MONOID k-FUNCTORS

1. Introduction and notation

In these notes we study algebraic structures on the category of k-functors with special emphasis on monoid objects.
If R is a ring, then we denote by $R^{\times}$ its multplicative monoid.

2. Algebraic structures in the category of k-functors

In the sequel we assume that the reader is familiar with notions of a monoid, group etc. in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 2-5].

Definition 2.1. *A monoid (group, abelian group, ring) k-functor* is a monoid (group, abelian group, ring) object in the category of k-functors.

Example 2.2. Let $\mathfrak{X}$ be a k-functor such that $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{X})$ exists. Then $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{X})$ is a monoid k-functor with respect to composition of morphisms.

Example 2.3. Basic example of a ring k-functor is a k-functor $\mathfrak{K}$ given by

$$\mathfrak{K}(A) = k,\ \mathfrak{K}(f) = 1_k$$

for any k-algebra A and morphism f of k-algebras. It can be described as a constant k-functor ([ML98, page 67]) corresponding to k.

Definition 2.4. Let $\mathfrak{R}$ be a ring k-functor. Then we denote by $\mathfrak{R}^{\times}$ the k-subfunctor of $\mathfrak{R}$ defined by

$$\mathfrak{R}^{\times}(A) = \mathfrak{R}(A)^{\times}$$

for every k-algebra A. We call $\mathfrak{R}^{\times}$ *the multiplicative monoid k-functor of* $\mathfrak{R}$.

Definition 2.5. Let $\mathfrak{A}$ be a commutative ring k-functor. *An $\mathfrak{A}$-algebra* is an $\mathfrak{A}$-algebra object in the category of k-functors.

Definition 2.6. Let $\mathfrak{R}$ be a ring k-functor. Suppose that $\mathfrak{M}$ is an abelian group k-functor and there exists a morphism $\mathfrak{R}\times\mathfrak{M}\rightarrow\mathfrak{M}$ of k-functors that for each k-algebra A makes $\mathfrak{M}(A)$ into an $\mathfrak{R}(A)$-module. Then we say that $\mathfrak{M}$ is *a module k-functor over* $\mathfrak{R}$.

Definition 2.7. Let $\mathfrak{R}$ be an ring k-functor and let $\mathfrak{M}_1,\mathfrak{M}_2$ be module k-functors over $\mathfrak{R}$. Suppose that $\sigma:\mathfrak{M}_1\rightarrow\mathfrak{M}_2$ is a morphism of abelian group k-functors such that the diagram

$$\begin{tikzcd}
\mathfrak{R}\times\mathfrak{M}_1 \arrow[r, "1_{\mathfrak{R}}\times\sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{R}\times\mathfrak{M}_2 \arrow[d, "\alpha_2"] \\
\mathfrak{M}_1 \arrow[r, "\sigma"'] & \mathfrak{M}_2
\end{tikzcd}$$

is commutative, where $\alpha_i:\mathfrak{R}\times\mathfrak{M}_i\rightarrow\mathfrak{M}_i$ define $\mathfrak{R}$-module structure on $\mathfrak{M}_i$ for $i=1,2$. Then σ is *a morphism of modules over* $\mathfrak{R}$.

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be module k-functors over $\mathfrak{R}$. We denote by

$$\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{M}_1,\mathfrak{M}_2)$$

as a class of all morphisms of modules $\mathfrak{M}_1\rightarrow\mathfrak{M}_2$ over $\mathfrak{R}_A$.

Definition 2.8. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be module k-functors over $\mathfrak{R}$. Assume that $\mathrm{Hom}_{\mathfrak{R}_A}\left((\mathfrak{M}_1)_A, (\mathfrak{M}_2)_A\right)$ is a set for every k-algebra A. Then we define a k-subfunctor $\mathcal{H}om_{\mathfrak{R}}(\mathfrak{M}_1, \mathfrak{M}_2)$ of internal hom of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ by formula

$$\mathbf{Alg}_k \ni A \mapsto \mathrm{Hom}_{\mathfrak{R}_A}\left((\mathfrak{M}_1)_A, (\mathfrak{M}_2)_A\right) \in \mathbf{Set}$$

We call $\mathcal{H}om_{\mathfrak{R}}(\mathfrak{M}_1, \mathfrak{M}_2)$ *a k-functor of module morphisms of* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$.

