

CATEGORIES OF PRESHEAVES

1. INTRODUCTION – SET THEORETICAL BACKGROUND

These notes deal with properties of categories of presheaves. For some arguments and also for the underlying set-theoretic setup we use Grothendieck universes [ML98, page 22]. This implies that our arguments rely on stronger foundational assumptions than the usual Zermelo-Frankel axioms. Grothendieck universes can be defined within Zermelo-Frankel set theory as follows.

Definition 1.1. Let U be a set. We say that U is *Grothendieck universe* if the following conditions are satisfied.

- (1) The set ω of finite ordinals in the sense of von Neumann is an element of U .
- (2) If $y \in U$ and $x \in y$, then $x \in U$.
- (3) If $x \in U$, then $\mathcal{P}(x) \in U$ and $\bigcup x \in U$.
- (4) If $x \in U$, $y \subseteq U$ and $f : x \rightarrow y$ is surjective, then $y \in U$.

If U is a Grothendieck universe, then the pair (U, \in) forms a model for Zermelo-Frankel theory. This implies that the existence of Grothendieck universes is independent from Zermelo-Frankel axioms. In this notes we extend the usual Zermelo-Frankel system by adding the following Tarski axiom.

Every set is an element of some Grothendieck universe.

This new formal system is called *Tarski-Grothendieck set theory*. Let U be a Grothendieck universe. We denote by \mathbf{Set}_U a category whose objects are elements of U and whose morphisms are maps of sets.

Definition 1.2. Let U be a Grothendieck universe. A category \mathcal{C} is *U-small* if classes of objects and morphisms of \mathcal{C} are members of U .

Definition 1.3. Let U be a Grothendieck universe. A category \mathcal{C} is *locally U-small* if for any pair X, Y of objects of \mathcal{C} we have $\text{Mor}_{\mathcal{C}}(X, Y) \in U$.

Throughout this notes we fix a Grothendieck universe U . Elements of U are called sets. We use term *class* for arbitrary sets (also these ones outside U). We denote \mathbf{Set}_U by \mathbf{Set} . By (locally) small category we mean (locally) U -small category.

2. CREATION OF LIMITS AND COLIMITS

Definition 2.1. Let $F : \mathcal{C} \rightarrow \mathcal{X}$, $D : I \rightarrow \mathcal{C}$ be functors. Suppose that $(X, \{f_i\}_{i \in I})$ is a cone in \mathcal{X} for the composition $F \cdot D$. We say that a cone $(Z, \{g_i\}_{i \in I})$ in \mathcal{C} for D is a *lift* of $(X, \{f_i\}_{i \in I})$ if $F(Z) = X$ and $F(g_i) = f_i$ for every $i \in I$.

Definition 2.2. Let $F : \mathcal{C} \rightarrow \mathcal{X}$, $D : I \rightarrow \mathcal{C}$ be functors. We say that F *creates limits for D* if every limiting cone for $F \cdot D$ has a unique lift to a cone for D and this lift is a limiting cone for D .

Definition 2.3. Let $F : \mathcal{C} \rightarrow \mathcal{X}$ be a functor. We say that:

- (1) F *creates limits* if F creates limits for all functors $D : I \rightarrow \mathcal{C}$.
- (2) F *creates small limits* if F creates limits for all functors $D : I \rightarrow \mathcal{C}$ with I being small category.

- (3) *F creates finite limits* if *F* creates limits for all functors $D : I \rightarrow \mathcal{C}$ with *I* being category with finitely many objects and arrows.

Some extra material on creation of limits can be found in [ML98, V.1]. By the usual arrow inverting one defines the notion of creation of colimits.

Now we prove an important result. First we need to introduce some notation. Suppose that \mathcal{C} and \mathcal{X} are categories. Then we denote by $\mathbf{Fun}(\mathcal{C}, \mathcal{X})$ the category with functors $\mathcal{C} \rightarrow \mathcal{X}$ as objects and natural transformations between them as morphisms. We also denote by $|\mathcal{C}|$ the category having the same objects as \mathcal{C} but with only identities as a morphism. There exists the canonical functor $|\mathcal{C}| \rightarrow \mathcal{C}$ that induces identity map on objects. The next result describes limits and colimits in functor categories.

