INTEGRATION

1. Introduction

In these notes we develop theory of Bochner-Lebesgue integral. Our exhibition is to some extent different from the standard one. The first step is typical - we start with integration of nonnegative functions and we prove monotone convergence. Then we introduce Lebesgue's spaces and prove their completeness. Lebesgue's dominated convergence is presented as a result about convergence in Lebesgue's spaces. After this we introduce integral as a linear operator on Lebesgue's spaces. Final section resolves certain disambiguity in our notions.

Most of the theory of Lebesgue's spaces (this does not embrace Bochner's integral itself due to obvious reasons) works for Banach spaces defined over fields with absolute values. The reader may always assume for hers convenience that the field is either $\mathbb{C}$ or $\mathbb{R}$.

2. Measurable spaces and measures

In this section we introduce fundamental notions.

Definition 2.1. Let X be a set and let Σ be a family of its subsets. Suppose that Σ is closed under countable unions, complements and contains X. Then Σ is *a σ-algebra of subsets of X*.

Definition 2.2. A pair (X, Σ) consisting of a set X together with a σ-algebra Σ of its subsets is *a measurable space*.

Remark 2.3. Let X be a set and let $\{\Sigma_i\}_{i\in I}$ be a class of σ-algebras of subsets of X. Then

$$\bigcap_{i\in I}\Sigma_i$$

is a σ-algebra of subsets of X.

Remark 2.4. Let X be a set and let $\mathcal{F}$ be a family of subsets of X. Then by Remark 2.3 there exists the smallest (with respect to incusion) σ-algebra containing $\mathcal{F}$. We denote it by $\sigma(\mathcal{F})$ and call it *the σ-algebra generated by $\mathcal{F}$*.

Definition 2.5. Let (X, Σ) and (Y, Δ) be measurable spaces. A map $f : X \to Y$ is *measurable* if $f^{-1}(B) \in \Sigma$ for every $B \in \Delta$.

Definition 2.6. Let (X, Σ) be a measurable space and let $\mu : \Sigma \to [0, +\infty]$ be a function. Suppose that

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

for every sequence $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint sets in Σ. Then μ is *a measure on* Σ.

Definition 2.7. A tuple (X, Σ, μ) consisting of a measurable space (X, Σ) and a measure μ on Σ is called *a space with measure*.

The next two notions relate measurable world with topological world. Reader trained in category theory may notice that we define a functor below.

Definition 2.8. Let X be a topological space. Then the σ-algebra $\mathcal{B}(X)$ generated by all open sets of X is called *the σ-algebra of Borel subsets of X*.

Definition 2.9. Let (X, Σ) be a measurable space and let Y be a topological space. A map $f : X \to Y$ is *measurable* if f is a measurable map $(X, \Sigma) \to (Y, \mathcal{B}(Y))$, where $\mathcal{B}(Y)$ is the σ-algebra of Borel sets on Y.

Finally we use the following notation related to pointwise limits of maps sequences.

Definition 2.10. Let X be a set and let Y be a topological space. Consider a map $f : X \to Y$ and a sequence $\{f_n : X \to Y\}_{n\in\mathbb{N}}$ of maps. If

$$f(x) = \lim_{n\to+\infty} f_n(x)$$

then $\{f_n\}_{n\in\mathbb{N}}$ is *pointwise convergent to* f. In this case we write

$$f = \lim_{n\to+\infty} f_n$$

3. Measurable functions with values in extended real line

Let $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ be the completion of $\mathbb{R}$ to linearly ordered set with the smallest and the greatest elements. Clearly $\overline{\mathbb{R}}$ is a complete linear order. Addition is partially defined operation on $\overline{\mathbb{R}}$ given by the following rules

$$(+\infty) + r = +\infty = r + (+\infty),\ (-\infty) + r = -\infty = r + (-\infty)$$

for every $r \in \mathbb{R}$. Moreover, $\overline{\mathbb{R}}$ with order topology is the two point compactification of $\mathbb{R}$.

Let $\{f_n : X \to \overline{\mathbb{R}}\}_{n\in\mathbb{N}}$ be a sequence of functions on a set X. We define functions

$$\sup_{n\in\mathbb{N}} f_n,\ \inf_{n\in\mathbb{N}} f_n : X \to \overline{\mathbb{R}}$$

by formulas

$$\left(\sup_{n\in\mathbb{N}} f_n\right)(x) = \sup_{n\in\mathbb{N}} f_n(x),\ \left(\inf_{n\in\mathbb{N}} f_n\right)(x) = \inf_{n\in\mathbb{N}} f_n(x)$$

for every $x \in X$. We define functions

$$\limsup_{n\to+\infty} f_n = \inf_{m\in\mathbb{N}} \sup_{n\geq m} f_n,\ \liminf_{n\to+\infty} f_n = \sup_{m\in\mathbb{N}} \inf_{n\geq m} f_n$$

If

$$\liminf_{n\to+\infty} f_n = \limsup_{n\to+\infty} f_n$$

then $\{f_n\}_{n\in\mathbb{N}}$ is pointwise convergent.

Let X be a set and let $f, g : X \to \overline{\mathbb{R}}$ be functions. We write $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.

Definition 3.1. Let X be a set and let $f : X \to \overline{\mathbb{R}}$ be a function. We say that f is *nonnegative* if $f \geq 0$.

Definition 3.2. Let X be a set. A sequence $\{f_n : X \to \overline{\mathbb{R}}\}_{n\in\mathbb{N}}$ is *nondecreasing* if $f_n \leq f_m$ for all pairs $n, m \in \mathbb{N}$ such that $n \leq m$.

Proposition 3.3. *Let (X, Σ) be a measurable space and let $\{f_n : X \to \overline{\mathbb{R}}\}_{n\in\mathbb{N}}$ be a sequence of measurable functions. Then functions*

$$\sup_{n\in\mathbb{N}} f_n,\ \inf_{n\in\mathbb{N}} f_n$$

are measurable.

Proof. Note that $\inf_{n\in\mathbb{N}} f_n = -\sup_{n\in\mathbb{N}} (-f_n)$. Thus it suffices to prove the proposition for $\sup_{n\in\mathbb{N}} f_n$. Fix $r \in \mathbb{R}$ and note that

$$\{x \in X \mid \sup_{n\in\mathbb{N}} f_n(x) > r\} = \bigcup_{q\in\mathbb{Q}, q>r} \bigcup_{n\in\mathbb{N}} \{x \in X \mid f_n(x) \geq q\}$$

Therefore, we derive that $f = \sup_{n\in\mathbb{N}} f_n$ satisfies $f^{-1}\left((r,+\infty]\right) \in \Sigma$ for every $r \in \mathbb{R}$. Family of all left-open intervals in $\overline{\mathbb{R}}$ generate $\mathcal{B}(\overline{\mathbb{R}})$. Hence f is measurable. □

Corollary 3.4. *Let (X,Σ) be a measurable space and let $\{f_n : X \to \overline{\mathbb{R}}\}_{n\in\mathbb{N}}$ be a sequence of measurable functions. Then functions*

$$\liminf_{n\to+\infty} f_n,\ \limsup_{n\to+\infty} f_n$$

are measurable. In particular, if $\{f_n(x)\}_{n\in\mathbb{N}}$ is convergent for every $x \in X$, then also

$$\lim_{n\to+\infty} f_n$$

is measurable.

