LINEARLY REDUCTIVE GROUPS

1. Motivation – linear representations of compact topological groups

In this section we fix a compact topological group $\mathbf{G}$. Assume that $\rho : \mathbf{G} \to \mathrm{GL}_n(\mathbb{C})$ is a continuous homomorphism i.e. a complex, n-dimensional linear representation of $\mathbf{G}$. For every $g \in \mathbf{G}$ we get a matrix

$$\rho(g) = \big[c_{ij}(g)\big]_{1 \leq i,j \leq n}$$

For i, j function $c_{ij} : \mathbf{G} \to \mathbb{C}$ is a continuous complex valued function. Alternatively suppose that $\{e_1, e_2, ..., e_n\}$ is the standard basis of $\mathbb{C}^n$ on which $\mathrm{GL}_n(\mathbb{C})$ act. Then c_{ij} is equal to a function

$$\mathbf{G} \ni g \mapsto \langle g \cdot e_i, e_j \rangle \in \mathbb{C}$$

Fix now $g_1, g_2 \in \mathbf{G}$ and note that

$$\big[c_{ij}(g_2 \cdot g_1)\big]_{1 \leq i,j \leq n} = \rho(g_2 \cdot g_1) = \rho(g_2) \cdot \rho(g_1) = \left[\sum_{k=1}^{n} c_{ik}(g_2) \cdot c_{kj}(g_1)\right]_{1 \leq i,j \leq n}$$

Hence

$$c_{ij}(g_2 \cdot g_1) = \sum_{k=1}^{n} c_{ik}(g_2) \cdot c_{kj}(g_1)$$

for every $1 \leq i, j \leq n$. This implies that $\sum_{1 \leq i,j \leq n} \mathbb{C} \cdot c_{ij} \subseteq \mathcal{L}^2(\mathbf{G}, \mathbb{C})$ is a linear $\mathbf{G} \times \mathbf{G}^{\mathrm{op}}$-subrepresentation of the regular representation $\mathcal{L}^2(\mathbf{G}, \mathbb{C})$. We call it *the matrix coefficients of* ρ.

2. Matrix coefficients of a representation

Proposition 2.1. *Let $\mathfrak{X}$ be a monoid k-functor and let V be a finitely generated, projective k-module. Fix a morphism of monoids $\rho : \mathfrak{X} \to \mathcal{L}_V$. Fix k-algebra A and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. For every A-algebra B and $x \in \mathfrak{X}_A(B)$ we consider the formula*

$$c_{v,w}(x) = \langle \rho_A(x) \cdot v_B, w_B \rangle$$

Then $c_{v,w}$ defines a regular function on $\mathfrak{X}_A$ for every k-algebra A.

Proof. Suppose that $f : B \to C$ is a morphism of A-algebras and pick $x \in \mathfrak{X}_A(B)$. Since ρ_A is natural and $w : A \otimes_k V \to A$ is a morphism of A-modules, we derive that the diagram

$$\begin{tikzcd}
V_B \arrow[r, "\rho_A(x)"] \arrow[d, "1_{V_A} \otimes_A f"'] & V_B \arrow[r, "w_B"] \arrow[d, "1_{V_A} \otimes_A f"] & B \arrow[d, "f"] \\
V_C \arrow[r, "\rho_A(\mathfrak{X}_A(f)(x))"'] & V_C \arrow[r, "w_C"'] & C
\end{tikzcd}$$

is commutative. Hence

$$c_{v,w}\big(\mathfrak{X}_A(f)(x)\big) = \langle \rho_A\big(\mathfrak{X}_A(f)(x)\big) \cdot v_C, w_C \rangle = f\big(\langle \rho_A(x) \cdot v_B, w_B \rangle\big) = f\big(c_{v,w}(x)\big)$$

and this implies that $c_{v,w} : \mathfrak{X}_A \to \mathbb{A}^1_A$ is natural. □

Definition 2.2. Let $\mathfrak{X}$ be a monoid k-functor and let (V,ρ) be its representation with finitely generated, projective underlying k-module V. Fix k-algebra A and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. Then the regular function $c_{v,w}$ on $\mathfrak{X}_A$ is called *the matrix coefficient of v and w.*

Proposition 2.3. *Let $\mathfrak{X}$ be a monoid k-functor and let (V,ρ) be its representation with finitely generated projective underlying k-module V. Then the following assertions holds.*

(1) *For every k-algebra A map*

$$(A \otimes_k V) \times \left(A \otimes_k V^{\vee}\right) \ni (v,w) \mapsto c_{v,w} \in \mathrm{Mor}_A\left(\mathfrak{X}_A, \mathbb{A}^1_A\right)$$

is A-bilinear.

(2) *The collection of maps*

$$\left\{ (A \otimes_k V) \times \left(A \otimes_k V^{\vee}\right) \ni (v,w) \mapsto c_{v,w} \in \mathrm{Mor}_A\left(\mathfrak{X}_A, \mathbb{A}^1_A\right) \right\}_{A \in \mathbf{Alg}_k}$$

gives rise to a morphism of k-functors

$$V_{\mathrm{a}} \times V_{\mathrm{a}}^{\vee} \longrightarrow \mathcal{M}\mathrm{or}_k\left(\mathfrak{X}, \mathbb{A}^1_k\right)$$

