DIFFERENTIABILITY

1. Introduction

In these notes we collect basic results on derivatives of functions defined on open subsets of real or complex Banach spaces.
Symbol $\mathbb{K}$ denotes the base field which is either $\mathbb{R}$ or $\mathbb{C}$.

2. Bounded multilinear forms on normed spaces

In this section we fix a positive integer n and consider normed spaces $\mathfrak{D}_1, ..., \mathfrak{D}_n, \mathfrak{X}$ over $\mathbb{K}$. Let $\pi_i : \mathfrak{D}_1 \times ... \times \mathfrak{D}_n \to \mathfrak{D}_i$ be the projection onto i-th axis.

Definition 2.1. Let $L : \mathfrak{D}_1 \times ... \times \mathfrak{D}_n \to \mathfrak{X}$ be a $\mathbb{K}$-multilinear form. Suppose that there exists $c \in \mathbb{R}_+$ such that

$$\|L(\mathbf{x})\| \leq c \cdot \|\pi_1(\mathbf{x})\| \cdot ... \cdot \|\pi_n(\mathbf{x})\|$$

for every $\mathbf{x} \in \mathfrak{D}_1 \times ... \times \mathfrak{D}_n$. Then L is *bounded* $\mathbb{K}$*-multilinear form*.

The following result characterizes bounded multilinear forms.

Theorem 2.2. *Let $L : \mathfrak{D}_1 \times ... \times \mathfrak{D}_n \to \mathfrak{X}$ be a $\mathbb{K}$-multilinear form. Then the following assertions are equivalent.*

- **(i)** *L is continuous.*
- **(ii)** *L is continuous at zero n-tuple in $\mathfrak{D}_1 \times ... \times \mathfrak{D}_n$.*
- **(iii)** *L is bounded.*

Proof. Left for the reader as an exercise. □

Definition 2.3. Let $L : \mathfrak{D}_1 \times ... \times \mathfrak{D}_n \to \mathfrak{X}$ be a bounded $\mathbb{K}$-multilinear form. Then we define

$$\|L\| = \sup\left\{\|L(\mathbf{x})\| \,\middle|\, \mathbf{x} \in \mathfrak{D}_1 \times ... \times \mathfrak{D}_n \text{ and } \|\pi_i(\mathbf{x})\| = 1 \text{ for all } i\right\}$$

and call it *the operator norm of L*.

Fact 2.4. *Let $L : \mathfrak{D}_1 \times ... \times \mathfrak{D}_n \to \mathfrak{X}$ be a bounded $\mathbb{K}$-multilinear form. Then*

$$\|L(\mathbf{x})\| \leq \|L\| \cdot \|\pi_1(\mathbf{x})\| \cdot ... \cdot \|\pi_n(\mathbf{x})\|$$

for every $\mathbf{x} \in \mathfrak{D}_1 \times ... \times \mathfrak{D}_n$.

Proof. Left for the reader as an exercise. □

Theorem 2.5. *Let $L(\mathfrak{D}_1, ..., \mathfrak{D}_n; \mathfrak{X})$ be a $\mathbb{K}$-vector space of bounded $\mathbb{K}$-multilinear forms $\mathfrak{D}_1 \times ... \times \mathfrak{D}_n \to \mathfrak{X}$ with respect to operations defined pointwise. Suppose that $\mathfrak{X}$ is a Banach space over $\mathbb{K}$. Then*

$$L(\mathfrak{D}_1, ..., \mathfrak{D}_n; \mathfrak{X}) \ni L \mapsto \|L\| \in \mathbb{R}_+ \cup \{0\}$$

is a norm which makes $L(\mathfrak{D}_1, ..., \mathfrak{D}_n; \mathfrak{X})$ into a Banach space over $\mathbb{K}$.

Proof. We left as an exercise the proof that operator norm is well defined vector space norm on $L\left(\mathfrak{D}_1, ..., \mathfrak{D}_n; \mathfrak{X}\right)$. Consider a Cauchy's sequence $\{L_m\}_{m\in\mathbb{N}}$ with respect to operator norm. Fix $\mathbf{x} \in \mathfrak{D}_1 \times ... \times \mathfrak{D}_n$. Then by Fact 2.4

$$\|\left(L_m - L_k\right)(\mathbf{x})\| \leq \|L_m - L_k\| \cdot \|\pi_1(\mathbf{x})\| \cdot ... \cdot \|\pi_n(\mathbf{x})\|$$

for every $m, k \in \mathbb{N}$. This implies that $\left\{L_m\left(\mathbf{x}\right)\right\}_{m\in\mathbb{N}}$ is a Cauchy's sequence in $\mathfrak{X}$. Since $\mathfrak{X}$ is a Banach space over $\mathbb{K}$, we derive that this sequence is convergent. We define