Proof. Follows directly from Proposition 3.3 and definitions. □

Remark 3.5. Let X be a set and let A be its subsets. We denote by $\mathbb{1}_A$ a $\{0,1\}$-valued function on X equal to 1 for $x \in A$ and equal to zero otherwise.

Proposition 3.6. *Let (X,Σ) be a measurable space and let $f : X \to \overline{\mathbb{R}}$ be a nonnegative, measurable function. Then there exists a nondecreasing sequence $\{s_n : X \to \overline{\mathbb{R}}\}_{n\in\mathbb{N}}$ of nonnegative, measurable functions such that $s_n(X)$ is a finite subset of $\mathbb{R}$ for every $n \in \mathbb{N}$ and $\{s_n\}_{n\in\mathbb{N}}$ is pointwise convergent to f. Moreover, $s_n \le f$ for every $n \in \mathbb{N}$.*

Proof. For every $n \in \mathbb{N}$ and integer $0 \le k < n \cdot 2^n$ we define

$$A_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right)$$

Then $A_{n,k}$ is a measurable set. We define

$$s_n(x) = \sum_{k=0}^{n\cdot 2^{n-1}} \frac{k}{2^n} \cdot \mathbb{1}_{A_{n,k}}$$

Then each $s_n : X \to \overline{\mathbb{R}}$ is a nonnegative, measurable function such that $s_n(X)$ is a finite subset of $\mathbb{R}$. Moreover, we have

$$|s_n(x) - f(x)| \le \frac{1}{2^n}$$

for every $x \in X$ such that $f(x) \le n$. It follows that

$$f = \lim_{n\to+\infty} s_n$$

By definition of s_n we have $s_n \le f$ for each $n \in \mathbb{N}$. This completes the proof. □

4. Lebesgue's integral of nonnegative functions

Definition 4.1. Let (X,Σ,μ) be a space with measure. A measurable function $s : X \to \overline{\mathbb{R}}$ such that $s(X)$ is a finite subset of $\mathbb{R}$ and

$$\mu\left(\left\{x \in X \,\middle|\, s(x) \neq 0\right\}\right) \in \mathbb{R}$$

is *a μ-simple function*.

Definition 4.2. Let (X,Σ,μ) be a space with measure and $s : X \to \overline{\mathbb{R}}$ be a nonnegative, μ-simple function. Then

$$\int_X s\, d\mu = \sum_{y\in\overline{\mathbb{R}}} y \cdot \mu\left(s^{-1}(y)\right)$$

is *the integral of s with respect to μ*.

Fact 4.3. *Let (X, Σ, μ) be a space with measure and $s_1, s_2 : X \to \overline{\mathbb{R}}$ be nonnegative, μ-simple functions. Then the following assertions hold.*

(1) *If $a, b \in \mathbb{R}$ and $a, b \geq 0$, then $as_1 + bs_2$ is a nonnegative, μ-simple function and*

$$\int_X (as_1 + bs_2)\, d\mu = a \int_X s_1\, d\mu + b \int_X s_2\, d\mu$$

(2) *If $s_1 \leq s_2$, then*

$$\int_X s_1\, d\mu \leq \int_X s_2\, d\mu$$

Proof. Left for the reader as an exercise. □

Definition 4.4. Let $f : X \to \overline{\mathbb{R}}$ be a nonnegative, measurable function on a space (X, Σ, μ) with measure. Then we define

$$\int_X f\, d\mu = \sup \left\{ \int_X s\, d\mu \,\middle|\, s \text{ is a nonnegative, } \mu\text{-simple function and } s \leq f \right\}$$

We call it *the integral of f with respect to μ*.

Fact 4.5. *Let $f, g : X \to \overline{\mathbb{R}}$ be a nonnegative, measurable functions on a space (X, Σ, μ) with measure. If $f \leq g$, then*

$$\int_X f\, d\mu \leq \int_X g\, d\mu$$

Proof. Left for the reader as an exercise. □

Theorem 4.6 (Monotone Convergence Theorem). *Let $\{f_n : X \to \overline{\mathbb{R}}\}_{n \in \mathbb{N}}$ be a sequence of nonnegative, measurable functions on a space (X, Σ, μ) with measure. Assume that $\{f_n\}_{n \in \mathbb{N}}$ is nondecreasing and let f be a nonnegative function which is a limit of $\{f_n\}_{n \in \mathbb{N}}$. Then $f : X \to \overline{\mathbb{R}}$ is a nonnegative, measurable function and*

$$\lim_{n \to +\infty} \int_X f_n\, d\mu = \int_X f\, d\mu$$

Proof. By Corollary 3.4 function f is measurable. It is also nonnegative. By Fact 4.5 we deduce that

$$\int_X f_n\, d\mu \leq \int_X f_{n+1}\, d\mu \leq \int_X f\, d\mu$$

for every $n \in \mathbb{N}$ and hence

$$\lim_{n \to +\infty} \int f_n d\mu \leq \int f d\mu$$

Fix a number $\alpha \in (0, 1)$. Pick a μ-simple, nonnegative function $s : X \to \overline{\mathbb{R}}$ such that $s \leq f$. Consider the set

$$A_n = \left\{ x \in X \,\middle|\, f_n(x) < \alpha s(x) \right\}$$

Then $A_n \in \Sigma$ for every $n \in \mathbb{N}$. Since $\{f_n\}_{n \in \mathbb{N}}$ is nondecreasing sequence, we derive that $\{A_n\}_{n \in \mathbb{N}}$ is nonincreasing sequence of sets. Since $s(X)$ is a finite subset of $\mathbb{R}$ and

$$s(x) \leq f(x) = \lim_{n \to +\infty} f_n(x)$$

we derive that

$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset,\ A_1 \subseteq \left\{ x \in X \,\middle|\, s(x) \neq 0 \right\}$$

In particular, $\mu(A_1) \in \mathbb{R}$ and

$$\lim_{n \to +\infty} \mu(A_n) = 0$$

We have inequality

$$\alpha \int_X s\,d\mu = \int_X \alpha s\,d\mu = \int_X \mathbb{1}_{X\setminus A_n} \cdot (\alpha \cdot s)\;d\mu + \int_X \mathbb{1}_{A_n} \cdot (\alpha \cdot s)\;d\mu \leq$$

$$\leq \int_X f_n\,d\mu + \mu(A_n) \cdot \sup_{x\in X}(\alpha s(x)) = \int_X f_n\,d\mu + \mu(A_n) \cdot \sup_{x\in X}(\alpha s(x))$$

By virtue of

$$\lim_{n\to+\infty} \mu(A_n) = 0$$

we have

$$\alpha \int_X s\,d\mu \leq \lim_{n\to+\infty} \int_X f_n\,d\mu$$

Since s is an arbitrary nonnegative and μ-simple function such that $s \leq f$, we deduce that

$$\alpha \int_X f\,d\mu \leq \lim_{n\to+\infty} \int_X f_n\,d\mu$$

Finally for $\alpha \to 1$ we obtain

$$\int_X f\,d\mu \leq \lim_{n\to+\infty} \int_X f_n\,d\mu$$

and this completes the proof. □

The theorem above is a reason why Lebesgue's integration theory is such a powerful tool.

