BOREL MEASURES ON LOCALLY COMPACT SPACES

1. Borel measures on locally compact spaces

For a topological space X we denote by $\mathcal{B}(X)$ the σ-algebra of all open subsets of X.

Definition 1.1. Let X be a Hausdorff topological space and let $\mu : \mathcal{B}(X) \to [0, +\infty]$ be a measure.

(1) If $\mu(K) \in \mathbb{R}$ for every compact subset K of X, then μ is *finite on compact sets*.

(2) Suppose that for every open subset U of X we have

$$\mu(U) = \sup\left\{\mu(K)\,\middle|\,K \text{ compact subset of } X \text{ contained in } U\right\}$$

then μ is *inner regular on open sets*.

(3) Suppose that for every Borel subset A of X we have

$$\mu(A) = \sup\left\{\mu(K)\,\middle|\,K \text{ compact subset of } X \text{ contained in } A\right\}$$

then μ is *inner regular*.

(4) We say that μ is *outer regular* if for every A in $\mathcal{B}(X)$ we have

$$\mu(A) = \inf\left\{\mu(U)\,\middle|\,U \text{ is open in } X \text{ and contains } A\right\}$$

Finally μ is *a regular Borel measure* if it is finite on compact sets, inner regular on open sets and outer regular.

Definition 1.2. Let X be a locally compact space. Then X is σ-compact if there exists a family $\{K_n\}_{n\in\mathbb{N}}$ of compact subsets such that $X = \bigcup_{n\in\mathbb{N}} K_n$.

Theorem 1.3. *Let X be a locally compact space. Let $\mathcal{K}$ be a family of compact subsets of X satisfying the following conditions.*

(1) *$\mathcal{K}$ contains empty set.*

(2) *If K in $\mathcal{K}$ and $U_0, U_1, ..., U_n$ are open subsets of X such that*

$$K \subseteq \bigcup_{n=0}^{k} U_n$$

then there exist $K_0, K_1, ..., K_n$ in $\mathcal{K}$ such that $K_n \subseteq U_n$ for every $n \leq k$ and

$$K = \bigcup_{n=0}^{k} K_n$$

(3) *If K is a compact subset of X, then there exists a compact subset L of $\mathcal{K}$ such that $K \subseteq L$.*

Suppose next that h is a real valued function on $\mathcal{K}$ such that the following assertions hold.

(1) *For every subset K in $\mathcal{K}$ we have $h(K) \geq 0$, $h(\varnothing) = 0$.*

(2) *If $K \subseteq L$ are compact subsets in $\mathcal{K}$, then $h(K) \subseteq h(L)$.*

(3) *If K, L are subsets in $\mathcal{K}$, then*

$$h(K \cup L) \leq h(K) + h(L)$$

and if $K \cap L = \varnothing$, then the equality holds.

For an open subset U of X we define

$$\mu^*(U) = \sup_{K\in\mathcal{K},\, K\subseteq U} h(K)$$

and for arbitrary subset A of X we define

$$\mu^*(A) = \inf\left\{\mu^*(U)\,\middle|\, U \textit{ is an open subset of } X \textit{ containing } A\right\}$$

Then μ^* is a well defined outer measure on X, Borel subsets are μ^*-measurable and $\mu = \mu^*_{|\mathcal{B}(X)}$ is a regular Borel measure. Moreover, if X is σ-compact, then μ is inner regular.

Proof of the theorem. Note that μ^* is well defined. Indeed, if U and V are open subsets of X such that $U \subseteq V$, then $\sup_{K\in\mathcal{K},\, K\subseteq U} h(K) \leq \sup_{K\in\mathcal{K},\, K\subseteq V} h(K)$ and hence it makes sense to define

$$\mu^*(U) = \sup_{K\in\mathcal{K},\, K\subseteq U} h(K)$$

and

$$\mu^*(A) = \inf\left\{\mu^*(U)\,\middle|\, U \text{ is an open subset of } X \text{ containing } A\right\}$$

for arbitrary subset A of X. Now we check that μ^* is an outer measure. By definition and corresponding properties of h we have $\mu^*(\varnothing) = 0$ and μ^* is monotone. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of X such that $\mu^*(A_n) \in \mathbb{R}$ for every $n \in \mathbb{N}$. Fix $\epsilon > 0$ and for each $n \in \mathbb{N}$ we pick an open subset U_n such that $A_n \subseteq U_n$ and

$$\mu^*(U_n) \leq \mu^*(A_n) + \frac{\epsilon}{2^{n+2}}$$

There exists a compact subset $K \in \mathcal{K}$ of $\bigcup_{n\in\mathbb{N}} U_n$ such that

$$\mu^*\left(\bigcup_{n\in\mathbb{N}} U_n\right) \leq h(K) + \frac{\epsilon}{2}$$

Since K is compact, there exists $k \in \mathbb{N}$ such that $K \subseteq \bigcup_{n=0}^{k} U_n$. By property of $\mathcal{K}$ there exist compact sets $K_0, K_1, ..., K_k$ such that $K_n \subseteq U_n$ and $K = \bigcup_{n=0}^{k} K_n$. Thus we have

$$\mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \mu^*\left(\bigcup_{n\in\mathbb{N}} U_n\right) \leq h(K) + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \sum_{n=0}^{k} h(K_n) \leq$$
$$\leq \frac{\epsilon}{2} + \sum_{n\in\mathbb{N}} \mu^*(U_n) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n) + \frac{\epsilon}{2} + \sum_{n\in\mathbb{N}} \frac{\epsilon}{2^{n+2}} = \sum_{n\in\mathbb{N}} \mu^*(A_n) + \epsilon$$

