FIBERED CATEGORIES AND EQUIVARIANT OBJECTS

1. INTRODUCTION

In these notes we often work with two distinct categories. In order to make our notation clear we denote by $h^{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ the Yoneda embedding for category $\mathcal{C}$. In particular, if X is an object of $\mathcal{C}$, then $h^{\mathcal{C}}_X$ is a presheaf associated with X.

2. FIBERED CATEGORIES

Definition 2.1. Let $p : \mathcal{E} \to \mathcal{B}$ be a functor and let $\phi : \xi \to \eta$ be a morphism of $\mathcal{E}$. Suppose that for every object ζ of $\mathcal{E}$ the square of classes

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{E}}(\zeta,\xi) \arrow[r, "\mathrm{Mor}_{\mathcal{E}}(1_{\zeta},\phi)"] \arrow[d, "\text{induced by } p"'] & \mathrm{Mor}_{\mathcal{E}}(\zeta,\eta) \arrow[d, "\text{induced by } p"] \\
\mathrm{Mor}_{\mathcal{B}}\big(p(\zeta),p(\xi)\big) \arrow[r, "\mathrm{Mor}_{\mathcal{B}}(1_{p(\zeta)},p(\phi))"'] & \mathrm{Mor}_{\mathcal{B}}\big(p(\zeta),p(\eta)\big)
\end{tikzcd}$$

is cartesian. Then ϕ is *a cartesian morphism of p*.

One can rephrase definition above in terms of presheaves as follows. Morphism $\phi : \xi \to \eta$ is cartesian with respect to p if the following square

$$\begin{tikzcd}
h^{\mathcal{E}}_{\xi} \arrow[r, "h^{\mathcal{E}}_{\phi}"] \arrow[d, "\text{induced by } p"'] & h^{\mathcal{E}}_{\eta} \arrow[d, "\text{induced by } p"] \\
h^{\mathcal{B}}_{p(\xi)} \cdot p \arrow[r, "\left(h^{\mathcal{B}}_{p(\phi)}\right)_p"'] & h^{\mathcal{B}}_{p(\eta)} \cdot p
\end{tikzcd}$$

is cartesian in the category $\widehat{\mathcal{E}}$.

Fact 2.2. *Let $p : \mathcal{E} \to \mathcal{B}$ be a functor, let $f : X \to Y$ be a morphism of $\mathcal{B}$ and let η be an object of $\mathcal{E}$. Suppose that $\phi_1 : \xi_1 \to \eta, \phi_2 : \xi_2 \to \eta$ are morphisms of $\mathcal{E}$ that are cartesian with respect to p and assume that $p(\phi_1) = p(\phi_2)$. Then there exists a unique morphism $\theta : \xi_1 \to \xi_2$ such that $\phi_1 = \phi_2 \cdot \theta$. Moreover, θ is an isomorphism.*

Proof. We use the presheaf reformulation of a definition of cartesian morphisms of p. It implies that there exists a unique natural transformation $\sigma : h^{\mathcal{E}}_{\xi_1} \to h^{\mathcal{E}}_{\xi_2}$ such that $h^{\mathcal{E}}_{\phi_1} = h^{\mathcal{E}}_{\phi_2} \cdot \sigma$. Moreover, σ is a natural isomorphism. Since $h^{\mathcal{E}} : \mathcal{E} \to \widehat{\mathcal{E}}$ is full and faithful, we derive that there exists a unique morphism $\theta : \xi_1 \to \xi_2$ such that $h^{\mathcal{E}}_{\theta} = \sigma$. Then θ satisfies the assertion. □

Definition 2.3. Let $p : \mathcal{E} \to \mathcal{B}$ be a functor, let $f : X \to Y$ be a morphism of $\mathcal{B}$ and let η be an object of $\mathcal{E}$ such that $p(\eta) = Y$. A pair (ξ, ϕ) such that ξ is an object of $\mathcal{E}$ and $\phi : \xi \to \eta$ is a morphism of $\mathcal{E}$ is called *a pullback of η along f* if the following conditions are satisfied.

(1) $p(\phi) = f$

(2) ϕ is cartesian morphism of p.

Note that Fact 2.2 implies that pullbacks are unique up to a unique isomorphism.

Definition 2.4. Let $p : \mathcal{E} \to \mathcal{B}$ be a functor. Then p is *a fibered category* if and only if for every morphism $f : X \to Y$ of $\mathcal{B}$ and every object η of $\mathcal{E}$ such that $p(\eta) = Y$ there exists a pullback of η along f. If $p : \mathcal{E} \to \mathcal{B}$ is a fibered category, then we say that *$\mathcal{E}$ is fibered over $\mathcal{B}$ with respect to p.*

Now we give some examples of fibered categories. The first is a prototypical for the notion of a cartesian category. It shows that any category $\mathcal{B}$ with fiber products gives rise in a canonical way to a fibered category over $\mathcal{B}$ with cartesian arrows as cartesian squares in $\mathcal{B}$.

Example 2.5 (the fibered category of arrows)**.** Let $\mathcal{B}$ be a category. We define the category $\mathrm{Arr}(\mathcal{B})$ of arrows of $\mathcal{B}$ as follows. Objects of $\mathrm{Arr}(\mathcal{B})$ are morphisms $\pi : \tilde{X} \to X$ of $\mathcal{B}$. Now if $\pi : \tilde{X} \to X$ and $\psi : \tilde{Y} \to Y$ are objects of $\mathrm{Arr}(\mathcal{B})$, then a morphism $\pi \to \psi$ is a pair (f, ϕ) such that $f : X \to Y$ and $\phi : \tilde{X} \to \tilde{Y}$ are morphisms in $\mathcal{B}$ making the square

$$\begin{tikzcd} \tilde{X} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \tilde{Y} \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutative. There exists a functor $p_{\mathrm{Arr}} : \mathrm{Arr}(\mathcal{B}) \to \mathcal{B}$ given by formula $p_{\mathrm{Arr}}\big((f, \phi)\big) = f$. Suppose now that $f : X \to Y$ and $\psi : \tilde{Y} \to Y$ are morphisms of $\mathcal{B}$ and there exists a commutative square

$$\begin{tikzcd} \tilde{X} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \tilde{Y} \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

It is a direct consequence of the definition that (f, ϕ) is a cartesian morphisms of p_{Arr} if and only if the square above is cartesian. Thus p_{Arr} is a fibered category provided that $\mathcal{B}$ admits fiber products.

