GEOMETRIC INVARIANT THEORY

1. INTRODUCTION

These notes present core results in geometric invariant theory. We mostly follow monography [Mumford et al., 1994]. We extensively use the language of schemes.

Throughout these notes we fix a field k and a group scheme $\mathbf{G}$ over k with the identity $e : \operatorname{Spec} k \to \mathbf{G}$ and the multiplication $\mu : \mathbf{G} \times_k \mathbf{G} \to \mathbf{G}$. We also fix a k-scheme X equipped with an action of $\mathbf{G}$ determined by morphism $a : \mathbf{G} \times_k X \to X$.

2. BASIC PROPERTIES OF QUOTIENTS

The following result gives scheme-theoretic criterion for topological quotient in the case of group scheme actions.

Proposition 2.1. *Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Assume that q is submersive and the morphism $\mathbf{G} \times_k X \to X \times_Y X$ induced by a and pr_X is surjective. Then the diagram*

$$\mathbf{G} \times_k X \underset{\operatorname{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{\ q\ } Y$$

is a cokernel in the category topological spaces.

Proof. Let π_1 and π_2 be distinct projections $X \times_Y X \to X$. Pick points x_1 and x_2 in X such that $q(x_1) = q(x_2)$. Then there exists a field extension K over k such that $k(x_1) \subseteq K$ and $k(x_2) \subseteq K$. These give rise to K-points $\overline{x_1}$ and $\overline{x_2}$ of X such that their images under q is the same K-point of Y. Since we have an identification

$$\left(X \times_Y X \right)(K) = X(K) \times_{Y(K)} X(K)$$

induced by π_1 and π_2, we derive that there exists a K-point $\overline{z}$ of $X \times_Y X$ such that $\pi_1(\overline{z}) = \overline{x_1}$ and $\pi_2(\overline{z}) = \overline{x_2}$. Let z be the point of $X \times_Y X$ corresponding to $\overline{z}$. Then $\pi_1(z) = x_1$ and $\pi_2(z) = x_2$. By assumption a and pr_X induce surjection $\mathbf{G} \times_k X \twoheadrightarrow X \times_Y X$. Thus there exists a point u of $\mathbf{G} \times_k X$ such that $a(u) = x_1$ and $\operatorname{pr}_X(u) = x_2$. Thus x_1 and x_2 are identified by an equivalence relation on the underlying set of X which is determined by the pair (a, pr_X). Therefore, fibers of q are equivalence classes with respect to this relation. Since q is submersive, this implies that the diagram

$$\mathbf{G} \times_k X \underset{\operatorname{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{\ q\ } Y$$

is a cokernel in the category topological spaces. $\qquad\square$

Now we prove a series results concerning fpqc descent. For this we fix a k-scheme Y with the trivial action of $\mathbf{G}$ and a $\mathbf{G}$-equivariant morphism $q : X \to Y$. Let $g : Y' \to Y$ be a morphism of k-schemes and consider the cartesian square

$$
\begin{array}{ccc}
X' & \xrightarrow{\;g'\;} & X \\
{\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\
Y' & \xrightarrow{\;g\;} & Y
\end{array}
$$

of k-schemes. Note that X' admits a unique action a' of $\mathbf{G}$ such that the square above consists of $\mathbf{G}$-equivariant morphism (we consider g as a $\mathbf{G}$-equivariant morphism between trivial $\mathbf{G}$-schemes).

Fact 2.2. *Suppose that g is faithfully flat and quasi-compact. Assume that q' is (universally) submersive. Then q is (universally) submersive.*

Proof. It suffices to prove that submersive morphisms have descent property. This follows from the fact that g (as faithfully flat and quasi-compact morphism) and q' are submersive. Details are left for the reader. $\qquad\square$

Fact 2.3. *Suppose that g is faithfully flat and quasi-compact. Then the canonical morphism $X' \times_{Y'} X' \to X \times_Y X$ is faithfully flat and quasi-compact and there is the cartesian square*

$$
\begin{array}{ccc}
\mathbf{G} \times_k X' & \longrightarrow & \mathbf{G} \times_k X \\
\downarrow & & \downarrow \\
X' \times_{Y'} X' & \longrightarrow & X \times_Y X
\end{array}
$$

in which the left vertical arrow is induced by $\langle a', \mathrm{pr}_{X'} \rangle : \mathbf{G} \times_k X' \to X' \times_k X'$, the right vertical arrow is induced by $\langle a, \mathrm{pr}_X \rangle : \mathbf{G} \times_k X \to X \times_k X$ and the bottom horizontal morphism is the canonical morphism.

Proof. Note that squares

$$
\begin{array}{ccc}
X' \times_{Y'} X' & \longrightarrow & X' \times_Y X' \\
\downarrow & & \downarrow \\
Y' & \xrightarrow{\;g\;} & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
X' \times_Y X' & \longrightarrow & X \times_Y X \\
\downarrow & & \downarrow \\
X' \times_k X' & \xrightarrow{\;g' \times_k g'\;} & X \times_k X
\end{array}
$$

are cartesian. Since both g and $g' \times_k g'$ are faithfully flat and quasi-compact, we derive that both morphisms $X' \times_{Y'} X' \to X' \times_Y X'$ and $X' \times_Y X' \to X \times_Y X$ are faithfully flat and quasi-compact. Then their composition i.e. the canonical morphism $X' \times_{Y'} X' \to X \times_Y X$ is faithfully flat and quasi-compact. $\qquad\square$

Finally we need the following notion

Definition 2.4. Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Consider a pair

$$
q_* \mathcal{O}_X \overset{q_* a^{\#}}{\underset{q_* \mathrm{pr}_X^{\#}}{\rightrightarrows}} q_* \left(\mathrm{pr}_X \right)_* \mathcal{O}_{\mathbf{G} \times_k X} = q_* a_* \mathcal{O}_{\mathbf{G} \times_k X}
$$

of morphisms of sheaves of rings on Y. Suppose that $q^{\#} : \mathcal{O}_Y \to q_* \mathcal{O}_X$ is a kernel of this pair. Then $\mathcal{O}_Y$ *is the sheaf of $\mathbf{G}$-invariants for q.*

Proposition 2.5. *Suppose that g is faitfully flat and quasi-compact. Assume that q' is quasi-compact, semiseparated and $\mathcal{O}_{Y'}$ is the sheaf of $\mathbf{G}$-invariants for q'. Then $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q.*

Proof. We denote by a' the action of $\mathbf{G}$ on X'. First note that q is semiseparated and quasi-compact morphism as these classes of morphisms admit descent along quasi-compact and faithfully flat morphisms. Since q is quasi-compact, semiseparated and g is flat, we derive that for every quasi-coherent sheaf $\mathcal{F}$ on X the canonical morphism $q'_* g'^* \mathcal{F} \to g^* q_* \mathcal{F}$ is an isomorphism. Thus the diagram

$$\mathcal{O}_{Y'} \xrightarrow{q^\#} q'_* \mathcal{O}_{X'} \underset{q'_* \mathrm{pr}_{X'}^\#}{\overset{q'_* a'^\#}{\rightrightarrows}} q'_* \left(\mathrm{pr}_{X'}\right)_* \mathcal{O}_{\mathbf{G} \times_k X'} = q'_* a'_* \mathcal{O}_{\mathbf{G} \times_k X'}$$

is isomorphic to the diagram

$$g^* \mathcal{O}_Y \xrightarrow{g^* q^\#} g^* \left(q_* \mathcal{O}_X\right) \underset{g^* q_* \mathrm{pr}_X^\#}{\overset{g^* q_* a^\#}{\rightrightarrows}} g^* \left(q_* \left(\mathrm{pr}_X\right)_* \mathcal{O}_{\mathbf{G} \times_k X}\right) = g^* \left(q_* a_* \mathcal{O}_{\mathbf{G} \times_k X}\right)$$

Since $\mathcal{O}_{Y'}$ is the sheaf of $\mathbf{G}$-invariants for q', the first diagram is a kernel diagram. Hence the second is a kernel diagram. According to the fact that g is faithfully flat we deduce that the diagram

$$\mathcal{O}_Y \xrightarrow{q^\#} q_* \mathcal{O}_X \underset{q_* \mathrm{pr}_X^\#}{\overset{q_* a^\#}{\rightrightarrows}} q_* \left(\mathrm{pr}_X\right)_* \mathcal{O}_{\mathbf{G} \times_k X} = q_* a_* \mathcal{O}_{\mathbf{G} \times_k X}$$

is also a kernel diagram. Thus $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q. $\qquad\square$

3. CATEGORICAL AND GEOMETRIC QUOTIENTS

Definition 3.1. Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Suppose that the following conditions hold.

(1) q is submersive.

(2) The morphism $\mathbf{G} \times_k X \to X \times_Y X$ induced by $\langle a, \mathrm{pr}_x \rangle : \mathbf{G} \times_k X \to X \times_k X$ is surjective.

(3) $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariant for q.

Then q is *a geometric quotient of X*.

Corollary 3.2. *Let q be a geometric quotient of X. Then the diagram*

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel in the category of ringed spaces.