Proof. We left the proof of **(1)** to the reader.
We prove **(2)**. Consider k-algebra A and an A-algebra B with structural morphism $f : A \to B$. Fix $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. We prove that restriction of $c_{v,w} : \mathfrak{X}_A \to \mathbb{A}^1_A$ to the category $\mathbf{Alg}_B$ is c_{v_B,w_B}. For this pick a B-algebra C and an element $x \in \mathfrak{X}_A(C) = \mathfrak{X}_B(C)$. Note that

$$c_{v,w}(x) = \langle \rho_A(x) \cdot v_C, w_C \rangle = \langle \rho_B(x) \cdot v_C, w_C \rangle = \langle \rho_B(x) \cdot (v_B)_C, (w_B)_C \rangle = c_{v_B,w_B}(x)$$

and hence $c_{v,w|\mathbf{Alg}_B} = c_{v_B,w_B}$. Consider the square

$$\begin{tikzcd}
V_{\mathrm{a}}(A) \times V_{\mathrm{a}}^{\vee}(A) \arrow[r] \arrow[d, "V_a(f) \times V_a^{\vee}(f)"'] & \mathcal{M}\mathrm{or}_k\left(\mathfrak{X}, \mathbb{A}^1\right)(A) \arrow[d, "\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathbb{A}^1)(f)"] \\
V_{\mathrm{a}}(B) \times V_{\mathrm{a}}^{\vee}(B) \arrow[r] & \mathcal{M}\mathrm{or}_k\left(\mathfrak{X}, \mathbb{A}^1\right)(B)
\end{tikzcd}$$

in which both horizontal arrows are given by formula $(v,w) \mapsto c_{v,w}$. We proved that the square commutes. Since f is an arbitrary morphism of k-algebras, we conclude the assertion. □

Corollary 2.4. *Let $\mathfrak{X}$ be a monoid k-functor and let (V,ρ) be its representation with finitely generated projective underlying k-module V. Then there exists a morphism of k-functors*

$$\left(V \otimes_k V^{\vee}\right)_{\mathrm{a}} \xrightarrow{c} \mathcal{M}\mathrm{or}_k\left(\mathfrak{X}, \mathbb{A}^1_k\right)$$

given by formula

$$(A \otimes_k V) \otimes_A \left(A \otimes_k V^{\vee}\right) \ni (v,w) \mapsto c_{v,w} \in \mathrm{Mor}_A\left(\mathfrak{X}_A, \mathbb{A}^1_A\right)$$

Moreover, c is a morphism of k-functors equipped with $\mathfrak{X} \times \mathfrak{X}^{\mathrm{op}}$-actions.

Proof. The first part is an immediate consequence of Proposition 2.3. We prove that c is a morphism of k-functors equipped with $\mathfrak{X} \times \mathfrak{X}^{\mathrm{op}}$-actions. For this we fix a k-algebra k and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. Pick a morphism of k-algebras $f : A \to B$, $(y,z) \in \mathfrak{X}(A) \times \mathfrak{X}(A)^{\mathrm{op}}$ and $x \in \mathfrak{X}_A(B)$. Then we have

$$c_{\rho(y)\cdot v, w\cdot\rho(z)}(x) = \left\langle \rho_A(x) \cdot \left(\rho(y) \cdot v\right)_B, \left(w \cdot \rho(z)\right)_B \right\rangle =$$
$$= \left\langle \rho_A(x) \cdot \rho_A((\mathfrak{X}_A(f)(y))) \cdot v_B, w_B \cdot \rho_A\left(\mathfrak{X}_A(f)(z)\right) \right\rangle = w_B\big(\rho_A\left(\mathfrak{X}_A(f)(z)\right) \cdot \rho_A(x) \cdot \rho_A\left(\mathfrak{X}_A(f)(y)\right) \cdot v_B\big) =$$
$$= w_B\big(\rho_A\left(\mathfrak{X}_A(f)(z) \cdot x \cdot \mathfrak{X}_A(f)(y)\right) \cdot v_B\big) = \left\langle \rho_A\left(\mathfrak{X}_A(f)(z) \cdot x \cdot \mathfrak{X}_A(f)(y)\right) \cdot v_B, w_B \right\rangle =$$

$$= c_{v,w}\big(\mathfrak{X}_A(f)(z)\cdot x\cdot \mathfrak{X}_A(f)(y)\big)$$

and hence c is a morphism of k-functors equipped with actions of $\mathfrak{X}\times\mathfrak{X}^{\mathrm{op}}$. □

3. The category of linear representations

In this section we fix a monoid k-functor $\mathfrak{G}$. Note that there exists the forgetful functor $\mathbf{Rep}(\mathfrak{G}) \to \mathbf{Mod}(k)$ that sends each linear representation to its underlying k-module.

Theorem 3.1. *The forgetful functor*

$$\mathbf{Rep}(\mathfrak{G}) \longrightarrow \mathbf{Mod}(k)$$

creates small colimits.

Proof. Suppose that $I \ni i \mapsto (V_i, \rho_i) \in \mathbf{Rep}(\mathfrak{G})$ is a diagram of linear representations of $\mathfrak{G}$ indexed by some category I. Let V together with $u_i : V_i \to V$ for $i \in I$ be a colimit of the diagram $I \ni i \mapsto V_i \in \mathbf{Mod}(k)$.

Assume first that (V,ρ) is a structure of the linear representation of $\mathfrak{G}$ on V such that $u_i : V_i \to V$ for $i \in I$ becomes a cocone over the diagram $I \ni i \mapsto (V_i,\rho_i) \in \mathbf{Rep}(\mathfrak{G})$. For every k-algebra A the functor $A\otimes_k(-)$ preserves colimits and hence $1_A \otimes_k u_i$ for $i \in I$ is a colimit of the diagram $I \ni i \mapsto 1_A\otimes_k V_i \in \mathbf{Mod}(A)$. For each $i \in I$ we have an action $\rho_i^A : \mathfrak{G}(A) \to \mathrm{Hom}_A\left(A\otimes_k V_i, A\otimes_k V_i\right)$ of $\mathfrak{G}(A)$ on $A\otimes_k V_i$ and we may view the diagram $I \ni i \mapsto 1_A\otimes_k V_i \in \mathbf{Mod}(A)$ as a diagram in the category of A-modules equipped with $\mathfrak{G}(A)$-actions given by A-module morphisms. We refer to this category as to category of A-linear $\mathfrak{G}(A)$-actions. Now the forgetful functor

