

GEOMETRY OF k -FUNCTORS

1. INTRODUCTION

In these notes we provide functorial approach to algebraic geometry. Our aim is to show that functorial and geometrical techniques are interrelated in a very efficient way.

Throughout these notes k is a fixed commutative ring and \mathbf{Alg}_k denote the category of commutative k -algebras. If A, B are k -algebras, then we denote by $\mathrm{Mor}_k(A, B)$ the set of all morphisms $A \rightarrow B$ of k -algebras. Similarly if X, Y are k -schemes (i.e. schemes together with morphism to $\mathrm{Spec} k$), then we denote by $\mathrm{Mor}_k(X, Y)$ the set of all morphisms $X \rightarrow Y$ of k -schemes (morphisms of schemes that preserve structure morphisms to $\mathrm{Spec} k$).

2. k -FUNCTORS

Definition 2.1. The category $\mathbf{Fun}(\mathbf{Alg}_k, \mathbf{Set})$ of copresheaves on \mathbf{Alg}_k is called *the category of k -functors*.

If \mathfrak{X} and \mathfrak{Y} are k -functors, then we denote by $\mathrm{Mor}_k(\mathfrak{X}, \mathfrak{Y})$ the class of morphisms $\mathfrak{X} \rightarrow \mathfrak{Y}$ of k -functors. If $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a morphism of k -functors, then for every k -algebra A we denote by σ^A the corresponding component of σ .

Let \mathfrak{X} and \mathfrak{Y} be A -functors for some k -algebra A . Then we denote by $\mathrm{Mor}_A(\mathfrak{X}, \mathfrak{Y})$ the class of morphisms of A -functors $\mathfrak{X} \rightarrow \mathfrak{Y}$. For every A -algebra B and a morphism $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$ of A -functors we denote by $\mathfrak{X}_B, \mathfrak{Y}_B, \sigma_B$ the restrictions $\mathfrak{X}|_{\mathbf{Alg}_B}, \mathfrak{Y}|_{\mathbf{Alg}_B}, \sigma|_{\mathbf{Alg}_B}$ of these entities to the category of B -algebras.

Fact 2.2. Let \mathfrak{X} and \mathfrak{Y} be k -functors. Assume that A is a k -algebra, B is an A -algebra, C is an B -algebra. Then the composition of maps of classes

$$\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A) \xrightarrow{\sigma \mapsto \sigma_B} \mathrm{Mor}_B(\mathfrak{X}_B, \mathfrak{Y}_B) \xrightarrow{\sigma \mapsto \sigma_C} \mathrm{Mor}_C(\mathfrak{X}_C, \mathfrak{Y}_C)$$

equals

$$\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A) \xrightarrow{\sigma \mapsto \sigma_C} \mathrm{Mor}_C(\mathfrak{X}_C, \mathfrak{Y}_C)$$

Proof. Left to the reader. □

Definition 2.3. Let \mathfrak{X} and \mathfrak{Y} be k -functors and suppose that for every k -algebra A the class $\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set. We define

$$\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(A) = \mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$$

for every k -algebra A . This is a k -functor. Indeed, for every k -algebra A and A -algebra B we can compose a morphism $\sigma : \mathfrak{X}_A \rightarrow \mathfrak{Y}_A$ of k -functors with the forgetful functor $\mathbf{Alg}_B \rightarrow \mathbf{Alg}_A$. This induces a map

$$\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(A) \ni \sigma \mapsto \sigma_B \in \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(B)$$

and according to Fact 2.2 these maps make $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})$ a k -functor. The k -functor $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})$ is called *a hom k -functor of \mathfrak{X} and \mathfrak{Y}* .

3. ZARISKI LOCAL k -FUNCTORS AND ZARISKI SHEAVES

In this part we use the notion of a Grothendieck topology on a category. For this notion we refer the reader to [Mon19b].

Definition 3.1. Let $\{f_i : X_i \rightarrow X\}_{i \in I}$ be a family of morphisms of k -schemes. We say that $\{f_i\}_{i \in I}$ is a *Zariski covering* of X if the following conditions are satisfied.

- (1) For every $i \in I$ morphism f_i is an open immersion of schemes.
- (2) Morphism $\coprod_{i \in I} X_i \rightarrow X$ induced by $\{f_i\}_{i \in I}$ is surjective.

The collection of all Zariski coverings on \mathbf{Sch}_k is a Grothendieck pretopology.

Definition 3.2. We call the Grothendieck topology generated by the pretopology consisting of Zariski coverings on \mathbf{Sch}_k the *Zariski topology* on \mathbf{Sch}_k . A presheaf on \mathbf{Sch}_k that is a sheaf with respect to Zariski topology on \mathbf{Sch}_k is called a *Zariski sheaf*.

Let \mathfrak{X} be a presheaf on the category of k -schemes. Recall that by [Mon19b, Theorem 3.5] \mathfrak{X} is a Zariski sheaf if and only if for every k -scheme X and for every Zariski covering $\{f_i : X_i \rightarrow X\}$ of X the diagram

$$\mathfrak{X}(X) \xrightarrow{\langle \mathfrak{X}(f_i) \rangle_{i \in I}} \prod_{i \in I} \mathfrak{X}(X_i) \xrightleftharpoons[\langle \mathfrak{X}(f'_{ij}) \cdot pr_j \rangle_{(i,j)}]{\langle \mathfrak{X}(f''_{ij}) \cdot pr_i \rangle_{(i,j)}} \prod_{(i,j) \in I \times I} \mathfrak{X}(X_i \times_X X_j)$$

is a kernel of a pair of arrows, where for every $(i, j) \in I \times I$ morphisms f'_{ij} and f''_{ij} form a cartesian square

$$\begin{array}{ccc} X_i \times_X X_j & \xrightarrow{f''_{ij}} & X_j \\ f'_{ij} \downarrow & & \downarrow f_j \\ X_i & \xrightarrow{f_i} & X \end{array}$$

Now we repeat this definitions for k -algebras and k -functors.

