ALGEBRAIZATION OF FORMAL M-SCHEMES

1. Some 2-categorical limits

Consider a category $\mathcal{C}$ and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Our goal is to construct certain 2-categorical limit associated with a pair $(\mathcal{C}, T)$. Consider pairs (X, u) consisting of an object X of $\mathcal{C}$ and an isomorphism $u : T(X) \to X$ in $\mathcal{C}$. If (X, u) and (Y, w) are two such pairs, then a morphism $f : (X, u) \to (Y, u)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that the following square

$$\begin{tikzcd} T(X) \arrow[r, "u"] \arrow[d, "T(f)"'] & X \arrow[d, "f"] \\ T(Y) \arrow[r, "w"'] & Y \end{tikzcd}$$

is commutative. This data give rise to a category $\mathcal{C}(T)$. There exists a forgetful functor $\pi : \mathcal{C}(T) \to \mathcal{C}$ that sends a morphism $f : (X, u) \to (Y, w)$ to $f : X \to Y$. Moreover, there exists a natural isomorphism $\sigma : T \cdot \pi \Rightarrow \pi$ such that the component of σ on an object (X, u) of $\mathcal{C}(T)$ is u. The next result states that the data above form a certain 2-categorical limit.

Theorem 1.1. *Let $(\mathcal{C}, T)$ be a pair consiting of a category and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Suppose that $\mathcal{D}$ is a category, $P : \mathcal{D} \to \mathcal{C}$ is a functor and $\tau : T \cdot P \Rightarrow P$ is a natural isomorphisms. Then there exists a unique functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$.*

Proof. Suppose that $F : \mathcal{D} \to \mathcal{C}(T)$ is a functor such that $P = \pi \cdot F$ and $\sigma_F = \tau$. Pick an object X of $\mathcal{D}$. Then we have $\pi \cdot F(X) = P(X)$ and $\sigma_{F(X)} = \tau_X$. This implies that

$$F(X) = \left(P(X), \tau_X : T(P(X)) \to P(X)\right)$$

Next if $f : X \to Y$ is a morphism in $\mathcal{D}$, then we derive that $\pi(F(f)) = P(f)$. Hence $F(f) = P(f)$. This implies that there exists at most one functor F satisfying the properties above. Note also that formulas

$$F(X) = \left(P(X), \tau_X : T(P(X)) \to P(X)\right), F(f) = P(f)$$

for an object X in $\mathcal{D}$ and a morphism $f : X \to Y$ in $\mathcal{D}$, give rise to a functor that satisfy $P = \pi \cdot F$ and $\sigma_F = \tau$. This establishes existence and the uniqueness of F. □

Assume now that the pair $(\mathcal{C}, T)$ consists of a monoidal category $\mathcal{C}$ and a monoidal endofunctor T. Then there exists a canonical monoidal structure on $\mathcal{C}(T)$. We define $(-) \otimes_{\mathcal{C}(T)} (-)$ by formula

$$(X, u) \otimes_{\mathcal{C}(T)} (Y, w) = \left(X \otimes_{\mathcal{C}} Y, (u \otimes_{\mathcal{C}} w) \cdot m_{X,Y}\right)$$

where

$$m_{X,Y} : T\left(X \otimes_{\mathcal{C}} Y\right) \to T(X) \otimes_{\mathcal{C}} T(Y)$$

is the tensor preserving isomorphism of T. We also define the unit

$$I_{\mathcal{C}(T)} = \left(I, T(I) \cong I\right)$$

where isomorphism $T(I) \cong I$ is precisely the unit preserving isomorphism of the monoidal functor T. The associativity natural isomorphism for $(-) \otimes_{\mathcal{C}(T)} (-)$ and right, left units for $I_{\mathcal{C}(T)}$ in $\mathcal{C}(T)$ are associavity natural isomorphism and right, left units for $\mathcal{C}$, respectively. The structure makes a functor $\pi : \mathcal{C}(T) \to \mathcal{C}$ strict monoidal and σ a monoidal natural isomorphism. The next result states that the data with these extra monoidal structure form a 2-categorical limit in the 2-category of monoidal categories.

Theorem 1.2. *Let $(\mathcal{C}, T)$ be a pair consiting of a monoidal category and its monoidal endofunctor $T : \mathcal{C} \to \mathcal{C}$. Suppose that $\mathcal{D}$ is a monoidal category, $P : \mathcal{D} \to \mathcal{C}$ is a monoidal functor and $\tau : T \cdot P \Rightarrow P$ is a monoidal natural isomorphisms. Then there exists a unique monoidal functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$ as monoidal functors and monoidal transformations.*

Proof. Note that F must be defined as it was described in the proof of Theorem 1.1. Namely we must have

$$F(X) = (P(X), \tau_X : T(P(X)) \to P(X)), F(f) = P(f)$$

for an object X in $\mathcal{C}$ and a morphism $f : X \to Y$ in $\mathcal{C}$.
Suppose now that F admits a structure of a monoidal functor such that $P = \pi \cdot F$ as monoidal functors. Let

$$\left\{m^F_{X,Y} : F(X \otimes_{\mathcal{D}} Y) \to F(X) \otimes_{\mathcal{C}(T)} F(Y)\right\}_{X,Y \in \mathcal{C}}, \phi^F : F(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$$

be the data forming that structure. Since π is a strict monoidal functor and $P = \pi \cdot F$ as monoidal functors, we derive that for any objects X, Y of $\mathcal{C}$

$$\pi(m^F_{X,Y}) : P(X \otimes_{\mathcal{D}} Y) \to P(X) \otimes_{\mathcal{C}} P(Y)$$

is the tensor preserving isomorphism $m^P_{X,Y} : P(X \otimes_{\mathcal{D}} Y) \to P(X) \otimes_{\mathcal{C}} P(Y)$ of the monoidal functor P. By the same argument

$$\pi(\phi_F) : P(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$$

is the unit preserving isomorphism $\phi^P : P(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$ of P. Thus we deduce that for any objects X, Y of $\mathcal{C}$ we have $m^F_{X,Y} = m^P_{X,Y}$ and $\phi^F = \phi^P$. This implies that there exists at most one monoidal functor F such that $P = \pi \cdot F$ as monoidal functors.
On the other hand define $m^F_{X,Y} = m^P_{X,Y}$ for objects X, Y in $\mathcal{C}$ and $\phi^F = \phi^P$. We check now that F equipped with these data is a monoidal functor. Fix objects X, Y in $\mathcal{C}$. The square