$$\Big\{\text{the category of } A\text{-linear } \mathfrak{G}(A)\text{-actions}\Big\} \longrightarrow \mathbf{Mod}(A)$$

creates small limits. Indeed, the category on the right hand side is isomorphic with the category $\mathbf{Mod}\left(A[\mathfrak{G}(A)]\right)$ of left modules over the monoid A-algebra $A[\mathfrak{G}(A)]$ and the forgetful functor

$$\mathbf{Mod}\left(A[\mathfrak{G}(A)]\right) \longrightarrow \mathbf{Mod}(A)$$

creates small colimits. This implies that $\rho^A : \mathfrak{G}(A) \to \mathrm{Hom}_A(A\otimes_k V, A\otimes_k V)$ must be a unique morphism of monoids such that $1_A\otimes_k u_i$ for every $i \in I$ is a morphism of A-modules with A-linear action of $\mathfrak{G}(A)$. This implies that ρ is unique and hence (V,ρ) is unique lift of $(V,\{u_i\}_{i\in I})$ to $\mathbf{Rep}(\mathfrak{G})$. This shows the uniqueness of a lift.

For the existence assume for given k-algebra A that $\rho^A : \mathfrak{G}(A) \to \mathrm{Hom}_A(A\otimes_k V, A\otimes_k V)$ is a unique morphism of monoids such that $1_A\otimes_k u_i$ for every $i \in I$ is a morphism of A-modules with A-linear action of $\mathfrak{G}(A)$. Note that ρ^A exists because the forgetful functor

$$\Big\{\text{the category of } A\text{-linear } \mathfrak{G}(A)\text{-actions}\Big\} \longrightarrow \mathbf{Mod}(A)$$

creates small colimits. Denote $\rho = \{\rho^A\}_{A\in\mathbf{Alg}_k}$. We verify that ρ is a morphism of k-functors $\rho : \mathfrak{G} \to \mathcal{L}_V$. For this consider morphism $f : A \to B$ of k-algebras and the commutative square

$$\begin{tikzcd} A \otimes_k V_i \arrow[r, "1_A \otimes_k u_i"] \arrow[d, "f \otimes_k 1_{V_i}"'] & A \otimes_k V \arrow[d, "f \otimes_k 1_V"] \\ B \otimes_k V_i \arrow[r, "1_B \otimes_k u_i"'] & B \otimes_k V \end{tikzcd}$$

defined for every $i \in I$. Note that the top row of the square is a morphism of A-modules with A-linear $\mathfrak{G}(A)$-actions. Similarly interpreting $B \otimes_k V_i$ and $B \otimes_k V$ as A-modules with A-linear actions of $\mathfrak{G}(A)$ given by $\rho_i^B \cdot \mathfrak{G}(f)$ and $\rho^B \cdot \mathfrak{G}(f)$, respectively, we derive that the square consists of A-modules with A-linear actions of $\mathfrak{G}(A)$ and all maps preserve actions except possibly $f \otimes_k 1_V$. Since $A \otimes_k V$ together with $1_A \otimes_k u_i$ for $i \in I$ is a colimit of $I \ni i \mapsto 1_A \otimes_k V_i \in \mathbf{Mod}(A)$ in the category of A-modules, we deduce that $f \otimes_k 1_V$ is the only morphism of A-modules making squares commutative for all $i \in I$. Since $A \otimes_k V$ with ρ^A and $1_A \otimes_k u_i$ for $i \in I$ is a colimit of the same diagram, but interpreted as a diagram of A-modules with A-linear action of $\mathfrak{G}(A)$-modules, we derive from uniqueness of $f \otimes_k 1_V$ that it must also preserve $\mathfrak{G}(A)$-action. Hence $(f \otimes_k 1_V) \cdot \rho^A = \rho^B \cdot \mathfrak{G}(f)$. Thus ρ is a morphism of k-functors. By definition of ρ^A for each k-algebra A, we derive that it is a morphism of monoid k-functors. Hence (V, ρ) is a linear representation of $\mathfrak{G}$ and again by componentwise definition of ρ we deduce that (V, ρ) is a colimit of the diagram $I \ni i \mapsto (V_i, \rho_i) \in \mathbf{Rep}(\mathfrak{G})$. □

Theorem 3.2. *Let A be a commutative ring. The following assertions are equivalent.*

- **(i)** Spec A *is a Hausdorff space.*
- **(ii)** *Every prime ideal of A is maximal.*
- **(iii)** *Every $A/\mathcal{N}$-module is flat, where $\mathcal{N}$ is a nilradical of A.*
- **(iv)** *Every finitely generated ideal of A is generated by an idempotent.*

Lemma 3.2.1. *Let A be a commutative ring and M be an A-module. Then M is flat if and only if $M_{\mathfrak{p}}$ is flat for all $\mathfrak{p} \in \operatorname{Spec} A$.*

Proof of the lemma. For every $\mathfrak{p} \in \operatorname{Spec} A$ we have a natural isomorphism

$$M_{\mathfrak{p}} \otimes_A (-) \cong (M \otimes_A (-))_{\mathfrak{p}}$$

Now the statement follows from the fact that a chain complex of A-modules is exact if and only if it is exact after localization in every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ □

Lemma 3.2.2. *Let A be a local ring such that each A-module is flat. Then A is a field.*

Proof of the lemma. Let $\mathfrak{m}$ be a maximal ideal of A and k be a residue field. Pick finitely generated ideal $\mathfrak{a} \subseteq \mathfrak{m}$. Consider the canonical exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow A \xrightarrow{a \mapsto a \bmod \mathfrak{a}} A/\mathfrak{a} \longrightarrow 0$$