Since ϵ is an arbitrary positive number, we derive that

$$\mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n)$$

Note that this inequality is obvious when there exists $n \in \mathbb{N}$ such that $\mu^*(A_n) = +\infty$. Thus the inequality above holds for arbitrary countable family of subsets of X. Therefore, μ^* is an outer measure. Next we use Carathéodory construction [**?**, Theorem 3.2] and check that Borel sets are μ^*-measurable. For this consider a subset E of X and let U be an open subset of X. We show that

$$\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$$

Clearly the inequality $\leq$ holds and hence if $\mu^*(E) = +\infty$, then the equality holds regardless of U. Thus we may assume that $\mu^*(E) \in \mathbb{R}$. Fix $\epsilon > 0$ and consider open subset V such that $E \subseteq V$ and $\mu^*(V) \leq \mu^*(E) + \frac{\epsilon}{2}$. Next let $K \subseteq U \cap V$ be an element of $\mathcal{K}$ such that $\mu^*(U \cap V) \leq h(K) + \frac{\epsilon}{4}$. Let $L \in \mathcal{K}$ be subset of $V \setminus K$ such that $\mu^*(V \setminus K) \leq \mu^*(L) + \frac{\epsilon}{4}$. We have

$$\mu^*(E) \leq \mu^*(E \cap U) + \mu^*(E \setminus U) \leq \mu^*(V \cap U) + \mu^*(V \setminus U) \leq \mu^*(V \cap U) + \mu^*(V \setminus K) \leq$$
$$\leq \left(h(K) + \frac{\epsilon}{4}\right) + \left(h(L) + \frac{\epsilon}{4}\right) = h(K) + h(L) + \frac{\epsilon}{2} = h(K \cup L) + \frac{\epsilon}{2} \leq \mu^*(V) + \frac{\epsilon}{2} \leq \mu^*(E) + \epsilon$$

and since $\epsilon > 0$ was arbitrary, we derive that

$$\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$$

Hence this equality holds for every subset E of X and every open subset U of X. Thus open subsets of X are μ^*-measurable. Hence $\mathcal{B}(X)$ consists of μ^*-measurable subsets. Next we denote $\mu = \mu^*_{|\mathcal{B}(X)}$. This is a measure. By definition of μ^* measure μ is outer regular. Moreover, for every $K \in \mathcal{K}$ if U is an open subset containing K, then

$$h(K) \leq \mu(K) \leq \mu(U)$$

Thus $\mu(U) = \sup_{K\in\mathcal{K},\,K\subseteq U} \mu(K)$ and μ is inner regular on open sets. Consider open subset U of X such that $\mathbf{cl}(U)$ is compact. Then there exists L in $\mathcal{K}$ such that $\mathbf{cl}(U) \subseteq L$. For every subset $K \subseteq U$ in $\mathcal{K}$ we have $h(K) \leq h(L)$ and hence

$$\mu(U) = \sup_{K\in\mathcal{K},\,K\subseteq U} h(K) \leq h(L) \in \mathbb{R}$$

This proves that every open subset U with compact closure satisfies $\mu(U) \in \mathbb{R}$. Since X is locally compact, this implies that μ is finite on compact sets. Thus μ is a regular Borel measure.
Now we assume that X is σ-compact. Let $X = \bigcup_{n\in\mathbb{N}} K_n$, where K_n is compact for $n \in \mathbb{N}$. We may assume that sequence $\{K_n\}_{n\in\mathbb{N}}$ is nondecreasing. Pick Borel subset A of X. Since μ is outer regular, we derive that

$$\mu(K_n \setminus A) = \inf\left\{\mu\left(U \cap K_n\right) \,\middle|\, U \text{ is an open subset of } X \text{ containing } K_n \setminus A\right\}$$

Thus

$$\mu(K_n \cap A) = \sup\left\{\mu(K) \,\middle|\, K \text{ is a compact subset of } X \text{ contained in } K_n \cap A\right\}$$

We have

$$\mu(A) = \sup_{n\in\mathbb{N}} \mu(K_n \cap A) = \sup_{n\in\mathbb{N}} \left(\sup\left\{\mu(K) \,\middle|\, K \text{ is a compact subset of } X \text{ contained in } K_n \cap A\right\}\right) =$$

$$= \sup\left\{\mu(K) \,\middle|\, K \text{ is a compact subset of } X \text{ contained in } A\right\}$$

Therefore, μ is inner regular. □

Corollary 1.4. *Let X be a locally compact space. Suppose next that $\mathcal{K}$ is the family of all compact subsets of X and $h : \mathcal{K} \to \mathbb{R}$ is a function as in Theorem 1.3. Then the thesis of Theorem 1.3 holds.*

Proof. It suffices to prove if K is a compact subset of a sum $\bigcup_{n=0}^{k} U_n$ of open subsets of X, then there exist compact subsets $K_0, K_1, ..., K_k$ of X such that $K_n \subseteq U_n$ for every $n \leq k$ and $K = \bigcup_{n=0}^{k} K_n$. Let x be a point of K and pick an open neighbourhood U_x of this point such that $\mathbf{cl}(U_x)$ is compact and $U_x \subseteq U_n$ for some n. Since K is compact, there exist $x_1, ..., x_m$ in K such that

$$K \subseteq \bigcup_{i=1}^{m} U_{x_i}$$

Define

$$K_n = K \cap \bigcup_{\left\{i\in\{1,...,m\}\,|\,\mathbf{cl}(U_{x_i})\subseteq U_n\right\}} \mathbf{cl}(U_{x_i})$$

By definition $K_n \subseteq U_n$ for every $n \leq k$ and $K = \bigcup_{n=0}^{k} K_n$. □