Theorem 4.7 (Fatou's lemma). *Let $\{f_n : X \to \overline{\mathbb{R}}\}_{n\in\mathbb{N}}$ be a sequence of nonnegative, measurable functions on a space (X,Σ,μ) with measure. Then*

$$\int_X \liminf_{n\to+\infty} f_n\,d\mu \leq \liminf_{n\to+\infty} \int_X f_n\,d\mu$$

Proof. For every $m \in \mathbb{N}$ denote $\inf_{n\geq m} f_n$ by g_m. Corollary 3.4 implies that $\{g_m : X \to \overline{\mathbb{R}}\}_{m\in\mathbb{N}}$ is a nondecreasing sequence of nonnegative, measurable functions on (X,Σ). By Theorem 4.6 we have

$$\lim_{m\to+\infty} \int_X \inf_{n\geq m} f_n = \lim_{m\to+\infty} \int_X g_m\,d\mu = \int_X \lim_{m\to+\infty} g_m\,d\mu = \int_X \liminf_{n\to+\infty} f_n\,d\mu$$

Hence

$$\int_X \liminf_{n\to+\infty} f_n\,d\mu = \lim_{m\to+\infty} \int_X \inf_{n\geq m} f_n \leq \liminf_{n\to+\infty} \int_X f_n\,d\mu$$

□

Proposition 4.8. *Let $f,g : X \to \overline{\mathbb{R}}$ be a nonnegative, measurable functions on a space (X,Σ,μ) with measure. Fix numbers $a,b \in \{0\}\cup\mathbb{R}_+$. Then the function $af+bg$ is measurable and*

$$\int_X (af+bg)\;d\mu = a\int_X f\,d\mu + b\int_X g\,d\mu$$

Proof. By Proposition 3.6 there exist nondecreasing sequences $\{s_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ of nonnegative, measurable functions such that

(1) $s_n(X), t_n(X)$ are finite subsets of $\mathbb{R}$ for each $n \in \mathbb{N}$.

(2)

$$f(x) = \lim_{n\to+\infty} s_n(x),\; g(x) = \lim_{n\to+\infty} t_n(x)$$

(3) $s_n \leq f$, $t_n \leq g$ for all $n \in \mathbb{N}$.

It follows that

$$\lim_{n\to+\infty}(as_n+bt_n)=af+bg$$

Thus $af+bg$ is measurable by Corollary 3.4. By definition

$$a\int_X f\,d\mu+b\int_X g\,d\mu\leq\int_X (af+bg)\,d\mu$$

Hence if one of the integrals

$$\int_X f\,d\mu,\ \int_X g\,d\mu$$

is infinite, then the assertion holds. Suppose that both integrals are finite. Then $\{s_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ consist of nonnegative, μ-simple functions. By Theorem 4.6 and Fact 4.3 we have

$$\int_X(af+bg)\,d\mu=\lim_{n\to+\infty}\int_X(as_n+bt_n)\,d\mu=\lim_{n\to+\infty}\left(a\int_X s_n\,d\mu+b\int_X t_n\,d\mu\right)=$$

$$=a\left(\lim_{n\to+\infty}\int_X s_n\,d\mu\right)+b\left(\lim_{n\to+\infty}\int_X t_n\,d\mu\right)=a\int_X f\,d\mu+b\int_X g\,d\mu$$

□

5. Hölder and Minkowski integral inequalities

Theorem 5.1 (Hölder). *Let (X,Σ,μ) be a space with measure and let $p,q\in(1,+\infty)$ satisfy*

$$\frac{1}{p}+\frac{1}{q}=1$$

If $f,g:X\to\overline{\mathbb{R}}$ are nonnegative Σ-measurable functions, then

$$\int_X f\cdot g\,d\mu\leq\left(\int_X f^p\,d\mu\right)^{\frac{1}{p}}\cdot\left(\int_X g^q\,d\mu\right)^{\frac{1}{q}}$$

For the proof we need the following lemma.

Lemma 5.1.1. *Let a,b be nonnegative extended real numbers and let $p,q\in(1,+\infty)$ satisfy*

$$\frac{1}{p}+\frac{1}{q}=1$$

Then

$$a^{\frac{1}{p}}\cdot b^{\frac{1}{q}}\leq\frac{a}{p}+\frac{b}{q}$$

Proof of the lemma. Without loss of generality we may assume that $a,b\in\mathbb{R}_+$. Next the inequality in the question is equivalent with

$$\frac{\ln a}{p}+\frac{\ln b}{q}\leq\ln\left(\frac{a}{p}+\frac{b}{q}\right)$$

and this inequality is an instance of Jensen's inequality, since

$$\frac{1}{p}+\frac{1}{q}=1$$

and logarithm is concave. □

Proof of the theorem. We may assume that

$$\left(\int_X f^p\,d\mu\right)^{\frac{1}{p}},\left(\int_X g^q\,d\mu\right)^{\frac{1}{q}}\in\mathbb{R}_+$$

By Lemma 5.1.1 we have

$$\frac{f(x)\cdot g(x)}{\left(\int_X f^p\,d\mu\right)^{\frac{1}{p}}\cdot\left(\int_X g^q\,d\mu\right)^{\frac{1}{q}}} = \left(\frac{f(x)^p}{\int_X f^p\,d\mu}\right)^{\frac{1}{p}}\cdot\left(\frac{g(x)^q}{\int_X f^q\,d\mu}\right)^{\frac{1}{q}} \leq \frac{1}{p}\cdot\frac{f(x)^p}{\int_X f^p\,d\mu}+\frac{1}{q}\cdot\frac{g(x)^q}{\int_X g^q\,d\mu}$$

for every $x\in X$. Integrating both sides with respect to μ yields

$$\frac{\int_X f\cdot g\,d\mu}{\left(\int_X f^p\,d\mu\right)^{\frac{1}{p}}\cdot\left(\int_X g^q\,d\mu\right)^{\frac{1}{q}}}\leq\frac{1}{p}+\frac{1}{q}=1$$

and hence the inequality in the statement holds. □

Corollary 5.2 (Minkowski)**.** *Let (X,Σ,μ) be a space with measure and let $p\in[1,+\infty)$. Suppose that $f,g: X\to\overline{\mathbb{R}}$ are nonnegative Σ-measurable functions. Then*

$$\left(\int_X (f+g)^p\,d\mu\right)^{\frac{1}{p}}\leq\left(\int_X f^p\,d\mu\right)^{\frac{1}{p}}+\left(\int_X g^p\,d\mu\right)^{\frac{1}{p}}$$