Definition 3.3. Let $\{f_i : A \rightarrow A_i\}_{i \in I}$ be a family of morphisms of k -algebras. We say that $\{f_i\}_{i \in I}$ is a *Zariski covering* of A if the following conditions are satisfied.

- (1) For every $i \in I$ morphism $\mathrm{Spec} f_i$ is an open immersion of schemes.
- (2) Morphism $\coprod_{i \in I} \mathrm{Spec} A_i \rightarrow \mathrm{Spec} A$ induced by $\{\mathrm{Spec} f_i\}_{i \in I}$ is surjective.

The collection of all Zariski coverings on \mathbf{Alg}_k induces on its opposite category \mathbf{Aff}_k of affine k -schemes a Grothendieck pretopology.

Definition 3.4. We call the Grothendieck topology generated by the pretopology consisting of Zariski coverings on \mathbf{Aff}_k the *Zariski topology* on \mathbf{Aff}_k . A k -functor that is a sheaf with respect to Zariski topology on \mathbf{Aff}_k is called a *Zariski local k -functor*.

Let \mathfrak{X} be a k -functor. Again by [Mon19b, Theorem 3.5] \mathfrak{X} is a Zariski local k -functor if and only if for every k -algebra A and for every Zariski covering $\{f_i : A \rightarrow A_i\}$ of A the diagram

$$\mathfrak{X}(A) \xrightarrow{\langle \mathfrak{X}(f_i) \rangle_{i \in I}} \prod_{i \in I} \mathfrak{X}(A_i) \xrightleftharpoons[\langle \mathfrak{X}(f'_{ij}) \cdot pr_j \rangle_{(i,j)}]{\langle \mathfrak{X}(f''_{ij}) \cdot pr_i \rangle_{(i,j)}} \prod_{(i,j) \in I \times I} \mathfrak{X}(A_i \otimes_A A_j)$$

is a kernel of a pair of arrows, where for every $(i, j) \in I \times I$ morphisms f'_{ij} and f''_{ij} form a cocartesian square

$$\begin{array}{ccc}
 A & \xrightarrow{f_j} & A_j \\
 f_i \downarrow & & \downarrow f'_{ji} \\
 A_i & \xrightarrow{f'_{ij}} & A_i \otimes_A A_j
 \end{array}$$

Now we state the main result of this section.

Theorem 3.5. *Let*

$$\widehat{\mathbf{Sch}}_k \longrightarrow \text{the category of } k\text{-functors}$$

be the restriction of presheaves on \mathbf{Sch}_k to copresheaves on \mathbf{Alg}_k (k -functors) induced by the contravariant functor $\text{Spec} : \mathbf{Alg}_k \rightarrow \mathbf{Sch}_k$. Then it induces an equivalence of categories between Zariski sheaves on \mathbf{Sch}_k and Zariski local k -functors.

Proof. Note that \mathbf{Aff}_k with Zariski topology is a dense subsite ([Mon19b, definition 4.4]) of \mathbf{Sch}_k with Zariski topology. Hence the result is a special case of a more general theorem [Mon19b, Theorem 4.6]. \square

Proposition 3.6. *Let $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a monomorphism of k -functors and \mathfrak{Y} be a Zariski local k -functor. Assume that for every k -algebra A and every morphism $\tau : \mathfrak{P}_{\text{Spec } A} \rightarrow \mathfrak{Y}$ of k -functors there exist a Zariski local k -functor \mathfrak{Z} that fits into a cartesian square*

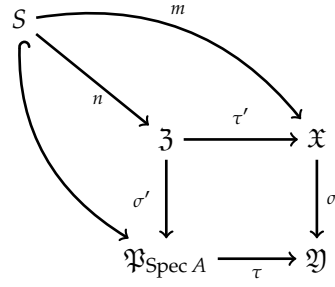
$$\begin{array}{ccc}
 \mathfrak{Z} & \longrightarrow & \mathfrak{X} \\
 \downarrow & & \downarrow \sigma \\
 \mathfrak{P}_{\text{Spec } A} & \xrightarrow{\tau} & \mathfrak{Y}
 \end{array}$$

Then \mathfrak{X} is a Zariski local k -functor.

Proof. Suppose that A is a k -algebra and S is a covering sieve on A with respect to Zariski topology. Recall that by [Mon19b, page 2] we may consider S as a subcopresheaf of $\mathfrak{P}_{\text{Spec } A}$. Suppose that $\tau : \mathfrak{P}_{\text{Spec } A} \rightarrow \mathfrak{Y}$ and $m : S \rightarrow \mathfrak{X}$ are morphisms of k -functors such that $\sigma \cdot m$ is equal to the composition of $S \hookrightarrow \mathfrak{P}_{\text{Spec } A}$ with τ . Next there exists a Zariski local k -functor \mathfrak{Z} that fits into a cartesian square

$$\begin{array}{ccc}
 \mathfrak{Z} & \xrightarrow{\tau'} & \mathfrak{X} \\
 \sigma' \downarrow & & \downarrow \sigma \\
 \mathfrak{P}_{\text{Spec } A} & \xrightarrow{\tau} & \mathfrak{Y}
 \end{array}$$

of k -functors. By universal property of cartesian squares there exists a unique morphism $n : S \rightarrow \mathfrak{Z}$ of k -functors such that the diagram



is commutative. Since \mathfrak{Z} is Zariski local, there exists a morphism $\rho : \mathfrak{P}_{\text{Spec } A} \rightarrow \mathfrak{Z}$ such that $\rho|_S = n$. Then $(\tau' \cdot \rho)|_S = \tau' \cdot n = m$ and hence matching family m admits an amalgamation. Since σ is a monomorphism, this suffices to prove that \mathfrak{X} is a Zariski local k -functor. \square

