ALGEBRAIC MONOIDS

1. THE UNIT GROUP OF AN ALGEBRAIC MONOID

We start by proving result on generic finiteness.

Theorem 1.1. *Let $f : X \rightarrow Y$ be a dominant morphism of finite type between irreducible schemes. Suppose that η is a generic point and assume that the generic fiber $f^{-1}(\eta)$ is finite. Then there exists an open and nonempty subset V of Y such that the restriction $f^{-1}(V) \rightarrow V$ of V is finite.*

For the proof we need the following local version of the theorem.

Lemma 1.1.1. *Let A be a ring such that* $\operatorname{Spec} A$ *is irreducible and let B be an A-algebra of finite type. Suppose that a unique minimal prime ideal $\mathfrak{p}$ of A is nilpotent and $k(\mathfrak{p}) \otimes_A B$ is finite over $k(\mathfrak{p})$, where $k(\mathfrak{p})$ denotes the residue field of $\mathfrak{p}$ in A. Then there exists nonzero s in A such that B_s is a finite A_s-module.*

Proof of the lemma. Let $b_1, ..., b_n$ be generators of B as an A-algebra. Then

$$\overline{b_i} = b_i \operatorname{mod} \mathfrak{p}B$$

for $1 \leq i \leq n$ are generators of $B/\mathfrak{p}B$ as an $A/\mathfrak{p}$ algebra. Since $k(\mathfrak{p}) \otimes_A B$ is finite over $k(\mathfrak{p})$ for each i there exists positive integer m_i and a polynomial

$$f_i(x) = s_{m_i} x^{m_i} + s_{m_i - 1} x^{m_i - 1} + ... + s_0 \in (A/\mathfrak{p})[x]$$

such that $s_{m_i} \neq 0$ and $f_i(\overline{b_i}) = 0$. Let $s \in A$ be an element such that

$$\overline{s} = s \operatorname{mod} \mathfrak{p} = s_{m_1} \cdot s_{m_2} \cdot ... \cdot s_{m_n}$$

Clearly s is nonzero and $B_s/(\mathfrak{p}B)_s = (B/\mathfrak{p}B)_s$ is a finite A_s-algebra. Hence there exists a finite A_s-submodule M of B_s such that

$$B_s = M + (\mathfrak{p}B)_s = M + \mathfrak{p}B_s$$

Since $\mathfrak{p}$ is nilpotent, there exists $N \in \mathbb{N}$ such that $\mathfrak{p}^N = 0$. Thus

$$B_s = M + \mathfrak{p}B_s = M + \mathfrak{p}M + \mathfrak{p}^2 B_s = ... = M + \mathfrak{p}M + ... + \mathfrak{p}^{N-1}M + \mathfrak{p}^N B_s = M + \mathfrak{p}M + ... + \mathfrak{p}^{N-1}M$$

is a finite A_s-module. $\square$

Proof of the theorem. Pick an open, nonempty, affine neighborhood W of η. Since f is of finite type, we derive that

$$f^{-1}(W) = \bigcup_{i=1}^{n} U_i$$

where each U_i is nonempty open affine subscheme of X and moreover, the morphism $U_i \rightarrow V$ induced by f is of finite type. According to Lemma 1.1.1 for each i there exists an open, affine and nonempty subscheme $W_i \subseteq W$ such that the morphism $f^{-1}(W_i) \cap U_i \rightarrow W_i$ induced by f is finite. Thus replacing W by the intersection of $W_1, ..., W_n$ we may assume that each $U_i \rightarrow W$ is finite. Consider

$$F = f^{-1}(W) \setminus \left(\bigcap_{i=1}^{n} U_i\right)$$

Then F is a closed subset of $f^{-1}(W)$ and it does not contain the generic point ξ of X. Since each restriction $U_i \rightarrow W$ of f is finite, we derive that $f\big(U_i \cap F\big)$ is closed in W for every $1 \leq i \leq n$ and does not contain $\eta = f(\xi)$ (f is dominant). Thus $f(F)$ is a closed subset of W and $\eta \notin f(F)$.