$$\begin{tikzcd}
T\left(P\left(X \otimes_{\mathcal{D}} Y\right)\right) \arrow[r, "\tau_{X \otimes_{\mathcal{C}} Y}"] \arrow[d, "T\left(m^P_{X,Y}\right)"'] & P\left(X \otimes_{\mathcal{C}} Y\right) \arrow[d, "m^P_{X,Y}"] \\
T\left(P(X) \otimes_{\mathcal{C}} P(Y)\right) \arrow[r, "(\tau_X \otimes_{\mathcal{C}} \tau_Y) \cdot m^T_{P(X),P(Y)}"'] & P(X) \otimes_{\mathcal{C}} P(Y)
\end{tikzcd}$$

is commutative due to the fact that $\tau : T \cdot P \Rightarrow P$ is a monoidal natural isomorphisms. This implies that $m^F_{X,Y}$ is a morphism in $\mathcal{C}(T)$. It follows that $m^F_{X,Y}$ is a natural isomorphism and due to the definition of associativity in $\mathcal{C}(T)$, we derive its compatibility with $m^F_{X,Y}$. Similarly, since the square

$$\begin{tikzcd}
T\left(P\left(I_{\mathcal{D}}\right)\right) \arrow[r, "\tau_{I_{\mathcal{D}}}"] \arrow[d, "T\left(\phi^P\right)"'] & P\left(I_{\mathcal{D}}\right) \arrow[d, "\phi^P"] \\
T\left(I_{\mathcal{C}}\right) \arrow[r, "\phi^T"'] & I_{\mathcal{C}}
\end{tikzcd}$$

is commutative, we deduce that ϕ^F is a morphism in $\mathcal{C}(T)$. By definition of left and right unit in $\mathcal{C}(T)$, we derive their compatibility with ϕ^F. This finishes the verification of the fact that F with $\{m^F_{X,Y}\}_{X,Y \in \mathcal{C}}$ and ϕ^F is a monoidal functor. Definitions of $\{m^F_{X,Y}\}_{X,Y \in \mathcal{C}}$ and ϕ^F show that the identities $P = \pi \cdot F$ holds on the level of monoidal structures. Since the 2-forgetful functor from

2-category of monoidal categories into 2-category of categories is faithful on 2-cells, the identity $\sigma_F = \tau$ of natural isomorphisms is also the identity of monoidal natural isomorphisms. □

Theorem 1.3. *Let $(\mathcal{C}, T)$ be a pair consiting of a category and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Assume that T preserves colomits. Then the following assertions hold.*

(1) *$\pi : \mathcal{C}(T) \to \mathcal{C}$ creates colimits.*

(2) *Suppose that $\mathcal{D}$ is a category, $P : \mathcal{D} \to \mathcal{C}$ a functor preserving small colimits and $\tau : T \cdot P \Rightarrow P$ a natural isomorphisms. Then the unique functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$ preserves small colimits.*

Proof. Let I be a small category and $D : I \to \mathcal{C}(T)$ be a diagram such that the composition $\pi \cdot D : I \to \mathcal{C}$ admits a colimit given by cocone $(X, \{g_i\}_{i \in I})$. Since T preserves colimits, we derive that $(T(X), \{T(u_i)\}_{i \in I})$ is a colimit of $T \cdot \pi \cdot D : I \to \mathcal{C}$. Now $\sigma_D : T \cdot \pi \cdot D \to \pi \cdot D$ is a natural isomorphism. Hence there exists a unique arrow $u : T(X) \to X$ such that $u \cdot T(g_i) = g_i \cdot \sigma_{D(i)}$ for $i \in I$. Clearly u is an isomorphism and hence (X, u) is an object of $\mathcal{C}(T)$. Moreover, the family $\{g_i\}_{i \in I}$ together with (X, u) is a colimiting cocone over D. This proves **(1)**. Now **(2)** is a consequence of **(1)**. □

Now we apply the results above to certain more general diagrams of categories.

Definition 1.4. A diagram

$$\dots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \dots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

of categories and functors is called *a telescope of categories*.

Definition 1.5. Let

$$\dots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \dots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

be a telescope of monoidal categories and monoidal cocontinuous functors. Then *a 2-categorical limit of the telescope* consists of a monoidal category $\mathcal{C}$, a family of monoidal cocontinuous functors $\{\pi_n : \mathcal{C} \to \mathcal{C}_n\}_{n \in \mathbb{N}}$ and a family of monoidal natural isomorphisms $\{\sigma_n : F_{n+1} \cdot \pi_{n+1} \Rightarrow \pi_n\}_{n \in \mathbb{N}}$ such that the following universal property holds. For any monoidal category $\mathcal{D}$, family $\{P_n : \mathcal{D} \to \mathcal{C}_n\}_{n \in \mathbb{N}}$ of cocontinuous monoidal functors and a family $\{\tau_n : F_n P_{n+1} \Rightarrow P_n\}_{n \in \mathbb{N}}$ of monoidal natural isomorphisms there exists a unique monoidal cocontinuous functor $F : \mathcal{D} \to \mathcal{C}$ satisfying $P_n = \pi_n \cdot F$ and $(\sigma_n)_F = \tau_n$ for every $n \in \mathbb{N}$.

Corollary 1.6. *Let*

$$\dots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \dots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

be a telescope of monoidal categories and monoidal cocontinuous functors. Then its 2-limit exists.

Proof. We decompose the task of constructing its 2-limit as follows. First note that one may form a product $\mathcal{C} = \prod_{n \in \mathbb{N}} \mathcal{C}_n$. Next the functors $\{F_n\}_{n \in \mathbb{N}}$ induce an endofunctor $T = \prod_{n \in \mathbb{N}} F_n \times t$, where $\mathbf{1}$ is the terminal category (it has single object and single identity arrow) and $t : \mathcal{C}_0 \to \mathbf{1}$ is the unique functor. Consider the category $\mathcal{C}(T)$. We define $\{\pi_n : \mathcal{C}(T) \to \mathcal{C}_n\}_{n \in \mathbb{N}}$ to be a family of functors given by coordinates of $\pi : \mathcal{C}(T) \to \mathcal{C}$ and $\{\sigma_n : F_n \cdot \pi_{n+1} \Rightarrow \pi_n\}_{n \in \mathbb{N}}$ to be a family of natural isomorphisms given by coordinates of $\sigma : \pi \cdot T \Rightarrow \pi$. Now this data form a 2-limit of the telescope by compilation of Theorem 1.2 and Theorem 1.3. □

2. **M**-EQUIVARIANT QUASI-COHERENT SHEAVES

LS TODO: Tu trzeba zdefiniować i następnie opisać przypadek schematu z lokalnie liniowym działaniem, bo on jest najważniejszy

3. FORMAL **M**-SCHEMES

This section is devoted to introducing some notions from formal geometry that are central in this notes.