Proof. Due to the fact that $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q it suffices to prove that

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

is the cokernel in the category of topological spaces. This follows from Proposition 2.1. $\qquad\square$

Definition 3.3. Let $q : X \to Y$ be a morphism of k-schemes such that the diagram

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \overset{q}{\longrightarrow} Y$$

is a cokernel in the category of k-schemes. Then $q : X \to Y$ is *a categorical quotient of X.*

Fact 3.4. *Every geometric quotient is categorical.*

Proof. Categorical quotient is a cokernel in the category of k-schemes. On the other hand geometric quotient is a cokernel in the category of locally ringed spaces and hence it also satisfies cokernel property in its full subcategory of k-schemes. Thus every geometric quotient is categorical. $\qquad\square$

Let $q : X \to Y$ be a morphism of k-schemes such that $q \cdot \mathrm{pr}_X = q \cdot a$. For a morphism $g : Y' \to Y$ of k-schemes consider the cartesian square

$$
\begin{array}{ccc}
X' & \overset{g'}{\longrightarrow} & X \\
{\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\
Y' & \underset{g}{\longrightarrow} & Y
\end{array}
$$

Then there exists a unique action $a' : \mathbf{G} \times_k X' \to X'$ of $\mathbf{G}$ on X' such that the square above consists of $\mathbf{G}$-equivariant morphism (we consider Y, Y' as $\mathbf{G}$-schemes equipped with trivial $\mathbf{G}$-actions). Keeping this in mind we have the following.

Definition 3.5. A morphism $q : X \to Y$ is *a uniform categorical (geometric) quotient of X* if for every flat morphism $g : Y' \to Y$ of k-schemes a base change $q' : X' \to Y'$ of q along g is a categorical (geometric) quotient of X'.

Fact 3.6. *Let Y be a k-scheme with the trivial action of $\mathbf{G}$ and let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism. Then the following assertions are equivalent.*

  (i) *There exists an open cover $\mathcal{V}$ of Y such that for every V in $\mathcal{V}$ morphism $q^{-1}(V) \to V$ is a uniform categorical (geometric) quotient.*

  (ii) *q is a uniform categorical (geometric) quotient.*

  (iii) *For every affine k-scheme Y' and a flat morphism $g : Y' \to Y$ of k-schemes a base change $q' : X' \to Y'$ of q along g is a categorical (geometric) quotient.*

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 3.7. A morphism $q : X \to Y$ is *a universal categorical (geometric) quotient of X* if for every morphism $g : Y' \to Y$ of k-schemes a base change $q' : X' \to Y'$ of q along g is a categorical (geometric) quotient of X'.

Corollary 3.8. *Let $g : Y' \to Y$ be a faithfully flat and quasi-compact morphism. Suppose that q' is a geometric quotient, then q is a geometric quotient.*

Proof. This follows from Facts 2.2, 2.3 and Proposition 2.5. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the next result we give a simple example of a universal geometric quotient.

Proposition 3.9. *Suppose that* $\mathbf{G}$ *is a quasi-compact group scheme over* k. *Let* Y *be a* k-*scheme and consider* $\mathbf{G} \times_k Y$ *with the action of* $\mathbf{G}$ *induced by the regular action on the left factor. Then* $\mathrm{pr}_Y : \mathbf{G} \times_k Y \to Y$ *is a universal geometric quotient.*

Proof. Clearly pr_Y is univerally submersive (it is even universally open). Let $\mu : \mathbf{G} \times_k \mathbf{G} \to \mathbf{G}$ be the multiplication morphism and let $\pi_{23} : \mathbf{G} \times_k \mathbf{G} \times Y \to \mathbf{G} \times_k Y$ be the projection on the last two factors. Then the morphism

$$\mathbf{G} \times_k \mathbf{G} \times_k Y \to (\mathbf{G} \times_k Y) \times_Y (\mathbf{G} \times_k Y) = \mathbf{G} \times_k \mathbf{G} \times_k Y$$

induced by $\langle \mu \times_k 1_Y, \pi_{23} \rangle : \mathbf{G} \times_k \mathbf{G} \times_k Y \to (\mathbf{G} \times_k Y) \times_k (\mathbf{G} \times_k Y)$ is an isomorphism. We show that $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for pr_Y. For this pick an affine open subset V of Y. It suffices to check that the diagram

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{\mathrm{pr}_Y^{\#}} \Gamma\left(\mathbf{G} \times_k V, \mathcal{O}_{\mathbf{G} \times_k Y}\right) \underset{\pi_{23}^{\#}}{\overset{(\mu \times_k 1_Y)^{\#}}{\rightrightarrows}} \Gamma\left(\mathbf{G} \times_k \mathbf{G} \times_k V, \mathcal{O}_{\mathbf{G} \times_k \mathbf{G} \times_k Y}\right)$$

is a kernel. Since $\mathbf{G}$ is quasi-compact and separated (every group k-scheme is separated), we derive that the diagram above is isomorphic with

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{f \mapsto 1 \otimes f} \Gamma(\mathbf{G}, \mathcal{O}_\mathbf{G}) \otimes_k \Gamma(V, \mathcal{O}_Y) \underset{\chi \otimes f \mapsto 1 \otimes \chi \otimes f}{\overset{\chi \otimes f \mapsto \mu^{\#}(\chi) \otimes f}{\rightrightarrows}} \Gamma(\mathbf{G}, \mathcal{O}_\mathbf{G}) \otimes_k \Gamma(\mathbf{G}, \mathcal{O}_\mathbf{G}) \otimes_k \Gamma(V, \mathcal{O}_Y)$$

Thus the first diagram is the kernel diagram if $f \mapsto 1 \otimes f$ induces an isomorphism of $\Gamma(V, \mathcal{O}_Y)$ with subspace of $\Gamma(\mathbf{G}, \mathcal{O}_\mathbf{G}) \otimes_k \Gamma(V, \mathcal{O}_Y)$ given by formula

$$\left\{ \chi \in \Gamma(\mathbf{G}, \mathcal{O}_\mathbf{G}) \,\middle|\, \mu^{\#}(\chi) = 1 \otimes \chi \right\} \otimes_k \Gamma(V, \mathcal{O}_Y)$$

Hence it suffices to prove that

$$\left\{ \chi \in \Gamma(\mathbf{G}, \mathcal{O}_\mathbf{G}) \,\middle|\, \mu^{\#}(\chi) = 1 \otimes \chi \right\} = \text{constant functions on } \mathbf{G}$$

For this pick a k-algebra A and let $g : \operatorname{Spec} A \to \mathbf{G}$ be an A-point. Next let $e : \operatorname{Spec} A \to \mathbf{G}$ be an A-point of $\mathbf{G}$ which corresponds to the identity element of $\mathbf{G}$. Suppose that a regular function χ in $\mathbf{G}$ satisfies $\mu^{\#}(\chi) = 1 \otimes \chi$. Then

$$g^{\#}(\chi) = \langle g, e \rangle^{\#} \mu^{\#}(\chi) = \langle g, e \rangle^{\#} (1 \otimes \chi) = e^{\#}(\chi)$$

Recall that e is given by the composition of the structural morphism $\operatorname{Spec} A \to \operatorname{Spec} k$ and the k-point $\operatorname{Spec} k \to \mathbf{G}$ determined by the identity of $\mathbf{G}$. Thus $g^{\#}(\chi)$ is an element of k. Since this follows for every $g : \operatorname{Spec} A \to \mathbf{G}$, we derive that χ is a constant function. This completes the proof of our claim that

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{\mathrm{pr}_Y^{\#}} \Gamma\left(\mathbf{G} \times_k V, \mathcal{O}_{\mathbf{G} \times_k Y}\right) \underset{\pi_{23}^{\#}}{\overset{(\mu \times_k 1_Y)^{\#}}{\rightrightarrows}} \Gamma\left(\mathbf{G} \times_k \mathbf{G} \times_k V, \mathcal{O}_{\mathbf{G} \times_k \mathbf{G} \times_k Y}\right)$$

is the kernel diagram and hence $\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for pr_Y. Therefore, we proved that pr_Y is a geometric quotient of $\mathbf{G} \times_k Y$. Consider any morphism $Y' \to Y$. Then base change of pr_Y along this morphism is $\mathrm{pr}_{Y'}$. We conclude that pr_Y is a universal geometric quotient for $\mathbf{G} \times_k Y$. $\square$
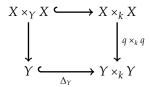
4. Geometric quotients of separated actions

Definition 4.1. The action of $\mathbf{G}$ on X is *separated* if the morphism $\langle a, \mathrm{pr}_X \rangle : \mathbf{G} \times_k X \to X \times_k X$ has closed set-theoretic image.