Hence $V = W \setminus f(F)$ is an open neighborhood of η and $f^{-1}(V) \subseteq \bigcap_{i=1}^{n} U_i$. Thus the restriction $f^{-1}(V) \to V$ of f is finite. □

Theorem 1.2. *Let $\mathbf{M}$ be a geometrically integral algebraic monoid k-scheme. Suppose that $\mathbf{G}$ is a group of units of $\mathbf{M}$ and $i : \mathbf{G} \hookrightarrow \mathbf{M}$ is the canonical monomorphism. Then i is an open immersion.*

Proof. Assume that k is algebraically closed. Denote by $\mu : \mathbf{M} \times_k \mathbf{M} \to \mathbf{M}$ and $e : \operatorname{Spec} k \to \mathbf{M}$ the multiplication and the unit, respectively. Since $\mathbf{M}$ is integral and of finite type over k, we derive that $\mathbf{M} \times_k \mathbf{M}$ is integral and

$$\dim(\mathbf{M} \times_k \mathbf{M}) = 2 \cdot \dim(\mathbf{M})$$

Moreover, μ is surjective (which can be checked on k-functors of points). Pick any irreducible component Z of $\mu^{-1}(e)$. By [Görtz and Wedhorn, 2010, Lemma 14.109] we deduce

$$\dim(Z) \geq \dim\left(\mu^{-1}(\eta)\right)$$

where η is the generic point of $\mathbf{M}$. Since

$$\dim\left(\mu^{-1}(\eta)\right) = \dim(\mathbf{M} \times_k \mathbf{M}) - \dim(\mathbf{M}) = 2 \cdot \dim(\mathbf{M}) - \dim(\mathbf{M}) = \dim(\mathbf{M})$$

we deduce that $\dim(Z) \geq \dim(\mathbf{M})$. Moreover, we have $\mathbf{G} \cong \mu^{-1}(e)$ as k-schemes and this isomorphism is given by the restriction $\pi : \mu^{-1}(e) \to \mathbf{G}$ to $\mu^{-1}(e)$ of the projection $\mathrm{pr} : \mathbf{M} \times_k \mathbf{M} \to \mathbf{M}$ on the first factor (this can be checked on k-functors of points). Hence $\mathbf{G}$ is of finite type over k as it is isomorphic with a closed subscheme of $\mathbf{M} \times_k \mathbf{M}$ and each irreducible component Z of $\mathbf{G}$ is of dimension at least $\dim(\mathbf{M})$. Now we fix an irreducible component Z of $\mathbf{G}$ and consider it as a closed subscheme of $\mathbf{G}$ with reduced structure. Then the morphism $i_{|Z} : Z \hookrightarrow \mathbf{M}$ is a monomorphism of finite type and $\dim(Z) \geq \dim(\mathbf{M})$. Hence $i_{|Z}$ is dominant. Since i is a monomorphism, this implies that $\mathbf{G}$ has only one irreducible component and $i : \mathbf{G} \hookrightarrow \mathbf{M}$ is dominant. By Theorem 1.1 there exists an open and nonempty subset V of $\mathbf{M}$ such that the morphism $i^{-1}(V) \hookrightarrow V$ induced by i is finite. Finite monomorphisms are closed immersions and dominant, closed immersions with integral scheme as a codomain are isomorphisms. Thus $i^{-1}(V) \to V$ is an isomorphism. Now pick a k-point g of $\mathbf{G}$. Since $\mathbf{G}$ is a group k-scheme, we derive that $g \cdot (-) : \mathbf{M} \to \mathbf{M}$ is an automorphism of the k-scheme $\mathbf{M}$. This implies that $i^{-1}(g \cdot V) \to g \cdot V$ is an isomorphism. This holds for every k-point of $\mathbf{G}$ and

$$i(\mathbf{G}) \subseteq \bigcup_{g \in \mathbf{G}(k)} g \cdot V$$

where $\mathbf{G}(k)$ is the set of k-points of $\mathbf{G}$. Therefore, i is an open immersion.
If k is not algebraically closed, then we pick an algebraically closed extension K of k and consider $1_{\operatorname{Spec} K} \times_k i$. This is an open immersion according to the case considered above. By faithfully flat descent i is an open immersion. □

The more general result for algebraically closed fields is [Brion, 2014, Theorem 1].