4. SCHEMES AND THEIR FUNCTORS OF POINTS

Let X be a k -scheme. We define a k -functor \mathfrak{P}_X by formula

$$\mathfrak{P}_X(A) = \text{Mor}_k(\text{Spec } A, X)$$

That is \mathfrak{P}_X is the restriction of the presheaf on \mathbf{Sch}_k represented by X to the category \mathbf{Alg}_k along the functor $\text{Spec} : \mathbf{Alg}_k \rightarrow \mathbf{Sch}_k$. Next if $f : X \rightarrow Y$ is a morphism of k -schemes, then \mathfrak{P}_f is the restriction of a morphism of presheaves on \mathbf{Sch}_k represented by f to the category of k -algebras along $\text{Spec} : \mathbf{Alg}_k \rightarrow \mathbf{Sch}_k$. Thus we have a functor

$$\mathbf{Sch}_k \xrightarrow{\mathfrak{P}} \text{the category of } k\text{-functors}$$

Fact 4.1. *Functor*

$$\mathbf{Sch}_k \xrightarrow{\mathfrak{P}} \text{the category of } k\text{-functors}$$

is full, faithful and its image consists of Zariski local k -functors. Moreover, \mathfrak{P} preserves limits.

Proof. Note that the presheaf h_X on \mathbf{Sch}_k represented by X is a Zariski sheaf. Indeed, this just rephrases standard fact that morphism of schemes can be glued in Zariski topology. Next according to Theorem 3.5 the functor $\text{Spec} : \mathbf{Alg}_k \rightarrow \mathbf{Sch}_k$ induces an equivalence between the category of Zariski sheaves and the category of local Zariski k -functors. Thus \mathfrak{P}_X is a local Zariski k -functor and \mathfrak{P} it is full and faithful. Note that Yoneda embedding $h : \mathbf{Sch}_k \rightarrow \widehat{\mathbf{Sch}_k}$ and the functor

$$\widehat{\mathbf{Sch}_k} \xrightarrow{\text{induced by Spec}} \text{the category of } k\text{-functors}$$

preserve limits. Thus their composition \mathfrak{P} also preserves limits. \square

Definition 4.2. Let X be a k -scheme. Then \mathfrak{P}_X is called *the k -functor of points of X* .

Finally note that for every k -algebra A we have an identification $\mathfrak{P}_{\text{Spec } A} = \text{Hom}_k(A, -)$ and this identification is natural with respect to A . In other words $\mathfrak{P} \cdot \text{Spec}$ is the (co)Yoneda embedding of \mathbf{Alg}_k into the category of k -functors.

Suppose now that A is a k -algebra and $\mathfrak{a} \subseteq A$ is an ideal. Then we define $V(\mathfrak{a}) = \text{Spec } A/\mathfrak{a}$ as a closed subscheme $\text{Spec } A$ induced by the quotient morphism $A \rightarrow A/\mathfrak{a}$. We define an open subscheme $D(\mathfrak{a}) = \text{Spec } A \setminus V(\mathfrak{a})$ of $\text{Spec } A$.

Definition 4.3. Let $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of k -functors. Assume that for every k -algebra A and every morphism $\tau : \mathfrak{P}_{\text{Spec } A} \rightarrow \mathfrak{Y}$ of k -functors there exist an ideal \mathfrak{a} in A and a morphism $\tau' : \mathfrak{P}_{D(\mathfrak{a})} \rightarrow \mathfrak{X}$ of k -functors such that the square

$$\begin{array}{ccc} \mathfrak{P}_{D(\mathfrak{a})} & \xrightarrow{\tau'} & \mathfrak{X} \\ \downarrow & & \downarrow \sigma \\ \mathfrak{P}_{\text{Spec } A} & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

is cartesian. Then σ is an open immersion of k -functors.

Fact 4.4. The class of open immersions of k -functors is closed under base change and composition.

Proof. Left to the reader. □

Definition 4.5. Let \mathfrak{X} be a k -functor and $\{\sigma_i : \mathfrak{X}_i \rightarrow \mathfrak{X}\}_{i \in I}$ be a family of open immersions. Then for every k -algebra A and $x \in \mathfrak{X}(A)$ we have a family of ideals $\{\mathfrak{a}_i\}_{i \in I}$ defined by cartesian squares

$$\begin{array}{ccc} \mathfrak{P}_{D(\mathfrak{a}_i)} & \xrightarrow{\tau'} & \mathfrak{X}_i \\ \downarrow & & \downarrow \sigma_i \\ \mathfrak{P}_{\text{Spec } A} & \xrightarrow{\tau} & \mathfrak{X} \end{array}$$

in which bottom vertical morphism $\tau : \mathfrak{P}_{\text{Spec } A} \rightarrow \mathfrak{X}$ corresponds to x . We say that $\{\sigma_i\}_{i \in I}$ is an open cover of \mathfrak{X} if for every k -algebra A and $x \in \mathfrak{X}(A)$ we have

$$\text{Spec } A = \bigcup_{i \in I} D(\mathfrak{a}_i)$$

or in other words $A = \sum_{i \in I} \mathfrak{a}_i$.

Theorem 4.6. Let \mathfrak{X} be a k -functor. Then the following are equivalent.

