FILTERS IN TOPOLOGY

1. Introduction

In these short notes we study filters of subsets with their applications to topological spaces. Filters were introduced in [Cartan, 1937] as an effective tool in studying general topological spaces. Here we recapitulate some of Cartan's results. In particular, we give a concise proof of Tychonoff's theorem on compact spaces. For introduction to topological spaces we refer to [Monygham, 2024].

2. Filters

Definition 2.1. Let X be a set and let $\mathcal{F}$ be a nonempty family of subsets of X. Assume that the following assertions hold.

(1) $\mathcal{F}$ is closed under finite intersections.

(2) If F_1 and F_2 are subsets of X such that $F_1 \in \mathcal{F}$ and $F_1 \subseteq F_2$, then $F_2 \in \mathcal{F}$.

Then $\mathcal{F}$ is a *filter* on X.

We note the following fact.

Fact 2.2. *Let X be a set and let $\{\mathcal{F}_i\}_{i\in I}$ be a family of filters on X. Then*

$$\bigcap_{i\in I} \mathcal{F}_i$$

is a filter on X.

Proof. Left for the reader as an exercise. □

Definition 2.3. Let X be a set and let $\mathcal{F}$ be a filter on X. Assume that $\emptyset \notin \mathcal{F}$. Then $\mathcal{F}$ is *proper*.

Filters are functorial as it is displayed in the following notion.

Definition 2.4. Let $\mathcal{F}$ be a filter on a set X and let $f : X \to Y$ be a map of sets. Then a filter

$$f(\mathcal{F}) = \left\{Z \subseteq Y \,\middle|\, \text{there exists } F \in \mathcal{F} \text{ such that } f(F) \subseteq Z\right\}$$

on Y is the *image* of $\mathcal{F}$ under f.

Let us note the following result.

Fact 2.5. *Let $\mathcal{F}$ be a filter on a set X and let $f : X \to Y$ be a map of sets. If $\mathcal{F}$ is proper, then $f(\mathcal{F})$ is proper.*

Proof. Left for the reader as an exercise. □

Now we introduce the notion of ultrafilter and prove its properties.

Definition 2.6. Let X be a set and let $\mathcal{F}$ be a proper filter on X. Suppose that $\mathcal{F}$ is maximal with respect to inclusion among proper filters on X. Then $\mathcal{F}$ is an *ultrafilter* on X.

Proposition 2.7. *Let X be a set and let $\mathcal{F}$ be a proper filter on X. The following assertions are equivalent.*

(i) *$\mathcal{F}$ is an ultrafilter on X.*

(ii) *For each subset F of X either $F \in \mathcal{F}$ or $X \setminus F \in \mathcal{F}$.*

Proof. Assume that $\mathcal{F}$ is an ultrafilter and let F be a subset of X. Suppose that $F \notin \mathcal{F}$. Then the smallest filter containing $\{F\} \cup \mathcal{F}$, which exists according to Fact 2.2, is not a proper filter. This implies that there exists $F' \in \mathcal{F}$ such that $F \cap F' = \emptyset$. Since $F' \subseteq X \setminus F$ and $\mathcal{F}$ is a filter, we derive that $X \setminus F \in \mathcal{F}$. This proves that **(i)** $\Rightarrow$ **(ii)**.

Suppose that for each subset F of X either $F \in \mathcal{F}$ or $X \setminus F \in \mathcal{F}$. Consider a filter $\tilde{\mathcal{F}}$ such that $\mathcal{F} \subsetneq \tilde{\mathcal{F}}$. If $F \in \tilde{\mathcal{F}} \setminus \mathcal{F}$, then $X \setminus F \in \mathcal{F}$ and hence $\emptyset = F \cap (X \setminus F) \in \tilde{\mathcal{F}}$. This implies that $\tilde{\mathcal{F}}$ is not a proper filter. Thus $\mathcal{F}$ is an ultrafilter on X. This completes the proof of **(ii)** $\Rightarrow$ **(i)**. □

Corollary 2.8. *Let $f : X \to Y$ be a map of sets and let $\mathcal{F}$ be an ultrafilter of subsets of X. Then $f(\mathcal{F})$ is an ultrafilter.*

Proof. Filter $f(\mathcal{F})$ is proper according to Fact 2.5. Fix a subset F of Y. By Proposition 2.7 either $f^{-1}(F) \in \mathcal{F}$ or $f^{-1}(Y \setminus F) \in \mathcal{F}$. Thus either $F \in f(\mathcal{F})$ or $Y \setminus F \in f(\mathcal{F})$. Proposition 2.7 implies that $f(\mathcal{F})$ is an ultrafilter. □

The following result uses axiom of choice.

