**CHARGES AND THEIR INTEGRALS**

## 1. Charges with values in extended real line

**Definition 1.1.** Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu : \Sigma \rightarrow \overline{\mathbb{R}}$ be a function. Suppose that $\mu(\emptyset) = 0$ and

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

for every pair of disjoint sets $A, B \in \Sigma$. Then $\mu$ is *a charge on* $\Sigma$.

**Fact 1.2.** *Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Suppose that $\mu : \Sigma \rightarrow \overline{\mathbb{R}}$ is a charge. Then the image of $\mu$ is not a superset of $\{-\infty, +\infty\}$.*

*Proof.* Left for the reader as an exercise. □

**Example 1.3.** For each $n \in \mathbb{N}_+$ we denote the subset of $\mathbb{N}$ consisting of consecutive numbers from $0$ to $n-1$ by $[n]$. Let $A \subseteq \mathbb{N}$ be a subset. We define *the upper density of $A$* and *the lower density of $A$* as the following numbers respectively

$$\overline{d}(A) = \limsup_{n \rightarrow +\infty} \frac{|A \cap [n]|}{n},\ \underline{d}(A) = \liminf_{n \rightarrow +\infty} \frac{|A \cap [n]|}{n}$$

If $\overline{d}(A) = \underline{d}(A)$ for some $A \subseteq \mathbb{N}$, then we denote their value by $d(A)$ and *the density of $A$*. We set

$$\Sigma = \left\{ A \subseteq \mathbb{N} \,\middle|\, d(A) \text{ exists } \right\}$$

Then $\Sigma$ is an algebra of subsets of $\mathbb{N}$. Moreover, $d$ is a real and nonnegative charge on $\Sigma$.

**Example 1.4.** For the notion of ultrafilter we refer to [Monygham, 2022]. Let $X$ be a set and let $\mathcal{F}$ be an ultrafilter of subsets of $X$ Consider a function given by formula

$$\mu(A) = \begin{cases} 1 & \text{if } A \in \mathcal{F} \\ 0 & \text{otherwise} \end{cases}$$

for every $A \subseteq X$. Then $\mu$ is a $\{0,1\}$-valued charge on the algebra of all subsets of $X$.

**Example 1.5.** Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers such that the series

$$\sum_{n \in \mathbb{N}} a_n$$

is convergent. Let $\Sigma$ be an algebra of all finite and cofinite subsets in $\mathbb{N}$. We define

$$\mu(A) = \sum_{n \in A} a_n$$

for every $A \in \Sigma$. Then $\mu : \Sigma \rightarrow \overline{\mathbb{R}}$ is a charge.

**Definition 1.6.** Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu$ be a charge on $\Sigma$. If $\mu(A) \in \mathbb{R}$ for every $A \in \Sigma$, then $\mu$ is *a real charge on* $\Sigma$.

**Definition 1.7.** Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu$ be a charge on $\Sigma$. If $\mu(A) \in [0, +\infty]$ for every $A \in \Sigma$, then $\mu$ is *a nonnegative charge on* $\Sigma$.

**Definition 1.8.** Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu$ be a charge on $\Sigma$. If there exists $\kappa \in \mathbb{R}$ such that $\mu(A) \geq \kappa$ for every $A \in \Sigma$, then $\mu$ is *bounded from below*.

**Definition 1.9.** Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu$ be a charge on $\Sigma$. If there exists $\kappa \in \mathbb{R}$ such that $\mu(A) \leq \kappa$ for every $A \in \Sigma$, then $\mu$ is *bounded from above*.

**Definition 1.10.** Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu$ be a charge on $\Sigma$. If $\mu$ is bounded from below and from above, then $\mu$ is *bounded*.

**Example 1.11.** Charges defined in Examples 1.3 and 1.4 are real, bounded and nonnegative.

**Example 1.12.** Consider a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that the series

$$\sum_{n\in\mathbb{N}} a_n$$

is convergent, but not absolutely convergent. Then the charge defined by $\{a_n\}_{n\in\mathbb{N}}$ as in Example 1.5 is real but not bounded from below or above.

Now we prove important Jordan decomposition for charges. Our approach closely follows Stanisław Saks [Saks, 1937].

**Theorem 1.13** (Jordan decomposition)**.** *Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu : \Sigma \rightarrow \overline{\mathbb{R}}$ be a charge. For every $A \in \Sigma$ set*

$$\mu_+(A) = \sup\left\{\mu(B)\,\middle|\, B \in \Sigma \text{ and } B \subseteq A\right\},\ \mu_-(A) = \sup\left\{-\mu(B)\,\middle|\, B \in \Sigma \text{ and } B \subseteq A\right\}$$

*Then the following assertions hold.*

**(1)** *$\mu_+$ and $\mu_-$ are nonnegative charges on $\Sigma$.*

**(2)** *For every $A \in \Sigma$ set*

$$|\mu|(A) = \sup\left\{\sum_{P\in\mathbb{P}} |\mu(P)|\,\middle|\, \mathbb{P} \text{ is a finite partition of } A \text{ onto sets in } \Sigma\right\}$$

*Then $|\mu|$ is a nonnegative charge on $\Sigma$ and*

$$|\mu|(A) = \mu_+(A) + \mu_-(A)$$

*for every $A \in \Sigma$.*

**(3)** *If $\mu$ is bounded from below, then $\mu_-$ is a bounded charge and*

$$\mu(A) = \mu_+(A) - \mu_-(A)$$

*for every $A \in \Sigma$.*

**(4)** *If $\mu$ is bounded from above, then $\mu_+$ is a bounded charge and*

$$\mu(A) = \mu_+(A) - \mu_-(A)$$

*for every $A \in \Sigma$.*

*Proof.* We left for the reader the proof of **(1)**.