Since k is a flat A-module, we derive that the sequence

$$0 \longrightarrow k \otimes_A \mathfrak{a} \longrightarrow k \xrightarrow{\alpha \mapsto \alpha \bmod \mathfrak{a}k} k/\mathfrak{a}k \longrightarrow 0$$

is exact. Since $\mathfrak{a}k = 0$ because $\mathfrak{a} \subseteq \mathfrak{m}$, we deduce from the short exact sequence that $k \otimes_A \mathfrak{a} = 0$. By Nakayama lemma this implies that $\mathfrak{a} = 0$ ($\mathfrak{a}$ is finitely generated over A). Thus every finitely generated A-submodule of $\mathfrak{m}$ is trivial. Thus $\mathfrak{m} = 0$ and hence A is a field. □

4. Results on affine monoids

Definition 4.1. Let $\mathfrak{G}$ be a monoid k-functor. We say that $\mathfrak{G}$ is *a monoid k-functor with zero* if there exists a k-point $\mathbf{o}$ of $\mathfrak{G}$ such that the following two morphisms

$$\mathbf{1} \times \mathfrak{G} \xrightarrow{\mathbf{o} \times 1_{\mathfrak{G}}} \mathfrak{G} \times \mathfrak{G} \xrightarrow{\mathrm{mul}} \mathfrak{G} \qquad \mathfrak{G} \times \mathbf{1} \xrightarrow{1_{\mathfrak{G}} \times \mathbf{o}} \mathfrak{G} \times \mathfrak{G} \xrightarrow{\mathrm{mul}} \mathfrak{G}$$

where $\mathrm{mul} : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ is the multiplication on $\mathfrak{G}$, factor through $\mathbf{o}$. If this is the case, then $\mathbf{o}$ is called *the zero of* $\mathfrak{G}$.

Definition 4.2. Let $\mathfrak{G}$ be a monoid k-functor. For each k-algebra A we denote by $\mathfrak{G}^*(A)$ the group of units of $\mathfrak{G}(A)$. This gives rise to a subgroup k-functor $\mathfrak{G}^*$ of $\mathfrak{G}$. We call $\mathfrak{G}^*$ *the group of units of* $\mathfrak{G}$.

Now we describe the universal property of the group of units. Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{G}$ be a group k-functor. Suppose that $\sigma : \mathfrak{G} \to \mathfrak{G}$ is a morphism of monoid k-functors. Then σ factors through $\mathfrak{G}^*$.

Proposition 4.3. *Let* $\mathbf{M}$ *be an affine k-monoid scheme and denote by* $\mathfrak{G}$ *the k-monoid functor that represents* $\mathbf{M}$*. Then* $\mathfrak{G}^*$ *is representable by an affine k-group scheme. Moreover, if* $\mathbf{M}$ *is an affine integral k-monoid scheme of finite type over k, then* $\mathfrak{G}^*$ *is an open k-subfunctor of* $\mathfrak{G}$.

5. Diagonalisable monoid k-schemes

Consider an abstract commutative monoid Γ. Consider the monoid k-algebra $k[\Gamma]$. Recall that $k[\Gamma]$ as a free k-vector space over k and its elements can be uniquely written as

$$\sum_{\gamma \in \Gamma} k_\gamma \cdot \gamma$$

where almost all k_γ are zero for $\gamma \in \Gamma$. Next the k-algebra $k[\Gamma]$ admits a structure of a commutative bialgebra with a comultiplication given by

$$k[\Gamma] \ni \sum_{\gamma \in \Gamma} k_\gamma \cdot \gamma \to \sum_{\gamma \in \Gamma} k_\gamma \cdot (\gamma \otimes \gamma) \in k[\Gamma] \otimes_k k[\Gamma]$$

and a counit

$$k[\Gamma] \ni \sum_{\gamma \in \Gamma} k_\gamma \cdot \gamma \mapsto \sum_{\gamma \in \Gamma} k_\gamma \in k$$

This makes $\operatorname{Spec} k[\Gamma]$ into a monoid k-scheme. We denote this monoid k-scheme by $\mathbf{D}_\Gamma$. For an alternative description note that we have identifications

$$\mathfrak{P}_{\mathbf{D}_\Gamma}(A) \cong \mathrm{Mor}_k\left(k[\Gamma], A\right) \cong \mathbf{Mon}\left(\Gamma, A^\times\right)$$

natural in k-algebra A, where the right hand side denotes the set of morphisms of monoids from Γ to the multiplicative monoid $A^\times$ of A. The k-functor

$$\mathbf{Alg}_k \ni A \mapsto \mathbf{Mon}\left(\Gamma, A^\times\right) \in \mathbf{Set}$$

is a monoid k-functor with respect to multiplication of monoid homomorphisms in $\mathbf{Mon}\left(\Gamma, A^\times\right)$ for every k-algebra A. Hence the identification above makes the functor of points $\mathfrak{P}_{\mathbf{D}_\Gamma}$ into the monoid k-functor and induces precisely the bialgebra structure on $k[\Gamma]$ described above.
Note that if $g : \Gamma_1 \to \Gamma_2$ is a morphism of commutative monoids, then $k[g] : k[\Gamma_1] \to k[\Gamma_2]$ is a morphism of bialgebras (with respect to the structure described above). We denote $\operatorname{Spec} k[g]$ by $\mathbf{D}_g$.

Definition 5.1. Let $\mathbf{M}$ be a monoid k-scheme. We say that $\mathbf{M}$ is *diagonalisable* if there exists an abstract commutative monoid Γ such that $\mathbf{M}$ is visomorphic to $\mathbf{D}_\Gamma$ as a monoid k-scheme.

Now we prove the following important result.

Theorem 5.2. *Suppose that k is commutative ring such that* $\operatorname{Spec} k$ *is connected (i.e. k has no nontrivial idempotents). Consider the functor*

$$\begin{tikzcd} \Gamma_1 \arrow[d, "g"'] & & \mathbf{D}_{\Gamma_1} \\ \Gamma_2 \arrow[r, phantom, "\longmapsto"] & & \mathbf{D}_{\Gamma_2} \arrow[u, "\mathbf{D}_g"'] \end{tikzcd}$$

defined on the category of commutative monoids and with values in the category of monoid schemes over k. This functor preserves finite products and induces an equivalence of categories between abstract commutative monoids and diagonalisable monoid schemes over k.