Proof. The case $p=1$ follows from Proposition 4.8. Thus we assume that $p\in(1,+\infty)$. Suppose that $p\in[1,+\infty)$. Note that if $q\in[1,+\infty)$ satisfies

$$\frac{1}{p}+\frac{1}{q}=1$$

then $q=\frac{p}{p-1}$. Hence by Theorem 5.1 we have

$$\int_X f\cdot(f+g)^{p-1}\,d\mu\leq\left(\int_X f^p\,d\mu\right)^{\frac{1}{p}}\cdot\left(\int_X (f+g)^{(p-1)\cdot q}\right)^{\frac{1}{q}}=\left(\int_X f^p\,d\mu\right)^{\frac{1}{p}}\cdot\left(\int_X (f+g)^p\right)^{1-\frac{1}{p}}$$

and

$$\int_X g\cdot(f+g)^{p-1}\,d\mu\leq\left(\int_X g^p\,d\mu\right)^{\frac{1}{p}}\cdot\left(\int_X (f+g)^{(p-1)\cdot q}\right)^{\frac{1}{q}}=\left(\int_X g^p\,d\mu\right)^{\frac{1}{p}}\cdot\left(\int_X (f+g)^p\right)^{1-\frac{1}{p}}$$

Thus

$$\int_X (f+g)^p\,d\mu=\int_X (f+g)\cdot(f+g)^{p-1}\,d\mu=\int_X f\cdot(f+g)^{p-1}\,d\mu+\int_X g\cdot(f+g)^{p-1}\,d\mu\leq$$
$$\leq\left(\int_X f^p\,d\mu\right)^{\frac{1}{p}}\cdot\left(\int_X (f+g)^p\right)^{1-\frac{1}{p}}+\left(\int_X g^p\,d\mu\right)^{\frac{1}{p}}\cdot\left(\int_X (f+g)^p\right)^{1-\frac{1}{p}}$$

dividing both sides by

$$\left(\int_X (f+g)^p\right)^{1-\frac{1}{p}}$$

yields

$$\left(\int_X (f+g)^p\,d\mu\right)^{\frac{1}{p}}\leq\left(\int_X f^p\,d\mu\right)^{\frac{1}{p}}+\left(\int_X g^p\,d\mu\right)^{\frac{1}{p}}$$

This completes the proof. □

Finally in the next section we also use the following integral inequality.

Proposition 5.3. *Let (X,Σ,μ) be a space with measure and let $p\in[1,+\infty)$. Suppose that $f,g: X\to\overline{\mathbb{R}}$ are nonnegative functions measurable with respect to μ. Then*

$$\int_X (f+g)^p\,d\mu\leq C_p\cdot\left(\int_X f^p\,d\mu+\int_X g^p\,d\mu\right)$$

where

$$C_p = \begin{cases} 1 & \text{if } p \in (0,1) \\ 2^p & \text{if } p \in [1,+\infty) \end{cases}$$

Proof. Pick nonnegative numbers $a, b \in \overline{\mathbb{R}}$. Then

$$(a+b)^p \leq C_p \cdot (a^p + b^p)$$

where

$$C_p = \begin{cases} 1 & \text{if } p \in (0,1) \\ 2^p & \text{if } p \in [1,+\infty) \end{cases}$$

Application of Fact 4.5 completes the proof. □

6. Strongly measurable functions

In this section we introduce a class of measurable functions which form the basis of integration in Banach spaces.

Proposition 6.1. *Let (Y, d) be a metric space and let (X, Σ) be a measurable space. Suppose that a sequence $\{f_n : X \to Y\}_{n\in\mathbb{N}}$ of measurable functions is pointwise convergent to some function $f : X \to Y$. Then f is measurable.*

Proof. Let U be an open subset of Y. We define

$$U_k = \left\{ y \in Y \,\middle|\, \operatorname{dist}(y, X \setminus U) > 2^{-k} \right\}$$

for every $k \in \mathbb{N}$. Then $\{U_k\}_{k\in\mathbb{N}}$ are open subsets of Y. We have

$$f^{-1}(U) = \bigcup_{k\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} f_n^{-1}(U_k)$$

and the left hand side is clearly an element of Σ. Hence preimages of open subsets of Y under f are in Σ. Since σ-algebra $\mathcal{B}(Y)$ is generated by open sets, we derive the assertion. □

We fix a field $\mathbb{K}$ together with an absolute value $|-|$. Suppose that Y is a normed vector space over $\mathbb{K}$ and suppose that $\|-\|$ is its norm. Let X be a set and let $f : X \to Y$ be a function. We define a nonnegative function $\|f\| : X \to \overline{\mathbb{R}}$ by formula

$$\|f\|(x) = \|f(x)\|$$

for every $x \in X$.

Definition 6.2. Let Y be a normed vector space over $\mathbb{K}$ and let (X, Σ) be a measurable space. A function $f : X \to Y$ is *strongly measurable* if it is measurable and $f(X)$ is a separable subspace of Y.

Proposition 6.3. *Let Y be a normed vector space over $\mathbb{K}$ and let (X, Σ) be a measurable space. Suppose that a sequence $\{f_n : X \to Y\}_{n\in\mathbb{N}}$ of strongly measurable functions is pointwise convergent to some function $f : X \to Y$. Then f is strongly measurable.*

Proof. According to Proposition 6.1 function f is measurable. Moreover, we have

$$f(X) \subseteq \mathbf{cl}\left(\bigcup_{n\in\mathbb{N}} f_n(X) \right)$$

and hence $f(X)$ is a separable subspace of Y. Thus f is strongly measurable. □

Proposition 6.4. *Let $n \in \mathbb{N}$ and let $Y_0, ..., Y_n$ be normed vector spaces over $\mathbb{K}$. Suppose that (X, Σ) is a measurable space and $f_i : X \to Y_i$ for $0 \le i \le n$ are strongly measurable functions. Then the function*

$$X \ni x \mapsto \Big(f_0(x), ..., f_n(x)\Big) \in \prod_{i=0}^{n} Y_i$$

is strongly measurable.

Proof. Note that the family of open subsets of

$$\prod_{i=0}^{n} f_i(X)$$

is contained in σ-algebra generated by sets

$$\prod_{i=0}^{n} \big(U_i \cap f_i(X)\big)$$

where U_i is an open subset of Y_i for $0 \le i \le n$. Indeed, this is a consequence of the fact that $f_i(X)$ are separable for $0 \le i \le n$. It follows that the function in question is measurable. Since finite product of separable metric spaces is separable, we derive that its image is separable. Hence the function in the statement is strongly measurable. □

Corollary 6.5. *Let Y be a normed space over $\mathbb{K}$ and let (X, Σ) be a measurable space. Let $f, g : X \to Y$ be strongly measurable functions. Then*

$$\alpha f + \beta g$$

is strongly measurable for all $\alpha, \beta \in \mathbb{K}$.