Definition 3.1. Let $\mathbf{M}$ be a monoid k-scheme. *A formal* $\mathbf{M}$*-scheme* consists of a sequence $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ of $\mathbf{M}$-schemes together with $\mathbf{M}$-equivariant closed immersions

$$Z_0 \hookrightarrow Z_1 \hookrightarrow \dots \hookrightarrow Z_n \hookrightarrow Z_{n+1} \hookrightarrow \dots$$

satisfying the following assertions.

(1) We have $Z_0 = Z_n^{\mathbf{M}}$ scheme-theoretically for every $n \in \mathbb{N}$.

(2) Let $\mathcal{I}_n$ be an ideal of $\mathcal{O}_{Z_n}$ defining Z_0. Then for every $m \leq n$ the subscheme $Z_m \subset Z_n$ is defined by $\mathcal{I}_n^{m+1}$.

Example 3.2. Let $\mathbf{M}$ be a monoid k-scheme and let Z be a $\mathbf{M}$-scheme. Consider a quasi-coherent ideal $\mathcal{I}$ of fixed point subscheme $Z^{\mathbf{M}}$ of Z. Then for every $n \in \mathbb{N}$ ideal $\mathcal{I}^n$ is $\mathbf{M}$-equivariant and hence

$$V(\mathcal{I}) \hookrightarrow V(\mathcal{I}^2) \hookrightarrow \dots \hookrightarrow V(\mathcal{I}^n) \hookrightarrow \dots$$

is a formal $\mathbf{M}$-scheme. We denote it by $\widehat{Z}$.

Definition 3.3. Let $\mathbf{M}$ be a monoid k-scheme. Suppose that $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ and $\mathcal{W} = \{W_n\}_{n\in\mathbb{N}}$ are formal $\mathbf{M}$-schemes. Then *a morphism* $f : \mathcal{Z} \to \mathcal{W}$ *of formal* $\mathbf{M}$*-schemes* consists of a family of $\mathbf{M}$-equivariant morphisms $f = \{f_n : Z_n \to W_n\}_{n\in\mathbb{N}}$ such that the diagram

$$\begin{tikzcd}
Z_0 \arrow[r, hook] \arrow[d, "f_0"'] & Z_1 \arrow[r, hook] \arrow[d, "f_1"'] & \dots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[d, "f_n"'] & Z_{n+1} \arrow[r, hook] \arrow[d, "f_{n+1}"'] & \dots \\
W_0 \arrow[r, hook] & W_1 \arrow[r, hook] & \dots \arrow[r, hook] & W_n \arrow[r, hook] & W_{n+1} \arrow[r, hook] & \dots
\end{tikzcd}$$

is commutative.

Since group k-scheme is also a monoid k-scheme, definitions above can be applied to group k-schemes.

Definition 3.4. Let $\mathbf{G}$ be a group k-scheme. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{G}$-scheme. Then there we have the corresponding telescope of monoidal categories

$$\dots \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_{n+1}) \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_n) \longrightarrow \dots \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_2) \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_1) \longrightarrow \mathfrak{Qcoh}_{\mathbf{G}}(Z_0)$$

and cocontinuous monoidal functors given by restricting $\mathbf{G}$-equivariant quasi-coherent sheaves to closed $\mathbf{G}$-subschemes. Then we define *a category* $\mathfrak{Qcoh}(\mathcal{Z})$ *of quasi-coherent sheaves on* $\mathcal{Z}$ as a monoidal category which is a 2-limit of the telescope above. This category is defined uniquely up to a monoidal equivalence.

Let Z be a $\mathbf{G}$-scheme and let $\mathcal{I}$ be a quasi-coherent ideal of $Z^{\mathbf{G}}$. We have a commutative diagram

$$\begin{tikzcd}
V(\mathcal{I}) \arrow[r, hook] \arrow[rrr, hook, bend right, start anchor=south, end anchor=west] & V(\mathcal{I}^2) \arrow[r, hook] \arrow[rrd, hook] & \ldots \arrow[r, hook] & V(\mathcal{I}^n) \arrow[r, hook] \arrow[ld, hook'] & \ldots \\
 & & & Z &
\end{tikzcd}$$

in the category of **G**-schemes. Thus restriction functors $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}_{\mathbf{G}}(V(\mathcal{I}^n))$ for $n \in \mathbb{N}$ induce a unique cocontinuous monoidal functor $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}(\widehat{Z})$.

Definition 3.5. Let Z be a **G**-scheme. Then a unique cocontinuous monoidal functor $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}(\widehat{Z})$ is called *the comparison functor*.

Definition 3.6. Let **M** be a monoid k-scheme with group of units **G**. Let $\mathcal{Z} = \{Z_n\}_{n \in \mathbb{N}}$ be a formal **M**-scheme. A **M**-scheme Z is called *an algebraization of* $\mathcal{Z}$ if the following two conditions are satisfied.

(1) $\mathcal{Z}$ is isomorphic to $\widehat{Z}$ in the category of formal **M**-schemes.

(2) The comparison functor $\mathfrak{Qcoh}_{\mathbf{G}}(Z) \to \mathfrak{Qcoh}\left(\widehat{Z}\right)$ is an equivalence of monoidal categories.

4. Locally linear M-schemes

Definition 4.1. Let **M** be a monoid k-scheme and let X be a **M**-scheme. Suppose that each point of X admits an open affine **M**-stable neighborhood. Then we say that X is *a locally linear* **M**-*scheme*.

Proposition 4.2. *Let* **M** *be an affine monoid k-scheme and let X be a* **M**-*scheme. Suppose that there exists a quasi-coherent* **M**-*equivariant ideal* $\mathcal{I}$ *on* X *with nilpotent sections. Consider an open subset* U *of* X. *Then the following are equivalent.*

(1) U *is* **M**-*stable.*

(2) $U \cap V(\mathcal{I})$ *is* **M**-*stable.*

Proof. Let $\alpha : \mathbf{M} \times X \to X$ be the action of **M** on X. Fix open subset U of X. If U is **M**-stable, then $U \cap V(\mathcal{I})$ is **M**-stable. So suppose that $U \cap V(\mathcal{I})$ is **M**-stable. Since $\mathcal{I}$ has nilpotent sections and **M** is affine, we derive that closed immersions $U \cap V(\mathcal{I}) \hookrightarrow U$ and $\mathbf{M} \times (U \cap V(\mathcal{I})) \hookrightarrow \mathbf{M} \times U$ induce homeomorphisms on topological spaces. Consider the commutative diagram

$$\begin{tikzcd}
\mathbf{M} \times U \arrow[r, "\alpha_{|U \cap V(\mathcal{I})}"] & X \\
\mathbf{M} \times (U \cap V(\mathcal{I})) \arrow[r] \arrow[u, hook] & U \cap V(\mathcal{I}) \arrow[u, hook]
\end{tikzcd}$$

where the bottom horizontal arrow is the induced action on $U \cap V(\mathcal{I})$ and vertical morphisms are homeomorphisms. The commutativity of the diagram implies that $\alpha\,(\mathbf{M} \times U)$ is contained set-theoretically in U. Since U is open in X, we derive that morphism of schemes $\alpha_{|\mathbf{M} \times U}$ factors through U. Hence U is **M**-stable. □

