

# GEOMETRY OF $k$ -FUNCTORS

## 1. INTRODUCTION

In these notes we provide functorial approach to algebraic geometry. Our aim is to show that functorial and geometrical techniques are interrelated in a very efficient way.

Throughout these notes  $k$  is a fixed commutative ring and  $\mathbf{Alg}_k$  denote the category of commutative  $k$ -algebras. If  $A, B$  are  $k$ -algebras, then we denote by  $\mathrm{Mor}_k(A, B)$  the set of all morphisms  $A \rightarrow B$  of  $k$ -algebras. Similarly if  $X, Y$  are  $k$ -schemes (i.e. schemes together with morphism to  $\mathrm{Spec} k$ ), then we denote by  $\mathrm{Mor}_k(X, Y)$  the set of all morphisms  $X \rightarrow Y$  of  $k$ -schemes (morphisms of schemes that preserve structure morphisms to  $\mathrm{Spec} k$ ).

## 2. $k$ -FUNCTORS

**Definition 2.1.** The category  $\mathbf{Fun}(\mathbf{Alg}_k, \mathbf{Set})$  of copresheaves on  $\mathbf{Alg}_k$  is called *the category of  $k$ -functors*.

If  $\mathfrak{X}$  and  $\mathfrak{Y}$  are  $k$ -functors, then we denote by  $\mathrm{Mor}_k(\mathfrak{X}, \mathfrak{Y})$  the class of morphisms  $\mathfrak{X} \rightarrow \mathfrak{Y}$  of  $k$ -functors. If  $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$  is a morphism of  $k$ -functors, then for every  $k$ -algebra  $A$  we denote by  $\sigma^A$  the corresponding component of  $\sigma$ .

Let  $\mathfrak{X}$  and  $\mathfrak{Y}$  be  $A$ -functors for some  $k$ -algebra  $A$ . Then we denote by  $\mathrm{Mor}_A(\mathfrak{X}, \mathfrak{Y})$  the class of morphisms of  $A$ -functors  $\mathfrak{X} \rightarrow \mathfrak{Y}$ . For every  $A$ -algebra  $B$  and a morphism  $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$  of  $A$ -functors we denote by  $\mathfrak{X}_B, \mathfrak{Y}_B, \sigma_B$  the restrictions  $\mathfrak{X}|_{\mathbf{Alg}_B}, \mathfrak{Y}|_{\mathbf{Alg}_B}, \sigma|_{\mathbf{Alg}_B}$  of these entities to the category of  $B$ -algebras.

**Fact 2.2.** Let  $\mathfrak{X}$  and  $\mathfrak{Y}$  be  $k$ -functors. Assume that  $A$  is a  $k$ -algebra,  $B$  is an  $A$ -algebra,  $C$  is an  $B$ -algebra. Then the composition of maps of classes

$$\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A) \xrightarrow{\sigma \mapsto \sigma_B} \mathrm{Mor}_B(\mathfrak{X}_B, \mathfrak{Y}_B) \xrightarrow{\sigma \mapsto \sigma_C} \mathrm{Mor}_C(\mathfrak{X}_C, \mathfrak{Y}_C)$$

equals

$$\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A) \xrightarrow{\sigma \mapsto \sigma_C} \mathrm{Mor}_C(\mathfrak{X}_C, \mathfrak{Y}_C)$$

*Proof.* Left to the reader. □

**Definition 2.3.** Let  $\mathfrak{X}$  and  $\mathfrak{Y}$  be  $k$ -functors and suppose that for every  $k$ -algebra  $A$  the class  $\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$  is a set. We define

$$\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(A) = \mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$$

for every  $k$ -algebra  $A$ . This is a  $k$ -functor. Indeed, for every  $k$ -algebra  $A$  and  $A$ -algebra  $B$  we can compose a morphism  $\sigma : \mathfrak{X}_A \rightarrow \mathfrak{Y}_A$  of  $k$ -functors with the forgetful functor  $\mathbf{Alg}_B \rightarrow \mathbf{Alg}_A$ . This induces a map

$$\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(A) \ni \sigma \mapsto \sigma_B \in \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(B)$$

and according to Fact 2.2 these maps make  $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})$  a  $k$ -functor. The  $k$ -functor  $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})$  is called *a hom  $k$ -functor of  $\mathfrak{X}$  and  $\mathfrak{Y}$* .

3. ZARISKI LOCAL  $k$ -FUNCTORS AND ZARISKI SHEAVES

In this part we use the notion of a Grothendieck topology on a category. For this notion we refer the reader to [Mon19a].

**Definition 3.1.** Let  $\{f_i : X_i \rightarrow X\}_{i \in I}$  be a family of morphisms of  $k$ -schemes. We say that  $\{f_i\}_{i \in I}$  is a *Zariski covering* of  $X$  if the following conditions are satisfied.

- (1) For every  $i \in I$  morphism  $f_i$  is an open immersion of schemes.
- (2) Morphism  $\coprod_{i \in I} X_i \rightarrow X$  induced by  $\{f_i\}_{i \in I}$  is surjective.

The collection of all Zariski coverings on  $\mathbf{Sch}_k$  is a Grothendieck pretopology.

**Definition 3.2.** We call the Grothendieck topology generated by the pretopology consisting of Zariski coverings on  $\mathbf{Sch}_k$  the *Zariski topology on  $\mathbf{Sch}_k$* . A presheaf on  $\mathbf{Sch}_k$  that is a sheaf with respect to Zariski topology on  $\mathbf{Sch}_k$  is called a *Zariski sheaf*.

Let  $\mathfrak{X}$  be a presheaf on the category of  $k$ -schemes. Recall that by [Mon19a, Theorem 3.5]  $\mathfrak{X}$  is a Zariski sheaf if and only if for every  $k$ -scheme  $X$  and for every Zariski covering  $\{f_i : X_i \rightarrow X\}$  of  $X$  the diagram

$$\mathfrak{X}(X) \xrightarrow{\langle \mathfrak{X}(f_i) \rangle_{i \in I}} \prod_{i \in I} \mathfrak{X}(X_i) \xrightleftharpoons[\langle \mathfrak{X}(f''_{ij}) \cdot pr_j \rangle_{(i,j)}]{\langle \mathfrak{X}(f'_{ij}) \cdot pr_i \rangle_{(i,j)}} \prod_{(i,j) \in I \times I} \mathfrak{X}(X_i \times_X X_j)$$

is a kernel of a pair of arrows, where for every  $(i, j) \in I \times I$  morphisms  $f'_{ij}$  and  $f''_{ij}$  form a cartesian square

$$\begin{array}{ccc} X_i \times_X X_j & \xrightarrow{f''_{ij}} & X_j \\ f'_{ij} \downarrow & & \downarrow f_j \\ X_i & \xrightarrow{f_i} & X \end{array}$$

Now we repeat this definitions for  $k$ -algebras and  $k$ -functors.

**Definition 3.3.** Let  $\{f_i : A \rightarrow A_i\}_{i \in I}$  be a family of morphisms of  $k$ -algebras. We say that  $\{f_i\}_{i \in I}$  is a *Zariski covering* of  $A$  if the following conditions are satisfied.

- (1) For every  $i \in I$  morphism  $f_i$  is an open immersion of schemes.
- (2) Morphism  $\coprod_{i \in I} \text{Spec } A_i \rightarrow \text{Spec } A$  induced by  $\{f_i\}_{i \in I}$  is surjective.

The collection of all Zariski coverings on  $\mathbf{Alg}_k$  induces on its opposite category  $\mathbf{Aff}_k$  of affine  $k$ -schemes a Grothendieck pretopology.

**Definition 3.4.** We call the Grothendieck topology generated by the pretopology consisting of Zariski coverings on  $\mathbf{Aff}_k$  the *Zariski topology on  $\mathbf{Aff}_k$* . A  $k$ -functor that is a sheaf with respect to Zariski topology on  $\mathbf{Aff}_k$  is called a *Zariski local  $k$ -functor*.