Definition 2.6. Suppose that $p_1 : \mathcal{E}_1 \to \mathcal{B}$ and $p_2 : \mathcal{E}_2 \to \mathcal{B}$ are fibered categories. Then a functor $F : \mathcal{E}_1 \to \mathcal{E}_2$ is *a morphism of fibered categories* if the following two assertions are satisfied.

(1) $p_1 = F \cdot p_2$ or in other words F is a functor over $\mathcal{B}$.

(2) Image under F of a cartesian morphism of p_1 is a cartesian morphism of p_2.

Next example is closely related to the previous one, but is of more topological flavour.

Example 2.7 (the fibered category vector bundles)**.** Let **Top** be the category of topological spaces. We define a subcategory $\mathbf{VectBund}_{\mathbb{R}}$ of $\mathrm{Arr}(\mathbf{Top})$ of vector bundles as follows. Objects of $\mathbf{VectBund}_{\mathbb{R}}$ are topological $\mathbb{R}$-vector bundles $\pi : \mathcal{V} \to X$. Now if $\pi : \mathcal{V} \to X$ and $\psi : \mathcal{W} \to Y$ are topological $\mathbb{R}$-vector bundles, then a morphism $\pi \to \psi$ is a pair (f, ϕ) such that $f : X \to Y$ is a continuous map and $\phi : \mathcal{V} \to \mathcal{W}$ is a continuous making the square

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \mathcal{W} \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutative and moreover, ϕ induces an $\mathbb{R}$-linear map on fibers i.e. for each point x in X map ϕ induces an $\mathbb{R}$-linear map $\pi^{-1}(x) \to \psi^{-1}\left(f(x)\right)$. Since topological vector bundles are stable under continuous change of base, we obtain a fibered category $\mathbf{VectBund}_{\mathbb{R}} \to \mathbf{Top}$ as the restriction of $p_{\mathrm{Arr}} : \mathrm{Arr}(\mathbf{Top}) \to \mathbf{Top}$. Thus we have a commutative triangle

$$\begin{tikzcd} \mathbf{VectBund}_{\mathbb{R}} \arrow[rr, hook] \arrow[dr] & & \mathrm{Arr}(\mathbf{Top}) \arrow[dl, "p_{\mathrm{Arr}}"] \\ & \mathbf{Top} & \end{tikzcd}$$

According to Example 2.5 the inclusion $\mathbf{VectBund}_{\mathbb{R}} \hookrightarrow \mathrm{Arr}(\mathbf{Top})$ is a morphism of fibered categories.

3. Example: Principial Bundles

We devote this section to another important example of a fibered category. We fix a category with finite limits $\mathcal{B}$ and a group object G of $\mathcal{B}$.

Definition 3.1. Let $\mathcal{P}$ be an object of $\mathcal{B}$ equipped with an action of G, let T be an object of $\mathcal{B}$ with trivial action of G and let $\pi : \mathcal{P} \to T$ be a G-equivariant morphism (with respect to these G-actions). Consider a sieve S on T. Suppose for every arrow $g : \widetilde{T} \to T$ in S there exists a G-equivariant isomorphism $\phi_f : G \times \widetilde{T} \to f^*\mathcal{P}$ satisfying $\mathrm{pr}_{\widetilde{T}} = \psi \cdot \phi_f$, where

$$\begin{tikzcd} f^*\mathcal{P} \arrow[r, "\phi"] \arrow[d, "\psi"'] & \mathcal{P} \arrow[d, "\pi"] \\ \widetilde{T} \arrow[r, "f"'] & T \end{tikzcd}$$

is a cartesian square in $\mathcal{B}$ and $\mathrm{pr}_{\widetilde{T}} : G \times \widetilde{T} \to \widetilde{T}$ is the projection. Then we say that S *trivializes* π.

In the remaining part of this section we fix a Grothendieck topology $\mathcal{J}$ on $\mathcal{B}$.

Definition 3.2. Let $\mathcal{P}$ be an object of $\mathcal{B}$ equipped with an action of G, let T be an object of $\mathcal{B}$ with trivial action of G and let $\pi : \mathcal{P} \to T$ be a G-equivariant morphism (with respect to these G-actions). Suppose that there exists a covering sieve S in $\mathcal{J}(T)$ that trivializes π. Then π is called *a principial G-bundle with respect to $\mathcal{J}$*.