Proposition 2.9. *Let X be a set and let $\mathcal{F}$ be a proper filter on X. Then there exists an ultrafilter $\tilde{\mathcal{F}}$ on X such that $\mathcal{F} \subseteq \tilde{\mathcal{F}}$.*

Proof. Consider the family

$$\mathbf{F} = \{\mathcal{G} \,|\, \mathcal{G} \text{ is a proper filter on } X \text{ and } \mathcal{F} \subseteq \mathcal{G}\}$$

Note that $\mathbf{F}$ is nonempty because $\mathcal{F} \in \mathbf{F}$. The inclusion of filters introduces partial order on $\mathbf{F}$ and if $\mathrm{L} \subseteq \mathbf{F}$ is a linearly ordered subset, then

$$\bigcup \mathrm{L}$$

is a proper filter. Hence each chain in $(\mathbf{F}, \subseteq)$ admits an upper bound. Zorn's lemma implies that $(\mathbf{F}, \subseteq)$ has a maximal element $\tilde{\mathcal{F}}$. Clearly $\tilde{\mathcal{F}}$ is an ultrafilter on X and $\mathcal{F} \subseteq \tilde{\mathcal{F}}$. □

3. Filters and convergence in topological spaces

Definition 3.1. Let X be a topological space and let $\mathcal{F}$ be a proper filter on X. Consider a point x in X. Suppose that for every open neighborhood U of x we have $U \in \mathcal{F}$. Then $\mathcal{F}$ *converges* to x in X.

Proposition 3.2. *Let X, Y be topological spaces and let $f : X \to Y$ be a map of sets. Let x be a point in X. Then the following assertions are equivalent.*

- **(i)** *f is continuous at x.*
- **(ii)** *If $\mathcal{F}$ is a proper filter on X convergent to x, then $f(\mathcal{F})$ converges to $f(x)$.*
- **(iii)** *If $\mathcal{F}$ is an ultrafilter on X convergent to x, then $f(\mathcal{F})$ converges to $f(x)$.*

Proof. Suppose that f is a continuous at x. Fix a proper filter $\mathcal{F}$ on X convergent to x. Fix an open neighborhood V of $f(x)$ in Y. Since f is continuous at x, there exists an open neighborhood U of x such that $f(U) \subseteq V$. Note that $U \in \mathcal{F}$ and hence $V \in f(\mathcal{F})$. Since V is arbitrary open neighborhood of $f(x)$ in Y, we derive that $f(\mathcal{F})$ converges to $f(x)$ in Y. This proves the implication **(i)** $\Rightarrow$ **(ii)**.

The implication **(ii)** $\Rightarrow$ **(iii)** follows from definition of an ultrafilter.

Suppose now that **(iii)** holds. Consider an open neighborhood V of $f(x)$ in Y. Assume that for every open neighborhood U of x in X the set $U \setminus f^{-1}(V)$ is nonempty. Let $\mathcal{F}$ be a filter generated by all sets of the form $U \setminus f^{-1}(V)$ where U is an open neighborhood of x. Then $\mathcal{F}$ is a proper filter on X. Next by Proposition 2.9 there exists an ultrafilter $\tilde{\mathcal{F}}$ on X which contains $\mathcal{F}$. Since $\mathcal{F}$

converges to x in X, we derive that $\tilde{\mathcal{F}}$ converges to x in X. Thus $f(\tilde{\mathcal{F}})$ converges to $f(x)$ in Y. Note that

$$f\left(X \setminus f^{-1}(V)\right) \in f(\tilde{\mathcal{F}})$$

This implies that $Y \setminus V \in f(\tilde{\mathcal{F}})$ and hence $V \notin f(\tilde{\mathcal{F}})$. It follows that the filter $f(\tilde{\mathcal{F}})$ cannot converge to $f(x)$ in Y. We arrive at contradiction. This means that there exists an open neighborhood U of x in X such that $U \subseteq f^{-1}(V)$. This proves that f is continuous at x. We infer **(iii)** $\Rightarrow$ **(i)**. □

Definition 3.3. Let X be a topological space and let $\mathcal{F}$ be a proper filter on X. Consider a point x in X. Suppose that for every open neighborhood U of x and every $F \in \mathcal{F}$ we have $F \cap U \neq \emptyset$. Then x is an *accumulation point* of $\mathcal{F}$.