Corollary 1.3. *Let $\mathbf{M}$ be a geometrically integral, algebraic monoid over k. Then the inclusion $\mathbf{G} \hookrightarrow \mathbf{M}$ of the group of units is schematically dense open immersion.*

Let us also note the following theorem.

Theorem 1.4 ([Demazure and Gabriel, 1970, Chapitre 2, &2, Corollaire 3.6])**.** *Let $\mathbf{M}$ be an affine, algebraic monoid k-scheme. Suppose that $\mathbf{G}$ is a group of units of $\mathbf{M}$. Then the following results holds.*

(1) *There exists a finite dimensional vector space V over k and a closed immersion*

$$\mathbf{M} \hookrightarrow \operatorname{Spec} Sym(V \otimes_k V^{\vee}) = \mathbf{L}(V)$$

of algebraic monoids.

(2) *There exists a regular function f on $\mathbf{M}$ such that canonical morphism $\mathbf{G} \hookrightarrow \mathbf{M}$ is the inclusion of open subscheme of $\mathbf{M}$ on which f is nonzero.*

The converse is also true.

Theorem 1.5 ([Brion, 2014, Theorem 2]). *Let* $\mathbf{M}$ *be a geometrically integral algebraic monoid over a field* k *and let* $\mathbf{G}$ *be an group of units of* $\mathbf{M}$*. If* $\mathbf{G}$ *is affine, then* $\mathbf{M}$ *is affine.*

Definition 1.6. Let $\mathbf{M}$ be a geometrically integral algebraic monoid over k and let $\mathbf{G}$ be its group of units. If $\mathbf{G}$ is (linearly) reductive, then $\mathbf{M}$ is called *a (linearly) reductive monoid over* k.

By definition every reductive group is affine. Hence using Theorem 1.5 we deduce the following result.

Corollary 1.7. *Let* $\mathbf{M}$ *be a reductive monoid over* k*. Then* $\mathbf{M}$ *is affine.*

2. Toric monoids

Definition 2.1. Let T be a torus over k and let $\overline{T}$ be a geometrically integral, algebraic monoid having T as the group of units. Then $\overline{T}$ is *a toric monoid over* k.

Corollary 2.2. *Let* $\overline{T}$ *be a toric monoid over* k*. Then* $\overline{T}$ *is a linearly reductive monoid over* k*.*

Proof. This follows from [Monygham, 2020, Corollary 10.4] □

Theorem 2.3. *Let* $\overline{T}$ *be a toric monoid over* k *with group of units* T *and let* K *be an algebraically closed extension of* k*. Suppose that* N *is a dimension of* T*.*

(1) *The group of characters of* T_K *is isomorphic to* $\mathbb{Z}^N$ *and there exists an abstract submonoid* S *of* $\mathbb{Z}^N$ *such that the open immersion*
$$T_K = \mathrm{Spec}\left(\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m\right) \hookrightarrow \mathrm{Spec}\left(\bigoplus_{m\in S} K\cdot\chi^m\right) = \overline{T}_K$$
is induced by the inclusion $S \hookrightarrow \mathbb{Z}^N$.

(2) *Let* $\{V_\lambda\}_{\lambda\in\mathbf{Irr}(T)}$ *be a set of irreducible representation of* T *such that* V_λ *is in isomorphism class* λ*. For every* λ *there exists a finite subset* A_λ *of* $\mathbb{Z}^N$ *such that*
$$K\otimes_k V_\lambda = \bigoplus_{m\in A_\lambda} K\cdot\chi^m$$
If λ *is in* $\mathbf{Irr}(\overline{T})$*, then* A_λ *is a subset of* S*. Moreover, we have*
$$\mathbb{Z}^N = \coprod_{\lambda\in\mathbf{Irr}(T)} A_\lambda$$
and $A_{\lambda_0} = \{0\}$*, where* λ_0 *is the class of the trivial representation of* T*.*

(3) *If* $\overline{T}$ *has a zero, then there exists a homomorphism* $f : \mathbb{Z}^N \to \mathbb{Z}$ *of abelian groups such that* $f_{|S\smallsetminus\{0\}} > 0$*. In particular,* f *induces a closed immersion*
$$\mathrm{Spec}\, K\times_k \mathbb{G}_m = \mathrm{Spec}\, K[\mathbb{Z}] \hookrightarrow \mathrm{Spec}\left(\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m\right) = T_K$$
of group K*-schemes that extends to a zero preserving closed immersion* $\mathbb{A}^1_K \hookrightarrow \overline{T}_K$ *of monoid* K*-schemes.*

