RADON-NIKODYM THEOREM, HAHN-JORDAN DECOMPOSITION AND LEBESGUE DECOMPOSITION

1. Introduction

This notes are devoted to some more advanced notions in measure theory. Tools presented here are indispensable in probability theory and statistics. We refer to [Monygham, 2018] for extensive introduction to measure theory.

2. Signed and Complex Measures

In this section we define extension of the usual notion of measure. Denote by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ the topological space obtained as a two-point compactification of the line $\mathbb{R}$. Addition is partially defined operation on $\overline{\mathbb{R}}$ given by the following rules

$$(+\infty) + r = +\infty = r + (+\infty),\ (-\infty) + r = -\infty = r + (-\infty)$$

for every $r \in \mathbb{R}$

Definition 2.1. Let (X, Σ) be a measurable space. *A signed measure on* (X, Σ) is a function $\nu : \Sigma \rightarrow \overline{\mathbb{R}}$ such that $\nu(\emptyset) = 0$ and

$$\nu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \nu(A_n)$$

for every family $\{A_n\}_{n \in \mathbb{N}}$ of pairwise disjoint subsets of Σ. We also say that ν is *a real measure on* (X, Σ) if it is signed measure and takes values in $\mathbb{R}$.

Definition 2.2. Let (X, Σ) be a measurable space. *A complex measure* is a function $\nu : \Sigma \rightarrow \mathbb{C}$ such that $\nu(\emptyset) = 0$ and

$$\nu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \nu(A_n)$$

for every family $\{A_n\}_{n \in \mathbb{N}}$ of pairwise disjoint subsets of Σ.

Definition 2.3. Let (X, Σ) be a measurable space and let μ, ν be two measures (either complex or signed) on (X, Σ). Suppose that for every set A in Σ we have

$$\mu(A) = 0 \Rightarrow \nu(A) = 0$$

Then we write $\nu \ll \mu$ and say that ν is *absolutely continuous with respect to* μ.

Definition 2.4. Let (X, Σ) be a measurable space and let μ, ν be two measures (either complex or signed) on (X, Σ). Suppose that there exists a set $S \in \Sigma$ such that

$$\mu(A \cap S) = 0,\ \nu(A \setminus S) = 0$$

for every $A \in \Sigma$. Then we write $\nu \perp \mu$ and say that ν is *singular with respect to* μ.

3. Hahn-Jordan Decomposition

Theorem 3.1 (Hahn-Jordan decomposition)**.** *Let (X, Σ) be a measurable space and $\nu : \Sigma \rightarrow \overline{\mathbb{R}}$ be a signed measure. Then there exists the unique pair of measures $\nu_+, \nu_- : \Sigma \rightarrow [0, +\infty]$ such that*

$$\nu = \nu_+ - \nu_-$$

and $\nu_+ \perp \nu_-$.

For the proof we need the following notion.

Definition 3.2. Let (X, Σ, ν) be a space with signed measure. A set $A \in \Sigma$ is *positive* if for every subset B of A such that $B \in \Sigma$ we have inequality $\nu(B) \geq 0$.

Lemma 3.2.1. *Let $B \in \Sigma$ be a set such that $\nu(B) \in \mathbb{R}$ and $\nu(B) > 0$. Then there exists a positive set $C \subseteq B$ such that $\nu(B) \leq \nu(C)$.*

Proof of the lemma. First note that all sets $A \in \Sigma$ contained in B have finite measure (we left the proof as an exercise for the reader). For every subset $A \in \Sigma$ contained in B we define

$$\delta(A) = \max\left\{\frac{1}{2}\inf\left\{\nu(D)\,\middle|\,D \text{ is a subset of } A \text{ in } \Sigma\right\}, -1\right\}$$

Note that $\delta(A) \leq 0$. Now we define a sequence $\{D_n\}_{n\in\mathbb{N}}$ of disjoint subsets of B and members of Σ. This is done recursively as follows. If $D_0, ..., D_n$ are defined, then we pick D_{n+1} as a subset of $B \setminus (D_0 \cup ... \cup D_n)$ in Σ such that

$$\nu(D_{n+1}) \leq \delta\big(B \setminus (D_0 \cup ... \cup D_n)\big)$$

Let

$$C = B \setminus \bigcup_{n\in\mathbb{N}} D_n$$

be a subset of B. Clearly $C \in \Sigma$ and for every $n \in \mathbb{N}$ we have

$$\delta(C) \geq \delta\big(B \setminus (D_0 \cup ... \cup D_n)\big)$$

Thus

$$\nu(C) = \nu(B) - \sum_{n\in\mathbb{N}} \nu(D_n) \geq \nu(B) - \sum_{n\in\mathbb{N}} \delta\big(B \setminus (D_0 \cup ... \cup D_n)\big) = \nu(B) - \sum_{n\in\mathbb{N}} \delta(C)$$