Proof. This is a consequence of Proposition 6.4 and the fact that Y is topological vector space over $\mathbb{K}$. Details are left for the reader. □

Theorem 6.6. *Let Y be a normed space over $\mathbb{K}$ and let (X, Σ) be a measurable space. Let $f : X \to Y$ be a function. Then the following assertions are equivalent.*

- **(i)** *f is strongly measurable.*
- **(ii)** *There exists a sequence $\{s_n : X \to Y\}_{n \in \mathbb{N}}$ of measurable functions pointwise convergent to f such that $s_n(X) \subseteq Y$ is finite and the inequality*

$$||f - s_n|| \le ||f||$$

holds for every $n \in \mathbb{N}$.

For the proof we need the following lemma.

Lemma 6.6.1. *Let $n, k \in \mathbb{N}$ satisfy $k \le n$. Then*

$$\Big\{(r_0, ..., r_n) \in \mathbb{R}^{n+1} \,\Big|\, \min_{0 \le i \le n} r_i < r_j \text{ for } j < k \text{ and } r_k = \min_{0 \le i \le n} r_i\Big\} \subseteq \mathbb{R}^{n+1}$$

is a Borel subset.

Proof of the lemma. Left for the reader. □

Proof of the theorem. Suppose that f is strongly measurable. Consider a countable subset $\{y_k\}_{k \in \mathbb{N}}$ of Y which closure contains $f(X)$ and assume that y_0 is zero in Y. From Proposition 6.4 we deduce that the function

$$X \ni x \mapsto \Big(\|y_0 - f(x)\|, ..., \|y_n - f(x)\|\Big) \in \mathbb{R}^{n+1}$$

is measurable for each $n \in \mathbb{N}$. Thus by Lemma 6.6.1 the set

$$A_{n,k} = \left\{x \in X \,\middle|\, \min_{0 \leq i \leq n} \|y_i - f(x)\| < \|y_j - f(x)\| \text{ for } j < k \text{ and } \|y_k - f(x)\| = \min_{0 \leq i \leq n} \|y_i - f(x)\|\right\}$$

is in Σ for all $k, n \in \mathbb{N}$ such that $k \leq n$. For $n \in \mathbb{N}$ we define a function $s_n : X \to Y$ by formula

$$s_n(x) = \sum_{k=0}^{n} y_k \cdot \mathbb{1}_{A_{n,k}}$$

Note that s_n is measurable, $s_n(X)$ is finite and

$$\|s_n(x) - f(x)\| = \min_{0 \leq i \leq n} \|y_i - f(x)\|$$

for every $x \in X$. Thus

$$\lim_{n \to +\infty} s_n = f$$

and $\|s_n - f\| \leq \|f\|$. This completes the proof of **(i)** $\Rightarrow$ **(ii)**.

Suppose now that there exists a sequence $\{s_n : X \to Y\}_{n \in \mathbb{N}}$ of measurable functions pointwise convergent to f such that $s_n(X) \subseteq Y$ is finite. Then Proposition 6.3 asserts that f is strongly measurable. This proves that **(ii)** $\Rightarrow$ **(i)**. □

7. Lebesgue spaces

In this section we fix a positive real number p and a Banach space Y with norm $\|-\|$ over a field $\mathbb{K}$ with absolute value $|-|$.

Definition 7.1. Let $f : X \to Y$ be a strongly measurable function on a space (X, Σ, μ) with measure. Then

$$\|f\|_p = \left(\int_X \|f\|^p \, d\mu\right)^{\frac{1}{p}}$$

is *the p-norm of f with respect to μ*.

Definition 7.2. Let $f : X \to Y$ be a strongly measurable function on a space (X, Σ, μ) with measure. If

$$\|f\|_p \in \mathbb{R}$$

then f is *p-th power integrable with respect to μ* or shortly *p-th power μ-integrable*.

Definition 7.3. Let (X, Σ, μ) be a space with measure. Then the set of all Y-valued, p-th power μ-integrable functions is denoted by $L^p(\mu, Y)$ and is called *the Lebesgue space of p-th power μ-integrable functions for Y*.

By Corollary 6.5 the set of all strongly measurable, Y-valued functions on a space (X, Σ) is a $\mathbb{K}$-vector space with respect to the usual operations. Our next goal is to show that $L^p(\mu, Y)$ is a $\mathbb{K}$-vector subspace of this space and to introduce the topology on $L^p(\mu, Y)$ which makes it into a topological vector space over $\mathbb{K}$.

Proposition 7.4. *Let (X, Σ, μ) be a space with measure. Then $L^p(\mu, Y)$ is a $\mathbb{K}$-vector subspace of the $\mathbb{K}$-vector space of all strongly measurable functions on (X, Σ). Moreover, the following assertions hold.*

(1) *If $p \in (0, 1)$, then*

$$L^p(\mu, Y) \times L^p(\mu, Y) \ni (f, g) \mapsto \int_X \|f - g\|^p \, d\mu \in \mathbb{R}_+ \cup \{0\}$$

is a translation invariant pseudometric on $L^p(\mu, Y)$.

(2) *If $p \in [1, +\infty)$, then*

$$\|-\|_p : L^p(\mu, Y) \to \mathbb{R}_+ \cup \{0\}$$

is a seminorm.

Proof. Note that if $f \in L^p(\mu, Y)$ and $\alpha \in \mathbb{K}$, then

$$\|\alpha \cdot f\|_p = |\alpha| \cdot \|f\|_p$$

Hence if $f \in L^p(\mu, Y)$, then also $\alpha \cdot f \in L^p(\mu, Y)$ and moreover, the function $\|-\|_p$ is positively homogeneous. Next we separately handle cases $p \in (0,1)$ and $p \in [1, +\infty)$.

Suppose that $p \in (0,1)$. Then by Proposition 5.3

$$\int_X (f+g)^p \, d\mu \leq \int_X f^p \, d\mu + \int_X g^p \, d\mu$$

for any two nonnegative strongly measurable functions $f, g : X \to \overline{\mathbb{R}}$ on (X, Σ, μ). Hence

$$f, g \in L^p(\mu, Y) \Leftarrow f + g \in L^p(\mu, Y)$$

and

$$L^p(\mu, Y) \times L^p(\mu, Y) \ni (f, g) \mapsto \int_X \|f - g\|^p \, d\mu \in \mathbb{R}_+ \cup \{0\}$$

is a translation invariant pseudometric on $L^p(\mu, Y)$. This completes the proof for this case.

Suppose now that $p \in [1, +\infty)$. This case follows from Corollary 5.2. □

For now on we consider $L^p(\mu, Y)$ as a topological vector $\mathbb{K}$-space with respect to topology described in Proposition 7.4.