If $\mathfrak{M}$ is a module k-functor over some ring k-functor $\mathfrak{R}$, then we denote (if it exists) $\mathcal{H}om_{\mathfrak{R}}(\mathfrak{M}, \mathfrak{M})$ by $\mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$.

Example 2.9. Let $\mathfrak{M}$ be a module over a ring k-functor $\mathfrak{R}$. Assume that $\mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$ exists. Then $\mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$ is a ring k-functor with respect to composition of morphisms of modules as the multiplication and canonically defined addition of module morphisms.
If $\mathfrak{R}$ is a commutative ring k-functor, then $\mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$ admits additional strucutre of a $\mathfrak{R}$-algebra k-functor induced via a unique morphism $\mathfrak{R} \to \mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$ of ring k-functors that sends $1 \mapsto 1_{\mathfrak{M}}$.

3. Global regular functions on a k-functor

Recall the ring k-functor $\mathfrak{K}$ from Example 2.3. Note that a $\mathfrak{K}$-algebra $\mathfrak{A}$ can be viewed as a functor $\mathfrak{A} : \mathbf{Alg}_k \to \mathbf{Alg}_k$.

Definition 3.1. The $\mathfrak{K}$-algebra $\mathfrak{O}_k$ represented by the identity functor on $\mathbf{Alg}_k$ is called *the structure $\mathfrak{K}$-algebra*.

Let $|-| : \mathbf{Alg}_k \to \mathbf{Set}$ be the forgetful k-functor. Note that $|-|$ is the underlying k-functor of $\mathfrak{K}$-algebra $\mathfrak{O}_k$. Recall that the affine line $\mathbb{A}^1_k$ is an affine k-scheme having k-algebra of polynomials with one variable as a k-algebra of regular functions.

Fact 3.2. *Let* $|-| : \mathbf{Alg}_k \to \mathbf{Set}$ *be the forgetful k-functor. Then we have natural isomorphism*

$$\mathfrak{P}_{\mathbb{A}^1_k} \cong |-|$$

Proof. Let B be a k-algebra. We have the following chain of identifications

$$\mathfrak{P}_{\mathbb{A}^1_k}(B) = \mathrm{Mor}_k\left(\mathrm{Spec}\, B, \mathbb{A}^1_k\right) = \mathrm{Mor}_k\left(\mathrm{Spec}\, B, \mathrm{Spec}\, k[x]\right) = \mathrm{Mor}_k\left(k[x], B\right) = |B|$$

natural in B. □

In particular, since $|-|$ carries the structure $\mathfrak{K}$-algebra $\mathfrak{O}_k$, we derive that $\mathfrak{P}_{\mathbb{A}^1_k}$ admits a structure of $\mathfrak{K}$-algebra isomorphic to $\mathfrak{O}_k$.
No we introduce regular functions on a k-functors.

Definition 3.3. Let $\mathfrak{X}$ be a k-functor and assume that $\mathrm{Mor}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$ is a set. Then $\mathrm{Mor}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$ is a k-algebra with respect to the structure induced by $\mathfrak{O}_k$. We call this k-algebra *the k-algebra of global regular functions on* $\mathfrak{X}$. Its elements are called *global regular functions on* $\mathfrak{X}$.

Definition 3.4. Let $\mathfrak{X}$ be a k-functor. Suppose that A is a k-algebra, $x \in \mathfrak{X}(A)$ and $f \in \mathrm{Mor}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$. The element $f(x) \in A$ is called *the value of f on point x*.

For given k-functor $\mathfrak{X}$ we describe k-algebra operations on $\mathrm{Mor}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$ in terms of values of its elements on points of $\mathfrak{X}$. For this consider $\alpha \in k$ and $f, g \in \mathrm{Mor}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$. We have formulas

$$(f+g)(x) = f(x) + g(x),\ (f \cdot g)(x) = f(x) \cdot g(x),\ (\alpha \cdot f)(x) = \alpha \cdot f(x)$$

in which right hand side are k-algebra operations in A.