Since $\nu(C) \in \mathbb{R}$, we derive that $\delta(C) = 0$. This implies that C is a positive set and $\nu(C) \geq \nu(B)$. □

Proof of the theorem. Assume that for every $A \in \Sigma$ we have $\nu(A) \in \mathbb{R} \cup \{-\infty\}$. Now consider

$$\alpha = \sup\left\{\nu(C)\,\middle|\,C \text{ is positive}\right\}$$

We can find a nondecreasing sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of nonnegative real numbers that converges to α and such that for every $n \in \mathbb{N}$ there exists a positive set C_n with $\nu(C_n) = \alpha_n$. Now pick $P = \bigcup_{n\in\mathbb{N}} C_n$. Obviously P is positive and $\nu(P) = \alpha$. In particular, $\alpha \in \mathbb{R}$. Assume that there exists $B \in \Sigma$ such that $B \subseteq X \setminus P$ and $\nu(B) > 0$. According to Lemma 3.2.1 we deduce that there exists a positive set C inside B such that $\nu(B) \leq \nu(C)$. Then we get

$$\alpha = \nu(P) < \nu(P) + \nu(C) = \nu(P \cup C)$$

and $P \cup C$ is positive. This contradicts the definition of α. Hence for every $B \subseteq X \setminus P$ such that $B \in \Sigma$ we have $\nu(B) \leq 0$. Thus measures

$$\nu_+(A) = \nu(A \cap P),\ \nu_-(A) = -\nu(A \setminus P)$$

defined for $A \in \Sigma$ satisfy the assertion of the theorem. This finishes the proof of the Hahn-Jordan decomposition under the assumption that $\nu(A) \in \mathbb{R} \cup \{-\infty\}$ for all $A \in \Sigma$.
If $\nu(A) \in \mathbb{R} \cup \{+\infty\}$ for every $A \in \Sigma$, then we apply the result above for $-\nu$. Finally the case $\nu(A_1) = -\infty$ and $\nu(A_2) = +\infty$ for some $A_1, A_2 \in \Sigma$ yields to the contradiction. Hence Hahn-Jordan decomposition is proved. □

Corollary 3.3. *Let (X, Σ) be a measurable space and $\nu : \Sigma \to \overline{\mathbb{R}}$ be a signed measure. Then either ν_+ or ν_- is finite.*

Proof. According to Theorem 3.1 we have $\nu = \nu_+ - \nu_-$ and both ν_+, ν_- are nonnegative measures such that $\nu_+ \perp \nu_-$. We cannot have $\nu_+(X) = \nu_-(X) = +\infty$, because then $\nu(X)$ would be undefined in $\overline{\mathbb{R}}$. This implies that either $\nu_+(X) \in \mathbb{R}$ or $\nu_-(X) \in \mathbb{R}$. □

4. Lebesgue decomposition

Definition 4.1. Let (X, Σ) be a measurable space and $\mu : \Sigma \to \overline{\mathbb{R}}$ be a signed measure. We say that μ is σ-finite if there exists a decomposition

$$X = \bigcup_{n\in\mathbb{N}} X_n$$

onto pairwise disjoint elements of Σ such that $\mu(X_n) \in \mathbb{R}$ for every $n \in \mathbb{N}$.

Theorem 4.2 (Lebesgue decomposition)**.** *Let (X, Σ) be a measurable space and let μ be a σ-finite, measure on (X, Σ). Suppose that ν is either a signed and σ-finite measure or a complex measure on (X, Σ). Then there exists a unique decomposition*

$$\nu = \nu_s + \nu_a$$

of measure ν such that $\nu_s \perp \mu$ and $\nu_a \ll \mu$.