Now we define a subcategory $\mathbb{B}G$ of $\mathrm{Arr}(\mathcal{B})$ (Example 2.5) that depends on the site $(\mathcal{B}, \mathcal{J})$. Its objects are principial G-bundles with respect to $\mathcal{J}$ and if $\pi : \mathcal{P} \to T$ and $\psi : Q \to Z$ are principial G-bundles with respect to $\mathcal{J}$, then a morphism $\pi \to \psi$ is a pair (f, ϕ) such that $f : T \to Z$ and $\phi : \mathcal{P} \to Q$ are morphisms in $\mathcal{B}$ such that ϕ is G-equivariant and the square

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\phi"] \arrow[d, "\pi"'] & Q \arrow[d, "\psi"] \\ T \arrow[r, "f"'] & Z \end{tikzcd}$$

is commutative. We have a functor $p_{G,\mathcal{J}} : \mathbb{B}G \to \mathcal{B}$ given by the restriction of p_{Arr} to $\mathbb{B}G$. In other words if (f, ϕ) is a morphism of $\mathbb{B}G$, then $p_{G,\mathcal{J}}\big((f,\phi)\big) = f$. Let $\psi : Q \to Z$ be a principial G-bundle with respect to $\mathcal{J}$ and let $f : T \to Z$ be a morphism. Consider the cartesian square

$$\begin{tikzcd} f^*Q \arrow[r, "\phi"] \arrow[d, "\pi"'] & Q \arrow[d, "\psi"] \\ T \arrow[r, "f"'] & Z \end{tikzcd}$$

in $\mathcal{B}$. Then by the universal property there exists a unique action of G on f^*Q such that the square above consists of G-equivariant morphisms (T, Z are equipped with trivial G-actions). Moreover, with respect to this action $\psi : f^*Q \to T$ becomes a principial G-bundle with respect to $\mathcal{J}$. Indeed, if S is in $\mathcal{J}(Z)$ and S trivializes ψ, then its pullback f^*S trivializes π and is an element of $\mathcal{J}(T)$ (by definition of a Grothendieck topology). This shows that $p_{G,\mathcal{J}} : \mathbb{B}G \to \mathcal{B}$ is a fibered category. Moreover, we have a commutative triangle

$$\begin{tikzcd} \mathbb{B}G \arrow[rr, hook] \arrow[dr, "p_{G,\mathcal{J}}"'] & & \mathrm{Arr}(\mathcal{B}) \arrow[dl, "p_{\mathrm{Arr}}"] \\ & \mathcal{B} & \end{tikzcd}$$

and by Example 2.5 the inclusion $\mathbb{B}G \hookrightarrow \mathrm{Arr}(\mathcal{B})$ is a morphism of fibered categories.

Definition 3.3. $p_{G,\mathcal{J}} : \mathbb{B}G \to \mathcal{B}$ is called *the fibered category of principial G-bundles on* $(\mathcal{B}, \mathcal{J})$.

From now suppose that X is an object of $\mathcal{B}$ equipped with an action $a : G \times X \to X$ of G. We define a category $[X/G]$ as follows. Its objects are pairs (π, ϕ), that can be presented by diagrams

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\alpha"] \arrow[d, "\pi"'] & X \\ T & \end{tikzcd}$$

such that π is a principal G-bundle with respect to $\mathcal{J}$ and ϕ is a G-equivariant morphism. Suppose that $(\pi : \mathcal{P} \to T, \alpha : \mathcal{P} \to X)$ and $(\psi : \mathcal{Q} \to Z, \beta : \mathcal{Q} \to X)$ are two such objects. Then a morphism $(\pi, \alpha) \to (\psi, \beta)$ is a morphism $(f, \phi) : \pi \to \psi$ in $\mathbb{B}G$ such that $\alpha = \beta \cdot f$. Clearly this makes $[X/G]$ into a subcategory of $\mathbb{B}G$. We denote by $p_{G,\mathcal{J},X} : [X/B] \to \mathcal{B}$ the restriction of the functor $p_{G,\mathcal{J}} : \mathbb{B}G \to \mathcal{B}$. By description of cartesian morphisms of $p_{G,\mathcal{J}}$ we deduce that $p_{G,\mathcal{J},X}$ is a fibered category. We have a commutative triangle

$$\begin{tikzcd} {[X/G]} \arrow[rr, hook] \arrow[dr, "p_{G,\mathcal{J},X}"'] & & \mathbb{B}G \arrow[dl, "p_{G,\mathcal{J}}"] \\ & \mathcal{B} & \end{tikzcd}$$

and the inclusion $\mathbb{B}G \hookrightarrow \mathrm{Arr}(\mathcal{B})$ is a morphism of fibered categories. Note that if $\mathbf{1}$ is a terminal object of $\mathcal{B}$ equipped with trivial action of G, then we have a canonical isomorphism $[\mathbf{1}/G] \cong \mathbb{B}G$ of categories over $\mathcal{B}$.

Definition 3.4. $p_{G,\mathcal{J},X} : \mathbb{B}G \to \mathcal{B}$ is called *the quotient fibered category of G-object X on* $(\mathcal{B}, \mathcal{J})$.

4. Pseudo-functors and fibered categories of elements

Pseudo-functors are certain non-strict 2-functors. In this section we introduce a procedure that enables to construct a fibered category out of a pseudo-functor. We start by defining this notion.

Definition 4.1. Let $\mathcal{B}$ be a category. Consider the tuple of collections

$$F = \left(\{F(X)\}_{X \in \mathrm{Ob}(\mathcal{B})}, \{F(f)\}_{f \in \mathrm{Mor}(\mathcal{B})}, \{\Theta^{f,g}\}_{f,g \in \mathrm{Mor}(\mathcal{B}), \mathrm{cod}(f) = \mathrm{dom}(g)}, \{\epsilon^X\}_{X \in \mathrm{Ob}(\mathcal{B})}\right)$$

of the following data.

(1) For each object X of $\mathcal{B}$ a category $F(X)$.

(2) For each arrow $f : X \to Y$ a functor $F(f) : F(Y) \to F(X)$.

(3) For each object X of $\mathcal{B}$ a natural isomorphism $\epsilon^X : 1_{F(X)} \to F(1_X)$.

(4) For any two composable morphisms $f : X \to Y$ and $g : Y \to Z$ of $\mathcal{B}$ a natural isomorphism $\Theta^{g,f} : F(f) \cdot F(g) \to F(g \cdot f)$

Suppose that these data are subject to the following conditions.