Example 3.5. Let $\mathfrak{X}$ be a k-functor and assume that $\mathcal{M}or_k(\mathfrak{X}, \mathfrak{O}_k)$ exists. Fix k-algebra A. Note that $\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{O}_A)$ is an A-algebra of global regular functions on $\mathfrak{X}_A$. Moreover, if B is an A-algebra, then

$$\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{O}_A) \ni f \mapsto f_B \in \mathrm{Mor}_B(\mathfrak{X}_B, \mathfrak{O}_B)$$

is a morphism of A-algebras. This implies that $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{O}_k)$ admits a canonical structure of an $\mathfrak{O}_k$-algebra k-functor.

4. Actions of monoid k-functors

In the sequel we assume that the reader is familiar with notion of an action of a monoid object in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 5].

Let $\mathfrak{G}$ be a monoid k-functor and $\mathfrak{X}$ be a k-functor together with an action $\alpha : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X}$. Next assume that k-functor $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ exists. By Example 2.2 it is a monoid k-functor. We define a morphism $\rho : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ of k-functors by formula $\rho(x) = \alpha_x$. Note that by discussion preceding [Mon19, Theorem 2.7] and by [Mon19, Corollary 2.9], we deduce that ρ is a well defined morphism of k-functors. We show now that ρ is a morphism of monoids. For this pick k-algebra A and $x, y \in \mathfrak{G}(A)$. Since α is an action, we deduce that $\alpha_{x\cdot y} = \alpha_x \cdot \alpha_y$ and hence also

$$\rho(x \cdot y) = \alpha_{x\cdot y} = \alpha_x \cdot \alpha_y = \rho(x) \cdot \rho(y)$$

Therefore, ρ is a morphism of monoid k-functors. This shows how to construct a morphism of monoid k-functors ρ from an action α of $\mathfrak{G}$.

Theorem 4.1. *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{X}$ be a k-functor such that $\mathcal{I}\mathrm{so}_k(\mathfrak{X}, \mathfrak{X})$ exists. Suppose that*

$$\Big\{\textit{actions of } \mathfrak{G} \textit{ on } \mathfrak{X}\Big\} \longrightarrow \Big\{\textit{Morphisms } \rho : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X}) \textit{ of monoid k-functors}\Big\}$$

is a map of classes described above. Then it is bijection.

Proof. Our goal is to construct the inverse of the map. Recall [Mon19, Theorem 2.7] and substitute in that Theorem $\mathfrak{J} = \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$. Consider maps

$$\Phi : \Big\{\text{families } \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X} \text{ of morphisms}\Big\} \to \mathrm{Mor}_k\left(\mathfrak{G}, \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})\right)$$

and

$$\Psi : \mathrm{Mor}_k\left(\mathfrak{G}, \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})\right) \to \Big\{\text{families } \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X} \text{ of morphisms}\Big\}$$

in that Theorem. Then the map in the statement above is the restriction of Φ to $\mathfrak{G}$-actions on $\mathfrak{X}$ on the right and morphisms $\mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ of monoid k-functors on the left. Since by [Mon19, Theorem 2.7] maps Φ and Ψ are mutually inverse, it suffices to check that Ψ sends a morphism $\rho : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ of monoids to an action of $\mathfrak{G}$ on $\mathfrak{X}$. For this denote $\Psi(\rho)$ by α. Consider k-algebra A and A-points $x, y \in \mathfrak{G}(A)$, $z \in \mathfrak{X}(A)$. Then

$$\alpha\left(y, \alpha(x, z)\right) = \rho(y)\left(\rho(x)(z)\right) = \left(\rho(y) \cdot \rho(x)\right)(z) = \rho\left(x \cdot y\right)(z) = \alpha\left(x \cdot y, z\right)$$