Theorem 4.2. *Let $q : X \to Y$ be a geometric quotient of X. Assume that q is universally submersive. Then the following assertions are equivalent.*

  (i) *The action of $\mathbf{G}$ on X is separated.*

  (ii) *Y is separated.*

Proof. We have a cartesian square

$$
\begin{array}{ccc}
X \times_Y X & \hookrightarrow & X \times_k X \\
\downarrow & & \downarrow{\scriptstyle q \times_k q} \\
Y & \underset{\Delta_Y}{\hookrightarrow} & Y \times_k Y
\end{array}
$$

It follows that $X \times_Y X \hookrightarrow X \times_k X$ is a locally closed immersion. Since q is a geometric quotient, we derive that $\langle a, \mathrm{pr}_X \rangle$ factors as a surjective morphism $\mathbf{G} \times_k X \twoheadrightarrow X \times_Y X$ followed by the immersion $X \times_Y X \hookrightarrow X \times_k X$. Thus the action of $\mathbf{G}$ on X is separated if and only if $X \times_Y X$ is a closed subscheme of $X \times_k X$. Since q is universally submersive, we derive that $q \times_k q$ is submersive. As the square above is cartesian we derive that $\Delta_Y(Y) \subseteq Y \times_k Y$ is closed if and only if $X \times_Y X \subseteq X \times_k X$ is closed. Therefore, Y is separated if and only if the action of $\mathbf{G}$ on X is separated. $\qquad\square$

5. Geometric quotients of free actions and principal bundles

Definition 5.1. The action of $\mathbf{G}$ on X is *free* if the morphism $\langle a, \mathrm{pr}_X \rangle : \mathbf{G} \times_k X \to X \times_k X$ is a closed immersion.

Definition 5.2. Let x be a k-point of X. We consider x as a morphism $\mathrm{Spec}\, k \to X$. Suppose that *the orbit morphism $\mathbf{G} \to X$ of x* given by the composition

$$
\mathbf{G} = \mathbf{G} \times_k \mathrm{Spec}\, k \xhookrightarrow{\;1_{\mathbf{G}} \times_k x\;} \mathbf{G} \times_k X \longrightarrow X
$$

is a closed immersion. Then the action of $\mathbf{G}$ on X has *a closed free orbit at x*.

Fact 5.3. *If the action of $\mathbf{G}$ on X is free, then every k-point of X has a closed free orbit.*

The following result states that over special type of local complete noetherian k-algebras geometric quotients of free actions correspond to trivial $\mathbf{G}$-bundles.

Theorem 5.4. *Suppose that k is an algebraically closed field and $\mathbf{G}$ is a smooth algebraic group over k. Let $q : X \to Y$ be a geometric quotient and a morphism locally of finite type and let Y be the spectrum of a complete local noetherian k-algebra such that the residue field of the closed point of Y is k. Then the following assertions hold.*

  (1) *If x is a k-point of X which has a closed free orbit, then there exists a $\mathbf{G}$-equivariant, étale and surjective morphism $f : \mathbf{G} \times_k Y \to X$ such that the triangle*

is commutative and the morphism

$$Y = \operatorname{Spec} k \times_k Y \xleftarrow{e \times_k 1_Y} \mathbf{G} \times_k Y \xrightarrow{f} X$$

is a section of q.

(2) *If the action of $\mathbf{G}$ on X is free, then f is an isomorphism.*

The proof relies on two algebraic lemmas. The first describe free actions in context of complete rings (it is [Mumford et al., 1994, lemma on page 18]) and the second is a version of Hensel's lemma.

Lemma 5.4.1. *Let $(R, \mathfrak{m}, k)$ be a complete local noetherian k-algebra and let $\sigma : R \to R[[x_1, ..., x_n]]$ be a local morphism into a ring of formal power series over R. Assume that the composition*

$$R \xrightarrow{\sigma} R[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod (x_1, ..., x_n)} R$$

is the identity and the composition

$$R \xrightarrow{\sigma} R[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod \mathfrak{m}} (R/\mathfrak{m})[[x_1, ..., x_n]] = k[[x_1, ..., x_n]]$$

is surjective. Consider elements $y_1, ..., y_n$ of R such that $\sigma(y_i) \bmod \mathfrak{m} = x_i$ for $i = 1, ..., n$. Then the composition

$$R \xrightarrow{\sigma} R[[x_1, ..., x_n]] \xrightarrow{f \mapsto f \bmod (y_1, ..., y_n)} (R/(y_1, ..., y_n))[[x_1, ..., x_n]]$$

is an isomorphism.

Proof of the lemma. For convienience let ϕ denote the morphism given by the rule $r \mapsto \sigma(r) \bmod (y_1, ..., y_n)$. Also denote $R/(y_1, ..., y_n)$ by S. According to assumptions we have

$$\sigma(y_i) = x_i + y_i + \sum_{j=1}^{n} x_j \cdot \mathfrak{m}[[x_1, ..., x_n]]$$

for each i. Thus $\phi(y_i) = \sum_{j=1}^{n} f_{ij} \cdot x_j$ where $f_{ij} \in S$ are elements such that the matrix $\left[f_{ij}\right]_{1 \le i,j \le n}$ is invertible in S. Hence

$$S[[x_1, ..., x_n]] = S[[\phi(y_1), ..., \phi(y_n)]]$$

and ϕ composed with $S[[\phi(y_1), ..., \phi(y_n)]] \twoheadrightarrow S$ is the quotient morphism $R \twoheadrightarrow S$. From this observations we derive that ϕ is surjective. It remains to prove that it is injective. Consider z in R such that $\phi(z) = 0$. Suppose that $z \in (y_1, ..., y_n)^m$ for some $m \in \mathbb{N}$. Write

$$z = \sum_{\alpha \in \Lambda} c_\alpha \cdot y_1^{\alpha_1} ... y_n^{\alpha_n}$$

for some $c_\alpha \in R$ where $\Lambda = \{\alpha \in \mathbb{N}^n \mid \alpha_1 + ... + \alpha_n = m\}$. Hence

$$0 = \phi(z) = \sum_{\alpha \in \Lambda} \phi(c_\alpha) \cdot \phi(y_1)^{\alpha_1} ... \phi(y_n)^{\alpha_n}$$

Thus $\phi(c_\alpha) \in (\phi(y_1), ..., \phi(y_n))$ for every $\alpha \in \Lambda$. Since ϕ composed with $S[[\phi(y_1), ..., \phi(y_n)]] \twoheadrightarrow S$ is the quotient morphism $R \twoheadrightarrow S$, we derive that

$$c_\alpha \bmod (y_1, ..., y_n) = \phi(c_\alpha) \bmod (\phi(y_1), ..., \phi(y_n)) = 0$$

for every $\alpha \in \Lambda$. Thus $c_\alpha \in (y_1, ..., y_n)$ for every $\alpha \in \Lambda$, which implies that $z \in (y_1, ..., y_n)^{m+1}$. Thus we proved that

$$\phi(z) = 0 \text{ and } z \in (y_1, ..., y_n)^m \Rightarrow z \in (y_1, ..., y_n)^{m+1}$$

By $\mathfrak{m}$-adic completeness of R this implies that $\phi(z) = 0$ if and only if $z = 0$. Hence ϕ is also injective. $\square$

Lemma 5.4.2. *Let $(R, \mathfrak{m})$ be a complete local noetherian ring and let $R \to S$ be a local morphism into a local noetherian ring. Suppose that there exists a finitely generated R-submodule N of S such that*

$$S = N + \mathfrak{m}S$$

Then $S = N$.

Proof of the lemma. Pick s in S. Since $S = N + \mathfrak{m}S$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $x_n \in \mathfrak{m}^n N$ and

$$s - \sum_{i \leq n} x_i \in \mathfrak{m}^{n+1}S$$

According to the assumption that $(R, \mathfrak{m})$ is complete with respect to $\mathfrak{m}$-adic topology and N is finitely generated over R, we deduce that N is complete with respect to $\mathfrak{m}$-adic topology. Hence there exists a unique element x in N such that

$$x = \sum_{n \in \mathbb{N}} x_n$$

where above series is convergent with respect to $\mathfrak{m}$-adic topology. Note also that

$$x - \sum_{i \leq n} x_i \in \mathfrak{m}^{n+1}N$$

for every $n \in \mathbb{N}$. Thus we have

$$s - x = \left(s - \sum_{i \leq n} x_i\right) - \left(x - \sum_{i \leq n} x_i\right) \in \mathfrak{m}^{n+1}S + \mathfrak{m}^{n+1}N = \mathfrak{m}^{n+1}S$$

for every $n \in \mathbb{N}$. Hence

$$x - s \in \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n S$$

Since $R \to S$ is local morphism and S is a local ring, we deduce that $\mathfrak{m}S$ is contained in the maximal ideal of S. By assumptions S is noetherian. Therefore, S is separated with respect to $\mathfrak{m}$-adic topology. This implies that

$$0 = \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n S$$

Thus $s - x = 0$ and we infer that s is an element of N. This completes the proof that $S = N$. $\square$

In what follows we shall denote by $\mathbf{G}x$ the closed subscheme determined by the orbit morphism $\mathbf{G} \to X$ of a k-point x of X which has a closed free orbit. For readers convienience we include the following lemmas, which have topological content.