Fix $A \in \Sigma$. Let $\mathbb{P}$ be a finite partition of $A$ onto a sets in $\Sigma$. Consider families

$$\mathbb{P}_+ = \left\{P \in \mathbb{P}\,\middle|\, \mu(P) > 0\right\},\ \mathbb{P}_- = \left\{P \in \mathbb{P}\,\middle|\, \mu(P) \leq 0\right\}$$

Clearly $\mathbb{P} = \mathbb{P}_+ \cup \mathbb{P}_-$ and $\mathbb{P}_+ \cap \mathbb{P}_- = \emptyset$. Moreover, we have

$$\sum_{P\in\mathbb{P}} |\mu(P)| = \sum_{P\in\mathbb{P}_+} \mu(P) - \sum_{P\in\mathbb{P}_-} \mu(P) = \mu\left(\bigcup_{P\in\mathbb{P}_+} P\right) - \mu\left(\bigcup_{P\in\mathbb{P}_-} P\right) \leq \mu_+(A) + \mu_-(A)$$

and thus $|\mu|(A) \leq \mu_+(A) + \mu_-(A)$ for every $A \in \Sigma$.

Again fix arbitrary $A \in \Sigma$. There exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ of subsets of $A$ contained in $\Sigma$ such that $\mu(B_n) \geq 0$ for every $n \in \mathbb{N}$ and $\{\mu(B_n)\}_{n\in\mathbb{N}}$ is convergent to $\mu_+(A)$. Similarly there exists a sequence $\{C_n\}_{n\in\mathbb{N}}$ of subsets of $A$ contained in $\Sigma$ such that $\mu(C_n) < 0$ for every $n \in \mathbb{N}$ and $\{\mu(C_n)\}_{n\in\mathbb{N}}$ is convergent to $-\mu_-(A)$. For each $n \in \mathbb{N}$ we define

$$\mathcal{S}_n = \big\{B_n \setminus C_n, B_n \cap C_n, C_n \setminus B_n, A \setminus (B_n \cup C_n)\big\}$$

and

$$\tilde{B}_n = \bigcup \big\{S \in \mathcal{S}_n \,|\, \mu(S) > 0\big\},\ \tilde{C}_n = \bigcup \big\{S \in \mathcal{S}_n \,|\, \mu(S) \leq 0\big\}$$

Then $A = \tilde{B}_n \cup \tilde{C}_n$, $\tilde{B}_n \cap \tilde{C}_n = \emptyset$, $\mu(\tilde{B}_n) \geq \mu(B_n)$, $\mu(\tilde{C}_n) \leq \mu(C_n)$ for every $n \in \mathbb{N}$. It follows from inequalities that $\{\mu(\tilde{B}_n)\}_{n\in\mathbb{N}}$ is convergent to $\mu_+(A)$ and $\{\mu(\tilde{C}_n)\}_{n\in\mathbb{N}}$ is convergent to $-\mu_-(A)$.

Now we have

$$\mu_+(A) + \mu_-(A) = \lim_{n\to+\infty}\big(\mu(\tilde{B}_n) - \mu(\tilde{C}_n)\big) = \lim_{n\to+\infty}\big(|\mu(\tilde{B}_n)| + |\mu(\tilde{C}_n)|\big) \leq |\mu|(A)$$

Hence $\mu_+(A) + \mu_-(A) \leq |\mu|(A)$ for every $A \in \Sigma$. This completes the proof of **(2)**.

Now in order to prove **(3)** assume that $\mu$ is bounded from below. Then clearly $\mu_-$ is bounded. Fix $A \in \Sigma$. As above there exist sequences $\{\tilde{B}_n\}_{n\in\mathbb{N}}$ and $\{\tilde{C}_n\}_{n\in\mathbb{N}}$ of subsets of $A$ contained in $\Sigma$ such that $A = \tilde{B}_n \cup \tilde{C}_n$, $\tilde{B}_n \cap \tilde{C}_n = \emptyset$ and

$$\mu_+(A) = \lim_{n\to+\infty}\mu(\tilde{B}_n),\ \mu_-(A) = \lim_{n\to+\infty}\mu(\tilde{C}_n)$$

Using the fact that $\mu_-(A) \in \mathbb{R}$ we derive

$$\mu(A) = \lim_{n\to+\infty}\big(\mu(\tilde{B}_n) + \mu(\tilde{C}_n)\big) = \mu_+(A) - \mu_-(A)$$

Since $A \in \Sigma$ is arbitrary, we deduced **(3)**.

The proof of **(4)** is analogical to the proof of **(3)** and is omited. □

**Example 1.14.** If $\mu$ is the charge from Example 1.12, then for every cofinite $A \subseteq \mathbb{N}$ we have $\mu_+(A) = +\infty$ and $\mu_-(A) = +\infty$. Thus $\mu_+ - \mu_-$ is undefined.

## 2. $\sigma$-ADDITIVE CHARGES AND SIGNED MEASURES

**Definition 2.1.** Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu : \Sigma \to \overline{\mathbb{R}}$ be a charge. Suppose that for every sequence $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\Sigma$ such that

$$\bigcup_{n\in\mathbb{N}} A_n \in \Sigma$$

the equality

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}}\mu(A_n)$$

holds. Then $\mu$ is *a $\sigma$-additive charge on* $\Sigma$.

For the sake of giving a counterexample we first prove the following result.