- (i) \mathfrak{X} is isomorphic with functor of points of some k -scheme.
- (ii) \mathfrak{X} is a Zariski local k -functor and there exists an open cover $\{\sigma_i : \mathfrak{P}_{X_i} \rightarrow \mathfrak{X}\}_{i \in I}$ of k -functors for some family $\{X_i\}_{i \in I}$ of k -schemes.
- (iii) \mathfrak{X} is a Zariski local k -functor and there exists an open cover $\{\sigma_i : \mathfrak{P}_{\text{Spec } A_i} \rightarrow \mathfrak{X}\}_{i \in I}$ of k -functors for some family $\{A_i\}_{i \in I}$ of k -algebras.

The proof depends on two lemmas. Check [Mon19b, Definition 7.1] for the notion of a locally surjective morphism.

Lemma 4.6.1. Let $f : X \rightarrow Y$ be a morphism of k -schemes. Suppose that f is surjective morphism and an open immersion locally on X . Then \mathfrak{P}_f is a locally surjective morphism of Zariski local k -functors.

Proof of the lemma. Let A be a k -algebra and $g : \text{Spec } A \rightarrow Y$ be a morphism of k -schemes. Since f is surjective and an open immersion locally on X , there exist a Zariski cover $\{f_i : A \rightarrow A_i\}_{i \in I}$ and a family $\{g_i : \text{Spec } A_i \rightarrow X\}_{i \in I}$ of morphisms of k -schemes such that $f \cdot g_i = g \cdot \text{Spec } f_i$ for every $i \in I$. This implies that $\mathfrak{P}_f(g_i) = \mathfrak{P}_Y(f_i)(g)$ for every $i \in I$. Thus \mathfrak{P}_f is a locally surjective morphism of Zariski local k -functors. □

Lemma 4.6.2. Let $X = \coprod_{i \in I} X_i$, $R = \coprod_{i, j \in I} R_{ij}$ be disjoint sums of k -schemes and let $p, q : R \rightarrow X$ be morphisms of k -schemes such that the following conditions are satisfied.

- (1) For any $i, j \in I$ morphism $p|_{R_{ij}}$ induces an open immersion $R_{ij} \hookrightarrow X_i$ and morphism $q|_{R_{ij}}$ induces an open immersion $R_{ij} \hookrightarrow X_j$.
- (2) For every $i \in I$ morphisms $p|_{R_{ii}}$ and $q|_{R_{ii}}$ are equal and induce an isomorphisms $R_{ii} \rightarrow X_i$.
- (3) Triple (R, p, q) is an equivalence relation on X in the category of k -schemes.

Then there exist a k -scheme Y and a morphism $f : X \rightarrow Y$ of k -schemes such that

$$\mathfrak{P}_R \begin{array}{c} \xrightarrow{\mathfrak{P}_p} \\ \xrightarrow{\mathfrak{P}_q} \end{array} \mathfrak{P}_X \xrightarrow{\mathfrak{P}_f} \mathfrak{P}_Y$$

is a cokernel of a pair $(\mathfrak{P}_p, \mathfrak{P}_q)$ in the category of Zariski local k -functors.

Proof of the lemma. Let

$$R \begin{array}{c} \xrightarrow{p} \\ \xrightarrow{q} \end{array} X \xrightarrow{f} Y$$

be a cokernel in the category of ringed spaces. It exists according to [Mon19c, Remark 2.3]. Moreover, [Mon19c, Theorem 3.2] states that for every $i \in I$ subset $f(X_i)$ is open in Y and we have an isomorphism of ringed spaces $X_i \cong f(X_i)$ induced by f . Therefore, Y is a k -scheme and $f : X \rightarrow Y$ is a morphism of k -schemes.

Now we verify that \mathfrak{P}_f is the quotient in the category of Zariski local k -functors. For this note that we proved above that f is open immersion of k -schemes locally on X and it is surjective. Thus by Lemma 4.6.1 we derive that \mathfrak{P}_f is a locally surjective morphism of Zariski local k -functors. Therefore ([Mon19b, Theorem 7.3]), it suffices to show that the square

$$\begin{array}{ccc} \mathfrak{P}_R & \xrightarrow{\mathfrak{P}_q} & \mathfrak{P}_X \\ \mathfrak{P}_p \downarrow & & \downarrow \mathfrak{P}_f \\ \mathfrak{P}_X & \xrightarrow{\mathfrak{P}_f} & \mathfrak{P}_Y \end{array}$$

is cartesian. Since \mathfrak{P} preserves limits (Fact 4.1), we derive that it suffices to check that

$$\begin{array}{ccc} R & \xrightarrow{q} & X \\ p \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

is cartesian square of k -schemes. By [Mon19c, Remark 2.3] we have $R_{ij} = X_i \times_Y X_j$ for every $i, j \in I$ and hence

$$X \times_Y X = \left(\coprod_{i \in I} X_i \right) \times_Y \left(\coprod_{i \in I} X_i \right) = \coprod_{i, j \in I} (X_i \times_Y X_j) = \coprod_{i, j \in I} R_{ij} = R$$

Thus the result follows. \square

Proof of the theorem. If (i) holds, then we may assume that $\mathfrak{X} = \mathfrak{P}_Y$ for some k -scheme Y . Fact 4.1 states that \mathfrak{P}_Y is a Zariski local k -functor and clearly $1_{\mathfrak{P}_Y} : \mathfrak{P}_Y \rightarrow \mathfrak{P}_Y$ is an open cover. Thus (i) \Rightarrow (ii).

Every functor of points of a k -scheme admits open cover by functors of points of affine k -schemes. Indeed, it suffices to take open affine subschemes that cover given k -scheme and apply \mathfrak{P} . This implies that every open cover of a k -functor \mathfrak{X} by functors of points of k -schemes admits refinement by open cover of functors of points of affine k -schemes. Therefore, implication (ii) \Rightarrow (iii) holds.