Lemma 5.4.3. *Let $q : X \to Y$ be a geometric quotient and assume that Y is the spectrum of a local k-algebra such that the residue field of the closed point o of Y is k. Let x be a k-point of X with free closed orbit, then $q^{-1}(o) = \mathbf{G}x$ as closed subschemes of X.*

Proof of the lemma. Morphism q induces the morphism of residue fields $k(q(x)) \hookrightarrow k(x) = k$ over k. This implies that $k(q(x)) = k$ and hence $q(x)$ is a k-point of Y. Note that o is the unique k-point of Y. Thus $q(x) = o$. Clearly $q^{-1}(o)$ is a closed **G**-stable subscheme of X (it is the preimage of o under **G**-equivariant q), that contains x. Since **G**x is the smallest closed **G**-stable subscheme of X containing x, we deduce that **G**$x \subseteq q^{-1}(o)$ scheme-theoretically. Consider the pair of arrows

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X$$

Passing to functors of points we obtain that $a^{-1}(\mathbf{G}x) = \mathrm{pr}_X(\mathbf{G}.x)$. Since q is the cokernel of the pair (a, pr_X) in the category of topological spaces, we deduce that there exists a closed subset Z of Y such that $q^{-1}(Z) = \mathbf{G}x$. Clearly $o \in Z$ and hence $q^{-1}(o) \subseteq \mathbf{G}x$ set-theoretically. On the other hand above we proved that $\mathbf{G}x \subseteq q^{-1}(o)$ scheme-theoretically. This can only happen if $q^{-1}(o) = \mathbf{G}x$ as closed subschemes of X. $\qquad\square$

Lemma 5.4.4. *Let $q : X \to Y$ be a geometric quotient and assume that Y is the spectrum of a local k-algebra such that the residue field of the closed point o of Y is k. Let U be an open **G**-stable subset of X which contain a k-point. Then $U = X$.*

Proof of the lemma. Consider the pair of arrows

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X$$

Since U is **G**-stable open subset of X, we derive that $\mathrm{pr}_X^{-1}(U) = a^{-1}(U)$. Next by definition q is the cokernel of the above pair in the category of topological spaces. Hence there exists an open subset V of Y such that $U = q^{-1}(V)$. Since U contains a k-point of X, we deduce as in Lemma 5.4.3 that $o \in V$. Thus $V = Y$ and finally $U = q^{-1}(V) = X$. $\qquad\square$

Proof of the theorem. We first prove **(1)**. Denote by o the closed point of Y. Assume that x is a k-point of X which has a closed free orbit. Consider the surjective morphism $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{\mathbf{G},e}$ induced by the orbit morphism $\mathbf{G} \hookrightarrow X$ of x. Since **G** is smooth over k, the ring $\mathcal{O}_{\mathbf{G},e}$ is regular. Pick a system of parameters $x_1, ..., x_n$ of $\mathcal{O}_{\mathbf{G},e}$ and let $y_1, ..., y_n$ be elements of $\mathcal{O}_{X,x}$ such that y_i is send to x_i by the morphism $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{\mathbf{G},e}$ for $1 \le i \le n$. Define S to be the quotient ring $\mathcal{O}_{X,x}/(y_1, ..., y_n)$. The morphism q induces the morphism $q^{\#} : \mathcal{O}_{Y,o} \to \mathcal{O}_{X,x}$ and hence the morphism $\mathcal{O}_{Y,o} \to S$. By Lemma 5.4.3 we have

$$S/\mathfrak{m}_o S = k$$

where $\mathfrak{m}_o$ is the maximal ideal of $\mathcal{O}_{Y,o}$. According to Lemma 5.4.2 we derive that $\mathcal{O}_{Y,o} \to S$ is surjective. Let $f : \mathbf{G} \times_k \mathrm{Spec}\, S \to X$ be the unique **G**-equivariant morphism induced by the surjection $\mathcal{O}_{X,x} \twoheadrightarrow S$. We have a commutative square

$$\begin{array}{ccc}
\mathbf{G} \times_k \mathrm{Spec}\, S & \overset{f}{\longrightarrow} & X \\
{\scriptstyle \mathrm{pr}_{\mathrm{Spec}\, S}} \downarrow & & \downarrow {\scriptstyle q} \\
\mathrm{Spec}\, S & \underset{j}{\hookrightarrow} & Y
\end{array}$$

where j is a closed immersion induced by $\mathcal{O}_{Y,o} \twoheadrightarrow S$. According to assumptions q is locally of finite type. Moreover, **G** is an algebraic group over k and hence $\mathrm{pr}_{\mathrm{Spec}\, S}$ is locally of finite type. These two assertions together with the fact that $\mathrm{Spec}\, S \hookrightarrow Y$ is a closed immersion of noetherian

schemes (and thus is of finite type) imply that f is locally of finite type. Then by Lemma 5.4.1 we deduce that f induces an isomorphism

$$\widehat{\mathcal{O}_{X,x}} \longrightarrow \hat{S}[[x_1, ..., x_n]] = \hat{S} \hat{\otimes}_k \widehat{\mathcal{O}_{\mathbf{G},e}}$$

of complete local rings. Since f is locally of finite type, it follows that f is étale at a k-point of $\mathbf{G} \times_k \operatorname{Spec} S$ determined by the unique k-point of $\operatorname{Spec} S$ and $e \in \mathbf{G}$. Let U be an étale locus of f. It contains a k-point and hence it is nonempty. Moreover, U is open (it is étale locus) subset of X. Since f is $\mathbf{G}$-equivariant, we derive that U is $\mathbf{G}$-stable. Similarly $f(U)$ is open $\mathbf{G}$-stable subset of X and $x \in f(U)$. Thus by Lemma 5.4.4 we deduce that

$$U = \mathbf{G} \times_k \operatorname{Spec} S, \ f(U) = X$$

Therefore, f is étale and surjective. Now we pullback $q : X \to Y$ along the closed immersion $\operatorname{Spec} S \hookrightarrow Y$. We obtain a cartesian square

$$
\begin{array}{ccc}
\tilde{X} & \overset{\tilde{j}}{\hookrightarrow} & X \\
{\scriptstyle \tilde{q}} \downarrow & & \downarrow {\scriptstyle q} \\
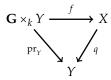\operatorname{Spec} S & \underset{j}{\hookrightarrow} & Y
\end{array}
$$

Then f factors as a morphism $\mathbf{G} \times_k \operatorname{Spec} S \to \tilde{X}$ followed by a closed immersion $\tilde{j}$. Since f is étale and surjective, we deduce that $\tilde{j}$ is étale and surjective. This implies that $\tilde{j}$ is an isomorphism of schemes. By definition of geometric quotient we derive that the diagram

$$\mathcal{O}_Y \overset{q^\#}{\longrightarrow} q_* \mathcal{O}_X \underset{q_* \operatorname{pr}_X^\#}{\overset{q_* a^\#}{\rightrightarrows}} q_* \left(\operatorname{pr}_X \right)_* \mathcal{O}_{\mathbf{G} \times_k X} = q_* a_* \mathcal{O}_{\mathbf{G} \times_k X}$$

is the kernel in the category of sheaves on Y. Hence $q^\# : \mathcal{O}_Y \to q_* \mathcal{O}_X$ is a monomorphism of sheaves. On the other hand we have

$$q^\# = j_* q_* \left(\tilde{j}^{-1} \right)^\# \cdot j_* \tilde{q}^\# \cdot j^\#$$

and thus $j^\#$ is a monomorphism. Since j is a closed immersion, we infer that j is an isomorphism. Therefore, we can identify $\operatorname{Spec} S$ with Y. Then f is a morphism which makes the triangle

$$
\begin{array}{ccc}
\mathbf{G} \times_k Y & \overset{f}{\longrightarrow} & X \\
 & {\scriptstyle \operatorname{pr}_Y} \searrow \quad \swarrow {\scriptstyle q} & \\
 & Y &
\end{array}
$$

commutative. This completes the proof of **(1)**.

For the proof of **(2)** consider the section $s : Y \hookrightarrow X$ described in **(1)**. Then f fits into a cartesian square

$$\begin{array}{ccc} \mathbf{G} \times_k Y & \xrightarrow{\ f\ } & X \times_Y Y = X \\ {\scriptstyle 1_\mathbf{G} \times_Y s}\big\downarrow & & \big\downarrow {\scriptstyle 1_X \times_Y s} \\ \mathbf{G} \times_k X & \xhookrightarrow[\ \phi\ ]{} & X \times_Y X \end{array}$$

where ϕ is a closed immersion induced by the closed immersion $\langle a, \mathrm{pr}_X \rangle : \mathbf{G} \times_k X \hookrightarrow X \times_k X$ (the action of $\mathbf{G}$ on X is free). Thus f is a closed immersion. By **(1)** it is étale and surjective. Therefore, f is an isomorphism. $\qquad\square$

Remark 5.5. We expect that Theorem 5.4 holds for prime spectra of strictly henselian rings.

Now we introduce sufficient condition for smoothness of geometric quotient in case of locally algebraic k-schemes.

Corollary 5.6. *Suppose that* $\mathbf{G}$ *is a smooth algebraic group over k. Let $q : X \to Y$ be a morphism of finite type between k-schemes locally of finite type. Assume that q is a uniform geometric quotient of X and x is a k-point of X with closed free orbit. Then q is smooth at x.*

Proof. Since smoothness descent along faithfully flat morphisms, we may assume that k is algebraically closed. Let $y = q(x)$. Then y is a k-point of Y. Now $1_{\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}}} \times_k q$ is a geometric quotient and $\widehat{\mathcal{O}_{Y,y}}$ is a complete local noetherian k-algebra with k as a residue field. Moreover, x is a k-point of $\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}} \times_k X$ with a closed free orbit. By Theorem 5.4 we deduce that $1_{\mathrm{Spec}\,\widehat{\mathcal{O}_{Y,y}}} \times_k q$ is smooth. From descent for smoothness we infer that q is smooth at x. $\qquad\square$

Definition 5.7. Let $q : X \to Y$ be a $\mathbf{G}$-equivariant morphism into a k-scheme Y equipped with the trivial $\mathbf{G}$-action. Suppose that q is faithfully flat, quasi-compact morphism and the square

$$\begin{array}{ccc} \mathbf{G} \times_k X & \xrightarrow{\ a\ } & X \\ {\scriptstyle \mathrm{pr}_X}\big\downarrow & & \big\downarrow {\scriptstyle q} \\ X & \xrightarrow[\ q\ ]{} & Y \end{array}$$

is cartesian. Then q is *a principal* $\mathbf{G}$*-bundle*.