**Proposition 2.2.** *Let $\Sigma$ be an algebra of subsets of $\mathbb{N}$ which contains each finite subset of $\mathbb{N}$ and a family $\{d\cdot\mathbb{N}\}_{d\in\mathbb{N}_+}$. Suppose that $\mu$ is a charge on $\Sigma$ such that*

$$\mu(d\cdot\mathbb{N}) = \frac{1}{d}$$

*for every $d \in \mathbb{N}_+$. Then $\mu$ is not $\sigma$-additive.*

*Proof.* Suppose that $\mu$ is a charge on $\Sigma$ such that

$$\mu(d\cdot\mathbb{N}) = \frac{1}{d}$$

for every $d \in \mathbb{N}_+$. Assume that $d_1, ..., d_s \in \mathbb{N}_+$ are pairwise coprime. Then inclusion-exclusion principle implies that

$$\mu\left(\bigcup_{k=1}^{s} d_k \cdot \mathbb{N}\right) = 1 - \prod_{i=1}^{s}\left(1 - \frac{1}{d_i}\right)$$

Let $\mathbb{P}$ be the set of all primes. For each $n \in \mathbb{N}_+$ let $\nu_p(n) \in \mathbb{N}$ be the exponent of $p \in \mathbb{P}$ in prime factorization of $n$. Fix now a sequence $\alpha = \{\alpha_p\}_{p \in \mathbb{P}}$ of elements in $\mathbb{N}_+$ such that $\alpha_p = 1$ for all but finitely many $p \in \mathbb{P}$. Consider the set

$$\Gamma_\alpha = \left\{n \in \mathbb{N}_+ \,\middle|\, \nu_p(n) \geq \alpha_p \text{ for some } p \in \mathbb{P}\right\}$$

Clearly $\Gamma_\alpha$ is cofinite and

$$\Gamma_\alpha = \bigcup_{p \in \mathbb{P}} p^{\alpha_p} \cdot \mathbb{N}$$

If $\mu$ is $\sigma$-additive, then

$$\mu(\Gamma_\alpha) = \lim_{N \to +\infty} \mu\left(\bigcup_{p<N} p^{\alpha_p} \cdot \mathbb{N}\right) = 1 - \lim_{N \to +\infty} \prod_{p<N}\left(1 - \frac{1}{p^{\alpha_p}}\right) = 1$$

Now for fixed $n \in \mathbb{N} \cap (1, +\infty)$ we pick $\alpha = \{\alpha_p\}_{p \in \mathbb{P}}$ and $\beta = \{\beta_p\}_{p \in \mathbb{P}}$ such that

$$\alpha_p = \begin{cases} \nu_p(n) & \text{if } \nu_p(n) > 0 \\ 1 & \text{otherwise} \end{cases}$$

for each $p \in \mathbb{P}$ and

$$\beta_p = \begin{cases} \nu_p(n) + 1 & \text{if } \nu_p(n) > 0 \\ 1 & \text{otherwise} \end{cases}$$

Then $\mu(\Gamma_\alpha) = \mu(\Gamma_\beta) = 1$ and hence $\mu(\{n\}) = \mu\left(\Gamma_\alpha \setminus \Gamma_\beta\right) = 0$. This holds for all $n \in \mathbb{N} \cap (1, +\infty)$. Moreover, by $\sigma$-additivity it follows that

$$\mu(\{0\}) = \mu\left(\bigcap_{n \in \mathbb{N}} 2^n \cdot \mathbb{N}\right) = \lim_{n \to +\infty} \mu(2^n \cdot \mathbb{N}) = \lim_{n \to +\infty} \frac{1}{2^n} = 0$$

and hence

$$\mu(2 \cdot \mathbb{N}) = \sum_{n \in \mathbb{N}} \mu(\{2 \cdot n\}) = 0$$

This contradicts the fact that $\mu(2 \cdot \mathbb{N}) \neq 0$. □

**Example 2.3.** Let $d$ be the density charge defined in Example 1.3. Then Proposition 2.2 implies that $d$ is not $\sigma$-additive.

**Proposition 2.4.** *Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu : \Sigma \to \overline{\mathbb{R}}$ be a $\sigma$-additive charge. Then $\mu_+, \mu_-$ and $|\mu|$ are $\sigma$-additive charges.*

*Proof.* Suppose that $\{A_n\}_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint subsets in $\Sigma$ such that

$$A = \bigcup_{n \in \mathbb{N}} A_n \in \Sigma$$

Let $B \in \Sigma$ be a subset of $A$. Since $\mu$ is $\sigma$-additive, we derive

$$\mu(B) = \sum_{n \in \mathbb{N}} \mu(A_n \cap B) \leq \sum_{n \in \mathbb{N}} \mu_+(A_n)$$

Thus $\mu_+(A) \leq \sum_{n\in\mathbb{N}} \mu_+(A_n)$. On the other hand pick a family $\{B_n\}_{n\in\mathbb{N}}$ of sets in $\Sigma$ such that $B_n \subseteq A_n$ and $\mu(B_n) \geq 0$ for each $n \in \mathbb{N}$. Then

$$\sum_{n\in\mathbb{N}} \mu(B_n) = \lim_{N\to+\infty} \sum_{n\leq N} \mu(B_n) = \lim_{N\to+\infty} \mu\left(\bigcup_{n\leq N} B_n\right) \leq \mu^+(A)$$

and hence $\sum_{n\in\mathbb{N}} \mu_+(A_n) \leq \mu_+(A)$. This proves that $\mu_+$ is $\sigma$-additive.

Since $(-\mu)_+ = \mu_-$ and $-\mu$ is $\sigma$-additive, we derive that $\mu_-$ is $\sigma$-additive by the case considered above.

