

LINEARLY REDUCTIVE GROUPS

1. MOTIVATION – LINEAR REPRESENTATIONS OF COMPACT TOPOLOGICAL GROUPS

In this section we fix a compact topological group \mathbf{G} . Assume that $\rho : \mathbf{G} \rightarrow \mathrm{GL}_n(\mathbb{C})$ is a continuous homomorphism i.e. a complex, n -dimensional linear representation of \mathbf{G} . For every $g \in \mathbf{G}$ we get a matrix

$$\rho(g) = [c_{ij}(g)]_{1 \leq i, j \leq n}$$

For i, j function $c_{ij} : \mathbf{G} \rightarrow \mathbb{C}$ is a continuous complex valued function. Alternatively suppose that $\{e_1, e_2, \dots, e_n\}$ is the standard basis of \mathbb{C}^n on which $\mathrm{GL}_n(\mathbb{C})$ act. Then c_{ij} is equal to a function

$$\mathbf{G} \ni g \mapsto \langle g \cdot e_i, e_j \rangle \in \mathbb{C}$$

Fix now $g_1, g_2 \in \mathbf{G}$ and note that

$$[c_{ij}(g_2 \cdot g_1)]_{1 \leq i, j \leq n} = \rho(g_2 \cdot g_1) = \rho(g_2) \cdot \rho(g_1) = \left[\sum_{k=1}^n c_{ik}(g_2) \cdot c_{kj}(g_1) \right]_{1 \leq i, j \leq n}$$

Hence

$$c_{ij}(g_2 \cdot g_1) = \sum_{k=1}^n c_{ik}(g_2) \cdot c_{kj}(g_1)$$

for every $1 \leq i, j \leq n$. This implies that $\sum_{1 \leq i, j \leq n} \mathbb{C} \cdot c_{ij} \subseteq \mathcal{L}^2(\mathbf{G}, \mathbb{C})$ is a linear $\mathbf{G} \times \mathbf{G}^{\mathrm{op}}$ -subrepresentation of the regular representation $\mathcal{L}^2(\mathbf{G}, \mathbb{C})$. We call it *the matrix coefficients of ρ* .

2. MATRIX COEFFICIENTS OF A REPRESENTATION

Proposition 2.1. *Let \mathfrak{X} be a monoid k -functor and let V be a finitely generated, projective k -module. Fix a morphism of monoids $\rho : \mathfrak{X} \rightarrow \mathcal{L}_V$. Fix k -algebra A and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^\vee$. For every A -algebra B and $x \in \mathfrak{X}_A(B)$ we consider the formula*

$$c_{v,w}(x) = \langle \rho_A(x) \cdot v_B, w_B \rangle$$

Then $c_{v,w}$ defines a regular function on \mathfrak{X}_A for every k -algebra A .

Proof. Suppose that $f : B \rightarrow C$ is a morphism of A -algebras and pick $x \in \mathfrak{X}_A(B)$. Since ρ_A is natural and $w : A \otimes_k V \rightarrow A$ is a morphism of A -modules, we derive that the diagram

$$\begin{array}{ccccc} V_B & \xrightarrow{\rho_A(x)} & V_B & \xrightarrow{w_B} & B \\ 1_{V_A} \otimes_A f \downarrow & & \downarrow 1_{V_A} \otimes_A f & & \downarrow f \\ V_C & \xrightarrow{\rho_A(\mathfrak{X}_A(f)(x))} & V_C & \xrightarrow{w_C} & C \end{array}$$

is commutative. Hence

$$c_{v,w}(\mathfrak{X}_A(f)(x)) = \langle \rho_A(\mathfrak{X}_A(f)(x)) \cdot v_C, w_C \rangle = f(\langle \rho_A(x) \cdot v_B, w_B \rangle) = f(c_{v,w}(x))$$

and this implies that $c_{v,w} : \mathfrak{X}_A \rightarrow \mathbb{A}_A^1$ is natural. □

Definition 2.2. Let \mathfrak{X} be a monoid k -functor and let (V, ρ) be its representation with finitely generated, projective underlying k -module V . Fix k -algebra A and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^\vee$. Then the regular function $c_{v,w}$ on \mathfrak{X}_A is called *the matrix coefficient of v and w* .