Let  $\mathfrak{X}$  be a  $k$ -functor. Again by [Mon19a, Theorem 3.5]  $\mathfrak{X}$  is a Zariski local  $k$ -functor if and only if for every  $k$ -algebra  $A$  and for every Zariski covering  $\{f_i : A \rightarrow A_i\}$  of  $A$  the diagram

$$\mathfrak{X}(A) \xrightarrow{\langle \mathfrak{X}(f_i) \rangle_{i \in I}} \prod_{i \in I} \mathfrak{X}(A_i) \xrightleftharpoons[\langle \mathfrak{X}(f''_{ij}) \cdot pr_j \rangle_{(i,j)}]{\langle \mathfrak{X}(f'_{ij}) \cdot pr_i \rangle_{(i,j)}} \prod_{(i,j) \in I \times I} \mathfrak{X}(A_i \otimes_A A_j)$$

is a kernel of a pair of arrows, where for every  $(i, j) \in I \times I$  morphisms  $f'_{ij}$  and  $f''_{ij}$  form a cocartesian square

$$\begin{array}{ccc} A & \xrightarrow{f_j} & A_j \\ f_i \downarrow & & \downarrow f'_{ji} \\ A_i & \xrightarrow{f'_{ij}} & A_i \otimes_A A_j \end{array}$$

Now we state the main result of this section.

**Theorem 3.5.** *Let*

$$\widehat{\mathbf{Sch}}_k \longrightarrow \text{the category of } k\text{-functors}$$

*be the restriction of presheaves on  $\mathbf{Sch}_k$  to copresheaves on  $\mathbf{Alg}_k$  ( $k$ -functors) induced by the contravariant functor  $\text{Spec} : \mathbf{Alg}_k \rightarrow \mathbf{Sch}_k$ . Then it induces an equivalence of categories between Zariski sheaves on  $\mathbf{Sch}_k$  and Zariski local  $k$ -functors.*

*Proof.* Note that  $\mathbf{Aff}_k$  with Zariski topology is a dense subsite ([Mon19a, definition 4.4]) of  $\mathbf{Sch}_k$  with Zariski topology. Hence the result is a special case of a more general theorem [Mon19a, Theorem 4.6].  $\square$

#### 4. SCHEMES AND THEIR FUNCTORS OF POINTS

Let  $X$  be a  $k$ -scheme. We define a  $k$ -functor  $\mathfrak{P}_X$  by formula

$$\mathfrak{P}_X(A) = \text{Mor}_k(\text{Spec } A, X)$$

That is  $\mathfrak{P}_X$  is the restriction of the presheaf on  $\mathbf{Sch}_k$  represented by  $X$  to the category  $\mathbf{Alg}_k$  along the functor  $\text{Spec} : \mathbf{Alg}_k \rightarrow \mathbf{Sch}_k$ . Next if  $f : X \rightarrow Y$  is a morphism of  $k$ -schemes, then  $\mathfrak{P}_f$  is the restriction of a morphism of presheaves on  $\mathbf{Sch}_k$  represented by  $f$  to the category of  $k$ -algebras along  $\text{Spec} : \mathbf{Alg}_k \rightarrow \mathbf{Sch}_k$ . Thus we have a functor

$$\mathbf{Sch}_k \xrightarrow{\mathfrak{P}} \text{the category of } k\text{-functors}$$

v

**Fact 4.1.** *Functor*

$$\mathbf{Sch}_k \xrightarrow{\mathfrak{P}} \text{the category of } k\text{-functors}$$

*is full, faithful and its image consists of Zariski local  $k$ -functors. Moreover,  $\mathfrak{P}$  preserves limits.*

*Proof.* Note that the presheaf  $h_X$  on  $\mathbf{Sch}_k$  represented by  $X$  is a Zariski sheaf. Indeed, this just rephrases standard fact that morphism of schemes can be glued in Zariski topology. Next according to Theorem 3.5 the functor  $\text{Spec} : \mathbf{Alg}_k \rightarrow \mathbf{Sch}_k$  induces an equivalence between the category of Zariski sheaves and the category of local Zariski  $k$ -functors. Thus  $\mathfrak{P}_X$  is a local Zariski  $k$ -functor and  $\mathfrak{P}$  it is full and faithful. Note that Yoneda embedding  $h : \mathbf{Sch}_k \rightarrow \widehat{\mathbf{Sch}}_k$  and the functor

$$\widehat{\mathbf{Sch}}_k \xrightarrow{\text{induced by Spec}} \text{the category of } k\text{-functors}$$

preserve limits. Thus their composition  $\mathfrak{P}$  also preserves limits.  $\square$

**Definition 4.2.** Let  $X$  be a  $k$ -scheme. Then  $\mathfrak{P}_X$  is called *the  $k$ -functor of points of  $X$* .

Finally note that for every  $k$ -algebra  $A$  we have an identification  $\mathfrak{P}_{\text{Spec } A} = \text{Hom}_k(A, -)$  and this identification is natural with respect to  $A$ . In other words  $\mathfrak{P} \cdot \text{Spec}$  is the (co)Yoneda embedding of  $\mathbf{Alg}_k$  into the category of  $k$ -functors.

Suppose now that  $A$  is a  $k$ -algebra and  $\mathfrak{a} \subseteq A$  is an ideal. Then we define  $V(\mathfrak{a}) = \text{Spec } A/\mathfrak{a}$  as a closed subscheme  $\text{Spec } A$  induced by the quotient morphism  $A \rightarrow A/\mathfrak{a}$ . We define an open subscheme  $D(\mathfrak{a}) = \text{Spec } A \setminus V(\mathfrak{a})$  of  $\text{Spec } A$ .

**Definition 4.3.** Let  $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$  be a morphism of  $k$ -functors. Assume that for every  $k$ -algebra  $A$  and every morphism  $\tau : \mathfrak{P}_{\text{Spec } A} \rightarrow \mathfrak{Y}$  of  $k$ -functors there exist an ideal  $\mathfrak{a}$  in  $A$  and a morphism  $\tau' : \mathfrak{P}_{D(\mathfrak{a})} \rightarrow \mathfrak{X}$  of  $k$ -functors such that the square

$$\begin{array}{ccc} \mathfrak{P}_{D(\mathfrak{a})} & \xrightarrow{\tau'} & \mathfrak{X} \\ \downarrow & & \downarrow \sigma \\ \mathfrak{P}_{\text{Spec } A} & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

is cartesian. Then  $\sigma$  is an *open immersion of  $k$ -functors*.

**Fact 4.4.** *The class of open immersions of  $k$ -functors is closed under base change and composition.*

*Proof.* Left to the reader.  $\square$

**Definition 4.5.** Let  $\mathfrak{X}$  be a  $k$ -functor and  $\{\sigma_i : \mathfrak{X}_i \rightarrow \mathfrak{X}\}_{i \in I}$  be a family of open immersions. Then for every  $k$ -algebra  $A$  and  $x \in \mathfrak{X}(A)$  we have a family of ideals  $\{\mathfrak{a}_i\}_{i \in I}$  defined by cartesian squares

$$\begin{array}{ccc} \mathfrak{P}_{D(\mathfrak{a}_i)} & \xrightarrow{\tau'} & \mathfrak{X}_i \\ \downarrow & & \downarrow \sigma_i \\ \mathfrak{P}_{\text{Spec } A} & \xrightarrow{\tau} & \mathfrak{X} \end{array}$$

in which bottom vertical morphism  $\tau : \mathfrak{P}_{\text{Spec } A} \rightarrow \mathfrak{X}$  corresponds to  $x$ . We say that  $\{\sigma_i\}_{i \in I}$  is an *open cover of  $\mathfrak{X}$*  if for every  $k$ -algebra  $A$  and  $x \in \mathfrak{X}(A)$  we have

$$\text{Spec } A = \bigcup_{i \in I} D(\mathfrak{a}_i)$$

or in other words  $A = \sum_{i \in I} \mathfrak{a}_i$ .

**Theorem 4.6.** *Let  $\mathfrak{X}$  be a  $k$ -functor. Then the following are equivalent.*

- (i)  $\mathfrak{X}$  is isomorphic with functor of points of some  $k$ -scheme.
- (ii)  $\mathfrak{X}$  is a Zariski local  $k$ -functor and there exists an open cover  $\{\sigma_i : \mathfrak{P}_{X_i} \rightarrow \mathfrak{X}\}_{i \in I}$  of  $k$ -functors for some family  $\{X_i\}_{i \in I}$  of  $k$ -schemes.
- (iii)  $\mathfrak{X}$  is a Zariski local  $k$ -functor and there exists an open cover  $\{\sigma_i : \mathfrak{P}_{\text{Spec } A_i} \rightarrow \mathfrak{X}\}_{i \in I}$  of  $k$ -functors for some family  $\{A_i\}_{i \in I}$  of  $k$ -algebras.

The proof depends on two lemmas. Check [Mon19a, Definition 7.1] for the notion of a locally surjective morphism.