Remark 7.5. Note that the sequence $\{f_n\}_{n \in \mathbb{N}}$ of elements of $L^p(\mu, Y)$ converges to $f \in L^p(\mu, Y)$ if and only if

$$\lim_{n \to +\infty} \|f_n - f\|_p \, d\mu = 0$$

and the space $L^p(\mu, Y)$ carries translation invariant pseudometric.

Theorem 7.6 (Riesz). *Let (X, Σ, μ) be a space with measure. If $\{f_n : X \to Y\}_{n \in \mathbb{N}}$ is a Cauchy sequence of elements of $L^p(\mu, Y)$, then there exist an increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ of natural numbers and a function $f : X \to Y$ which is p-the power μ-integrable such that*

$$\lim_{k \to +\infty} f_{n_k}(x) = f(x)$$

for all x outside some set in Σ of measure μ equal to zero. Moreover, $\{f_n\}_{n \in \mathbb{N}}$ converges to f in $L^p(\mu, Y)$.

Proof. We consider an increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ of natural numbers such that

$$\int_X \|f_{n_{k+1}} - f_{n_k}\|^p \, d\mu \leq 4^{-k}$$

for every $k \in \mathbb{N}$. For $k \in \mathbb{N}$ consider a set

$$A_k = \{x \in X \,|\, \|f_{n_{k+1}}(x) - f_{n_k}(x)\|^p \geq 2^{-k}\}$$

in Σ. Then

$$2^{-k} \cdot \mu(A_k) \leq \int_X \|f_{n_{k+1}} - f_{n_k}\|^p \, d\mu \leq 4^{-k}$$

Hence $\mu(A_k) \leq 2^{-k}$ for each $k \in \mathbb{N}$. For $m \in \mathbb{N}$ we define

$$B_m = \bigcup_{k=m}^{+\infty} A_k$$

Then

$$\mu(B_m) = \mu\left(\bigcup_{k=m}^{+\infty} A_k\right) \leq \sum_{k=m}^{+\infty} \mu(A_k) \leq \sum_{k=m}^{+\infty} 2^{-k} = 2^{1-m}$$

and $\{B_m\}_{m \in \mathbb{N}}$ is a nonincreasing sequence of subsets of Σ. This proves that

$$B = \bigcap_{m \in \mathbb{N}} B_m$$

satisfy $\mu(B) = 0$. For $x \notin B_m$ we have

$$\sum_{k=m}^{+\infty} \|f_{n_{k+1}}(x) - f_{n_k}(x)\| \leq \sum_{k=m}^{+\infty} 2^{-kp} = \left(\frac{1}{2^p}\right)^m \cdot \frac{2^p}{2^p - 1}$$

Since Y is a Banach space, we deduce that for $x \notin B_m$ series

$$f_{n_0}(x) + \sum_{k \in \mathbb{N}} \left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)$$

is convergent. Therefore, it is also convergent for $x \notin B$. We define $f : X \to Y$ as a sum of the series for $x \notin B$ and $f(x) = 0$ for $x \in B$. Then

$$\lim_{k \to +\infty} f_{n_k}(x) = f(x)$$

for $x \notin B$. Hence

$$\lim_{k \to +\infty} \mathbb{1}_{X \setminus B} \cdot f_{n_k} = f$$

and Proposition 6.3 asserts that f is a strongly measurable function. Moreover, by Theorem 4.6 and Proposition 5.3 we have

$$\int_X \|f\|^p \, d\mu = \int_X \mathbb{1}_{X \setminus B} \cdot \|f_{n_0} + \sum_{k \in \mathbb{N}} (f_{n_{k+1}} - f_{n_k})\|^p \, d\mu \leq \int_X \left(\|f_{n_0}\| + \sum_{k \in \mathbb{N}} \|f_{n_{k+1}} - f_{n_k}\|\right)^p d\mu \leq$$

$$\leq C_p \cdot \left(\int_X \|f_{n_0}^p\| \, d\mu + \sum_{k \in \mathbb{N}} \int_X \|f_{n_{k+1}} - f_{n_k}\|^p \, d\mu\right) \leq C_p \cdot \left(\int_X \|f_{n_0}\|^p \, d\mu + \sum_{k \in \mathbb{N}} 4^{-k}\right)$$

where C_p is some positive constant depending only on p. Thus $f \in L^p(\mu, Y)$. Again by Theorem 4.6 and Proposition 5.3 we have

$$\int_X \|f - f_{n_m}\|^p \, d\mu = \int_X \mathbb{1}_{X \setminus B} \cdot \|\sum_{k=m}^{+\infty} (f_{n_{k+1}} - f_{n_k})\|^p \, d\mu \leq \int_X \|\sum_{k=m}^{+\infty} (f_{n_{k+1}} - f_{n_k})\|^p \, d\mu \leq$$

$$\leq C_p \cdot \int_X \sum_{k=m}^{\infty} \|f_{n_{k+1}} - f_{n_k}\|^p \, d\mu = C_p \cdot \sum_{k=m}^{+\infty} \int_X \|f_{n_{k+1}} - f_{n_k}\|^p \, d\mu = \sum_{k=m}^{+\infty} 4^{-k} = 4^{-m} \cdot \frac{4}{3}$$

where C_p is some positive constant depending only on p. Therefore, $\{f_{n_k}\}_{k \in \mathbb{N}}$ converges to f in $L^p(\mu, Y)$. Since $\{f_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^p(\mu, Y)$ with a subsequence convergent to f, we derive that $\{f_n\}_{n \in \mathbb{N}}$ converges to f in $L^p(\mu, Y)$. □

Next theorem is a criterion connecting pointwise convergence and convergence in $L^p(\mu, Y)$.

Theorem 7.7 (Lebesgue's dominated convergence theorem)**.** *Let (X, Σ, μ) be a space with measure and let $\{f_n : X \to Y\}_{n \in \mathbb{N}}$ be a sequence of p-th power μ-integrable functions. Suppose that $f : X \to Y$ is a pointwise limit of $\{f_n\}_{n \in \mathbb{N}}$ and assume that there exists nonnegative, measurable function $g : X \to \overline{\mathbb{R}}$ such that $\|f_n\|^p \leq g$ holds for every $n \in \mathbb{N}$ and*

$$\int_X g \, d\mu \in \mathbb{R}$$

Then $f \in L^p(\mu, Y)$ and $\{f_n\}_{n \in \mathbb{N}}$ converges to f in $L^p(\mu, Y)$.

For the proof we need the following result.

