

QUOTIENTS OF ALGEBRAIC GROUPS

1. INTRODUCTION

Throughout this notes k denote a field and \mathbf{G} denote a group scheme over k . We denote by e the identity of \mathbf{G} . We also fix a k -scheme X equipped with an action of \mathbf{G} determined by morphism $a : \mathbf{G} \times_k X \rightarrow X$.

2. BASIC PROPERTIES OF SCHEME GROUP QUOTIENTS

The following result gives scheme-theoretic criterion for topological quotient in the case of group scheme actions.

Proposition 2.1. *Let Y be a k -scheme with the trivial action of \mathbf{G} and let $q : X \rightarrow Y$ be a \mathbf{G} -equivariant morphism. Assume that q is submersive and the morphism $\mathbf{G} \times_k X \rightarrow X \times_Y X$ induced by a and pr_X is surjective. Then the diagram*

$$\mathbf{G} \times_k X \xrightarrow[\text{pr}_X]{a} X \xrightarrow{q} Y$$

is a cokernel in the category topological spaces.

Proof. Let π_1 and π_2 be distinct projections $X \times_Y X \rightarrow X$. Pick points x_1 and x_2 in X such that $q(x_1) = q(x_2)$. Then there exists a field extension K over k such that $k(x_1) \subseteq K$ and $k(x_2) \subseteq K$. These give rise to K -points \bar{x}_1 and \bar{x}_2 of X such that their images under q is the same K -point of Y . Since we have an identification

$$(X \times_Y X)(K) = X(K) \times_{Y(K)} X(K)$$

induced by π_1 and π_2 , we derive that there exists a K -point \bar{z} of $X \times_Y X$ such that $\pi_1(\bar{z}) = \bar{x}_1$ and $\pi_2(\bar{z}) = \bar{x}_2$. Let z be the point of $X \times_Y X$ corresponding to \bar{z} . Then $\pi_1(z) = x_1$ and $\pi_2(z) = x_2$. By assumption a and pr_X induce surjection $\mathbf{G} \times_k X \twoheadrightarrow X \times_Y X$. Thus there exists a point u of $\mathbf{G} \times_k X$ such that $a(u) = x_1$ and $\text{pr}_X(u) = x_2$. Thus x_1 and x_2 are identified by an equivalence relation on the underlying set of X which is determined by the pair (a, pr_X) . Therefore, fibers of q are equivalence classes with respect to this relation. Since q is submersive, this implies that the diagram

$$\mathbf{G} \times_k X \xrightarrow[\text{pr}_X]{a} X \xrightarrow{q} Y$$

is a cokernel in the category topological spaces. □

Now we prove a series results concerning fpqc descent. For this we fix a k -scheme Y with the trivial action of \mathbf{G} and a \mathbf{G} -equivariant morphism $q : X \rightarrow Y$. Let $g : Y' \rightarrow Y$ be a morphism of k -schemes and consider the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ q' \downarrow & & \downarrow q \\ Y' & \xrightarrow{g} & Y \end{array}$$

of k -schemes. Note that X' admits a unique action a' of \mathbf{G} such that the square above consists of \mathbf{G} -equivariant morphism (we consider g as a \mathbf{G} -equivariant morphism between trivial \mathbf{G} -schemes).

Fact 2.2. *Suppose that g is faithfully flat and quasi-compact. Assume that q' is (universally) submersive. Then q is (universally) submersive.*

Proof. It suffices to prove that submersive morphisms have descent property. This follows from the fact that g (as faithfully flat and quasi-compact morphism) and q' are submersive. Details are left for the reader. \square

Fact 2.3. *Suppose that g is faithfully flat and quasi-compact. Then the canonical morphism $X' \times_{Y'} X' \rightarrow X \times_Y X$ is faithfully flat and quasi-compact and there is the cartesian square*

$$\begin{array}{ccc} \mathbf{G} \times_k X' & \longrightarrow & \mathbf{G} \times_k X \\ \downarrow & & \downarrow \\ X' \times_{Y'} X' & \longrightarrow & X \times_Y X \end{array}$$

in which the left vertical arrow is induced by $\langle a', \text{pr}_{X'} \rangle : \mathbf{G} \times_k X' \rightarrow X' \times_k X'$, the right vertical arrow is induced by $\langle a, \text{pr}_X \rangle : \mathbf{G} \times_k X \rightarrow X \times_k X$ and the bottom horizontal morphism is the canonical morphism.

Proof. Note that squares

$$\begin{array}{ccc} X' \times_{Y'} X' & \longrightarrow & X' \times_Y X' \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{g} & Y \end{array} \quad \begin{array}{ccc} X' \times_Y X' & \longrightarrow & X \times_Y X \\ \downarrow & & \downarrow \\ X' \times_k X' & \xrightarrow{g' \times_k g'} & X \times_k X \end{array}$$

are cartesian. Since both g and $g' \times_k g'$ are faithfully flat and quasi-compact, we derive that both morphisms $X' \times_{Y'} X' \rightarrow X' \times_Y X'$ and $X' \times_Y X' \rightarrow X \times_Y X$ are faithfully flat and quasi-compact. Then their composition i.e. the canonical morphism $X' \times_{Y'} X' \rightarrow X \times_Y X$ is faithfully flat and quasi-compact. \square

Finally we need the following notion

Definition 2.4. Let Y be a k -scheme with the trivial action of \mathbf{G} and let $q : X \rightarrow Y$ be a \mathbf{G} -equivariant morphism. Consider a pair

$$q_* \mathcal{O}_X \xrightleftharpoons[q_* \text{pr}_X^\#]{q_* a^\#} q_* (\text{pr}_X)_* \mathcal{O}_{\mathbf{G} \times_k X} = q_* a_* \mathcal{O}_{\mathbf{G} \times_k X}$$

of morphisms of sheaves of rings on Y . Suppose that $q^\# : \mathcal{O}_Y \rightarrow q_* \mathcal{O}_X$ is a kernel of this pair. Then \mathcal{O}_Y is the sheaf of \mathbf{G} -invariants for q .

Proposition 2.5. *Suppose that g is faithfully flat and quasi-compact. Assume that q' is quasi-compact, semiseparated and $\mathcal{O}_{Y'}$ is the sheaf of \mathbf{G} -invariants for q' . Then \mathcal{O}_Y is the sheaf of \mathbf{G} -invariants for q .*

Proof. We denote by a' the action of \mathbf{G} on X' . First note that q is semiseparated and quasi-compact morphism as these classes of morphisms admit descent along quasi-compact and faithfully flat

morphisms. Since q is quasi-compact, semiseparated and g is flat, we derive that for every quasi-coherent sheaf \mathcal{F} on X the canonical morphism $q'_* g'^* \mathcal{F} \rightarrow g^* q_* \mathcal{F}$ is an isomorphism. Thus the diagram

$$\mathcal{O}_{Y'} \xrightarrow{q^\#} q'_* \mathcal{O}_{X'} \xrightarrow[q'_* \text{pr}_{X'}^\#]{q'_* a'^\#} q'_* (\text{pr}_{X'})_* \mathcal{O}_{\mathbf{G} \times_k X'} = q'_* a'_* \mathcal{O}_{\mathbf{G} \times_k X'}$$

is isomorphic to the diagram

$$g^* \mathcal{O}_Y \xrightarrow{g^* q^\#} g^* (q_* \mathcal{O}_X) \xrightarrow[g^* q_* \text{pr}_X^\#]{g^* q_* a^\#} g^* (q_* (\text{pr}_X)_* \mathcal{O}_{\mathbf{G} \times_k X}) = g^* (q_* a_* \mathcal{O}_{\mathbf{G} \times_k X})$$

Since $\mathcal{O}_{Y'}$ is the sheaf of \mathbf{G} -invariants for q' , the first diagram is a kernel diagram. Hence the second is a kernel diagram. According to the fact that g is faithfully flat we deduce that the diagram

$$\mathcal{O}_Y \xrightarrow{q^\#} q_* \mathcal{O}_X \xrightarrow[q_* \text{pr}_X^\#]{q_* a^\#} q_* (\text{pr}_X)_* \mathcal{O}_{\mathbf{G} \times_k X} = q_* a_* \mathcal{O}_{\mathbf{G} \times_k X}$$

is also a kernel diagram. Thus \mathcal{O}_Y is the sheaf of \mathbf{G} -invariants for q . □

3. CATEGORICAL AND GEOMETRIC QUOTIENTS

Definition 3.1. Let Y be a k -scheme with the trivial action of \mathbf{G} and let $q : X \rightarrow Y$ be a \mathbf{G} -equivariant morphism. Suppose that the following conditions hold.