Proof. Suppose that Γ_1, Γ_2 are commutative monoids and $f : k[\Gamma_1] \to k[\Gamma_2]$ is a morphism of bialgebras over k. Let Δ_1, ξ_1 and Δ_2, ξ_2 be comultiplications and counits for $k[\Gamma_1], k[\Gamma_2]$, respectively. Fix $\gamma \in \Gamma_1$ and suppose that $f(\gamma) = \sum_{\gamma' \in \Gamma_2} k_{\gamma'} \cdot \gamma'$. The fact that f is a morphism of bialgebras over k implies that

$$\Delta_2\left(f\left(\gamma\right)\right) = \left(f \otimes_k f\right)\left(\Delta_1(\gamma)\right) = \left(f \otimes_k f\right)\left(\gamma \otimes_k \gamma\right) = f(\gamma) \otimes_k f(\gamma)$$

Substituting $\sum_{\gamma' \in \Gamma_2} k_{\gamma'} \cdot \gamma'$ for $f(\gamma)$ we deduce that

$$\sum_{\gamma' \in \Gamma_2} k_{\gamma'} \cdot \left(\gamma' \otimes \gamma'\right) = \sum_{\gamma' \in \Gamma_2} \sum_{\gamma'' \in \Gamma_2} k_{\gamma'} \cdot k_{\gamma''} \cdot \left(\gamma' \otimes \gamma''\right)$$

Thus we derive that

$$k_{\gamma'} \cdot k_{\gamma''} = \begin{cases} 0 & \text{if } \gamma' \neq \gamma'' \\ k_{\gamma'} & \text{if } \gamma' = \gamma'' \end{cases}$$

Since there are no nontrivial idempotents in k, this implies that $k_{\gamma'} = 0, 1$ for each $\gamma' \in \Gamma_2$. Again by the fact that f is a morphism of k-bialgebras, we derive that

$$\xi_1(\gamma) = \xi_2\left(f(\gamma)\right)$$

Substituting $\sum_{\gamma' \in \Gamma_2} k_{\gamma'} \cdot \gamma'$ for $f(\gamma)$ yields that

$$\sum_{\gamma' \in \Gamma_2} k_{\gamma'} = 1$$

Combining this with previously established fact that $k_{\gamma'} = 0, 1$ for each $\gamma' \in \Gamma_2$ we deduce that there exists precisely one $\gamma' \in \Gamma_2$ such that $f(\gamma) = \gamma'$. This proves that $f(\Gamma_1) \subseteq \Gamma_2$. Since f preserves multiplication and unit, we deduce that $f = k[g]$ for some homomorphism of abstract monoids $g : \Gamma_1 \to \Gamma_2$. Thus the functor described in the statement is full.
It is also clearly faithful. Indeed, for two distinct morphisms of monoids $g_1, g_2 : \Gamma_1 \to \Gamma_2$ we have $k[g_1] \neq k[g_2]$ and hence $\operatorname{Spec} k[g_1] \neq \operatorname{Spec} k[g_2]$.
By definition of diagonalisable monoid the image of the functor is an essential subcategory of the category of diagonalisable k-schemes.
Finally, consider commutative monoids Γ_1, Γ_2 and note that isomorphism

$$k[\Gamma_1 \times \Gamma_2] \ni \sum_{(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2} k_{(\gamma_1, \gamma_2)} \cdot (\gamma_1, \gamma_2) \mapsto \sum_{(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2} k_{(\gamma_1, \gamma_2)} \cdot \gamma_1 \otimes \gamma_2 \in k[\Gamma_1] \otimes_k k[\Gamma_2]$$

is a morphism of k-bialgebras. This implies that the functor described in the statement preserves binary products. The functor preserves terminal objects, since k is a monoid k-algebra for trivial (zero) commutative monoid. $\square$

6. Representations of diagonalisable monoid k-schemes

Definition 6.1. Let Γ be a commutative monoid and let $\mathbf{D}_{\Gamma}$ be the corresponding monoid k-scheme. Suppose that V is a representation of $\mathbf{D}_{\Gamma}$ with respect to a morphism of monoid k-functors given by

$$\mathfrak{P}_{\mathbf{D}_{\Gamma}}(A) = \mathbf{Mod}\left(\Gamma, A^{\times}\right) \ni f \mapsto f(\gamma)\cdot(-) \in \mathcal{L}_V(A)$$

where γ is a fixed element of Γ. Then V is called *a representation of* $\mathbf{D}_{\Gamma}$ *of weight* γ.

Fact 6.2. *Let Γ be a commutative monoid and let γ be its element. Suppose that V is a representation of $\mathbf{D}_{\Gamma}$ of weight γ. Then V can be equivalently described as a comodule over $k[\Gamma]$ with respect to the following coaction*

$$V_{\gamma} \ni v \mapsto \gamma \otimes v \in k[\Gamma] \otimes_k V_{\gamma}$$

Proof. Denote by $\rho : \mathfrak{P}_{\mathbf{D}_{\Gamma}} \to \mathcal{L}_V$ the morphism of monoid k-functors that makes a V into a representation of $\mathbf{D}_{\Gamma}$. Then $\rho\left(1_{\mathbf{D}_{\Gamma}}\right)$ is a morphism of $k[\Gamma]$-modules

$$k[\Gamma] \otimes_k V \ni 1 \otimes v \mapsto \gamma \otimes v \in k[\Gamma] \otimes_k V$$

We obtain the coaction of $k[\Gamma]$ on V corresponding to ρ by transforming morphism $\rho\left(1_{\mathbf{D}_{\Gamma}}\right)$ via the canonical isomorphism

$$\mathrm{Hom}_{k[\Gamma]}\left(k[\Gamma] \otimes_k V, k[\Gamma] \otimes_k V\right) \cong \mathrm{Hom}_k\left(V, k[\Gamma] \otimes_k V\right)$$