$$L\left(\mathbf{x}\right) = \lim_{m\to+\infty} L_m\left(\mathbf{x}\right)$$

We also have

$$\left|\|L_m\| - \|L_k\|\right| \leq \|L_m - L_k\|$$

for every $m, k \in \mathbb{N}$. Thus $\{\|L_m\|\}_{m\in\mathbb{N}}$ is a Cauchy's sequence in $\mathbb{R}$ and hence it is convergent in $\mathbb{R}$. Note that we have

$$\|L(\mathbf{x})\| = \lim_{m\to+\infty}\|L_m\left(\mathbf{x}\right)\| \leq \left(\lim_{m\to+\infty}\|L_m\|\right) \cdot \|\pi_1(\mathbf{x})\| \cdot ... \cdot \|\pi_n(\mathbf{x})\|$$

Therefore, $L : \mathfrak{D}_1 \times ... \times \mathfrak{D}_n \to \mathfrak{X}$ is a bounded $\mathbb{K}$-multilinear form. We claim that L is the limit of $\{L_m\}_{m\in\mathbb{N}}$ with respect to operator norm. For the proof fix $\mathbf{x} \in \mathfrak{D}_1 \times ... \times \mathfrak{D}_n$ such that $\|\pi_1(\mathbf{x})\| = ... = \|\pi_n(\mathbf{x})\| = 1$. Then

$$\|\left(L - L_m\right)(\mathbf{x})\| \leq \|L(\mathbf{x}) - L_k(\mathbf{x})\| + \|L_k - L_m\|$$

Thus we have

$$\|\left(L - L_m\right)(\mathbf{x})\| \leq \limsup_{k\to+\infty}\|L_k - L_m\|$$

Since the left hand side does not depend on $\mathbf{x}$, we deduce that

$$\|L - L_m\| \leq \limsup_{k\to+\infty}\|L_k - L_m\|$$

Invoking once again the assumption that $\{\|L_m\|\}_{m\in\mathbb{N}}$ is Cauchy's sequence we infer

$$\lim_{m\to+\infty}\|L - L_m\| \leq \lim_{m\to+\infty}\limsup_{k\to+\infty}\|L_k - L_m\| = 0$$

This completes the proof. □

Proposition 2.6. *The canonical map $L\left(\mathfrak{D}_1, ..., \mathfrak{D}_{n-1}; L\left(\mathfrak{D}_n, \mathfrak{X}\right)\right) \to L\left(\mathfrak{D}_1, ..., \mathfrak{D}_n; \mathfrak{X}\right)$ which sends L in $L\left(\mathfrak{D}_1, ..., \mathfrak{D}_{n-1}; L\left(\mathfrak{D}_n, \mathfrak{X}\right)\right)$ to a $\mathbb{K}$-multilinear form given by formula*

$$\mathfrak{D}_1 \times ... \times \mathfrak{D}_n \ni \mathbf{x} \mapsto L(\pi_1(\mathbf{x}), ..., \pi_{n-1}(\mathbf{x}))(\pi_n(\mathbf{x})) \in \mathfrak{X}$$

is an isometry of normed spaces.

Proof. Left for the reader as an exercise. □

3. Fréchet derivative

In this section we introduce derivatives and prove their basic properties. We fix Banach spaces $\mathfrak{D}, \mathfrak{X}$ over $\mathbb{K}$. Let U be an open subset of $\mathfrak{D}$ and let V be an open subset of $\mathfrak{X}$.

Fact 3.1. *Let x be a point in U and let $f : U \to V$ be a function. Suppose that $L_i : \mathfrak{D} \to \mathfrak{X}$ for $i = 1, 2$ are $\mathbb{K}$-linear maps. If both functions*

$$\left\{h \in \mathfrak{D}\,\middle|\, h \neq 0 \text{ and } x + h \in U\right\} \ni h \mapsto \frac{f(x+h) - f(x) - L_i(h)}{\|h\|} \in \mathfrak{X}$$

tend to zero as $h \to 0$, then $L_1 = L_2$.

Proof. By assumption the function

$$\left\{h \in \mathfrak{D} \,\middle|\, h \neq 0 \text{ and } x+h \in U\right\} \ni h \mapsto (L_1 - L_2)\left(\frac{h}{\|h\|}\right) \in \mathfrak{X}$$

tends to zero as $h \to 0$. This implies that $L_1 - L_2$ sends each vector of the unit sphere in $\mathfrak{D}$ to zero. Thus $L_1 - L_2 = 0$ and hence $L_1 = L_2$. □

Definition 3.2. Let x be a point in U. A function $f : U \to V$ is *differentiable at point x* if there exists a continuous $\mathbb{K}$-linear map $L : \mathfrak{D} \to \mathfrak{X}$ such that the function

$$\left\{h \in \mathfrak{D} \,\middle|\, h \neq 0 \text{ and } x+h \in U\right\} \ni h \mapsto \frac{f(x+h) - f(x) - L(h)}{\|h\|} \in \mathfrak{X}$$

tends to zero as $h \to 0$. Moreover, the unique continuous $\mathbb{K}$-linear map L is *the derivative of f at x*.