- (1) q is submersive.
- (2) The morphism $\mathbf{G} \times_k X \rightarrow X \times_Y X$ induced by $\langle a, \text{pr}_X \rangle : \mathbf{G} \times_k X \rightarrow X \times_k X$ is surjective.
- (3) \mathcal{O}_Y is the sheaf of \mathbf{G} -invariant for q .

Then q is a *geometric quotient* of X .

Corollary 3.2. Let q be a geometric quotient of X . Then the diagram

$$\mathbf{G} \times_k X \xrightarrow[\text{pr}_X]{a} X \xrightarrow{q} Y$$

is a cokernel in the category of ringed spaces.

Proof. Due to the fact that \mathcal{O}_Y is the sheaf of \mathbf{G} -invariants for q it suffices to prove that

$$\mathbf{G} \times_k X \xrightarrow[\text{pr}_X]{a} X \xrightarrow{q} Y$$

is the cokernel in the category of topological spaces. This follows from Proposition 2.1. □

Definition 3.3. Let $q : X \rightarrow Y$ be a morphism of k -schemes such that the diagram

$$\mathbf{G} \times_k X \xrightarrow[\text{pr}_X]{a} X \xrightarrow{q} Y$$

is a cokernel in the category of k -schemes. Then $q : X \rightarrow Y$ is a *categorical quotient* of X .

Fact 3.4. *Every geometric quotient is categorical.*

Proof. Categorical quotient is a cokernel in the category of k -schemes. On the other hand geometric quotient is a cokernel in the category of locally ringed spaces and hence it also satisfies cokernel property in its full subcategory of k -schemes. Thus every geometric quotient is categorical. \square

Let $q : X \rightarrow Y$ be a morphism of k -schemes such that $q \cdot \text{pr}_X = q \cdot a$. For a morphism $g : Y' \rightarrow Y$ of k -schemes consider the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ q' \downarrow & & \downarrow q \\ Y' & \xrightarrow{g} & Y \end{array}$$

Then there exists a unique action $a' : \mathbf{G} \times_k X' \rightarrow X'$ of \mathbf{G} on X' such that the square above consists of \mathbf{G} -equivariant morphism (we consider Y, Y' as \mathbf{G} -schemes equipped with trivial \mathbf{G} -actions). Keeping this in mind we have the following.

Definition 3.5. A morphism $q : X \rightarrow Y$ is a *uniform categorical (geometric) quotient* of X if for every flat morphism $g : Y' \rightarrow Y$ its base change $q' : X' \rightarrow Y'$ is a categorical (geometric) quotient of X' .

Definition 3.6. A morphism $q : X \rightarrow Y$ is a *universal categorical (geometric) quotient* of X if for every morphism $g : Y' \rightarrow Y$ its base change $q' : X' \rightarrow Y'$ is a categorical (geometric) quotient of X' .

Corollary 3.7. *Let $g : Y' \rightarrow Y$ be a faithfully flat and quasi-compact morphism. Suppose that q' is a geometric quotient, then q is a geometric quotient.*

Proof. This follows from Facts 2.2, 2.3 and Proposition 2.5. \square

In the next result we give a simple example of a universal geometric quotient.

Proposition 3.8. *Suppose that \mathbf{G} is a quasi-compact group scheme over k . Let Y be a k -scheme and consider $\mathbf{G} \times_k Y$ with the action of \mathbf{G} induced by the regular action on the left factor. Then $\text{pr}_Y : \mathbf{G} \times_k Y \rightarrow Y$ is a universal geometric quotient.*

Proof. Clearly pr_Y is universally submersive (it is even universally open). Let $\mu : \mathbf{G} \times_k \mathbf{G} \rightarrow \mathbf{G}$ be the multiplication morphism and let $\pi_{23} : \mathbf{G} \times_k \mathbf{G} \times Y \rightarrow \mathbf{G} \times_k Y$ be the projection on the last two factors. Then the morphism

$$\mathbf{G} \times_k \mathbf{G} \times_k Y \rightarrow (\mathbf{G} \times_k Y) \times_Y (\mathbf{G} \times_k Y) = \mathbf{G} \times_k \mathbf{G} \times_k Y$$

induced by $\langle \mu \times_k 1_Y, \pi_{23} \rangle : \mathbf{G} \times_k \mathbf{G} \times_k Y \rightarrow (\mathbf{G} \times_k Y) \times_k (\mathbf{G} \times_k Y)$ is an isomorphism. We show that \mathcal{O}_Y is the sheaf of \mathbf{G} -invariants for pr_Y . For this pick an affine open subset V of Y . It suffices to check that the diagram

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{\text{pr}_Y^\#} \Gamma(\mathbf{G} \times_k V, \mathcal{O}_{\mathbf{G} \times_k Y}) \xrightleftharpoons[\pi_{23}^\#]{(\mu \times_k 1_Y)^\#} \Gamma(\mathbf{G} \times_k \mathbf{G} \times_k V, \mathcal{O}_{\mathbf{G} \times_k \mathbf{G} \times_k Y})$$

is a kernel. Since \mathbf{G} is quasi-compact and separated (every group k -scheme is separated), we derive that the diagram above is isomorphic with

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{f \mapsto 1 \otimes f} \Gamma(\mathbf{G}, \mathcal{O}_{\mathbf{G}}) \otimes_k \Gamma(V, \mathcal{O}_Y) \xrightarrow[\chi \otimes f \mapsto 1 \otimes \chi \otimes f]{\chi \otimes f \mapsto \mu^\#(\chi) \otimes f} \Gamma(\mathbf{G}, \mathcal{O}_{\mathbf{G}}) \otimes_k \Gamma(\mathbf{G}, \mathcal{O}_{\mathbf{G}}) \otimes_k \Gamma(V, \mathcal{O}_Y)$$

Thus the first diagram is the kernel diagram if $f \mapsto 1 \otimes f$ induces an isomorphism of $\Gamma(V, \mathcal{O}_Y)$ with subspace of $\Gamma(\mathbf{G}, \mathcal{O}_{\mathbf{G}}) \otimes_k \Gamma(V, \mathcal{O}_Y)$ given by formula

$$\{\chi \in \Gamma(\mathbf{G}, \mathcal{O}_{\mathbf{G}}) \mid \mu^\#(\chi) = 1 \otimes \chi\} \otimes_k \Gamma(V, \mathcal{O}_Y)$$

Hence it suffices to prove that

$$\{\chi \in \Gamma(\mathbf{G}, \mathcal{O}_{\mathbf{G}}) \mid \mu^\#(\chi) = 1 \otimes \chi\} = \text{constant functions on } \mathbf{G}$$