Therefore, α is an action of $\mathfrak{G}$ on $\mathfrak{X}$. □

Proposition 4.2. *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{X}_1, \mathfrak{X}_2$ be k-functors such that $\mathcal{M}\mathrm{or}_k(\mathfrak{X}_1, \mathfrak{X}_1), \mathcal{M}\mathrm{or}_k(\mathfrak{X}_2, \mathfrak{X}_2)$ exist. Suppose that $\alpha_1 : \mathfrak{G} \times \mathfrak{X}_1 \to \mathfrak{X}_1$, $\alpha_2 : \mathfrak{G} \times \mathfrak{X}_2 \to \mathfrak{X}_2$ are actions of $\mathfrak{G}$, respectively. Suppose that $\sigma : \mathfrak{X}_1 \to \mathfrak{X}_2$ is a morphism of k-functors. Then the following assertions are equivalent.*

(i) *The square*

$$\begin{tikzcd} \mathfrak{G} \times \mathfrak{X}_1 \arrow[r, "1_{\mathfrak{G}} \times \sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{G} \times \mathfrak{X}_2 \arrow[d, "\alpha_2"] \\ \mathfrak{X}_1 \arrow[r, "\sigma"'] & \mathfrak{X}_2 \end{tikzcd}$$

is commutative.

(ii) *For every k-algebra A and $x \in \mathfrak{G}(A)$ we have*

$$\sigma_A \cdot \rho_1(x) = \rho_2(x) \cdot \sigma_A$$

where $\rho_1 : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}_1, \mathfrak{X}_1)$ and $\rho_2 : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}_2, \mathfrak{X}_2)$ are morphism of monoid k-functors corresponding to α_1 and α_2, respectively.

Proof. Conditions expressed in **(i)** and **(ii)** are directly translatable to each other by virtue of the bijection in Theorem 4.1. □

Definition 4.3. Let $\mathfrak{G}$ be a monoid k-functor and let $(\mathfrak{X}_1, \alpha_1)$, $(\mathfrak{X}_2, \alpha_2)$ be k-functors with actions of $\mathfrak{G}$. Suppose that $\sigma : \mathfrak{X}_1 \to \mathfrak{X}_2$ is a morphism k-functors such that the square

$$\begin{tikzcd} \mathfrak{G} \times \mathfrak{X}_1 \arrow[r, "1_{\mathfrak{G}} \times \sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{G} \times \mathfrak{X}_2 \arrow[d, "\alpha_2"] \\ \mathfrak{X}_1 \arrow[r, "\sigma"'] & \mathfrak{X}_2 \end{tikzcd}$$

is commutative. Then σ is called *an $\mathfrak{G}$-equivariant morphism*.

5. Modules over ring k-functor

Let $\mathfrak{A}$ be a commutative ring k-functor and let $\mathfrak{R}$ be a $\mathfrak{A}$-algebra k-functor. This means that there exists a morphism $\mathfrak{A} \to \mathfrak{R}$ of ring k-functors and for every k-algebra A induced morphism $\mathfrak{A}(A) \to \mathfrak{R}(A)$ sends $\mathfrak{A}(A)$ to the center of a ring $\mathfrak{R}(A)$. Fix a module $\mathfrak{M}$ over $\mathfrak{A}$. Next assume that k-functor $\mathcal{E}nd_{\mathfrak{A}}(\mathfrak{M})$ exists. Recall that by Example 2.9 it is a ring k-functor.

Definition 5.1. In the setting above suppose that $\alpha : \mathfrak{R} \times \mathfrak{M} \to \mathfrak{M}$ is a morphism of k-functors. Suppose that α makes $\mathfrak{M}$ into $\mathfrak{R}$-module and moreover, for every k-algebra A and for every point $x \in \mathfrak{R}(A)$ morphism α_x is a morphism of $\mathfrak{A}_A$-modules. Then α is called *a $\mathfrak{A}$-linear $\mathfrak{R}$-action on $\mathfrak{M}$*.