Theorem 2.4. *Let \mathcal{C}, \mathcal{X} be a categories. Then the functor $\mathbf{Fun}(\mathcal{C}, \mathcal{X}) \rightarrow \mathbf{Fun}(|\mathcal{C}|, \mathcal{X})$ induced by the precomposition with the functor $|\mathcal{C}| \rightarrow \mathcal{C}$ creates all limits and colimits.*

Proof. We prove that this functor creates limits. Creation of colimits can be handled similarly. Let *I* be a category. For every object *i* in *I* consider a functor $F_i : \mathcal{C} \rightarrow \mathcal{X}$ and for every arrow $\alpha : i \rightarrow j$ in *I* consider a natural transformation $F_\alpha : F_i \rightarrow F_j$. Suppose that these data gives rise to a functor $I \rightarrow \mathbf{Fun}(\mathcal{C}, \mathcal{X})$. Each limiting cone over the composition of $I \rightarrow \mathbf{Fun}(\mathcal{C}, \mathcal{X})$ and $\mathbf{Fun}(\mathcal{C}, \mathcal{X}) \rightarrow \mathbf{Fun}(|\mathcal{C}|, \mathcal{X})$ consists of a family of objects $\{F(X)\}_{X \in \mathcal{C}}$ of \mathcal{X} parametrized by objects of \mathcal{C} and a family $\{f_{i,X}\}_{i \in I, X \in \mathcal{C}}$ of arrows in \mathcal{X} parametrized by objects of $I \times \mathcal{C}$ such that the following assertion hold.

- (*) For every $X \in \mathcal{C}$ a pair $(F(X), \{f_{i,X}\}_{i \in I})$ is a limiting cone for a functor $I \rightarrow \mathcal{X}$ given by $i \mapsto F_i(X)$ and $\alpha \mapsto F_\alpha(X)$ for any object *i* and arrow α in *I*.

We now show that there exists a unique lift of a pair $(\{F(X)\}_{X \in \mathcal{C}}, \{f_{i,X}\}_{i \in I, X \in \mathcal{C}})$ to a cone $(F, \{f_i\}_{i \in I})$ over the functor $I \rightarrow \mathbf{Fun}(\mathcal{C}, \mathcal{X})$ described by data $(\{F_i\}_{i \in I}, \{F_\alpha\}_{\alpha \in \mathbf{Mor}(I)})$. For this pick an arrow $f : X \rightarrow Y$. Then by (*) there exists a unique arrow $F(f) : F(X) \rightarrow F(Y)$ such that every square

$$\begin{array}{ccc} F(Y) & \xrightarrow{f_{i,Y}} & F_i(Y) \\ \uparrow F(f) & & \uparrow F_i(f) \\ F(X) & \xrightarrow{f_{i,X}} & F_i(X) \end{array}$$

for every $i \in I$ is commutative. Suppose that $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are arrows in \mathcal{C} . Then

$$f_{i,Z} \cdot F(g \cdot f) = F_i(g \cdot f) \cdot f_{i,X} = F_i(g) \cdot F_i(f) \cdot f_{i,X} = F_i(g) \cdot f_{i,Y} \cdot F(f) = f_{i,Z} \cdot F(g) \cdot F(f)$$

According to (*) we deduce that $F(g \cdot f) = F(g) \cdot F(f)$. Similarly we prove that $F(1_X) = 1_{F(X)}$. Hence there exists a unique functor $F : \mathcal{C} \rightarrow \mathcal{X}$ that extends object mapping $\{F(X)\}_{X \in \mathcal{C}}$ and such that $\{f_i : F \rightarrow F_i\}_{i \in I}$ becomes a collection of natural transformations of functors. Therefore, $(F, \{f_i\}_{i \in I})$ is a unique lift of $(\{F(X)\}_{X \in \mathcal{C}}, \{f_{i,X}\}_{i \in I, X \in \mathcal{C}})$ to a cone over the functor $I \rightarrow \mathbf{Fun}(\mathcal{C}, \mathcal{X})$ described by data $(\{F_i\}_{i \in I}, \{F_\alpha\}_{\alpha \in \mathbf{Mor}(I)})$. Now we prove that the cone $(F, \{f_i\}_{i \in I})$ is limiting. For this assume that $(G, \{g_i\}_{i \in I})$ is a cone over the functor $I \rightarrow \mathbf{Fun}(\mathcal{C}, \mathcal{X})$ described by data $(\{F_i\}_{i \in I}, \{F_\alpha\}_{\alpha \in \mathbf{Mor}(I)})$. By (*) we derive that for every $X \in \mathcal{C}$ there exists a unique morphism $\tau_X : G(X) \rightarrow F(X)$ such that

$$\begin{array}{ccc}
G(X) & \xrightarrow{\tau_X} & F(X) \\
& \searrow g_{i,X} & \swarrow f_{i,X} \\
& F_i(X) &
\end{array}$$

It suffices to verify that a collection $\{\tau_X\}_{X \in \mathcal{C}}$ is a natural transformation of functors $G \rightarrow F$. For this pick $f : X \rightarrow Y$. Then

$$f_{i,Y} \cdot F(f) \cdot \tau_X = F_i(f) \cdot f_{i,X} \cdot \tau_X = F_i(f) \cdot g_{i,X} = g_{i,Y} \cdot G(f) = f_{i,Y} \cdot \tau_Y \cdot G(f)$$

for every $i \in I$. According to (\star) we deduce that $F(f) \cdot \tau_X = \tau_Y \cdot G(f)$. Since f is arbitrary, we derive that $\{\tau_X\}_{X \in \mathcal{C}}$ is a natural transformation of functors $G \rightarrow F$. \square

Let \mathcal{C}, \mathcal{X} be categories. For every object $X \in \mathcal{C}$ we denote by $\text{ev}_X : \mathbf{Fun}(\mathcal{C}, \mathcal{X}) \rightarrow \mathcal{X}$ the functor that sends $F \in \mathbf{Fun}(\mathcal{C}, \mathcal{X})$ to $F(X)$ and $f : F \rightarrow G$ in $\mathbf{Fun}(\mathcal{C}, \mathcal{X})$ to $f_X : F(X) \rightarrow G(X)$.