For this pick a k -algebra A and let $g : \text{Spec } A \rightarrow \mathbf{G}$ be an A -point. Next let $e : \text{Spec } A \rightarrow \mathbf{G}$ be an A -point of \mathbf{G} which corresponds to the identity element of \mathbf{G} . Suppose that a regular function χ in \mathbf{G} satisfies $\mu^\#(\chi) = 1 \otimes \chi$. Then

$$g^\#(\chi) = \langle g, e \rangle^\# \mu^\#(\chi) = \langle g, e \rangle^\#(1 \otimes \chi) = e^\#(\chi)$$

Recall that e is given by the composition of the structural morphism $\text{Spec } A \rightarrow \text{Spec } k$ and the k -point $\text{Spec } k \rightarrow \mathbf{G}$ determined by the identity of \mathbf{G} . Thus $g^\#(\chi)$ is an element of k . Since this follows for every $g : \text{Spec } A \rightarrow \mathbf{G}$, we derive that χ is a constant function. This completes the proof of our claim that

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{\text{pr}_Y^\#} \Gamma(\mathbf{G} \times_k Y, \mathcal{O}_{\mathbf{G} \times_k Y}) \xrightarrow[\pi_{23}^\#]{(\mu \times_k 1_Y)^\#} \Gamma(\mathbf{G} \times_k \mathbf{G} \times_k Y, \mathcal{O}_{\mathbf{G} \times_k \mathbf{G} \times_k Y})$$

is the kernel diagram and hence \mathcal{O}_Y is the sheaf of \mathbf{G} -invariants for pr_Y . Therefore, we proved that pr_Y is a geometric quotient of $\mathbf{G} \times_k Y$. Consider any morphism $Y' \rightarrow Y$. Then base change of pr_Y along this morphism is $\text{pr}_{Y'}$. We conclude that pr_Y is a universal geometric quotient for $\mathbf{G} \times_k Y$. \square

4. GEOMETRIC QUOTIENTS OF SEPARATED ACTIONS

Definition 4.1. The action of \mathbf{G} on X is *separated* if the morphism $\langle a, \text{pr}_X \rangle : \mathbf{G} \times_k X \rightarrow X \times_k X$ has closed set-theoretic image.

Theorem 4.2. Let $q : X \rightarrow Y$ be a geometric quotient of X . Assume that q is universally submersive. Then the following assertions are equivalent.

- (i) The action of \mathbf{G} on X is separated.
- (ii) Y is separated.

Proof. We have a cartesian square

$$\begin{array}{ccc} X \times_Y X & \hookrightarrow & X \times_k X \\ \downarrow & & \downarrow q \times_k q \\ Y & \xrightarrow{\Delta_Y} & Y \times_k Y \end{array}$$

It follows that $X \times_Y X \hookrightarrow X \times_k X$ is a locally closed immersion. Since q is a geometric quotient, we derive that $\langle a, \text{pr}_X \rangle$ factors as a surjective morphism $\mathbf{G} \times_k X \twoheadrightarrow X \times_Y X$ followed by the immersion $X \times_Y X \hookrightarrow X \times_k X$. Thus the action of \mathbf{G} on X is separated if and only if $X \times_Y X$ is a closed subscheme of $X \times_k X$. Since q is universally submersive, we derive that $q \times_k q$ is submersive. As

the square above is cartesian we derive that $\Delta_Y(Y) \subseteq Y \times_k Y$ is closed if and only if $X \times_Y X \subseteq X \times_k X$ is closed. Therefore, Y is separated if and only if the action of \mathbf{G} on X is separated. \square

5. GEOMETRIC QUOTIENTS OF FREE ACTIONS AND PRINCIPAL BUNDLES

Definition 5.1. The action of \mathbf{G} on X is *free* if the morphism $\langle a, \text{pr}_X \rangle : \mathbf{G} \times_k X \rightarrow X \times_k X$ is a closed immersion.

Definition 5.2. Let x be a k -point of X . Suppose that the orbit morphism $\mathbf{G} \rightarrow X$ of x given by the composition

$$\mathbf{G} = \mathbf{G} \times_k \text{Spec } k \xrightarrow{\text{induced by } x} \mathbf{G} \times_k X \longrightarrow X$$

is a closed immersion. Then the action of \mathbf{G} on X has a *closed free orbit* at x .

Fact 5.3. If the action of \mathbf{G} on X is free, then every k -point of X has a closed free orbit.

The following result states that over special type of local complete noetherian k -algebras geometric quotients of free actions correspond to trivial \mathbf{G} -bundles.

Theorem 5.4. Suppose that k is an algebraically closed field and \mathbf{G} is a smooth algebraic group over k . Let $q : X \rightarrow Y$ be a geometric quotient locally of finite type and let Y be the spectrum of a complete local noetherian k -algebra such that the residue field of the closed point of Y is k . Then the following assertions hold.

- (1) If x is a k -point of X which has a closed free orbit, then there exists a \mathbf{G} -equivariant, étale and surjective morphism $f : \mathbf{G} \times_k Y \rightarrow X$ such that the triangle

$$\begin{array}{ccc} \mathbf{G} \times_k Y & \xrightarrow{f} & X \\ \text{pr}_Y \searrow & & \swarrow q \\ & Y & \end{array}$$

is commutative and the morphism

$$Y = \text{Spec } k \times_k Y \xrightarrow{e \times_k 1_Y} \mathbf{G} \times_k Y \xrightarrow{f} X$$

is a section of q .

- (2) If the action of \mathbf{G} on X is free, then f is an isomorphism.

The proof relies on two algebraic lemmas. The first describe free actions in context of complete rings (it is [Mumford et al., 1994, lemma on page 18]) and the second is a version of Hensel's lemma.

Lemma 5.4.1. Let (R, \mathfrak{m}, k) be a complete local noetherian k -algebra and let $\sigma : R \rightarrow R[[x_1, \dots, x_n]]$ be a local morphism into a ring of formal power series over R . Assume that the composition

$$R \xrightarrow{\sigma} R[[x_1, \dots, x_n]] \xrightarrow{f \mapsto f \bmod (x_1, \dots, x_n)} R$$

is the identity and the composition

$$R \xrightarrow{\sigma} R[[x_1, \dots, x_n]] \xrightarrow{f \mapsto f \bmod \mathfrak{m}} (R/\mathfrak{m})[[x_1, \dots, x_n]] = k[[x_1, \dots, x_n]]$$

is surjective. Consider elements y_1, \dots, y_n of R such that $\sigma(y_i) \bmod \mathfrak{m} = x_i$ for $i = 1, \dots, n$. Then the composition

$$R \xrightarrow{\sigma} R[[x_1, \dots, x_n]] \xrightarrow{f \mapsto f \bmod (y_1, \dots, y_n)} (R/(y_1, \dots, y_n))[[x_1, \dots, x_n]]$$

is an isomorphism.

Proof of the lemma. For convenience let ϕ denote the morphism given by the rule $r \mapsto \sigma(r) \bmod (y_1, \dots, y_n)$. Also denote $R/(y_1, \dots, y_n)$ by S . According to assumptions we have

$$\sigma(y_i) = x_i + y_i + \sum_{j=1}^n x_j \cdot \mathfrak{m}[[x_1, \dots, x_n]]$$

for each i . Thus $\phi(y_i) = \sum_{j=1}^n f_{ij} \cdot x_j$ where $f_{ij} \in S$ are elements such that the matrix $[f_{ij}]_{1 \leq i, j \leq n}$ is invertible in S . Hence

$$S[[x_1, \dots, x_n]] = S[[\phi(y_1), \dots, \phi(y_n)]]$$

and ϕ composed with $S[[\phi(y_1), \dots, \phi(y_n)]] \rightarrow S$ is the quotient morphism $R \rightarrow S$. From this observations we derive that ϕ is surjective. It remains to prove that it is injective. Consider z in R such that $\phi(z) = 0$. Suppose that $z \in (y_1, \dots, y_n)^m$ for some $m \in \mathbb{N}$. Write

$$z = \sum_{\alpha \in \Lambda} c_\alpha \cdot y_1^{\alpha_1} \dots y_n^{\alpha_n}$$

for some $c_\alpha \in R$ where $\Lambda = \{\alpha \in \mathbb{N}^n \mid \alpha_1 + \dots + \alpha_n = m\}$. Hence

$$0 = \phi(z) = \sum_{\alpha \in \Lambda} \phi(c_\alpha) \cdot \phi(y_1)^{\alpha_1} \dots \phi(y_n)^{\alpha_n}$$