Suppose that a k -functor \mathfrak{X} is Zariski local and $\{\sigma_i : \mathfrak{P}_{\text{Spec } A_i} \rightarrow \mathfrak{X}\}_{i \in I}$ is an open cover of \mathfrak{X} . Note that for every $i, j \in I$ there exist a k -scheme R_{ij} and open immersions $p_{ij} : R_{ij} \hookrightarrow \text{Spec } A_i$, $q_{ij} : R_{ij} \hookrightarrow \text{Spec } A_j$ such that the square

$$\begin{array}{ccc} \mathfrak{P}_{R_{ij}} & \xrightarrow{\mathfrak{P}_{q_{ij}}} & \mathfrak{P}_{\text{Spec } A_j} \\ \mathfrak{P}_{p_{ij}} \downarrow & & \downarrow \sigma_j \\ \mathfrak{P}_{\text{Spec } A_i} & \xrightarrow{\sigma_i} & \mathfrak{X} \end{array}$$

is cartesian. Consider k -scheme $X = \coprod_{i \in I} \text{Spec } A_i$ and morphism $\sigma : \mathfrak{P}_X \rightarrow \mathfrak{X}$ induced by $\{\sigma_i\}_{i \in I}$. Moreover, consider k -scheme $R = \coprod_{i, j \in I} R_{ij}$ and morphisms $p, q : R \rightarrow X$ induced by $\{p_{ij}\}_{i, j \in I}$ and $\{q_{ij}\}_{i, j \in I}$, respectively. Note that the square

$$\begin{array}{ccc} \mathfrak{P}_R & \xrightarrow{\mathfrak{P}_q} & \mathfrak{P}_X \\ \mathfrak{P}_p \downarrow & & \downarrow \sigma \\ \mathfrak{P}_X & \xrightarrow{\sigma} & \mathfrak{X} \end{array}$$

is cartesian and hence $(\mathfrak{P}_R, \mathfrak{P}_p, \mathfrak{P}_q)$ is an equivalence relation. By Lemma 4.6.2 there exist a k -scheme Y and a morphism $f : X \rightarrow Y$ such that

$$\mathfrak{P}_R \xrightarrow[\mathfrak{P}_q]{\mathfrak{P}_p} \mathfrak{P}_X \xrightarrow{\mathfrak{P}_f} \mathfrak{P}_Y$$

is a cokernel of $(\mathfrak{P}_p, \mathfrak{P}_q)$. Moreover, σ is locally surjective morphism of Zariski local k -functors and hence also

$$\mathfrak{P}_R \xrightarrow[\mathfrak{P}_q]{\mathfrak{P}_p} \mathfrak{P}_X \xrightarrow{\sigma} \mathfrak{X}$$

is a cokernel of $(\mathfrak{P}_p, \mathfrak{P}_q)$. Thus \mathfrak{P}_Y is isomorphic with \mathfrak{X} . This proves (iii) \Rightarrow (i). \square

5. REPRESENTABLE MORPHISMS OF k -FUNCTORS

Definition 5.1. Let $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of k -functors. Assume that for every k -algebra A and every morphism $\tau : \mathfrak{P}_{\text{Spec } A} \rightarrow \mathfrak{Y}$ of k -functors there exist a k -scheme X , a morphism $f : X \rightarrow \text{Spec } A$ and a morphism $\tau' : \mathfrak{P}_X \rightarrow \mathfrak{X}$ of k -functors such that the square

$$\begin{array}{ccc}
\mathfrak{P}_X & \xrightarrow{\tau'} & \mathfrak{X} \\
\mathfrak{P}_f \downarrow & & \downarrow \sigma \\
\mathfrak{P}_{\mathrm{Spec} A} & \xrightarrow{\tau} & \mathfrak{Y}
\end{array}$$

is cartesian. Then σ is a representable morphism of k -functors.

Fact 5.2. *The class of representable morphisms of k -functors is closed under base change and composition.*

Proof. Left to the reader. □

Proposition 5.3. *Let $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a representable morphism of Zariski local k -functors. Fix a k -scheme Y and a morphism $\tau : \mathfrak{P}_Y \rightarrow \mathfrak{Y}$. Then there exist a k -scheme X , a morphism $f : X \rightarrow Y$ and a morphism $\tau' : \mathfrak{P}_X \rightarrow \mathfrak{X}$ such that the square*

$$\begin{array}{ccc}
\mathfrak{P}_X & \xrightarrow{\tau'} & \mathfrak{X} \\
\mathfrak{P}_f \downarrow & & \downarrow \sigma \\
\mathfrak{P}_Y & \xrightarrow{\tau} & \mathfrak{Y}
\end{array}$$

is cartesian.

Proof. Let

$$\begin{array}{ccc}
\mathfrak{Z} & \xrightarrow{\tau'} & \mathfrak{X} \\
\sigma' \downarrow & & \downarrow \sigma \\
\mathfrak{P}_Y & \xrightarrow{\tau} & \mathfrak{Y}
\end{array}$$

be a cartesian square. According to [Mon19b, Theorem 2.12] k -functor \mathfrak{Z} is Zariski local. Suppose that $\{f_i : \mathrm{Spec} A_i \rightarrow Y\}_{i \in I}$ is an open cover of Y . Then $\{\mathfrak{P}_{f_i} : \mathfrak{P}_{\mathrm{Spec} A_i} \rightarrow \mathfrak{P}_Y\}_{i \in I}$ is an open cover of \mathfrak{P}_Y and hence its base change $\{\tau_i : \mathfrak{Z}_i \rightarrow \mathfrak{Z}\}_{i \in I}$ is an open cover of \mathfrak{Z} . Since σ is representable, we deduce that \mathfrak{Z}_i is a functor of points of some k -scheme for $i \in I$. Now by Theorem 4.6 we derive that there exists a k -scheme X such that \mathfrak{Z} is isomorphic with \mathfrak{P}_X . This proves the result. □