Corollary 2.5. *Let \mathcal{C}, \mathcal{X} and I be categories and let $D : I \rightarrow \mathbf{Fun}(\mathcal{C}, \mathcal{X})$ be a functor. Suppose that for every $X \in \mathcal{C}$ the functor $\text{ev}_X \cdot D : I \rightarrow \mathcal{X}$ admits a limit (colimit). Then D admits a limit (colimit). Moreover, suppose that $(F, \{f_i\}_{i \in I})$ is a cone (cocone) over D . Then the following are equivalent.*

- (i) $(F, \{f_i\}_{i \in I})$ is a limiting cone (colimiting cocone) over D .
- (ii) $(F, \{f_i\}_{i \in I})$ is a cone (cocone) over D and for every $X \in \mathcal{C}$ the pair $(F(X), \{f_{i,X}\}_{i \in I})$ is a limiting cone (colimiting cocone) over $\text{ev}_X \cdot D$.

Proof. The assumption that for every $X \in \mathcal{C}$ the functor $\text{ev}_X \cdot D : I \rightarrow \mathcal{X}$ admits a limit (colimit) implies that the composition of D with the functor $\mathbf{Fun}(\mathcal{C}, \mathcal{X}) \rightarrow \mathbf{Fun}(|\mathcal{C}|, \mathcal{X})$ induced by the canonical functor $|\mathcal{C}| \rightarrow \mathcal{C}$ admits a limit (colimit). Now by Theorem 2.4 we derive that the functor $\mathbf{Fun}(\mathcal{C}, \mathcal{X}) \rightarrow \mathbf{Fun}(|\mathcal{C}|, \mathcal{X})$ creates limits and colimits. Hence D admits a limit (colimit). More precisely there exists a limiting cone (colimiting cocone) $(F, \{f_i\}_{i \in I})$ over D such that for every $X \in \mathcal{C}$ the pair $(F(X), \{f_{i,X}\}_{i \in I})$ is a limiting cone (colimiting cocone) over $\text{ev}_X \cdot D$. Since any two limiting cones (colimiting cocones) over given functor are isomorphic, we deduce that (i) \Rightarrow (ii). On the other hand if $(F, \{f_i\}_{i \in I})$ is a cone (cocone) over D and for every $X \in \mathcal{C}$ the pair $(F(X), \{f_{i,X}\}_{i \in I})$ is a limiting cone (colimiting cocone) over $\text{ev}_X \cdot D$, then, according to the fact that $\mathbf{Fun}(\mathcal{C}, \mathcal{X}) \rightarrow \mathbf{Fun}(|\mathcal{C}|, \mathcal{X})$ creates limits and colimits, we derive that $(F, \{f_i\}_{i \in I})$ is a limiting cone (colimiting cocone) over D . Thus (ii) \Rightarrow (i) holds. \square

3. PRESHEAVES

Definition 3.1. Let \mathcal{C} be a locally small category. We denote by $\widehat{\mathcal{C}}$ the category $\mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ and we call it *the category of presheaves on \mathcal{C}* .

Definition 3.2. Let \mathcal{C} be a locally small category. For every object $X \in \mathcal{C}$ we define $h_X = \text{Mor}_{\mathcal{C}}(-, X)$. We call it *the presheaf represented by X* . Next for every morphism $f : X \rightarrow Y$ in \mathcal{C} we define a natural transformation $h_f : h_X \rightarrow h_Y$ given by formula

$$\text{Mor}_{\mathcal{C}}(Z, X) \ni g \mapsto f \cdot g \in \text{Mor}_{\mathcal{C}}(Z, Y)$$

This defines a functor $h : \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ called *the Yoneda embedding of \mathcal{C}* .

Theorem 3.3 (Yoneda lemma). *Let \mathcal{C} be a locally small category. For every object $X \in \mathcal{C}$ and a presheaf $F \in \widehat{\mathcal{C}}$ map*

$$\text{Mor}_{\widehat{\mathcal{C}}}(h_X, F) \rightarrow F(X)$$

given by formula $p \mapsto p(1_X)$ is a bijection natural in both X and F .