Thus $\phi(c_\alpha) \in (\phi(y_1), \dots, \phi(y_n))$ for every $\alpha \in \Lambda$. Since ϕ composed with $S[[\phi(y_1), \dots, \phi(y_n)]] \rightarrow S$ is the quotient morphism $R \rightarrow S$, we derive that

$$c_\alpha \bmod (y_1, \dots, y_n) = \phi(c_\alpha) \bmod (\phi(y_1), \dots, \phi(y_n)) = 0$$

for every $\alpha \in \Lambda$. Thus $c_\alpha \in (y_1, \dots, y_n)$ for every $\alpha \in \Lambda$, which implies that $z \in (y_1, \dots, y_n)^{m+1}$. Thus we proved that

$$\phi(z) = 0 \text{ and } z \in (y_1, \dots, y_n)^m \Rightarrow z \in (y_1, \dots, y_n)^{m+1}$$

By \mathfrak{m} -adic completeness of R this implies that $\phi(z) = 0$ if and only if $z = 0$. Hence ϕ is also injective. \square

Lemma 5.4.2. *Let (R, \mathfrak{m}) be a complete local noetherian ring and let $R \rightarrow S$ be a local morphism into a local noetherian ring. Suppose that there exists a finitely generated R -submodule N of S such that*

$$S = N + \mathfrak{m}S$$

Then $S = N$.

Proof of the lemma. Pick s in S . Since $S = N + \mathfrak{m}S$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $x_n \in \mathfrak{m}^n N$ and

$$s - \sum_{i \leq n} x_i \in \mathfrak{m}^{n+1} S$$

According to the assumption that (R, \mathfrak{m}) is complete with respect to \mathfrak{m} -adic topology and N is finitely generated over R , we deduce that N is complete with respect to \mathfrak{m} -adic topology. Hence there exists a unique element x in N such that

$$x = \sum_{n \in \mathbb{N}} x_n$$

where above series is convergent with respect to \mathfrak{m} -adic topology. Note also that

$$x - \sum_{i \leq n} x_i \in \mathfrak{m}^{n+1} N$$

for every $n \in \mathbb{N}$. Thus we have

$$s - x = \left(s - \sum_{i \leq n} x_i \right) - \left(x - \sum_{i \leq n} x_i \right) \in \mathfrak{m}^{n+1}S + \mathfrak{m}^{n+1}N = \mathfrak{m}^{n+1}S$$

for every $n \in \mathbb{N}$. Hence

$$x - s \in \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n S$$

Since $R \rightarrow S$ is local morphism and S is a local ring, we deduce that $\mathfrak{m}S$ is contained in the maximal ideal of S . By assumptions S is noetherian. Therefore, S is separated with respect to \mathfrak{m} -adic topology. This implies that

$$0 = \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n S$$

Thus $s - x = 0$ and we infer that s is an element of N . This completes the proof that $S = N$. \square

In what follows we shall denote by $\mathbf{G}x$ the closed subscheme determined by the orbit morphism $\mathbf{G} \rightarrow X$ of a k -point x of X which has a closed free orbit. For readers convenience we include the following lemmas, which have topological content.

Lemma 5.4.3. *Let $q : X \rightarrow Y$ be a geometric quotient and assume that Y is the spectrum of a local k -algebra such that the residue field of the closed point o of Y is k . Let x be a k -point of X with free closed orbit, then $q^{-1}(o) = \mathbf{G}x$ as closed subschemes of X .*

Proof of the lemma. Morphism q induces the morphism of residue fields $k(q(x)) \hookrightarrow k(x) = k$ over k . This implies that $k(q(x)) = k$ and hence $q(x)$ is a k -point of Y . Note that o is the unique k -point of Y . Thus $q(x) = o$. Clearly $q^{-1}(o)$ is a closed \mathbf{G} -stable subscheme of X (it is the preimage of o under \mathbf{G} -equivariant q), that contains x . Since $\mathbf{G}x$ is the smallest closed \mathbf{G} -stable subscheme of X containing x , we deduce that $\mathbf{G}x \subseteq q^{-1}(o)$ scheme-theoretically. Consider the pair of arrows

$$\mathbf{G} \times_k X \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{\text{pr}_X} \end{array} X$$

Passing to functors of points we obtain that $a^{-1}(\mathbf{G}x) = \text{pr}_X^{-1}(\mathbf{G}x)$. Since q is the cokernel of the pair (a, pr_X) in the category of topological spaces, we deduce that there exists a closed subset Z of Y such that $q^{-1}(Z) = \mathbf{G}x$. Clearly $o \in Z$ and hence $q^{-1}(o) \subseteq \mathbf{G}x$ set-theoretically. On the other hand above we proved that $\mathbf{G}x \subseteq q^{-1}(o)$ scheme-theoretically. This can only happen if $q^{-1}(o) = \mathbf{G}x$ as closed subschemes of X . \square

Lemma 5.4.4. *Let $q : X \rightarrow Y$ be a geometric quotient and assume that Y is the spectrum of a local k -algebra such that the residue field of the closed point o of Y is k . Let U be an open \mathbf{G} -stable subset of X which contain a k -point. Then $U = X$.*

Proof of the lemma. Consider the pair of arrows

$$\mathbf{G} \times_k X \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{\text{pr}_X} \end{array} X$$

Since U is \mathbf{G} -stable open subset of X , we derive that $\text{pr}_X^{-1}(U) = a^{-1}(U)$. Next by definition q is the cokernel of the above pair in the category of topological spaces. Hence there exists an open subset V of Y such that $U = q^{-1}(V)$. Since U contains a k -point of X , we deduce as in Lemma 5.4.3 that $o \in V$. Thus $V = Y$ and finally $U = q^{-1}(V) = X$. \square

Proof of the theorem. We first prove (1). Denote by o the closed point of Y . Assume that x is a k -point of X which has a closed free orbit. Consider the surjective morphism $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{G,e}$ induced by the orbit morphism $\mathbf{G} \hookrightarrow X$ of x . Since \mathbf{G} is smooth over k , the ring $\mathcal{O}_{G,e}$ is regular. Pick a system of parameters x_1, \dots, x_n of $\mathcal{O}_{G,e}$ and let y_1, \dots, y_n be elements of $\mathcal{O}_{X,x}$ such that y_i is sent to x_i by the morphism $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{G,e}$ for $1 \leq i \leq n$. Define S to be the quotient ring $\mathcal{O}_{X,x}/(y_1, \dots, y_n)$. The morphism q induces the morphism $q^\# : \mathcal{O}_{Y,o} \rightarrow \mathcal{O}_{X,x}$ and hence the morphism $\mathcal{O}_{Y,o} \rightarrow S$. By Lemma 5.4.3 we have

$$S/\mathfrak{m}_o S = k$$

where \mathfrak{m}_o is the maximal ideal of $\mathcal{O}_{Y,o}$. According to Lemma 5.4.2 we derive that $\mathcal{O}_{Y,o} \rightarrow S$ is surjective. Let $f : \mathbf{G} \times_k \text{Spec } S \rightarrow X$ be the unique \mathbf{G} -equivariant morphism induced by the surjection $\mathcal{O}_{X,x} \twoheadrightarrow S$. We have a commutative square