(1) For every arrow $f : X \to Y$ in $\mathcal{B}$ we have
$$1_{F(f)} = \Theta^{f,1_X} \cdot \epsilon^X_{F(f)}, \; 1_{F(f)} = \Theta^{1_Y,f} \cdot F(f)\left(\epsilon^Y\right)$$

(2) For any three morphisms $f : X \to Y, g : Y \to Z, h : Z \to W$ of $\mathcal{B}$ the square of functors and natural isomorphisms

$$\begin{tikzcd} F(f)\cdot F(g)\cdot F(h) \arrow[r, "F(f)(\Theta^{h,g})"] \arrow[d, "\Theta^{g,f}_{F(h)}"'] & F(f)\cdot F(h\cdot g) \arrow[d, "\Theta^{h\cdot g,f}"] \\ F(g\cdot f)\cdot F(h) \arrow[r, "\Theta^{h,g\cdot f}"'] & F(h\cdot g\cdot f) \end{tikzcd}$$

is commutative.

Then F is called *a pseudo-functor on $\mathcal{B}$*

Now we show how to construct a fibered category from a pseudo-functor. Suppose that $\mathcal{B}$ is a category and
$$F = \left(\{F(X)\}_{X\in \mathrm{Ob}(\mathcal{B})}, \{F(f)\}_{f\in \mathrm{Mor}(\mathcal{B})}, \{\Theta^{f,g}\}_{f,g\in \mathrm{Mor}(\mathcal{B}),\, \mathrm{cod}(f)=\mathrm{dom}(g)}, \{\epsilon^X\}_{X\in \mathrm{Ob}(\mathcal{B})}\right)$$
is a pseudo-functor on $\mathcal{B}$. We define a category $\int_{\mathcal{B}} F$. Its objects are pairs (X, ξ) such that X is an object of $\mathcal{B}$ and ξ is an object of $F(X)$. If (X,ξ) and (Y,η) are objects of $\int_{\mathcal{B}} F$, then a morphism between these objects is a pair (f,σ) such that $f : X \to Y$ is a morphism of $\mathcal{B}$ and $\sigma : \xi \to F(f)(\eta)$ is a morphism of $F(X)$. Now suppose that $(f,\sigma) : (X,\xi) \to (Y,\eta)$ and $(g,\tau) : (Y,\eta) \to (Z,\zeta)$ are morphisms of $\int_{\mathcal{B}} F$. Then we define their composition by formula
$$(g,\tau)\cdot(f,\sigma) = \left(g\cdot f, \Theta^{g,f}_{\zeta} \cdot F(f)\,(\tau)\cdot\sigma\right)$$

Fact 4.2. *$\int_{\mathcal{B}} F$ is a well defined category.*

Proof. We first verify that the composition of morphisms in $\int_{\mathcal{B}} F$ is associative. Suppose that $(f,\sigma) : (X,\xi) \to (Y,\eta), (g,\tau) : (Y,\eta) \to (Z,\zeta), (h,\rho) : (Z,\zeta) \to (W,\omega)$ are morphisms of $\int_{\mathcal{B}} F$. Then
$$\left((h,\rho)\cdot(g,\tau)\right)\cdot(f,\sigma) = \left(h\cdot g, \Theta^{h,g}_{\omega}\cdot F(g)\,(\rho)\cdot\tau\right)\cdot(f,\sigma) =$$
$$= \left(h\cdot g\cdot f, \Theta^{h\cdot g,f}_{\omega}\cdot F(f)\big(\Theta^{h,g}_{\omega}\cdot F(g)\,(\rho)\cdot\tau\big)\cdot\sigma\right) = \left(h\cdot g\cdot f, \Theta^{h\cdot g,f}_{\omega}\cdot F(f)\big(\Theta^{h,g}_{\omega}\big)\cdot F(f)\big(F(g)\,(\rho)\,\big)\cdot F(f)\big(\tau\big)\cdot\sigma\right)$$
and
$$(h,\rho)\cdot\left((g,\tau)\cdot(f,\sigma)\right) = (h,\rho)\cdot\left(g\cdot f, \Theta^{g,f}_{\zeta}\cdot F(f)\big(\tau\big)\cdot\sigma\right) =$$
$$= \left(h\cdot g\cdot f, \Theta^{h,g\cdot f}_{\omega}\cdot F(g\cdot f)\big(\rho\big)\cdot\Theta^{g,f}_{\zeta}\cdot F(f)\big(\tau\big)\cdot\sigma\right) = \left(h\cdot g\cdot f, \Theta^{h,g\cdot f}_{\omega}\cdot\Theta^{g,f}_{F(h)(\omega)}\cdot F(f)\big(F(g)\,(\rho)\,\big)\cdot F(f)\big(\tau\big)\cdot\sigma\right)$$
Since $\Theta^{h\cdot g,f}_{\omega}\cdot F(f)\big(\Theta^{h,g}_{\omega}\big) = \Theta^{h,g\cdot f}_{\omega}\cdot\Theta^{g,f}_{F(h)(\omega)}$, we deduce that
$$\left((h,\rho)\cdot(g,\tau)\right)\cdot(f,\sigma) = (h,\rho)\cdot\left((g,\tau)\cdot(f,\sigma)\right)$$

and hence the composition in $\int_{\mathcal{B}} F$ is associative. Next we prove that for each object (X,ξ) of $\int_{\mathcal{B}} F$ there exists an identity morphism. We claim that $(1_X, \epsilon^X_\xi) : (X,\xi) \to (X,\xi)$ is the identity. Indeed, for morphisms $(f,\sigma) : (X,\xi) \to (Y,\eta)$ and $(g,\tau) : (Z,\zeta) \to (X,\xi)$ we have

$$(f,\sigma)\cdot(1_X,\epsilon^X_\xi) = \left(f, \Theta^{f,1_X}_\eta \cdot F(1_X)(\sigma)\cdot \epsilon^X_\xi\right) = \left(f, \Theta^{f,1_X}_\eta \cdot \epsilon^X_{F(f)(\eta)} \cdot \sigma\right) = (f,\sigma)$$

and

$$(1_X,\epsilon^X_\xi)\cdot(g,\tau) = \left(g, \Theta^{1_X,g}_\xi \cdot F(g)\left(\epsilon^X_\xi\right)\cdot\tau\right) = (g,\tau)$$