Proposition 2.3. Let \mathfrak{X} be a monoid k -functor and let (V, ρ) be its representation with finitely generated projective underlying k -module V . Then the following assertions holds.

(1) For every k -algebra A map

$$(A \otimes_k V) \times (A \otimes_k V^\vee) \ni (v, w) \mapsto c_{v,w} \in \text{Mor}_A(\mathfrak{X}_A, \mathbb{A}_A^1)$$

is A -bilinear.

(2) The collection of maps

$$\{(A \otimes_k V) \times (A \otimes_k V^\vee) \ni (v, w) \mapsto c_{v,w} \in \text{Mor}_A(\mathfrak{X}_A, \mathbb{A}_A^1)\}_{A \in \mathbf{Alg}_k}$$

gives rise to a morphism of k -functors

$$V_a \times V_a^\vee \longrightarrow \text{Mor}_k(\mathfrak{X}, \mathbb{A}_k^1)$$

Proof. We left the proof of (1) to the reader.

We prove (2). Consider k -algebra A and an A -algebra B with structural morphism $f : A \rightarrow B$. Fix $v \in A \otimes_k V$, $w \in A \otimes_k V^\vee$. We prove that restriction of $c_{v,w} : \mathfrak{X}_A \rightarrow \mathbb{A}_A^1$ to the category \mathbf{Alg}_B is c_{v_B, w_B} . For this pick a B -algebra C and an element $x \in \mathfrak{X}_A(C) = \mathfrak{X}_B(C)$. Note that

$$c_{v,w}(x) = \langle \rho_A(x) \cdot v_C, w_C \rangle = \langle \rho_B(x) \cdot v_C, w_C \rangle = \langle \rho_B(x) \cdot (v_B)_C, (w_B)_C \rangle = c_{v_B, w_B}(x)$$

and hence $c_{v,w}|_{\mathbf{Alg}_B} = c_{v_B, w_B}$. Consider the square

$$\begin{array}{ccc} V_a(A) \times V_a^\vee(A) & \longrightarrow & \text{Mor}_k(\mathfrak{X}, \mathbb{A}^1)(A) \\ \downarrow V_a(f) \times V_a^\vee(f) & & \downarrow \text{Mor}_k(\mathfrak{X}, \mathbb{A}^1)(f) \\ V_a(B) \times V_a^\vee(B) & \longrightarrow & \text{Mor}_k(\mathfrak{X}, \mathbb{A}^1)(B) \end{array}$$

in which both horizontal arrows are given by formula $(v, w) \mapsto c_{v,w}$. We proved that the square commutes. Since f is an arbitrary morphism of k -algebras, we conclude the assertion. \square

Corollary 2.4. Let \mathfrak{X} be a monoid k -functor and let (V, ρ) be its representation with finitely generated projective underlying k -module V . Then there exists a morphism of k -functors

$$(V \otimes_k V^\vee)_a \xrightarrow{c} \text{Mor}_k(\mathfrak{X}, \mathbb{A}_k^1)$$

given by formula

$$(A \otimes_k V) \otimes_A (A \otimes_k V^\vee) \ni (v, w) \mapsto c_{v,w} \in \text{Mor}_A(\mathfrak{X}_A, \mathbb{A}_A^1)$$

Moreover, c is a morphism of k -functors equipped with $\mathfrak{X} \times \mathfrak{X}^{\text{op}}$ -actions.

Proof. The first part is an immediate consequence of Proposition 2.3. We prove that c is a morphism of k -functors equipped with $\mathfrak{X} \times \mathfrak{X}^{\text{op}}$ -actions. For this we fix a k -algebra k and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^\vee$. Pick a morphism of k -algebras $f : A \rightarrow B$, $(y, z) \in \mathfrak{X}(A) \times \mathfrak{X}(A)^{\text{op}}$ and $x \in \mathfrak{X}_A(B)$. Then we have

$$\begin{aligned} c_{\rho(y) \cdot v, w \cdot \rho(z)}(x) &= \langle \rho_A(x) \cdot (\rho(y) \cdot v)_B, (w \cdot \rho(z))_B \rangle = \\ &= \langle \rho_A(x) \cdot \rho_A((\mathfrak{X}_A(f)(y))) \cdot v_B, w_B \cdot \rho_A(\mathfrak{X}_A(f)(z)) \rangle = w_B(\rho_A(\mathfrak{X}_A(f)(z)) \cdot \rho_A(x) \cdot \rho_A(\mathfrak{X}_A(f)(y))) \cdot v_B = \\ &= w_B(\rho_A(\mathfrak{X}_A(f)(z)) \cdot x \cdot \mathfrak{X}_A(f)(y)) \cdot v_B = \langle \rho_A(\mathfrak{X}_A(f)(z)) \cdot x \cdot \mathfrak{X}_A(f)(y) \cdot v_B, w_B \rangle = \end{aligned}$$

$$= c_{v,w}(\mathfrak{X}_A(f)(z) \cdot x \cdot \mathfrak{X}_A(f)(y))$$

and hence c is a morphism of k -functors equipped with actions of $\mathfrak{X} \times \mathfrak{X}^{\text{op}}$. \square

3. ALGEBRA OF REGULAR FUNCTIONS OF A k -FUNCTOR

Example 3.1. For every k -algebra A we denote by $|A|$ its underlying set. We denote by \mathbb{A}_k^1 a k -functor given by assignment $\mathbb{A}_k^1(A) = |A|$ for every A . We call \mathbb{A}_k^1 the *affine line over k* . Let $k[x]$ be a polynomial k -algebra with variable x . For every k -algebra A map of sets

$$\text{Mor}_k(k[x], A) \ni f \mapsto f(x) \in |A|$$

is a bijection. The family of such maps gives rise to an isomorphism of k -functors

$$\text{Mor}_k(\text{Spec}(-), \text{Spec} k[x]) \cong \text{Mor}_k(k[x], -) \cong \mathbb{A}_k^1$$

and hence \mathbb{A}_k^1 is representable by an affine k -scheme $\text{Spec} k[x]$.

Definition 3.2. Let \mathfrak{X} be a k -functor. Consider $\alpha \in k$ and $f, g \in \text{Mor}_k(\mathfrak{X}, \mathbb{A}_k^1)$. Then for every k -algebra A and $x \in \mathfrak{X}(A)$ formulas

$$(f + g)(x) = f(x) + g(x), (f \cdot g)(x) = f(x) \cdot g(x), (\alpha \cdot f)(x) = \alpha \cdot f(x)$$

define k -algebra operations on the class $\text{Mor}_k(\mathfrak{X}, \mathbb{A}_k^1)$. We call them *pointwise k -algebra operations*. In particular, if $\text{Mor}_k(\mathfrak{X}, \mathbb{A}_k^1)$ is a set, then pointwise k -algebras operations on this set give rise to the k -algebra of regular functions on \mathfrak{X} .

4. k -FUNCTORS OF MONOIDS AND THEIR LINEAR REPRESENTATIONS

In the sequel we assume that the reader is familiar with notions of a monoid, group etc. in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 2-5].

Definition 4.1. A *monoid (group) k -functor* is a monoid (group) object in the category of k -functors.

Next we introduce an important notion of a linear representation of a monoid k -functor. For this we define k -functors associated with modules over k and discuss their properties.

Example 4.2. Let V be a k -module. We define a k -functor V_a . We set

$$V_a(A) = A \otimes_k V, V_a(f) = f \otimes_k 1_V$$

for every k -algebra A and every morphism $f : A \rightarrow B$ of k -algebras. Moreover, V_a admits a structure of a commutative group k -functor. Indeed, $V_a(A)$ is a commutative group with respect to addition induced by its structure of A -module and $V_a(f) : V_a(A) \rightarrow V_a(B)$ preserves the addition.

Suppose now that V, W are k -modules and $\sigma : (V_a)_A \rightarrow (W_a)_A$ is a morphism of A -functors. Then for every A -algebra B we denote by $\sigma^B : B \otimes_k V \rightarrow B \otimes_k W$ the component of σ for B .

Definition 4.3. Let V, W be k -modules and let A be a k -algebra. A morphism $\sigma : (V_a)_A \rightarrow (W_a)_A$ of A -functors is *linear* if for every A -algebra B the component $\sigma^B : B \otimes_k V \rightarrow B \otimes_k W$ is a morphism of B -modules.

Next Fact characterizes linear morphism.

Fact 4.4. Let V, W be k -modules and let A be a k -algebra. Suppose that $\phi : A \otimes_k V \rightarrow A \otimes_k W$ is a morphism of A -modules. Then there exists a unique linear morphism $\sigma : (V_a)_A \rightarrow (W_a)_A$ of A -functors such that $\sigma^A = \phi$.

Proof. Note that if such σ exists, then by requirement $\sigma^A = \phi$ for every morphism $f : A \rightarrow B$ of k -algebras the following diagram

$$\begin{array}{ccc}
A \otimes_k V & \xrightarrow{\phi} & A \otimes_k W \\
f \otimes_k 1_V \downarrow & & \downarrow f \otimes_k 1_W \\
B \otimes_k V & \xrightarrow{\sigma^B} & B \otimes_k W
\end{array}$$

must commute. We make this into a definition of a morphism σ^B of B -modules. It is a matter of linear algebra that this diagram uniquely determines σ^B and also that $\sigma^A = \phi$. It remains to verify that $\sigma = \{\sigma^B\}_{B \in \mathbf{Alg}_A}$ defined in such a way is a morphism of A -functors. For this suppose that $f : A \rightarrow B$ and $g : B \rightarrow C$ are morphisms of k -algebras. Then we have

$$\begin{aligned}
\sigma_C \cdot (g \otimes_k 1_V) \cdot (f \otimes_k 1_V) &= \sigma_C \cdot ((g \cdot f) \otimes_k 1_V) = ((g \cdot f) \otimes_k 1_W) \cdot \phi = \\
&= (g \otimes_k 1_W) \cdot (f \otimes_k 1_V) \cdot \phi = (g \otimes_k 1_W) \cdot \sigma_B \cdot (f \otimes_k 1_V)
\end{aligned}$$

and hence $\sigma_C \cdot (g \otimes_k 1_V) = (g \otimes_k 1_W) \cdot \sigma_B$. Thus σ is a linear morphism of A -functors. \square

We restate Fact 4.4 in the form of the following result.

Corollary 4.5. *Let V, W be k -modules and A be a k -algebra. Consider the map*

$$\mathrm{Hom}_A(A \otimes_k V, A \otimes_k W) \longrightarrow \mathrm{Mor}_A((V_a)_A, (W_a)_A)$$

that sends morphism ϕ to a unique linear morphism $\sigma : (V_a)_A \rightarrow (W_a)_A$ of A -functors such that $\sigma^A = \phi$. Then this map is injective and its image consists of all linear morphisms of A -functors.

Example 4.6. Let V be a k -module. We define a k -functor \mathcal{L}_V . We set

$$\mathcal{L}_V(A) = \mathrm{Hom}_A(A \otimes_k V, A \otimes_k V)$$

for every k -algebra A . Next for every morphism $f : A \rightarrow B$ of k -algebras and a morphism $\phi : A \otimes_k V \rightarrow A \otimes_k V$ of A -modules we define $\mathcal{L}_V(f)(\phi)$ as a unique morphism of B -modules such that the diagram

$$\begin{array}{ccc}
A \otimes_k V & \xrightarrow{\phi} & A \otimes_k V \\
f \otimes_k 1_V \downarrow & & \downarrow f \otimes_k 1_V \\
B \otimes_k V & \xrightarrow{\mathcal{L}_V(\phi)} & B \otimes_k V
\end{array}$$

is commutative. Note also that $\mathcal{L}_V(A)$ is a monoid k -functor with respect to the usual composition of morphism of A -modules and $\mathcal{L}_V(f) : \mathcal{L}_V(A) \rightarrow \mathcal{L}_V(B)$ preserves this composition.

Remark 4.7. Corollary 4.5 implies that there are injective maps that make the square

$$\begin{array}{ccc}
\mathcal{L}_V(A) & \hookrightarrow & \mathrm{Mor}_A((V_a)_A, (V_a)_A) \\
\mathcal{L}_V(f) \downarrow & & \downarrow \sigma \mapsto \sigma_B \\
\mathcal{L}_V(B) & \hookrightarrow & \mathrm{Mor}_B((V_a)_B, (V_a)_B)
\end{array}$$

commutative for every morphism $f : A \rightarrow B$ of k -algebras. Also Corollary 4.5 shows that for every k -algebra A this identifies $\mathcal{L}_V(A)$ with a subset of the class $\mathrm{Mor}_A((V_a)_A, (V_a)_A)$ consisting of all linear morphism of A -functor.

The discussion below is partially an application of the main result in [Mon19a, section 6] (Remark 4.7 shows that \mathcal{L}_V is a subcopresheaf of internal endomorphisms of V_a and hence the machinery developed in the citation above can be applied), but for the reader's convenience we decide to include all essential details even if this requires repetition.

Let \mathfrak{X} be a monoid k -functor and let V be a k -module. Suppose that $\alpha : \mathfrak{X} \times V_a \rightarrow V_a$ is an action of \mathfrak{X} on V_a . Assume that A is a k -algebra and $x \in \mathfrak{X}(A)$. We denote by $i_x : \mathbf{1}_A \rightarrow \mathfrak{X}_A$ the morphism of A -functors corresponding to x by means of Fact ???. Since $\mathbf{1}_A$ is terminal A -functor, a morphism $\alpha_A \cdot (i_x \times 1_{(V_a)_A})$ is isomorphic to a morphism $\alpha_x : (V_a)_A \rightarrow (V_a)_A$ of A -functors. Suppose now that for any k -algebra A and point $x \in \mathfrak{X}(A)$ morphism α_x is linear. Then we define a morphism $\rho : \mathfrak{X} \rightarrow \mathcal{L}_V$ of k -functors by formula $\rho(x) = \alpha_x^A$. We first check that ρ really is a morphism of k -functors. For this fix morphism $f : A \rightarrow B$ of k -algebras and $x \in \mathfrak{X}(A)$. Then $\alpha_{\mathfrak{X}(f)(x)}$ is a morphism of B -functors isomorphic with $\alpha_B \cdot (i_{\mathfrak{X}(f)(x)} \times 1_{(V_a)_B})$ and since

$$\alpha_B \cdot (i_{\mathfrak{X}(f)(x)} \times 1_{(V_a)_B}) = \alpha_B \cdot (i_x \times 1_{(V_a)_A})_B = (\alpha_A \cdot (i_x \times 1_{(V_a)_A}))_B$$

we derive that $\alpha_{\mathfrak{X}(f)(x)} = (\alpha_x)_B$. This implies that

$$\rho(\mathfrak{X}(f)(x)) = \alpha_{\mathfrak{X}(f)(x)}^B = ((\alpha_x)_B)^B = \alpha_x^B = \mathcal{L}_V(f)(\rho(x))$$

and thus ρ is a morphism of k -functors. Now we show that ρ is a morphism of monoids. For this pick k -algebra A and $x, y \in \mathfrak{X}(A)$. Since α is an action, we deduce that $\alpha_{x \cdot y} = \alpha_x \cdot \alpha_y$ and hence also

$$\rho(x \cdot y) = \alpha_{x \cdot y}^A = \alpha_x^A \cdot \alpha_y^A = \rho(x) \cdot \rho(y)$$

Therefore, ρ is a morphism of monoid k -functors.

Theorem 4.8. *Let \mathfrak{X} be a monoid k -functor and let V be a k -module. Consider the following classes.*

- (1) *The class of actions $\alpha : \mathfrak{X} \times V_a \rightarrow V_a$ of \mathfrak{X} such that for any k -algebra A and point $x \in \mathfrak{X}(A)$ morphism α_x is linear.*
- (2) *The class of morphisms $\rho : \mathfrak{X} \rightarrow \mathcal{L}_V$ of monoid k -functors.*

Let α be an element of (1) and $\rho : \mathfrak{X} \rightarrow \mathcal{L}_V$ be the element of (2) such that $\rho(x) = \alpha_x^A$ for any k -algebra A and $x \in \mathfrak{X}(A)$. Then the correspondence $\alpha \mapsto \rho$ is a bijection between these classes.

Proof. We may refer to [Mon19a, Theorem 6.3], but for self-containment of the presentation let us give a direct proof of this important result. \square

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