Thus this coaction is given by formula

$$V \ni v \mapsto \gamma \otimes v \ni k[\Gamma] \otimes_k V$$

□

Fact 6.3. *Let Γ be a commutative monoid and let $\mathbf{D}_{\Gamma}$ be the corresponding monoid k-scheme. Suppose that V_1, V_2 are representations of $\mathbf{D}_{\Gamma}$ and assume that V_1, V_2 have weights γ_1, γ_2 with $\gamma_1 \neq \gamma_2$. Then*

$$\mathrm{Hom}_{\mathbf{D}_{\Gamma}}\left(V_1, V_2\right) = 0$$

Proof. This follows from Fact 6.2. □

Let Γ be a commutative monoid and let $\mathbf{D}_{\Gamma}$ be the corresponding monoid k-scheme. For every representation V of $\mathbf{D}_{\Gamma}$ and fixed γ in Γ define

$$V[\gamma] = \left\{v \in V \,\middle|\, d(v) = \gamma \otimes v\right\}$$

where $d : V \to k[\Gamma] \otimes_k V$ is the coaction. Then $V[\gamma]$ is a subrepresentation of V. Note that according to Fact 6.2 $V[\gamma]$ is a subrepresentation of V of weight γ.

Proposition 6.4. *Let Γ be a commutative monoid and let $\mathbf{D}_{\Gamma}$ be the corresponding monoid k-scheme. For every representation V of $\mathbf{D}_{\Gamma}$ we have a direct sum*

$$V = \bigoplus_{\gamma \in \Gamma} V[\gamma]$$

Proof. Let Δ, ξ be the comultiplication and the counit of $k[\Gamma]$, respectively. Let $d : V \to k[\Gamma] \otimes_k V$ be a coaction. Fix $v \in V$. Then we have a unique decomposition $d(v) = \sum_{\gamma \in \Gamma} \gamma \otimes v_{\gamma}$. Then

$$\sum_{\gamma \in \Gamma} \gamma \otimes \gamma \otimes v_{\gamma} = \left(\Delta \otimes_k 1_V\right)\left(d(v)\right) = \left(1_{k[\Gamma]} \otimes_k d\right)\left(d(v)\right) = \sum_{\gamma \in \Gamma} \gamma \otimes d(v_{\gamma})$$

This implies that $d(v_{\gamma}) = \gamma \otimes v_{\gamma}$ and hence $v_{\gamma} \in V[\gamma]$. On the other hand we have

$$v = \xi\left(d(v)\right) = \sum_{\gamma \in \Gamma} v_{\gamma}$$

Thus

$$v \in \sum_{\gamma \in \Gamma} V[\gamma]$$

Hence

$$V = \sum_{\gamma \in \Gamma} V[\gamma]$$

Moreover, suppose that $\sum_{\gamma \in \Gamma} v_\gamma = \sum_{\gamma \in \Gamma} v'_\gamma$ for some $v_\gamma, v'_\gamma \in V[\gamma]$. Then

$$\sum_{\gamma \in \Gamma} \gamma \otimes v_\gamma = d\left(\sum_{\gamma \in \Gamma} v_\gamma\right) = d\left(\sum_{\gamma \in \Gamma} v'_\gamma\right) = \sum_{\gamma \in \Gamma} \gamma \otimes v'_\gamma$$

and hence $v_\gamma = v'_\gamma$ for each $\gamma \in \Gamma$. This proves the direct decomposition of V as we claimed. □

Corollary 6.5. *Let k be a field. Suppose that Γ is a commutative monoid and let $\mathbf{D}_\Gamma$ be the corresponding monoid k-scheme. Then the category $\mathbf{Rep}(\mathbf{D}_\Gamma)$ is semisimple. Moreover, each irreducible representation of $\mathbf{D}_\Gamma$ is isomorphic to one-dimensional representation of weight γ for a unique $\gamma \in \Gamma$.*

Proof. This is a consequence of Fact 6.3 and Proposition 6.4. □

7. Diagonalisable group k-schemes

Let Γ be an abstract commutative group. Then in addition to k-bialgebra structure the k-algebra $k[\Gamma]$ admits an antipode map

$$k[\Gamma] \ni \sum_{\gamma \in \Gamma} k_\gamma \cdot \gamma \mapsto \sum_{\gamma \in \Gamma} k_\gamma \cdot \gamma^{-1} \in k[\Gamma]$$

That makes $k[\Gamma]$ into a commutative Hopf k-algebra. Thus $\mathbf{D}_\Gamma$ is a group k-scheme in this case. The forgetful functor $|-| : \mathbf{Ab} \to \mathbf{CMon}$ sending commutative (abelian) group to its underlying commutative monoid admits left adjoint $(-)_{\mathbf{Grp}} : \mathbf{CMon} \to \mathbf{Ab}$. Hence for every commutative monoid Γ there exists a universal commutative group $\Gamma_{\mathbf{Grp}}$ generated by Γ. This is used in the following result.

Proposition 7.1. *Let Γ be a commutative monoid. Then the canonical morphism $\Gamma \to \Gamma_{\mathbf{Grp}}$ induces a monomorphism of monoid k-schemes*

$$\mathbf{D}_{\Gamma_{\mathbf{Grp}}} \hookrightarrow \mathbf{D}_\Gamma$$

that identifies $\mathbf{D}_{\Gamma_{\mathbf{Grp}}}$ with $(\mathbf{D}_\Gamma)^*$.