Proof. Fix $p : h_X \rightarrow F$ for some $X \in \mathcal{C}$ and $F \in \widehat{\mathcal{C}}$. Denote $x = p(1_X)$. Next let $f : Y \rightarrow X$ be a morphism in \mathcal{C} . Since p is natural transformation, we derive that the diagram

$$\begin{array}{ccc} h_X(Y) & \xrightarrow{p_Y} & F(Y) \\ h_X(f) \uparrow & & \uparrow F(f) \\ h_X(X) & \xrightarrow{p_X} & F(X) \end{array}$$

is commutative. Thus $p_Y(f) = p_Y(h_X(f)(1_X)) = F(f)(x)$. This shows that for every object $Y \in \mathcal{C}$ and every morphism $f : Y \rightarrow X$ we have $p_Y(f) = F(f)(x)$. Hence p is uniquely determined by x . This proves that the map described in the statement is injective. Now we prove that it is surjective. For this fix an element $x \in F(X)$ and define $p : h_X \rightarrow F$ by formula $p_Y(f) = F(f)(x)$ for every morphism $f : Y \rightarrow X$ in \mathcal{C} . Consider morphisms $g : Z \rightarrow Y$ and $f : Y \rightarrow X$ in \mathcal{C} and note that

$$F(g)(p_Y(f)) = F(g) \cdot F(f)(x) = F(f \cdot g)(x) = p_Z(f \cdot g) = p_Z(h_X(g)(f))$$

Thus p is a morphism of presheaves and $p(1_X) = x$.

It remains to prove that the map in the statement is natural with respect to X and F . This is left to the reader as an exercise. \square

Corollary 3.4. *Let \mathcal{C} be a locally small category. The functor $h : \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ is full and faithful.*

Proof. Fully faithfulness follows from Theorem 3.3. \square

Now we investigate small limits and colimits in presheaf categories. For this fix a locally small category \mathcal{C} and $X \in \mathcal{C}$. We denote by $\text{ev}_X : \widehat{\mathcal{C}} \rightarrow \mathbf{Set}$ the functor that sends a presheaf F to $F(X)$ and a morphism $f : F \rightarrow G$ to f_X .

Corollary 3.5. *Fix a locally small category \mathcal{C} . Let I be a category and let $D : I \rightarrow \widehat{\mathcal{C}}$ be a functor. If I is a small category, then D admits a limit (colimit). Moreover, for a cone (cocone) $(F, \{f_i\}_{i \in I})$ over D the following assertions are equivalent.*

- (i) $(F, \{f_i\}_{i \in I})$ is a limiting cone (colimiting cocone) over D .
- (ii) $(F, \{f_i\}_{i \in I})$ is a cone (cocone) over D and for every $X \in \mathcal{C}$ the pair $(F(X), \{f_{i,X}\}_{i \in I})$ is a limiting cone (colimiting cocone) over $\text{ev}_X \cdot D$.

Proof. By [ML98, V.1, Theorem 1 and Exercise 8] we know that the category \mathbf{Set} admits both small limits and small colimits. Now it suffices to use Corollary 2.5. \square

Finally we add one notational remark. Let \mathcal{C} be a locally small category and F, G be presheaves on \mathcal{C} . Then we denote by $\text{Mor}_{\mathcal{C}}(F, G)$ the class of morphisms of presheaves with domain F and codomain G .

4. CLASSES OF GENERATORS

Definition 4.1. Let \mathcal{C} be a category. A class \mathcal{K} of objects of \mathcal{C} is called a *class of generators* for \mathcal{C} if for any pair of distinct and parallel arrows

$$X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y$$

there exists $Z \in \mathcal{K}$ and a morphism $h : Z \rightarrow X$ such that $f \cdot h \neq g \cdot h$.

Now we introduce special case of the notion of the class of generators of category. For this we need one more definition.

Definition 4.2. Let \mathcal{C} be a category and X be an object of \mathcal{C} . An object of \mathcal{C} over X is a morphism $f : Y \rightarrow X$ in \mathcal{C} . If $f_1 : Y_1 \rightarrow X$, $f_2 : Y_2 \rightarrow X$ are objects of \mathcal{C} over X , then a morphism over X between these objects consists of a morphism $f : Y_1 \rightarrow Y_2$ in \mathcal{C} such that the following triangle

$$\begin{array}{ccc} Y_1 & \xrightarrow{f} & Y_2 \\ & \searrow f_1 & \swarrow f_2 \\ & X & \end{array}$$

is commutative. This defines the category of objects of \mathcal{C} over X .