Therefore, $\int_{\mathcal{B}} F$ is a category. □

Next we define a functor $p_F : \int_{\mathcal{B}} F \to \mathcal{B}$ by formula

$$p_F\Big((f,\sigma) : (X,\xi) \to (Y,\tau)\Big) = f : X \to Y$$

This is clearly a well defined functor. Now we prove the following statement.

The functor $p_F : \int_{\mathcal{B}} F \to \mathcal{B}$ is a fibered category.

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{B}$ and η be an object of $F(Y)$. Thus (Y,η) is an object of $\int_{\mathcal{B}} F$. It suffices to show that (Y,η) admits a pullback along f. We claim that

$$\left(f, 1_{F(f)(\eta)}\right) : \left(X, F(f)(\eta)\right) \to (Y,\eta)$$

is a cartesian morphism of p_F that yields a pullback of η along f. To prove the claim consider an object (Z,ζ) of $\int_{\mathcal{B}} F$ and suppose that $(g,\tau) : (Z,\zeta) \to (Y,\eta)$ is a morphism of $\int_{\mathcal{B}} F$ such that g factors through f. Then there exists $h : Z \to X$ such that $f\cdot h = g$. Note that $\tau : \zeta \to F(g)(\eta)$. Since $g = f\cdot h$, we have

$$\tau = \Theta^{f,h}_\eta \cdot \left(\Theta^{f,h}_\eta\right)^{-1}\cdot\tau = \Theta^{f,h}_\eta \cdot F(h)\left(1_{F(f)(\eta)}\right)\cdot\left(\Theta^{f,h}_\eta\right)^{-1}\cdot\tau$$

and hence

$$(g,\tau) = \left(f, 1_{F(f)(\eta)}\right)\cdot\left(h, \left(\Theta^{f,h}_\eta\right)^{-1}\cdot\tau\right)$$

Thus (g,τ) factors through $\left(f,1_{F(f)(\eta)}\right)$ and the formula above shows that this factorization is unique. Hence $\left(f,1_{F(f)(\eta)}\right)$ is a cartesian morphism of p_F. □

Definition 4.3. Let $\mathcal{B}$ be a category and let F be a pseudo-functor on $\mathcal{B}$. A fibered category $p_F : \int_{\mathcal{B}} F \to \mathcal{B}$ constructed above is called *the fibered category of elements of the pseudo-functor F*.

It is possible to construct a pseudo-functor out of a fibered category. We will give a brief outline of this construction. For this we introduce notation that will be also used in other considerations.

Definition 4.4. Let $p : \mathcal{E} \to \mathcal{B}$ be a fibered category. For every object X of $\mathcal{B}$ we denote by $p^{-1}(X)$ a subcategory of $\mathcal{E}$ consisting of all morphisms $\phi : \xi \to \eta$ such that $p(\phi) = 1_X$. Then $p^{-1}(X)$ is called *the fiber of p over X*.

Suppose now that $p : \mathcal{E} \to \mathcal{B}$ is a fibered category. Let $f : X \to Y$ be a morphism. For every object η in $p^{-1}(Y)$ we pick its pullback $\tilde{f}_\eta : f^*\eta \to \eta$ along f. By universal property of cartesian morphisms we deduce that this induces a functor $f^* : p^{-1}(Y) \to p^{-1}(X)$. Universal property of cartesian morphisms implies also the following assertions.

(1) For each object X of $\mathcal{B}$ we may choose $(1_X)^* = 1_{p^{-1}(X)}$.

(2) For any two composable morphisms $f : X \to Y$ and $g : Y \to Z$ of $\mathcal{B}$ there exists a unique natural isomorphism $\Theta^{g,f} : f^* g^* \to (g \cdot f)^*$ of functors such that for every ζ in $p^{-1}(Z)$ we have commutative diagram

$$\begin{tikzcd} f^* g^* \zeta \arrow[r, "\tilde{f}_{g^*\zeta}"] \arrow[d, "\Theta^{g,f}_\zeta"'] & g^*\zeta \arrow[r, "\tilde{g}_\zeta"] & \zeta \arrow[d, "1_\zeta"] \\ (g \cdot f)^* \zeta \arrow[rr, "\widetilde{g \cdot f}_\zeta"'] & & \zeta \end{tikzcd}$$

From **(1)**, **(2)** and Fact 2.2 one can deduce that the collection

$$\left(\{p^{-1}(X)\}_{X \in \mathrm{Ob}(\mathcal{B})}, \{f^*\}_{f \in \mathrm{Mor}(\mathcal{B})}, \{\Theta^{f,g}\}_{f,g \in \mathrm{Mor}(\mathcal{B}), \mathrm{cod}(f) = \mathrm{dom}(g)}, \{1_{p^{-1}(X)}\}_{X \in \mathrm{Ob}(\mathcal{B})}\right)$$

is a pseudo-functor.

Remark 4.5. The construction of the fibered category of elements is a part of 2-equivalence between appropriately defined category of pseudo-functors on $\mathcal{B}$ and the category of fibered categories over $\mathcal{B}$.