**Lemma 4.6.1.** *Let  $f : X \rightarrow Y$  be a morphism of  $k$ -schemes. Suppose that  $f$  is surjective morphism and an open immersion locally on  $X$ . Then  $\mathfrak{P}_f$  is a locally surjective morphism of Zariski local  $k$ -functors.*

*Proof of the lemma.* Let  $A$  be a  $k$ -algebra and  $g : \text{Spec } A \rightarrow Y$  be a morphism of  $k$ -schemes. Since  $f$  is surjective and an open immersion locally on  $X$ , there exist a Zariski cover  $\{f_i : A \rightarrow A_i\}_{i \in I}$  and a family  $\{g_i : \text{Spec } A_i \rightarrow X\}_{i \in I}$  of morphisms of  $k$ -schemes such that  $f \cdot g_i = g \cdot \text{Spec } f_i$  for every  $i \in I$ . This implies that  $\mathfrak{P}_f(g_i) = \mathfrak{P}_Y(f_i)(g)$  for every  $i \in I$ . Thus  $\mathfrak{P}_f$  is a locally surjective morphism of Zariski local  $k$ -functors.  $\square$

**Lemma 4.6.2.** *Let  $X = \coprod_{i \in I} X_i, R = \coprod_{i,j \in I} R_{ij}$  be disjoint sums of  $k$ -schemes and let  $p, q : R \rightarrow X$  be morphisms of  $k$ -schemes such that the following conditions are satisfied.*

- (1) *For any  $i, j \in I$  morphism  $p|_{R_{ij}}$  induces an open immersion  $R_{ij} \hookrightarrow X_i$  and morphism  $q|_{R_{ij}}$  induces an open immersion  $R_{ij} \hookrightarrow X_j$ .*
- (2) *For every  $i \in I$  morphisms  $p|_{R_{ii}}$  and  $q|_{R_{ii}}$  are equal and induce an isomorphisms  $R_{ii} \rightarrow X_i$ .*
- (3) *Triple  $(R, p, q)$  is an equivalence relation on  $X$  in the category of  $k$ -schemes.*

*Then there exist a  $k$ -scheme  $Y$  and a morphism  $f : X \rightarrow Y$  of  $k$ -schemes such that*

$$\mathfrak{P}_R \begin{array}{c} \xrightarrow{\mathfrak{P}_p} \\ \xrightarrow{\mathfrak{P}_q} \end{array} \mathfrak{P}_X \xrightarrow{\mathfrak{P}_f} \mathfrak{P}_Y$$

*is a cokernel of a pair  $(\mathfrak{P}_p, \mathfrak{P}_q)$  in the category of Zariski local  $k$ -functors.*

*Proof of the lemma.* Let

$$R \begin{array}{c} \xrightarrow{p} \\ \xrightarrow{q} \end{array} X \xrightarrow{f} Y$$

be a cokernel in the category of ringed spaces. It exists according to [Mon19b, Remark 2.3]. Moreover, [Mon19b, Theorem 3.2] states that for every  $i \in I$  subset  $f(X_i)$  is open in  $Y$  and we have an isomorphism of ringed spaces  $X_i \cong f(X_i)$  induced by  $f$ . Therefore,  $Y$  is a  $k$ -scheme and  $f : X \rightarrow Y$  is a morphism of  $k$ -schemes.

Now we verify that  $\mathfrak{P}_f$  is the quotient in the category of Zariski local  $k$ -functors. For this note that we proved above that  $f$  is open immersion of  $k$ -schemes locally on  $X$  and it is surjective. Thus by Lemma 4.6.1 we derive that  $\mathfrak{P}_f$  is a locally surjective morphism of Zariski local  $k$ -functors. Therefore ([Mon19a, Theorem 7.3]), it suffices to show that the square

$$\begin{array}{ccc} \mathfrak{P}_R & \xrightarrow{\mathfrak{P}_q} & \mathfrak{P}_X \\ \mathfrak{P}_p \downarrow & & \downarrow \mathfrak{P}_f \\ \mathfrak{P}_X & \xrightarrow{\mathfrak{P}_f} & \mathfrak{P}_Y \end{array}$$

is cartesian. Since  $\mathfrak{P}$  preserves limits (Fact 4.1), we derive that it suffices to check that

$$\begin{array}{ccc}
R & \xrightarrow{q} & X \\
p \downarrow & & \downarrow f \\
X & \xrightarrow{f} & Y
\end{array}$$

is cartesian square of  $k$ -schemes. By [Mon19b, Remark 2.3] we have  $R_{ij} = X_i \times_Y X_j$  for every  $i, j \in I$  and hence

$$X \times_Y X = \left( \coprod_{i \in I} X_i \right) \times_Y \left( \coprod_{i \in I} X_i \right) = \coprod_{i, j \in I} (X_i \times_Y X_j) = \coprod_{i, j \in I} R_{ij} = R$$

Thus the result follows.  $\square$

*Proof of the theorem.* If (i) holds, then we may assume that  $\mathfrak{X} = \mathfrak{P}_Y$  for some  $k$ -scheme  $Y$ . Fact 4.1 states that  $\mathfrak{P}_Y$  is a Zariski local  $k$ -functor and clearly  $1_{\mathfrak{P}_Y} : \mathfrak{P}_Y \rightarrow \mathfrak{P}_Y$  is an open cover. Thus (i)  $\Rightarrow$  (ii).

Every functor of points of a  $k$ -scheme admits open cover by functors of points of affine  $k$ -schemes. Indeed, it suffices to take open affine subschemes that cover given  $k$ -scheme and apply  $\mathfrak{P}$ . This implies that every open cover of a  $k$ -functor  $\mathfrak{X}$  by functors of points of  $k$ -schemes admits refinement by open cover of functors of points of affine  $k$ -schemes. Therefore, implication (ii)  $\Rightarrow$  (iii) holds.

Suppose that a  $k$ -functor  $\mathfrak{X}$  is Zariski local and  $\{\sigma_i : \mathfrak{P}_{\text{Spec } A_i} \rightarrow \mathfrak{X}\}_{i \in I}$  is an open cover of  $\mathfrak{X}$ . Note that for every  $i, j \in I$  there exist a  $k$ -scheme  $R_{ij}$  and open immersions  $p_{ij} : R_{ij} \hookrightarrow \text{Spec } A_i$ ,  $q_{ij} : R_{ij} \hookrightarrow \text{Spec } A_j$  such that the square

$$\begin{array}{ccc}
\mathfrak{P}_{R_{ij}} & \xrightarrow{\mathfrak{P}_{q_{ij}}} & \mathfrak{P}_{\text{Spec } A_j} \\
\mathfrak{P}_{p_{ij}} \downarrow & & \downarrow \sigma_j \\
\mathfrak{P}_{\text{Spec } A_i} & \xrightarrow{\sigma_i} & \mathfrak{X}
\end{array}$$

is cartesian. Consider  $k$ -scheme  $X = \coprod_{i \in I} \text{Spec } A_i$  and morphism  $\sigma : \mathfrak{P}_X \rightarrow \mathfrak{X}$  induced by  $\{\sigma_i\}_{i \in I}$ . Moreover, consider  $k$ -scheme  $R = \coprod_{i, j \in I} R_{ij}$  and morphisms  $p, q : R \rightarrow X$  induced by  $\{p_{ij}\}_{i, j \in I}$  and  $\{q_{ij}\}_{i, j \in I}$ , respectively. Note that the square

$$\begin{array}{ccc}
\mathfrak{P}_R & \xrightarrow{\mathfrak{P}_q} & \mathfrak{P}_X \\
\mathfrak{P}_p \downarrow & & \downarrow \sigma \\
\mathfrak{P}_X & \xrightarrow{\sigma} & \mathfrak{X}
\end{array}$$

is cartesian and hence  $(\mathfrak{P}_R, \mathfrak{P}_p, \mathfrak{P}_q)$  is an equivalence relation. By Lemma 4.6.2 there exist a  $k$ -scheme  $Y$  and a morphism  $f : X \rightarrow Y$  such that

$$\mathfrak{P}_R \xrightarrow[\mathfrak{P}_q]{\mathfrak{P}_p} \mathfrak{P}_X \xrightarrow{\mathfrak{P}_f} \mathfrak{P}_Y$$

is a cokernel of  $(\mathfrak{P}_p, \mathfrak{P}_q)$ . Moreover,  $\sigma$  is locally surjective morphism of Zariski local  $k$ -functors and hence also

$$\mathfrak{P}_R \begin{array}{c} \xrightarrow{\mathfrak{P}_p} \\ \xrightarrow{\mathfrak{P}_q} \end{array} \mathfrak{P}_X \xrightarrow{\sigma} \mathfrak{X}$$

is a cokernel of  $(\mathfrak{P}_p, \mathfrak{P}_q)$ . Thus  $\mathfrak{P}_Y$  is isomorphic with  $\mathfrak{X}$ . This proves (iii)  $\Rightarrow$  (i).  $\square$

**Proposition 4.7.** *Let  $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$  be a monomorphism of  $k$ -functors and  $\mathfrak{Y}$  be a Zariski local  $k$ -functor. Assume that for every  $k$ -algebra  $A$  and every morphism  $\tau : \mathfrak{P}_{\text{Spec } A} \rightarrow \mathfrak{Y}$  of  $k$ -functors there exist a Zariski local  $k$ -functor  $\mathfrak{Z}$  that fits into a cartesian square*

$$\begin{array}{ccc} \mathfrak{Z} & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \sigma \\ \mathfrak{P}_{\text{Spec } A} & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

Then  $\mathfrak{X}$  is a Zariski local  $k$ -functor.