Now we use Theorem 5.4 to describe principal $\mathbf{G}$-bundles in the category of locally algebraic k-schemes.

Theorem 5.8. *Suppose that* $\mathbf{G}$ *is a smooth algebraic group over k. Let $q : X \to Y$ be a morphism of finite type between k-schemes locally of finite type. Then the following assertions are equivalent.*

   (i) *q is a universal geometric quotient and the action of $\mathbf{G}$ on X is free.*

   (ii) *q is a uniform geometric quotient and the action of $\mathbf{G}$ on X is free.*

   (iii) *q is a principal $\mathbf{G}$-bundle.*

Proof. Clearly **(i)** $\Rightarrow$ **(ii)**. Suppose that **(ii)** holds. Let $\overline{k}$ be an algebraic closure of k. Then $1_{\mathrm{Spec}\,\overline{k}} \times_k q$ is a uniform quotient and the action of $\mathrm{Spec}\,\overline{k} \times_k \mathbf{G}$ on $\mathrm{Spec}\,\overline{k} \times_k X$ induced by the action of $\mathbf{G}$ on X is free. Moreover, if $1_{\mathrm{Spec}\,\overline{k}} \times_k q$ is a principal $\mathrm{Spec}\,\overline{k} \times_k \mathbf{G}$-bundle, then q is a $\mathbf{G}$-bundle. This follows from the observation that property of being a principal bundle descents along faithfully flat and quasi-compact base extensions. Thus we may assume that k is algebraically closed. Next

we pick a k-point y of Y and consider base change $1_{\operatorname{Spec} \widehat{\mathcal{O}_{Y,y}}} \times_Y q$. This is a geometric quotient (because morphism $\operatorname{Spec} \widehat{\mathcal{O}_{Y,y}} \to Y$ is flat) and a morphism of finite type. Moreover, the action of $\mathbf{G}$ on $\operatorname{Spec} \widehat{\mathcal{O}_{Y,y}} \times_Y X$ is free. Since $\widehat{\mathcal{O}_{Y,y}}$ is a complete noetherian k-algebra with residue field k, we derive by Theorem 5.4 that $\operatorname{Spec} \widehat{\mathcal{O}_{Y,y}} \times_Y q$ is isomorphic as a $\mathbf{G}$-equivariant morphism with $\operatorname{pr}_{\operatorname{Spec} \widehat{\mathcal{O}_{Y,y}}}$. By faithfuly flat descent for flat morphism we deduce that q is flat at every point in the fiber $q^{-1}\left(\operatorname{Spec} \mathcal{O}_{Y,y}\right)$. Since y is an arbitrary k-point, it follows that q is flat at every point of X which specializes to a k-point. Every point of X is a generization of a k-point (X is locally of finite type and k is algebraically closed). Thus q is flat. It is also surjective (as it is a geometric quotient) and quasi-compact (it is of finite type). Therefore, it is faithfully flat and quasi-compact morphism. In order to obtain **(iii)** it remains to prove that the morphism $\Phi : \mathbf{G} \times_k X \to X \times_Y X$ induced by a and pr_X is an isomorphism. Note that it is a closed immersion (the action of $\mathbf{G}$ on X is closed). Moreover, $1_{\operatorname{Spec} \widehat{\mathcal{O}_{Y,y}}} \times_Y \Phi$ is an isomorphism due to the fact that $1_{\operatorname{Spec} \widehat{\mathcal{O}_{Y,y}}} \times_Y q$ is isomorphic as a $\mathbf{G}$-equivariant morphism with $\operatorname{pr}_{\operatorname{Spec} \widehat{\mathcal{O}_{Y,y}}}$. By faithfully flat descent we infer that $1_{\operatorname{Spec} \mathcal{O}_{Y,y}} \times_Y \Phi$ is an isomorphism. This holds for every k-point y in Y. Thus Φ induces an isomorphism $\mathcal{O}_{X \times_Y X, \Phi(z)} \to \mathcal{O}_{\mathbf{G} \times_k X, z}$ for every k-point z of $X \times_Y X$. Hence a closed immersion Φ is an isomorphism. This completes the proof of **(ii)** $\Rightarrow$ **(iii)**.

Assume now that **(iii)** holds. Then the square

$$
\begin{array}{ccc}
\mathbf{G} \times_k X & \xrightarrow{\;\;a\;\;} & X \\
{\scriptstyle \operatorname{pr}_X}\big\downarrow & & \big\downarrow{\scriptstyle q} \\
X & \xrightarrow[\;\;q\;\;]{} & Y
\end{array}
$$

is cartesian and q is faithfully flat and quasi-compact. By Proposition 3.9 morphism pr_X is a universal geometric quotient. According to Corollary 3.8 we derive that q is universal geometric quotient. Moreover, the cartesian square above shows that the morphism $\mathbf{G} \times_k X \to X \times_Y X$ induced by a and pr_X is an isomorphism. Thus the action of $\mathbf{G}$ on X is free. This is **(i)**. Hence **(iii)** $\Rightarrow$ **(i)** holds. $\square$

6. Good categorical quotients

We start by the following criterion for categorical quotients.

Theorem 6.1. *Let $q : X \to Y$ be a morphism into a k-scheme Y equipped with the trivial $\mathbf{G}$-action. Assume that the following assertions hold.*

(1) *q is $\mathbf{G}$-equivariant.*

(2) *$\mathcal{O}_Y$ is the sheaf of $\mathbf{G}$-invariants for q.*

(3) *If Z is a $\mathbf{G}$-stable closed subset of X, then $q(Z)$ is a closed subset of Y.*

(4) *If $\{Z_i\}_{i \in I}$ is a family of closed $\mathbf{G}$-stable subsets with the empty intersection, then the intersection $\{q(Z_i)\}_{i \in I}$ is empty.*

Then q is submersive and it is a categorical quotient of X.

Proof. Clearly $q(X)$ is closed in Y. Hence $V = Y \setminus q(X)$ is open. Moreover, $q^\# : \mathcal{O}_Y \to q_* \mathcal{O}_X$ is a monomorphism of sheaves of k-algebras. Thus we have a monomorphism $\mathcal{O}_V \hookrightarrow (q_* \mathcal{O}_X)_{q^{-1}(V)}$. We have $(q_* \mathcal{O}_X)_{q^{-1}(V)} = 0$ and hence $\mathcal{O}_V = 0$. This implies that $V = \varnothing$. Thus q is surjective. Suppose that Z is a subset of Y such that $q^{-1}(Z)$ is a closed subset of X. Then $q^{-1}(Z)$ is a $\mathbf{G}$-stable closed subset and hence $q\left(q^{-1}(Z)\right)$ is closed. Note that $q\left(q^{-1}(Z)\right) = Z$ because q is surjective.

Thus Z is closed. This completes the proof that q is submersive.

Now we show that q is a categorical quotient of X. For this pick a **G**-equivariant morphism $p : X \to Z$ where Z is a k-scheme with the trivial **G**-action. Consider open affine cover $\{W_i\}_{i \in I}$ of Z. Then $X \smallsetminus p^{-1}(W_i)$ is a closed **G**-stable closed for $i \in I$. Define $V_i = Y \smallsetminus q\left(X \smallsetminus p^{-1}(W_i)\right)$ for each i. Thus V_i is an open subset of Y for every $i \in I$. Moreover, we have

$$\bigcap_{i \in I} X \smallsetminus p^{-1}(W_i) = \varnothing$$

and hence $\{V_i\}_{i \in I}$ form an open cover of Y. Note that for every $i \in I$ we have $q^{-1}(V_i) \subseteq p^{-1}(V_i)$. Consider the composition

$$\Gamma\left(W_i, \mathcal{O}_Z\right) \xrightarrow{\;\;p^{\#}\;\;} \Gamma\left(p^{-1}(W_i), \mathcal{O}_X\right) \xrightarrow{\;\;f \mapsto f_{|q^{-1}(V_i)}\;\;} \Gamma\left(q^{-1}(V_i), \mathcal{O}_X\right)$$

for every i in I. Since the action of **G** on Z is trivial, we derive that the image of the morphism above consists of **G**-invariant functions on $q^{-1}(V_i)$. This means that the morphism above factors uniquely through $q^{\#}_{V_i} : \Gamma\left(V_i, \mathcal{O}_Y\right) \to \Gamma\left(q^{-1}(V_i), \mathcal{O}_X\right)$. Since W_i is affine for every i in I, we obtain a unique morphism $f_i : V_i \to W_i$ such that $f_i \cdot q_{|q^{-1}(V_i)} = p_{|q^{-1}(V_i)}$ for each i. By construction the family $\{f_i\}_{i \in I}$ glue to a morphism $f : Y \to Z$ such that $f \cdot q = p$. This morphism is unique due to the fact that f_i are unique for every i. This finishes the proof of the fact that q is a categorical quotient of X. $\qquad\square$

Definition 6.2. Let $q : X \to Y$ be a morphism into a k-scheme Y equipped with the trivial **G**-action. Suppose that q satisfies conditions **(1)-(4)** of Theorem 6.1. Then q is *a good categorical quotient of X*.