For every object X of a category \mathcal{C} we denote by \mathcal{C}/X the category of objects over X . Next suppose that X is an object of \mathcal{C} and \mathcal{K} is a subclass of the class of objects of \mathcal{C} . We denote by \mathcal{K}/X the full subcategory of \mathcal{C}/X that consists of morphisms $f : K \rightarrow X$ such that K is in \mathcal{K} . For every such class we denote by π_X the canonical functor $\mathcal{K}/X \rightarrow \mathcal{K}$ that sends every arrow $f : K \rightarrow X$ in \mathcal{K}/X to K . In the case of considerations in which multiple distinct classes are involved we specify more precise notation. Next suppose that $f : X \rightarrow Y$ is a morphism in a category \mathcal{C} . Then the composition with f induces a functor $\mathcal{C}/X \rightarrow \mathcal{C}/Y$. We denote this functor by \mathcal{C}/f . Now if \mathcal{K} is a class of objects of \mathcal{C} , then we denote by \mathcal{K}/f the functor $\mathcal{K}/X \rightarrow \mathcal{K}/Y$ induced by \mathcal{C}/f .

Definition 4.3. Let \mathcal{C} be a category and \mathcal{K} be a class of objects of \mathcal{C} . Suppose that for every object X of \mathcal{C} a pair

$$(X, \{f\}_{f \in \mathcal{K}/X})$$

is a colimiting cocone of a functor given as the composition of $\pi_X : \mathcal{K}/X \rightarrow \mathcal{K}$ with the inclusion functor $\mathcal{K} \hookrightarrow \mathcal{C}$. Then we call \mathcal{K} a dense class of generators for \mathcal{C} .

Let \mathcal{C} be a locally small category and \mathcal{K} be a class of objects of \mathcal{C} . We also denote by $\widehat{\mathcal{K}}$ the corresponding full subcategory of $\widehat{\mathcal{C}}$. We define a functor $\Gamma_{\mathcal{K}} : \mathcal{C} \rightarrow \widehat{\mathcal{K}}$ as the composition of the Yoneda embedding $\mathcal{C} \rightarrow \widehat{\mathcal{C}}$ with the restriction functor $\widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{K}}$.

Theorem 4.4. Let \mathcal{C} be a locally small category and \mathcal{K} be a class of objects of \mathcal{C} . Then the following are equivalent.

- (i) \mathcal{K} is a (dense) class of generators for \mathcal{C} .
- (ii) The functor

$$\Gamma_{\mathcal{K}} : \mathcal{C} \rightarrow \widehat{\mathcal{K}}$$

is (full and) faithful.

Proof. First we need to introduce some notation. For every object X of \mathcal{C} we denote by $F_X : \mathcal{K}/X \rightarrow \mathcal{C}$ the functor obtained as the composition of $\pi_X : \mathcal{K}/X \rightarrow \mathcal{K}$ with the inclusion functor $\mathcal{K} \hookrightarrow \mathcal{C}$. We also denote by Γ_X the value of Γ on X and for every object Y of \mathcal{C} we denote by $\text{Cocone}_Y(F_X)$ the class of cocones with Y as the vertex over the functor F_X . Finally if $g : X \rightarrow Y$ is a morphism of \mathcal{C} , then we denote by Γ_g a natural morphism $\Gamma_X \rightarrow \Gamma_Y$ induced by g . Suppose now that X and Y are objects of \mathcal{C} . Let $\sigma : \Gamma_X \rightarrow \Gamma_Y$ be a natural transformation. Then one can show that $\{\sigma(f)\}_{f \in \mathcal{K}/X}$ is a cocone of F_X with vertex in Y and moreover, the map

$$\text{Mor}_{\mathcal{K}}(\Gamma_X, \Gamma_Y) \ni \sigma \mapsto \{\sigma(f)\}_{f \in \mathcal{K}/X} \in \text{Cocone}_Y(F_X)$$

is bijective. We have a commutative triangle

$$\begin{array}{ccc}
\text{Mor}_{\mathcal{K}}(\Gamma_X, \Gamma_Y) & \xrightarrow{\sigma \mapsto \{\sigma(f)\}_{f \in \mathcal{K}/X}} & \text{Cocone}_Y(F_X) \\
& \nwarrow \scriptstyle g \mapsto \Gamma_g \quad \nearrow \scriptstyle g \mapsto \{g \cdot f\}_{f \in \mathcal{K}/X} & \\
& \text{Mor}_{\mathcal{C}}(X, Y) &
\end{array}$$

From this we derive that Γ is (full and) faithful if and only if

$$\text{Mor}_{\mathcal{C}}(X, Y) \ni g \mapsto \{g \cdot f\}_{f \in \mathcal{K}/X} \in \text{Cocone}_Y(F_X)$$

is (bijective) injective for any pair X, Y of objects in \mathcal{C} . This map is (bijective) injective for any pair X, Y of objects in \mathcal{C} if and only if \mathcal{K} is a class of (dense) generators for \mathcal{C} . This proves theorem. \square

Corollary 4.5. *Let \mathcal{C} be a locally small category. Then the class of representable presheaves $\{h_X\}_{X \in \mathcal{C}}$ is a dense class of generators for $\widehat{\mathcal{C}}$.*