According to Theorem 1.13 we have $|\mu| = \mu_+ + \mu_-$. Hence also $|\mu|$ is $\sigma$-additive. □

**Definition 2.5.** Let $X$ be a set and let $\Sigma$ be a $\sigma$-algebra of its subsets. Let $\mu : \Sigma \to \overline{\mathbb{R}}$ be a $\sigma$-additive charge. Then $\mu$ is *a signed measure on* $\Sigma$.

**Example 2.6.** Measures are defined in [Monygham, 2019]. Note that each measure is a nonnegative, signed measure.

The following notion plays central role in studying structure of signed measures.

**Definition 2.7.** Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu : \Sigma \to \overline{\mathbb{R}}$ be a charge. *A positive set for* $\mu$ is a set $P \in \Sigma$ such that

$$\mu(A \cap P) \geq 0,\ \mu(A \setminus P) \leq 0$$

for every $A \in \Sigma$.

**Example 2.8.** The charge in Example 1.12 does not have positive sets.

The following important result shows the existence of positive sets for signed measures.

**Theorem 2.9** (Hahn)**.** *Let $X$ be a set and let $\Sigma$ be a $\sigma$-algebra of its subsets. Let $\mu : \Sigma \to \overline{\mathbb{R}}$ be a signed measure. Then there exists a positive set for $\mu$.*

The proof proceeds by constructing approximations for a positive set.

**Lemma 2.9.1.** *Let $X$ be a set and let $\Sigma$ be a $\sigma$-algebra of its subsets. Let $\mu : \Sigma \to \overline{\mathbb{R}}$ be a signed measure. Suppose that $\mu(A) \geq 0$ for some $A \in \Sigma$. Then for each $\epsilon > 0$ there exists a subset $Q_\epsilon$ of $A$ such that the following assertions hold.*

- **(1)** *$Q_\epsilon \in \Sigma$ and $\mu(Q_\epsilon) \geq \mu(A)$.*
- **(2)** *If $B \in \Sigma$ and $B \subseteq Q_\epsilon$, then $\mu(B) \geq -\epsilon$.*

*Proof of the lemma.* Let $\mathfrak{F}$ be a family of all sets in $\Sigma$ contained in $A$. For any two sets $F_1, F_2 \in \mathfrak{F}$ we define

$$F_1 \sqsubseteq_\epsilon F_2$$

if and only if $F_2 \subsetneq F_1$ and $\mu(F_1 \setminus F_2) < -\epsilon$. Clearly $\sqsubseteq_\epsilon$ is transitive and antireflexive. Suppose that $\{F_n\}_{n\in\mathbb{N}}$ is a sequence of sets in $\mathfrak{F}$ which is a chain with respect to $\sqsubseteq_\epsilon$. Then

$$\bigcup_{n\in\mathbb{N}} (F_n \setminus F_{n+1}) \in \mathfrak{F}$$

and

$$\mu\left(\bigcup_{n\in\mathbb{N}} (F_n \setminus F_{n+1})\right) = \sum_{n\in\mathbb{N}} \mu(F_n \setminus F_{n+1}) < -\sum_{n\in\mathbb{N}} \epsilon$$

This contradicts the fact that $\mu(A) \geq 0$. Hence there are no infinite chains in $\mathfrak{F}$ with respect to $\sqsubseteq_\epsilon$. Thus there exists $Q_\epsilon \in \mathfrak{F}$ which is maximal with respect to $\sqsubseteq_\epsilon$ and is contained in a chain with respect to $\sqsubseteq_\epsilon$ which starts with $A$. Then $Q_\epsilon$ satisfies assertions. □

**Lemma 2.9.2.** *Let $X$ be a set and let $\Sigma$ be a $\sigma$-algebra of its subsets. Let $\mu : \Sigma \to \overline{\mathbb{R}}$ be a signed measure. Suppose that $\mu(A) > 0$ for some $A \in \Sigma$. Then there exists a subset $Q$ of $A$ such that the following assertions hold.*

**(1)** *$Q \in \Sigma$ and $\mu(Q) \geq \mu(A)$.*

**(2)** *If $B \in \Sigma$ and $B \subseteq Q$, then $\mu(B) \geq 0$.*

*Proof of the lemma.* We define a sequence $\{Q_n\}_{n\in\mathbb{N}}$ of sets in $\Sigma$ which are contained in $A$. We set $Q_0 = A$ and if $Q_n$ is defined for some $n \in \mathbb{N}$, then we pick $Q_{n+1} \subseteq Q_n$ such that $\mu(Q_n) \leq \mu(Q_{n+1})$ and

$$\mu(B) \geq -\frac{1}{n+1}$$

for every $B \in \Sigma$ and $B \subseteq Q_{n+1}$. This construction is possible due to Lemma 2.9.1. Define

$$Q = \bigcap_{n\in\mathbb{N}} Q_n$$

Then $Q \in \Sigma$ and $Q \subseteq A$. Since $\{\mu(Q_n)\}_{n\in\mathbb{N}}$ is nondecreasing and $Q_0 = A$, we derive

$$\mu(A) \leq \lim_{n\to+\infty} \mu(Q_n) = \mu(Q)$$

Now if $B \in \Sigma$ and $B \subseteq Q$, then

$$\mu(B) \geq -\frac{1}{n+1}$$

for every $n \in \mathbb{N}$. Thus $\mu(B) \geq 0$. This proves that $Q$ satisfies assertions. □

