1. Formal functors and representability

Example 1.1 (Formal schemes from algebraic ones)**.** Let Z be a $\mathbf{G}$-scheme and $\mathcal{I}$ be the ideal of $Z^{\mathbf{G}}$. Then $Z_n = V(\mathcal{I}^{n+1})$ is a closed $\mathbf{G}$-stable subscheme of Z for every $n \in \mathbb{N}$ and this yields to a formal $\mathbf{G}$-scheme $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$. We denote this formal $\mathbf{G}$-scheme by $\widehat{Z}$.

Now we define morphisms of formal $\mathbf{G}$-schemes.

Definition 1.2. Let $\mathcal{Z} = \{Z_n\}$ and $\mathcal{W} = \{W_n\}$ be formal $\mathbf{G}$-schemes. A *morphism* $\varphi : \mathcal{W} \to \mathcal{Z}$ *of formal* $\mathbf{G}$*-schemes* is a family of $\mathbf{G}$-equivariant morphisms $\varphi = \{\varphi_n : W_n \to Z_n\}$ such that for every $n \in \mathbb{N}$ we have a commutative square

$$\begin{tikzcd}
W_{n+1} \arrow[r, "\varphi_{n+1}"] & Z_{n+1} \\
W_n \arrow[u, hook] \arrow[r, "\varphi_n"'] & Z_n \arrow[u, hook]
\end{tikzcd}$$

Remark 1.3 (Morphisms of formal $\overline{\mathbf{G}}$-schemes are $\overline{\mathbf{G}}$-equivariant)**.** Let $\mathcal{W}$ and $\mathcal{Z}$ be formal $\overline{\mathbf{G}}$-schemes and consider their morphism $\varphi : \mathcal{W} \to \mathcal{Z}$ (as formal $\mathbf{G}$-schemes). Then for every $n \in \mathbb{N}$ the morphism $\varphi_n \colon W_n \to Z_n$ is $\overline{\mathbf{G}}$-equivariant. To see this, consider Diagram (1).

$$\begin{tikzcd}
W_n \arrow[r, "r_n"'] \arrow[rr, bend left, "\varphi_n"] \arrow[dr, "\pi_{W_n}"'] & W_0 \times_{Z_0} Z_n \arrow[r, "q_n"'] \arrow[d, "p_n"] & Z_n \arrow[d, "\pi_{Z_n}"] \\
& W_0 \arrow[r, "\varphi_0"'] & Z_0
\end{tikzcd} \tag{1}$$

Since W_0 and Z_0 are equipped with trivial $\overline{\mathbf{G}}$-actions, also the pullback $W_0 \times_{Z_0} Z_n$ is a $\overline{\mathbf{G}}$-scheme and q_n is $\overline{\mathbf{G}}$-equivariant. Recall that π_{Z_n}, π_{W_n} are affine morphisms. Therefore, p_n is affine. Hence r_n is a $\mathbf{G}$-equivariant morphism between $\overline{\mathbf{G}}$-schemes separated (even affine) over W_0. Thus r_n is $\overline{\mathbf{G}}$-equivariant.

Definition 1.4. A *locally linear* $\overline{\mathbf{G}}$*-scheme* is a $\overline{\mathbf{G}}$-scheme which admits an open cover by affine $\overline{\mathbf{G}}$-stable subschemes. The category of locally linear $\overline{\mathbf{G}}$-schemes consists of those schemes and $\overline{\mathbf{G}}$-equivariant morphisms.

Let Z be a locally linear $\overline{\mathbf{G}}$-scheme. By Proposition **??**, the map $\mathcal{D}_Z \to Z$ is an isomorphism. In particular, there is a canonical morphism $\pi_Z \colon Z \to Z^{\mathbf{G}}$, which is the multiplication by zero. For an affine open $\overline{\mathbf{G}}$-stable cover $\{V_i\}_i$ of Z, we have $V_i = \pi_Z^{-1}(\pi_Z(V_i))$ by Proposition **??**, hence the canonical morphism $\pi_Z \colon Z \to Z^{\mathbf{G}}$ is affine.