Proof. For every k-algebra we have an isomorphism of groups

$$\mathbf{Mon}\left(\Gamma, A^\times\right)^* \cong \mathbf{Mon}\left(\Gamma, A^*\right) \cong \mathbf{Mon}\left(\Gamma_{\mathbf{Grp}}, A^*\right) \cong \mathbf{Mon}\left(\Gamma_{\mathbf{Grp}}, A^\times\right)$$

natural in A. Note that this natural isomorphisms identifies $\mathfrak{P}^*_{\mathbf{D}_\Gamma}$ with $\mathfrak{P}_{\mathbf{D}_{\Gamma_{\mathbf{Grp}}}}$ by morphism induced by the unit $\Gamma \to \Gamma_{\mathbf{Grp}}$ of the adjunction $|-| \vdash (-)_{\mathbf{Grp}}$. □

Corollary 7.2. *Let $\mathbf{G}$ be a group k-scheme. Suppose that G is isomorphic to $\mathbf{D}_\Gamma$ as a monoid k-scheme for some commutative monoid Γ. Then Γ is a group.*

Proof. Suppose that $\mathbf{G} \cong \mathbf{D}_\Gamma$ as a monoid k-schemes. We derive that $\mathbf{D}_\Gamma$ is a group k-scheme. Hence $\mathbf{D}_{\Gamma_{\mathbf{Grp}}} \hookrightarrow \mathbf{D}_\Gamma$ is an isomorphism of monoid k-schemes. This implies that $\Gamma = \Gamma_{\mathbf{Grp}}$ and thus Γ is an abstract group. □

Definition 7.3. Let $\mathbf{G}$ be a group k-scheme. We say that $\mathbf{G}$ is *diagonalisable group k-scheme* if it is diagonalisable as a monoid scheme over k.

Example 7.4. Let $\mathbb{Z}$ be a commutative group of additive integers. We denote by $\mathbb{G}_m$ the monoid k-scheme $\mathbf{D}_\mathbb{Z}$. Note that $\mathbb{G}_m$ represents the group k-functor

$$\mathbf{Alg}_k \ni A \mapsto A^* \in \mathbf{Ab}$$

We call $\mathbb{G}_m$ *the multiplicative group over k*.

Definition 7.5. Let $\mathfrak{G}$ be a monoid k-functor. Then the morphisms $\mathfrak{G} \to \mathfrak{P}_{\mathbb{G}_m}$ of monoid k-functors are called *characters of* $\mathfrak{G}$. They form a group $\mathcal{X}(\mathfrak{G})$ called *the group of characters of* $\mathfrak{G}$.

Corollary 7.6. *Suppose that k is commutative ring such that* $\operatorname{Spec} k$ *is connected (i.e. k has no nontrivial idempotents). Functors*

$$\begin{tikzcd} \Gamma_1 \arrow[d, "g"'] & \longmapsto & \mathbf{D}_{\Gamma_1} \\ \Gamma_2 & & \mathbf{D}_{\Gamma_2} \arrow[u, "\mathbf{D}_g"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{G}_1 \arrow[d, "f"'] & \longmapsto & \mathcal{X}(\mathbf{G}_1) \\ \mathbf{G}_2 & & \mathcal{X}(\mathbf{G}_2) \arrow[u, "\mathcal{X}(f)"'] \end{tikzcd}$$

induce an equivalence between categories of abstract commutative groups and diagonalisable group schemes over k..

Proof. This is a consequence of Theorem 5.2. □

7.1. **Results on linear representations.**

Proposition 7.7. *Let* $\mathbf{M}$ *be an affine monoid k-scheme and let V be a representation of* $\mathbf{M}$. *Then for every k-algebra A the natural morphism of A-modules*

$$V^{\mathbf{M}} \otimes_k A \to (A \otimes_k V)^{\mathbf{M}_A}$$

is an isomorphism.

Proof. Note that we have a left exact sequence of k-vector spaces defining invariants

$$0 \longrightarrow V^{\mathbf{M}} \longrightarrow V \xrightarrow{\Delta - p} \Gamma(\mathbf{M}, \mathcal{O}_{\mathbf{M}}) \otimes_k V$$

where $\Delta : V \to \Gamma(\mathbf{M}, \mathcal{O}_{\mathbf{M}}) \otimes_k V$ is the coaction and $p : V \to \Gamma(\mathbf{M}, \mathcal{O}_{\mathbf{M}}) \otimes_k V$ is the trivial coaction defined by formula $p(v) = 1 \otimes v$ for every v in V. Now tensoring the sequence with k-algebra A yields a left exact sequence

$$0 \longrightarrow V^{\mathbf{M}} \otimes_k A \longrightarrow A \otimes_k V \xrightarrow{\Delta_A - p_A} \Gamma(\mathbf{M}_A, \mathcal{O}_{\mathbf{M}_A}) \otimes_A (A \otimes_k V)$$

where Δ_A is the coaction on $A \otimes_k V$ induced by Δ and p_A is the trivial coaction on $A \otimes_k V$. This shows that $V^{\mathbf{M}} \otimes_k A \to (A \otimes_k V)^{\mathbf{M}_A}$ is an isomorphism. □

Proposition 7.8. *Let* $\mathbf{G}$ *be an affine group k-scheme and let V, W be representations of* $\mathbf{G}$. *If V is finite dimensional, then for every k-algebra A the canonical morphism*

$$A \otimes_k \operatorname{Hom}_{\mathbf{G}}(V, W) \longrightarrow \operatorname{Hom}_{\mathbf{G}_A}(A \otimes_k V, A \otimes_k W)$$

is an isomorphism of A-modules.

