. Thus $\mathcal{A}_{n+1}[\lambda] \twoheadrightarrow \mathcal{A}_n[\lambda]$ is an isomorphism. □

Lemma 9.1.3. *We have $\mathcal{A}_0 = \mathcal{A}_0[\lambda_0]$ and for every $n > 0$ the surjection*

$$\mathcal{A}_{n+1}[\lambda_0] \longrightarrow\!\!\!\!\!\to \mathcal{A}_0[\lambda_0]$$

is an isomorphism.

Lemma 9.1.4. *Let λ be an element of $\mathbf{Irr}(T)$. Then the functor*

$$\mathfrak{Qcoh}_{\mathbf{G}}(Z_0) \ni \mathcal{F} \mapsto \mathcal{F}[\lambda] \in \mathfrak{Qcoh}_T(Z_0)$$

is exact.

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