INTRODUCTION TO MEASURE THEORY

1. Families of sets

In this section we study various families of sets that are important in the development of measure theory.

Definition 1.1. Let X be a set and $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of X. We define the following types of families.

(1) $\mathcal{F}$ is *an algebra* if it contains X and is closed under finite unions, intersections and completions.

(2) $\mathcal{F}$ is *a σ-algebra* if it is an algebra and is closed under countable unions.

(3) $\mathcal{F}$ is *a monotone family* if it is closed under unions of countable non-decreasing sequences and under intersections of countable non-increasing sequences.

(4) $\mathcal{F}$ is *a π-system* if it is closed under finite intersections.

(5) $\mathcal{F}$ is *a λ-system* if it contains X and is closed under complements and countable disjoint unions.

Fact 1.2. *Let X be a set and $\{\mathcal{F}_i\}_{i\in I}$ be a class of families subsets of X. Suppose that $\mathcal{F}_i$ is an algebra (σ-algebra, monotone family, π-system, λ-system) for every $i \in I$. Then the intersection $\bigcap_{i\in I} \mathcal{F}_i$ is an algebra (σ-algebra, monotone family, π-system, λ-system).*

Proof. Left as an exercise. □

Definition 1.3. Let $\mathcal{F}$ be a family of subsets of X. We denote by $\sigma(\mathcal{F})$, $\lambda(\mathcal{F})$ and $\mathcal{M}(\mathcal{F})$ intersections of all σ-algebras, λ-systems and monotone families containing $\mathcal{F}$, respectively. We call them *σ-algebra, λ-system and monotone family generated by $\mathcal{F}$*, respectively.

Theorem 1.4 (Dynkin's π-λ lemma)**.** *Let X be a set and $\mathcal{P}$ be a π-system of its subsets. Then $\lambda(\mathcal{P}) = \sigma(\mathcal{P})$.*

For the proof we need the following result.

Lemma 1.4.1. *Let $\mathcal{L}$ be a λ-system. Then for every $A \in \mathcal{L}$ family*

$$\mathcal{L}_A = \left\{B \in \mathcal{L} \,\middle|\, A \cap B \in \mathcal{L}\right\}$$

is a λ-system.

Proof of the lemma. Since $A \in \mathcal{L}$, we have $X \in \mathcal{L}_A$. Suppose now that $B \in \mathcal{L}_A$. Then $A \cap B \in \mathcal{L}$. Since $X \smallsetminus A \in \mathcal{L}$, we derive that also $(A \cap B) \cup (X \smallsetminus A) \in \mathcal{L}$ and hence

$$A \cap (X \smallsetminus B) = X \smallsetminus ((A \cap B) \cup (X \smallsetminus A)) \in \mathcal{L}$$

Thus $X \smallsetminus B \in \mathcal{L}_A$. Finally note that $\mathcal{L}_A$ is closed under countable disjoint unions. □

Proof of the theorem. Fix $A \in \mathcal{P}$. Define $\mathcal{L}_A$ as in Lemma 1.4.1 with $\mathcal{L} = \lambda(\mathcal{P})$. Then $\mathcal{L}_A$ is a λ-system. Moreover, $\mathcal{L}_A$ contains $\mathcal{P}$. Hence $\mathcal{L}_A = \lambda(\mathcal{P})$. This shows that $\lambda(\mathcal{P})$ is closed under intersections with members of $\mathcal{P}$. Now fix $A \in \lambda(\mathcal{P})$ and define $\mathcal{L}_A$ as in Lemma 1.4.1 with $\mathcal{L} = \lambda(\mathcal{P})$. Then $\mathcal{P} \subseteq \mathcal{L}_A$ and $\mathcal{L}_A$ is a λ-system. Thus $\mathcal{L}_A = \lambda(\mathcal{P})$. This proves that $\lambda(\mathcal{P})$ is a π-system. A π-system that is simultaneously a λ-system is a σ-algebra. Thus $\sigma(\mathcal{P}) \subseteq \lambda(\mathcal{P})$. Since it is clear that $\lambda(\mathcal{P}) \subseteq \sigma(\mathcal{P})$, we derive that $\lambda(\mathcal{P}) = \sigma(\mathcal{P})$. □

Theorem 1.5 (Halmos's lemma on monotone classes)**.** *Let X be a set and $\mathcal{A}$ be an algebra of its subsets. Then $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$.*

For the proof we need the following easy results. Their proofs are left to the reader.