Corollary 4.3. *Let* **M** *be an affine monoid k-scheme and let X be a* **M**-*scheme. Suppose that there exists a quasi-coherent* **M**-*equivariant ideal* $\mathcal{I}$ *on* X *such that* $\mathcal{I}^n = 0$ *for* $n \in \mathbb{N}$. *Consider an open subset* U *of* X. *Then the following are equivalent.*

(1) U *is* **M**-*stable and affine.*

(2) $U \cap V(\mathcal{I})$ *is* **M**-*stable and affine.*

Proof. Since $\mathcal{I}^n = 0$, we derive that U is affine if and only if $U \cap V(\mathcal{I})$ is affine. Combining this with Proposition 4.2, we deduce the result. □

Corollary 4.4. *Let $\mathbf{M}$ be an affine monoid k-scheme and let X be a $\mathbf{M}$-scheme. Suppose that there exists a quasi-coherent $\mathbf{M}$-equivariant ideal $\mathcal{I}$ on X such that $\mathcal{I}^n = 0$ for $n \in \mathbb{N}$. Then X is locally linear $\mathbf{M}$-scheme if and only if $V(\mathcal{I})$ is locally linear $\mathbf{M}$-scheme.*

Proof. This is a consequence of Corollary 4.3. □

5. Some results on formal $\mathbf{M}$-schemes

Corollary 5.1. *Let $\mathbf{M}$ be an affine monoid k-scheme and let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{G}$-scheme. Then Z_n is locally linear $\mathbf{G}$-scheme for every $n \in \mathbb{N}$.*

Proof. Let $\mathcal{I}_n$ be an ideal defining Z_0 in Z_n. Since $\mathcal{Z}$ is a formal $\mathbf{M}$-scheme, we derive that $\mathcal{I}_n^{n+1} = 0$ and Z_0 is locally linear $\mathbf{M}$-scheme. Thus we apply Corollary 4.4 and derive that Z_n is locally linear $\mathbf{M}$-scheme. □

We are particularly interested in formal $\mathbf{M}$-schemes for monoid $\mathbf{M}$ with zero. For this we need the following elementary result.

Proposition 5.2. *Let $\mathbf{M}$ be a monoid k-scheme with zero $\mathbf{o}$ and let X be a $\mathbf{M}$-scheme. Then the following results hold.*

(1) *The multiplication by zero $\mathbf{o}\cdot(-) : X \to X$ factors through $X^{\mathbf{M}}$ inducing a $\mathbf{M}$-equivariant retraction $r_{\mathbf{M}} : X \twoheadrightarrow X^{\mathbf{M}}$.*

(2) *If $\mathbf{N}$ is a submonoid k-scheme of $\mathbf{M}$ and $\mathbf{o}$ is a k-point of $\mathbf{N}$, then $r_{\mathbf{M}} = r_{\mathbf{N}}$.*

(3) *If $\mathbf{M}$ is affine and X is locally linear $\mathbf{M}$-scheme, then $r_{\mathbf{M}}$ is affine.*

Proof. The multiplication $\mathbf{o}\cdot(-) : \mathfrak{P}_X \to \mathfrak{P}_X$ factors as an $\mathfrak{P}_{\mathbf{M}}$-equivariant epimorphism $\mathfrak{P}_X \twoheadrightarrow \mathfrak{P}_{X^{\mathbf{M}}}$ composed with a closed immersion $\mathfrak{P}_{X^{\mathbf{M}}} \hookrightarrow \mathfrak{P}_X$. The $\mathfrak{P}_{\mathbf{M}}$-equivariant epimorphism $\mathfrak{P}_X \to \mathfrak{P}_{X^{\mathbf{M}}}$ corresponds to a $\mathbf{M}$-equivariant morphism $r_{\mathbf{M}} : X \to X^{\mathbf{M}}$ of k-schemes such that $r_{\mathbf{M}}$ restricted to $X^{\mathbf{M}}$ is the identity $1_{X^{\mathbf{M}}}$. This proves **(1)**.
For the proof of **(2)** note that $\mathbf{o}\cdot(-) : \mathfrak{P}_X \to \mathfrak{P}_X$ is defined similarly for $\mathbf{M}$ and $\mathbf{N}$ (provided that $\mathbf{o}$ is a k-point of $\mathbf{N}$). Thus $r_{\mathbf{M}} = r_{\mathbf{N}}$.
Suppose now that $\mathbf{M}$ is affine and X is locally linear $\mathbf{M}$-scheme. Consider the action $\alpha : \mathbf{M} \times X \to X$ of $\mathbf{M}$ on X. Since X is locally linear and $\mathbf{M}$ is affine, we derive that α is an affine morphism of k-schemes. Now $\mathbf{o}\cdot(-) : X \to X$ is given as a composition

$$X \xrightarrow{\cong} \mathbf{o} \times X \hookrightarrow \mathbf{M} \times X \xrightarrow{\alpha} X$$

The morphism above is affine (as a composition of affine morphisms). Since the composition of r with a closed immersion $X^{\mathbf{M}} \hookrightarrow X$ is $\mathbf{o} \times (-)$ and hence an affine morphism, we derive that r is affine. This proves **(3)**. □

Let us note the immediate consequence of this result.

Corollary 5.3. *Let $\mathbf{M}$ be an affine monoid k-scheme with zero and let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{M}$-scheme. Then $\mathcal{Z}$ is a part of the commutative diagram*

$$\begin{tikzcd} Z_0 \arrow[r, hook] \arrow[drr, bend right, two heads, "r_0 = 1_{Z_0}"'] & Z_1 \arrow[r, hook] \arrow[dr, two heads, "r_1"] & \dots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[dl, bend left, two heads, "r_n"] & \dots \\ & & Z_0 & & \end{tikzcd}$$

*in which vertical morphisms $r_n : Z_n \twoheadrightarrow Z_0$ are affine **M**-equivariant morphisms such that $r_{n|Z_0} = 1_{Z_0}$. Moreover, if **N** is a submonoid k-scheme of **M** containing the zero of **M**, then $\mathcal{Z}$ is a formal **N**-scheme.*

Proof. This is an immediate consequence of Corollary 5.1 and Proposition 5.2. □

6. Toruses and toric monoid k-schemes

Definition 6.1. Let T be an affine algebraic group over k. Suppose that there exists $n \in \mathbb{N}$ such that for every algebraically closed extension K of k there exists an isomorphism

$$T_K \cong \operatorname{Spec} K \times \underbrace{\mathbb{G}_m \times \mathbb{G}_m \times \dots \times \mathbb{G}_m}_{n \text{ times}}$$

of group schemes over K. Then T is called *a torus over k*.