Proof. Fix a k-algebra A. Since V is finite dimensional, for every k-algebra B there exists an isomorphism $B \otimes_k \operatorname{Hom}_k(V, W) \to \operatorname{Hom}_B(B \otimes_k V, B \otimes_k W)$ of B-modules natural in B. This implies that $\operatorname{Hom}_k(V, W)$ is a representation of $\mathbf{G}$ via the action given by formula

$$(g \cdot f)(v) = g \cdot f(g^{-1} \cdot v)$$

where $f \in \operatorname{Hom}_B(B \otimes_k V, B \otimes_k W)$, $v \in B \otimes_k V$ and $g \in \mathfrak{P}_{\mathbf{G}}(B)$. Similarly $\operatorname{Hom}_A(A \otimes_k V, A \otimes_k W)$ is a representation of $\mathbf{G}_K$ and the canonical isomorphism $A \otimes_k \operatorname{Hom}_k(V, W) \to \operatorname{Hom}_A(A \otimes_k V, A \otimes_k W)$

of A-modules is $\mathbf{G}_A$-equivariant. Now we apply Proposition 7.7 to derive a chain of isomorphisms

$$\mathrm{Hom}_A\left(A\otimes_k V, A\otimes_k W\right)^{\mathbf{G}_A}\cong\left(A\otimes_k \mathrm{Hom}_k(V,W)\right)^{\mathbf{G}_A}\cong A\otimes_k \mathrm{Hom}_k(V,W)^{\mathbf{G}}$$

of A-modules. Since we have identifications

$$\mathrm{Hom}_{\mathbf{G}_A}\left(A\otimes_k V, A\otimes_k W\right)\cong\mathrm{Hom}_A\left(A\otimes_k V, A\otimes_k W\right)^{\mathbf{G}_A},\ \mathrm{Hom}_{\mathbf{G}}\left(V,W\right)\cong\mathrm{Hom}_k\left(V,W\right)^{\mathbf{G}}$$

we deduce the statement. ☐

Proposition 7.9. *Let $\mathbf{G}$ be an affine group scheme over k and let V, W be $\mathbf{G}$-representation such that $\mathrm{Hom}_{\mathbf{G}}(U,W)=0$ for every finite dimensional $\mathbf{G}$-subrepresentation of V. Then for every k-algebra A we have*

$$\mathrm{Hom}_{\mathbf{G}_A}\left(A\otimes_k V, A\otimes_k W\right)=0$$

Proof. Let $\mathcal{F}$ be a set of all finite dimensional $\mathbf{G}$-subrepresentations of V. Since V is a $\mathbf{G}$-representation and $\mathbf{G}$ is an affine group k-scheme, we have

$$V=\mathrm{colim}_{U\in\mathcal{F}}\,U$$

Fix k-algebra A then we have identifications of A-modules

$$\begin{aligned}\mathrm{Hom}_{\mathbf{G}_A}\left(A\otimes_k V, A\otimes_k W\right)=\mathrm{Hom}_{\mathbf{G}_A}\left(A\otimes_k \mathrm{colim}_{U\in\mathcal{F}}U, A\otimes_k W\right)=\\ =\mathrm{Hom}_{\mathbf{G}_A}\left(\mathrm{colim}_{U\in\mathcal{F}}A\otimes_k U, A\otimes_k W\right)=\lim_{U\in\mathcal{F}}\mathrm{Hom}_{\mathbf{G}_A}\left(A\otimes_k U, A\otimes_k W\right)=\\ =\lim_{U\in\mathcal{F}}\Big(A\otimes_k \mathrm{Hom}_{\mathbf{G}}\left(U,W\right)\Big)=0\end{aligned}$$

where we apply Proposition 7.8. ☐

7.2. Linear algebraic monoids.

Proposition 7.10. *Let $\mathbf{M}$ be a monoid k-scheme. Then the k-functor of units $\mathfrak{P}^*_{\mathbf{M}}$ of $\mathfrak{P}_{\mathbf{M}}$ is representable by a group k-scheme $\mathbf{M}^*$. Moreover, if $\mathbf{M}$ is affine and of finite type over k, then $\mathbf{M}^*$ is an open subscheme of $\mathbf{M}$.*

Proof. Note that $\mathfrak{P}^*_{\mathbf{M}}$ fits into a cartesian square

$$\begin{tikzcd}\mathfrak{P}^*_{\mathbf{M}} \arrow[r] \arrow[d] & \mathbf{1} \arrow[d, "\mathfrak{P}_e"] \\ \mathfrak{P}_{\mathbf{M}}\times\mathfrak{P}_{\mathbf{M}} \arrow[r, "\mathfrak{P}_m"'] & \mathfrak{P}_{\mathbf{M}}\end{tikzcd}$$

where $m:\mathbf{M}\times\mathbf{M}\to\mathbf{M}$ is the multiplication and $e:\mathrm{Spec}\,k\to\mathbf{M}$ is the unit. Since the functor

$$\widehat{\mathbf{Sch}_k}\longrightarrow \textit{the category of k-functors}$$

preserves fiber products, we derive that $\mathfrak{P}^*_{\mathbf{M}}$ is isomorphic to $\mathfrak{P}_{\mathbf{M}^*}$, where $\mathbf{M}^*$ is a k-scheme defined by the cartesian diagram

$$\begin{tikzcd}\mathbf{M}^* \arrow[r] \arrow[d] & \mathrm{Spec}\,k \arrow[d, "e"] \\ \mathbf{M}\times\mathbf{M} \arrow[r, "m"'] & \mathbf{M}\end{tikzcd}$$

Since $\mathfrak{P}_{\mathbf{M}^*} \cong \mathfrak{P}^*_{\mathbf{M}}$, we deduce that $\mathbf{M}^*$ admits a structure of a group k-scheme.
Now suppose that $\mathbf{M}$ is affine monoid k-scheme of finite type over k. Then there exist a finite dimensional vector space V over k and a closed immersion $i : \mathbf{M} \to L(V)$ of monoid k-schemes. □

Definition 7.11. Let $\mathbf{M}$ be an affine monoid k-scheme. Suppose that the group $\mathbf{G}$ of units of $\mathbf{M}$ is an algebraic group over k and that the open immersion $\mathbf{G} \hookrightarrow \mathbf{M}$ is schematically dense. Then $\mathbf{M}$ is *a linear algebraic monoid over k*.

Definition 7.12. Let $\mathbf{M}$ be a linear algebraic monoid over k. Suppose that the group $\mathbf{G}$ of units of $\mathbf{M}$ is (linearly) reductive. Then $\mathbf{M}$ is *a (linearly) reductive monoid over k*.

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