We continue the discussion. We assume that we are given an $\mathfrak{A}$-linear $\mathfrak{R}$-action $\alpha : \mathfrak{R} \times \mathfrak{M} \to \mathfrak{M}$ on $\mathfrak{M}$. We define a morphism $\rho : \mathfrak{R} \to \mathcal{E}nd_{\mathfrak{A}}(\mathfrak{M})$ of k-functors by formula $\rho(x) = \alpha_x$. As in Section 4 we can prove that ρ is a morphism of ring k-functors. Now we have the following result.

Theorem 5.2. *Let $\mathfrak{R}$ be an algebra k-functor over commutative ring $\mathfrak{A}$ k-functor and let $\mathfrak{M}$ be a $\mathfrak{A}$-module such that $\mathcal{E}nd_{\mathfrak{A}}(\mathfrak{M})$ exists. Suppose that*

$$\Big\{\mathfrak{A} \text{ linear actions of } \mathfrak{R} \text{ on } \mathfrak{M}\Big\} \longrightarrow \Big\{\text{Morphisms } \rho : \mathfrak{R} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}) \text{ of ring k-functors}\Big\}$$

is a map of classes described above. Then it is bijection.

Proof. The proof is similar to the proof of Theorem 4.1. □

6. Monoid algebra $\mathfrak{O}_k[\mathfrak{G}]$ and its modules

Definition 6.1. Let $\mathfrak{G}$ be a monoid k-functor. Then we construct an $\mathfrak{O}_k$-algebra $\mathfrak{O}_k[\mathfrak{G}]$ as follows. For every k-algebra A we define

$$\mathfrak{O}_k[\mathfrak{G}](A) = A\big[\mathfrak{G}(A)\big]$$

where the right hand side is monoid A-algebra for the abstract monoid $\mathfrak{G}(A)$. The structure of monoid k-functor on $\mathfrak{G}$ and $\mathfrak{K}$-algebra $\mathfrak{O}_k$ makes $\mathfrak{O}_k[\mathfrak{G}]$ into a ring k-functor. Moreover, we have a morphism $\mathfrak{O}_k \to \mathfrak{O}_k[\mathfrak{G}]$ which for every k-algebra A is given by the canonical inclusion

$$A \hookrightarrow A\big[\mathfrak{G}(A)\big]$$

Thus $\mathfrak{O}_k[\mathfrak{G}]$ is $\mathfrak{O}_k$-algebra. We call $\mathfrak{O}_k[\mathfrak{G}]$ *a monoid $\mathfrak{O}_k$-algebra over $\mathfrak{G}$*.

Fact 6.2. *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{R}$ be an $\mathfrak{O}_k$-algebra k-functor. Then every morphism*

$$\sigma : \mathfrak{G} \to \mathfrak{R}^{\times}$$

of monoid k-functors admits a unique extension

$$\tilde{\sigma} : \mathfrak{O}_k[\mathfrak{G}] \to \mathfrak{R}$$

to a morphism of $\mathfrak{O}_k$-algebras.

Proof. This follows from the analogical universal property of algebras over abstract monoids (monoid algebras in **Set**). □

Definition 6.3. Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{M}$ be a module over $\mathfrak{O}_k$. Suppose that $\alpha : \mathfrak{G} \times \mathfrak{M} \to \mathfrak{M}$ is an action of $\mathfrak{G}$ such that for any k-algebra A and point $x \in \mathfrak{G}(A)$ morphism $\alpha_x : \mathfrak{M}_A \to \mathfrak{M}_A$ is a morphism of $\mathfrak{O}_A$-modules. Then α is called *a linear $\mathfrak{G}$-action on $\mathfrak{M}$.*

Suppose now that $\mathfrak{G}$ is a monoid k-functor and $\mathfrak{M}$ is a module $\mathfrak{O}_k$. Note that every linear $\mathfrak{G}$-action $\alpha : \mathfrak{G} \times \mathfrak{M} \to \mathfrak{M}$ extends uniquely to a $\mathfrak{O}_k$-linear action $\mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}$ of monoid $\mathfrak{O}_k$-algebra. This gives a bijection

$$\left\{\text{Linear actions of } \mathfrak{G} \text{ on } \mathfrak{M}\right\} \longrightarrow \left\{\mathfrak{O}_k\text{-linear actions } \mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}\right\}$$