5. Example: Quasi-coherent sheaves

Note that all examples of fibered categories given so far were fibered subcategories of the fibered category of arrows $p_{\mathrm{Arr}} : \mathrm{Arr}(\mathcal{B}) \to \mathcal{B}$ for a given category $\mathcal{B}$ with fibered-products. In this section we employ the procedure that produces a fibered category out of a pseudo-functor to obtain an important example of a category fibered over $\mathbf{Sch}_k$ (the category of schemes over a ring k), which is not of this type.

Let $f : X \to Y$ be a morphism of k-schemes. We have an adjunction

$$\begin{tikzcd} \mathfrak{Qcoh}(X) \arrow[r, bend left, "f_*"] \arrow[r, phantom, "\perp"] & \mathfrak{Qcoh}(Y) \arrow[l, bend left, "f^*"] \end{tikzcd}$$

It is determined by the bijection

$$\mathrm{Hom}_{\mathcal{O}_Y}(f^*\mathcal{G}, \mathcal{F}) \xrightarrow{\Phi^f_{\mathcal{G},\mathcal{F}}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, f_*\mathcal{F})$$

Suppose now that $f : X \to Y$ and $g : Y \to Z$ are morphisms of k-schemes. Since $(g \cdot f)_* = g_* \cdot f_*$, there exists a unique natural isomorphism $\Theta^{g,f} : f^* g^* \to (g \cdot f)^*$ such that for every quasi-coherent sheaf $\mathcal{F}$ in $\mathfrak{Qcoh}(X)$ and every quasi-coherent sheaf $\mathcal{H}$ in $\mathfrak{Qcoh}(Z)$ we have

$$\Phi^{g \cdot f}_{\mathcal{H},\mathcal{F}} = \Phi^g_{\mathcal{H}, f_*\mathcal{F}} \cdot \Phi^f_{g^*\mathcal{H},\mathcal{F}} \cdot \mathrm{Hom}_{\mathcal{O}_X}(\Theta^{g,f}_{\mathcal{H}}, 1_{\mathcal{F}})$$

Now we have the following result.

Fact 5.1. *Suppose that $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$ are morphism of k-schemes. Then the square*

$$\begin{tikzcd} f^*g^*h^* \arrow[r, "f^*\Theta^{h,g}"] \arrow[d, "\Theta^{g,f}_{h^*}"'] & f^*(h\cdot g)^* \arrow[d, "\Theta^{h\cdot g,f}"] \\ (g\cdot f)^*h^* \arrow[r, "\Theta^{h,g\cdot f}"'] & (h\cdot g\cdot f)^* \end{tikzcd}$$

of functors and natural isomorphisms is commutative.

Proof. Suppose that $\mathcal{F}$ is an object of $\mathfrak{Qcoh}(X)$ and $\mathcal{K}$ is an object of $\mathfrak{Qcoh}(W)$. Then

$$\Phi^h_{\mathcal{K},g_*f_*\mathcal{F}} \cdot \Phi^g_{h^*\mathcal{K},f_*\mathcal{F}} \cdot \Phi^f_{g^*h^*\mathcal{K},\mathcal{F}} \cdot \mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{g,f}_{h^*\mathcal{K}}, 1_{\mathcal{F}}\big) \cdot \mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h,g\cdot f}_{\mathcal{K}}, 1_{\mathcal{F}}\big) =$$
$$= \Phi^h_{\mathcal{K},g_*f_*\mathcal{F}} \cdot \Phi^{g\cdot f}_{h^*\mathcal{K},\mathcal{F}} \cdot \mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{g,f}_{h^*\mathcal{K}}, 1_{\mathcal{F}}\big) = \Phi^{h\cdot g\cdot f}_{\mathcal{K},\mathcal{F}}$$

and

$$\Phi^h_{\mathcal{K},g_*f_*\mathcal{F}} \cdot \Phi^g_{h^*\mathcal{K},f_*\mathcal{F}} \cdot \Phi^f_{g^*h^*\mathcal{K},\mathcal{F}} \cdot \mathrm{Hom}_{\mathcal{O}_X}\big(f^*\Theta^{h,g}_{\mathcal{K}}, 1_{\mathcal{F}}\big) \cdot \mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h\cdot g,f}_{\mathcal{K}}, 1_{\mathcal{F}}\big) =$$
$$= \Phi^h_{\mathcal{K},g_*f_*\mathcal{F}} \cdot \Phi^g_{h^*\mathcal{K},f_*\mathcal{F}} \cdot \mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h,g}_{\mathcal{K}}, 1_{f_*\mathcal{F}}\big) \cdot \Phi^f_{(h\cdot g)^*\mathcal{K},\mathcal{F}} \cdot \mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h\cdot g,f}_{\mathcal{K}}, 1_{\mathcal{F}}\big) =$$
$$= \Phi^{h\cdot g}_{\mathcal{K},f_*\mathcal{F}} \cdot \Phi^f_{(h\cdot g)^*\mathcal{K},\mathcal{F}} \cdot \mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h\cdot g,f}_{\mathcal{K}}, 1_{\mathcal{F}}\big) = \Phi^{h\cdot g\cdot f}_{\mathcal{K},\mathcal{F}}$$