Example 6.2. If $T \cong \underbrace{\mathbb{G}_m \times \mathbb{G}_m \times \dots \times \mathbb{G}_m}_{n \text{ times}}$, then T is a torus. We call toruses T of this form *split toruses*.

Example 6.3. Define

$$\mathbf{S}^1 = \operatorname{Spec} k[x,y]/(x^2 + y^2 - 1)$$

a scheme over k and let $\mathfrak{P}_{\mathbf{S}^1}$ be its functor of points. Then for every k-algebra A we have

$$\mathfrak{P}_{\mathbf{S}^1}(A) = \left\{(u,v) \in A \times A \,\middle|\, u^2 + v^2 = 1\right\}$$

There is also a morphism $\mathfrak{P}_{\mathbf{S}^1} \times \mathfrak{P}_{\mathbf{S}^1} \to \mathfrak{P}_{\mathbf{S}^1}$ of k-functors given by

$$\mathfrak{P}_{\mathbf{S}^1}(A) \times \mathfrak{P}_{\mathbf{S}^1}(A) \to \mathfrak{P}_{\mathbf{S}^1} \ni ((u_1,v_1),(u_2,v_2)) \mapsto (u_1u_2 - v_1v_2, u_1v_2 + u_2v_1) \in \mathfrak{P}_{\mathbf{S}^1}(A)$$

for every k-algebra A. This makes $\mathfrak{P}_{\mathbf{S}^1}$ into a group k-functor. Thus $\mathbf{S}^1$ with the group structure described above is an affine algebraic group over k. We call it *the circle group over k*.
Now suppose that $\operatorname{char}(k) = 2$ and K is an algebraically closed extension of k. Consider an element $i \in K$ such that $i^2 = -1$. For every K-algebra A we have a map

$$\mathfrak{P}_{\mathbf{S}^1}(A) \ni (u,v) \mapsto u + iv \in A^*$$

First note that this map is bijective. Indeed, its inverse is given by

$$A^* \ni a \mapsto \left(\frac{1}{2}(a + a^{-1}), \frac{1}{2i}(a - a^{-1})\right) \in \mathfrak{P}_{\mathbf{S}^1}(A)$$

Moreover, the map $\mathfrak{P}_{\mathbf{S}^1}(A) \to A^*$ is a homomorphism of abstract groups. Thus $\mathfrak{P}_{\mathbf{S}^1}$ resricted to the category $\mathbf{Alg}_K$ of K-algebras is isomorphic with $\mathfrak{P}_{\operatorname{Spec} K \times \mathbb{G}_m}$ as a group k-functor. Hence

$$\mathbf{S}^1_K \cong \operatorname{Spec} K \times \mathbb{G}_m$$

as algebraic group schemes over K. Hence $\mathbf{S}^1$ is a torus over k.
Now assume that $k = \mathbb{R}$. Then abstract groups

$$\mathfrak{P}_{\mathbf{S}^1}(\mathbb{R}) = \left\{z \in \mathbb{C} \,\middle|\, |z| = 1\right\} \subseteq \mathbb{C}^*,\ \mathbb{R}^*$$

are not isomorphic. Indeed, the left hand side group has infinite torsion subgroup and the right hand side group has torsion subgroup equal to $\{-1, 1\}$. This implies that over $\mathbb{R}$ algebraic groups $\mathbf{S}^1$ and $\mathbb{G}_m$ are not isomorphic. Hence $\mathbf{S}^1$ is not a split torus over $\mathbb{R}$.

Corollary 6.4. *Let T be a torus over k. Then T is a linearly reductive algebraic group.*

Definition 6.5. Let T be a torus over k and let $\overline{T}$ be a linearly reductive monoid having T as the group of units. Then $\overline{T}$ is *a toric monoid over k*

Theorem 6.6. *Let $\overline{T}$ be a toric monoid over k with group of units T and let K be an algebraically closed extension of k. Suppose that N is a dimension of T.*

(1) *The group of characters of T_K is isomorphic to $\mathbb{Z}^N$ and there exists an abstract submonoid S of $\mathbb{Z}^N$ such that the open immersion*

$$T_K = \operatorname{Spec}\left(\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m\right) \hookrightarrow \operatorname{Spec}\left(\bigoplus_{m\in S} K\cdot\chi^m\right) = \overline{T}_K$$

is induced by the inclusion $S \hookrightarrow \mathbb{Z}^N$.

(2) *Let $\{V_\lambda\}_{\lambda\in\mathbf{Irr}(T)}$ be a set of irreducible representation of T such that V_λ is in isomorphism class λ. For every λ there exists a finite subset A_λ of $\mathbb{Z}^N$ such that*

$$K\otimes_k V_\lambda = \bigoplus_{m\in A_\lambda} K\cdot\chi^m$$

If λ consists of irreducible representations of $\overline{T}$, then A_λ is a subset of S. Moreover, we have

$$\mathbb{Z}^N = \coprod_{\lambda\in\mathbf{Irr}(T)} A_\lambda$$

and $A_{\lambda_0} = \{0\}$, where λ_0 is a class of trivial representation of T.

(3) *If $\overline{T}$ has a zero, then there exists a homomorphism $f:\mathbb{Z}^N\to\mathbb{Z}$ of abelian groups such that $f_{|S\smallsetminus\{0\}} > 0$. Moreover, f induces a closed immersion*

$$\operatorname{Spec}K\times\mathbb{G}_m = \operatorname{Spec}K[\mathbb{Z}] \hookrightarrow \operatorname{Spec}\left(\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m\right) = T_K$$

of group K-schemes that extends to a zero preserving closed immersion $\mathbb{A}^1_K \hookrightarrow \overline{T}_K$ of monoid K-schemes.