*Proof.* Suppose that  $A$  is a  $k$ -algebra and  $S$  is a covering sieve on  $A$  with respect to Zariski topology. Recall that by [Mon19a, page 2] we may consider  $S$  as a subcopresheaf of  $\mathfrak{P}_{\text{Spec } A}$ . Suppose that  $\tau : \mathfrak{P}_{\text{Spec } A} \rightarrow \mathfrak{Y}$  and  $m : S \rightarrow \mathfrak{X}$  are morphisms of  $k$ -functors such that  $\sigma \cdot m$  is equal to the composition of  $S \hookrightarrow \mathfrak{P}_{\text{Spec } A}$  with  $\tau$ . Next there exists a Zariski local  $k$ -functor  $\mathfrak{Z}$  that fits into a cartesian square

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{\tau'} & \mathfrak{X} \\ \downarrow \sigma' & & \downarrow \sigma \\ \mathfrak{P}_{\text{Spec } A} & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

of  $k$ -functors. By universal property of cartesian squares there exists a unique morphism  $n : S \rightarrow \mathfrak{Z}$  of  $k$ -functors such that the diagram

$$\begin{array}{ccccc} S & & \xrightarrow{m} & & \mathfrak{X} \\ & \searrow n & & \searrow \tau' & \\ & \mathfrak{Z} & \xrightarrow{\tau'} & \mathfrak{X} & \\ & \downarrow \sigma' & & \downarrow \sigma & \\ & \mathfrak{P}_{\text{Spec } A} & \xrightarrow{\tau} & \mathfrak{Y} & \end{array}$$

is commutative. Since  $\mathfrak{Z}$  is Zariski local, there exists a morphism  $\rho : \mathfrak{P}_{\text{Spec } A} \rightarrow \mathfrak{Z}$  such that  $\rho|_S = n$ . Then  $(\tau' \cdot \rho)|_S = \tau' \cdot n = m$  and hence matching family  $m$  admits an amalgamation. Since  $\sigma$  is a monomorphism, this suffices to prove that  $\mathfrak{X}$  is a Zariski local  $k$ -functor.  $\square$

5. REPRESENTABLE MORPHISMS OF  $k$ -FUNCTORS

**Definition 5.1.** Let  $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$  be a morphism of  $k$ -functors. Assume that for every  $k$ -algebra  $A$  and every morphism  $\tau : \mathfrak{P}_{\text{Spec } A} \rightarrow \mathfrak{Y}$  of  $k$ -functors there exist a  $k$ -scheme  $X$ , a morphism  $f : X \rightarrow \text{Spec } A$  and a morphism  $\tau' : \mathfrak{P}_X \rightarrow \mathfrak{X}$  of  $k$ -functors such that the square

$$\begin{array}{ccc} \mathfrak{P}_X & \xrightarrow{\tau'} & \mathfrak{X} \\ \mathfrak{P}_f \downarrow & & \downarrow \sigma \\ \mathfrak{P}_{\text{Spec } A} & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

is cartesian. Then  $\sigma$  is a *representable morphism of  $k$ -functors*.

**Fact 5.2.** The class of representable morphisms of  $k$ -functors is closed under base change and composition.

*Proof.* Left to the reader. □

**Proposition 5.3.** Let  $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$  be a representable morphism of Zariski local  $k$ -functors. Fix a  $k$ -scheme  $Y$  and a morphism  $\tau : \mathfrak{P}_Y \rightarrow \mathfrak{Y}$ . Then there exist a  $k$ -scheme  $X$ , a morphism  $f : X \rightarrow Y$  and a morphism  $\tau' : \mathfrak{P}_X \rightarrow \mathfrak{X}$  such that the square

$$\begin{array}{ccc} \mathfrak{P}_X & \xrightarrow{\tau'} & \mathfrak{X} \\ \mathfrak{P}_f \downarrow & & \downarrow \sigma \\ \mathfrak{P}_Y & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

is cartesian.

*Proof.* Let

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{\tau'} & \mathfrak{X} \\ \sigma' \downarrow & & \downarrow \sigma \\ \mathfrak{P}_Y & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

be a cartesian square. According to [Mon19a, Theorem 2.12]  $k$ -functor  $\mathfrak{Z}$  is Zariski local. Suppose that  $\{f_i : \text{Spec } A_i \rightarrow Y\}_{i \in I}$  is an open cover of  $Y$ . Then  $\{\mathfrak{P}_{f_i} : \mathfrak{P}_{\text{Spec } A_i} \rightarrow \mathfrak{P}_Y\}_{i \in I}$  is an open cover of  $\mathfrak{P}_Y$  and hence its base change  $\{\tau_i : \mathfrak{Z}_i \rightarrow \mathfrak{Z}\}_{i \in I}$  is an open cover of  $\mathfrak{Z}$ . Since  $\sigma$  is representable, we deduce that  $\mathfrak{Z}_i$  is a functor of points of some  $k$ -scheme for  $i \in I$ . Now by Theorem 4.6 we derive that there exists a  $k$ -scheme  $X$  such that  $\mathfrak{Z}$  is isomorphic with  $\mathfrak{P}_X$ . This proves the result. □

**Definition 5.4.** Let  $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$  be a morphism of  $k$ -functors. Assume that for every  $k$ -algebra  $A$  and every morphism  $\tau : \mathfrak{P}_{\text{Spec } A} \rightarrow \mathfrak{Y}$  of  $k$ -functors there exist an ideal  $\mathfrak{a}$  in  $A$  and morphism  $\tau' : \mathfrak{P}_{V(\mathfrak{a})} \rightarrow \mathfrak{X}$  such that the square



$$\begin{array}{ccc} \mathfrak{P}_{V(\mathfrak{a})} = \mathfrak{P}_{\mathrm{Spec} A/\mathfrak{a}} & \xrightarrow{\tau'} & \mathfrak{X} \\ \mathfrak{P}_{\mathrm{Spec} q} \downarrow & & \downarrow \sigma \\ \mathfrak{P}_{\mathrm{Spec} A} & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

is cartesian, where  $q : A \rightarrow A/\mathfrak{a}$  is the quotient map. Then  $\sigma$  is a closed immersion of  $k$ -functors.

**Fact 5.5.** *The class of closed immersions of  $k$ -functors is closed under base change and composition.*

*Proof.* Left to the reader. □

**Proposition 5.6.** *Let  $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$  be a closed (open) immersion of  $k$ -functors. Fix a  $k$ -scheme  $Y$  and a morphism  $\tau : \mathfrak{P}_Y \rightarrow \mathfrak{Y}$ . Then there exist a  $k$ -scheme  $X$ , a closed (open) immersion  $f : X \rightarrow Y$  of schemes and a morphism  $\tau' : \mathfrak{P}_X \rightarrow \mathfrak{X}$  of  $k$ -functors such that the square*

$$\begin{array}{ccc} \mathfrak{P}_X & \xrightarrow{\tau'} & \mathfrak{X} \\ \mathfrak{P}_f \downarrow & & \downarrow \sigma \\ \mathfrak{P}_Y & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