Proposition 6.3. *Let $q : X \to Y$ be a morphism into a k-scheme Y equipped with the trivial **G**-action. Assume that X is quasi-compact and the following assertions hold.*

(1) *q is **G**-equivariant.*

(2) *$\mathcal{O}_Y$ is the sheaf of **G**-invariants for q.*

(3) *If Z is a **G**-stable closed subset of X, then $q(Z)$ is a closed subset of Y.*

(4) *If Z_1 and Z_2 are closed **G**-stable subsets with the empty intersection, then $q(Z_1) \cap q(Z_2) = \varnothing$.*

Then q is a good categorical quotient of X.

Proof. Suppose that $\{Z_i\}_{i \in I}$ is a family of closed **G**-stable subsets with the empty intersection. Since X is quasi-compact, there exists a finite subset $\{i_1, ..., i_n\} \subseteq I$ such that the family $\{Z_{i_1}, ..., Z_{i_n}\}$ has empty intersection. Then

$$\bigcap_{i \in I} q(Z_i) \subseteq \bigcap_{j=1}^{n} q(Z_{i_j}) = \varnothing$$

according to **(4)**. This implies that q is a good categorical quotient. $\qquad\square$

As in case of categorical and geometric quotients one can introduce the following notion.

Definition 6.4. A morphism $q : X \to Y$ is *a universal (uniform) good categorical quotient of X* if for every (flat) morphism $g : Y' \to Y$ of k-schemes a base change $q' : X' \to Y'$ of q along g is a good categorical quotient of X'.

7. Affine case

In this section we make the first important step towards existence of quotients by proving that good categorical quotients exists for affine k-schemes equipped with an action of geometrically reducitve groups.

Proposition 7.1. *Suppose that $\mathbf{G}$ is geometrically reductive group and X is an affine k-scheme. Let Z_1, Z_2 be nonempty closed $\mathbf{G}$-stable subsets of X such that $Z_1 \cap Z_2 = \varnothing$. Then there exists $\mathbf{G}$-invariant regular function f on X such that $f_{|Z_1} = 1$ and $f_{|Z_2} = 0$.*

Proof. By [Monygham, 2020, Corollary 5.4] we may consider Z_1 and Z_2 as a closed $\mathbf{G}$-stable subschemes of X. Since $Z_1 \cap Z_2 = \varnothing$, we have

$$\Gamma(Z_1 \cup Z_2, \mathcal{O}_X) = \Gamma(Z_1, \mathcal{O}_X) \times \Gamma(Z_2, \mathcal{O}_X)$$

In particular, there exists a regular invariant function

$$g \in \Gamma(Z_1, \mathcal{O}_X)^{\mathbf{G}} \times \Gamma(Z_2, \mathcal{O}_X)^{\mathbf{G}} = \Gamma(Z_1 \cup Z_2, \mathcal{O}_X)^{\mathbf{G}}$$

such that $g_{|Z_1} = 1$ and $g_{|Z_2} = 0$. Consider the canonical morphism

$$\Gamma(X, \mathcal{O}_X)^{\mathbf{G}} \longrightarrow \Gamma(Z_1 \cup Z_2, \mathcal{O}_X)^{\mathbf{G}} = \Gamma(Z_1, \mathcal{O}_X)^{\mathbf{G}} \times \Gamma(Z_2, \mathcal{O}_X)^{\mathbf{G}}$$

According to [Monygham, 2021, Theorem 2.4] there exists $f \in \Gamma(X, \mathcal{O}_X)$ and a positive integer r such that $f_{|Z_1 \cup Z_2} = g^r$. Then $f_{|Z_1} = 1$ and $f_{|Z_2} = 0$. $\qquad\square$

Theorem 7.2. *Suppose that X is an affine k-scheme and $\mathbf{G}$ is a geometrically reductive group. Let $Y = \operatorname{Spec}\Gamma(X, \mathcal{O}_X)^{\mathbf{G}}$ and let $q : X \to Y$ be the canonical morphism. Then q is a uniform good categorical quotient of X. Moreover, the following assertions hold.*

(1) *If X is of finite type over k, then Y is of finite type over k.*

(2) *If $\mathbf{G}$ is linearly reductive, then q is a universal good categorical quotient of X.*

(3) *q is universally submersive.*

For the proof we need to following results.

Lemma 7.2.1. *Let $\mathbf{G}$ be an algebraic group which acts on $\operatorname{Spec} A$ for some k-algebra A. Fix a flat $A^{\mathbf{G}}$-algebra B. Then the canonical morphism $B \to (A \otimes_{A^{\mathbf{G}}} B)^{\mathbf{G}}$ is an isomorphism of k-algebras.*

Proof of the lemma. For every linear representation V of $\mathbf{G}$ we have a left exact sequence

$$0 \longrightarrow V^{\mathbf{G}} \longrightarrow V \xrightarrow{x \mapsto c(x) - 1 \otimes x} k[\mathbf{G}] \otimes_k V$$

where $c : V \to k[\mathbf{G}] \otimes_k V$ is the coaction. Now we denote by d the coaction on A. Thus we have left exact sequences

$$0 \longrightarrow A^{\mathbf{G}} \otimes_{A^{\mathbf{G}}} B \longrightarrow A \otimes_{A^{\mathbf{G}}} B \xrightarrow{x \otimes 1 \mapsto d(x) \otimes 1 - 1 \otimes x \otimes 1} k[\mathbf{G}] \otimes_k A \otimes_{A^{\mathbf{G}}} B$$

and

$$0 \longrightarrow (A \otimes_{A^{\mathbf{G}}} B)^{\mathbf{G}} \longrightarrow A \otimes_{A^{\mathbf{G}}} B \xrightarrow{x \otimes 1 \mapsto d(x) \otimes 1 - 1 \otimes x \otimes 1} k[\mathbf{G}] \otimes_k A \otimes_{A^{\mathbf{G}}} B$$

Note that $A \otimes_{A^{\mathbf{G}}} B \ni x \otimes 1 \mapsto d(x) \otimes 1 \in k[\mathbf{G}] \otimes_k A \otimes_{A^{\mathbf{G}}} B$ is the coaction induced by c on the base change $A \otimes_{A^{\mathbf{G}}} B$. This implies that there is canonical isomorphism

$$B = A^{\mathbf{G}} \otimes_{A^{\mathbf{G}}} B \cong (A \otimes_{A^{\mathbf{G}}} B)^{\mathbf{G}}$$

$\square$

Lemma 7.2.2. *Let $\mathbf{G}$ be geometrically reductive group which acts on $\operatorname{Spec} A$ for some k-algebra A. If $f_1, ..., f_n \in A^{\mathbf{G}}$ and*

$$f \in \left(\sum_{i=1}^{n} A f_i \right) \cap A^{\mathbf{G}}$$

then there exists positive integer r such that

$$f^r \in \sum_{i=1}^{n} A^{\mathbf{G}} f_i$$

Moreover, if $\mathbf{G}$ is linearly reductive, then r can be chosen to be 1.

Proof of the lemma. Let $d : A \to k[\mathbf{G}] \otimes_k A$ be the coaction of $\mathbf{G}$ on A. The proof goes on induction on n. Write $f = a_1 f_1 + ... + a_n f_n$ for $a_1, ..., a_n \in A$. Consider $\mathfrak{a} = \operatorname{ann}(f_1) + A f_2 + ... + A f_n$. This is a $\mathbf{G}$-stable ideal in A. We show now that a_1 is $\mathbf{G}$-invariant modulo $\mathfrak{a}$. Indeed, we have

$$(1 \otimes f_1)(d(a_1) - 1 \otimes a_1) = d(f_1)d(a_1) - 1 \otimes f_1 a_1 = d(f) - 1 \otimes f = 0$$

Hence

$$d(a_1) - 1 \otimes a_1 \in k[\mathbf{G}] \otimes_k \operatorname{ann}(f_1) \subseteq k[\mathbf{G}] \otimes_k \mathfrak{a}$$

and this shows that a_1 is $\mathbf{G}$-invariant modulo $\mathfrak{a}$. Therefore, according to [Monygham, 2021, Theorem 2.4] there exists positive integer r and $a_1' \in A^{\mathbf{G}}$ such that $a_1^r - a_1' \in \mathfrak{a}$. Thus

$$f^r \in f_1^r a_1^r + A f_2 + ... + A f_n = f_1^r a_1' + A f_2 + ... + A f_n$$

Now if $n = 1$, then we have $f^r = f_1^r a_1' \in A^{\mathbf{G}} f_1$ and the assertion holds. On the other hand if $n \geq 2$, then we can apply inductive hypothesis to

$$f^r - f_1^r a_1' \in (A f_2 + ... + A f_n) \cap A^{\mathbf{G}}$$

and obtain that

$$(f^r - f_1^r a_1')^d \in A^{\mathbf{G}} f_2 + ... + A^{\mathbf{G}} f_n$$

for some positive integer d. Then

$$f^{rd} \in A^{\mathbf{G}} f_1 + A^{\mathbf{G}} f_2 + ... + A^{\mathbf{G}} f_n$$

and the assertion holds. $\square$

Lemma 7.2.3. *Let $\mathbf{G}$ be geometrically reductive group which acts on $\operatorname{Spec} A$ for some k-algebra A. Then the morphism $\operatorname{Spec} A \to \operatorname{Spec} A^{\mathbf{G}}$ is surjective.*