Proof. Suppose first that ν is finite measure. Consider

$$\alpha = \sup_{A\in\Sigma,\, \mu(A)=0} \nu(A)$$

Since ν is finite, we derive that $\alpha \in \mathbb{R}$. Consider a sequence $\{A_n\}_{n\in\mathbb{N}}$ such that $A_n \in \Sigma$, $\mu(A_n) = 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} \nu(A_n) = \alpha$. Define $S = \bigcup_{n\in\mathbb{N}} A_n$. Then $\mu(S) = 0$ and $\nu(S) = \alpha$. Moreover, if $A \in \Sigma$ and $A \cap S = \emptyset$, then $\mu(A) = 0$ implies $\nu(A) = 0$. Now we define $\nu_s(A) = \nu(A \cap S)$ and $\nu_a(A) = \nu(A \setminus S)$ for every $A \in \Sigma$. Then $\nu = \nu_s + \nu_a$ and $\nu_s \perp \mu$, $\nu_a \ll \mu$.
Now assume that ν is σ-finite measure. There exists a decomposition

$$X = \bigcup_{n\in\mathbb{N}} X_n$$

onto pairwise disjoint elements of Σ such that $\mu(X_n) \in \mathbb{R}$ for every $n \in \mathbb{N}$. We define $\nu_n(A) = \nu(A \cap X_n)$ for each $n \in \mathbb{N}$ and $A \in \Sigma$. Then ν_n is a finite measure. By the case above we find $\nu_n = \nu_{ns} + \nu_{na}$ and $\nu_{ns} \perp \mu$, $\nu_{na} \ll \mu$ for some measures on Σ. Now we define

$$\nu_s = \sum_{n\in\mathbb{N}} \nu_{ns},\ \nu_a = \sum_{n\in\mathbb{N}} \nu_{an}$$

Then $\nu = \nu_s + \nu_a$ and $\nu_s \perp \mu$, $\nu_a \ll \mu$.
Now consider the case when ν is σ-finite and signed measure. According to Theorem 3.1 we write $\nu = \nu_+ - \nu_-$ for measures ν_+, ν_- such that $\nu_+ \perp \nu_-$. Then ν_+, ν_- are σ-finite measures. According to previous case we can write $\nu_+ = \nu_{+s} + \nu_{+a}$, $\nu_- = \nu_{-s} + \nu_{-a}$ for measures such that $\nu_{+s} \perp \mu, \nu_{-s} \perp \mu, \nu_{+a} \ll \mu, \nu_{-a} \ll \mu$. Then $\nu_s = \nu_{+s} - \nu_{-s}, \nu_a = \nu_{+a} - \nu_{-a}$ are signed measures and $\nu_s \perp \mu, \nu_a \ll \mu$.
Finally assume that ν is complex. Then $\nu = \nu^r + i \cdot \nu^i$, where ν^r and ν^i are finite, signed measures. Form the case above we have decompositions

$$\nu^r = \nu^r_s + \nu^r_a,\ \nu^i = \nu^i_s + \nu^i_s$$

and $\nu^r_s \perp \mu$, $\nu^i_s \perp \mu$, $\nu^r_a \ll \mu$, $\nu^i_a \ll \mu$. Then complex measures

$$\nu_s = \nu^r_s + i \cdot \nu^i_s,\ \nu_a = \nu^r_a + i \cdot \nu^i_a$$

satisfy $\nu_s \perp \mu$, $\nu_a \ll \mu$. □

5. Radon-Nikodym theorem and derivatives

In this section we prove famous result of Radon and Nikodym.

Theorem 5.1 (Radon-Nikodym)**.** *Let (X, Σ) be a measurable space and let μ be a σ-finite, signed measure on (X, Σ). Suppose that $\nu \ll \mu$ for ν that is either complex measure or σ-finite, signed measure. Then there exists a measurable function $f : X \to \mathbb{C}$ such that*

$$\nu(A) = \int_A f d\mu$$

for every $A \in \Sigma$.

Proof for finite measures μ, ν. Fix $n \in \mathbb{N}$ and $k \in \mathbb{N}$. According to Theorem 3.1 there exists a set $P_{n,k} \in \Sigma$ such that

$$\left(\nu - \frac{k}{2^n} \cdot \mu\right)\left(A \cap P_{n,k}\right) \geq 0, \left(\nu - \frac{k}{2^n} \cdot \mu\right)\left(A \setminus P_{n,k}\right) \leq 0$$

for every $A \in \Sigma$. We may assume that $P_{n,0} = X$, $P_{n,k+1} \subseteq P_{n,k}$ and $P_{n,k} = P_{n+1,2k}$ for every $n, k \in \mathbb{N}$. Since $\nu \ll \mu$ and ν is finite, we derive that

$$\mu\left(\bigcap_{k \in \mathbb{N}} P_{n,k}\right) = \nu\left(\bigcap_{k \in \mathbb{N}} P_{n,k}\right) = 0$$

and may assume that this set is empty for every $n \in \mathbb{N}$. Pick $n \in \mathbb{N}$. We define a function $f_n : X \to \mathbb{C}$ by formula

$$f_n(x) = \sum_{k \in \mathbb{N}} \frac{k}{2^n} \cdot \chi_{P_{n,k} \setminus P_{n,k+1}}(x)$$