Therefore, we derive that

$$\Phi^h_{\mathcal{K},g_*f_*\mathcal{F}} \cdot \Phi^g_{h^*\mathcal{K},f_*\mathcal{F}} \cdot \Phi^f_{g^*h^*\mathcal{K},\mathcal{F}} \cdot \mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{g,f}_{h^*\mathcal{K}}, 1_{\mathcal{F}}\big) \cdot \mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h,g\cdot f}_{\mathcal{K}}, 1_{\mathcal{F}}\big) =$$
$$= \Phi^h_{\mathcal{K},g_*f_*\mathcal{F}} \cdot \Phi^g_{h^*\mathcal{K},f_*\mathcal{F}} \cdot \Phi^f_{g^*h^*\mathcal{K},\mathcal{F}} \cdot \mathrm{Hom}_{\mathcal{O}_X}\big(f^*\Theta^{h,g}_{\mathcal{K}}, 1_{\mathcal{F}}\big) \cdot \mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h\cdot g,f}_{\mathcal{K}}, 1_{\mathcal{F}}\big)$$

and hence

$$\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h,g\cdot f}_{\mathcal{K}} \cdot \Theta^{g,f}_{h^*\mathcal{K}}, 1_{\mathcal{F}}\big) = \mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{g,f}_{h^*\mathcal{K}}, 1_{\mathcal{F}}\big) \cdot \mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h,g\cdot f}_{\mathcal{K}}, 1_{\mathcal{F}}\big) =$$
$$= \mathrm{Hom}_{\mathcal{O}_X}\big(f^*\Theta^{h,g}_{\mathcal{K}}, 1_{\mathcal{F}}\big) \cdot \mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h\cdot g,f}_{\mathcal{K}}, 1_{\mathcal{F}}\big) = \mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h\cdot g,f}_{\mathcal{K}} \cdot f^*\Theta^{h,g}_{\mathcal{K}}, 1_{\mathcal{F}}\big)$$

Since this equality holds for every quasi-coherent sheaf $\mathcal{F}$ on X, we deduce that

$$\Theta^{h,g\cdot f}_{\mathcal{K}} \cdot \Theta^{g,f}_{h^*\mathcal{K}} = \Theta^{h\cdot g,f}_{\mathcal{K}} \cdot f^*\Theta^{h,g}_{\mathcal{K}}$$

for every quasi-coherent sheaf $\mathcal{K}$. This proves the assertion. □

Note that for every k-scheme X we may assume that $(1_X)_* = 1_{\mathfrak{Qcoh}(X)} = (1_X)^*$ and $\Phi^{1_X}_{\mathcal{G},\mathcal{F}} = \mathrm{Hom}_{\mathcal{O}_X}(1_{\mathcal{F}}, 1_{\mathcal{G}})$.

Fact 5.2. *Let $f : X \to Y$ and $g : Z \to X$ be morphisms of k-schemes. Then*

$$\Theta^{f,1_X} = 1_{f^*},\ \Theta^{1_X,g} = 1_{g^*}$$

Proof. Suppose that $\mathcal{F}$ is an object of $\mathfrak{Qcoh}(X)$ and $\mathcal{G}$ is an object of $\mathfrak{Qcoh}(Y)$. Then

$$\Phi^f_{\mathcal{G},\mathcal{F}} = \Phi^{f\cdot 1_X}_{\mathcal{G},\mathcal{F}} = \Phi^f_{\mathcal{G},\mathcal{F}} \cdot \Phi^{1_X}_{f^*\mathcal{G},\mathcal{F}} \cdot \mathrm{Hom}_{\mathcal{O}_X}\left(\Theta^{f,1_X}_{\mathcal{G}}, 1_{\mathcal{F}}\right) = \Phi^f_{\mathcal{G},\mathcal{F}} \cdot \mathrm{Hom}_{\mathcal{O}_X}\left(\Theta^{f,1_X}_{\mathcal{G}}, 1_{\mathcal{F}}\right)$$

and thus $\mathrm{Hom}_{\mathcal{O}_X}\left(\Theta^{f,1_X}_{\mathcal{G}}, 1_{\mathcal{F}}\right) = \mathrm{Hom}_{\mathcal{O}_X}\left(1_{f^*\mathcal{G}}, 1_{\mathcal{F}}\right)$. Since this holds for every quasi-coherent sheaf $\mathcal{F}$ on X, we derive that $\Theta^{f,1_X}_{\mathcal{G}} = 1_{f^*\mathcal{G}}$. Thus $\Theta^{f,1_X} = 1_{f^*}$.
Suppose that $\mathcal{H}$ is an object of $\mathfrak{Qcoh}(X)$ and $\mathcal{F}$ is an object of $\mathfrak{Qcoh}(Z)$. Then

$$\Phi^g_{\mathcal{H},\mathcal{F}} = \Phi^{1_X\cdot g}_{\mathcal{H},\mathcal{F}} = \Phi^{1_X}_{\mathcal{H},g_*\mathcal{F}} \cdot \Phi^g_{\mathcal{H},\mathcal{F}} \cdot \mathrm{Hom}_{\mathcal{O}_Z}\left(\Theta^{1_X,g}_{\mathcal{H}}, 1_{\mathcal{F}}\right) = \Phi^g_{\mathcal{H},\mathcal{F}} \cdot \mathrm{Hom}_{\mathcal{O}_Z}\left(\Theta^{1_X,g}_{\mathcal{H}}, 1_{\mathcal{F}}\right)$$

and thus $\mathrm{Hom}_{\mathcal{O}_Z}\left(\Theta^{1_X,g}_{\mathcal{H}}, 1_{\mathcal{F}}\right) = \mathrm{Hom}_{\mathcal{O}_Z}\left(1_{g^*\mathcal{H}}, 1_{\mathcal{F}}\right)$. Since this holds for every quasi-coherent sheaf $\mathcal{F}$ on Z, we derive that $\Theta^{1_X,g}_{\mathcal{H}} = 1_{g^*\mathcal{H}}$. Thus $\Theta^{1_X,g} = 1_{g^*}$. □

Now Facts 5.1 and 5.2 imply that the collection

$$\left(\{\mathfrak{Qcoh}(X)\}_{X\in\mathbf{Sch}_k},\{f^*\}_{f\in\mathrm{Mor}(\mathbf{Sch}_k)},\{\Theta^{f,g}\}_{f,g\in\mathrm{Mor}(\mathbf{Sch}_k),\,\mathrm{cod}(f)=\mathrm{dom}(g)},\{1_{1_{\mathfrak{Qcoh}(X)}}\}_{X\in\mathbf{Sch}_k}\right)$$

forms a pseudo-functor on $\mathbf{Sch}_k$.