Proof. We want to apply Theorem 4.4 to $\widehat{\mathcal{C}}$. Our issue is that in general $\widehat{\mathcal{C}}$ is not a locally small category. To fix this we must be specific and work with Grothendieck universes [ML98, page 22]. We assume (c.f. Section 1) that our base Grothendieck universe is U . Then $\mathbf{Set} = \mathbf{Set}_U$ is the category of U -small sets and \mathcal{C} is a locally U -small category. Next $\widehat{\mathcal{C}} = \mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set}_U)$ is a presheaf category. Now we fix another universe V that contains U and such that \mathcal{C} is V -small. We denote by \mathbf{Set}_V the category of V -small sets. We can apply Theorem 4.4 to a locally V -small category $\widehat{\mathcal{C}}$. Consider the composition of the Yoneda embedding $\widehat{\mathcal{C}} \rightarrow \mathbf{Fun}((\widehat{\mathcal{C}})^{\text{op}}, \mathbf{Set}_V)$ with the restriction $\mathbf{Fun}((\widehat{\mathcal{C}})^{\text{op}}, \mathbf{Set}_V) \rightarrow \mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set}_V)$ induced by the usual Yoneda embedding $h : \mathcal{C} \rightarrow \widehat{\mathcal{C}}$. The composition is isomorphic with the functor $\widehat{\mathcal{C}} = \mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set}_U) \rightarrow \mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set}_V)$ induced by the inclusion $\mathbf{Set}_U \hookrightarrow \mathbf{Set}_V$. Hence it is full and faithful. Now (replacing our base universe U by V) we can apply Theorem 4.4 to a locally V -small category $\widehat{\mathcal{C}}$ and derive the statement. \square

5. INTERNAL HOM

We start by making few remarks. Let \mathcal{C} be a locally small category and let X be an object of \mathcal{C} . Recall that $\pi_X : \mathcal{C}/X \rightarrow \mathcal{C}$ is a functor that sends morphism $f : Y \rightarrow X$ to Y . For every presheaf F on \mathcal{C} we denote by $F|_X$ the functor

$$F \cdot (\pi_X)^{\text{op}} : (\mathcal{C}/X)^{\text{op}} \rightarrow \mathbf{Set}$$

The map $F \mapsto F|_X$ extends to a functor $\widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}/X}$. Let $\mathbf{1}_X$ denote a presheaf on \mathcal{C}/X that assigns to every object over X a set with one element. According to Corollary 3.5 we derive that $\mathbf{1}_X$ is a terminal object in $\widehat{\mathcal{C}/X}$.

Fact 5.1. *Let \mathcal{C} be a category and let F be a presheaf on \mathcal{C} . Suppose that $x \in F(X)$ for some X in \mathcal{C} . Then x determines a morphism $\mathbf{1}_X \rightarrow F|_X$ that for every object f in \mathcal{C}/X sends a unique element of $\mathbf{1}_X(f)$ to $F(f)(x) \in F|_X(f)$. This gives rise to a bijection*

$$F(X) \cong \text{Mor}_{\mathcal{C}/X}(\mathbf{1}_X, F|_X)$$

Proof. We left to the reader as an exercise. \square

Let \mathcal{C} be a locally small category. If $f : X \rightarrow Y$ is a morphism in \mathcal{C} , then we have a functor $\widehat{\mathcal{C}/Y} \rightarrow \widehat{\mathcal{C}/X}$ induced by the precomposition with $(\mathcal{C}/f)^{\text{op}}$.

Definition 5.2. Let \mathcal{C} be a locally small category and let F, G be presheaves on \mathcal{C} . Assume that for every object X in \mathcal{C} the class $\text{Mor}_{\mathcal{C}/X}(F|_X, G|_X)$ is a set. We define

$$\text{Mor}_{\mathcal{C}}(F, G)(X) = \text{Mor}_{\mathcal{C}/X}(F|_X, G|_X)$$

for every X in \mathcal{C} . This is a presheaf on \mathcal{C} , since for every morphism $f : X \rightarrow Y$, we can compose a morphism $\sigma : F|_Y \rightarrow G|_Y$ of presheaves with $(\mathcal{C}/f)^{\text{op}}$ i.e. we have a map

$$\text{Mor}_{\mathcal{C}}(F, G)(Y) \ni \sigma \mapsto \sigma_{(\mathcal{C}/f)^{\text{op}}} \in \text{Mor}_{\mathcal{C}}(F, G)(X)$$

and these make $\text{Mor}_{\mathcal{C}}(F, G)$ a functor. The presheaf $\text{Mor}_{\mathcal{C}}(F, G)$ is called *an internal hom of F and G* .