*Proof of the theorem.* By Fact 1.2 and changing $\mu$ to $-\mu$ if necessary, we may assume that there is no set $A \in \Sigma$ such that $\mu(A) = +\infty$. Consider the family

$$\mathcal{P} = \left\{Q \in \Sigma \,\middle|\, \mu(B) \geq 0 \text{ for each } B \subseteq Q \text{ such that } B \in \Sigma\right\}$$

Denote by $\alpha$ the least upper bound of $\mu(Q)$ for $Q \in \mathcal{P}$. There exists a sequence $\{Q_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to+\infty} \mu(Q_n) = \alpha$$

Define

$$P = \bigcup_{n\in\mathbb{N}} Q_n$$

Then $P \in \mathcal{P}$ and $\mu(P) = \alpha$. Since by assumption $\mu(P)$ is finite, we derive that $\alpha \in \mathbb{R}$. Assume that there exists a set $A \in \Sigma$ such that $\mu(A) > 0$ and $A \subseteq X \setminus P$. Then by Lemma 2.9.2 there exists $Q \in \mathcal{P}$ such that $Q \subseteq A$ and $\mu(A) \leq \mu(Q)$. Then $Q \cup P \in \mathcal{P}$ and

$$\alpha = \mu(P) < \mu(P) + \mu(Q) = \mu(Q \cup P) \leq \alpha$$

This is a contradiction. Hence $P$ is a positive set for $\mu$. □

**Corollary 2.10.** *Let $X$ be a set and let $\Sigma$ be a $\sigma$-algebra of its subsets. Let $\mu : \Sigma \to \overline{\mathbb{R}}$ be a signed measure. Then $\mu$ is either bounded from below or from above.*

*Proof.* Indeed, let $P \in \Sigma$ be a positive set of $\mu$. Then $\mu_+(X) = \mu(P)$, $\mu_-(X) = \mu(X \setminus P)$ and both cannot be infinite by Fact 1.2. □

## 3. Complex charges and spaces of bounded charges

**Definition 3.1.** Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu : \Sigma \to \mathbb{C}$ be a function. Suppose that $\mu(\emptyset) = 0$ and

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

for every pair of disjoint sets $A, B \in \Sigma$. Then $\mu$ is *a complex charge on* $\Sigma$.

**Remark 3.2.** Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Each real charge on $\Sigma$ is a complex on $\Sigma$.

**Definition 3.3.** Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu : \Sigma \to \mathbb{C}$ be a charge. Suppose that for every sequence $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\Sigma$ such that

$$\bigcup_{n\in\mathbb{N}} A_n \in \Sigma$$

the equality

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

holds. Then $\mu$ is *a $\sigma$-additive complex charge on* $\Sigma$.

**Definition 3.4.** Let $X$ be a set and let $\Sigma$ be a $\sigma$-algebra of its subsets. Let $\mu : \Sigma \to \mathbb{C}$ be a charge. If $\mu$ is $\sigma$-additive, then $\mu$ is *a complex measure on* $\Sigma$.

**Fact 3.5.** *Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu : \Sigma \to \mathbb{C}$ be a charge. For every $A \in \Sigma$ we define*

$$|\mu|(A) = \sup\left\{ \sum_{P\in\mathbb{P}} |\mu(P)| \,\middle|\, \mathbb{P} \text{ is a finite partition of } A \text{ onto sets in } \Sigma\right\}$$

*Then $|\mu|$ is a nonnegative charge on $\Sigma$.*

*Moreover, if $\mu$ is $\sigma$-additive, then also $|\mu|$ is $\sigma$-additive.*

*Proof.* The fact that $|\mu|$ is a charge is left for the reader as an exercise.

Assume now that $\mu$ is $\sigma$-additive. Suppose that $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint subsets in $\Sigma$ such that

$$A = \bigcup_{n\in\mathbb{N}} A_n \in \Sigma$$

Pick a finite partition $\mathbb{P}$ of $A$ onto sets in $\Sigma$. Since $\mu$ is $\sigma$-additive, we derive that

$$\sum_{P\in\mathbb{P}} |\mu(P)| = \sum_{P\in\mathbb{P}} \left| \sum_{n\in\mathbb{N}} \mu(A_n \cap P)\right| \leq$$
$$\leq \sum_{P\in\mathbb{P}} \sum_{n\in\mathbb{N}} |\mu(A_n \cap P)| = \sum_{n\in\mathbb{N}} \sum_{P\in\mathbb{P}} |\mu(A_n \cap P)| \leq \sum_{n\in\mathbb{N}} |\mu|(A_n)$$

This proves that $|\mu|(A) \leq \sum_{n\in\mathbb{N}} |\mu|(A_n)$. On the other hand for each $n \in \mathbb{N}$ pick a finite partition $\mathbb{P}_n$ of $A_n$ onto a sets in $\Sigma$. Then

$$\sum_{n\in\mathbb{N}} \sum_{P\in\mathbb{P}_n} |\mu(P)| = \lim_{N\to+\infty} \sum_{n\leq N} \sum_{P\in\mathbb{P}_n} |\mu(P)| \leq$$
$$\leq \limsup_{N\to+\infty} \left( \sum_{n\leq N} \sum_{P\in\mathbb{P}_n} |\mu(P)| + \left|\mu\left(A \setminus \bigcup_{n\leq N} A_n\right)\right|\right) \leq |\mu|(A)$$

Hence $\sum_{n\in\mathbb{N}} |\mu|(A_n) \leq |\mu|(A)$. This completes the proof of $\sigma$-additivity of $\mu$. $\square$

**Theorem 3.6.** *Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu : \Sigma \to \mathbb{C}$ be a charge. Then the following assertions are equivalent.*