Next assume that k-functor $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ exists. By Example 2.9 it is an $\mathfrak{O}_k$-algebra k-functor. Next by Theorem 5.2 we have a bijection

$$\left\{\mathfrak{O}_k\text{-linear actions of } \mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}\right\} \longrightarrow \left\{\text{Morphisms } \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}) \text{ of } \mathfrak{O}_k\text{-algebras}\right\}$$

Finally Fact 6.2 implies that we have a bijection

$$\left\{\text{Morphisms } \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}) \text{ of } \mathfrak{O}_k\text{-algebras}\right\} \longrightarrow \left\{\text{Morphisms } \mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}) \text{ of monoids}\right\}$$

This chain of bijections sends a linear action $\alpha : \mathfrak{G} \times \mathfrak{M} \to \mathfrak{M}$ of $\mathfrak{G}$ to a morphism $\rho : \mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ of monoid k-functors given by $\rho(x) = \alpha_x$ for every $x \in \mathfrak{G}(A)$ and every k-algebra A. We proved the following result.

Proposition 6.4. *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{M}$ be a $\mathfrak{O}_k$-module such that $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ exists. Then the following classes are in canonical bijections described above.*

(1) *Linear actions of $\mathfrak{G}$ on $\mathfrak{M}$.*

(2) *$\mathfrak{O}_k$-linear actions $\mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}$. These are precisely $\mathfrak{O}_k[\mathfrak{G}]$-modules.*

(3) *Morphisms $\mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ of $\mathfrak{O}_k$-algebras.*

(4) *Morphisms $\mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ of monoids.*

Moreover, the bijection between class **(1)** *and* **(2)** *does not require the existence of $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$.*

Now in a similar manner we can describe morphisms.

Proposition 6.5. *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{M}_1$, $\mathfrak{M}_2$ be k-functors of $\mathfrak{O}_k$-modules such that $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1), \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$ exist. Suppose that $\alpha_1 : \mathfrak{G} \times \mathfrak{M}_1 \to \mathfrak{M}_1$, $\alpha_2 : \mathfrak{G} \times \mathfrak{M}_2 \to \mathfrak{M}_2$ are linear actions of $\mathfrak{G}$, respectively. Suppose that $\sigma : \mathfrak{M}_1 \to \mathfrak{M}_2$ is a morphism of modules over $\mathfrak{O}_k$. Then the following assertions are equivalent.*

(i) *The square*

$$\begin{tikzcd}
\mathfrak{G}\times\mathfrak{M}_1 \arrow[r, "1_{\mathfrak{G}}\times\sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{G}\times\mathfrak{M}_2 \arrow[d, "\alpha_2"] \\
\mathfrak{M}_1 \arrow[r, "\sigma"'] & \mathfrak{M}_2
\end{tikzcd}$$

is commutative.

(ii) *The square*

$$\begin{tikzcd}
\mathfrak{O}_k[\mathfrak{G}]\times\mathfrak{M}_1 \arrow[r, "1_{\mathfrak{O}_k[\mathfrak{G}]}\times\sigma"] \arrow[d, "\tilde{\alpha_1}"'] & \mathfrak{O}_k[\mathfrak{G}]\times\mathfrak{M}_2 \arrow[d, "\tilde{\alpha_2}"] \\
\mathfrak{M}_1 \arrow[r, "\sigma"'] & \mathfrak{M}_2
\end{tikzcd}$$

is commutative, where $\tilde{\alpha_1}$ and $\tilde{\alpha_2}$ are $\mathfrak{O}_k$-linear actions of $\mathfrak{O}_k[\mathfrak{G}]$ corresponding to α_1 and α_2, respectively. This states that σ is a morphism of $\mathfrak{O}_k[\mathfrak{G}]$-modules.