Proof. Since T is a torus, we derive that

$$T_K = \operatorname{Spec}K\times\underbrace{\mathbb{G}_m\times\mathbb{G}_m\times\ldots\times\mathbb{G}_m}_{N\text{ times}} = \operatorname{Spec}\left(\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m\right)$$

and hence

$$\overline{T}_K = \operatorname{Spec}\left(\bigoplus_{s\in S} K\cdot\chi^s\right)$$

for some abstract submonoid S of $\mathbb{Z}^N$. Moreover, the open immersion $T_K \hookrightarrow \overline{T}_K$ is induced by the inclusion $S\hookrightarrow\mathbb{Z}^N$. This proves **(1)**.
We have identification

$$k[T] = \bigoplus_{\lambda\in\mathbf{Irr}(T)} V_\lambda^{n_\lambda}$$

of T-representations, where $n_\lambda \in \mathbb{N}\smallsetminus\{0\}$ for each λ. Thus

$$\bigoplus_{m\in\mathbb{Z}^N} K\cdot\chi^m = K\otimes_k k[T] = \bigoplus_{\lambda\in\mathbf{Irr}(T)} (K\otimes_k V_\lambda)^{n_\lambda}$$

This implies that $n_\lambda = 1$ for every λ and moreover, we derive that

$$K\otimes_k V_\lambda = \bigoplus_{m\in A_\lambda} K\cdot\chi^m$$

for some finite set $A_\lambda \subseteq \mathbb{Z}^N$. We also have $A_{\lambda_0} = \{0\}$ and $A_\lambda \subseteq S \setminus \{0\}$ for $\lambda \in \mathbf{Irr}(\overline{T})$. This proves **(2)**.
Since $\overline{T}$ admits a zero, we derive that

$$\mathfrak{m} = \bigoplus_{m \in S \setminus \{0\}} K \cdot \chi^s \subseteq \bigoplus_{m \in \mathbb{Z}^N} K \cdot \chi^m$$

is an ideal. This implies that $S \setminus \{0\}$ is closed under addition. In particular, there exists a homomorphism of abelian groups $f : \mathbb{Z}^N \to \mathbb{Z}$ such that $f_{|S \setminus \{0\}} > 0$. This implies **(3)**. □

7. Commuting actions

Corollary 7.1. *Let* $\mathbf{G}$ *be an affine group scheme over* k *and let* $\mathfrak{G}$ *be a monoid* k*-functor. Denote by* Λ *the set of isomorphism classes of irreducible* $\mathbf{G}$*-representations. Suppose that* V *is a representation of both* $\mathbf{G}$ *and* $\mathfrak{G}$ *and assume that their actions on* V *commute. Assume that* V *is completely reducible as a* $\mathbf{G}$*-representation and consider the decomposition*

$$V = \bigoplus_{\lambda \in \Lambda} V[\lambda]$$

onto isotypic components with respect to the action of $\mathbf{G}$*. Then for every* λ *in* Λ *the subspace* $V[\lambda]$ *is a* $\mathfrak{G}$*-subrepresentation of* V*.*

Proof. Part of the structure V as the $\mathfrak{G}$-representation is the morphism $\rho : \mathfrak{G} \to \mathcal{L}_V$ of k-monoids. Fix k-algebra A and $g \in \mathfrak{G}(A)$. Since actions of $\mathbf{G}$ and $\mathfrak{G}$ on V commute, morphism $\rho(g) : A \otimes_k V \to A \otimes_k V$ of A-modules is a morphism of $\mathbf{G}_A$-representation. According to Proposition **??** we derive that

$$\mathrm{Hom}_{\mathbf{G}_A}\left(A \otimes_k V[\lambda_1], A \otimes_k V[\lambda_2]\right) = 0$$

for distinct $\lambda_1, \lambda_2 \in \Lambda$. Thus

$$\rho(g)\left(A \otimes_k V[\lambda]\right) \subseteq A \otimes_k V[\lambda]$$

for every λ in Λ. This holds for every k-algebra A and $g \in \mathfrak{G}(A)$. Hence $V[\lambda]$ is $\mathfrak{G}$-subrepresentation of V. □

8. Algebraization of formal **M**-schemes

In this section we prove the main algebraization result. The first step is to prove the following interesting theorem.

Theorem 8.1. *Let* $\mathbf{M}$ *be a Kempf monoid with unit group* $\mathbf{G}$ *and let* $\mathcal{Z} = \{Z_n\}_{n \in \mathbb{N}}$ *be a formal* $\mathbf{M}$*-scheme. Then there exists a locally linear* $\mathbf{M}$*-scheme* Z *equipped with an action of* $\mathbf{M}$ *such that* $\widehat{Z}$ *is isomorphic to* $\mathcal{Z}$*.*

Monoid $\mathbf{M}$ is affine and admits zero $\mathbf{o}$. Hence by Corollary 5.3 formal $\mathbf{M}$-scheme $\mathcal{Z}$ corresponds to a sequence of surjections

$$\ldots \twoheadrightarrow \mathcal{A}_{n+1} \twoheadrightarrow \mathcal{A}_n \twoheadrightarrow \ldots \twoheadrightarrow \mathcal{A}_1 \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$$

of quasi-coherent $\mathcal{O}_{Z_0}$-algebras with $\mathbf{M}$-linearization such that $\mathcal{A}_n^{\mathbf{M}} = \mathcal{A}_0$ for every $n \in \mathbb{N}$ and if $\mathcal{I}_n$ is the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_0$ in $\mathcal{A}_n$, then $\mathcal{I}_n^{m+1}$ is the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_m$ for $m \leq n$ and $n \in \mathbb{N}$. Since $\mathbf{M}$ is a Kempf monoid, there exists a closed subgroup T of the center $Z(\mathbf{G})$ such that T is a torus and the scheme-theoretic closure $\overline{T}$ of T in $\mathbf{M}$ contains the zero $\mathbf{o}$ of $\mathbf{M}$. We derive by Corollary 5.3 that $\mathcal{A}_n^{\overline{\mathbf{T}}} = \mathcal{A}_0$ for every $n \in \mathbb{N}$. By definition $\overline{T}$ is a toric monoid k-scheme with T as a group of units. We start with some lemmas.

Lemma 8.1.1. *Let λ be in $\mathbf{Irr}(\overline{T})$. Then there exists $n_\lambda \in \mathbb{N}$ such that for each $n > n_\lambda$ and any $\lambda_1, ..., \lambda_n \in \mathbf{Irr}(\overline{T}) \smallsetminus \{\lambda_0\}$ the representation*

$$\bigotimes_{i=1}^{n} V_{\lambda_i}$$

for $1 \leq i \leq n$ has trivial isotypic component of type λ. We have $n_{\lambda_0} = 0$, where λ_0 is an isomorphism type of the trivial representation of T.