*is cartesian.*

*Proof.* According to Fact 5.5 (Fact 4.4) pullback  $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{P}_Y \rightarrow \mathfrak{P}_Y$  of  $\sigma$  along  $\tau$  is a closed (open) immersion of  $k$ -functors. Since  $\mathfrak{P}_Y$  is a Zariski local  $k$ -functor by Fact 4.1 and closed (open) immersions are monomorphisms, we derive by Proposition 4.7 that a fiber-product  $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{P}_Y$  of  $\sigma$  and  $\tau$  is a Zariski local  $k$ -functor. Since closed (open) immersions of  $k$ -functors are representable, we deduce by Proposition 5.3 that there exists a  $k$ -scheme  $X$ , a morphism  $f : X \rightarrow Y$  of  $k$ -schemes and a morphism  $\tau' : \mathfrak{P}_X \rightarrow \mathfrak{X}$  of  $k$ -functors such that the square

$$\begin{array}{ccc} \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{P}_Y \cong \mathfrak{P}_X & \xrightarrow{\tau'} & \mathfrak{X} \\ \mathfrak{P}_f \downarrow & & \downarrow \sigma \\ \mathfrak{P}_Y & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

is cartesian and  $\mathfrak{P}_f$  is a closed (open) immersion of  $k$ -functors. Since the functor

$$\widehat{\mathrm{Sch}}_k \xrightarrow{\mathfrak{P}} \text{the category of } k\text{-functors}$$

preserves finite limits, it follows that for every open affine subset  $V$  of  $Y$  we have a cartesian square

$$\begin{array}{ccc} \mathfrak{P}_{f^{-1}(V)} & \hookrightarrow & \mathfrak{P}_X \\ \mathfrak{P}_{f_V} \downarrow & & \downarrow \mathfrak{P}_f \\ \mathfrak{P}_V & \hookrightarrow & \mathfrak{P}_Y \end{array}$$

where  $f_V : f^{-1}(V) \rightarrow V$  is the restriction of  $f$ . Next as  $\mathfrak{P}_f$  is a closed (open) immersion and  $V$  is affine, we derive that  $f_V$  is a closed (open) immersion of schemes. Since this holds for every affine open subset  $V$  of  $Y$ , we deduce that  $f$  is a closed (open) immersion.  $\square$

The next result is frequently used in the theory of *algebraic spaces*.

**Proposition 5.7.** *Let  $\mathfrak{Y}$  be a  $k$ -functor such that the diagonal  $\mathfrak{Y} \rightarrow \mathfrak{Y} \times \mathfrak{Y}$  is representable. Then every morphism  $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$  of  $k$ -functors is representable.*

*Proof.* Fix a morphism of  $k$ -functors  $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$ . Let  $Y$  be a  $k$ -scheme and let  $\tau : \mathfrak{P}_Y \rightarrow \mathfrak{Y}$  be a morphism of  $k$ -functors. Consider the cartesian square

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{\tau'} & \mathfrak{X} \\ \sigma' \downarrow & & \downarrow \sigma \\ \mathfrak{P}_Y & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

Then there exists a cartesian square

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{\quad} & \mathfrak{Y} \\ \downarrow & & \downarrow \text{diagonal} \\ \mathfrak{P}_Y \times \mathfrak{Y} & \xrightarrow{\tau \times \sigma} & \mathfrak{Y} \times \mathfrak{Y} \end{array}$$

Since the diagonal of  $\mathfrak{Y}$  is representable, we derive that  $\mathfrak{Z}$  is isomorphic with functor of points of some  $k$ -scheme. This finishes the proof.  $\square$

## 6. EXAMPLE: GRASSMANNIANS

In this section we use representability results to prove the existence of grassmannian  $k$ -scheme. We start by recalling the notion of quotient.

**Definition 6.1.** Let  $\mathcal{C}$  be a category and let  $X$  be an object of  $\mathcal{C}$ . Suppose that  $f_1 : X \rightarrow X_1$  and  $f_2 : X \rightarrow X_2$  are epimorphisms in  $\mathcal{C}$ . We say that  $f_1$  and  $f_2$  are *equivalent* if there exists a commutative triangle

$$\begin{array}{ccc} X_1 & \xrightarrow{\cong} & X_2 \\ f_1 \swarrow & & \searrow f_2 \\ & X & \end{array}$$

in  $\mathcal{C}$  in which horizontal arrow is an isomorphism. Class of epimorphisms with domain in  $X$  which are equivalent with respect to the relation above is called *a quotient of  $X$* .

**Definition 6.2.** Let  $V$  be a  $k$ -module and let  $n$  be a positive integer. For  $k$ -algebra  $A$  we define

$$\text{Grass}_{V,n}(A) = \left\{ \begin{array}{l} \text{Quotients of } A \otimes_k V \text{ represented by epimorphisms} \\ \text{with codomains that are projective } A\text{-modules of rank } n \end{array} \right\}$$

Note that if  $f : A \rightarrow B$  is a morphism of  $k$ -algebras (making  $B$  into an  $A$ -algebra), then the functor  $B \otimes_A (-)$  induces the canonical map

$$\text{Grass}_{V,n}(f) : \text{Grass}_{V,n}(A) \rightarrow \text{Grass}_{V,n}(B)$$

This makes  $\text{Grass}_{V,n}$  into a  $k$ -functor. We call it *the grassmannian  $k$ -functor of quotients of rank  $n$  of  $V$* .

**Theorem 6.3.** *Let  $V$  be a  $k$ -module and let  $n$  be a positive integer. Then the  $k$ -functor  $\text{Grass}_{V,n}$  is representable.*

We start with the following general result.

**Lemma 6.3.1.** *Let  $X$  be a locally ringed space and  $\phi : \mathcal{P} \rightarrow \mathcal{Q}$  be a morphism of  $\mathcal{O}_X$ -modules such that  $\mathcal{P}$  is of finite type and  $\mathcal{Q}$  is locally free of finite rank. Then for every point  $x$  in  $X$  the following assertions are equivalent.*

- (i)  $1_{k(x)} \otimes_{\mathcal{O}_{X,x}} \phi_x$  is an isomorphism of vector spaces over  $k(x)$ .
- (ii)  $\phi_x$  is an isomorphism of  $\mathcal{O}_{X,x}$ -modules.

Moreover, the subset

$$\{x \in X \mid \phi_x \text{ is an isomorphism of } \mathcal{O}_{X,x}\text{-modules}\}$$

of  $X$  is open.

*Proof of the lemma.* Suppose that  $\mathcal{K} = \ker(\phi)$ ,  $\mathcal{L} = \text{coker}(\phi)$ . Note first that  $\mathcal{L}$  is  $\mathcal{O}_X$ -module of finite type as the homomorphic image of  $\mathcal{Q}$ . Fix a point  $x$  in  $X$  such that  $1_{k(x)} \otimes_{\mathcal{O}_{X,x}} \phi_x$  is an isomorphisms of  $k(x)$  vector spaces. This implies that  $k(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x = 0$  and hence by Nakayama lemma we derive that  $\mathcal{L}_x = 0$ . Thus we have a short exact sequence

$$0 \longrightarrow \mathcal{K}_x \longrightarrow \mathcal{P}_x \xrightarrow{\phi_x} \mathcal{Q}_x \longrightarrow 0$$

Facts that  $\mathcal{Q}_x$  is finitely presented and  $\mathcal{P}_x$  is finitely generated over  $\mathcal{O}_{X,x}$  imply that  $\mathcal{K}_x$  is finitely generated over  $\mathcal{O}_{X,x}$ . Since  $\mathcal{Q}_x$  is free, we derive that the sequence above is split exact. Therefore, also the sequence

$$0 \longrightarrow k(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{K}_x \longrightarrow k(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{P}_x \xrightarrow{1_{k(x)} \otimes_{\mathcal{O}_{X,x}} \phi_x} k(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{Q}_x \longrightarrow 0$$

is exact and hence  $k(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{K}_x = 0$ . Nakayama lemma implies that  $\mathcal{K}_x = 0$ . Thus we derive that  $1_{k(x)} \otimes_{\mathcal{O}_{X,x}} \phi_x$  is an isomorphisms of  $k(x)$  vector spaces if and only if  $\phi_x$  is an isomorphisms of  $\mathcal{O}_{X,x}$ -modules. In other words

$$\begin{aligned} & \{x \in X \mid 1_{k(x)} \otimes_{\mathcal{O}_{X,x}} \phi_x \text{ is an isomorphism of vector spaces over } k(x)\} = \\ & = \{x \in X \mid \phi_x \text{ is an isomorphism of } \mathcal{O}_{X,x}\text{-modules}\} \end{aligned}$$

Note that

$$\{x \in X \mid \phi_x \text{ is an isomorphism of } \mathcal{O}_{X,x}\text{-modules}\} \subseteq \{x \in X \mid \phi_x \text{ is an epimorphism of } \mathcal{O}_{X,x}\text{-modules}\}$$

and

$$\{x \in X \mid \phi_x \text{ is an epimorphism of } \mathcal{O}_{X,x}\text{-modules}\} = X \setminus \text{supp}(\mathcal{L})$$