Definition 5.4. Let $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of k -functors. Assume that for every k -algebra A and every morphism $\tau : \mathfrak{P}_{\mathrm{Spec} A} \rightarrow \mathfrak{Y}$ of k -functors there exist an ideal \mathfrak{a} in A and morphism $\tau' : \mathfrak{P}_{V(\mathfrak{a})} \rightarrow \mathfrak{X}$ such that the square

$$\begin{array}{ccc}
\mathfrak{P}_{V(\mathfrak{a})} = \mathfrak{P}_{\mathrm{Spec} A/\mathfrak{a}} & \xrightarrow{\tau'} & \mathfrak{X} \\
\mathfrak{P}_{\mathrm{Spec} q} \downarrow & & \downarrow \sigma \\
\mathfrak{P}_{\mathrm{Spec} A} & \xrightarrow{\tau} & \mathfrak{Y}
\end{array}$$

is cartesian, where $q : A \rightarrow A/\mathfrak{a}$ is the quotient map. Then σ is a closed immersion of k -functors.

Fact 5.5. *The class of closed immersions of k -functors is closed under base change and composition.*

Proof. Left to the reader. □

Proposition 5.6. *Let $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a closed (open) immersion of k -functors. Fix a k -scheme Y and a morphism $\tau : \mathfrak{P}_Y \rightarrow \mathfrak{Y}$. Then there exist a k -scheme X , a closed (open) immersion $f : X \rightarrow Y$ of schemes and a morphism $\tau' : \mathfrak{P}_X \rightarrow \mathfrak{X}$ of k -functors such that the square*

$$\begin{array}{ccc} \mathfrak{P}_X & \xrightarrow{\tau'} & \mathfrak{X} \\ \mathfrak{P}_f \downarrow & & \downarrow \sigma \\ \mathfrak{P}_Y & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

is cartesian.

Proof. According to Fact 5.5 (Fact 4.4) pullback $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{P}_Y \rightarrow \mathfrak{P}_Y$ of σ along τ is a closed (open) immersion of k -functors. Since \mathfrak{P}_Y is a Zariski local k -functor by Fact 4.1 and closed (open) immersions are monomorphisms, we derive by Proposition 3.6 that a fiber-product $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{P}_Y$ of σ and τ is a Zariski local k -functor. Since closed (open) immersions of k -functors are representable, we deduce by Proposition 5.3 that there exists a k -scheme X , a morphism $f : X \rightarrow Y$ of k -schemes and a morphism $\tau' : \mathfrak{P}_X \rightarrow \mathfrak{X}$ of k -functors such that the square

$$\begin{array}{ccc} \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{P}_Y \cong \mathfrak{P}_X & \xrightarrow{\tau'} & \mathfrak{X} \\ \mathfrak{P}_f \downarrow & & \downarrow \sigma \\ \mathfrak{P}_Y & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

is cartesian and \mathfrak{P}_f is a closed (open) immersion of k -functors. Since the functor

$$\widehat{\text{Sch}}_k \xrightarrow{\mathfrak{P}} \text{the category of } k\text{-functors}$$

preserves finite limits, it follows that for every open affine subset V of Y we have a cartesian square

$$\begin{array}{ccc} \mathfrak{P}_{f^{-1}(V)} & \hookrightarrow & \mathfrak{P}_X \\ \mathfrak{P}_{f_V} \downarrow & & \downarrow \mathfrak{P}_f \\ \mathfrak{P}_V & \hookrightarrow & \mathfrak{P}_Y \end{array}$$

where $f_V : f^{-1}(V) \rightarrow V$ is the restriction of f . Next as \mathfrak{P}_f is a closed (open) immersion and V is affine, we derive that f_V is a closed (open) immersion of schemes. Since this holds for every affine open subset V of Y , we deduce that f is a closed (open) immersion. \square

The next result is frequently used in the theory of algebraic spaces.

Proposition 5.7. *Let \mathfrak{Y} be a k -functor such that the diagonal $\mathfrak{Y} \rightarrow \mathfrak{Y} \times \mathfrak{Y}$ is representable. Then every morphism $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$ of k -functors is representable.*

Proof. Fix a morphism of k -functors $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$. Let Y be a k -scheme and let $\tau : \mathfrak{P}_Y \rightarrow \mathfrak{Y}$ be a morphism of k -functors. Consider the cartesian square

$$\begin{array}{ccc}
\mathfrak{Z} & \xrightarrow{\tau'} & \mathfrak{X} \\
\sigma' \downarrow & & \downarrow \sigma \\
\mathfrak{P}_Y & \xrightarrow{\tau} & \mathfrak{Y}
\end{array}$$

Then there exists a cartesian square

$$\begin{array}{ccc}
\mathfrak{Z} & \longrightarrow & \mathfrak{Y} \\
\downarrow & & \downarrow \text{diagonal} \\
\mathfrak{P}_Y \times \mathfrak{Y} & \xrightarrow{\tau \times \sigma} & \mathfrak{Y} \times \mathfrak{Y}
\end{array}$$

Since the diagonal of \mathfrak{Y} is representable, we derive that \mathfrak{Z} is isomorphic with functor of points of some k -scheme. This finishes the proof. \square

6. CLOSED IMMERSIONS AND HOM k -FUNCTORS

Definition 6.1. Let X be a k -scheme. Suppose that there exists an open affine cover $X = \bigcup_{i \in I} X_i$ such that k -algebra $\Gamma(X_i, \mathcal{O}_{X_i})$ is free as a k -module. Then we say that X is a *locally free k -scheme*.