Clearly f_n is a measurable function with real nonnegative values. If $m, n \in \mathbb{N}$ and $n \leq m$, then we have

$$f_n(x) \leq f_m(x) \leq f_n(x) + \frac{1}{2^n}$$

Thus $\{f_n\}_{n \in \mathbb{N}}$ is a nondecreasing sequence of measurable functions convergent uniformly to a measurable function $f : X \to \mathbb{C}$. Moreover, for every $A \in \Sigma$ and $n \in \mathbb{N}$ we have

$$\nu(A) - \frac{1}{2^n}\mu(A) = \sum_{k \in \mathbb{N}} \nu\left(A \cap \left(P_{n,k} \setminus P_{n,k+1}\right)\right) - \frac{1}{2^n}\mu(A) \leq$$

$$\leq \sum_{k \in \mathbb{N}} \frac{k+1}{2^n}\mu\left(A \cap \left(P_{n,k} \setminus P_{n,k+1}\right)\right) - \frac{1}{2^n}\sum_{k \in \mathbb{N}} \mu\left(A \cap \left(P_{n,k} \setminus P_{n,k+1}\right)\right) \leq$$

$$\leq \sum_{k \in \mathbb{N}} \frac{k}{2^n}\mu\left(A \cap \left(P_{n,k} \setminus P_{n,k+1}\right)\right) \leq \sum_{k \in \mathbb{N}} \nu\left(A \cap \left(P_{n,k} \setminus P_{n,k+1}\right)\right) = \nu(A)$$

and since

$$\int_A f_n \, d\mu = \sum_{k \in \mathbb{N}} \frac{k}{2^n}\mu\left(A \cap \left(P_{n,k} \setminus P_{n,k+1}\right)\right)$$

we derive that

$$\nu(A) - \frac{1}{2^n}\mu(A) \leq \int_A f_n \, d\mu \leq \nu(A)$$

This inequality together with monotone convergence theorem imply that

$$\nu(A) = \lim_{n \to +\infty} \int_A f_n \, d\mu = \int_A f \, d\mu$$

This finishes the proof for finite measures ν, μ. □

Reduction to finite case. Assume now that ν and μ are σ-finite measures on (X, Σ). Then there exists a decomposition

$$X = \bigcup_{n \in \mathbb{N}} X_n$$

onto disjoint subsets in Σ such that $\nu(X_n) \in \mathbb{R}$ and $\mu(X_n) \in \mathbb{R}$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we define $\nu_n(A) = \nu(A \cap X_n)$ and $\mu_n(A) = \mu(A \cap X_n)$ for $A \in \Sigma$. Since $\nu \ll \mu$, we derive that $\nu_n \ll \mu_n$ for every $n \in \mathbb{N}$. Measures $\{\nu_n\}_{n \in \mathbb{N}}$ and $\{\mu_n\}_{n \in \mathbb{N}}$ are finite. By finite case of the theorem we deduce that for each $n \in \mathbb{N}$ there exists a measurable function $f_n : X \to \mathbb{C}$ such that

$$\nu_n(A) = \int_A f_n \, d\mu_n$$

for every $A \in \Sigma$. Note that f_n is real nonnegative and can be set equal to zero outside X_n. Thus

$$\nu_n(A) = \int_A f_n \, d\mu_n = \int_A f_n \, d\mu$$

for every $A \in \Sigma$. Therefore, we deduce that

$$\nu(A) = \sum_{n \in \mathbb{N}} \nu(A \cap X_n) = \sum_{n \in \mathbb{N}} \nu_n(A) = \sum_{n \in \mathbb{N}} \int_A f_n \, d\mu = \int_A \left(\sum_{n \in \mathbb{N}} f_n \right) d\mu$$

by monotone convergence theorem.
Next suppose that μ is a σ-finite, signed measure and ν is a σ-finite measure. According to Theorem 3.1 we may write $\mu = \mu_+ - \mu_-$ for measures μ_+, μ_- such that $\mu_+ \perp \mu_-$. There exists a set $P \in \Sigma$ such that $\mu_-(P) = \mu_+(X \setminus P) = 0$. We define $\nu_1(A) = \nu(A \cap P)$ and $\nu_2(A) = \nu(A \setminus P)$ for every $A \in \Sigma$. Then ν_1, ν_2 are σ-finite measures and $\nu_1 \ll \mu_+$, $\nu_2 \ll \mu_-$. By the case considered above there exist measurable functions $f_+ : X \to \mathbb{C}$, $f_- : X \to \mathbb{C}$ such that