Since  $\mathcal{L}$  is finitely generated, we derive that  $\text{supp}(\mathcal{L})$  is closed and  $X \setminus \text{supp}(\mathcal{L})$  is open. Now there is a short exact sequence

$$0 \longrightarrow \mathcal{K}_{|X \setminus \text{supp}(\mathcal{L})} \longrightarrow \mathcal{P}_{|X \setminus \text{supp}(\mathcal{L})} \xrightarrow{\phi_{|X \setminus \text{supp}(\mathcal{L})}} \mathcal{Q}_{|X \setminus \text{supp}(\mathcal{L})} \longrightarrow 0$$

It follows that  $\mathcal{K}_{|X \setminus \text{supp}(\mathcal{L})}$  is  $\mathcal{O}_X$ -module of finite type. Thus

$$\{x \in X \mid \phi_x \text{ is an isomorphism of } \mathcal{O}_{X,x}\text{-modules}\} = (X \setminus \text{supp}(\mathcal{L})) \setminus \text{supp}(\mathcal{K}_{|X \setminus \text{supp}(\mathcal{L})})$$

is an open subset of  $X$ .  $\square$

Let  $V$  be a free  $k$ -module and let  $n$  be a positive integer. Consider a morphism  $u : k^{\oplus n} \rightarrow V$  of  $k$ -modules. Now we define a  $k$ -subfunctor  $\text{Grass}_V^u$  of  $\text{Grass}_{V,n}$  by formula

$$\text{Grass}_V^u(A) = \left\{ \begin{array}{l} \text{Elements of } \text{Grass}_{V,n}(A) \text{ which are represented by epimorphisms } \phi : A \otimes_k V \rightarrow U \\ \text{such that the composition } \phi \cdot (1_A \otimes_k u) \text{ is an isomorphism} \end{array} \right\}$$

for every  $k$ -algebra. Next we proceed in steps.

**Lemma 6.3.2.** *Let  $V$  be a free  $k$ -module and let  $n$  be a positive integer. Then*

$$\{\text{Grass}_V^u \hookrightarrow \text{Grass}_{V,n}\}_{u \in \text{Hom}_k(k^{\oplus n}, V)}$$

*is an open cover of  $\text{Grass}_{V,n}$ .*

*Proof of the lemma.* Let  $A$  be a  $k$ -algebra. Consider a morphism  $\tau : \mathfrak{P}_{\text{Spec } A} \rightarrow \text{Grass}_{V,n}$  that corresponds to some quotient of  $A \otimes_k V$  that is represented by an epimorphism  $\phi : A \otimes_k V \rightarrow U$  of  $A$ -modules with projective  $A$ -module  $U$  of rank  $n$ . Let  $u : k^{\oplus n} \rightarrow V$  be a morphism of  $k$ -modules. Consider a cartesian square

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\quad} & \text{Grass}_V^u \\ \downarrow & & \downarrow \\ \mathfrak{P}_{\text{Spec } A} & \xrightarrow{\tau} & \text{Grass}_{V,n} \end{array}$$

Pick a  $k$ -algebra  $B$  and a morphism  $f : A \rightarrow B$  of  $k$ -algebras. Note that  $f$  makes  $B$  into an  $A$ -algebra. Then  $f \in \mathfrak{X}(B)$  if and only if  $(1_B \otimes_A \phi) \cdot (1_B \otimes_k u)$  is an isomorphism of  $B$ -modules. Thus by Lemma 6.3.1 we deduce that  $f \in \mathfrak{X}(B)$  if and only if  $\text{Spec } f : \text{Spec } B \rightarrow \text{Spec } A$  factors through an open subscheme

$$\begin{aligned} W_u &= \left\{ \mathfrak{q} \in \text{Spec } A \mid (\phi \cdot (1_A \otimes_k u))_{\mathfrak{q}} \text{ is an isomorphism of } A_{\mathfrak{q}}\text{-modules} \right\} = \\ &= \left\{ \mathfrak{q} \in \text{Spec } A \mid k(\mathfrak{q}) \otimes_{A_{\mathfrak{q}}} (\phi \cdot (1_A \otimes_k u))_{\mathfrak{q}} \text{ is an isomorphism of } k(\mathfrak{q})\text{-vector spaces} \right\} \end{aligned}$$

This implies that  $\mathfrak{X} \hookrightarrow \mathfrak{P}_{\text{Spec } A}$  is isomorphic to an open immersion  $\mathfrak{P}_{W_u} \hookrightarrow \mathfrak{P}_{\text{Spec } A}$ .

Pick now  $\mathfrak{q} \in \text{Spec } A$  and an epimorphism  $\theta : k^{\oplus I} \twoheadrightarrow V$  for some set  $I$ . Then there exist  $J \subseteq I$  with  $\text{card}(J) = n$  such that the restriction to  $k(\mathfrak{q})^{\oplus J}$  of the morphism

$$1_{k(\mathfrak{q})} \otimes_{A_{\mathfrak{q}}} (\phi \cdot (1_A \otimes \theta))_{\mathfrak{q}} : k(\mathfrak{q})^{\oplus I} \rightarrow k(\mathfrak{q}) \otimes_{A_{\mathfrak{q}}} U_{\mathfrak{q}}$$

is an isomorphism of  $k(\mathfrak{q})$ -vector spaces. Let  $u : k^{\oplus n} \rightarrow V$  be a morphism given as the composition of the canonical monomorphism  $k^{\oplus n} = k^{\oplus J} \hookrightarrow k^{\oplus I}$  with  $\theta$ . Then

$$\left( 1_{k(\mathfrak{q})} \otimes_{A_{\mathfrak{q}}} (\phi \cdot (1_A \otimes u))_{\mathfrak{q}} \right)$$

is an isomorphism of  $k(\mathfrak{q})$ -vector spaces. Note that module  $U_{\mathfrak{q}}$  is a free  $A_{\mathfrak{q}}$ -module of rank  $n$ . Hence by Lemma 6.3.1 we derive that

$$(\phi \cdot (1_A \otimes u))_{\mathfrak{q}}$$

is an isomorphism of  $A_{\mathfrak{q}}$ -modules. Thus  $\mathfrak{q} \in W_u$ . Since  $\mathfrak{q}$  is arbitrary, we deduce that

$$\mathrm{Spec} A = \bigcup_{u \in \mathrm{Hom}_k(k^{\oplus n}, V)} W_u$$

This finishes the proof.  $\square$

**Lemma 6.3.3.** *Let  $V$  be a  $k$ -module and let  $n$  be a positive integer. Suppose that  $u : k^{\oplus n} \rightarrow V$  is a morphism of  $k$ -modules. Then  $\mathrm{Grass}_V^u$  is representable by a  $k$ -scheme.*

*Proof of the lemma.* Pick  $k$ -algebra  $A$  and let  $\phi : A \otimes_k V \rightarrow U$  be a morphism of  $A$ -modules that represents an element of  $\mathrm{Grass}_V^u(A)$ . Let  $v$  be an inverse of  $A$ -module isomorphism  $\phi \cdot (1_A \otimes_k u)$ . Then  $\theta = v \cdot \phi : A \otimes_k V \rightarrow A^{\oplus n}$  represents the same element of  $\mathrm{Grass}_V^u(A)$  as  $\phi$ . Moreover, it is a unique representative of this element having the property that  $\theta \cdot (1_A \otimes_k u) = 1_{A^{\oplus n}}$ . Thus we derive that

$$\mathrm{Grass}_V^u(A) = \{ \theta : A \otimes_k V \rightarrow A^{\oplus n} \mid \theta \cdot (1_A \otimes_k u) = 1_{A^{\oplus n}} \}$$

is natural identification. Note that for every  $k$ -algebra we have a cartesian square of sets

$$\begin{array}{ccc} \mathrm{Grass}_V^u(A) & \xrightarrow{\quad} & \mathbf{1} \\ \downarrow & & \downarrow 1_{A^{\oplus n}} \\ \mathrm{Hom}_A(A \otimes_k V, A^{\oplus n}) & \xrightarrow{\mathrm{Hom}_A(u, 1_{A^{\oplus n}})} & \mathrm{Hom}_A(A^{\oplus n}, A^{\oplus n}) \end{array}$$