Proof of the lemma. Let K be an algebraically closed extension of k. Pick A_λ and f as in Theorem 6.6 and define

$$n_\lambda = \sup_{m \in A_\lambda} f(m)$$

We have

$$K \otimes_k V_{\lambda_1} \otimes_k ... \otimes_k V_{\lambda_n} = \bigoplus_{(m_1,...,m_n) \in A_{\lambda_1} \times ... \times A_{\lambda_n}} K \cdot \chi^{m_1+...+m_n}$$

and since $m_1, ...m_n \in A_{\lambda_1} \cup ... \cup A_{\lambda_n} \subseteq S \smallsetminus \{0\}$ we derive that

$$f(m_1 + ... + m_n) = f(m_1) + ... + f(m_n) \geq n > n_\lambda = \sup_{m \in A_\lambda} f(m)$$

This implies that V_λ is not an isotypic component of $V_{\lambda_1} \otimes_k ... \otimes_k V_{\lambda_n}$. □

Lemma 8.1.2. *Fix λ in $\mathbf{Irr}(\overline{T})$. Then $\mathcal{A}_{n+1}[\lambda] \twoheadrightarrow \mathcal{A}_n[\lambda]$ is an isomorphism for $n \geq n_\lambda$.*

Proof of the lemma. Since $\mathcal{A}_n^{\overline{T}} = \mathcal{A}_0$ and $\overline{T}$ is linearly reductive monoid, we derive that $\mathcal{I}_n[\lambda] = 0$ for $\lambda \notin \mathbf{Irr}(\overline{T}) \smallsetminus \{\lambda_0\}$. Fix $\lambda \in \mathbf{Irr}(\overline{T})$. By Lemma 8.1.1 we derive that

$$\left(\underbrace{\mathcal{I}_{n+1} \otimes_k \mathcal{I}_{n+1} \otimes_k ... \otimes_k \mathcal{I}_{n+1}}_{n+1 \text{ times}}\right)[\lambda] = 0$$

for $n \geq n_\lambda$. Note also that the image of the composition

$$\underbrace{\mathcal{I}_{n+1} \otimes_k \mathcal{I}_{n+1} \otimes_k ... \otimes_k \mathcal{I}_{n+1}}_{n \text{ times}} \twoheadrightarrow \underbrace{\mathcal{I}_{n+1} \otimes_{\mathcal{O}_{Z_0}} \mathcal{I}_{n+1} \otimes_{\mathcal{O}_{Z_0}} ... \otimes_{\mathcal{O}_{Z_0}} \mathcal{I}_{n+1}}_{n \text{ times}} \longrightarrow \mathcal{A}_{n+1}$$

is $\mathcal{I}_{n+1}^{n+1}$. Since the composition above is a morphism of sheaves with $\overline{T}$-linearization, we derive that $\mathcal{I}_{n+1}^{n+1}[\lambda] = 0$ for $n \geq n_\lambda$. Hence the kernel of $\mathcal{A}_{n+1}[\lambda] \twoheadrightarrow \mathcal{A}_n[\lambda]$ is trivial. □

Proof of Theorem. According to Corollary 7.1 and the fact that T is central in $\mathbf{M}$ we derive that $\mathcal{A}_n[\lambda]$ is a quasi-coherent sheaf with $\mathbf{M}$-linearization. For $\lambda \in \mathbf{Irr}(\overline{T})$ we define

$$\mathcal{A}[\lambda] = \mathcal{A}_n[\lambda]$$

where $n \geq n_\lambda$ as in Lemma 8.1.2. We set

$$\mathcal{A} = \bigoplus_{\lambda \in \mathbf{Irr}(\overline{T})} \mathcal{A}[\lambda]$$

Clearly $\mathcal{A}[\lambda_0] = \mathcal{A}_0 = \mathcal{O}_{Z_0}$ canonically (where λ_0 is the trivial T-representation), hence $\mathcal{A}$ is a quasi-coherent sheaf on Z_0 with $\mathbf{M}$-linearization. Actually $\mathcal{A} = \lim_{n \in \mathbb{N}} \mathcal{A}_n$ in the category of quasi-coherent sheaves with $\mathbf{M}$-linearization on Z_0. We construct the $\mathcal{O}_{Z_0}$-algebra structure on $\mathcal{A}$. For this pick $\lambda_1, \lambda_2 \in \mathbf{Irr}(\overline{T})$. Consider the irreducible representations V_{λ_1} and V_{λ_1} in classes λ_1 and λ_2, respectively. Suppose that $\eta_1, ..., \eta_s$ are finitely many classes in $\mathbf{Irr}(\overline{T})$ such that $V_{\lambda_1} \otimes_k V_{\lambda_2}$ can be completely decomposed onto irreducible representation in these classes. Since the image of the multiplication $\mathcal{A}_n[\lambda_1] \otimes_{\mathcal{O}_{Z_0}} \mathcal{A}_n[\lambda_2] \to \mathcal{A}_n$ on $\mathcal{A}_n$ is also the image of a morphism

$$\mathcal{A}_n[\lambda_1] \otimes_k \mathcal{A}_n[\lambda_2] \twoheadrightarrow \mathcal{A}_n[\lambda_1] \otimes_{\mathcal{O}_{Z_0}} \mathcal{A}_n[\lambda_2] \longrightarrow \mathcal{A}_n$$

we deduce that it is contained in $\bigoplus_{i=1}^s \mathcal{A}_n[\eta_i]$. By Lemma 8.1.2 all these multiplications for $n \geq \sup\{n_{\lambda_1}, n_{\lambda_2}, n_{\eta_1}, ..., n_{\eta_s}\}$ can be identified. Now we define

$$\mathcal{A}[\lambda_1] \otimes_{\mathcal{O}_{Z_0}} \mathcal{A}[\lambda_2] \to \bigoplus_{i=1}^{s} \mathcal{A}[\eta_i] \subseteq \mathcal{A}$$

as a morphism induced by the multiplication morphism for any $n \geq \sup\{n_{\lambda_1}, n_{\lambda_2}, n_{\eta_1}, ..., n_{\eta_s}\}$. This gives an $\mathcal{O}_{Z_0}$-algebra structure on $\mathcal{A}$, so $\mathcal{A}$ is in fact the limit of $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ is the category of quasi-coherent algebras with **M**-linearization. Note that from the description of $\mathcal{A}$ it follows that for every $n \in \mathbb{N}$ we have a surjective morphism $p_n : \mathcal{A} \twoheadrightarrow \mathcal{A}_n$ of algebras. We denote its kernel by $\mathcal{J}_n$ and we put $\mathcal{J} = \mathcal{J}_0$. The natural injection $\mathcal{O}_{Z_0} = \mathcal{A}_0 \to \mathcal{A}$ is a section of p_0, so that we have

$$\mathcal{J} = \bigoplus_{\lambda \in \mathbf{Irr}(\overline{T}) \setminus \{\lambda_0\}} \mathcal{A}[\lambda]$$

Recall that we denote by $\mathcal{I}_n$ the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$ for $n \in \mathbb{N}$. Then $\mathcal{I}_n = \mathcal{J}/\mathcal{J}_n$. Fix $m \in \mathbb{N}$ and consider $n \in \mathbb{N}$ such that $n \geq m$. Since $\mathcal{Z}$ is a formal **M**-scheme, the sheaf $\mathcal{I}_n^{m+1}$ is the kernel of the morphism $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_m$. Thus

$$\mathcal{J}_m/\mathcal{J}_n = \mathcal{I}_n^{m+1} = (\mathcal{J}^{m+1} + \mathcal{J}_n)/\mathcal{J}_n$$