**(i)** *There exists $\kappa \in \mathbb{R}_+$ such that*

$$|\mu(A)| \leq \kappa$$

*for every $A \in \Sigma$.*

**(ii)** *$|\mu|$ is a bounded charge.*

*Proof.* Assume that there exists $\kappa \in \mathbb{R}_+$ such that $|\mu(A)| \leq \kappa$ for every $A \in \Sigma$. For each $A \in \Sigma$ write

$$\mu(A) = \mu_r(A) + \sqrt{-1} \cdot \mu_i(A)$$

where $\mu_r(A), \mu_i(A) \in \mathbb{R}$. Then $\mu_r, \mu_i : \Sigma \to \mathbb{R}$ are real charges and $|\mu_r(A)|, |\mu_i(A)| \leq \kappa$ for every $A \in \Sigma$. Part **(2)** of Theorem 1.13 implies that $|\mu_r|, |\mu_i|$ are bounded. Note that

$$|\mu|(A) \leq |\mu_r|(A) + |\mu_i|(A)$$

for every $A \in \Sigma$. Hence $|\mu|$ is bounded. This proves that **(i)** $\Rightarrow$ **(ii)**.

Suppose now that $|\mu|$ is a bounded charge. Then there exists $\kappa \in \mathbb{R}_+$ such that $|\mu|(A) \leq \kappa$ for every $A \in \Sigma$. Since $|\mu|(A) \leq |\mu|(A)$ for every $A \in \Sigma$, we deduce that $|\mu(A)| \leq \kappa$ for each $A \in \Sigma$. This completes the proof of **(ii)** $\Rightarrow$ **(i)**. $\square$

**Definition 3.7.** Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu : \Sigma \to \mathbb{C}$ be a charge. If $|\mu|$ is bounded, then $\mu$ is *a bounded complex charge on* $\Sigma$.

**Definition 3.8.** Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Let $\mu : \Sigma \to \mathbb{C}$ be a charge. We define

$$\|\mu\| = |\mu|(X)$$

Then $\|\mu\|$ is *the total variation of* $\mu$.

**Theorem 3.9.** *Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Consider the set*

$$\mathrm{ba}(\Sigma, \mathbb{C}) = \big\{\mu : \Sigma \to \mathbb{C} \,\big|\, \mu \text{ is a bounded charge on } \Sigma\big\}$$

*Then the following assertions hold.*

**(1)** $\mathrm{ba}(\Sigma, \mathbb{C})$ *is a $\mathbb{C}$-linear space with respect to canonical operations of addition of charges and multiplication by complex scalars.*

**(2)** *Then*

$$\mathrm{ba}(\Sigma, \mathbb{C}) \ni \mu \mapsto \|\mu\| \in [0, +\infty)$$

*is a norm.*

**(3)** *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to $\|-\|$. Then $\{\mu_n\}_{n\in\mathbb{N}}$ is convergent to some $\mu \in \mathrm{ba}(\Sigma, \mathbb{C})$. Moreover, if $\{\mu_n\}_{n\in\mathbb{N}}$ are $\sigma$-additive, then $\mu$ is $\sigma$-additive.*

**(4)** *Let* $\mathrm{ba}(\Sigma, \mathbb{R})$ *be an $\mathbb{R}$-linear subspace of* $\mathrm{ba}(\Sigma, \mathbb{C})$ *that consists of real bounded charges. Then* $\mathrm{ba}(\Sigma, \mathbb{R})$ *is closed with respect to $\|-\|$.*

*Proof.* Proofs of **(1)** and **(2)** are left for the reader.

Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to $\|-\|$. For every $A \in \Sigma$ and each $n, m \in \mathbb{N}$ we have

$$|\mu_n(A) - \mu_m(A)| \leq \|\mu_n - \mu_m\|$$

Since $\mathbb{C}$ with the usual absolute value is complete, we derive that there exists $\mu(A) \in \mathbb{C}$ such that $\{\mu_n(A)\}_{n\in\mathbb{N}}$ converges to $\mu(A)$. Now pick at most countable family $\mathcal{F}$ of pairwise disjoint sets in $\Sigma$ such that

$$\bigcup_{F \in \mathcal{F}} F \in \Sigma$$

Suppose also that

$$\mu_n(A) = \sum_{F \in \mathcal{F}} \mu_n(F)$$

for every $n \in \mathbb{N}$. We define a measure $u$ on the power set of $\mathcal{F}$ by formula

$$u(Z) = |Z|$$

for every $Z \subseteq \mathcal{F}$. Let $L^1(u, \mathbb{C})$ is a space of complex valued functions defined on $\mathcal{F}$ which are integrable with respect to $u$. In particular, $L^1(u, \mathbb{C})$ is a Banach space over $\mathbb{C}$ with norm

$$\|f\|_1 = \int_{\mathcal{F}} f\, du = \sum_{F \in \mathcal{F}} |f(F)|$$

and integral

$$\int_{\mathcal{F}} f\, du = \sum_{F \in \mathcal{F}} f(F)$$

For the details we refer to [Monygham, 2019]. Since $\|\mu_n\|$ is finite for each $n \in \mathbb{N}$ by Theorem 3.6, we derive that the function $\mathcal{F} \ni F \mapsto \mu_n(F) \in \mathbb{C}$, which we denote by $f_n$, is an element of $L^1(u, \mathbb{C})$ for every $n \in \mathbb{N}$. Moreover, the distance of $f_n$ and $f_m$ in $L^1(u, \mathbb{C})$ is bounded by $\|\mu_n - \mu_m\|$ for all pairs $n, m \in \mathbb{N}$. Hence the sequence $\{f_n\}_{n \in \mathbb{N}}$ is convergent in $L^1(u, \mathbb{C})$. It is also pointwise convergent to a function $\mathcal{F} \ni F \mapsto \mu(F) \in \mathbb{C}$, which we denote by $f$. By general results in [Monygham, 2019] we deduce that $f$ is a limit of $\{f_n\}_{n \in \mathbb{N}}$ in $L^1(\mu, \mathbb{C})$ and from considerations above we have inequality

$$\|f - f_n\|_1 = \lim_{m \to +\infty} \|f_m - f_n\|_1 \leq \limsup_{m \to +\infty} \|\mu_n - \mu_m\|$$

Let us note some consequences of this fact.