Proof. Since T is a torus, we derive that
$$T_K = \mathrm{Spec}\, K\times_k \underbrace{\mathbb{G}_m\times_k\mathbb{G}_m\times_k\ldots\times_k\mathbb{G}_m}_{N\text{ times}} = \mathrm{Spec}\left(\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m\right)$$

and hence

$$\overline{T}_K = \operatorname{Spec}\left(\bigoplus_{s\in S} K\cdot\chi^s\right)$$

for some abstract submonoid S of $\mathbb{Z}^N$. Moreover, the open immersion $T_K \hookrightarrow \overline{T}_K$ is induced by the inclusion $S \hookrightarrow \mathbb{Z}^N$. This proves **(1)**.
We have identification

$$k[T] = \bigoplus_{\lambda\in\mathbf{Irr}(T)} V_\lambda^{n_\lambda}$$

of T-representations, where $n_\lambda \in \mathbb{N}\smallsetminus\{0\}$ for each λ. Thus

$$\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m = K\otimes_k k[T] = \bigoplus_{\lambda\in\mathbf{Irr}(T)} (K\otimes_k V_\lambda)^{n_\lambda}$$

This implies that $n_\lambda = 1$ for every λ and moreover, we derive that

$$K\otimes_k V_\lambda = \bigoplus_{m\in A_\lambda} K\cdot\chi^m$$

for some finite set $A_\lambda \subseteq \mathbb{Z}^N$. We also have $A_{\lambda_0} = \{0\}$ and $A_\lambda \subseteq S\smallsetminus\{0\}$ for $\lambda \in \mathbf{Irr}(\overline{T})$. This proves **(2)**.
Since $\overline{T}$ admits a zero, we derive that

$$\mathfrak{m} = \bigoplus_{m\in S\smallsetminus\{0\}} K\cdot\chi^s \subseteq \bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m$$

is an ideal. This implies that $S\smallsetminus\{0\}$ is closed under addition. In particular, there exists a homomorphism of abelian groups $f:\mathbb{Z}^N\to\mathbb{Z}$ such that $f_{|S\smallsetminus\{0\}} > 0$. This implies **(3)**. □

3. Kempf monoids

In this section we introduce important class of a monoid k-schemes, which contains all reductive monoids over k. We recall classical result concerning quotients with respect to actions of linearly reductive groups on affine algebraic schemes over k.

Theorem 3.1. *[[Bialynicki-Birula et al., 2013, Theorem 5.4 and discussion below its statement]] Let X be an affine k-scheme of finite type equipped with an action of a linearly reductive algebraic group $\mathbf{G}$. Consider the morphism $\pi : X \to Y$ of affine k-schemes induced by the inclusion $\Gamma(X,\mathcal{O}_X)^{\mathbf{G}} \hookrightarrow \Gamma(X,\mathcal{O}_X)$. Then the following assertions hold.*

- **(1)** *Y is of finite type over k.*
- **(2)** *If Z_1 and Z_2 are disjoint, $\mathbf{G}$-stable and closed subschemes of X, then $\pi(Z_1)$ and $\pi(Z_2)$ are disjoint.*
- **(3)** *π is surjective.*
- **(4)** *If we consider Y as a k-scheme with trivial $\mathbf{G}$-action, then π is $\mathbf{G}$-equivariant morphism.*
- **(5)** *If $p : X \to W$ is a $\mathbf{G}$-equivariant morphism and W is a k-scheme with trivial $\mathbf{G}$-action, then p uniquely factors through π.*

Now we are ready to prove the following result.