Proof of the lemma. Pick a prime ideal $\mathfrak{p} \in \operatorname{Spec} A^{\mathbf{G}}$. Consider $f \in A\mathfrak{p} \cap A^{\mathbf{G}}$. Then there exist $f_1, ..., f_n \in \mathfrak{p}$ such that

$$f \in (A f_1 + ... + A f_n) \cap A^{\mathbf{G}}$$

By Lemma 7.2.2 we have

$$f^r \in A^{\mathbf{G}} f_1 + ... + A^{\mathbf{G}} f_n \subseteq \mathfrak{p}$$

for some positive integer r. Since $\mathfrak{p}$ is a prime ideal, we derive that $f \in \mathfrak{p}$. Thus $A\mathfrak{p} \cap A^{\mathbf{G}} = \mathfrak{p}$. Thus we have an injective morphisms $A^{\mathbf{G}}/\mathfrak{p} \hookrightarrow A/A\mathfrak{p}$ of k-algebras. This implies that the morphism $k(\mathfrak{p}) \to k(\mathfrak{p}) \otimes_{A^{\mathbf{G}}} A$ is also injective, where $k(\mathfrak{p})$ is a residue field of $\mathfrak{p}$ in $A^{\mathbf{G}}$. We infer that the fiber of $\operatorname{Spec} A \to \operatorname{Spec} A^{\mathbf{G}}$ is nonempty. $\square$

Lemma 7.2.4. *Let* **G** *be geometrically reductive group which acts on* Spec A *for some k-algebra A. Suppose that $\mathfrak{a}$ is an ideal in $A^{\mathbf{G}}$. Then $(A/A\mathfrak{a})^{\mathbf{G}}$ is canonically isomorphic with $A^{\mathbf{G}}/\mathfrak{a}$.*

Proof of the lemma. Lemma 7.2.2 shows that $A\mathfrak{a} \cap A^{\mathbf{G}} = \mathfrak{a}$. Since **G** is linearly reductive, we have a canonical identification

$$(A/A\mathfrak{a})^{\mathbf{G}} = A^{\mathbf{G}}/(A\mathfrak{a})^{\mathbf{G}} = A^{\mathbf{G}}/A\mathfrak{a} \cap A^{\mathbf{G}} = A^{\mathbf{G}}/\mathfrak{a}$$

$\square$

Proof of the theorem. Since X is quasi-compact, we may verify conditions of Proposition 6.3. First let us denote by A the k-algebra of global regular functions $\Gamma(X, \mathcal{O}_X)$. Suppose that $V \subseteq \operatorname{Spec} A^{\mathbf{G}} = Y$ is an open affine subset. Then $B = \Gamma(V, \mathcal{O}_Y)$ is a flat $A^{\mathbf{G}}$-algebra and by Lemma 7.2.1 we have canonical isomorphism

$$B \cong (A \otimes_{A^{\mathbf{G}}} B)^{\mathbf{G}}$$

This implies that $\Gamma(V, \mathcal{O}_Y) \cong \Gamma(q^{-1}(V), \mathcal{O}_X)^{\mathbf{G}}$ and hence $\mathcal{O}_Y$ is the sheaf of **G**-invariants for q. Fix now a closed **G**-stable subset Z of X. By [Monygham, 2020, Corollary 5.4] there exists a **G**-stable ideal $\mathfrak{a} \subseteq A$ such that its vanishing set is equal to Z. Consider a commutative square

$$
\begin{array}{ccc}
\operatorname{Spec} A/\mathfrak{a} & \longrightarrow & \operatorname{Spec} A \\
\downarrow & & \downarrow \\
\operatorname{Spec} (A/\mathfrak{a})^{\mathbf{G}} & \longrightarrow & \operatorname{Spec} A^{\mathbf{G}}
\end{array}
$$

with canonically defined arrows. Note that $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} (A/\mathfrak{a})^{\mathbf{G}}$ is surjective (Lemma 7.2.3) and according to [Monygham, 2021, Theorem 2.4] morphism

$$\operatorname{Spec} (A/\mathfrak{a})^{\mathbf{G}} \to \operatorname{Spec} A^{\mathbf{G}}/\mathfrak{a} \cap A^{\mathbf{G}}$$

is surjective. Thus the set-theoretic image of $\operatorname{Spec} A/\mathfrak{a}$ under the map $\operatorname{Spec} A \to \operatorname{Spec} A^{\mathbf{G}}$ is a closed subset given by $\operatorname{Spec} A^{\mathbf{G}}/\mathfrak{a} \cap A^{\mathbf{G}}$. Hence $q(Z)$ is a closed subset of Y.

Fix now two closed **G**-stable subsets Z_1, Z_2 and assume that $Z_1 \cap Z_2 = \varnothing$. We claim that $q(Z_1) \cap q(Z_2) = \varnothing$. For this we may assume that Z_1, Z_2 are both nonempty. Proposition 7.1 implies that there exists $f \in \Gamma(X, \mathcal{O}_X)^{\mathbf{G}}$ such that $f_{|Z_1} = 1$ and $f_{|Z_2} = 0$. Then f viewed as a function on Y satisfies $f_{|q(Z_1)} = 1$ and $f_{|q(Z_2)} = 0$. Thus $q(Z_1) \cap q(Z_2) = \varnothing$.

This completes the proof that q is a good categorical quotient. Note that by Lemma 7.2.1 q is a uniform good categorical quotient.

If X is of finite type over k, then by [Monygham, 2021, Theorem 3.1] we deduce that Y which is the prime spectrum of $\Gamma(X, \mathcal{O}_X)^{\mathbf{G}}$ is of finite type over k.

If **G** is linearly reductive, then by Lemma 7.2.4 we deduce that good categorical quotients in affine case are stable under base changes along closed immersions. Lemma 7.2.1 shows that they are closed under flat base changes. Since every morphism can be factored as a flat morphism followed by a closed immersion, we derive that q is a universal good categorical quotient.

Finally if Z is a closed subscheme of Y, then the morphism $q^{-1}(Z) \to Z$ induced by q is submersive. Since q is a uniform good categorical quotient and every morphism can be factored as a flat morphism followed by a closed immersion, we deduce that q is universally submersive. $\square$

8. QUOTIENTS DETERMINED BY LINEARIZATION

We start by discussing some preliminary result concerning **G**-linearizations of quasi-coherent sheaves. We assume in this section that **G** is an affine group scheme over k.

Proposition 8.1. *Let $\mathcal{F}$ be a quasi-coherent sheaf on X and let $\tau : a^*\mathcal{F} \to \mathrm{pr}_X^*\mathcal{F}$ be a* **G***-linearization of $\mathcal{F}$. Suppose that X is quasi-compact and semiseparated. Then the morphism*

$$\Gamma(X,\mathcal{F}) \ni s \mapsto \tau(a^*s) \in \Gamma(\mathbf{G} \times_k X, \mathrm{pr}_X^*\mathcal{F}) = k[\mathbf{G}] \otimes_k \Gamma(X,\mathcal{F})$$

is a coaction of **G** *on $\mathcal{F}$.*

Proof. We denote the morphism in the statement by c. Fix $s \in \Gamma(X,\mathcal{F})$. Write

$$c(s) = \sum_{i=1}^{n} a_i \otimes s_i \in k[\mathbf{G}] \otimes_k \Gamma(X,\mathcal{F})$$

Then

$$\left(1_{k[\mathbf{G}]} \otimes_k c\right)(c(s)) = \sum_{i=1}^{n} a_i \otimes c(s_i) = \sum_{i=1}^{n} a_i \otimes \tau(a^*s_i) = \left(\mathrm{pr}_{23}^*\tau\right)\left(\sum_{i=1}^{n} a_i \otimes a^*s_i\right) =$$

$$= \left(\mathrm{pr}_{23}^*\tau\right)\left(\sum_{i=1}^{n}(1_{\mathbf{G}} \times_k a)^*(a_i \otimes s_i)\right) = \left(\mathrm{pr}_{23}^*\tau \cdot (1_{\mathbf{G}} \times_k a)^*\tau\right)\left((1_{\mathbf{G}} \times_k a)^* a^*s\right) =$$

$$= \left((\mu \times_k 1_X)^*\tau\right)\left((\mu \times_k 1_X)^* a^*s\right) = (\mu \times_k 1_X)^*(\tau(a^*s)) = \left(\Delta \otimes_k 1_{\Gamma(X,\mathcal{F})}\right)(c(s))$$

where $\Delta : k[\mathbf{G}] \to k[\mathbf{G}] \otimes_k k[\mathbf{G}]$ is the comultiplication. Moreover, we also have

$$\left(\xi \otimes_k 1_{\Gamma(X,\mathcal{F})}\right)(c(s)) = (e \times_k 1_X)^*(\tau(a^*s)) = \left((e \times_k 1_X)^*\tau\right)\left((e \times_k 1_X)^* a^*s\right) =$$

$$= \left((e \times_k 1_X)^* a^*s\right) = 1 \otimes s$$

where $\xi : k[\mathbf{G}] \to k$ is the counit. These imply that c is the coaction of $k[\mathbf{G}]$ on the space of global sections of $\mathcal{F}$. $\qquad\qquad\square$

Definition 8.2. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and let $\tau : a^*\mathcal{F} \to \mathrm{pr}_X^*\mathcal{F}$ be a **G**-linearization of $\mathcal{F}$. Suppose that X is quasi-compact and semiseparated. Then Proposition 8.1 shows that $\Gamma(X,\mathcal{F})$ is a linear representation of **G**. We call it *the linear representation induced by* **G***-linearization τ.*

Now we study properties of the linear representation induced by **G**-linearization in case of a line bundle.