Lemma 1.5.1. *Let $\mathcal{M}$ be a monotone family. Then for every $A \in \mathcal{M}$ family*

$$\mathcal{M}_A = \left\{B \in \mathcal{M} \,\middle|\, A \cap B \in \mathcal{M}\right\}$$

is monotone.

Lemma 1.5.2. *Let $\mathcal{M}$ be a monotone family. Then a family*

$$\mathcal{M}^c = \left\{A \in \mathcal{M} \,\middle|\, X \setminus A \in \mathcal{M}\right\}$$

is monotone.

Proof of the theorem. Fix $A \in \mathcal{A}$. Define $\mathcal{M}_A$ as in Lemma 1.5.1 with $\mathcal{M} = \mathcal{M}(\mathcal{A})$. Then $\mathcal{M}_A$ is a monotone family. Moreover, $\mathcal{M}_A$ contains $\mathcal{A}$. Hence $\mathcal{M}_A = \mathcal{M}(\mathcal{A})$. This shows that $\mathcal{M}(\mathcal{A})$ is closed under intersections with members of $\mathcal{A}$. Now fix $A \in \mathcal{M}(\mathcal{A})$ and define $\mathcal{M}_A$ as in Lemma 1.5.1 with $\mathcal{M} = \mathcal{M}(\mathcal{A})$. Then $\mathcal{A} \subseteq \mathcal{M}_A$ and $\mathcal{M}_A$ is a monotone family. Thus $\mathcal{M}_A = \mathcal{M}(\mathcal{A})$. This proves that $\mathcal{M}(\mathcal{A})$ is closed under finite intersections. According to Lemma 1.5.2 we derive that $\mathcal{M}(\mathcal{A})^c$ is a monotone family and contains $\mathcal{A}$. Hence $\mathcal{M}(\mathcal{A})^c = \mathcal{M}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$ is closed under complements. Therefore, $\mathcal{M}(\mathcal{A})$ is a σ-algebra. Thus $\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$. Since it is clear that $\mathcal{M}(\mathcal{A}) \subseteq \sigma(\mathcal{A})$, we derive that $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$. □

2. Measurable spaces and measures

Definition 2.1. A pair (X, Σ) consisting of a set X together with a σ-algebra Σ of its subsets is called *a measurable space*.

Definition 2.2. Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces. A function $f : X_1 \to X_2$ is called *a measurable map* if $f^{-1}(A) \in \Sigma_1$ for every $A \in \Sigma_1$.

Measurable spaces and their morphisms form a category.

Definition 2.3. Let X be a set and Σ be an algebra of its subsets. A function $\mu : \Sigma \to [0, +\infty]$ such that $\mu(\varnothing) = 0$ and

$$\mu\left(\bigcup_{n=0}^{m} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

for every family $\{A_n\}_{0 \le n \le m}$ of pairwise disjoint subsets in Σ is called *an additive function*. If in addition μ satisfies

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

for every family $\{A_n\}_{n \in \mathbb{N}}$ of pairwise disjoint subsets in Σ such that $\bigcup_{n \in \mathbb{N}} A_n \in \Sigma$, then μ is called *a σ-additive function*. Moreover, if $\mu : \Sigma \to [0, +\infty]$ is a σ-additive function and Σ is a σ-algebra, then μ is called *a measure*.

Definition 2.4. A tuple (X, Σ, μ) consisting of a measurable space (X, Σ) and a measure $\mu : \Sigma \to [0, +\infty]$ is called *a space with measure*.

Definition 2.5. Let (X, Σ, μ) be a space with measure. We say that it is *finite* if $\mu(X)$ is finite. We say that it is *σ-finite* if there exists a sequence $\{X_n\}_{n \in \mathbb{N}}$ of subsets of Σ such that $\mu(X_n)$ is finite for every $n \in \mathbb{N}$ and $X = \bigcup_{n \in \mathbb{N}} X_n$.

Theorem 2.6. *Let (X, Σ) be a measurable space and $\mu_1, \mu_2 : \Sigma \to [0, +\infty]$ be measures such that $\mu_1(X) = \mu_2(X)$ is finite. Suppose that $\mathcal{P}$ is a π-system of subsets of X such that $\Sigma = \sigma(\mathcal{P})$ and $\mu_1(A) = \mu_2(A)$ for every $A \in \mathcal{P}$. Then $\mu_1 = \mu_2$.*

Proof. Define $\mathcal{F} = \{A \in \Sigma \,|\, \mu_1(A) = \mu_2(A)\}$. Straightforward verification shows that $\mathcal{F}$ is a λ-system. By assumption $\mathcal{P} \subseteq \mathcal{F}$. Therefore, $\lambda(\mathcal{P}) \subseteq \mathcal{F}$. By Theorem 1.5 we deduce that $\Sigma = \sigma(\mathcal{P}) = \lambda(\mathcal{P}) \subseteq \mathcal{F} \subseteq \Sigma$. Hence $\mathcal{F} = \Sigma$. □

Definition 2.7. Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be spaces with measures. A function $f : X_1 \to X_2$ is called *a morphism of spaces with measures* if f is a morphism of measurable spaces and for every $A \in \Sigma_2$ we have equality $\mu_2(A) = \mu_1(f^{-1}(A))$.