Lemma 7.7.1. *Let $f, g : X \to \overline{\mathbb{R}}$ be a nonnegative, measurable functions on a space (X, Σ, μ) with measure. Suppose that $f \leq g$ and*

$$\int_X f \, d\mu, \int_X g \, d\mu \in \mathbb{R}$$

Then

$$\int_X (g - f) \, d\mu = \int_X g \, d\mu - \int_X f \, d\mu$$

Proof of the lemma. According to Proposition 4.8 we obtain that

$$\int_X g\, d\mu = \int_X ((g-f)+f)\, d\mu = \int_X (g-f)\, d\mu + \int_X f\, d\mu$$

Since integrals above are finite, we have

$$\int_X (g-f)\, d\mu = \int_X g\, d\mu - \int_X f\, d\mu$$

□

Proof of the theorem. Since $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to f, we deduce that f is strongly measurable. Moreover, a sequence $\left\{\|f\|_n : X \to \overline{\mathbb{R}}\right\}_{n\in\mathbb{N}}$ converges pointwise to $\|f\|$. Since $\|f_n\|^p \leq g$ holds for every $n \in \mathbb{N}$, we deduce that $\|f\|^p \leq g$. Thus

$$\int_X \|f\|^p\, d\mu \leq \int_X g\, d\mu \in \mathbb{R}$$

Hence $f \in L^p(\mu, Y)$. By Proposition 5.3 there exists some positive constant C_p such that $\|f - f_n\|^p \leq C_p \cdot 2g$ holds for every $n \in \mathbb{N}$. Thus by Theorem 4.7 and Lemma 7.7.1 we have

$$\int_X C_p \cdot 2g\, d\mu - \int_X \limsup_{n\to+\infty} \|f - f_n\|^p\, d\mu = \int_X \left(C_p \cdot 2g - \limsup_{n\to+\infty} \|f-f_n\|^p\right) d\mu =$$
$$= \int_X \liminf_{n\to+\infty} \left(C_p \cdot 2g - \|f-f_n\|^p\right)\, d\mu \leq \liminf_{n\to+\infty} \int_X \left(C_p \cdot 2g - \|f-f_n\|^p\right)\, d\mu =$$
$$= \int_X C_p \cdot 2g\, d\mu - \limsup_{n\to+\infty} \int_X \|f-f_n\|^p\, d\mu$$

Hence

$$\limsup_{n\to+\infty} \int_X \|f - f_n\|^p\, d\mu \leq \int_X \limsup_{n\to+\infty} \|f-f_n\|^p\, d\mu = 0$$

Thus we deduce that $\{f_n\}_{n\in\mathbb{N}}$ converges to f in $L^p(\mu, Y)$. □

It turns out that Lebesgue's space $L^p(\mu, Y)$ contains certain dense subspace which can be easily described. We shall define this space and then prove that in fact it is dense.

Definition 7.8. Let (X, Σ, μ) be a space with measure. A measurable function $s : X \to Y$ such that $s(X) \subseteq Y$ is finite and

$$\mu\left(\{x \in X \,|\, s(x) \neq 0\}\right) \in \mathbb{R}$$

is *μ-simple*. The set of all μ-simple, Y-valued functions defined on (X, Σ, μ) is denoted by $S(\mu, Y)$.

Theorem 7.9. *Let (X, Σ, μ) be a space with measure. For each $f \in L^p(\mu, Y)$ there exists a sequence $\{s_n : X \to Y\}_{n\in\mathbb{N}}$ of μ-simple functions and a nonnegative, measurable function $g : X \to \overline{\mathbb{R}}$ such that the following assertions hold.*

(1)

$$\int_X g\, d\mu \in \mathbb{R}$$

(2) $\|s_n\|^p \leq g$ *for every* $n \in \mathbb{N}$

(3) $\{s_n\}_{n\in\mathbb{N}}$ *converges pointwise to* f.

Proof. Clearly every μ-simple function is strongly measurable and p-th power μ-integrable. Moreover, μ-simple functions are closed under $\mathbb{K}$-vector space operations defined on the space of strongly measurable functions. Hence $S(\mu, Y) \subseteq L^p(\mu, Y)$ is a $\mathbb{K}$-linear subspace.
Suppose now that $f : X \to Y$ is a p-th power μ-integrable function. By Theorem 6.6 there exists a sequence $\{s_n : X \to Y\}_{n\in\mathbb{N}}$ of measurable functions pointwise convergent to f such that $s_n(X)$ is finite and the inequality

$$\|s_n - f\| \leq \|f\|$$

holds for every $n \in \mathbb{N}$. Let $g = 2^p \cdot \|f\|^p$. Then

$$\int_X g \, d\mu \in \mathbb{R}$$

Moreover, for every $n \in \mathbb{N}$ we have $\|s_n\|^p \leq g$. Hence s_n is μ-simple for every $n \in \mathbb{N}$. This completes the proof of the theorem. $\square$

Corollary 7.10. *The space $S(\mu, Y)$ is a dense $\mathbb{K}$-linear subspace of $L^p(\mu, Y)$.*

Proof. This is an immediate consequence of Theorems 7.7 and 7.9. $\square$

8. Bochner's integral

In this section $\mathbb{K}$ is either field $\mathbb{R}$ or $\mathbb{C}$ with their usual absolute values.

Definition 8.1. Let Y be a Banach space over $\mathbb{K}$ and let (X, Σ, μ) be a space with measure. For every $s \in S(\mu, Y)$ we define

$$\int_X s \, d\mu = \sum_{y \in Y} y \cdot \mu\left(s^{-1}(y)\right)$$

and we call it *the integral of s with respect to μ*.

Fact 8.2. *Let Y be a Banach space over $\mathbb{K}$ and let (X, Σ, μ) be a space with measure. Then*

$$S(\mu, Y) \ni s \mapsto \int_X s \, d\mu \in Y$$

is a $\mathbb{K}$-linear operator such that

$$\left\| \int_X s \, d\mu \right\| \leq \|s\|_1$$

Proof. We left the proof (direct calculation) for the reader as an exercise. $\square$

Let Y be a Banach space over $\mathbb{K}$ and let (X, Σ, μ) be a space with measure. By Theorem 7.9 space $S(\mu, Y)$ is a dense $\mathbb{K}$-linear subspace of $L^1(\mu, Y)$. By Theorem 7.6 space $L^1(\mu, Y)$ is complete and by Fact 8.2 operator

$$S(\mu, Y) \ni s \mapsto \int_X s \, d\mu \in Y$$

is a $\mathbb{K}$-linear operator with norm equal to one. These imply that there exists a unique $\mathbb{K}$-linear operator

$$L^1(\mu, Y) \ni f \mapsto \int_X f \, d\mu \in Y$$

with norm equal to one extending the integral on $S(\mu, Y)$.

Definition 8.3. Let Y be a Banach space over $\mathbb{K}$ and let (X, Σ, μ) be a space with measure. The operator

$$L^1(\mu, Y) \ni f \mapsto \int_X f \, d\mu \in Y$$

is called *the Bochner's integral with respect to μ*. For every $f \in L^1(\mu, Y)$ element

$$\int_X f \, d\mu \in Y$$

is called *the integral of f with respect to μ*.

Definition 8.4. Elements of $L^1(\mu, Y)$ are called *μ-integrable functions with values in Y*.