Theorem 3.2. *Let $\mathbf{M}$ be a reductive algebraic monoid over k and let $\mathbf{G}$ be a group of units of $\mathbf{M}$. Assume that $\mathbf{M}$ admits a zero $\mathbf{o}$. Then there exists a central torus T in $\mathbf{G}$ such that $\mathbf{o} \in \mathbf{cl}(T)$.*

Proof. By assumption $\mathbf{G}$ is a reductive group. According to [Milne, 2017, Corollary 17.62 and Notation 12.29] its centre $Z(\mathbf{G})$ is an algebraic group of multiplicative type and the largest subtorus T of $Z(\mathbf{G})$ is the solvable radical $R(\mathbf{G})$ of $\mathbf{G}$. In particular, the quotient group $\mathbf{G}/T$ has trivial solvable radical and hence it is a semisimple algebraic group. Now T is linearly reductive [Monygham, 2020, Corollary 10.4]. Thus by Theorem 3.1 we obtain a quotient $\pi : \mathbf{M} \twoheadrightarrow \mathbf{Q}$ of $\mathbf{M}$ by the action of T. Note also that T is central in $\mathbf{M}$ as it is central in $\mathbf{G}$. Next the fact that T is central in $\mathbf{M}$, the fact that $\mathbf{M}$ is geometrically integral and Theorem 3.1 imply that $\mathbf{Q}$ is a geometrically integral, affine and algebraic monoid k-scheme with zero. Moreover, π is a surjective morphism of algebraic monoids over k. According to Theorem 1.2 we derive that the group of units $\mathbf{Q}^*$ is the open subscheme of $\mathbf{Q}$. From the fact that $\mathbf{G} \hookrightarrow \mathbf{M}$ is dominant we derive that the restriction $\pi_{|\mathbf{G}} : \mathbf{G} \to \mathbf{Q}$ is dominant. Thus π induces a dominant morphism of geometrically integral algebraic groups $\mathbf{G} \to \mathbf{Q}^*$. Next [Monygham, 2020, Theorem 5.3] implies that $\pi(\mathbf{G}) = \mathbf{Q}^*$. Theorem 1.4 implies that there exists a closed immersion of monoids $i : \mathbf{Q} \hookrightarrow \mathbf{L}(V)$ for some finite dimensional vector k-space V. Thus $i \cdot \pi_{\mathbf{G}}$ composed with the determinant $\det : \mathbf{L}(V) \to \mathbb{G}_m$ is a character of $\mathbf{G}$ that factors through the quotient morphism $\mathbf{G} \twoheadrightarrow \mathbf{G}/T$, but $\mathbf{G}/T$ is a semisimple algebraic group and hence it has only trivial characters. Therefore, the character of $\mathbf{G}$ constructed above is trivial. Hence $i(\mathbf{Q}^*) = i \cdot \pi(\mathbf{G})$ is contained in the algebraic subgroup $\mathbf{SL}(V)$ of $\mathbf{L}(V)$. Next i induces a morphism of algebraic groups $\mathbf{Q}^* \hookrightarrow \mathbf{SL}(V)$ and by [Monygham, 2020, Theorem 5.3] we infer that $i(\mathbf{Q}^*)$ is closed in $\mathbf{SL}(V)$. Since $\mathbf{SL}(V)$ is closed in $\mathbf{L}(V)$, we derive that $i(\mathbf{Q}^*)$ is closed in $\mathbf{L}(V)$ and hence it is also closed in $\mathbf{Q}$. On the other hand we proved that is open in $\mathbf{Q}$. $\mathbf{Q}$ is (geometrically) integral and hence it is connected. Thus $\mathbf{Q}^* = \mathbf{Q}$ which means that $\mathbf{Q}$ is a group k-scheme. Moreover, $\mathbf{Q}$ is a monoid k-scheme with zero. Thus is only possible if $\mathbf{Q}$ is $\operatorname{Spec} k$. Therefore, the categorical quotient $\pi : \mathbf{M} \to \mathbf{Q}$ consists of a single k-rational point. Thus by **(2)** Theorem 3.1 the closure of every orbit of T in $\mathbf{M}$ contains the zero $\mathbf{o}$. In particular, $\mathbf{o} \in \mathbf{cl}(T)$. □

This theorem motivates the following definition.

Definition 3.3. Let $\mathbf{M}$ be a geometrically integral, affine algebraic monoid over k. Assume that $\mathbf{M}$ admits a zero $\mathbf{o}$ and let $\mathbf{G}$ is a group of units of $\mathbf{M}$. Suppose that there exists a central subtorus T of $\mathbf{G}$ such that its closure contains $\mathbf{o}$. Then we say that $\mathbf{M}$ is *a Kempf monoid over k*.