Spaces with measures and their morphisms form a category.

3. Outer measures and Carathéodory's construction

Definition 3.1. Let X be a set and $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ be a function. Suppose that $\mu^*(\varnothing) = 0$, $\mu^*(A) \leq \mu^*(B)$ for every subset A of a set B contained in X and

$$\mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n)$$

for every family $\{A_n\}_{n\in\mathbb{N}}$ of subsets of X. Then we say that μ^* is *an outer measure on X*.

Theorem 3.2 (Carathéodory's construction). *Let X be a set and $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ be an outer measure on X. We define a family of sets $\Sigma_{\mu^*} \subseteq \mathcal{P}(X)$ by condition*

$$A \in \Sigma_{\mu^*} \iff \forall_{E\subseteq X}\, \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$$

Then the following assertions hold.

(1) *Σ_{μ^*} is an σ-algebra of subsets of X.*

(2) *For every family $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint subsets of Σ_{μ^*} and every subset E of X we have*

$$\mu^*\left(E \cap \bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu^*(E \cap A_n)$$

In particular, $\mu^*_{|\Sigma_{\mu^*}}$ is a measure.

(3) *Every subset A of X such that $\mu^*(A) = 0$ is contained in Σ_{μ^*}. In particular, $\mu^*_{|\Sigma_{\mu^*}}$ is complete.*

The proof is encapsulated in two lemmas.

Lemma 3.2.1. *Σ_{μ^*} is an algebra of sets.*

Proof of the lemma. Clearly $\varnothing \in \Sigma_{\mu^*}$ and $A \in \Sigma_{\mu^*} \iff X \setminus A \in \Sigma_{\mu^*}$. It suffices to prove that Σ_{μ^*} is closed under unions. For a subset B of X we denote $X \setminus B$ by B^c. Now assume that $A_1, A_2 \in \Sigma_{\mu^*}$ and pick a subset E of X. Then

$$\mu^*(E) = \mu^*(E \cap A_1) + \mu^*(E \cap A_1^c) = \mu^*(E \cap A_1) + \mu^*(E \cap A_1^c \cap A_2) + \mu^*(E \cap A_1^c \cap A_2^c)$$

Since we have equalities

$$E \cap A_1 = (E \cap (A_1 \cup A_2)) \cap A_1,\ E \cap A_1^c \cap A_2 = (E \cap (A_1 \cup A_2)) \cap A_1^c$$

we derive that $\mu^*(E \cap (A_1 \cup A_2)) = \mu^*(E \cap A_1) + \mu^*(E \cap A_1^c \cap A_2)$. Similarly we have equality

$$E \cap A_1^c \cap A_2^c = E \cap (A_1 \cup A_2)^c$$

and hence $\mu^*(E \cap A_1^c \cap A_2^c) = \mu^*(E \cap (A_1 \cup A_2)^c)$. Therefore, we have

$$\mu^*(E) = \mu^*(E \cap (A_1 \cup A_2)) + \mu^*(E \cap (A_1 \cup A_2)^c)$$

Thus we proved that $A_1 \cup A_2 \in \Sigma$. Therefore, Σ_{μ^*} is a family of subsets of X closed under finite unions, complements and containing $\varnothing$. Thus Σ_{μ^*} is an algebra of sets. □

Lemma 3.2.2. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a family of pairwise disjoint subsets of Σ_{μ^*}. Then $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma_{\mu^*}$ and for every subset E of X there is an equality*

$$\mu^*\left(E\cap\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}}\mu^*(E\cap A_n)$$