Definition 1.5. Let $\mathcal{Z}$ be a formal $\overline{\mathbf{G}}$-scheme. An *algebraization* of $\mathcal{Z}$ is a $\overline{\mathbf{G}}$-scheme Z such that

(1) Z is a locally linear $\overline{\mathbf{G}}$-scheme.

(2) $\mathcal{Z}$ and $\widehat{Z}$ are isomorphic formal $\overline{\mathbf{G}}$-schemes.

By the above discussion, the morphism $\pi_Z \colon Z \to Z^{\mathbf{G}}$ is affine for any algebraization Z.

Theorem 1.6 (Algebraization of a formal $\overline{\mathbf{G}}$-scheme)**.** *Let* $\mathcal{Z} = \{Z_n\}$ *be a formal* $\overline{\mathbf{G}}$*-scheme. Then there exists a colimit*

$$Z = \operatorname{colim}_n Z_n$$

in the category of locally linear $\overline{\mathbf{G}}$-schemes and Z is the unique algebraization of $\mathcal{Z}$. If in addition $\mathcal{Z}$ is locally Noetherian, then π_Z is of finite type. If $\mathcal{Z}$ is locally Noetherian and Z_0 is of finite type, then also Z is of finite type.

Now we spell out the main idea of the proof: the $\overline{\mathbf{G}}$-scheme Z required in Theorem 1.6 is equal to $\operatorname{Spec}_{Z_0}\mathcal{A}$, where $\mathcal{A}$ is the limit of $\mathcal{A}_n$ *in the category of* $\overline{\mathbf{G}}$*-algebras*; in other words each isotypic component of $\mathcal{A}$ is the limit of isotypic components of $\mathcal{A}_n$. Our first goal is to prove a stabilization result. We denote by $\operatorname{Irr}(\mathbf{G})$ the set of isomorphism types of irreducible $\mathbf{G}$-representations and by $\operatorname{Irr}(\overline{\mathbf{G}}) \subset \operatorname{Irr}(\mathbf{G})$ the subset of $\overline{\mathbf{G}}$-representations. For $\lambda \in \operatorname{Irr}(\mathbf{G})$ and a quasi-coherent $\overline{\mathbf{G}}$-module $\mathcal{C}$ on Z_0 we denote by $\mathcal{C}[\lambda] \subset \mathcal{C}$ the $\overline{\mathbf{G}}$-submodule such that $H^0(U, \mathcal{C}[\lambda]) \subset H^0(U, \mathcal{C})$ is the union of all $\mathbf{G}$-subrepresentations of $H^0(U, \mathcal{C})$ isomorphic to λ (i.e., the isotypic component of λ).

Lemma 1.6.1 (stabilization on an isotypic component)**.** *Let $\lambda \in \operatorname{Irr}(\overline{\mathbf{G}})$. Then there exists a number $n_\lambda \in \mathbb{N}$ such that the following holds. Let $\mathcal{Z} = \{Z_n\}$ be a formal $\overline{\mathbf{G}}$-scheme and $\{\mathcal{A}_{n+1} \twoheadrightarrow \mathcal{A}_n\}$ be the associated sequence of quasi-coherent $\overline{\mathbf{G}}$-algebras. Then for every $n > n_\lambda$ the surjection*

$$\mathcal{A}_n[\lambda] \twoheadrightarrow \mathcal{A}_{n-1}[\lambda]$$

is an isomorphism. If $\lambda_0 \in \operatorname{Irr}(\overline{\mathbf{G}})$ is the trivial representation, then we may take $n_{\lambda_0} = 0$.