Let us note for the future reference the following reformulation of Theorem 3.2.

Corollary 3.4. *Let $\mathbf{M}$ be a reductive monoid over k. Then $\mathbf{M}$ is a Kempf monoid.*

Now we give an example of a Kempf monoid which is not reductive.

Example 3.5 (Kempf monoid with nonreductive group of units)**.** Let n be a positive integer. Consider the algebraic group $\mathbf{B}_n$ of invertible upper triangular $n \times n$ matrices. Let $\overline{\mathbf{B}}_n$ be the closure of $\mathbf{B}_n$ in the algebraic monoid of all $n \times n$ matrices $\mathbf{M}_n$. Then $\overline{\mathbf{B}}_n$ is an affine, geometrically integral algebraic monoid over k with zero (it contains zero matrix). Actually $\overline{\mathbf{B}}_n$ (or better to say its k-functor of points) consists of all upper triangular $n \times n$ matrices. The group of units of $\overline{\mathbf{B}}_n$ is $\mathbf{B}_n$ and hence it is solvable. Moreover, the center of $\mathbf{B}_n$ contains the one-dimensional split torus $\mathbb{G}_m$ consisting of scalar matrices. The closure of this torus in $\overline{\mathbf{B}}_n$ contains zero matrix and hence $\overline{\mathbf{B}}_n$ is the Kempf monoid.

Let us discuss some properties of Kempf monoids. We first note the following.

Proposition 3.6. *Let $\mathbf{M}$ be a Kempf monoid over k and let T be a central torus of $\mathbf{M}$ such that T contains $\mathbf{o}$. Then the closure $\overline{T}$ of T in $\mathbf{M}$ with reduced subscheme structure is a closed toric submonoid k-scheme of $\mathbf{M}$ containing zero.*

Proof. The multiplication μ on $\mathbf{M}$ induces a morphism $\mu_{|\overline{T}\times_k\overline{T}} : \overline{T} \times_k \overline{T} \to \mathbf{M}$. Since scheme-theoretic image of $\mu(T \times_k T)$ is contained in $\overline{T}$ and $T \times_k T$ is open and schematically dense in $\overline{T} \times_k \overline{T}$, we deduce that $\mu_{|\overline{T}\times_k\overline{T}}$ factors through closed subscheme $\overline{T}$. Thus μ restricts to a multiplication $\nu : \overline{T} \times_k \overline{T} \to \overline{T}$ and hence $\overline{T} \hookrightarrow \mathbf{M}$ is closed immersion of monoid k-schemes. Clearly $\overline{T}$ is geometrically integral as a scheme-theoretic closure of a geometrically integral scheme T. The fact that the zero $\mathbf{o}$ of $\mathbf{M}$ is contained in $\overline{T}$ follows by definition. □

Corollary 3.7. *Let* $\mathbf{M}$ *be a Kempf monoid over* k*. Fix an algebraically closed field* K *over* k*. Then there exists a closed immersion*

$$i : \mathbb{A}^1_K \hookrightarrow \operatorname{Spec} K \times_k \mathbf{M}$$

of monoid K*-schemes sending the zero of* $\mathbb{A}^1_K$ *to the zero of* $\mathbf{M}_K = \operatorname{Spec} K \times_k \mathbf{M}$.

Proof. This follows from Proposition 3.6 and **(3)** Theorem 2.3. □

Theorem 3.8. *Let* $\mathbf{M}$ *be a Kempf monoid over* k *with group* $\mathbf{G}$ *of units and let* $j : Z \hookrightarrow \mathbf{M}$ *be a locally closed* $\mathbf{G}$*-stable subscheme of* $\mathbf{M}$*. Then the following are equivalent.*

- **(i)** *For every* $n \in \mathbb{N}$ *the* n*-th infinitesimal neighborhood* $\mathbf{M}_n$ *of* $\mathbf{o}$ *in* $\mathbf{M}$ *is contained in* Z*.*
- **(ii)** j *is an isomorphism.*

We first consider the following special case.