These cartesian squares induce cartesian square of  $k$ -functors. Thus it suffices to prove that  $k$ -functors

$$\mathbf{Alg}_k \ni A \mapsto \mathrm{Hom}_A(A \otimes_k V, A^{\oplus n}) \in \mathbf{Set}, \quad \mathbf{Alg}_k \ni A \mapsto \mathrm{Hom}_A(A^{\oplus n}, A^{\oplus n}) \in \mathbf{Set}$$

are representable. For this it suffices to prove that for every  $k$ -module  $W$  a  $k$ -functor

$$\mathbf{Alg}_k \ni A \mapsto \mathrm{Hom}_A(A \otimes_k W, A^{\oplus n}) \in \mathbf{Set}$$

Indeed, we have chain of bijections

$$\begin{aligned} \underbrace{\mathfrak{P}_{\mathrm{Spec} \mathrm{Sym}(W) \times_k \dots \times_k \mathrm{Spec} \mathrm{Sym}(W)}(A)}_{n \text{ times}} &\cong \mathrm{Mor}_k(\mathrm{Spec} A, \mathrm{Spec} \mathrm{Sym}(W))^n \cong \\ &\cong \mathrm{Hom}_k(W, A)^n \cong \mathrm{Hom}_k(W, A^{\oplus n}) \cong \mathrm{Hom}_A(A \otimes_k W, A^{\oplus n}) \end{aligned}$$

natural in  $k$ -algebra  $A$ .  $\square$

**Lemma 6.3.4.** *Let  $X$  be a locally ringed space, let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}_X$ -modules and let  $\{U_i\}_{i \in I}$  be an open cover of  $X$ . Fix a positive integer  $n$ . Suppose that for each  $i \in I$  there is an epimorphism  $\phi_i : \mathcal{F}|_{U_i} \rightarrow \mathcal{O}_{U_i}^{\oplus n}$  such that for any  $i, j \in I$  there exists a commutative triangle*

$$\begin{array}{ccc} \mathcal{O}_{U_i \cap U_j}^{\oplus n} & \xrightarrow{\quad \cong \quad} & \mathcal{O}_{U_i \cap U_j}^{\oplus n} \\ \phi_j|_{U_i \cap U_j} \swarrow & & \searrow \phi_i|_{U_i \cap U_j} \\ & \mathcal{F}|_{U_i \cap U_j} & \end{array}$$

with dotted arrow that is an isomorphism of  $\mathcal{O}_{U_i \cap U_j}$ -modules. Then there exists a locally free sheaf  $\mathcal{U}$  of rank  $n$  and an epimorphism  $\phi : \mathcal{F} \rightarrow \mathcal{U}$  such that there are commutative triangles

$$\begin{array}{ccc}
\mathcal{U}|_{U_i} & \xrightarrow{\psi_i} & \mathcal{O}_{U_i}^{\oplus n} \\
\phi|_{U_i} \swarrow & & \nearrow \phi_i \\
\mathcal{F}|_{U_i} & & 
\end{array}$$

with isomorphisms  $\psi_i$  for every  $i \in I$ . Moreover,  $\phi$  with these properties determine a unique quotient of  $\mathcal{F}$ .

*Proof of the lemma.* By assumption for every pair  $i, j \in I$  there exists an isomorphism  $\theta_{ij}$  such that the triangle

$$\begin{array}{ccc}
\mathcal{O}_{U_i \cap U_j}^{\oplus n} & \xrightarrow{\theta_{ij}} & \mathcal{O}_{U_i \cap U_j}^{\oplus n} \\
\phi|_{U_i \cap U_j} \swarrow & & \nearrow \phi|_{U_i \cap U_j} \\
\mathcal{F}|_{U_i \cap U_j} & & 
\end{array}$$

is commutative. Note that  $\theta_{ij}$  is a unique isomorphism that makes the triangle commutative. Thus  $\{\theta_{ij}\}_{i,j \in I}$  satisfy cocycle condition. Hence there exists a unique locally free sheaf  $\mathcal{U}$  of rank  $n$  with  $\{\theta_{ij}\}_{i,j \in I}$  as the family of transition isomorphisms. Moreover,  $\{\phi_i\}_{i \in I}$  induce an epimorphism  $\phi : \mathcal{F} \twoheadrightarrow \mathcal{U}$ . This constructs  $\phi$  with properties as in the statement.  $\square$

*Proof of the theorem.* By Lemma 6.3.4 we derive that  $\text{Grass}_{V,n}$  is a Zariski local  $k$ -functor. Now the theorem follows from Theorem 4.6 in the light of Lemma 6.3.2 and Lemma 6.3.3.  $\square$

## 7. CLOSED IMMERSIONS AND HOM $k$ -FUNCTORS

**Definition 7.1.** Let  $X$  be a  $k$ -scheme. Suppose that there exists an open affine cover  $X = \bigcup_{i \in I} X_i$  such that  $k$ -algebra  $\Gamma(X_i, \mathcal{O}_{X_i})$  is free as a  $k$ -module. Then we say that  $X$  is a *locally free  $k$ -scheme*.

Next theorem is the main result of this section.

**Theorem 7.2.** Let  $j : \mathfrak{Y}' \rightarrow \mathfrak{Y}$  be a closed immersion of  $k$ -functors and  $X$  be a locally free  $k$ -scheme. Suppose that classes  $\text{Mor}_A((\mathfrak{P}_X)_A, \mathfrak{Y}_A)$  are sets for every  $k$ -algebra  $A$ . Then classes  $\text{Mor}_A((\mathfrak{P}_X)_A, \mathfrak{Y}'_A)$  are sets for every  $k$ -algebra  $A$  and the morphism

$$\text{Mor}_k(1_{\mathfrak{P}_X}, j) : \text{Mor}_k(\mathfrak{P}_X, \mathfrak{Y}') \rightarrow \text{Mor}_k(\mathfrak{P}_X, \mathfrak{Y})$$

is a closed immersion of  $k$ -functors.

It is useful to isolate crucial steps in the argument. For this we proceed by proving some lemmas.

**Lemma 7.2.1.** Suppose that  $A$  is a commutative ring. Let  $j : \mathfrak{Y}' \rightarrow \mathfrak{Y}$  be a closed immersion of  $A$ -functors and  $X$  be an affine  $A$ -scheme such that  $\Gamma(X, \mathcal{O}_X)$  is a free  $A$ -module. Assume that  $\tau : \mathfrak{P}_X \rightarrow \mathfrak{Y}$  is a morphism of  $A$ -functors. Then there exists an ideal  $\mathfrak{a} \subseteq A$  such that for every  $A$ -algebra  $B$  the restriction  $\tau_B$  factors through  $j_B$  if and only if the structure morphism  $f : A \rightarrow B$  of  $B$  satisfies  $\mathfrak{a} \subseteq \ker(f)$ .

*Proof of the lemma.* Since  $j$  is a closed immersion of  $A$ -functors and  $X$  is affine  $k$ -scheme there exists an affine  $A$ -scheme  $X'$ , a closed immersion  $j' : X' \rightarrow X$  of schemes and a cartesian square

$$\begin{array}{ccc} \mathfrak{P}_{X'} & \longrightarrow & \mathfrak{Y}' \\ \mathfrak{P}_{j'} \downarrow & & \downarrow j \\ \mathfrak{P}_X & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

of  $A$ -functors. Next let  $B$  be an  $A$ -algebra with the structure morphism  $f : A \rightarrow B$ . Then  $\tau_B$  factors through  $j_B$  if and only if the projection  $\mathrm{Spec} B \times_{\mathrm{Spec} A} X \rightarrow X$  induced by  $f$  factors through  $X'$ . Let  $A[X]$  be the  $A$ -algebra of global regular functions on  $X$  and let  $\mathfrak{J}$  be an ideal in  $A[X]$  such that  $A[X]/\mathfrak{J} = A[X']$  is the  $A$ -algebra of global regular functions of  $X'$ . With this notation we derive that the projection  $\mathrm{Spec} B \times_{\mathrm{Spec} A} X \rightarrow X$  induced by  $f$  factors through  $X'$  if and only if the morphism  $A[X] \rightarrow B \otimes_A A[X]$  induced by  $f$  sends every element of  $\mathfrak{J}$  to zero. Since  $A[X]$  is a free  $A$ -module, we write  $A[X] = A^{\oplus I}$  for some index set  $I$ . Then the morphism  $A[X] \rightarrow B \otimes_A A[X]$  induced by  $f$  is just  $f^{\oplus I} : A^{\oplus I} \rightarrow B^{\oplus I}$ . We have  $f^{\oplus I}(\mathfrak{J}) = 0$  if and only if  $(pr_i^B \cdot f^{\oplus I})(\mathfrak{J}) = 0$  for every  $i \in I$ , where  $pr_i^B : B^{\oplus I} \rightarrow B$  is the projection on  $i$ -th component. Pick  $i \in I$  and consider the commutative diagram

$$\begin{array}{ccc} A^{\oplus I} & \xrightarrow{f^{\oplus I}} & B^{\oplus I} \\ pr_i^A \downarrow & & \downarrow pr_i^B \\ A & \xrightarrow{f} & B \end{array}$$