$$\nu_1(A) = \int_A f_+ \, d\mu_+, \; \nu_2(A) = \int_A f_- \, d\mu_-$$

for every $A \in \Sigma$. Moreover, we may assume that f_+ is equal to zero outside P and f_- is equal to zero outside $X \setminus P$. From this we have

$$\nu(A) = \nu(A \cap P) + \nu(A \setminus P) = \nu_1(A) + \nu_2(A) = \int_A f_+ \, d\mu_+ + \int_A f_- \, d\mu_- =$$

$$= \int_A f_+ \, d\mu - \int_A f_- \, d\mu = \int_A (f_+ - f_-) \, d\mu$$

for every $A \in \Sigma$.
Assume now that both μ, ν are σ-finite, signed measures. In this situation we may write $\nu = \nu_+ - \nu_-$ for measures ν_+, ν_- such that $\nu_+ \perp \nu_-$. There exists a set $Q \in \Sigma$ such that $\nu_-(Q) = \nu_+(X \setminus Q) = 0$. We define $\mu_1(A) = \mu(A \cap Q)$ and $\mu_2(A) = \mu(A \setminus P)$ for every $A \in \Sigma$. Then μ_1, μ_2 are σ-finite measures and $\nu_+ \ll \mu_1$, $\nu_- \ll \mu_2$. By the case considered previously there exist measurable functions $f_+ : X \to \mathbb{C}$, $f_- : X \to \mathbb{C}$ such that

$$\nu_+(A) = \int_A f_+ \, d\mu_1, \; \nu_-(A) = \int_A f_- \, d\mu_2$$

for every $A \in \Sigma$. Moreover, we may assume that f_+ is equal to zero outside Q and f_- is equal to zero outside $X \setminus Q$. From this we have

$$\nu(A) = \nu_+(A) + \nu_-(A) = \int_A f_+ \, d\mu_1 + \int_A f_- \, d\mu_2 = \int_A f_+ \, d\mu - \int_A f_- \, d\mu = \int_A (f_+ - f_-) \, d\mu$$

for every $A \in \Sigma$.
Suppose that ν is complex measure. Write $\nu = \nu_r - i \cdot \nu_i$. Then both ν_r, ν_- are finite, signed measures. Moreover, we have $\nu_r \ll \mu, \nu_i \ll \mu$. There exist measurable functions $f_r : X \to \mathbb{C}$ and $f_i : X \to \mathbb{C}$ that are real valued and satisfy

$$\nu_r(A) = \int_A f_r \, d\mu, \; \nu_i(A) = \int_A f_i \, d\mu$$

for every $A \in \Sigma$. Thus

$$\nu(A) = \nu_r(A) + i \cdot \nu_i(A) = \int_A f_r \, d\mu + i \cdot \int_A f_i \, d\mu = \int_A (f_r + i \cdot f_i) \, d\mu$$

for every $A \in \Sigma$. □

Definition 5.2. Let (X, Σ) be a measurable space and let μ, ν be either signed or complex measures on (X, Σ). Suppose that $\nu \ll \mu$. Then a measurable function $f : X \to \mathbb{C}$ that for every $A \in \Sigma$ satisfies

$$\nu(A) = \int_A f d\mu$$

is called *a Radon-Nikodym derivative of ν with respect to μ*. It is sometimes denoted by

$$\frac{d\nu}{d\mu}$$

6. Applications of Radon-Nikodyn, Banach spaces of measures

Theorem 6.1. *Let μ be a complex measure on (X, Σ). Then there exists a measurable function $f : X \to \mathbb{C}$ such that*

$$\mu(A) = \int_A f \, d|\mu|$$

for every $A \in \Sigma$ and $|f(x)| = 1$ for every x in X.

For the proof we need the following result.

Lemma 6.1.1. *Let μ be a measure on (X, Σ). Suppose that $f : X \to \mathbb{C}$ is a measurable function and S is a convex subset of $\mathbb{C}$. Assume that for every $A \in \Sigma$ such that $\mu(A) > 0$, we have*

$$\frac{1}{\mu(A)} \int_A f \, d\mu \in S$$

Then $f(x) \in S$

Proof. $\square$

References

[Monygham, 2018] Monygham (2018). Introduction to measure theory. *github repository: "Monygham/Pedo-mellon-a-minno"*.