Next theorem is the main result of this section.

Theorem 6.2. Let $j : \mathfrak{Y}' \rightarrow \mathfrak{Y}$ be a closed immersion of k -functors and X be a locally free k -scheme. Suppose that classes $\text{Mor}_A((\mathfrak{P}_X)_A, \mathfrak{Y}_A)$ are sets for every k -algebra A . Then classes $\text{Mor}_A((\mathfrak{P}_X)_A, \mathfrak{Y}'_A)$ are sets for every k -algebra A and the morphism

$$\text{Mor}_k(1_{\mathfrak{P}_X}, j) : \text{Mor}_k(\mathfrak{P}_X, \mathfrak{Y}') \rightarrow \text{Mor}_k(\mathfrak{P}_X, \mathfrak{Y})$$

is a closed immersion of k -functors.

It is useful to isolate crucial steps in the argument. For this we proceed by proving some lemmas.

Lemma 6.2.1. Suppose that A is a commutative ring. Let $j : \mathfrak{Y}' \rightarrow \mathfrak{Y}$ be a closed immersion of A -functors and X be an affine A -scheme such that $\Gamma(X, \mathcal{O}_X)$ is a free A -module. Assume that $\tau : \mathfrak{P}_X \rightarrow \mathfrak{Y}$ is a morphism of A -functors. Then there exists an ideal $\mathfrak{a} \subseteq A$ such that for every A -algebra B the restriction τ_B factors through j_B if and only if the structure morphism $f : A \rightarrow B$ of B satisfies $\mathfrak{a} \subseteq \ker(f)$.

Proof of the lemma. Since j is a closed immersion of A -functors and X is affine k -scheme there exists an affine A -scheme X' , a closed immersion $j' : X' \rightarrow X$ of schemes and a cartesian square

$$\begin{array}{ccc}
\mathfrak{P}_{X'} & \longrightarrow & \mathfrak{Y}' \\
\downarrow \mathfrak{P}_{j'} & & \downarrow j \\
\mathfrak{P}_X & \xrightarrow{\tau} & \mathfrak{Y}
\end{array}$$

of A -functors. Next let B be an A -algebra with the structure morphism $f : A \rightarrow B$. Then τ_B factors through j_B if and only if the projection $\text{Spec } B \times_{\text{Spec } A} X \rightarrow X$ induced by f factors through X' . Let $A[X]$ be the A -algebra of global regular functions on X and let \mathfrak{J} be an ideal in $A[X]$ such that $A[X]/\mathfrak{J} = A[X']$ is the A -algebra of global regular functions of X' . With this notation we derive that the projection $\text{Spec } B \times_{\text{Spec } A} X \rightarrow X$ induced by f factors through X' if and only if the morphism $A[X] \rightarrow B \otimes_A A[X]$ induced by f sends every element of \mathfrak{J} to zero. Since $A[X]$ is a free

A -module, we write $A[X] = A^{\oplus I}$ for some index set I . Then the morphism $A[X] \rightarrow B \otimes_A A[X]$ induced by f is just $f^{\oplus I} : A^{\oplus I} \rightarrow B^{\oplus I}$. We have $f^{\oplus I}(\mathfrak{J}) = 0$ if and only if $(pr_i^B \cdot f^{\oplus I})(\mathfrak{J}) = 0$ for every $i \in I$, where $pr_i^B : B^{\oplus I} \rightarrow B$ is the projection on i -th component. Pick $i \in I$ and consider the commutative diagram

$$\begin{array}{ccc} A^{\oplus I} & \xrightarrow{f^{\oplus I}} & B^{\oplus I} \\ pr_i^A \downarrow & & \downarrow pr_i^B \\ A & \xrightarrow{f} & B \end{array}$$

In the diagram pr_i^A is the projection on i -th component. Diagram implies that $(pr_i^B \cdot f^{\oplus I})(\mathfrak{J}) = 0$ for every $i \in I$ if and only if $(f \cdot pr_i^A)(\mathfrak{J}) = 0$ for every $i \in I$. This is equivalent with the condition that $f(\mathfrak{a}) = 0$ for ideal \mathfrak{a} in A generated by $\sum_{i \in I} pr_i^A(\mathfrak{J})$. Thus the lemma is proved. \square

Lemma 6.2.2. *Suppose that A is a commutative ring. Let $j : \mathfrak{Y}' \rightarrow \mathfrak{Y}$ be a closed immersion of A -functors and X be an A -scheme with open cover*

$$X = \bigcup_{i \in I} X_i$$

Assume that $\tau : \mathfrak{P}_X \rightarrow \mathfrak{Y}$ is a morphism of A -functors. Fix an A -algebra B . Then τ_B factors through j_B if and only if $(\tau|_{\mathfrak{P}_{X_i}})_B$ factors through j_B for every $i \in I$.