(iii) *For every k-algebra A and $x\in\mathfrak{G}(A)$ we have*

$$\sigma_A\cdot\tilde{\rho_1}(x)=\tilde{\rho_2}(x)\cdot\sigma_A$$

where $\tilde{\rho_1}:\mathfrak{O}_k[\mathfrak{G}]\to\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1)$ and $\tilde{\rho_2}:\mathfrak{O}_k[\mathfrak{G}]\to\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$ are morphism of $\mathfrak{O}_k$-algebras corresponding to $\tilde{\alpha_1}$ and $\tilde{\alpha_2}$, respectively.

(iv) *For every k-algebra A and $x\in\mathfrak{G}(A)$ we have*

$$\sigma_A\cdot\rho_1(x)=\rho_2(x)\cdot\sigma_A$$

where $\rho_1:\mathfrak{O}_k[\mathfrak{G}]\to\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1)$ and $\rho_2:\mathfrak{O}_k[\mathfrak{G}]\to\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$ are morphism of monoid k-functors restricting $\tilde{\rho_1}$ and $\tilde{\rho_2}$, respectively.

The equivalence of **(1)** *and* **(2)** *does not require the existence of $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1)$ and $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$.*

Proof. Conditions expressed in **(i)-(iv)** are directly translatable to each other by virtue of bijections in Proposition 6.4. □

Let $\mathfrak{G}$ be a monoid k-functor. We denote by **Mod** $(\mathfrak{O}_k[\mathfrak{G}])$ the category of $\mathfrak{O}_k[\mathfrak{G}]$-modules.

7. Regular functions as a module over monoid k-functor

Let $\mathfrak{G}$ be a monoid k-functor. In this section we discuss important example of a $\mathfrak{O}_k[\mathfrak{G}]$-module. Fix a k-functor $\mathfrak{X}$ for which $\mathcal{M}or_k(\mathfrak{X},\mathfrak{O}_k)$ exists. Let $\alpha:\mathfrak{G}\times\mathfrak{X}\to\mathfrak{X}$ be an action of $\mathfrak{G}$ on $\mathfrak{X}$. According to [Mon19, Corollary 2.12] we deduce that α corresponds to a unique morphism of k-functors $\rho:\mathfrak{G}\to\mathcal{I}so_k(\mathfrak{X})$. For every k-algebra A and $x\in\mathfrak{G}(A)$ we have $\rho(x)=\alpha_x$. Moreover, ρ is a morphism of k-monoids (this is a consequence of the fact that α is an action). Next we have a map of sets

$$\mathrm{Mor}_A\left(\mathfrak{X}_A,(\mathfrak{O}_k)_A\right)\ni f\mapsto f\cdot\rho(x)\in\mathrm{Mor}_A\left(\mathfrak{X}_A,(\mathfrak{O}_k)_A\right)$$

For every A-algebra B and every point $y\in\mathfrak{X}(B)$ we have

$$(f\cdot\rho(x))(y)=f\left(\rho(x)(y)\right)$$

From this description it follows that the map $f\mapsto f\cdot\rho(x)$ is a morphism of A-algebras.

Lemma 7.0.1. *Let Γ be a consistent set of formulas of $\mathcal{L}$. Then there exist a variable extension $\mathcal{M}$ of $\mathcal{L}$ and a set Δ of formulas of $\mathcal{M}$ such that the following assertions hold.*

(1) $\Gamma\subseteq\Delta$.

(2) *If* $(\neg\forall x\phi) \in \Gamma$*, then there exists a variable* z *in* $\mathcal{M}$ *substitutable for* x *in* ϕ *such that* $(\neg[\phi]^x_z) \in \Delta$.

(3) Δ *is consistent in* $\mathcal{M}$.

Proof of the lemma. We define variable extension $\mathcal{M}$ of $\mathcal{L}$. We enlarge $\mathbf{V}_{\mathcal{L}}$ to $\mathbf{V}_{\mathcal{M}}$ by adding for each formula of the form $(\neg\forall x\phi) in \Gamma$ a unique variable $z_{\phi,x}$. Hence

$$\mathbf{V}_{\mathcal{M}} = \mathbf{V}_{\mathcal{L}} \cup \left\{ z_{\phi,x} \,\middle|\, \text{if } (\neg\forall x\phi) \in \Gamma \text{ for some } x \in \mathbf{V}_{\mathcal{L}} \text{ and a formula } \phi \text{ of } \mathcal{L} \right\}$$