In the diagram  $pr_i^A$  is the projection on  $i$ -th component. Diagram implies that  $(pr_i^B \cdot f^{\oplus I})(\mathfrak{J}) = 0$  for every  $i \in I$  if and only if  $(f \cdot pr_i^A)(\mathfrak{J}) = 0$  for every  $i \in I$ . This is equivalent with the condition that  $f(\mathfrak{a}) = 0$  for ideal  $\mathfrak{a}$  in  $A$  generated by  $\sum_{i \in I} pr_i^A(\mathfrak{J})$ . Thus the lemma is proved.  $\square$

**Lemma 7.2.2.** *Suppose that  $A$  is a commutative ring. Let  $j : \mathfrak{Y}' \rightarrow \mathfrak{Y}$  be a closed immersion of  $A$ -functors and  $X$  be an  $A$ -scheme with open cover*

$$X = \bigcup_{i \in I} X_i$$

*Assume that  $\tau : \mathfrak{P}_X \rightarrow \mathfrak{Y}$  is a morphism of  $A$ -functors. Fix an  $A$ -algebra  $B$ . Then  $\tau_B$  factors through  $j_B$  if and only if  $(\tau|_{\mathfrak{P}_{X_i}})_B$  factors through  $j_B$  for every  $i \in I$ .*

*Proof of the lemma.* If  $\tau_B$  factors through  $j_B$ , then also  $(\tau|_{\mathfrak{P}_{X_i}})_B$  factors through  $j_B$  for every  $i \in I$ . It suffices to prove the converse. So suppose that  $(\tau|_{\mathfrak{P}_{X_i}})_B$  factors through  $j_B$  for every  $i \in I$ . Since  $j$  is a closed immersion of  $A$ -functors and  $X$  is an  $A$ -scheme, Proposition 5.6 implies that there exists a cartesian square

$$\begin{array}{ccc} \mathfrak{P}_{X'} & \longrightarrow & \mathfrak{Y}' \\ \mathfrak{P}_{j'} \downarrow & & \downarrow j \\ \mathfrak{P}_X & \xrightarrow{\tau} & \mathfrak{Y} \end{array}$$

where  $j' : X' \rightarrow X$  is a closed immersion of  $A$ -schemes. For each  $i \in I$  let  $j'_i : j'^{-1}(X_i) \rightarrow X_i$  be the restriction of  $j'$ . We have the induced cartesian square

$$\begin{array}{ccc}
\mathfrak{P}_{j'^{-1}(X_i)} & \longrightarrow & \mathfrak{Y}' \\
\mathfrak{P}_{j'_i} \downarrow & & \downarrow j \\
\mathfrak{P}_{X_i} & \xrightarrow{\tau|_{\mathfrak{P}_{X_i}}} & \mathfrak{Y}
\end{array}$$

Now  $(\tau|_{\mathfrak{P}_{X_i}})_B$  factors through  $j_B$ . This implies that  $(\mathfrak{P}_{j'_i})_B$  admits a section for every  $i \in I$ . Then  $(\mathfrak{P}_{j'_i})_B$  is an isomorphism for every  $i \in I$ . Thus  $j'_i \times_{\text{Spec } A} 1_{\text{Spec } B}$  is an isomorphism for every  $i \in I$  and hence  $j' \times_{\text{Spec } A} 1_{\text{Spec } B}$  is an isomorphism of  $B$ -schemes. This means that  $\tau_B$  factors through  $j_B$ .  $\square$

*Proof of the theorem.* Let  $A$  be a  $k$ -algebra. The restriction functor  $(-)|_{\mathbf{Alg}_A} = (-)_A$  preserves all closed immersions. Thus  $j_A$  is a closed immersion of  $A$ -functors and hence we derive that  $j_A : \mathfrak{Y}'_A \rightarrow \mathfrak{Y}_A$  is a monomorphism of  $A$ -functors. Thus we have an injective map of classes

$$\text{Mor}_A(1_{(\mathfrak{P}_X)_A}, j_A) : \text{Mor}_A((\mathfrak{P}_X)_A, \mathfrak{Y}'_A) \hookrightarrow \text{Mor}_A((\mathfrak{P}_X)_A, \mathfrak{Y}_A)$$

Hence if  $\text{Mor}_A((\mathfrak{P}_X)_A, \mathfrak{Y}_A)$  is a set, then  $\text{Mor}_A((\mathfrak{P}_X)_A, \mathfrak{Y}'_A)$  is a set. All these facts imply that both internal homs

$$\text{Mor}_k(\mathfrak{P}_X, \mathfrak{Y}'), \text{Mor}_k(\mathfrak{P}_X, \mathfrak{Y})$$

exist and morphism  $\text{Mor}_k(1_{\mathfrak{P}_X}, j)$  of  $k$ -functors is a monomorphism. Our task is to prove that it is a closed immersion. For this consider a  $k$ -algebra  $A$  and a morphism  $\sigma : \mathfrak{P}_{\text{Spec } A} \rightarrow \text{Mor}_k(\mathfrak{P}_X, \mathfrak{Y})$  of  $k$ -functors that sends  $1_A$  to some morphism  $\tau : (\mathfrak{P}_X)_A \rightarrow \mathfrak{Y}_A$  of  $A$ -functors. Consider a cartesian square

$$\begin{array}{ccc}
\mathfrak{U} & \longrightarrow & \text{Mor}_k(\mathfrak{P}_X, \mathfrak{Y}') \\
\downarrow & & \downarrow \text{Mor}_k(1_{\mathfrak{P}_X}, j) \\
\mathfrak{P}_{\text{Spec } A} & \xrightarrow{\sigma} & \text{Mor}_k(\mathfrak{P}_X, \mathfrak{Y})
\end{array}$$

Since  $\text{Mor}_k(1_{\mathfrak{P}_X}, j)$  is a monomorphism, we may consider  $\mathfrak{U}$  as a  $k$ -subfunctor of  $\mathfrak{P}_{\text{Spec } A}$ . For every  $k$ -algebra  $B$  subset  $\mathfrak{U}(B) \subseteq \text{Mor}_k(A, B) = \text{Mor}_k(\text{Spec } B, \text{Spec } A)$  consists of  $A$ -algebras  $B$  with structure morphisms  $f : A \rightarrow B$  such that  $\tau_B$  factors through  $j_B : \mathfrak{Y}'_B \rightarrow \mathfrak{Y}_B$ . Since  $X$  is a locally free  $k$ -scheme, we deduce that  $(\mathfrak{P}_X)_A$  is a functor of points of a locally free  $A$ -scheme

$$\text{Spec } A \times_{\text{Spec } k} X$$

Pick an open affine cover  $\cup_{i \in I} X_i$  of this  $A$ -scheme such that  $\Gamma(X_i, \mathcal{O}_X)$  is a free  $A$ -module. Now Lemma 7.2.2 implies that  $\tau_B$  factors through  $j_B$  if and only if  $(\tau|_{X_i})_B$  factors through  $j_B$  for every  $i \in I$ . Next by Lemma 7.2.1 we deduce that  $(\tau|_{X_i})_B$  factors through  $j_B$  for given  $i \in I$  if and only if  $f(\mathfrak{a}_i) = 0$  for some ideal  $\mathfrak{a}_i \subseteq A$  independent of  $f$ . Thus  $\mathfrak{U}$  consists of all morphisms  $f : A \rightarrow B$  of  $k$ -algebras such that  $f(\mathfrak{a}) = 0$  where  $\mathfrak{a} = \sum_{i \in I} \mathfrak{a}_i$ . Therefore,  $\mathfrak{U} \hookrightarrow \mathfrak{P}_{\text{Spec } A}$  is isomorphic with  $\mathfrak{P}_{V(\mathfrak{a})} = \mathfrak{P}_{\text{Spec } A/\mathfrak{a}} \hookrightarrow \mathfrak{P}_{\text{Spec } A}$  induced by the quotient map  $A \rightarrow A/\mathfrak{a}$  and hence  $\text{Mor}_k(1_{\mathfrak{P}_X}, j)$  is a closed immersion of  $k$ -functors.  $\square$

## REFERENCES

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