$$\begin{array}{ccc} \mathbf{G} \times_k \text{Spec } S & \xrightarrow{f} & X \\ \text{pr}_{\text{Spec } S} \downarrow & & \downarrow q \\ \text{Spec } S & \xrightarrow{j} & Y \end{array}$$

where j is a closed immersion induced by $\mathcal{O}_{Y,o} \twoheadrightarrow S$. According to assumptions q is locally of finite type. Moreover, \mathbf{G} is an algebraic group over k and hence $\text{pr}_{\text{Spec } S}$ is locally of finite type. These two assertions together with the fact that $\text{Spec } S \hookrightarrow Y$ is a closed immersion of noetherian schemes (and thus is of finite type) imply that f is locally of finite type. Then by Lemma 5.4.1 we deduce that f induces an isomorphism

$$\widehat{\mathcal{O}_{X,x}} \longrightarrow \hat{S}[[x_1, \dots, x_n]] = \hat{S} \hat{\otimes}_k \widehat{\mathcal{O}_{G,e}}$$

of complete local rings. Since f is locally of finite type, it follows that f is étale at a k -point of $\mathbf{G} \times_k \text{Spec } S$ determined by the unique k -point of $\text{Spec } S$ and $e \in \mathbf{G}$. Let U be an étale locus of f . It contains a k -point and hence it is nonempty. Moreover, U is open (it is étale locus) subset of X . Since f is \mathbf{G} -equivariant, we derive that U is \mathbf{G} -stable. Similarly $f(U)$ is open \mathbf{G} -stable subset of X and $x \in f(U)$. Thus by Lemma 5.4.4 we deduce that

$$U = \mathbf{G} \times_k \text{Spec } S, f(U) = X$$

Therefore, f is étale and surjective. Now we pullback $q : X \rightarrow Y$ along the closed immersion $\text{Spec } S \hookrightarrow Y$. We obtain a cartesian square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{j}} & X \\ \tilde{q} \downarrow & & \downarrow q \\ \text{Spec } S & \xrightarrow{j} & Y \end{array}$$

Then f factors as a morphism $\mathbf{G} \times_k \text{Spec } S \rightarrow \tilde{X}$ followed by a closed immersion \tilde{j} . Since f is étale and surjective, we deduce that \tilde{j} is étale and surjective. This implies that \tilde{j} is an isomorphism of schemes. By definition of geometric quotient we derive that the diagram

$$\mathcal{O}_Y \xrightarrow{q^\#} q_* \mathcal{O}_X \xrightarrow[q_* \text{pr}_X^\#]{q_* a^\#} q_* (\text{pr}_X)_* \mathcal{O}_{\mathbf{G} \times_k X} = q_* a_* \mathcal{O}_{\mathbf{G} \times_k X}$$

is the kernel in the category of sheaves on Y . Hence $q^\# : \mathcal{O}_Y \rightarrow q_* \mathcal{O}_X$ is a monomorphism of sheaves. On the other hand we have

$$q^\# = j_* q_* (\tilde{j}^{-1})^\# \cdot j_* \tilde{q}^\# \cdot j^\#$$

and thus $j^\#$ is a monomorphism. Since j is a closed immersion, we infer that j is an isomorphism. Therefore, we can identify $\text{Spec } S$ with Y . Then f is a morphism which makes the triangle

$$\begin{array}{ccc} \mathbf{G} \times_k Y & \xrightarrow{f} & X \\ \text{pr}_Y \searrow & & \swarrow q \\ & Y & \end{array}$$

commutative. This completes the proof of (1).

For the proof of (2) consider the section $s : Y \hookrightarrow X$ described in (1). Then f fits into a cartesian square

$$\begin{array}{ccc} \mathbf{G} \times_k Y & \xrightarrow{f} & X \times_Y Y = X \\ 1_{\mathbf{G}} \times_Y s \downarrow & & \downarrow 1_X \times_Y s \\ \mathbf{G} \times_k X & \xrightarrow{\phi} & X \times_Y X \end{array}$$

where ϕ is a closed immersion induced by the closed immersion $\langle a, \text{pr}_X \rangle : \mathbf{G} \times_k X \hookrightarrow X \times_k X$ (the action of \mathbf{G} on X is free). Thus f is a closed immersion. By (1) it is étale and surjective. Therefore, f is an isomorphism. \square

Remark 5.5. We expect that Theorem 5.4 holds for prime spectra of strictly henselian rings.

Now we introduce sufficient condition for smoothness of geometric quotient in case of locally algebraic k -schemes.

Corollary 5.6. *Suppose that \mathbf{G} is a smooth algebraic group over k . Let $q : X \rightarrow Y$ be a morphism of finite type between k -schemes locally of finite type. Assume that q is a uniform geometric quotient of X and x is a k -point of X with closed free orbit. Then q is smooth at x .*

Proof. Since smoothness descent along faithfully flat morphisms, we may assume that k is algebraically closed. Let $y = q(x)$. Then y is a k -point of Y . Now $1_{\text{Spec } \widehat{\mathcal{O}_{Y,y}}} \times_k q$ is a geometric quotient and $\widehat{\mathcal{O}_{Y,y}}$ is a complete local noetherian k -algebra with k as a residue field. Moreover, x is a k -point of $\text{Spec } \widehat{\mathcal{O}_{Y,y}} \times_k X$ with a closed free orbit. By Theorem 5.4 we deduce that $1_{\text{Spec } \widehat{\mathcal{O}_{Y,y}}} \times_k q$ is smooth. From descent for smoothness we infer that q is smooth at x . \square

Definition 5.7. Let $q : X \rightarrow Y$ be a \mathbf{G} -equivariant morphism into a k -scheme Y equipped with the trivial \mathbf{G} -action. Suppose that q is faithfully flat, quasi-compact morphism and the square

$$\begin{array}{ccc} \mathbf{G} \times_k X & \xrightarrow{a} & X \\ \text{pr}_X \downarrow & & \downarrow q \\ X & \xrightarrow{q} & Y \end{array}$$

is cartesian. Then q is a principal \mathbf{G} -bundle.

Now we use Theorem 5.4 to describe principal \mathbf{G} -bundles in the category of locally algebraic k -schemes.

Theorem 5.8. *Suppose that \mathbf{G} is a smooth algebraic group over k . Let $q : X \rightarrow Y$ be a morphism of finite type between k -schemes locally of finite type. Then the following assertions are equivalent.*

- (i) q is a universal geometric quotient and the action of \mathbf{G} on X is free.
- (ii) q is a uniform geometric quotient and the action of \mathbf{G} on X is free.
- (iii) q is a principal \mathbf{G} -bundle.