Theorem 3.4. *Let X be a topological space. Then the following assertions are equivalent.*

- **(i)** *Each ultrafilter on X is convergent to some point of X.*
- **(ii)** *Each proper filter on X admits an accumulation point.*
- **(iii)** *X is a quasi-compact topological space.*

Proof. Suppose that **(i)** holds. Let $\mathcal{F}$ be a proper filter on X. By Proposition 2.9 there exists an ultrafilter $\tilde{\mathcal{F}}$ that contains $\mathcal{F}$. By assumption $\tilde{\mathcal{F}}$ is convergent to some point x of X. Then x is an accumulation point of $\mathcal{F}$. Hence **(i)** $\Rightarrow$ **(ii)**.

Assume that **(ii)**. Pick a centered family $\mathcal{F}$ of closed subsets of X. Define

$$\tilde{\mathcal{F}} = \left\{F \,\middle|\, F \text{ contains some finite intersection of sets in } \mathcal{F}\right\}$$

Then $\tilde{\mathcal{F}}$ is a proper filter on X. By assumption $\tilde{\mathcal{F}}$ admits an accumulation point – say $x \in X$. Pick now $F \in \mathcal{F}$. Then for every open neighborhood U of x we have $U \cap F \neq \emptyset$. Since F is closed in X, we derive that $x \in F$. Thus x is contained in the intersection of $\mathcal{F}$. This implies that X is quasi-compact. This completes the proof of **(ii)** $\Rightarrow$ **(iii)**.

Assume that X is quasi-compact and suppose that $\mathcal{F}$ is an ultrafilter on X. Suppose that $\mathcal{F}$ is not convergent. Then for every $x \in X$ there exists open neighborhood U_x of x in X such that $U_x \notin \mathcal{F}$. Since X is quasi-compact, we deduce that there exist $n \in \mathbb{N}_+$ and $x_1, ..., x_n \in X$ such that

$$X = \bigcup_{i=1}^{n} U_{x_i}$$

According to Proposition 2.7 we derive that $X \setminus U_x \in \mathcal{F}$ for every $x \in X$. Hence

$$\bigcap_{i=1}^{n} (X \setminus U_{x_i}) \in \mathcal{F}$$

On the other hand we have

$$\bigcap_{i=1}^{n} (X \setminus U_{x_i}) = X \setminus \bigcup_{i=1}^{n} U_{x_i} = \emptyset$$

This is contradiction. Thus the implication **(iii)** $\Rightarrow$ **(i)** holds. □

4. Tychonoff's theorem

The following result is a celebrated theorem due to Tychonoff.

Theorem 4.1. *Let $\{X_i\}_{i \in I}$ be a family of quasi-compact topological spaces. Then the product*

$$\prod_{i \in I} X_i$$

is quasi-compact.

Proof. We denote $\prod_{i\in I} X_i$ by X. For each i in I we denote by $pr_i : X \to X_i$ the canonical projection onto i-th factor. Suppose that X_i is quasi-compact for every $i \in I$. Pick an ultrafilter $\mathcal{F}$ on X. Fix i in I. According to Corollary 2.8 the filter $pr_i(\mathcal{F})$ is an ultrafilter. Since X_i is quasi-compact, we derive that $pr_i(\mathcal{F})$ is convergent to some point $x_i \in X_i$. Let x be a point of X such that $pr_i(x) = x_i$ for each $i \in I$. Fix finite subset $\{i_1, ..., i_n\} \subseteq I$. Consider open neighborhood U_j of x_{i_j}. Then $U_{i_j} \in pr_{i_j}(\mathcal{F})$ for each j and hence $pr_{i_j}^{-1}(U_{i_j}) \in \mathcal{F}$ for each j. Since $\mathcal{F}$ is a filter, we derive that

$$\prod_{j=1}^{n} U_{i_j} \times \prod_{i\in I\setminus\{i_1,...,i_n\}} X_i = \bigcap_{j=1}^{n} pr_{i_j}^{-1}(U_{i_j}) \in \mathcal{F}$$

This implies that $\mathcal{F}$ is convergent to x in X. Thus every ultrafilter in X is convergent and hence Theorem 3.4 shows that X is quasi-compact. □

Theorem 4.2. *Let $\{X_i\}_{i\in I}$ be a family of nonempty topological spaces. If the product*

$$\prod_{i\in I} X_i$$

is quasi-compact, then X_i is quasi-compact for every $i \in I$.

Proof. We denote $\prod_{i\in I} X_i$ by X. For each i in I we denote by $pr_i : X \to X_i$ the canonical projection onto i-th factor. Assume that X is quasi-compact. Since $X_i \neq \emptyset$ for every $i \in I$, we derive that $pr_i : X \to X_i$ is a continuous and surjective map for every $i \in I$. Hence for each $i \in I$ space X_i is quasi-compact as an image of a quasi-compact space under continuous map. □

5. Extensions of continuous maps

Definition 5.1. Let X be a topological space. Suppose that for every $x \in X$ and every closed subset F of X does not containing x there exist disjoint open subsets U and V such that $x \in U$ and $F \subseteq V$. Then X is *regular*.