Definition 5.3. *The fibered category of quasi-coherent sheaves on* $\mathbf{Sch}_k$ is the fibered category of elements of the pseudo-functor

$$\left(\{\mathfrak{Qcoh}(X)\}_{X\in\mathbf{Sch}_k},\{f^*\}_{f\in\mathrm{Mor}(\mathbf{Sch}_k)},\{\Theta^{f,g}\}_{f,g\in\mathrm{Mor}(\mathbf{Sch}_k),\,\mathrm{cod}(f)=\mathrm{dom}(g)},\{1_{1_{\mathfrak{Qcoh}(X)}}\}_{X\in\mathbf{Sch}_k}\right)$$

Proposition 5.4. *Let X,Y be objects of $\mathcal{B}$ equipped with G-actions. Suppose that there exists a functor F which makes the triangle*

$$\begin{tikzcd}
{[X/G]} \arrow[rr, "F"] \arrow[dr, "p_{G,\mathcal{J},X}"'] & & {[Y/G]} \arrow[dl, "p_{G,\mathcal{J},Y}"] \\
& \mathbb{B}G &
\end{tikzcd}$$

commutative. Then the following assertions hold.

(1) *F is a morphism of fibered categories $p_{G,\mathcal{J},X}$ and $p_{G,\mathcal{J},Y}$.*

(2) *There exists a unique G-equivariant morphism $\Phi : X \to Y$ such that F is induced by Φ. That is F sends*

$$(\pi:\mathcal{P}\to T,\alpha:\mathcal{P}\to X)$$

to

$$(\pi:\mathcal{P}\to T,\Phi\cdot\alpha:\mathcal{P}\to Y)$$

6. Equivariant objects in a fibered category

Definition 6.1. Let $p:\mathcal{E}\to cB$ be a fibered category, let $M:\mathcal{B}^{\mathrm{op}}\to\mathbf{Mon}$ be a presheaf of monoids on $\mathcal{B}$ and let X be an object of $\mathcal{B}$ such that the presheaf $h^{\mathcal{B}}_X$ admits an action of M given by the morphism $\alpha : M\times h^{\mathcal{B}}_X\to h^{\mathcal{B}}_X$. Consider an object ξ in $p^{-1}(X)$. Suppose that there is an action $\beta : M\cdot p\times h^{\mathcal{E}}_{\xi}\to h^{\mathcal{E}}_{\xi}$ of a monoid presheaf $M\cdot p$ on $h^{\mathcal{E}}_{\xi}$ such that the square

$$\begin{tikzcd}
M\cdot p\times h^{\mathcal{E}}_{\xi} \arrow[r, "\beta"] \arrow[d, "1_{M\cdot p}\times(\text{induced by } p)"'] & h^{\mathcal{E}}_{\xi} \arrow[d, "\text{induced by } p"] \\
M\cdot p\times h^{\mathcal{B}}_X\cdot p \arrow[r, "\alpha_p"'] & h^{\mathcal{B}}_X\cdot p
\end{tikzcd}$$

is commutative. Then a pair (ξ,β) is called *an M-equivariant object over α.*

Definition 6.2. Suppose that (ξ_1,β_1) and (ξ_2,β_2) are objects over X with M-equivariant structures. Then a morphism $\phi:\xi_1\to\xi_2$ in $\mathcal{E}$ is *M-equivariant* if the square

$$\begin{tikzcd}
M\cdot p\times h^{\mathcal{E}}_{\xi_1} \arrow[r, "\beta_1"] \arrow[d, "1_{M\cdot p}\times h^{\mathcal{E}}_{\phi}"'] & h^{\mathcal{E}}_{\xi_1} \arrow[d, "\phi"] \\
M\cdot p\times h^{\mathcal{E}}_{\xi_2} \arrow[r, "\beta_2"'] & h^{\mathcal{E}}_{\xi_2}
\end{tikzcd}$$

is commutative.

In particular, we have a category $p^{-1}(X)_M$ of M-equaivariant objects over X and M-equivariant morphisms.
Now we assume that M is representable by some monoid object $\mathbf{M}$ of $\mathcal{B}$. It turns out that in this special case one can The following result will be extensively used in this section.

Proposition 6.3. *Let $p : \mathcal{E} \to \mathcal{B}$ be a fibered category, let X, Y be objects of $\mathcal{B}$ and let ξ be an object of $\mathcal{E}$ in $p^{-1}(X)$. Suppose that $\mathcal{B}$ admits finite products. Then there exists a commutative triangle*

$$\begin{tikzcd}
h^{\mathcal{B}}_Y \cdot p \times h^{\mathcal{E}}_\xi \arrow[rr, "\cong"] \arrow[dr, "pr_{h^{\mathcal{E}}_\xi}"'] & & h^{\mathcal{E}}_{pr_X^* \xi} \arrow[dl, "h^{\mathcal{E}}_{(\widetilde{pr_X})_\xi}"] \\
& h^{\mathcal{E}}_\xi &
\end{tikzcd}$$

in which top row is an isomorphism, where $(\widetilde{pr_X})_\xi : pr_X^* \xi \to \xi$ is a cartesian arrow over $pr_X : Y \times X \to X$.

Proof. $\square$