Both $\mathcal{J}_m$ and $\mathcal{J}^{m+1}$ are $\mathbf{Irr}(\overline{T})$-graded and for given $\lambda \in \mathbf{Irr}(\overline{T})$ and for $n \geq n_\lambda$ the isotypic component $\mathcal{J}_n[\lambda]$ is zero by Lemma 8.1.2. Hence $\mathcal{J}_m = \mathcal{J}^{m+1}$ for every $m \in \mathbb{N}$. We define

$$Z = \operatorname{Spec}_{Z_0}\mathcal{A}$$

and we denote by $\pi : Z \to Z_0$ the structural morphism. The scheme Z inherits a **M**-action from $\mathcal{A}$. For every $n \in \mathbb{N}$ the zero-set of $\mathcal{J}^{n+1}$ in $\mathcal{A}$ is a **M**-scheme isomorphic to $Z_n = \operatorname{Spec}_{Z_0}\mathcal{A}_n$. Hence $\mathcal{Z}$ is isomorphic to $\widehat{Z}$. □

Theorem 8.2. *Let* **M** *be a Kempf monoid and let Z be a locally noetherian* **M***-scheme. If $\widehat{Z}$ is locally noetherian, then the canonical morphism $\pi : Z \to Z_0$ is of finite type. In particular, Z is locally noetherian.*

9. Nibylandia

LS TODO: Tutaj szanowni państwo kończy się uporządkowany świat i zaczyna się...

Now a **G**-coherent sheaf on $\mathcal{Z}$ corresponds to a sequence

$$\ldots \twoheadrightarrow \mathcal{M}_{n+1} \twoheadrightarrow \mathcal{M}_n \twoheadrightarrow \ldots \twoheadrightarrow \mathcal{M}_1 \twoheadrightarrow \mathcal{M}_0$$

such that $\mathcal{M}_n$ is a coherent sheaf on $\mathcal{A}_0$ with **G**-action and the kernel of $\mathcal{M}_{n+1} \twoheadrightarrow \mathcal{M}_n$ is $\mathcal{I}_{n+1,n} \cdot \mathcal{M}_{n+1}$ for every $n \in \mathbb{N}$.

9.1. Kempf monoids.

Definition 9.1. Let **M** be a monoid k-scheme. Suppose that the following conditions hold.

LS TODO: Tutaj trzeba zdefiniować monoidy Kempfa. Najpierw trzeba porządnie spisać dowód algebraizacji, żeby mieć poprawną definicję

(1) **M** is affine, geometrically connected and geometrically normal.

(2) There exists zero **o** in **M**.

(3) There exists a torus T over k contained in the center of **M** such that the closure $\mathbf{cl}(T)$ of T in **M** contains **o**.

Then **M** is called *Kempf monoid*.

LS TODO: (or rather Jelisiejew :D)

Let $\mathbf{M}$ be a Kempf monoid and let $\mathbf{G}$ be its group of units. If V is a representation of $\mathbf{G}$ and λ is a class in Λ, then we denote by $V[\lambda] \subseteq V$ the sum of all irreducible T-subpresentations of V of isomorphism type λ. Since T is a central subgroup of $\mathbf{G}$, we derive by Proposition **??** that $V[\lambda]$ is a $\mathbf{G}$-representation of V.
Suppose that Z is a k-scheme with trivial action of $\mathbf{M}$. If $\mathcal{F}$ is a quasi-coherent sheaf on Z equipped with $\mathbf{G}$-action, then we denote by $\mathcal{F}[\lambda]$ a sheaf given by

$$U \mapsto \mathcal{F}(U)[\lambda]$$

for every open affine subset U of Z. Then $\mathcal{F}[\lambda] \subseteq \mathcal{F}$ is a $\mathbf{G}$-quasi-coherent subsheaf of $\mathcal{F}$.

LS TODO: Locally noether-ian trzeba włączyć i jechać z koksami dalej

Proof. Assume that each scheme Z_n is locally Noetherian over k. Then $\mathcal{I}_n$ is a coherent $\mathcal{A}_n$-module, thus $\mathcal{I}_n^i/\mathcal{I}^{i+1}$ is a coherent $\mathcal{A}_0$-module for all i. The series

$$0 = \mathcal{I}_n^{n+1} \subset \mathcal{I}^n \subset \ldots \subset \mathcal{I} \subset \mathcal{A}_n$$

has coherent subquotients, hence $\mathcal{A}_n$ is a coherent $\mathcal{O}_{Z_n}$-algebra. Thus $\mathcal{A}[\lambda]$ is a coherent $\mathcal{O}_{Z_0}$-module for every $\lambda \in \mathrm{Irr}(\overline{\mathbf{G}})$. The claim that π is of finite type is local on $Z^{\mathbf{G}}$, hence we may assume that $Z^{\mathbf{G}}$ is quasi-compact. The sheaf $\mathcal{J}/\mathcal{J}^2 \subseteq \mathcal{A}_1$ is coherent so there exists a finite set $\lambda_1, \ldots, \lambda_r \in \mathrm{Irr}(\overline{\mathbf{G}}) \setminus \{\lambda_0\}$ such that the morphism

$$\bigoplus_{i=1}^{r} \mathcal{A}[\lambda_i] \to \mathcal{J}/\mathcal{J}^2$$

induced by $\mathcal{A} \twoheadrightarrow \mathcal{A}_2$ is surjective. Let $\mathcal{B} \subset \mathcal{A}$ be the quasi-coherent $\mathcal{O}_{Z_0}$-subalgebra generated by the coherent subsheaf $\mathcal{M} := \bigoplus_{i=1}^r \mathcal{A}[\lambda_i] \subseteq \mathcal{A}$. Let $\overline{k}$ be an algebraic closure of k and let $\mathcal{A}' = \mathcal{A} \otimes \overline{k}$. Fix a Kempf's torus over $\overline{k}$ and the associated grading $\mathcal{A}' = \bigoplus_{i \geq 0} \mathcal{A}'[i]$ as in Corollary **??**. Then $\mathcal{J} = \bigoplus_{i \geq 1} \mathcal{A}'[i]$ is a graded ideal and $\mathcal{J}/\mathcal{J}^2$ is generated by the graded (coherent) subsheaf $\mathcal{M}' = \bigoplus_{i=1}^r \mathcal{A}'[\lambda_i]$. By graded Nakayama's lemma, the ideal $\mathcal{J}$ itself is generated by (the elements of) $\mathcal{M}'$. Then by induction on the degree, $\mathcal{A}'$ is generated by $\mathcal{M}'$ as an algebra. In other words, $\mathcal{A}' = \mathcal{B} \otimes \overline{k}$. Thus also $\mathcal{A} = \mathcal{B}$ and so $\mathcal{A}$ is of finite type over $\mathcal{O}_{Z_0}$. □

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