Proposition 8.3. *Suppose that X is quasi-compact and semiseparated. Let $\mathcal{L}$ be a line bundle on X and let $\tau : a^*\mathcal{L} \to \mathrm{pr}_X^*\mathcal{L}$ be a* **G***-linearization of $\mathcal{L}$. Then the following assertions hold.*

(1) *If $s \in \Gamma(X,\mathcal{L})$ is* **G***-invariant with respect to the structure of linear representation of* **G** *induced by τ, then the open subscheme*

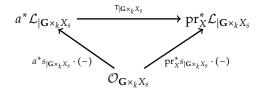$$X_s = \left\{x \in X \,\middle|\, s(x) \neq 0\right\}$$

of X is **G***-stable.*

(2) *If $t, s \in \Gamma(X,\mathcal{L})$ are* **G***-invariant with respect to the structure of linear representation of* **G** *induced by τ, then the regular function $\frac{t}{s} \in \Gamma(X_s, \mathcal{O}_X)$ is* **G***-invariant.*

Proof. Suppose that $s \in \Gamma(X,\mathcal{L})$ is **G**-invariant with respect to the structure of linear representation of **G** induced by τ. Then $\tau(a^*s) = \mathrm{pr}_X^*s$. Since τ is an isomorphism of line bundles on $\mathbf{G} \times_k X$, nonvanishing sets of $a^*s \in \Gamma(\mathbf{G} \times_k X, a^*\mathcal{L})$ and $\mathrm{pr}_X^*s \in \Gamma(\mathbf{G} \times_k X, \mathrm{pr}_X^*\mathcal{L})$ coincide. Next the nonvanishing set of a^*s is $a^{-1}(X_s)$. On the other hand the nonvanishing set of pr_X^*s is $\mathrm{pr}_X^{-1}(X_s)$. Therefore, $a^{-1}(X_s) = \mathrm{pr}_X^{-1}(X_s)$ and hence X_s is open **G**-stable subscheme of X. This completes the proof of **(1)**.

Suppose that $t, s \in \Gamma(X,\mathcal{L})$ are **G**-invariant. Clearly $(\mathcal{O}_X)_{|X_s} \to \mathcal{L}_{|X_s}$ given by multiplication by s is an isomorphism. Recall that $\frac{t}{s}$ is a unique element $r \in \Gamma(X_s, \mathcal{O}_X)$ such that $r \cdot s_{|X_s} = t_{|X_s}$. Since X_s is **G**-invariant, r is **G**-invariant if

$$a^*r = \mathrm{pr}_X^*r$$

Since s is **G**-invariant, we have a commutative triangle

$$a^*\mathcal{L}_{|\mathbf{G}\times_k X_s} \xrightarrow{\ \tau_{|\mathbf{G}\times_k X_s}\ } \mathrm{pr}_X^*\mathcal{L}_{|\mathbf{G}\times_k X_s}$$

$$a^*s_{|\mathbf{G}\times_k X_s}\cdot(-) \qquad\qquad \mathrm{pr}_X^*s_{|\mathbf{G}\times_k X_s}\cdot(-)$$

$$\mathcal{O}_{\mathbf{G}\times_k X_s}$$

in which all morphisms are isomorphisms. By **G**-invariance of s and t we have

$$\mathrm{pr}_X^*s_{|\mathbf{G}\times_k X_s}\cdot a^*r = \tau\left(a^*s_{|\mathbf{G}\times_k X_s}\right)\cdot a^*r = \tau\left(a^*s_{|\mathbf{G}\times_k X_s}\cdot a^*r\right) =$$

$$= \tau(a^*t_{|\mathbf{G}\times_k X_s}) = \mathrm{pr}_X^*t_{|\mathbf{G}\times_k X_s} = \mathrm{pr}_X^*s_{|\mathbf{G}\times_k X_s}\cdot\mathrm{pr}_X^*r$$

and hence $a^*r = \mathrm{pr}_X^*r$. This finishes the proof of **(2)**.   □

The following notion introduced by Mumford in [Mumford et al., 1994] is fundamental.

Definition 8.4. Let $\mathcal{L}$ be a line bundle on X and let $\tau : a^*\mathcal{L} \to \mathrm{pr}_X^*\mathcal{L}$ be a **G**-linearization of $\mathcal{L}$. Consider a point x in X. Then we say that:

(1) x *is semistable with respect to* τ *if there exists a* **G**-*invariant section* $s \in \Gamma(X,\mathcal{L}^{\otimes n})$ *with respect to* $\tau^{\otimes n}$ *for some* n *such that* X_s *is affine and contains* x.

(2) x *is stable with respect to* τ *if there exists a* **G**-*invariant section* $s \in \Gamma(X,\mathcal{L}^{\otimes n})$ *with respect to* $\tau^{\otimes n}$ *for some* n *such that* X_s *is affine, contains* x *and the action of* **G** *on* X_s *has closed orbits for geometric points.*

We also denote by

$$X^{ss}(\tau),\ X^s(\tau)$$

sets of semistable and stable points of X with respect to τ, respectively.

Theorem 8.5. *Suppose that* **G** *is geometrically reductive and* X *is of finite type over* k. *Let* $\mathcal{L}$ *be a line bundle on* X *which admits a* **G**-*linearization* $\tau : a^*\mathcal{L} \to \mathrm{pr}_X^*\mathcal{L}$. *Then there exists a uniform good categorical quotient* $q : X^{ss}(\tau) \to Y$ *of* $X^{ss}(\tau)$ *by* **G**. *Moreover, the following assertions hold.*

(1) *q is affine and universally submersive.*

(2) *There exists an ample line bundle* $\mathcal{M}$ *on* Y *such that* $q^*\mathcal{M} = \mathcal{L}^{\otimes n}$ *for some* n.

(3) *There exists an open subscheme* $\tilde{Y}$ *of* Y *such that* $q^{-1}(\tilde{Y}) = X^s(\tau)$ *and the morphism* $X^s(\tau) \to \tilde{Y}$ *induced by* q *is a uniform geometric quotient of* $X^s(\tau)$ *by* **G**.

Definition 8.6. Let K be a field extension of k and suppose that $\overline{x}$ is a K-point of X. We consider $\overline{x}$ as a morphism $\mathrm{Spec}\,K \to X$. Then the morphism

$$\mathbf{G}\times_k\mathrm{Spec}\,K \xhookrightarrow{\ 1_{\mathbf{G}}\times_k\overline{x}\ } \mathbf{G}\times_k X \xrightarrow{\ a\ } X$$

is called *the orbit morphism of* $\overline{x}$.

Proposition 8.7. *Let* Y *be a* k-*scheme with the trivial action of* **G** *and let* $q : X \to Y$ *be a* **G**-*equivariant morphism. Then the following assertions are equivalent.*

(i) *Fix a point* y *in* Y. *Consider a geometric point* $\overline{y}$ *such that* $k(\overline{y})$ *is algebraically closed field extension of* $k(y)$. *If* $k(\overline{y})$ *has sufficiently large trancendence degree over* k, *then the morphism* $\mathbf{G}\times_k\mathrm{Spec}\,k(\overline{y}) \to X_{\overline{y}}$ *induced by the orbit morphism of*

(ii) *The morphism* $\mathbf{G}\times_k X \to X\times_Y X$ *induced by* $\langle a,\mathrm{pr}_x\rangle : \mathbf{G}\times_k X \to X\times_k X$ *is surjective.*

Proof. We start by proving the implication **(i)** $\Rightarrow$ **(ii)**. Assume that **(i)** holds. Consider a point z in $X \times_Y X$. Let y be a point of Y such that $q\left(\mathrm{pr}_X(z)\right) = y = q\left(a(z)\right)$. Fix an algebraically closed extension K of $k(z)$. Then there exists a K-point $\bar{z}$ of $X \times_Y X$ with z as the underlying point. Suppose that $\bar{x} = \mathrm{pr}_X(\bar{z})$. By possibly enlarging K we may assume according to **(i)** that the set-theoretic image of the orbit morphism $o_{\bar{x}} : \mathbf{G} \times_k \mathrm{Spec}\, K \to X$ of $\bar{x}$ equals to $q^{-1}(y)$. In particular, $\bar{u} = a(\bar{z})$ is contained in the set-theoretic image of $o_{\bar{x}}$. $\square$

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