We prove now some properties of Bochner's integral. We start with noting the following.

Corollary 8.5. *Let Y be a Banach space over $\mathbb{K}$ and let (X, Σ, μ) be a space with measure. Suppose that $\{f_n : X \to Y\}_{n \in \mathbb{N}}$ is a sequence of μ-integrable functions convergent in $L^1(\mu, Y)$ to some μ-integrable function $f : X \to Y$. Then*

$$\lim_{n \to +\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

in Y.

Proof. By definition Bochner's integral is continuous with respect to $\|-\|_1$. □

Next we discuss linearity and convexity of integral.

Proposition 8.6. *Let (X, Σ, μ) be a space with measure. Let Y, Z be Banach spaces over $\mathbb{K}$ and let $T : Y \to Z$ be a $\mathbb{K}$-linear and continuous map of Banach spaces. Then the following assertions hold.*

(1) *T induces a $\mathbb{K}$-linear and continuous map*

$$L^1(\mu, Y) \ni f \mapsto T \cdot f \in L^1(\mu, Z)$$

(2) *The formula*

$$\int_X T \cdot f \, d\mu = T\left(\int_X f \, d\mu\right)$$

holds for every $f \in L^1(\mu, Y)$.

Proof. Note that

$$\|T \cdot f\|_1 = \int_X \|T \cdot f\| \, d\mu \leq \int_X \|T\| \cdot \|f\| \, d\mu \leq \|T\| \cdot \int_X \|f\| \, d\mu = \|T\| \cdot \|f\|_1$$

for every $f \in L^1(\mu, Y)$. This proves that **(1)** holds.

For the proof of **(2)** we fix a μ-simple function $s \in \mathcal{S}(\mu, Y)$. Then we have $T \cdot s \in \mathcal{S}(\mu, Z)$ and

$$\int_X T \cdot s \, d\mu = \sum_{z \in Z} z \cdot \mu\left(s^{-1}\left(T^{-1}(z)\right)\right) = \sum_{z \in Z} z \cdot \sum_{T(y) = z} \mu\left(s^{-1}(y)\right) =$$

$$= \sum_{z \in Z} \sum_{T(y) = z} T(y) \cdot \mu\left(s^{-1}(y)\right) = \sum_{y \in Y} T(y) \cdot \mu\left(s^{-1}(y)\right) =$$

$$= T\left(\sum_{y \in Y} y \cdot \mu\left(s^{-1}(y)\right)\right) = T\left(\int_X s \, d\mu\right)$$

Hence for every $s \in \mathcal{S}(\mu, Y)$ we have

$$\int_X T \cdot s \, d\mu = T\left(\int_X s \, d\mu\right)$$

Theorem 7.9 together with **(1)** and continuity of integral with respect to μ imply that

$$\int_X T \cdot f \, d\mu = T\left(\int_X f \, d\mu\right)$$

for every $f \in L^1(\mu, Y)$. □

Proposition 8.7. *Let Y be a Banach space over $\mathbb{K}$ and let (X, Σ, μ) be a space with measure. Suppose that C is a convex subset of Y. Let $f \in L^1(\mu, Y)$ be a function such that $\mu\left(f^{-1}(C)\right) \in \mathbb{R}_+$. Then*

$$\frac{1}{\mu\left(f^{-1}(C)\right)} \cdot \int_{f^{-1}(C)} f \, d\mu \in \mathbf{cl}\,(C)$$

Proof. Denote by c the mean value in question. If $c \notin \mathbf{cl}\,(C)$, then according to separation theorem proved in [Monygham, 2023] there exists an $\mathbb{R}$-linear continuous map $g : Y \to \mathbb{R}$ and $y \in Y$ such that we have $g(c-y) \subseteq \mathbb{R}_-$ and $g\left(\mathbf{cl}\,(C) - y\right) \subseteq \mathbb{R}_+$. Since $\mu\left(f^{-1}(C)\right) \in \mathbb{R}_+$, we have

$$\frac{1}{\mu\left(f^{-1}(C)\right)} \cdot \int_{f^{-1}(C)} g \cdot f\, d\mu > g(y)$$

On the other hand according to Proposition 8.6 we derive that

$$\frac{1}{\mu\left(f^{-1}(C)\right)} \cdot \int_{f^{-1}(C)} g \cdot f\, d\mu = g\left(\frac{1}{\mu\left(f^{-1}(C)\right)} \cdot \int_{f^{-1}(C)} f\, d\mu\right) = g(c) < g(y)$$

This is a contradiction. Thus $c \in \mathbf{cl}\,(C)$. □

9. Lebesgue integral of scalar functions

First we compare Bochner's integration with Lebesgue's integration of nonnegative functions. We introduce precise terminology.

Definition 9.1. Let X be a set and let $f : X \to \mathbb{C}$ be a function. If $f(x) \in \mathbb{R}$ for every $x \in X$, then we say that f is *real valued*. If in addition $f(x) \geq 0$ for every $x \in X$, then f is *nonnegative*.

As careful reader may notice there is certain ambiguity in theory developed so far. Indeed, if (X, Σ, μ) is a space with measure and $f : X \to \mathbb{C}$ is a μ-integrable, nonnegative function, then we have a twofold interpretation of

$$\int_X f\, d\mu$$

Firstly, if we consider f as a nonnegative, μ-measurable function with values in $\overline{\mathbb{R}}$, then we may consider integral of this nonnegative function described as in Section 4. On the other hand it may be considered as the Bochner integral of f with respect to μ as defined in Section 8. We explain now why these two numbers are equal. For this note that there is no ambiguity in definitions of simple functions and their integrals between Section 4 on the one hand and Sections 7, 8 on the other. By Proposition 3.6 there exists a nondecreasing sequence of nonnegative, μ-simple functions $\{s_n : X \to \mathbb{C}\}_{n \in \mathbb{N}}$ which is pointwise convergent to f. By Theorem 4.6 we have

$$\int_X f\, d\mu = \lim_{n \to +\infty} \int_X s_n\, d\mu$$

where we understand the left hand side as the integral in the sense of Section 4. On the other hand by Theorem 7.7 the sequence $\{s_n\}_{n \in \mathbb{N}}$ converges to f also in $L^1(\mu, \mathbb{C})$. Hence by Corollary 8.5 we deduce that

$$\int_X f\, d\mu = \lim_{n \to +\infty} \int_X s_n\, d\mu$$

where we understand the left hand side as the Bochner integral of f with respect to μ. Thus the two numbers are equal.

Let (X, Σ, μ) be a space with measure. In case of $\mathbb{C}$ or $\mathbb{R}$ valued μ-integrable function f on X its Bochner integral

$$\int_X f\, d\mu$$

is also called the Lebesgue integral of f with respect to μ.

References

[Monygham, 2023] Monygham (2023). Hahn-banach theorem. *github repository: "Monygham/Pedo-mellon-a-minno"*.