Proof of Lemma 1.6.1. The claims are preserved under field extension, so we may assume our field is algebraically closed (hence perfect) so we may use the Kempf's torus. Fix a grading on $k[\overline{\mathbf{G}}]$ induced by a Kempf's torus for k as in Corollary **??**. Denote by $A_\lambda \subseteq \mathbb{N}$ the set of weights which appear in $k[\mathbf{G}]_\lambda$. Since $\dim_k k[\mathbf{G}]_\lambda$ is finite by Proposition **??**, the set A_λ is finite. Put

$$n_\lambda = \sup A_\lambda.$$

Fix $n > n_\lambda$ and let $\mathcal{I}_n = \ker(\mathcal{A}_n \to \mathcal{A}_0)$. Then we have a decomposition with respect to the chosen torus

$$\mathcal{A}_n = \bigoplus_{i \geq 0} (\mathcal{A}_n)[i],$$

By Corollary **??**, we have $\mathcal{I}_n = \oplus_{i\geq 1}(\mathcal{A}_n)[i]$. Since $n > n_\lambda$ we have

$$\mathcal{I}_n^n \subset \bigoplus_{i \geq n} (\mathcal{A}_n)[i] \subseteq \bigoplus_{i \notin A_\lambda} (\mathcal{A}_n)[i]$$

Hence, $\mathcal{I}_n^n[\lambda] = 0$. But $\mathcal{I}_n^n[\lambda] = \ker(\mathcal{A}_n[\lambda] \to \mathcal{A}_{n-1}[\lambda])$, thus $\mathcal{A}_n[\lambda] \to \mathcal{A}_{n-1}[\lambda]$ is an isomorphism. Finally note that $A_{\lambda_0} = \{0\}$. This implies that $n_{\lambda_0} = 0$. ☐

Proof of Theorem 1.6. Let $\mathcal{A}_n$ be the quasi-coherent $\overline{\mathbf{G}}$-algebras as in (**??**). For $\lambda \in \operatorname{Irr}(\overline{\mathbf{G}})$ we define $\mathcal{A}[\lambda] := \mathcal{A}_n[\lambda]$, where $n \geq n_\lambda$ as in Lemma 1.6.1.

$$\mathcal{A} = \bigoplus_{\lambda \in \operatorname{Irr}(\overline{\mathbf{G}})} \mathcal{A}[\lambda] = \bigoplus_{\lambda \in \operatorname{Irr}(\overline{\mathbf{G}})} \mathcal{A}_{n_\lambda}[\lambda].$$

Clearly $\mathcal{A}[\lambda_0] = \mathcal{A}_0 = \mathcal{O}_{Z_0}$ canonically (where λ_0 is the trivial representation), hence $\mathcal{A}$ is an $\mathcal{O}_{Z_0}$-module. Actually $\mathcal{A} = \lim_n \mathcal{A}_n$ in the category of quasi-coherent $\overline{\mathbf{G}}$-modules on Z_0. We construct the algebra structure on $\mathcal{A}$. For this pick $\eta_1, \eta_2 \in \operatorname{Irr}(\overline{\mathbf{G}})$. Fix the finite set $\{\lambda_1, \dots, \lambda_s\} \subseteq \operatorname{Irr}(\overline{\mathbf{G}})$ of representations which appear in $k[\overline{\mathbf{G}}]_{\eta_1} \otimes_k k[\overline{\mathbf{G}}]_{\eta_2}$. Then, for every $n \in \mathbb{N}$, we have the multiplication

$$\mathcal{A}_n[\eta_1] \otimes_k \mathcal{A}_n[\eta_2] \to \mathcal{A}_n[\eta_1] \cdot \mathcal{A}_n[\eta_2] \subseteq \bigoplus_{i=1}^{s} \mathcal{A}_n[\lambda_i]$$

and by Lemma 1.6.1 these morphisms can be identified for $n \geq \sup\{n_{\eta_1}, n_{\eta_2}, n_{\lambda_1}, ..., n_{\lambda_s}\}$. We define

$$\mathcal{A}[\eta_1] \otimes_k \mathcal{A}[\eta_2] \to \bigoplus_{i=1}^{s} \mathcal{A}[\lambda_i] \subseteq \mathcal{A}$$

