## 1. Introduction

Throughout this notes $k$ denote a field and $\mathbf{G}$ denote a group scheme over $k$. We also fix a $k$-scheme $X$ equipped with an action of $\mathbf{G}$ determined by morphism $a : \mathbf{G} \times_k X \to X$.

## 2. Categorical and geometric quotients

**Definition 2.1.** Let $q : X \to Y$ be a morphism of $k$-schemes such that the diagram

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

is a cokernel in the category of $k$-schemes. Then $q : X \to Y$ is *a categorical quotient of* $X$.

**Definition 2.2.** Consider a cokernel

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

in the category of locally ringed spaces over $k$. If $Y$ is a scheme, then $q : X \to Y$ is *a geometric quotient of* $X$.

**Fact 2.3.** *Every geometric quotient is categorical.*

*Proof.* Categorical quotient is a cokernel in the category of $k$-schemes. On the other hand geometric quotient is a cokernel in the category of locally ringed spaces and hence it also satisfies cokernel property in its full subcategory of $k$-schemes. Thus every geometric quotient is categorical. □

**Corollary 2.4.** *Let $q : X \to Y$ be a morphism of schemes. The following assertions are equivalent.*

**(i)** *The diagram*

$$\mathbf{G} \times_k X \underset{\mathrm{pr}_X}{\overset{a}{\rightrightarrows}} X \xrightarrow{q} Y$$

*is a cokernel diagram of underlying topological spaces and the diagram*

$$\mathcal{O}_Y \xrightarrow{q^{\#}} q_*\mathcal{O}_X \underset{q_*\mathrm{pr}_X^{\#}}{\overset{q_*a^{\#}}{\rightrightarrows}} q_*\left(\mathrm{pr}_X\right)_* \mathcal{O}_{\mathbf{G}\times_k X} = q_* a_* \mathcal{O}_{\mathbf{G}\times_k X}$$

*is a kernel diagram in the category of sheaves on $Y$.*

**(ii)** *$q$ is a geometric quotient of $X$.*

*Proof.* This is a consequence of [Monygham, 2019, Theorem 2.9]. □

Let $q : X \to Y$ be a morphism of $k$-schemes such that $q \cdot \mathrm{pr}_X = q \cdot a$. For a morphism $g : Y' \to Y$ of $k$-schemes consider the cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Then there exists a unique action $a' : \mathbf{G} \times_k X' \to X'$ of $\mathbf{G}$ on $X'$ such that the square above consists of $\mathbf{G}$-equivariant morphism (we consider $Y, Y'$ as $\mathbf{G}$-schemes equipped with trivial $\mathbf{G}$-actions). Keeping this in mind we have the following.

**Definition 2.5.** A morphism $q : X \to Y$ is *a universal categorical (geometric) quotient of* $X$ if for every morphism $g : Y' \to Y$ its base change $q' : X' \to Y'$ is a categorical (geometric) quotient of $X'$.

## 3. Types of actions and criterion for smoothness of universal geometric quotients

**Definition 3.1.** The action of $\mathbf{G}$ on $X$ is

**(1)** *separated* if the morphism $(a, \mathrm{pr}_X) : \mathbf{G} \times_k X \to X \times_k X$ has closed set-theoretic image,

**(2)** *free* if the morphism $(a, \mathrm{pr}_X) : \mathbf{G} \times_k X \to X \times_k X$ is a closed immersion.

**Theorem 3.2.** *Let $q : X \to Y$ be a geometric quotient of $X$. If the action of $\mathbf{G}$ on $X$ is separated and $X$ is a separated $k$-scheme, then $Y$ is separated.*

## References

[Monygham, 2019] Monygham (2019). Locally ringed spaces. *github repository: "Monygham/Pedo-mellon-a-minno"*.