Remark 3.3. Notion of differentiability defined above is named by some authors *Fréchet differentiability* after french mathematician Maurice Fréchet.

Remark 3.4. Let x be a point in U and let $f : U \to V$ be a function differentiable at point x. Then the derivative of f at x is usually denoted by $f'(x)$.

Fact 3.5. *Let x be a point in U and let $f : U \to V$ be a function differentiable at x. Then f is continuous at x.*

Proof. The set $\left\{h \in \mathfrak{D} \,\middle|\, x+h \in U\right\}$ contains a neighborhood of zero in $\mathfrak{D}$ and consider the function $\phi_f(h)$ defined on this set and given by formula

$$f(x+h) - f(x) = f'(x)(h) + \phi_f(h) \cdot \|h\|$$

Since f is differentiable at x, we have

$$\lim_{h \to 0} \phi_f(h) = 0$$

and thus

$$\lim_{h \to 0} \big(f(x+h) - f(x)\big) = \lim_{h \to 0} \big(f'(x)(h) + \phi_f(h) \cdot \|h\|\big) = 0$$

This implies that f is continuous at x. □

Definition 3.6. A function $f : U \to V$ is *differentiable* if it is differentiable at each point of U.

4. Chain rule

Chain rule is a basic tools for calculating Fréchet derivatives of a superposition of two functions.

Theorem 4.1. *Let $U \subseteq \mathfrak{D}$, $V \subseteq \mathfrak{X}$, $W \subseteq \mathfrak{Z}$ be open subsets of Banach spaces over $\mathbb{K}$ and let $f : U \to V$, $g : V \to W$ be functions. Suppose that f is differentiable at some point x in U and g is differentiable at $f(x)$. Then $g \cdot f$ is differentiable at x and the chain rule*

$$(g \cdot f)'(x) = g'(f(x)) \cdot f'(x)$$

holds.

Proof. Let L be derivative of f at x and let K be a derivative of g at $f(x)$. For h in $\mathfrak{D}$ such that $x + h \in U$ define $\phi_f(h)$ by formula

$$f(x+h) - f(x) - L(h) = \phi_f(h) \cdot \|h\|$$

Similarly for s in $\mathfrak{X}$ such that $f(x) + s \in V$ define $\phi_g(s)$ by formula

$$g(f(x)+s) - g(f(x)) - K(s) = \phi_g(s) \cdot \|s\|$$

Now pick nonzero h in $\mathfrak{D}$ such that $x+h \in U$ and $f(x+h) \in V$. Then

$$\begin{aligned}
& g\left(f(x+h)\right) - g\left(f(x)\right) - K\left(L(h)\right) = \\
& = g\left(f(x+h) - f(x) + f(x)\right) - g\left(f(x)\right) - K\left(L(h)\right) = \\
& = K\left(f(x+h) - f(x)\right) + \phi_g\left(f(x+h) - f(x)\right) \cdot \|f(x+h) - f(x)\| - K\left(L(h)\right) = \\
& = K\left(\phi_f(h)\right) \cdot \|h\| + \phi_g\left(f(x+h) - f(x)\right) \cdot \|f(x+h) - f(x)\| = \\
& = K\left(\phi_f(h)\right) \cdot \|h\| + \phi_g\left(f(x+h) - f(x)\right) \cdot \|L(h) + \phi_f(h) \cdot \|h\|\|
\end{aligned}$$

Thus

$$\begin{aligned}
& \left\| \frac{g\left(f(x+h)\right) - g\left(f(x)\right) - K\left(L(h)\right)}{\|h\|} \right\| \leq \\
& \leq \left\| K\left(\phi_f(h)\right) \right\| + \|\phi_g\left(f(x+h) - f(x)\right)\| \cdot \left(\|L\| + \|\phi_f(h)\|\right)
\end{aligned}$$

Note that for $h \to 0$ we have

$$K\left(\phi_f(h)\right) \to 0$$

according to differentiability of f at x and

$$\phi_g\left(f(x+h) - f(x)\right) \to 0$$

which follows from continuity of f at x and differentiability of g at $f(x)$. This implies that

$$\lim_{h \to 0} \frac{g\left(f(x+h)\right) - g\left(f(x)\right) - K\left(L(h)\right)}{\|h\|} = 0$$

and hence the proof is completed. □

5. Lagrange's mean value theorem

The main topic of this section is very useful result, which connects derivatives with local change of a function.