as a morphism induced by the multiplication morphism for any $n \geq \sup\{n_{\eta_1}, n_{\eta_2}, n_{\lambda_1}, \ldots, n_{\lambda_s}\}$. This gives an $\mathcal{O}_{Z_0}$-algebra structure on $\mathcal{A}$, so $\mathcal{A}$ is in fact the limit of $\mathcal{A}_n$ is the category of $\overline{\mathbf{G}}$-algebras. Note that from the description of $\mathcal{A}$ it follows that for every $n \in \mathbb{N}$ we have a surjective morphism $p_n : \mathcal{A} \twoheadrightarrow \mathcal{A}_n$ of $\overline{\mathbf{G}}$-algebras. We denote its kernel by $\mathcal{J}_n$ and we put $\mathcal{J} := \mathcal{J}_0$. The natural injection $\mathcal{O}_{Z_0} = \mathcal{A}_0 \to \mathcal{A}$ is a section of p_0, so that we have

$$\mathcal{J} = \bigoplus_{\lambda \in \mathrm{Irr}(\overline{\mathbf{G}}) \setminus \{\lambda_0\}} \mathcal{A}[\lambda].$$

We also denote by $\mathcal{I}_n$ the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$ for $n \in \mathbb{N}$. Then $\mathcal{I}_n = \mathcal{J}/\mathcal{J}_n$. Fix $m \in \mathbb{N}$ and consider $n \in \mathbb{N}$ such that $n \geq m$. Since $\mathcal{Z}$ is a formal $\overline{\mathbf{G}}$-scheme, the sheaf $\mathcal{I}_n^{m+1}$ is the kernel of the morphism $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_m$. Thus

$$\mathcal{J}_m/\mathcal{J}_n = \mathcal{I}_n^{m+1} = (\mathcal{J}^{m+1} + \mathcal{J}_n)/\mathcal{J}_n.$$

Both $\mathcal{J}_m$ and $\mathcal{J}^{m+1}$ are $\mathrm{Irr}(\overline{\mathbf{G}})$-graded and for given $\lambda \in \mathrm{Irr}(\overline{\mathbf{G}})$ and $n \gg 0$ the isotypic component $\mathcal{J}_n[\lambda]$ is zero by Lemma 1.6.1. Hence $\mathcal{J}_m = \mathcal{J}^{m+1}$ for every $m \in \mathbb{N}$. We define

$$Z = \mathrm{Spec}_{Z_0}(\mathcal{A})$$

and we denote by $\pi : Z \to Z_0$ the structural morphism. The scheme Z inherits a $\overline{\mathbf{G}}$-action from $\mathcal{A}$. For every $n \in \mathbb{N}$ the zero-set of $\mathcal{J}^{n+1} \subseteq \mathcal{A}$ is a $\overline{\mathbf{G}}$-scheme isomorphic to Z_n. Hence $\mathcal{Z}$ is isomorphic to $\widehat{Z}$. Thus Z is an algebraization of $\mathcal{Z}$. Since $\mathcal{A} = \lim \mathcal{A}_n$, we have $Z = \mathrm{colim}\, Z_n$ in the category of locally linear $\overline{\mathbf{G}}$-schemes.

It remains to prove uniqueness of algebraization. Let $Z' = \mathrm{Spec}_{Z_0}\mathcal{A}'$ be an algebraization of $\mathcal{Z} = \{Z_n\}$. Then $Z_n \hookrightarrow Z'$, so by the universal property of colimit, we obtain a $\overline{\mathbf{G}}$-morphism $Z \to Z'$, corresponding to $\mathcal{A}' \to \mathcal{A}$. It induces epimorphisms $\mathcal{A}' \twoheadrightarrow \mathcal{A}_n$ for all n. For each $\lambda \in \mathrm{Irr}(\overline{\mathbf{G}})$, the composition

$$\mathcal{A}'[\lambda] \to \mathcal{A}[\lambda] \simeq \mathcal{A}_{n_\lambda}[\lambda]$$

is an epimorphism, hence $\mathcal{A}' \to \mathcal{A}$ is an epimorphism. The kernel of $\mathcal{A}' \to \mathcal{A}$ is equal to

$$\bigcap_n \ker(\mathcal{A}' \to \mathcal{A}_n) = \bigcap_n \ker(\mathcal{A}' \to \mathcal{A}_0)^n.$$

To prove that this kernel is zero, we may enlarge the field to an algebraically closed field, so the result follows from Corollary **??**.