Proof of the lemma. We prove that $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma_{\mu^*}$. For this observe that we have

$$\mu^*(E) \le \mu^*\left(E\cap\bigcup_{n\in\mathbb{N}} A_n\right) + \mu^*\left(E\smallsetminus\bigcup_{n\in\mathbb{N}} A_n\right) \le \sum_{n\in\mathbb{N}}\mu^*(E\cap A_n) + \mu^*\left(E\smallsetminus\bigcup_{n\in\mathbb{N}} A_n\right) =$$

$$= \lim_{m\to+\infty}\left(\mu^*\left(E\cap\bigcup_{n=0}^{m} A_n\right)+\mu^*\left(E\smallsetminus\bigcup_{n\in\mathbb{N}} A_n\right)\right) \le \lim_{m\to+\infty}\left(\mu^*\left(E\cap\bigcup_{n=0}^{m} A_n\right)+\mu^*\left(E\smallsetminus\bigcup_{n=0}^{m} A_n\right)\right) = \mu^*(E)$$

and the last equality holds, since $\bigcup_{n=0}^{m} A_n \in \Sigma_{\mu^*}$ by Lemma 3.2.1. This implies that we have equalities everywhere above. Hence

$$\mu^*\left(E\cap\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}}\mu^*(E\cap A_n)$$

and $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma_{\mu^*}$. □

Proof of the theorem. Lemma 3.2.1 and Lemma 3.2.2 imply that Σ_{μ^*} is a σ-algebra and statement **(2)** holds. It suffices to verify that statement **(3)** holds. For this pick a subset A of X such that $\mu^*(A) = 0$. Then for every subset E of X we have

$$\mu^*(E) \le \mu^*(E\cap A) + \mu^*(E\smallsetminus A) = \mu^*(E\smallsetminus A) \le \mu^*(E)$$

Hence $\mu^*(E) = \mu^*(E\cap A)+\mu^*(E\smallsetminus A)$ and thus $A\in\Sigma_{\mu^*}$. □

Next result is a general tool of constructing measures.

Theorem 3.3 (Carathéodory extension)**.** *Let X be a set and Σ be some algebra of its subsets. Suppose that $\mu : \Sigma \to [0,+\infty]$ is a σ-additive function. Now for every subset A in X we define*

$$\mu^*(A) = \inf\left\{\sum_{n\in\mathbb{N}}\mu(A_n)\,\middle|\, A_n\in\Sigma \text{ for every } n\in\mathbb{N} \text{ and } A\subseteq\bigcup_{n\in\mathbb{N}} A_n\right\}$$

Then $\mu^* : \mathcal{P}(X)\to[0,+\infty]$ is an outer measure, $\Sigma\subseteq\Sigma_{\mu^*}$ and $\mu^*_{|\Sigma} = \mu$. Moreover, if $\mu(X)$ is finite, then $\mu^*_{|\sigma(\Sigma)}$ is a unique extension of Σ to a measure on $\sigma(\Sigma)$.

Proof. Standard verification shows that μ^* is an outer measure. Note that

$$\mu^*(A) = \inf\left\{\sum_{n\in\mathbb{N}}\mu(A_n)\,\middle|\,\{A_n\}_{n\in\mathbb{N}} \text{ is a family of pairwise disjoint subsets of } \Sigma \text{ and } A\subseteq\bigcup_{n\in\mathbb{N}} A_n\right\}$$

for every subset A of X. Let A be element of A and let E be an arbitrary subset of X. Fix $\epsilon > 0$. By the remark above there exists a family $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint elements of Σ such that

$$E\subseteq\bigcup_{n\in\mathbb{N}} A_n,\ \sum_{n\in\mathbb{N}}\mu(A_n)\le\mu^*(E)+\epsilon$$

By definition of μ^* we have $\mu^*(E\cap A)\le\sum_{n\in\mathbb{N}}\mu(A_n\cap A)$, $\mu^*(E\smallsetminus A)\le\sum_{n\in\mathbb{N}}\mu(A_n\smallsetminus A)$ and hence

$$\mu^*(E)\le\mu^*(E\cap A)+\mu^*(E\smallsetminus A)\le\sum_{n\in\mathbb{N}}\mu(A_n\cap A)+\sum_{n\in\mathbb{N}}\mu(A_n\smallsetminus A) =$$

$$= \sum_{n\in\mathbb{N}}\left(\mu(A_n\cap A)+\mu(A_n\smallsetminus A)\right) = \sum_{n\in\mathbb{N}}\mu(A_n)\le\mu^*(E)+\epsilon$$