- From the convergence of integrals with respect to $u$ we deduce

$$\mu\left(\bigcup_{F \in \mathcal{F}} F\right) = \lim_{n \to +\infty} \mu_n\left(\bigcup_{F \in \mathcal{F}} F\right) = \lim_{n \to +\infty} \sum_{F \in \mathcal{F}} \mu_n(F) = \sum_{F \in \mathcal{F}} \mu(F)$$

- The convergence in $\|-\|_1$ implies that

$$\sum_{F \in \mathcal{F}} |\mu(F)| = \lim_{n \to +\infty} \sum_{F \in \mathcal{F}} |\mu_n(F)| \leq \sup_{n \in \mathbb{N}} \|\mu_n\|$$

- From the inequality above we derive that

$$\sum_{F \in \mathcal{F}} |(\mu - \mu_n)(F)| = \|f - f_n\|_1 \leq \limsup_{m \to +\infty} \|\mu_m - \mu_n\|$$

Note that these assertions hold for every family $\mathcal{F}$ which satisfies the conditions specified above. Hence from the first assertion it follows that $\mu$ is a charge and if $\{\mu_n\}_{n \in \mathbb{N}}$ are $\sigma$-additive, then also $\mu$ is $\sigma$-additive. Next the second statement shows that $\mu$ is bounded. From the last assertion we deduce that $\mu$ is a limit of $\{\mu_n\}_{n \in \mathbb{N}}$ with respect to $\|-\|$. This completes the proof of **(3)**.

The proof of **(4)** follows from the investigation of the proof of **(3)** above. The details are left for the reader. □

**Corollary 3.10.** *Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. Consider the set*

$$\mathrm{bca}(\Sigma, \mathbb{C}) = \left\{ \mu : \Sigma \to \mathbb{C} \,\middle|\, \mu \text{ is a bounded and } \sigma\text{-additive charge on } \Sigma \right\}$$

*Then* $\mathrm{bca}(\Sigma, \mathbb{C})$ *is a* $\mathbb{C}$*-linear subspace of* $\mathrm{ba}(\Sigma, \mathbb{C})$ *closed with respect to total variation norm.*

*Proof.* Closedness follows from Theorem 3.9. The fact that $\mathrm{bca}(\Sigma, \mathbb{C})$ is $\mathbb{C}$-linear subspace of $\mathrm{ba}(\Sigma, \mathbb{C})$ is left as an exercise for the reader. □

**Remark 3.11.** Let $X$ be a set and let $\Sigma$ be an algebra of its subsets. We have the following diagram of Banach spaces and their inclusions.

$$\begin{tikzcd}
 & \mathrm{ba}(\Sigma,\mathbb{C}) & \\
\mathrm{bca}(\Sigma,\mathbb{C}) \arrow[ur, hook] & & \mathrm{ba}(\Sigma,\mathbb{R}) \arrow[ul, hook'] \\
 & \mathrm{ba}(\Sigma,\mathbb{C}) \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

In the diagram $\mathrm{cba}(\Sigma,\mathbb{R})$ is the intersection of $\mathrm{ba}(\Sigma,\mathbb{R})$ and $\mathrm{bca}(\Sigma,\mathbb{C})$ i.e. a Banach space over $\mathbb{R}$ of all real, bounded and $\sigma$-additive charges on $\Sigma$.

## 4. Space of essentially bounded functions

In this section we extend the notion of Lebesgue space to $p = +\infty$. We fix a Banach space $Y$ with norm $\|-\|$ over a field $\mathbb{K}$ with absolute value $|-|$.

**Definition 4.1.** Let $f : X \to Y$ be a strongly measurable function on a space $(X, \Sigma, \mu)$ with measure. Then

$$\|f\|_\infty = \sup\left\{r \in \mathbb{R}_+ \cup \{0\} \,\middle|\, \mu\left(\{x \in X \,|\, \|f(x)\| \geq r\}\right) > 0\right\}$$

is *the essential supremum of $f$ with respect to $\mu$*.

**Proposition 4.2.** *Let $(X, \Sigma, \mu)$ be a space with measure. Then*

**(1)** *If $\alpha \in \mathbb{K}$ and $f : X \to Y$ is a strongly measurable function on $(X, \Sigma)$, then*

$$\|\alpha \cdot f\|_\infty = |\alpha| \cdot \|f\|_\infty$$

**(2)** *If $f, g : X \to Y$ are strongly measurable functions on $(X, \Sigma)$, then*

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

*Proof.* Fix $\alpha \in \mathbb{K} \setminus \{0\}$ and $f : X \to Y$ be a strongly measurable function on $(X, \Sigma)$. Then

$$\{x \in X \,|\, \|(\alpha \cdot f)(x)\| \geq r\} = \left\{x \in X \,\middle|\, \|f(x)\| \geq \frac{r}{|\alpha|}\right\}$$

for every $r \in \mathbb{R}_+ \cup \{0\}$. Hence

$$\|\alpha \cdot f\|_\infty = \sup\left\{r \in \mathbb{R}_+ \cup \{0\} \,\middle|\, \mu\left(\{x \in X \,|\, \|(\alpha \cdot f)(x)\| \geq r\}\right) > 0\right\} =$$

$$= \sup\left\{r \in \mathbb{R}_+ \cup \{0\} \,\middle|\, \mu\left(\left\{x \in X \,\middle|\, \|f(x)\| \geq \frac{r}{|\alpha|}\right\}\right) > 0\right\} =$$

$$= |\alpha| \cdot \sup\left\{r \in \mathbb{R}_+ \cup \{0\} \,\middle|\, \mu\left(\{x \in X \,|\, \|f(x)\| \geq r\}\right) > 0\right\} = |\alpha| \cdot \|f\|_\infty$$