Proof. Clearly (i) \Rightarrow (ii). Suppose that (ii) holds. Let \bar{k} be an algebraic closure of k . Then $1_{\text{Spec } \bar{k}} \times_k q$ is a uniform quotient and the action of $\text{Spec } \bar{k} \times_k \mathbf{G}$ on $\text{Spec } \bar{k} \times_k X$ induced by the action of \mathbf{G} on X is free. Moreover, if $1_{\text{Spec } \bar{k}} \times_k q$ is a principal $\text{Spec } \bar{k} \times_k \mathbf{G}$ -bundle, then q is a \mathbf{G} -bundle. This follows from the observation that property of being a principal bundle descends along faithfully flat and quasi-compact base extensions. Thus we may assume that k is algebraically closed. Next we pick a k -point y of Y and consider base change $1_{\text{Spec } \widehat{\mathcal{O}_{Y,y}}} \times_Y q$. This is a geometric quotient (because morphism $\text{Spec } \widehat{\mathcal{O}_{Y,y}} \rightarrow Y$ is flat) and a morphism of finite type. Moreover, the action of \mathbf{G} on $\text{Spec } \widehat{\mathcal{O}_{Y,y}} \times_Y X$ is free. Since $\widehat{\mathcal{O}_{Y,y}}$ is a complete noetherian k -algebra with residue field k , we derive by Theorem 5.4 that $\text{Spec } \widehat{\mathcal{O}_{Y,y}} \times_Y q$ is isomorphic as a \mathbf{G} -equivariant morphism with $\text{pr}_{\text{Spec } \widehat{\mathcal{O}_{Y,y}}}$. By faithfully flat descent for flat morphism we deduce that q is flat at every point in the fiber $q^{-1}(\text{Spec } \mathcal{O}_{Y,y})$. Since y is an arbitrary k -point, it follows that q is flat at every point of X which specializes to a k -point. Every point of X is a generization of a k -point (X is locally of finite type and k is algebraically closed). Thus q is flat. It is also surjective (as it is a geometric quotient) and quasi-compact (it is of finite type). Therefore, it is faithfully flat and quasi-compact morphism. In order to obtain (iii) it remains to prove that the morphism $\Phi : \mathbf{G} \times_k X \rightarrow X \times_Y X$ induced by a and pr_X is an isomorphism. Note that it is a closed immersion (the action of \mathbf{G} on X is closed). Moreover, $1_{\text{Spec } \widehat{\mathcal{O}_{Y,y}}} \times_Y \Phi$ is an isomorphism due to the fact that $1_{\text{Spec } \widehat{\mathcal{O}_{Y,y}}} \times_Y q$ is isomorphic as a \mathbf{G} -equivariant morphism with $\text{pr}_{\text{Spec } \widehat{\mathcal{O}_{Y,y}}}$. By faithfully flat descent we infer that $1_{\text{Spec } \mathcal{O}_{Y,y}} \times_Y \Phi$ is an isomorphism. This holds for every k -point y in Y . Thus Φ induces an isomorphism $\mathcal{O}_{X \times_Y X, \Phi(z)} \rightarrow \mathcal{O}_{\mathbf{G} \times_k X, z}$ for every k -point z of $X \times_Y X$. Hence a closed immersion Φ is an isomorphism. This completes the proof of (ii) \Rightarrow (iii). Assume now that (iii) holds. Then the square

$$\begin{array}{ccc} \mathbf{G} \times_k X & \xrightarrow{a} & X \\ \text{pr}_X \downarrow & & \downarrow q \\ X & \xrightarrow{q} & Y \end{array}$$

is cartesian and q is faithfully flat and quasi-compact. By Proposition 3.8 morphism pr_X is a universal geometric quotient. According to Corollary 3.7 we derive that q is universal geometric quotient. Moreover, the cartesian square above shows that the morphism $\mathbf{G} \times_k X \rightarrow X \times_Y X$ induced by a and pr_X is an isomorphism. Thus the action of \mathbf{G} on X is free. This is (i). Hence (iii) \Rightarrow (i) holds. \square

6. GEOMETRICALLY REDUCTIVE GROUPS AND NAGATA'S THEOREM

We start by proving the following result which give yet another characterization of linearly reductive groups.

Theorem 6.1. *Let \mathbf{G} be a smooth affine algebraic group over k . Then the following assertions are equivalent.*

- (i) \mathbf{G} is linearly reductive.
- (ii) For every finitely dimensional linear representation V of \mathbf{G} and for every nonzero \mathbf{G} -invariant element v in V there exists a \mathbf{G} -invariant linear function $f : V \rightarrow k$ such that $f(v) \neq 0$.

We need the following easy result.

Lemma 6.1.1. *Let \mathbf{G} be an algebraic group over k which satisfies (ii). Suppose that V is a finitely dimensional representation of \mathbf{G} . Then the map*

$$\mathrm{Hom}_k(V, k)^{\mathbf{G}} \ni f \mapsto f|_{V^{\mathbf{G}}} \in \mathrm{Hom}_k(V^{\mathbf{G}}, k)$$

is an isomorphism of vector spaces over k .

Proof of the lemma. The image of the map in the statement is a k -vector subspace $W \subseteq \mathrm{Hom}_k(V^{\mathbf{G}}, k)$ such that for every nonzero element v in $V^{\mathbf{G}}$ there exists f in W such that $f(v) \neq 0$ (this is a consequence of (ii)). It follows that W cannot be proper subspace of $\mathrm{Hom}_k(V^{\mathbf{G}}, k)$. Hence the map in the statement is an epimorphism. Now fix a nonzero \mathbf{G} -invariant linear function $f : V \rightarrow k$. By (ii) there exists a \mathbf{G} -invariant linear function $w : \mathrm{Hom}_k(V, k) \rightarrow k$ such that $w(f) = 0$. Note that the canonical isomorphism

$$V \cong \mathrm{Hom}_k(\mathrm{Hom}_k(V, k), k)$$

of k -vector spaces is a morphism of representations of \mathbf{G} . Thus w is defined in terms of evaluation in some \mathbf{G} -invariant vector v in V . Therefore, $f(v) \neq 0$ and hence $f|_{V^{\mathbf{G}}} \neq 0$. Thus the map described in the statement is also a monomorphism. \square

Proof of the theorem. Suppose that (i) holds. Consider a \mathbf{G} -invariant nonzero vector v in a finitely dimensional representation V of \mathbf{G} . Then $k \cdot v \subseteq V$ is a \mathbf{G} -subrepresentation. Since \mathbf{G} is linearly reductive, there exists a morphism of \mathbf{G} -representations which is a left inverse of $k \cdot v \hookrightarrow V$. This morphism can be identified with a \mathbf{G} -invariant linear function $f : V \rightarrow k$ such that $f(v) \neq 0$. Hence (i) \Rightarrow (ii).

Now suppose that (ii) holds. Pick an epimorphism $\theta : V \twoheadrightarrow W$ of finitely dimensional representations V of \mathbf{G} . Assume that there exists a nonzero \mathbf{G} -invariant vector w in W such that $w \notin \theta(V^{\mathbf{G}})$. By Lemma 6.1.1 there exists f in $\mathrm{Hom}_k(W, k)^{\mathbf{G}}$ such that $f|_{\theta(V^{\mathbf{G}})} = 0$ and $f(w) \neq 0$. Then $f \cdot \theta$ is a nonzero element of $\mathrm{Hom}(V, k)^{\mathbf{G}}$ such that $(f \cdot \theta)|_{V^{\mathbf{G}}} = 0$. This is impossible according to Lemma 6.1.1. Hence $\theta^{\mathbf{G}} : V^{\mathbf{G}} \rightarrow W^{\mathbf{G}}$ is an epimorphism. Now assume that $\theta : V \twoheadrightarrow W$ is an epimorphism of arbitrary linear representations of \mathbf{G} . Since \mathbf{G} is affine, every linear representation of \mathbf{G} is rational (i.e. it is a sum of its finitely dimensional subrepresentations). This together with the finitely dimensional case considered above imply that $\theta^{\mathbf{G}} : V^{\mathbf{G}} \rightarrow W^{\mathbf{G}}$ is an epimorphism. Thus the functor $(-)^{\mathbf{G}} : \mathbf{Rep}(\mathbf{G}) \rightarrow \mathbf{Vect}_k$ is exact. \square

The result above motivates the following notion.

Definition 6.2. Let \mathbf{G} be a smooth affine algebraic group. Suppose that for every finitely dimensional representation V of \mathbf{G} and for every nonzero \mathbf{G} -invariant vector v of V there exists a homogenous \mathbf{G} -invariant polynomial $f : V \rightarrow k$ such that $f(v) \neq 0$. Then \mathbf{G} is *geometrically reductive*.

We state here the following celebrated result.

Theorem 6.3. *If \mathbf{G} is reductive, then it is geometrically reductive.*

The result above is due to Haboush and its proof can be found in [Haboush, 1975].