Since $\epsilon > 0$ is arbitrary, we derive that $\mu^*(E) = \mu^*(E\cap A)+\mu^*(E\smallsetminus A)$ and hence $A\in\Sigma_{\mu^*}$. Thus $\Sigma\subseteq\Sigma_{\mu^*}$. Once again fix $A\in\Sigma$. Then for every family $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint elements of Σ such that $A\subseteq\bigcup_{n\in\mathbb{N}} A_n$ we have $\mu(A) = \sum_{n\in\mathbb{N}}\mu(A_n\cap A)\le\sum_{n\in\mathbb{N}}\mu(A_n)$ and thus $\mu(A)\le\mu^*(A)$. Obviously $\mu^*(A)\le\mu(A)$. Therefore, for every $A\in\Sigma$ we have $\mu(A) = \mu^*(A)$. Together with

$\Sigma \subseteq \Sigma_{\mu^*}$ this implies that $\mu^*_{|\sigma(\Sigma)}$ is a measure that extends μ. Now we prove the uniqueness of extension under the assumption that $\mu(X)$ is finite. This follows immediately from Theorem 2.6. □

4. Outer metric measures

Definition 4.1. Let X be a topological space. The σ-algebra $\mathcal{B}_X$ generated by all open sets of X is called *the σ-algebra of Borel subsets of X*.

Definition 4.2. Let (X,d) be a metric space and $\mu^* : \mathcal{P}(X) \to [0,+\infty]$ be an outer measure. We say that μ^* is *a metric outer measure* if

$$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$$

for any two subsets E_1, E_2 of X with $\mathrm{dist}(E_1,E_2) = \inf_{x_1 \in E_1,\, x_2 \in E_2} d(x_1,x_2) > 0$.

Theorem 4.3 (Carathéodory)**.** *Let (X,d) be a metric space and $\mu^* : \mathcal{P}(X) \to [0,+\infty]$ be an outer metric measure on X. Then the σ-algebra $\mathcal{B}_X$ of Borel subsets of X is contained in Σ_{μ^*}.*

Proof. Let U be an open subset of X. Define $F = X \smallsetminus U$ and $U_n = \{x \in X \mid \mathrm{dist}(x,F) > \frac{1}{2^n}\}$ for $n \in \mathbb{N}$. Then $\{U_n\}_{n\in\mathbb{N}}$ form an ascending family of open sets and $U = \bigcup_{n\in\mathbb{N}} U_n$. Fix now a subset E of X such that $\mu^*(E) \in \mathbb{R}$. We define $E_n = E \cap U_n$ for every $n \in \mathbb{N}$. Since μ^* is an outer metric measure, we derive that

$$\mu^*\left(\bigcup_{n=0}^{m} E_{2n+1} \smallsetminus E_{2n}\right) = \sum_{n=0}^{m} \mu^*(E_{2n+1} \smallsetminus E_{2n}),\ \mu^*\left(\bigcup_{n=1}^{m} E_{2n} \smallsetminus E_{2n-1}\right) = \sum_{n=1}^{m} \mu^*(E_{2n} \smallsetminus E_{2n-1})$$

for every positive integer m. Thus we derive

$$\sum_{n\in\mathbb{N}} \mu^*(E_{2n+1} \smallsetminus E_{2n}) \leq \mu^*(E) \in \mathbb{R},\ \sum_{n\in\mathbb{N}} \mu^*(E_{2n} \smallsetminus E_{2n-1}) \leq \mu^*(E) \in \mathbb{R}$$

Hence we have $\sum_{n\in\mathbb{N}} \mu^*(E_{n+1} \smallsetminus E_n) \leq 2 \cdot \mu^*(E) \in \mathbb{R}$. Using the fact that μ^* is an outer measure, we derive that

$$\mu(E_m) \leq \mu^*(E \cap U) \leq \mu^*(E_m) + \sum_{n \geq m} \mu^*(E_{n+1} \smallsetminus E_n)$$

for every $m \in \mathbb{N}$. Hence these inequalities yield $\lim_{m\to+\infty} \mu^*(E_m) = \mu^*(E \cap U)$. Now we have $\mu^*(E_m) + \mu^*(E \smallsetminus U) \leq \mu^*(E) \leq \mu^*(E \cap U) + \mu^*(E \smallsetminus U)$ for every $m \in \mathbb{N}$. The first inequality holds due to the fact that μ^* is an outer metric measure. We derive that $\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \smallsetminus U)$. Note that if $\mu^*(E) = +\infty$, then inequality $\mu^*(E) \leq \mu^*(E \cap U) + \mu^*(E \smallsetminus U)$ must be equality. Hence for every subset E of X we have $\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \smallsetminus U)$. This implies that $U \in \Sigma_{\mu^*}$. Since U is an arbitrary open subset of X, we deduce that $\mathcal{B}_X \subseteq \Sigma_{\mu^*}$. □