Let F, G and H be presheaves on a locally small category \mathcal{C} and assume that $\text{Mor}_{\mathcal{C}}(F, G)$ exists. Fix a morphism of presheaves $\sigma : H \times F \rightarrow G$. Pick an object X in \mathcal{C} and $x \in H(X)$. Let $i_x : \mathbf{1}|_X \rightarrow H|_X$ be a morphism determined by $x \in H(X)$ as in Fact 5.1. Then $\sigma|_X \cdot (i_x \times 1_{F|_X})$ yields a morphism $\tau_x : F|_X \rightarrow G|_X$. Suppose now that $f : Y \rightarrow X$ is a morphism in \mathcal{C} . We have

$$\left(\sigma|_X \cdot (i_x \times 1_{F|_X}) \right)_{(\mathcal{C}/f)^{\text{op}}} = (\sigma|_X)_{(\mathcal{C}/f)^{\text{op}}} \cdot \left((i_x)_{(\mathcal{C}/f)^{\text{op}}} \times (1_{F|_X})_{(\mathcal{C}/f)^{\text{op}}} \right) = \sigma|_Y \cdot (i_{F(f)(x)} \times 1_{F|_Y})$$

because $(i_x)_{(\mathcal{C}/f)^{\text{op}}} = i_{F(f)(x)}$. This implies that $(\tau_x)_{(\mathcal{C}/f)^{\text{op}}} = \tau_{F(f)(x)}$. Hence $\tau : H \rightarrow \text{Mor}_{\mathcal{C}}(F, G)$ given by

$$H(X) \ni x \mapsto \tau_x \in \text{Mor}_{\mathcal{C}/X}(F|_X, G|_X)$$

is a morphism of presheaves. This defines a map of classes

$$\text{Mor}_{\mathcal{C}}(H \times F, G) \ni \sigma \mapsto \tau \in \text{Mor}_{\mathcal{C}}(H, \text{Mor}_{\mathcal{C}}(F, G))$$

Theorem 5.3. Let \mathcal{C} be a locally small category and F, G be presheaves on \mathcal{C} . Assume that for every object X in \mathcal{C} the class $\text{Mor}_{\mathcal{C}/X}(F|_X, G|_X)$ is a set. Then the map

$$\text{Mor}_{\mathcal{C}}(H \times F, G) \rightarrow \text{Mor}_{\mathcal{C}}(H, \text{Mor}_{\mathcal{C}}(F, G))$$

described above is a bijection natural in H .

Proof. The fact that the map in the statement is natural in H is left to the reader as an exercise. Pick an object X in \mathcal{C} . We verify now that the map

$$\text{Mor}_{\mathcal{C}}(h_X \times F, G) \rightarrow \text{Mor}_{\mathcal{C}}(h_X, \text{Mor}_{\mathcal{C}}(F, G))$$

is a bijection. Pick a morphism $\sigma : h_X \times F \rightarrow G$ of presheaves and suppose that $\tau : h_X \rightarrow \text{Mor}_{\mathcal{C}}(F, G)$ is its value under the discussed map. According to Yoneda lemma (Theorem 3.3) τ is uniquely determined by its value on 1_X . We denote this value by ρ . Thus it suffices to prove that

$$\text{Mor}_{\mathcal{C}}(h_X \times F, G) \ni \sigma \mapsto \rho \in \text{Mor}_{\mathcal{C}/X}(F|_X, G|_X)$$

is bijective. We retrieve ρ by means of procedure described before the statement of this theorem. Firstly 1_X according to Fact 5.1 determines a morphism $i : \mathbf{1}|_X \rightarrow (h_X)|_X$. Now $\rho \in \text{Mor}_{\mathcal{C}/X}(F|_X, G|_X)$ is isomorphic with $\sigma|_X \cdot (i \times 1_{F|_X})$. Hence for every $f : Y \rightarrow X$ and $y \in F(Y)$ we have

$$\rho_f(y) = \sigma_Y(f, y)$$

This implies that σ and ρ are mutually determined and thus

$$\text{Mor}_{\mathcal{C}}(h_X \times F, G) \rightarrow \text{Mor}_{\mathcal{C}}(h_X, \text{Mor}_{\mathcal{C}}(F, G))$$

is a bijection.

Now we prove the general case. We know that the map

$$\text{Mor}_{\mathcal{C}}(H \times F, G) \rightarrow \text{Mor}_{\mathcal{C}}(H, \text{Mor}_{\mathcal{C}}(F, G))$$

is natural in H and is bijective when H is a representable presheaf. Now the following statements hold.

- (1) Every presheaf is canonically the colimit of representable presheaves by Corollary 4.5.
- (2) The functor $(-) \times F : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}}$ preserves colimits (this follows from cartesian closedness of **Set** [ML98, page 98] and Corollary 3.5).
- (3) Suppose that V is a Grothendieck universe that contains the base universe U and such that $\widehat{\mathcal{C}}$ is V -locally small. Then the functor

$$\mathrm{Mor}_{\mathcal{C}}(-, \mathrm{Mor}_{\mathcal{C}}(F, G)) : \widehat{\mathcal{C}} \rightarrow \mathbf{Set}_V$$

preserves colimits [ML98, V.4, Theorem 1].