Assume that each scheme Z_n is locally Noetherian over k. Then $\mathcal{I}_n$ is a coherent $\mathcal{A}_n$-module, thus $\mathcal{I}_n^i/\mathcal{I}^{i+1}$ is a coherent $\mathcal{A}_0$-module for all i. The series

$$0 = \mathcal{I}_n^{n+1} \subset \mathcal{I}^n \subset \ldots \subset \mathcal{I} \subset \mathcal{A}_n$$

has coherent subquotients, hence $\mathcal{A}_n$ is a coherent $\mathcal{O}_{Z_n}$-algebra. Thus $\mathcal{A}[\lambda]$ is a coherent $\mathcal{O}_{Z_0}$-module for every $\lambda \in \mathrm{Irr}(\overline{\mathbf{G}})$. The claim that π is of finite type is local on $Z^{\mathbf{G}}$, hence we may assume that $Z^{\mathbf{G}}$ is quasi-compact. The sheaf $\mathcal{J}/\mathcal{J}^2 \subseteq \mathcal{A}_1$ is coherent so there exists a finite set $\lambda_1, \ldots, \lambda_r \in \mathrm{Irr}(\overline{\mathbf{G}}) \setminus \{\lambda_0\}$ such that the morphism

$$\bigoplus_{i=1}^{r} \mathcal{A}[\lambda_i] \to \mathcal{J}/\mathcal{J}^2$$

induced by $\mathcal{A} \twoheadrightarrow \mathcal{A}_2$ is surjective. Let $\mathcal{B} \subset \mathcal{A}$ be the quasi-coherent $\mathcal{O}_{Z_0}$-subalgebra generated by the coherent subsheaf $\mathcal{M} := \bigoplus_{i=1}^r \mathcal{A}[\lambda_i] \subseteq \mathcal{A}$. Let $\overline{k}$ be an algebraic closure of k and let $\mathcal{A}' = \mathcal{A} \otimes \overline{k}$. Fix a Kempf's torus over $\overline{k}$ and the associated grading $\mathcal{A}' = \bigoplus_{i \geq 0} \mathcal{A}'[i]$ as in Corollary **??**. Then $\mathcal{J} = \bigoplus_{i \geq 1} \mathcal{A}'[i]$ is a graded ideal and $\mathcal{J}/\mathcal{J}^2$ is generated by the graded (coherent) subsheaf $\mathcal{M}' = \bigoplus_{i=1}^r \mathcal{A}'[\lambda_i]$. By graded Nakayama's lemma, the ideal $\mathcal{J}$ itself is generated by (the elements of) $\mathcal{M}'$. Then by induction on the degree, $\mathcal{A}'$ is generated by $\mathcal{M}'$ as an algebra. In other words, $\mathcal{A}' = \mathcal{B} \otimes \overline{k}$. Thus also $\mathcal{A} = \mathcal{B}$ and so $\mathcal{A}$ is of finite type over $\mathcal{O}_{Z_0}$. □

With the proof of Theorem 1.6 in hand, we can easily algebraize also equivariant mappings between formal schemes.