Corollary 5.2. *Each completely regular space is regular.*

We can apply filters to prove the following interesting and useful result. For this we introduce the following notion.

Definition 5.3. Let X be a topological space and let Z be a dense subspace of X. Fix a point x in X. Then a family

$$\{F \subseteq Z \,|\, U \cap Z \subseteq F \text{ for some open neighborhood } U \text{ of } x\}$$

is the *trace* on Z of the neighborhood filter of x.

Fact 5.4. *Let X be a topological space and let Z be a dense subspace of X. Then the trace on Z of the neighborhood filter of x is a proper filter on Z for every $x \in X$.*

Proof. Left for the reader. □

Theorem 5.5. *Let X and Y be topological spaces and let Z be a dense subspace of X. Suppose that Y is regular and $f : Z \to Y$ is a continuous map. Let $\mathcal{F}_x$ the trace of the neighborhood filter of x on Z for each $x \in X$. Consider the following assertions.*

(i) *There exists a continuous map $\tilde{f} : X \to Y$ extending f.*

(ii) *$f(\mathcal{F}_x)$ is convergent in Y for every $x \in X$.*

Then **(i)** $\Rightarrow$ **(ii)** *and if Y is regular, then* **(ii)** $\Rightarrow$ **(i)** *holds.*

For the proof we need the following result.

Lemma 5.5.1. *Let X and Y be topological spaces and let Z be a dense subspace of X. Suppose $f : Z \to Y$ is a continuous map. Let U be an open subset of X and let $x \in U$ be a point such that $f(\mathcal{F}_x)$ is convergent in Y. Then $f(\mathcal{F}_x)$ converges to a point of* $\mathbf{cl}\,(f(U \cap Z))$.

Proof of the lemma. Suppose that $f(\mathcal{F}_x)$ converges to $y \in Y$. Let V be an open neighborhood of y in Y. Then $V \in f(\mathcal{F}_x)$ and hence $f^{-1}(V) \in \mathcal{F}_x$. Since $U \cap Z \in \mathcal{F}_x$ and $\mathcal{F}_x$ is a proper filter by Fact 5.4, we derive that $f^{-1}(V) \cap (U \cap Z)$ is nonempty. Thus $V \cap f(U \cap Z)$ is nonempty. This proves that y is in the closure of $f(U \cap Z)$. □

Proof of the theorem. Suppose that there exists a continuous extension $\tilde{f} : X \to Y$ of f. Let $\tilde{\mathcal{F}}_x$ be the filter on X generated by $\mathcal{F}_x$ for each $x \in X$. Then $\tilde{\mathcal{F}}_x$ is convergent to x. Hence $\tilde{f}\left(\tilde{\mathcal{F}}_x\right)$ is convergent to $\tilde{f}(x)$ in Y. Since $f(\mathcal{F}_x) = \tilde{f}\left(\tilde{\mathcal{F}}_x\right)$, we deduce that $f(\mathcal{F}_x)$ is convergent to $\tilde{f}(x)$ in Y. This completes the proof of **(i)** $\Rightarrow$ **(ii)**.

Now suppose that Y is regular and $f(\mathcal{F}_x)$ is convergent in Y for every $x \in X$. We define $\tilde{f}(x)$ as a limit in Y of $f(\mathcal{F}_x)$ for all $x \in X \setminus Z$ and $\tilde{f}(z) = f(z)$ for each $z \in Z$. It follows that $\tilde{f} : X \to Y$ is an extension of f such that $\tilde{f}(x)$ is a limit in Y of $f(\mathcal{F}_x)$ for all $x \in X$. It suffices to check that $\tilde{f}$ is continuous. Pick x in X and let V be an arbitrary open neighborhood of $\tilde{f}(x)$ in Y. Since Y is regular, there exists an open neighborhood W of $\tilde{f}(x)$ in Y such that $\mathbf{cl}(W) \subseteq V$. Note that $W \in f(\mathcal{F}_x)$ and hence $f^{-1}(W) \in \mathcal{F}_x$. By definition of trace there exists open neighborhood U of x such that $U \cap Z \subseteq f^{-1}(W)$. Lemma 5.5.1 implies that

$$\tilde{f}(U) \subseteq \mathbf{cl}\,(f(U \cap Z)) \subseteq \mathbf{cl}\,(W) \subseteq V$$

This shows that $\tilde{f}$ is continuous. We infer that **(ii)** $\Rightarrow$ **(i)**. □

References

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