It follows that

$$\|\alpha \cdot f\|_\infty = |\alpha| \cdot \|f\|_\infty$$

for every $\alpha \in \mathbb{K} \setminus \{0\}$. For $\alpha = 0$ this also holds for trivial reasons. Hence **(1)** is proved.

Suppose that $f, g : X \to Y$ are strongly measurable functions on $(X, \Sigma)$. Assume that $r \in \mathbb{R}_+$ is such that

$$\|f\|_\infty + \|g\|_\infty < r$$

We may pick $r_f, r_g \in \mathbb{R}_+$ such that $r_f + r_g = r$ and $\|f\|_\infty < r_f$ and $\|g\|_\infty < r_g$. Then

$$\{x \in X \,|\, \|(f+g)(x)\| \geq r\} \subseteq \{x \in X \,|\, \|f(x)\| + \|g(x)\| \geq r_f + r_g\} \subseteq$$
$$\subseteq \{x \in X \,|\, \|f(x)\| \geq r_f\} \cup \{x \in X \,|\, \|g(x)\| \geq r_g\}$$

Since $\|f\|_\infty < r_f$ and $\|g\|_\infty < r_g$, we deduce that

$$\mu\left(\{x \in X \,|\, \|f(x)\| \geq r_f\}\right) = \mu\left(\{x \in X \,|\, \|g(x)\| \geq r_g\}\right) = 0$$

This implies that

$$\mu\left(\{x \in X \,|\, \|(f+g)(x)\| \geq r\}\right) = 0$$

and thus $\|f+g\|_\infty < r$. This proves that

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

if right hand side is finite. Clearly the inequality holds if the right hand side is infinite. This completes the proof of **(2)**. □

**Definition 4.3.** Let $f : X \to Y$ be a strongly measurable function on a space $(X, \Sigma, \mu)$ with measure. If

$$\|f\|_\infty \in \mathbb{R}$$

then $f$ is *essentially bounded with respect to* $\mu$ or shortly $\mu$*-essentially bounded*.

**Definition 4.4.** Let $(X, \Sigma, \mu)$ be a space with measure. Then the set of all $Y$-valued and $\mu$-essentially bounded functions is denoted by $L^\infty(\mu, Y)$ and is called *the Lebesgue space of* $\mu$*-essentially bounded functions for* $Y$.

**Corollary 4.5.** *Let* $(X, \Sigma, \mu)$ *be a space with measure. Then* $L^\infty(\mu, Y)$ *is a* $\mathbb{K}$*-vector subspace of the* $\mathbb{K}$*-vector space of all strongly measurable functions on* $(X, \Sigma)$ *and*

$$\|-\|_\infty : L^\infty(\mu, Y) \to \mathbb{R}_+ \cup \{0\}$$

*is a seminorm.*

*Proof.* This follows immediately from Proposition 4.2. □

**Theorem 4.6** (Riesz)**.** *Let* $(X, \Sigma, \mu)$ *be a space with measure and let* $\{f_n : X \to Y\}_{n \in \mathbb{N}}$ *be a Cauchy sequence of elements of* $L^\infty(\mu, Y)$*. Then* $\{f_n\}_{n \in \mathbb{N}}$ *converges in* $L^\infty(\mu, Y)$*.*

*Proof.* Consider an increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ of natural numbers such that

$$\|f_n - f_m\|_\infty \leq 2^{-k}$$

for every $n, m \geq n_k$ and for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ sets

$$A_k = \bigcup_{n=n_k}^{+\infty} \bigcup_{m=n_k}^{+\infty} \{x \in X \,|\, \|f_n(x) - f_m(x)\| > 2^{-k}\}$$

and

$$B_k = \{x \in X \,|\, \|f_k(x)\| > \|f_k\|_\infty\}$$

are in $\Sigma$ and have measure $\mu$ equal to zero. Hence

$$A = \bigcup_{k \in \mathbb{N}} (A_k \cup B_k)$$

have measure $\mu$ equal to zero. Now $\{f_{n|X \setminus A}\}_{n \in \mathbb{N}}$ is a sequence of bounded functions which is Cauchy with respect to uniform norm. Since $Y$ is complete with respect to $\|-\|$, sequence $\{f_{n|X \setminus A}\}_{n \in \mathbb{N}}$ converges uniformly to some function $X \setminus A \to Y$. We extend this function to a function $f : X \to Y$ by setting it equal to zero on $A$. Note that $f$ is strongly measurable by

Proposition **??**. Moreover, $\{f_{n|X\setminus A}\}_{n\in\mathbb{N}}$ converges uniformly to $f_{|X\setminus A}$. Thus $f_{X\setminus A}$ is bounded and hence $f \in L^\infty(\mu, Y)$. For the same reason $f$ is a limit of $\{f_n\}_{n\in\mathbb{N}}$ in $L^\infty(\mu, Y)$. $\square$

## 5. Integration with respect to charges

## References

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