Proposition 1.7 (Algebraization of morphisms of formal $\overline{\mathbf{G}}$-schemes)**.** *Let $\mathcal{W} = \{W_n\}$ and $\mathcal{Z} = \{Z_n\}$ be formal $\overline{\mathbf{G}}$-schemes. Let W and Z be algebraizations of $\mathcal{W}$ and $\mathcal{Z}$ respectively (see Theorem 1.6). Then every $\overline{\mathbf{G}}$-morphism $\widehat{\varphi}\colon \mathcal{W} \to \mathcal{Z}$ is the formalization of a unique $\overline{\mathbf{G}}$-equivariant morphism $\varphi\colon W \to Z$.*

Proof. The map $\widehat{\varphi}$ induces maps $W_n \to Z_n \hookrightarrow Z$. By Theorem 1.6, the scheme W is a colimit of W_n in the category of locally linear $\overline{\mathbf{G}}$-schemes. By the universal property of the colimit, we obtain a unique $\overline{\mathbf{G}}$-equivariant morphism $W \to Z$. □

2. Formal $\mathbf{M}$-schemes

Let $\mathbf{M}$ be a k-monoid scheme.

Definition 2.1. Let X be a $\mathbf{M}$-scheme. We say that X is *locally linear* $\mathbf{M}$*-scheme* if there exists an open cover of X consisting of affine and $\mathbf{M}$-stable subchemes of X.

Definition 2.2. *A formal* $\mathbf{M}$*-scheme* consists of a sequence $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ of $\mathbf{M}$-schemes together with $\mathbf{M}$-equivariant closed immersions

$$Z_0 \hookrightarrow Z_1 \hookrightarrow \dots \hookrightarrow Z_n \hookrightarrow Z_{n+1} \hookrightarrow \dots$$

(1) $\mathbf{M}$-scheme Z_0 is locally linear.

(2) Let $\mathcal{I}_n$ be an ideal of $\mathcal{O}_{Z_n}$ defining Z_0. Then for every $m \le n$ the subscheme $Z_m \subset Z_n$ is defined by $\mathcal{I}_n^{m+1}$.

3. Thick subcategories

Definition 3.1. Let $\mathcal{C}$ be an abelian category and let $\mathcal{S}$ be its full subcategory. Suppose that for every exact sequence in $\mathcal{C}$

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

we have $X \in \mathcal{S}$ if and only if $X', X'' \in \mathcal{S}$. Then $\mathcal{S}$ is called *a thick subcategory of* $\mathcal{C}$.

Definition 3.2. A category $\mathcal{C}$ is called *well-powered* if the class of subobjects of X is a set for every object X in $\mathcal{C}$.

Proposition 3.3. *Let $\mathcal{C}$ be an* **Ab**3*-category and $\mathcal{S}$ be a thick subcategory. Assume that $\mathcal{S}$ is closed under small direct sums. Then for every object X in $\mathcal{C}$ there exists a unique subobject $S(X)$ such that for every morphism $f : Y \to X$ in $\mathcal{C}$ with Y in $\mathcal{S}$ we have $f(Y) \subseteq S(X)$.*

Proof. One can prove the result by invoking appropriate adjoint functor theorems [Mac Lane, 1998, Chapter V, Sections 5 and 6]. For self-containment we present the complete proof below.
Fix an object X of $\mathcal{C}$. Since $\mathcal{C}$ is well-powered, the class $\{Y_i\}_{i\in I}$ of subobjects of X that belong to $\mathcal{S}$ is a set. Since $\mathcal{S}$ is closed under small direct sums we derive that $\sum_{i\in I} Y_i \subseteq X$ is in $\mathcal{S}$. Indeed, this is the image of the canonical morphism

$$\bigoplus_{i\in I} Y_i \longrightarrow X$$

and since $\mathcal{S}$ is a thick subcategory closed under small direct sums, we deduce that this image is an object of $\mathcal{S}$. Thus $S(X) = \sum_{i\in I} Y_i$ is the largest subobject of X contained in $\mathcal{S}$. This implies the statement. □

Definition 3.4. Let $\mathcal{C}$ be an **Ab**3-category and $\mathcal{S}$ be a thick subcategory. Assume that $\mathcal{S}$ is closed under small direct sums. Then

References

[Mac Lane, 1998] Mac Lane, S. (1998). *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition.