LEBESGUE SPACES AND THEIR DUALS

1. Introduction

In these notes we are concerned with study of duals of Lebesgue spaces of scalar valued functions. For this purpose we introduce in the first section important classes of measure spaces, which are of indepedent interest. The second section discusses identifies L^q as dual to L^p for $p \in (1, +\infty)$ provided that

$$\frac{1}{p} + \frac{1}{q} = 1$$

Next we study L^{∞} as a dual to L^1 under localizability assumption. We rely on the material developed in [Monygham, 2018a], [Monygham, 2019] and [Monygham, 2018b].

2. Localizable measure spaces

We start with a series of definitions.

Definition 2.1. Let (X, Σ, μ) be a space with measure. We define binary relation $\sqsubseteq_{\mu}$ on domain Σ as follows

$$A \sqsubseteq_{\mu} B \Leftrightarrow \mu(A \setminus B) = 0$$

for all $A, B \in \Sigma$. Clearly Σ together with $\sqsubseteq_{\mu}$ is a preorder.

Definition 2.2. Let (X, Σ, μ) be a space with measure. If $\left(\Sigma, \sqsubseteq_{\mu}\right)$ admits least upper bounds for arbitrary subfamilies of Σ, then μ is *a Dedekind complete measure*.

Proposition 2.3. *Each σ-finite measure is Dedekind complete.*

Proof. Let μ be a measure on (X, Σ) and assume first that it is finite. Fix an arbitrary subfamily $\mathfrak{I}$ of Σ. consider

$$s = \sup_{I \in \mathfrak{I}} \mu(I)$$

Then there exists a sequence $\{I_n\}_{n \in \mathbb{N}}$ of elements in $\mathfrak{I}$ such that $I_n \subseteq I_{n+1}$ for every $n \in \mathbb{N}$ and $\mu(I_n) \to m$ for $n \to +\infty$. Define

$$S = \bigcup_{n \in \mathbb{N}} I_n$$

Then $S \in \Sigma$ and $\mu(S) = s$. Moreover, $\mu(I \setminus S) = 0$ for every $I \in \mathfrak{I}$. It follows that $I \sqsubseteq_{\mu} S$ for every $I \in \mathfrak{I}$. On the other hand if $T \in \Sigma$ is such that $I \sqsubseteq_{\mu} T$ for every $I \in \mathfrak{I}$. Then $\mu(I_n \setminus T) = 0$ for every $n \in \mathbb{N}$. Hence $\mu(S \setminus T) = 0$ and thus $S \sqsubseteq_{\mu} T$. This proves that S is a least upper bound of $\mathfrak{I}$ in Σ with respect to $\sqsubseteq_{\mu}$. Therefore, μ is Dedekind complete.

In order to prove the result for σ-finite measures note that each σ-finite measure is a sum of countably many finite measures and apply the finite case proved above. The details are left for the reader. □

Definition 2.4. Let (X, Σ, μ) be a space with measure and let $\mathcal{F}$ be a family of $\mathbb{C}$-valued functions defined on some subsets of X. For each f in $\mathcal{F}$ we denote by D_f the domain of f. Suppose that the following assertions hold.

(1) $D_f \in \Sigma$ for each $f \in \mathcal{F}$.

(2) Functions in $\mathcal{F}$ are measurable.

(3) If $f_1, f_2 \in \mathcal{F}$, then $f_{1|D_{f_1}\cap D_{f_2}}$ and $f_{2|D_{f_1}\cap D_{f_2}}$ are equal μ-almost everywhere.

Then $\mathcal{F}$ is *a μ-local family*.

The next theorem is an important result concerning Dedekind complete measures.

Theorem 2.5. *Let (X, Σ, μ) be a space with Dedekind complete measure and let $\mathcal{F}$ be a μ-local family. Then there exists a measurable $\mathbb{C}$-valued function F on (X, Σ) such that $F_{|D_f}$ and f are equal μ-almost everywhere for each $f \in \mathcal{F}$.*

For the proof we need the following special case of our result.

Lemma 2.5.1. *Let (X, Σ, μ) be a space with Dedekind complete measure and let $\mathcal{F}$ be a μ-local family. If functions in $\mathcal{F}$ are $\{0, 1\}$-valued, then there exists a measurable and $\{0, 1\}$-valued function F on (X, Σ) such that $F_{|D_f}$ and f are equal μ-almost everywhere for each $f \in \mathcal{F}$.*

Proof of the lemma. We define $A_f = f^{-1}(1)$ for $f \in \mathcal{F}$. Clearly $\{A_f\}_{f\in\mathcal{F}}$ is a family of sets in Σ. Let A be a least upper bound of $\{A_f\}_{f\in\mathcal{F}}$ with respect to $\sqsubseteq_\mu$. We claim for every $f \in \mathcal{F}$ sets $A \cap D_f$ and A_f differ by the set of measure μ equal to zero. In order to prove the claim note that $\left(A \setminus D_f\right) \cup A_f$ is an upper bound of $\{A_f\}_{f\in\mathcal{F}}$ with respect to $\sqsubseteq_\mu$. Hence

$$A \sqsubseteq_\mu \left(A \setminus D_f\right) \cup A_f$$

It follows $A \cap D_f \sqsubseteq_\mu A_f$. On the other hand $A_f \sqsubseteq_\mu A \cap D_f$. Thus $A \cap D_f$ and A_f are equivalent in $\left(\Sigma, \sqsubseteq_\mu\right)$. This proves the claim. Now it follows from that claim that $F = \mathbb{1}_A$ satisfies the assertion. □

Proof of the theorem. It suffices to prove the result under the additional assumption that all functions in $\mathcal{F}$ take values in nonnegative reals. Indeed, the theorem for $\mathbb{C}$-valued μ-local families can be reduced to the case of $\mathbb{R}$-valued families by means of decomposing each function in the family on its real and imaginary parts and the statement for $\mathbb{R}$-valued μ-local families in turn reduces to the result for nonnegative μ-local families.

Let us then assume that all functions in $\mathcal{F}$ take values in nonnegative real numbers. For each $n, k \in \mathbb{N}$ and $f \in \mathcal{F}$ we define

$$A_{k,n,f} = \left\{ x \in X \,\middle|\, \frac{k}{2^n} \leq f(x) \leq \frac{k+1}{2^n} \right\}$$

For fixed $n, k \in \mathbb{N}$ family $\left\{ \mathbb{1}_{A_{k,n,f}\,|D_f} \right\}_{f\in\mathcal{F}}$ is μ-local. By Lemma 2.5.1 it follows that there exist $A_{k,n} \in \Sigma$ such that functions $\mathbb{1}_{A_{k,n}\,|D_f}$ and $\mathbb{1}_{A_{k,n,f}\,|D_f}$ are equal μ-almost everywhere for each $f \in \mathcal{F}$. Fix $n \in \mathbb{N}$ and define a function

$$s_n(x) = \begin{cases} \sum_{k\in\mathbb{N}} \frac{k}{2^n} \cdot \mathbb{1}_{A_{k,n}}(x) & \text{if the series is finite} \\ 0 & \text{otherwise} \end{cases}$$

Then s_n is nonnegative valued and measurable function on (X, Σ). Similarly, for each $f \in \mathcal{F}$ consider a function

$$s_{n,f} = \sum_{k\in\mathbb{N}} \frac{k}{2^n} \cdot \mathbb{1}_{A_{k,n,f}\,|D_f}$$

Note that $s_{n,f}$ is measurable and defined on D_f for every $f \in \mathcal{F}$. Moreover, $s_{n|D_f}$ and $s_{n,f}$ are equal μ-almost everywhere for all $f \in \mathcal{F}$. Next we set

$$F(x) = \begin{cases} \lim_{n\to+\infty} s_n(x) & \text{if the limit exists and is finite} \\ 0 & \text{otherwise} \end{cases}$$

Since $\{s_n\}_{n\in\mathbb{N}}$ are measurable and nonnegative valued functions on (X,Σ), we deduce that F is measurable and nonnegative valued function on (X,Σ). Observe that

$$f = \lim_{n\to+\infty} s_{n,f}$$

for each $f \in \mathcal{F}$. This implies that $F_{|D_f}$ and f are equal μ-almost everywhere for each $f \in \mathcal{F}$. $\square$

The converse of Theorem 2.5 may be proved under some additional and mild assumption. We introduce it now as a separate notion, since it plays important role in taxonomy of measure spaces.

Definition 2.6. Let (X,Σ,μ) be a space with measure. Suppose that for every $B \in \Sigma$ with $\mu(B) > 0$ there exists $A \in \Sigma$, $A \subseteq B$ such that $\mu(A) \in \mathbb{R}_+$. Then μ is *a semifinite measure*.

The following fact relates semifinitness and essential containment.

Fact 2.7. *Let (X,Σ,μ) be a space with semifinite measure and let $A, B \in \Sigma$ be sets. If*

$$A \cap E \sqsubseteq_\mu B \cap E$$

for every $E \in \Sigma$ such that $\mu(E)$ is finite, then $A \sqsubseteq_\mu B$.

Proof. Suppose that $A \not\sqsubseteq_\mu B$. Then $\mu(A \setminus B) > 0$. By semifinitness of μ there exists $E \in \Sigma$ such that $\mu(E) \in \mathbb{R}_+$ and $E \subseteq A \setminus B$. Then

$$E \subseteq (A\setminus B)\cap E = (A\cap E)\setminus(B\cap E)$$

and hence $A \cap E \not\sqsubseteq_\mu B \cap E$. $\square$

Now we prove the aforementioned converse of Theorem 2.5.