Proof of the lemma. If τ_B factors through j_B , then also $(\tau|_{\mathfrak{P}_{X_i}})_B$ factors through j_B for every $i \in I$. It suffices to prove the converse. So suppose that $(\tau|_{\mathfrak{P}_{X_i}})_B$ factors through j_B for every $i \in I$. Since j is a closed immersion of A -functors and X is an A -scheme, Proposition 5.6 implies that there exists a cartesian square

$$\begin{array}{ccc} \mathfrak{P}_{X'} & \longrightarrow & \mathfrak{Y}' \\ \mathfrak{P}_{j'} \downarrow & & \downarrow j \\ \mathfrak{P}_X & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

where $j' : X' \rightarrow X$ is a closed immersion of A -schemes. For each $i \in I$ let $j'_i : j'^{-1}(X_i) \rightarrow X_i$ be the restriction of j' . We have the induced cartesian square

$$\begin{array}{ccc} \mathfrak{P}_{j'^{-1}(X_i)} & \longrightarrow & \mathfrak{Y}' \\ \mathfrak{P}_{j'_i} \downarrow & & \downarrow j \\ \mathfrak{P}_{X_i} & \xrightarrow{\tau|_{\mathfrak{P}_{X_i}}} & \mathfrak{Y} \end{array}$$

Now $(\tau|_{\mathfrak{P}_{X_i}})_B$ factors through j_B . This implies that $(\mathfrak{P}_{j'_i})_B$ admits a section for every $i \in I$. Then $(\mathfrak{P}_{j'_i})_B$ is an isomorphism for every $i \in I$. Thus $j'_i \times_{\text{Spec } A} 1_{\text{Spec } B}$ is an isomorphism for every $i \in I$ and hence $j' \times_{\text{Spec } A} 1_{\text{Spec } B}$ is an isomorphism of B -schemes. This means that τ_B factors through j_B . \square

Proof of the theorem. Let A be a k -algebra. The restriction functor $(-)|_{\mathbf{Alg}_A} = (-)_A$ preserves all closed immersions. Thus j_A is a closed immersion of A -functors and hence we derive that $j_A : \mathfrak{Y}'_A \rightarrow \mathfrak{Y}_A$ is a monomorphism of A -functors. Thus we have an injective map of classes

$$\mathrm{Mor}_A(1_{(\mathfrak{P}_X)_A}, j_A) : \mathrm{Mor}_A((\mathfrak{P}_X)_A, \mathfrak{Y}'_A) \hookrightarrow \mathrm{Mor}_A((\mathfrak{P}_X)_A, \mathfrak{Y}_A)$$

Hence if $\mathrm{Mor}_A((\mathfrak{P}_X)_A, \mathfrak{Y}_A)$ is a set, then $\mathrm{Mor}_A((\mathfrak{P}_X)_A, \mathfrak{Y}'_A)$ is a set. All these facts imply that both internal homs

$$\mathrm{Mor}_k(\mathfrak{P}_X, \mathfrak{Y}'), \mathrm{Mor}_k(\mathfrak{P}_X, \mathfrak{Y})$$

exist and morphism $\mathrm{Mor}_k(1_{\mathfrak{P}_X}, j)$ of k -functors is a monomorphism. Our task is to prove that it is a closed immersion. For this consider a k -algebra A and a morphism $\sigma : \mathfrak{P}_{\mathrm{Spec} A} \rightarrow \mathrm{Mor}_k(\mathfrak{P}_X, \mathfrak{Y})$ of k -functors that sends 1_A to some morphism $\tau : (\mathfrak{P}_X)_A \rightarrow \mathfrak{Y}_A$ of A -functors. Consider a cartesian square

$$\begin{array}{ccc} \mathfrak{U} & \longrightarrow & \mathrm{Mor}_k(\mathfrak{P}_X, \mathfrak{Y}') \\ \downarrow & & \downarrow \mathrm{Mor}_k(1_{\mathfrak{P}_X}, j) \\ \mathfrak{P}_{\mathrm{Spec} A} & \xrightarrow{\sigma} & \mathrm{Mor}_k(\mathfrak{P}_X, \mathfrak{Y}) \end{array}$$

Since $\mathrm{Mor}_k(1_{\mathfrak{P}_X}, j)$ is a monomorphism, we may consider \mathfrak{U} as a k -subfunctor of $\mathfrak{P}_{\mathrm{Spec} A}$. For every k -algebra B subset $\mathfrak{U}(B) \subseteq \mathrm{Mor}_k(A, B) = \mathrm{Mor}_k(\mathrm{Spec} B, \mathrm{Spec} A)$ consists of A -algebras B with structure morphisms $f : A \rightarrow B$ such that τ_B factors through $j_B : \mathfrak{Y}'_B \rightarrow \mathfrak{Y}_B$. Since X is a locally free k -scheme, we deduce that $(\mathfrak{P}_X)_A$ is a functor of points of a locally free A -scheme

$$\mathrm{Spec} A \times_{\mathrm{Spec} k} X$$

Pick an open affine cover $\bigcup_{i \in I} X_i$ of this A -scheme such that $\Gamma(X_i, \mathcal{O}_X)$ is a free A -module. Now Lemma 6.2.2 implies that τ_B factors through j_B if and only if $(\tau|_{X_i})_B$ factors through j_B for every $i \in I$. Next by Lemma 6.2.1 we deduce that $(\tau|_{X_i})_B$ factors through j_B for given $i \in I$ if and only if $f(\mathfrak{a}_i) = 0$ for some ideal $\mathfrak{a}_i \subseteq A$ independent of f . Thus \mathfrak{U} consists of all morphisms $f : A \rightarrow B$ of k -algebras such that $f(\mathfrak{a}) = 0$ where $\mathfrak{a} = \sum_{i \in I} \mathfrak{a}_i$. Therefore, $\mathfrak{U} \hookrightarrow \mathfrak{P}_{\mathrm{Spec} A}$ is isomorphic with $\mathfrak{P}_{V(\mathfrak{a})} = \mathfrak{P}_{\mathrm{Spec} A/\mathfrak{a}} \hookrightarrow \mathfrak{P}_{\mathrm{Spec} A}$ induced by the quotient map $A \rightarrow A/\mathfrak{a}$ and hence $\mathrm{Mor}_k(1_{\mathfrak{P}_X}, j)$ is a closed immersion of k -functors. \square

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