We define

$$\Delta = \Gamma \cup \left\{ (\neg[\phi]^x_{z_{\phi,x}}) \,\middle|\, \text{if } (\neg\forall x\phi) \in \Gamma \right\}$$

Then Δ is a set of formulas of $\mathcal{M}$ and **(1)** and **(2)** are satisfied. Suppose that Δ is inconsistent set of formulas of $\mathcal{M}$. By compactness of $\vdash_{\mathcal{M}}$ we derive that Δ has a finite inconsistent (with respect to $\vdash_{\mathcal{M}}$) subset. By Lemma **??** we derive that Γ is consistent set with respect to $\vdash_{\mathcal{M}}$. Thus there exists a finite subset

$$\left\{ \left(\neg[\phi_1]^{x_1}_{z_{\phi_1,x_1}}\right), \dots, \left(\neg[\phi_n]^{x_n}_{z_{\phi_n,x_n}}\right), \left(\neg[\phi_{n+1}]^{x_{n+1}}_{z_{\phi_{n+1},x_{n+1}}}\right) \right\} \subseteq \left\{ \left(\neg[\phi]^x_{z_{\phi,x}}\right) \,\middle|\, \text{if } (\neg\forall x\phi) \in \Gamma \right\}$$

such that its union with Γ is inconsistent with respect to $\vdash_{\mathcal{M}}$ and the union

$$\Xi = \Gamma \cup \left\{ \left(\neg[\phi_1]^{x_1}_{z_{\phi_1,x_1}}\right), \dots, \left(\neg[\phi_n]^{x_n}_{z_{\phi_n,x_n}}\right), \left(\neg[\phi_{n+1}]^{x_{n+1}}_{z_{\phi_{n+1},x_{n+1}}}\right) \right\}$$

is consistent with respect to $\vdash_{\mathcal{M}}$. Write $\phi = \phi_{n+1}$ and $x = x_{n+1}$. We have that $\Xi \cup \{\left(\neg[\phi]^x_{z_{\phi,x}}\right)\}$ is inconsistent with respect to $\vdash_{\mathcal{M}}$. We denote by $\perp$ any formula of $\mathcal{M}$ that is a negation of a tautology of $\mathcal{M}$. Then

$$\Xi \cup \{\left(\neg[\phi]^x_{z_{\phi,x}}\right)\} \vdash_{\mathcal{M}} \perp$$

By Theorem **??** we derive that

$$\Xi \vdash_{\mathcal{M}} \left(\left(\neg[\phi]^x_{z_{\phi,x}}\right) \to \perp \right)$$

Since formula

$$\left(\left((\neg[\phi]^x_{z_{\phi,x}}) \to \perp \right) \to [\phi]^x_{z_{\phi,x}} \right)$$

is a tautology of $\mathcal{M}$, we deduce that $\Xi \vdash_{\mathcal{M}} [\phi]^x_{z_{\phi,x}}$. By Theorem **??** we derive that $\Xi \vdash_{\mathcal{M}} \forall z_{\phi,x}[\phi]^x_{z_{\phi,x}}$ (indeed, $z_{\phi,x}$ does not occurre as a free variable in any formula of Ξ). By Lemma **??** we have $\forall z_{\phi,x}[\phi]^x_{z_{\phi,x}} \vdash_{\mathcal{M}} \forall x\phi$ and hence by Theorem **??** we have $\vdash_{\mathcal{M}} (\forall z_{\phi,x}[\phi]^x_{z_{\phi,x}} \to \forall x\phi)$. Thus $\Xi \vdash_{\mathcal{M}} \forall x\phi$. On the other hand $(\neg\forall x\phi) \in \Gamma$ and hence this formula is an element of Ξ. Thus

$$\Xi \vdash_{\mathcal{M}} \forall x\phi, \Xi \vdash_{\mathcal{M}} (\neg\forall x\phi)$$

and therefore, Ξ is inconsistent with respect to $\vdash_{\mathcal{M}}$. This is a contradiction. □

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