Therefore, we derive that the map in the question is bijective for every presheaf H . \square

6. SUBPRESHEAVES OF INTERNAL HOM

Let \mathcal{C} be a locally small category and let F, G be presheaves on \mathcal{C} . The requirement that $\mathrm{Mor}_{\mathcal{C}/X}(F|_X, G|_X)$ is a set for every object X in \mathcal{C} is a serious limitation of Theorem 5.3. In this section we explain a useful result which addresses this issue.

Definition 6.1. Let \mathcal{C} be a locally small category and let F, G, J be presheaves on \mathcal{C} . Suppose that for every object X in \mathcal{C} there exists an inclusion of classes $J(X) \subseteq \mathrm{Mor}_{\mathcal{C}/X}(F|_X, G|_X)$ such that the square of maps (horizontal arrows in the square are inclusions) of classes

$$\begin{array}{ccc} J(Y) & \hookrightarrow & \mathrm{Mor}_{\mathcal{C}/Y}(F|_Y, G|_Y) \\ J(f) \downarrow & & \downarrow \sigma \mapsto \sigma_{(\mathcal{C}/f)^{\mathrm{op}}} \\ J(X) & \hookrightarrow & \mathrm{Mor}_{\mathcal{C}/X}(F|_X, G|_X) \end{array}$$

is commutative for every morphism $f : X \rightarrow Y$ in \mathcal{C} . Then we say that J is a *subpresheaf of internal hom* of F and G .

Let \mathcal{C} be a locally small category, F, G, H be presheaves on \mathcal{C} . Fix a morphism $\sigma : H \times F \rightarrow G$ of presheaves. Recall from the previous section that for every object X in \mathcal{C} and x in $H(X)$ we denote by $i_x : 1|_X \rightarrow H|_X$ a unique morphism determined by x (Fact 5.1). Next we denote by $\tau_x : F|_X \rightarrow G|_X$ a unique morphism isomorphic with $\sigma|_X \cdot (i_x \times 1_{F|_X})$.

Definition 6.2. Let \mathcal{C} be a locally small category, F, G, H be presheaves on \mathcal{C} and assume that J is a subpresheaf of internal hom of F and G . Then a morphism of presheaves $\sigma : H \times F \rightarrow G$ is called a *family of J -morphisms parametrized by H* if for every object X in \mathcal{C} and every x in $H(X)$ we have $\tau_x \in J(X)$.

We continue discussion started before the definition. Let us now assume that $\sigma : H \times F \rightarrow G$ is a family of J -morphisms parametrized by H for some subpresheaf J of internal hom of F and G . Then $\tau : H \rightarrow J$ given by

$$H(X) \ni x \mapsto \tau_x \in J(X)$$

is a morphism of presheaves. The proof is identical to the proof of the analogous statement preceding Theorem 5.3. This gives rise to a map of classes

$$\{\text{families of } J\text{-morphisms parametrized by } H\} \ni \sigma \mapsto \tau \in \mathrm{Mor}_{\mathcal{C}}(H, J)$$

Theorem 6.3. Let \mathcal{C} be a locally small category and F, G be presheaves on \mathcal{C} . Assume that J is a subpresheaf of internal hom of F and G . Then the map

$$\{\text{families of } J\text{-morphisms parametrized by } H\} \rightarrow \mathrm{Mor}_{\mathcal{C}}(H, J)$$

described above is a bijection natural in H .

Proof. We enlarge our base universe U to a Grothendieck universe V such that \mathcal{C} is V -small. Then $\text{Mor}_{\mathcal{C}}(F, G) \in \mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set}_V)$ and J is a legitimate subobject of $\text{Mor}_{\mathcal{C}}(F, G)$ in $\mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set}_V)$. For every $H \in \widehat{\mathcal{C}} = \mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set}_U) \subseteq \mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set}_V)$ we have a bijection

$$\text{Mor}_{\mathcal{C}}(H \times F, G) \rightarrow \text{Mor}_{\mathcal{C}}(H, \text{Mor}_{\mathcal{C}}(F, G))$$

natural in H . This follows according to Theorem 5.3 applied to the enlarged category of presheaves $\mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set}_V)$. Finally this bijection induces a bijection

$$\{\text{families of } J\text{-morphisms parametrized by } H\} \rightarrow \text{Mor}_{\mathcal{C}}(H, J)$$

on its subclasses, which is natural in H and is given by the rule described in the discussion preceding the statement of the theorem. \square

7. REMARKS ON CATEGORIES OF COPRESHEAVES

Definition 7.1. Let \mathcal{C} be a locally small category. The category $\mathbf{Fun}(\mathcal{C}, \mathbf{Set})$ is called *the category of copresheaves on \mathcal{C}* .

All results stated above for categories of presheaves hold for categories of copresheaves by virtue of the identification

$$\mathbf{Fun}(\mathcal{C}, \mathbf{Set}) = \mathbf{Fun}((\mathcal{C}^{\text{op}})^{\text{op}}, \mathbf{Set})$$

REFERENCES

- [ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.