Theorem 2.8. *Let (X,Σ,μ) be a space with semifinite measure. Assume that for each μ-local family $\mathcal{F}$ there exists a measurable $\mathbb{C}$-valued function F on (X,Σ) such that $F_{|D_f}$ and f are equal μ-almost everywhere for each $f \in \mathcal{F}$. Then μ is a Dedekind complete measure.*

Proof of the theorem. Suppose that $\mathfrak{I}$ is an arbitrary subfamily in Σ. According to Proposition 2.3 for each set $E \in \Sigma$ with $\mu(E) \in \{0\} \cup \mathbb{R}_+$ there exists a set $S_E \in \Sigma$ such that S_E is a least upper bound of

$$\mathfrak{I}_E = \{I \cap E \,|\, I \in \mathfrak{I}\}$$

with respect to $\sqsubseteq_\mu$. Let $\mathcal{E}$ be a family of all sets in Σ with finite measure μ. Then $\{\mathbb{1}_{S_E|E}\}_{E\in\mathcal{E}}$ is a μ-local family of functions. Hence there exists a measurable $\mathbb{C}$-valued function F on (X,Σ) such that $F_{|E}$ and $\mathbb{1}_{S_E|E}$ are equal μ-almost everywhere for each $E \in \mathcal{E}$. Pick $S = F^{-1}(1)$. Since $S \cap E$ and S_E differ by the set of measure μ equal to zero, we derive that $S \cap E$ is a least upper bound of $\mathfrak{I}_E$ with respect to $\sqsubseteq_\mu$ for every $E \in \mathcal{E}$. Fact 2.7 implies that S is a least upper bound of $\mathfrak{I}$ with respect to $\sqsubseteq_\mu$. $\square$

Definition 2.9. Let (X,Σ,μ) be a space with a semifinite and Dedekind complete measure. Then μ is *localizable*.

3. Dual spaces to L^p for $p \in (1,+\infty)$

Let (X,Σ,μ) be a space with measure and let p be a real in $(1,+\infty)$. Define $q \in (1,+\infty)$ to be the unique number which satisfies

$$\frac{1}{p} + \frac{1}{q} = 1$$

Assume that $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$ with their usual absolute values. We start by stating the following consequence of Hölder inequality.

Proposition 3.1. *Let $g : X \to \mathbb{K}$ be a Σ-measurable function and let f be a function in $L^p(\mu, \mathbb{K})$. Then*

$$\int_X |g \cdot f| \, d\mu \leq \|g\|_q \cdot \|f\|_p$$

In particular, if $g \in L^q(\mu, \mathbb{K})$, then $g \cdot f \in L^1(\mu, \mathbb{K})$.

Proof. We left the details to the reader. □

Next we prove so called extremal equality.

Proposition 3.2. *Let g be a function in $L^q(\mu, \mathbb{K})$. Then*

$$\|g\|_q = \sup \left\{ \left| \int_X g \cdot f \, d\mu \right| \in \mathbb{R}_+ \cup \{0\} \, \middle| \, f \in L^p(\mu, \mathbb{K}) \text{ such that } \|f\|_p = 1 \right\}$$

Proof. By Proposition 3.1 it suffices to prove that

$$\|g\|_q \leq \sup \left\{ \left| \int_X g \cdot f \, d\mu \right| \in \mathbb{R}_+ \cup \{0\} \, \middle| \, f \in L^p(\mu, \mathbb{K}) \text{ such that } \|f\|_p = 1 \right\}$$

under the assumption that $\|g\|_q \neq 0$. Define

$$f(x) = \begin{cases} \|g\|_q^{1-q} \cdot \frac{|g(x)|^q}{g(x)} & \text{if } g(x) \neq 0 \\ 0 & \end{cases}$$

for $x \in X$. Then $f \in L^p(\mu, \mathbb{K})$. To be precise we have

$$\|f\|_p = \left(\int_X \|g\|_q^{(1-q)\cdot p} \cdot |g|^{(q-1)\cdot p} \, d\mu \right)^{\frac{1}{p}} = \left(\int_X \|g\|_q^{-q} \cdot |g|^q \, d\mu \right)^{\frac{1}{p}} = \left(\|g\|_q^{-q} \cdot \int_X \cdot |g|^q \, d\mu \right)^{\frac{1}{p}} = 1$$

Note that

$$\left| \int_X g \cdot f \, d\mu \right| = \int_X \|g\|_q^{(1-q)} \cdot |g|^q \, d\mu = \|g\|_q^{(1-q)} \cdot \int_X |g|^q \, d\mu = \|g\|_q^{(1-q)} \cdot \|g\|_q^q = \|g\|_q$$

and this completes the proof. □

The following theorem is the main result of this section.

Theorem 3.3. *Let $\Lambda : L^p(\mu, \mathbb{K}) \to \mathbb{K}$ be a continuous $\mathbb{K}$-linear map. Then there exists $g \in L^q(\mu, \mathbb{K})$ such that*

$$\Lambda(f) = \int_X g \cdot f \, d\mu$$

for every $f \in L^p(\mu, \mathbb{K})$. Moreover, g is uniquely defined up to a set of measure μ equal to zero.

We start by the following observation.

Lemma 3.3.1. *Let $\Lambda : L^p(\mu, \mathbb{K}) \to \mathbb{K}$ be a continuous $\mathbb{K}$-linear map. For each set $S \in \Sigma$ we define*

$$\Lambda_S(f) = \Lambda \left(\mathbb{1}_S \cdot f \right)$$

for every $f \in L^p(\mu, \mathbb{K})$. Then the following assertions hold.

(1) *$\Lambda_S : L^p(\mu, \mathbb{K}) \to \mathbb{K}$ is a continuous $\mathbb{K}$-linear map.*

(2) *The inequality*

$$\|\Lambda_S\| \leq \|\Lambda_T\| \leq \|\Lambda\|$$

holds for each $S, T \in \Sigma$ such that $S \subseteq T$.

(3) *There exists a σ-finite subset S in Σ such that $\|\Lambda_S\| = \|\Lambda\|$.*

Proof of the lemma. Assertions **(1)** and **(2)** are left for the reader as exercises.

We prove **(3)**. Suppose that $f \in L^p(\mu, \mathbb{K})$ satisfies $\|f\|_p \leq 1$. Then there exists a nondecreasing sequence $\{S_n\}_{n\in\mathbb{N}}$ of sets in Σ such that $\mu(S_n)$ is finite for every $n \in \mathbb{N}$ and $\{\mathbb{1}_{S_n} \cdot f\}_{n\in\mathbb{N}}$ converges to f in $L^p(\mu, \mathbb{R})$. Hence

$$\Lambda(f) = \lim_{n\to+\infty} \Lambda_{S_n}(f)$$

It follows that

$$\|\Lambda\| = \sup\left\{ \|\Lambda_S\| \,\middle|\, S \in \Sigma \text{ such that } \mu(S) \text{ is finite} \right\}$$

Hence there exists a nondecreasing sequence $\{S_n\}_{n\in\mathbb{N}}$ of sets in Σ such that $\mu(S_n)$ is finite for every $n \in \mathbb{N}$ and

$$\|\Lambda\| = \lim_{n\to+\infty} \|\Lambda_{S_n}\|$$

Then the union

$$S = \bigcup_{n\in\mathbb{N}} S_n$$

is a σ-finite set in Σ and satisfies $\|\Lambda\| = \|\Lambda_S\|$. □

We prove the theorem by gradually considering more general cases.

Proof. Assume that μ is finite measure. Then

$$\Sigma \ni A \mapsto \Lambda(\mathbb{1}_A) \in \mathbb{K}$$

is a $\mathbb{K}$-valued measure absolutely continuous with respect to μ. According to Radon-Nikodym there exists $g \in L^1(\mu, \mathbb{K})$ such that

$$\Lambda\left(\mathbb{1}_A\right) = \int_X g \cdot \mathbb{1}_A \, d\mu$$

for every A in Σ. It follows that

$$\Lambda(f) = \int_X g \cdot f \, d\mu$$

for every $f \in L^\infty(\mu, \mathbb{K})$. For each $n \in \mathbb{N}_+$ define $A_n = \left\{x \in X \,\middle|\, |g(x)| \leq n\right\}$ and consider a measurable and bounded function $f_n : X \to \mathbb{K}$ given by formula

$$f_n(x) = \begin{cases} \mathbb{1}_{A_n}(x) \cdot \frac{|g(x)|^q}{g(x)} & \text{if } g(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\int_X \mathbb{1}_{A_n} \cdot |g|^q \, d\mu = \int_X g \cdot f_n \, d\mu = \Lambda\left(f_n\right) \leq \|\Lambda\| \cdot \|f_n\|_p =$$

$$= \|\Lambda\| \cdot \|\mathbb{1}_{A_n} \cdot |g|^{q-1}\|_p = \|\Lambda\| \cdot \left(\int_X \mathbb{1}_{A_n} \cdot \left(|g|^{q-1}\right)^p d\mu\right)^{\frac{1}{p}} = \|\Lambda\| \cdot \left(\int_X \mathbb{1}_{A_n} \cdot |g|^q \, d\mu\right)^{\frac{1}{p}}$$

and thus

$$\left(\int_X \mathbb{1}_{A_n} \cdot |g|^q \, d\mu\right)^{\frac{1}{q}} \leq \|\Lambda\|$$

By monotone convergence we have

$$\|g\|_q = \lim_{n\to+\infty} \left(\int_X \mathbb{1}_{A_n} \cdot |g|^q \, d\mu\right)^{\frac{1}{q}} \leq \|\Lambda\|$$

Hence $g \in L^q(\mu, \mathbb{K})$. It follows that

$$L^p(\mu, \mathbb{K}) \ni f \mapsto \int_X g \cdot f \, d\mu \in \mathbb{K}$$

is continuous $\mathbb{K}$-linear map, which coincides with Λ on the space of μ-simple functions. Since μ-simple functions are dense in $L^p(\mu,\mathbb{K})$, we derive that

$$\Lambda(f) = \int_X g \cdot f \, d\mu$$

for every $f \in L^p(\mu,\mathbb{K})$.

Next assume that μ is σ-finite measure. Since μ is σ-finite, there exist a nondecreasing sequence $\{X_n\}_{n\in\mathbb{N}}$ of sets in Σ such that their union is X and $\mu(X_n)$ is finite for every $n \in \mathbb{N}$. According to the case considered above and Lemma 3.3.1 for each $n \in \mathbb{N}$ there exists $g_n \in L^q(\mu,\mathbb{K})$ such that

$$\Lambda_{X_n}(f) = \int_X g_n \cdot f \, d\mu$$

for every $f \in L^p(\mu,\mathbb{K})$. We may also assume that $g_{n|X\setminus X_n} = 0$ and $g_{n+1|X_n} = g_{n|X_n}$ for every $n \in \mathbb{N}$. Let g be a pointwise limit of a sequence $\{g_n\}_{n\in\mathbb{N}}$. Then $g : X \to \mathbb{K}$ is a measurable with respect to Σ. Moreover, we have $g_n = \mathbb{1}_{X_n} \cdot g$ for each $n \in \mathbb{N}$. Proposition 3.2 and monotone convergence imply that

$$\|g\|_q = \lim_{n\to+\infty} \|g_n\|_q = \lim_{n\to+\infty} \|\Lambda_{X_n}\| \leq \|\Lambda\|$$

This implies that $g \in L^q(\mu,\mathbb{K})$. Fix $f \in L^p(\mu,\mathbb{K})$. Then sequence $\{\mathbb{1}_{X_n} \cdot f\}_{n\in\mathbb{N}}$ converges to f in $L^p(\mu,\mathbb{K})$ and hence

$$\Lambda(f) = \lim_{n\to+\infty} \Lambda(\mathbb{1}_{X_n} \cdot f) = \lim_{n\to+\infty} \Lambda_{X_n}(f)$$

On the other hand by dominated convergence theorem

$$\int_X g \cdot f \, d\mu = \lim_{n\to+\infty} \int_X g_n \cdot f \, d\mu = \lim_{n\to+\infty} \Lambda_{X_n}(f)$$

This completes the proof for σ-finite case.

According to Lemma 3.3.1 there exists a σ-finite set S in Σ such that $\|\Lambda_S\| = \|\Lambda\|$. According to previous case there exists $g \in L^q(\mu,\mathbb{K})$ such that

$$\Lambda_S(f) = \int_X g \cdot f \, d\mu$$

for every $f \in L^p(\mu,\mathbb{K})$. We may also assume that $g_{|X\setminus S} = 0$. Suppose now that T is a σ-finite set in Σ such that $S \subseteq T$. Then there exists $g_T \in L^q(\mu,\mathbb{K})$ such that

$$\Lambda_T(f) = \int_X g_T \cdot f \, d\mu$$

for every $f \in L^p(\mu,\mathbb{K})$. We may assume that $\mathbb{1}_S \cdot g_T = g$. Proposition 3.2 implies that

$$\|\Lambda\| = \|\Lambda_S\| = \|g\|_q \leq \|g_T\|_q \leq \|\Lambda_T\| \leq \|\Lambda\|$$

Thus $\|g\|_q = \|g_T\|_q$ and this proves that $g_T = g$ up to set of measure μ equal to zero. Fix now $f \in L^p(\mu,\mathbb{K})$ and consider

$$T = \{x \in X \,|\, f(x) \neq 0\} \cup S$$

Then T is a σ-finite set in Σ and $S \subseteq T$. Hence

$$\Lambda(f) = \Lambda_T(f) = \int_X g_T \cdot f \, d\mu = \int_X g \cdot f \, d\mu$$

Since $f \in L^p(\mu,\mathbb{K})$ is arbitrary, the proof is completed. $\square$

4. Dual to L^1

Let (X, Σ, μ) be a space with measure. Assume that $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$ with their usual absolute values. We begin by proving the version of Hölder inequality for L^∞-norm.

Proposition 4.1. *Let $g : X \to \mathbb{K}$ be a Σ-measurable function and let f be a function in $L^1(\mu, \mathbb{K})$. Then*

$$\int_X |g \cdot f| \, d\mu \leq \|g\|_\infty \cdot \|f\|_1$$

In particular, if $g \in L^\infty(\mu, \mathbb{K})$, then $g \cdot f \in L^1(\mu, \mathbb{K})$.

Proof. Note that the set

$$\{x \in X \,|\, \|g\|_\infty < |g(x)|\}$$

is in Σ and is of measure μ zero. Thus

$$\int_X |g \cdot f| \, d\mu \leq \int_X |g| \cdot |f| \, d\mu \leq \|g\|_\infty \cdot \|f\|_1$$

This completes the proof. □

Next we prove the version extremal equality.

Proposition 4.2. *Let g be a function in $L^\infty(\mu, \mathbb{K})$. Then*

$$\sup \left\{ \left| \int_X g \cdot f \, d\mu \right| \in \mathbb{R}_+ \cup \{0\} \,\middle|\, f \in L^1(\mu, \mathbb{K}) \text{ such that } \|f\|_1 = 1 \right\} \leq \|g\|_\infty$$

If μ is semifinite, then

$$\sup \left\{ \left| \int_X g \cdot f \, d\mu \right| \in \mathbb{R}_+ \cup \{0\} \,\middle|\, f \in L^1(\mu, \mathbb{K}) \text{ such that } \|f\|_1 = 1 \right\} =$$

$$= \sup \left\{ \left| \int_X g \cdot f \, d\mu \right| \in \mathbb{R}_+ \cup \{0\} \,\middle|\, f \in L^1(\mu, \mathbb{K}) \cap L^\infty(\mu, \mathbb{K}) \text{ such that } \|f\|_1 = 1 \right\} = \|g\|_\infty$$

Proof. Proposition 4.1 implies that

$$\left| \int_X g \cdot f \, d\mu \right| \leq \int_X |g| \cdot |f| \, d\mu \leq \|g\|_\infty \cdot \|f\|_1 = \|g\|_\infty$$

for every $f \in L^1(\mu, \mathbb{K})$. Hence

$$\sup \left\{ \left| \int_X g \cdot f \, d\mu \right| \in \mathbb{R}_+ \cup \{0\} \,\middle|\, f \in L^1(\mu, \mathbb{K}) \text{ such that } \|f\|_1 = 1 \right\} \leq \|g\|_\infty$$

This proves the first part of the assertion.

Assume now that μ is semifinite. For each $r \in \mathbb{R}_+$ we denote

$$A_r = \{x \in X \,|\, |g(x)| \geq r\}$$

Fix now $r \in \mathbb{R}_+$ such that $\mu(A_r) > 0$. Since μ is semifinite, there exists $B_r \in \Sigma$ such that B_r is a subset of A_r and $m_r = \mu(B_r)$ is finite. We define a function

$$f_r(x) = \begin{cases} \frac{1}{m_r} \cdot \frac{|g(x)|}{g(x)} & \text{if } x \in B_r \\ 0 & \text{otherwise} \end{cases}$$

Then $f_r \in L^1(\mu, \mathbb{K}) \cap L^\infty(\mu, \mathbb{K})$ and $\|f_r\|_1 = 1$. We have

$$\left| \int_X g \cdot f_r \, d\mu \right| = \int_X g \cdot f_r \, d\mu = \int_{B_r} \frac{1}{m_r} \cdot |g| \, d\mu \geq r$$

Thus

$$\|g\|_\infty \leq \sup\left\{\left|\int_X g\cdot f\,d\mu\right| \in \mathbb{R}_+\cup\{0\}\,\middle|\, f\in L^1(\mu,\mathbb{K})\cap L^\infty(\mu,\mathbb{K}) \text{ such that } \|f\|_1 = 1\right\}$$

This completes the proof. □

According to Proposition 4.2 for each $g \in L^\infty(\mu,\mathbb{K})$ the map

$$L^1(\mu,\mathbb{K}) \ni f \mapsto \int_X g\cdot f\,d\mu \in \mathbb{K}$$

is continuous and $\mathbb{K}$-linear. We denote it by $\Phi(g)$. Then $\Phi : L^\infty(\mu,\mathbb{K}) \to (L^\infty(\mu,\mathbb{K}))^*$ is well defined $\mathbb{K}$-linear map of topological vector space over $\mathbb{K}$. The remaining part of this section is devoted to investigation of properties of Φ.

Theorem 4.3. *Φ is an isometry if and only if μ is semifinite.*

Proof. Proposition 4.2 implies that if μ is semifinite, then $\|\Phi(g)\| = \|g\|_\infty$ for every $g\in L^\infty(\mu,\mathbb{K})$. Hence if μ is semifinite, then Φ is an isometry.

Now suppose that Φ is an isometry. Pick a set $B\in\Sigma$ such that $\mu(B) = +\infty$. Then $\|\Phi(\mathbb{1}_B)\| = \|\mathbb{1}_B\|_\infty = 1$. It follows that

$$1 = \sup\left\{\left|\int_X \mathbb{1}_B\cdot f\,d\mu\right| \in \mathbb{R}_+\cup\{0\}\,\middle|\, f\in L^1(\mu,\mathbb{K}) \text{ such that } \|f\|_1 = 1\right\}$$

In particular, there exist $f\in L^1(\mu,\mathbb{K})$ with $\|f\|_1 = 1$ such that

$$0 < \left|\int_X \mathbb{1}_B\cdot f\,d\mu\right| = \left|\int_B f\,d\mu\right| \leq \int_B |f|\,d\mu \leq \|f\|_1$$

For each $n\in\mathbb{N}$ we set

$$A_n = \left\{x\in B\,\middle|\, |f(x)| > \frac{1}{n+1}\right\}$$

Then $A_n\in\Sigma$ for each $n\in\mathbb{N}$ and there exists $n_0\in\mathbb{N}$ such that $\mu(A_{n_0}) > 0$. Since $f\in L^1(\mu,\mathbb{K})$, we derive that $\mu(A_{n_0})$ is finite. According to definition $A_{n_0}\subseteq B$. Therefore, μ is semifinite. □

Theorem 4.4. *Φ is surjective isometry if and only if μ is localizable.*

Proof. Let $\Lambda : L^1(\mu,\mathbb{K}) \to \mathbb{K}$ be a continuous $\mathbb{K}$-linear map.

Assume first that μ is finite measure. Then

$$\Sigma \ni A \mapsto \Lambda(\mathbb{1}_A) \in \mathbb{K}$$

is a $\mathbb{K}$-valued measure absolutely continuous with respect to μ. According to Radon-Nikodym there exists $g\in L^1(\mu,\mathbb{K})$ such that

$$\Lambda\left(\mathbb{1}_A\right) = \int_X g\cdot\mathbb{1}_A\,d\mu$$

for every A in Σ. It follows that

$$\Lambda(f) = \int_X g\cdot f\,d\mu$$

for every $f\in L^\infty(\mu,\mathbb{K})$. Proposition 4.2 shows that

$$\|g\|_\infty = \sup\left\{\left|\int_X g\cdot f\,d\mu\right| \in \mathbb{R}_+\cup\{0\}\,\middle|\, f\in L^1(\mu,\mathbb{K})\cap L^\infty(\mu,\mathbb{K}) \text{ such that } \|f\|_1 = 1\right\} =$$

$$= \sup\left\{\left|\Lambda(f)\right| \in \mathbb{R}_+\cup\{0\}\,\middle|\, f\in L^1(\mu,\mathbb{K})\cap L^\infty(\mu,\mathbb{K}) \text{ such that } \|f\|_1 = 1\right\} \leq \|\Lambda\|$$

and hence $g \in L^{\infty}(\mu, \mathbb{K})$. Since $\Phi(g)$ and Λ coincide on μ-simple functions, we derive that $\Phi(g) = \Lambda$.

Now assume that μ is arbitrary localizable measure. Let $\mathcal{E}$ be a family of all subsets in Σ with finite measure μ. For each $E \in \mathcal{E}$ we consider $\Lambda_E : L^1(\mu, \mathbb{K}) \rightarrow \mathbb{K}$ given by formula $\Lambda_E(f) = \Lambda(\mathbb{1}_E \cdot f)$ for every $f \in L^1(\mu, \mathbb{K})$. Then Λ_E is a continuous $\mathbb{K}$-linear map and $\|\Lambda_E\| \leq \|\Lambda\|$ for each $E \in \mathcal{E}$. By the case proved above for each $E \in \mathcal{E}$ there exists $g_E \in L^{\infty}(\mu, \mathbb{K})$ such that $\Lambda_E = \Phi(g_E)$, $g_{E|X\setminus E} = 0$ and $\|g_E\|_{\infty} = \|\Lambda_E\|$. Theorem 4.3 and semifinitness of μ imply that if $E_1, E_2 \in \mathcal{E}$, then $g_{E_1|E_1 \cap E_2}$ and $g_{E_1|E_1 \cap E_2}$ are equal μ-almost everywhere. Since μ is Dedekind complete, Theorem 2.5 implies that there exists a measurable function $g : X \rightarrow \mathbb{K}$ such that $g_{|E}$ and $g_{E|E}$ are equal μ-almost everywhere for each $E \in \mathcal{E}$. Fix $r \in \mathbb{R}_{+}$ such that $r > \|\Lambda\|$ and assume that the set

$$A_r = \{x \in X \,|\, |g(x)| \geq r\}$$

is of positive measure μ. By semifinitness of μ there exists a set $B_r \in \Sigma$ such that $B_r \subseteq A_r$ and $\mu(B_r) \in \mathbb{R}_{+}$. Note that $B_r \in \mathcal{E}$. Since $g_{B_r|B_r}$ and $g_{|B_r}$ coincide μ-almost everywhere, we derive that the set

$$\{x \in X \,|\, |g_{B_r}(x)| \geq r\}$$

is of positive measure μ. On the other hand Theorem 4.3 shows that $\|g_{B_r}\|_{\infty} = \|\Lambda_{B_r}\| \leq \|\Lambda\|$. Since $r > \|\Lambda\|$, we derive contradiction. Hence $\mu(A_r) = 0$ for every $r > \|\Lambda\|$. This shows that $\|g\|_{\infty} \leq \|\Lambda\|$ and hence $g \in L^{\infty}(\mu, \mathbb{K})$. Pick now $f \in L^1(\mu, \mathbb{K})$. There exist a nondecreasing sequence $\{E_n\}_{n \in \mathbb{N}}$ of disjoint sets in $\mathcal{E}$ such that

$$\{x \in X \,|\, f(x) \neq 0\} = \bigcup_{n \in \mathbb{N}} E_n$$

Then the sequence $\{\mathbb{1}_{E_n} \cdot f\}_{n \in \mathbb{N}}$ converges to f in $L^1(\mu, \mathbb{K})$ and hence we have

$$\Lambda(f) = \lim_{n \to +\infty} \Lambda\left(\mathbb{1}_{E_n} \cdot f\right) = \lim_{n \to +\infty} \int_X g_{E_n} \cdot f \, d\mu = \lim_{n \to +\infty} \int_X g \cdot \mathbb{1}_{E_n} \cdot f \, d\mu$$

On the other hand by dominated convergence

$$\lim_{n \to +\infty} \int_X g \cdot \mathbb{1}_{E_n} \cdot f \, d\mu = \int_X g \cdot f \, d\mu$$

Thus

$$\Lambda(f) = \int_X g \cdot f \, d\mu$$

Since f is arbitrary element of $L^1(\mu, \mathbb{K})$, we derive that Λ coincides with $\phi(g)$. This together with Theorem 4.3 proves that if μ is localizable, then Φ is surjective isometry.

Now suppose that Φ is a surjective isometry. Pick a family of subsets $\mathfrak{I}$ of Σ. Let $\mathcal{S}$ be the family of all σ-finite subsets of X with respect to μ. Suppose that $S \in \mathcal{S}$. According to Proposition 2.3 the family

$$\mathfrak{I}_S = \{I \cap S \,|\, I \in \mathfrak{I}\}$$

admits a least upper bound with respect to $\sqsubseteq_{\mu}$. Denote this upper bound by $S_{\mathfrak{I}}$. If $S, T \in \mathcal{S}$ and $S \cap T$, then $\mathbb{1}_{S_{\mathfrak{I}}}$ and $\mathbb{1}_{T_{\mathfrak{I}}|S}$ are equal μ-everywhere. Now for each $f \in L^1(\mu, \mathbb{K})$ we define

$$S(f) = \{x \in X \,|\, f(x) \neq 0\}$$

Clearly $S(f) \in \mathcal{S}$ for every $f \in L^1(\mu, \mathbb{K})$. Now we define a map $\Lambda : L^1(\mu, \mathbb{K}) \rightarrow \mathbb{K}$ by formula

$$\Lambda(f) = \int_X \mathbb{1}_{S(f)_{\mathfrak{I}}} \cdot f \, d\mu$$

for every $f \in L^1(\mu, \mathbb{K})$. Fix now $f \in L^1(\mu, \mathbb{K})$ and assume that $T \in \mathcal{S}$ and $S(f) \subseteq T$. Since $\mathbb{1}_{S(f)_{\mathfrak{I}}}$ and $\mathbb{1}_{T_{\mathfrak{I}}|S(f)}$ are equal μ-everywhere, we derive that

$$\Lambda(f) = \int_X \mathbb{1}_{T_{\mathfrak{I}}} \cdot f \, d\mu$$

Fix now $\alpha_1, \alpha_2 \in \mathbb{K}$ and $f_1, f_2 \in L^1(\mu, \mathbb{K})$. Suppose that $T \in \mathcal{S}$ contains $S(f_1) \cup S(f_2)$. Then

$$\Lambda(\alpha_1 \cdot f_1 + \alpha_2 \cdot f_2) = \int_X \mathbb{1}_{T_{\mathfrak{J}}} \cdot (\alpha_1 \cdot f_1 + \alpha_2 \cdot f_2)\, d\mu =$$
$$= \alpha_1 \cdot \int_X \mathbb{1}_{T_{\mathfrak{J}}} \cdot f_1\, d\mu + \alpha_2 \cdot \int_X \mathbb{1}_{T_{\mathfrak{J}}} \cdot f_2\, d\mu = \alpha_1 \cdot \Lambda(f_1) + \alpha_2 \cdot \Lambda(f_2)$$

This proves that Λ is $\mathbb{K}$-linear. Now assume that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of elements of $L^1(\mu, \mathbb{K})$ which converge to $f \in L^1(\mu, \mathbb{K})$. Pick a set $T \in \mathcal{S}$ which contains all sets $\{S(f_n)\}_{n\in\mathbb{N}}$ and set $S(f)$. Then

$$\lim_{n\to+\infty} \Lambda(f_n) = \lim_{n\to+\infty} \int_X \mathbb{1}_{T_{\mathfrak{J}}} \cdot f_n\, d\mu = \int_X \mathbb{1}_{T_{\mathfrak{J}}} \cdot f\, d\mu = \Lambda(f)$$

Hence Λ is continuous. Since Φ is surjective, there exists $g \in L^\infty(\mu, \mathbb{K})$ such that $\Phi(g) = \Lambda$. Note that for every set $E \in \Sigma$ such that $\mu(E)$ is finite and for every $f \in L^1(\mu, \mathbb{K})$ we have

$$\int_X \mathbb{1}_{E_{\mathfrak{J}}} \cdot f\, d\mu = \Lambda(\mathbb{1}_E \cdot f) = \int_X g \cdot \mathbb{1}_E \cdot f\, d\mu$$

It follows that $\mathbb{1}_{E_{\mathfrak{J}}}$ and g are equal μ-everywhere for each $E \in \Sigma$ such that $\mu(E)$ is finite. Hence $g^{-1}(1) \cap E$ is a least upper bound of $\mathfrak{J}_E$ with respect to $\sqsubseteq_\mu$ for every $E \in \Sigma$ with $\mu(E)$ finite. Since Φ is isometry, Theorem 4.3 shows that μ is semifinite. Now Fact 2.7 implies that $g^{-1}(1)$ is a least upper bound of $\mathfrak{J}$ with respect to $\sqsubseteq_\mu$. Since $\mathfrak{J}$ is arbitrary, we derive that μ is Dedekind complete. Hence μ is localizable. □

5. Dual to L^∞

References

[Monygham, 2018a] Monygham (2018a). Introduction to measure theory. *github repository: "Monygham/Pedo-mellon-a-minno"*.

[Monygham, 2018b] Monygham (2018b). Radon-nikodym theorem, hahn-jordan decomposition and lebesgue decomposition. *github repository: "Monygham/Pedo-mellon-a-minno"*.

[Monygham, 2019] Monygham (2019). Integration. *github repository: "Monygham/Pedo-mellon-a-minno"*.