Lemma 3.8.1. *Let* U *be an open* $\mathbf{G}$*-stable subscheme of* $\mathbf{M}$*. If* $\mathbf{o}$ *is a point of* U*, then* $U = \mathbf{M}$*.*

Proof of the lemma. Fix $i : \mathbb{A}^1_K \hookrightarrow \operatorname{Spec} K \times_k \mathbf{M}$ as in Corollary 3.7. Denote

$$\operatorname{Spec} K \times_k \mathbf{M},\ \operatorname{Spec} K \times_k \mathbf{G},\ \operatorname{Spec} K \times_k U$$

by $\mathbf{M}_K$, $\mathbf{G}_K$, U_K, respectively. Note that $i\left(\mathbb{G}_{m,K}\right) \subseteq \mathbf{G}_K$. Fix a field L over K and a morphism $j : \operatorname{Spec} L \hookrightarrow \mathbf{M}_K$. Next consider the composition

$$\begin{tikzcd} \mathbb{A}^1_L = \mathbb{A}^1_K \times_K \operatorname{Spec} L \arrow[r, hook, "i \times_K j"] \arrow[rr, bend left, "f"] & \mathbf{M}_K \times_k \mathbf{M}_K \arrow[r, hook, "\mu_K"'] & \mathbf{M}_K \end{tikzcd}$$

where the second morphism $\mu_K : \mathbf{M}_K \times_k \mathbf{M}_K \to \mathbf{M}_K$ is the multiplication. Clearly f is $\mathbb{G}_{m,L}$-equivariant. Hence $f^{-1}(U_K)$ is an open $\mathbb{G}_{m,L}$-stable subscheme of $\mathbb{A}^1_L$. It contains the zero of $\mathbb{A}^1_L$ because $\mathbf{o}_K \in U_K$ by assumption. Since the only open $\mathbb{G}_{m,L}$-stable subscheme of $\mathbb{A}^1_L$ containing the zero is $\mathbb{A}^1_L$, we derive that $f^{-1}(U_K) = \mathbb{A}^1_L$. Thus the image of j is in U_K. Hence $U_K = \mathbf{M}_K$ because $j : \operatorname{Spec} L \to \mathbf{M}_K$ and L are arbitrary. By faithfully flat descent, we derive that $U = \mathbf{M}$. □

Proof of the theorem. Assume that **(i)** holds. Since $\mathbf{o}$ is a point in Z, we have a surjective morphism $j^{\#} : \mathcal{O}_{\mathbf{M},\mathbf{o}} \twoheadrightarrow \mathcal{O}_{Z,\mathbf{o}}$ of local rings. Both schemes $Z, \mathbf{M}$ are noetherian and hence we have a commutative square

$$\begin{tikzcd} \widehat{\mathcal{O}_{\mathbf{M},\mathbf{o}}} \arrow[r, "\widehat{j^{\#}}"] & \widehat{\mathcal{O}_{Z,\mathbf{o}}} \\ \mathcal{O}_{\mathbf{M},\mathbf{o}} \arrow[u, hook] \arrow[r, two heads, "j^{\#}"'] & \mathcal{O}_{Z,\mathbf{o}} \arrow[u, hook] \end{tikzcd}$$

where vertical morphisms are injective. Since $\mathbf{M}_n \subseteq Z$ for every $n \in \mathbb{N}$, we derive that $\widehat{j^{\#}}$ is an isomorphism. Hence $j^{\#}$ is injective and thus it is an isomorphism. This implies that there exists an open neighborhood V of $\mathbf{o}$ in $\mathbf{M}$ such that $V \subseteq Z$. Let $\mathbf{G} \cdot V$ be the open subscheme of $\mathbf{M}$ defined as the image of $\mathbf{G} \times_k V$ under the left action $\mathbf{G} \times_k \mathbf{M} \to \mathbf{M}$. This is $\mathbf{G}$-stable open subscheme of $\mathbf{M}$. From the fact that j is $\mathbf{G}$-equivariant, we deduce that $\mathbf{G} \cdot V \subseteq Z$. By Lemma 3.8.1 we infer that $\mathbf{G} \cdot V = \mathbf{M}$ because $\mathbf{o} \in V \subseteq \mathbf{G} \cdot V$. This shows that $Z = \mathbf{M}$. Thus we have **(i)** $\Rightarrow$ **(ii)**.
The implication **(ii)** $\Rightarrow$ **(i)** is obvious. □

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