The following theorem shows that geometric reductivity admits up to an integral extension the same property as linear reductivity (see also Remark 6.5 below).

Theorem 6.4. *Suppose that \mathbf{G} is geometrically reductive. Let A be a k -algebra such that $\text{Spec } A$ admits an action of \mathbf{G} and let \mathfrak{a} be a \mathbf{G} -stable ideal of A . We consider $A^{\mathbf{G}}/A^{\mathbf{G}} \cap \mathfrak{a}$ as a k -subalgebra of $(A/\mathfrak{a})^{\mathbf{G}}$ by means of the canonical inclusion $A^{\mathbf{G}}/A^{\mathbf{G}} \cap \mathfrak{a} \hookrightarrow A/\mathfrak{a}$. For every element $x \in (A/\mathfrak{a})^{\mathbf{G}}$ there exists positive integer r such that $x^r \in A^{\mathbf{G}}/A^{\mathbf{G}} \cap \mathfrak{a}$.*

Proof. Let $d : A \rightarrow k[\mathbf{G}] \otimes_k A$ be the coaction of \mathbf{G} on A . Pick an element $x_0 \in A$ which maps to x modulo \mathfrak{a} . Consider finitely dimensional vector subspace $V \subseteq A$ over k such that V is a \mathbf{G} -subrepresentation of A and $x_0 \in V$. Since x is x_0 modulo \mathfrak{a} , we derive that $c(x_0) - 1 \otimes x_0$ is in ideal of $k[\mathbf{G}] \otimes_k A$ generated by $k[\mathbf{G}] \otimes_k \mathfrak{a}$. Thus $W = k \cdot x_0 + V \cap \mathfrak{a} \subseteq A$ is finitely dimensional \mathbf{G} -subrepresentation of A . Let $\lambda : W \rightarrow k$ be a k -linear form such that $\lambda(x_0) = 1$ and $\lambda|_{V \cap \mathfrak{a}} = 0$. Since \mathbf{G} is geometrically reductive there exists $f \in \text{Sym}_r(W)^{\mathbf{G}}$ such that $f(\lambda) = 1$. Since the canonical morphism $\text{Sym}_r(W) \rightarrow A$ is a morphism of representations of \mathbf{G} , we deduce that f is mapped under this morphism to some \mathbf{G} -invariant element y in A . Note that f is sum of an r -th symmetric power of x_0 and some element of $\text{Sym}_r(V \cap \mathfrak{a})$. Thus $y \bmod \mathfrak{a} = x^r$. Hence $x^r \in A^{\mathbf{G}}/A^{\mathbf{G}} \cap \mathfrak{a}$. \square

Remark 6.5. Let \mathbf{G} be an algebraic group \mathbf{G} which acts on $\text{Spec } A$ for some k -algebra A and let \mathfrak{a} be a \mathbf{G} -stable ideal of A . Then the sequence

$$0 \longrightarrow \mathfrak{a}^{\mathbf{G}} \longrightarrow A^{\mathbf{G}} \longrightarrow (A/\mathfrak{a})^{\mathbf{G}}$$

is left exact and it induces a monomorphism $A^{\mathbf{G}}/\mathfrak{a} \cap A^{\mathbf{G}} = A^{\mathbf{G}}/\mathfrak{a}^{\mathbf{G}} \hookrightarrow (A/\mathfrak{a})^{\mathbf{G}}$. If \mathbf{G} is linearly reductive, then the sequence is exact and this monomorphism is an isomorphism. Theorem 6.4 states that if \mathbf{G} is geometrically reductive, then the monomorphism $A^{\mathbf{G}}/\mathfrak{a} \cap A^{\mathbf{G}} \hookrightarrow (A/\mathfrak{a})^{\mathbf{G}}$ is integral.

Now we are going to formulate the main result of this section.

Theorem 6.6. *Suppose that \mathbf{G} is geometrically reductive. Let A be a finitely generated k -algebra such that $\text{Spec } A$ admits an action of \mathbf{G} . Then $A^{\mathbf{G}}$ is finitely generated k -algebra.*

The theorem above was first proved by Nagata and here we follow Nagata's original proof. In the argument we denote the coaction of $k[\mathbf{G}]$ on A by $d : A \rightarrow k[\mathbf{G}] \otimes_k A$. The proof relies on a series of partial results.

Lemma 6.6.1. *Let $A \hookrightarrow B$ be an integral morphism of k -algebras and suppose that B is finitely generated over k . Then A is finitely generated.*

Proof of the lemma. Suppose that b_1, \dots, b_r are generators of B as a k -algebra. For every $1 \leq i \leq r$ we have a polynomial relation

$$b_i^{n_i} + a_{i,n_i-1}b_i^{n_i-1} + \dots + a_{i,1}b_i + a_{i,0} = 0$$

where $n_i > 0$ and $a_{i,j} \in A$ for $0 \leq j \leq n_i - 1$. Suppose that \tilde{A} is a k -subalgebra of A generated by $a_{i,j}$ for $1 \leq i \leq r$ and $0 \leq j \leq n_i - 1$. Then B is finite over \tilde{A} . Since $\tilde{A} \subseteq A \subseteq B$ and \tilde{A} is noetherian, we derive that A is finite over \tilde{A} . Hence A is finitely generated over k . \square

Lemma 6.6.2. *Suppose that \mathbf{G} is geometrically reductive. Let A be a k -algebra such that $\text{Spec } A$ admits an action of \mathbf{G} . Assume that A contains \mathbf{G} -invariant zero divisor and that for every proper \mathbf{G} -stable ideal \mathfrak{a} of A the k -algebra $(A/\mathfrak{a})^{\mathbf{G}}$ is finitely generated over k . Then $A^{\mathbf{G}}$ is finitely generated over k .*

Proof of the lemma. Let f be a \mathbf{G} -invariant zero divisor of A . By assumption both k -algebras $(A/fA)^{\mathbf{G}}$ and $(A/\text{ann}(f))^{\mathbf{G}}$ are finitely generated over k . Now by combination of Lemma 6.6.1 and Theorem 6.4 we obtain that $A^{\mathbf{G}}/A^{\mathbf{G}} \cap fA$ and $A^{\mathbf{G}}/A^{\mathbf{G}} \cap \text{ann}(f)$ are finitely generated over k . Let B be a finitely generated k -subalgebra of $A^{\mathbf{G}}$ which maps surjectively onto $A^{\mathbf{G}}/A^{\mathbf{G}} \cap fA$ and $A^{\mathbf{G}}/A^{\mathbf{G}} \cap \text{ann}(f)$. Let u_1, \dots, u_n be elements in A such that the image of $B \cdot u_1 + \dots + B \cdot u_n \subseteq A$ modulo $\text{ann}(f)$ contains a finite B -module $(A/\text{ann}(f))^{\mathbf{G}}$. Fix $a \in A^{\mathbf{G}}$. Since B maps surjectively onto $A^{\mathbf{G}}/A^{\mathbf{G}} \cap fA$, there exist $b \in B$ and $c \in A$ such that $a - b = fc$. Note that $fc \in A^{\mathbf{G}}$ and thus

$$(1 \otimes f)(d(c) - 1 \otimes c) = 0$$

This implies that c is sent to $(A/\text{ann}(f))^{\mathbf{G}}$ modulo $\text{ann}(f)$. Then $c \in B \cdot u_1 + \dots + B \cdot u_n$. Hence $a - b \in B \cdot fu_1 + \dots + B \cdot fu_n$. Therefore, $a \in B[f u_1, \dots, f u_n]$. This completes the proof that $A^{\mathbf{G}}$ is finitely generated over k . \square

Lemma 6.6.3. *Suppose that \mathbf{G} is geometrically reductive. Let $A = \bigoplus_{n \in \mathbb{N}} A_n$ be a \mathbb{N} -graded k -algebra such that A admits an action of \mathbf{G} . Assume that A_n is a \mathbf{G} -subrepresentation of A for every $n \in \mathbb{N}$ and that for every proper \mathbf{G} -stable homogenous ideal \mathfrak{a} of A the k -algebra $(A/\mathfrak{a})^{\mathbf{G}}$ is finitely generated over k . If A contains \mathbf{G} -invariant zero divisor, then $A^{\mathbf{G}}$ is finitely generated over k .*

Proof of the lemma. Let f be a \mathbf{G} -invariant zero divisor of A . We may pick f such that it is homogenous. Then both ideals fA and $\text{ann}(f)$ are homogenous, \mathbf{G} -stable and proper in A . Now we proceed as in the proof of Lemma 6.6.2. \square

Proof of the theorem. We first prove the theorem in case of \mathbb{N} -graded k -algebras and then reduce the general case to this graded case.

Assume that $A = \bigoplus_{n \in \mathbb{N}} A_n$ is \mathbb{N} -graded in such a way that $A_0 = k$ and A_n is a \mathbf{G} -subrepresentation of A for every $n \in \mathbb{N}$. Since A is finitely generated over k and by virtue of noetherian induction, we assume that $(A/\mathfrak{a})^{\mathbf{G}}$ is finitely generated over k for every homogenous \mathbf{G} -stable proper ideal \mathfrak{a} of A . If there are \mathbf{G} -invariant zero divisors of A , then by Lemma 6.6.3 we deduce that $A^{\mathbf{G}}$ is finitely generated over k . So we may assume that $A^{\mathbf{G}}$ contains no zero divisors of A . Pick a nonzero homogenous element $f \in A^{\mathbf{G}}$ of positive degree. If there are no such elements, then $A^{\mathbf{G}} = A_0 = k$ and the result holds. So we may assume that such an element exists. Note that it is noninvertible. Consider $x \in A$ such that $fx \in A^{\mathbf{G}}$. Then

$$0 = d(fx) - 1 \otimes fx = d(f) \cdot d(x) - (1 \otimes f) \cdot (1 \otimes x) = (1 \otimes f)(d(x) - 1 \otimes x)$$

Since f is not a zero divisor in A , we derive that $1 \otimes f$ is not a zero divisor in $k[\mathbf{G}] \otimes_k A$. Thus $d(x) = 1 \otimes x$ and $x \in A^{\mathbf{G}}$. This shows that $fA \cap A^{\mathbf{G}} = fA^{\mathbf{G}}$. By Theorem 6.4 $(A/fA)^{\mathbf{G}}$ is integral over $A^{\mathbf{G}}/fA \cap A^{\mathbf{G}} = A^{\mathbf{G}}/fA^{\mathbf{G}}$. Note that $(A/fA)^{\mathbf{G}}$ is finitely generated over k by inductive assumption. According to Lemma 6.6.1 we obtain that $A^{\mathbf{G}}/fA^{\mathbf{G}}$ is finitely generated over k . Clearly

$$A^{\mathbf{G}} = \bigoplus_{n \in \mathbb{N}} A_n^{\mathbf{G}}$$

and hence $A^{\mathbf{G}}/fA^{\mathbf{G}}$ inherits \mathbb{N} -grading from A . The ideal generated by elements of positive degree $(A^{\mathbf{G}}/fA^{\mathbf{G}})_+$ is finitely generated (as is every ideal in noetherian ring). Hence also

$$(A^{\mathbf{G}})_+ = \bigoplus_{n \in \mathbb{N}_+} A_n^{\mathbf{G}}$$

is finitely generated (generating set consists of lifts of generators of $(A^{\mathbf{G}}/fA^{\mathbf{G}})_+$ and f). This implies that $A^{\mathbf{G}}$ is finitely generated over $A_0^{\mathbf{G}} = k$.

Now assume that A is an arbitrary finitely generated k -algebra. By noetherian induction we may assume that $(A/\mathfrak{a})^G$ is finitely generated over k for every proper G -stable ideal \mathfrak{a} of A . Pick a finitely dimensional G -subrepresentation V of A which contains some finite set of generators of A as a k -algebra. Define $S = \text{Sym}(V)$ and $S_n = \text{Sym}_n(V)$ for every $n \in \mathbb{N}$. Then S is \mathbb{N} -graded, $S_0 = k$ and G acts on $\text{Spec } S$ in such a way that S_n is a G -subrepresentation of S for every S . By the case considered above S^G is finitely generated over k . The canonical (induced by $V \hookrightarrow A$) surjective morphism $S \twoheadrightarrow A$ of k -algebras is also a morphism of representations of G . Let I be its kernel. Then I is a G -stable ideal of S . By Theorem 6.4 we derive that $A^G = (S/I)^G$ is integral over its finitely generated k -subalgebra $S^G/I \cap S^G$. Moreover, by Lemma 6.6.2 we may assume that A^G does not contain zero divisors of A . In particular, it is an integral domain. Hence $S^G/I \cap S^G$ is a domain. Let $\overline{A^G}$ be the integral closure of $S^G/I \cap S^G$ in the field $Q(A^G)$ of fractions of A^G . Since $\overline{A^G}$ is integral over A^G , Lemma 6.6.1 shows that it suffices to prove that $\overline{A^G}$ is finitely generated over k . For this we show that $\overline{A^G}$ is a finite $S^G/I \cap S^G$ -module. Since fields are Nagata rings (see [Eisenbud, 1995, Corollary 13.13]), we may reduce this question to proving that $Q(A^G)$ is a finite extension of the field $Q(S^G/I \cap S^G)$ of fractions of $S^G/I \cap S^G$. Since $Q(S^G/I \cap S^G) \subseteq Q(A^G)$ is algebraic (due to the fact that $S^G/I \cap S^G \hookrightarrow A^G$ is integral), it suffices to show that $Q(A^G)$ is finitely generated field over $Q(S^G/I \cap S^G)$. For this pick a set T of nonzero divisors of A . Note that T is a multiplicative subset of A . Fix a maximal ideal $\mathfrak{m} \subseteq T^{-1}A$. Since nonzero elements of $\mathfrak{m} \cap A^G$ are zero divisors of A , we derive that $\mathfrak{m} \cap A^G = 0$. Thus $Q(A^G)$ is a subfield of $T^{-1}A/\mathfrak{m}$. The inclusion $A \hookrightarrow T^{-1}A$ induces an isomorphism between the field of fractions $Q(A/\mathfrak{m} \cap A)$ and the field $T^{-1}A/\mathfrak{m}$. By our assumption A is finitely generated over k . Thus $Q(A/\mathfrak{m} \cap A)$ is a field finitely generated over k . It follows that $Q(A^G)$ is a field finitely generated over k . This implies that $Q(A^G)$ is a field finitely generated over $Q(S^G/I \cap S^G)$. This completes the proof that A^G is a finitely generated k -algebra. \square

7. GOOD CATEGORICAL QUOTIENTS

We first introduce the following important notion.

Definition 7.1. Let Z be a closed subset of X . Suppose that $\text{pr}_X^{-1}(Z) = a^{-1}(Z)$. Then Z is a G -stable closed subset of X .

Theorem 7.2. Let $q : X \rightarrow Y$ be a morphism into a k -scheme Y equipped with the trivial G -action. Assume that the following assertions hold.

- (1) q is G -equivariant.
- (2) \mathcal{O}_Y is the sheaf of G -invariants for q .
- (3) If Z is a G -stable closed subset of X , then $q(Z)$ is a closed subset of Y .
- (4) If $\{Z_i\}_{i \in I}$ is a family of closed G -stable subsets with the empty intersection, then the intersection $\{q(Z_i)\}_{i \in I}$ is empty.

Then q is submersive and it is a categorical quotient of X .

Proof.

\square

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