HAHN-BANACH THEOREM

1. Introduction

In this notes we study Hahn-Banach theorem and its consequences. Our main goal is separation theorem for normed spaces.
Throughout the notes $\mathbb{K}$ is either topological field $\mathbb{R}$ or topological field $\mathbb{C}$.

2. Hahn-Banach theorem

We start by introducing certain notions concerning real maps defined on $\mathbb{R}$-vector spaces.

Definition 2.1. Let V be an $\mathbb{R}$-vector space. A map $p : V \to \mathbb{R}$ is *subadditive* if

$$p(v_1 + v_2) \leq p(v_1) + p(v_2)$$

for any vectors v_1, v_2 in V.

Definition 2.2. Let V be an $\mathbb{R}$-vector space. A map $p : V \to \mathbb{R}$ is *positive homogeneous* if

$$p(\alpha \cdot v) = \alpha \cdot p(v)$$

for every $\alpha \in \mathbb{R}_+$ and every v in V.

The following is central result of these notes.

Theorem 2.3 (Hahn-Banach)**.** *Let V be an $\mathbb{R}$-vector space and let $p : V \to \mathbb{R}$ be a subadditive and positive homogeneous map. Suppose that W is an $\mathbb{R}$-subspace of V and $f : W \to \mathbb{R}$ is an $\mathbb{R}$-linear map such that*

$$f(w) \leq p(w)$$

for every w in W. Then there exists $\mathbb{R}$-linear map $\tilde{f} : V \to \mathbb{R}$ such that $\tilde{f}_{|W} = f$ and $\tilde{f}(v) \leq p(v)$ for every v in V.

The heart of the proof is the following result.

Lemma 2.3.1. *Let V be an $\mathbb{R}$-vector space and let $p : V \to \mathbb{R}$ be a subadditive and positive homogeneous map. Suppose that W is an $\mathbb{R}$-subspace of V and $f : W \to \mathbb{R}$ is an $\mathbb{R}$-linear map such that*

$$f(w) \leq p(w)$$

for every w in W. Then for every vector $\tilde{v} \in V \setminus W$ there exists $\mathbb{R}$-linear map $\tilde{f} : W + \mathbb{R} \cdot \tilde{v} \to \mathbb{R}$ such that $\tilde{f}_{|W} = f$ and $\tilde{f}(v) \leq p(v)$ for every v in $W + \mathbb{R} \cdot \tilde{v}$.

Proof of the lemma. We claim that the set of $\lambda \in \mathbb{R}$ such that for every $\gamma \in \mathbb{R}$ and every $w \in W$ the following condition is satisfied

$$f(w) + \gamma \cdot \lambda \leq p\big(w + \gamma \cdot \tilde{v}\big)$$

is nonempty. In order to prove this we analyze this condition. Note that for $\gamma = 0$ the condition holds by assumption of the theorem. Thus we may assume that $\gamma \neq 0$. Let $\alpha = |\gamma|$. Now we consider two cases.

- For $\gamma > 0$ the condition is equivalent to

$$\lambda \leq p\left(\frac{w}{\alpha} + \tilde{v}\right) - f\left(\frac{w}{\alpha}\right)$$

  Since W is an $\mathbb{R}$-vector space, it can be equivalently stated as

$$\lambda \leq p\left(w + \tilde{v}\right) - f\left(w\right)$$

for every $w \in W$.

- For $\gamma < 0$ the condition is equivalent to
$$-p\left(\frac{w}{\alpha} - \tilde{v}\right) + f\left(\frac{w}{\alpha}\right) \leq \lambda$$
We invoke the fact that W is an $\mathbb{R}$-vector space one again and obtain equivalent condition
$$-p\left(w - \tilde{v}\right) + f\left(w\right) \leq \lambda$$
for every $w \in W$.

Thus in order to prove our claim it suffices to prove that
$$\sup_{w\in W} -p\left(w - \tilde{v}\right) + f\left(w\right) \leq \inf_{w\in W} p\left(w + \tilde{v}\right) - f\left(w\right)$$
Therefore, it suffices to prove that
$$p\left(w_1 - \tilde{v}\right) + f\left(w_1\right) \leq p\left(w_2 + \tilde{v}\right) - f\left(w_2\right)$$
for any $w_1, w_2 \in W$. Fix arbitrary $w_1, w_2 \in W$. The inequality
$$p\left(w_1 - \tilde{v}\right) + f\left(w_1\right) \leq p\left(w_2 + \tilde{v}\right) - f\left(w_2\right)$$
is equivalent to
$$f(w_1 + w_2) \leq p\left(w_2 + \tilde{v}\right) + p\left(w_1 - \tilde{v}\right)$$
which holds according to
$$f(w_1 + w_2) \leq p(w_1 + w_2) = p\left(w_2 + \tilde{v} + w_1 - \tilde{v}\right) \leq p\left(w_2 + \tilde{v}\right) + p\left(w_1 - \tilde{v}\right)$$
Thus the claim is proved. We infer the statement from the claim as follows. Pick $\lambda \in \mathbb{R}$ such that
$$f(w) + \gamma \cdot \lambda \leq p\big(w + \gamma \cdot \tilde{v}\big)$$
for every $\gamma \in \mathbb{R}$ and every $w \in W$. Then define $\tilde{f} : W + \mathbb{R} \cdot \tilde{v} \to \mathbb{R}$ by $\tilde{f}(w + \gamma \cdot \tilde{v}) = f(w) + \gamma \cdot \lambda$ for every $w \in W$ and $\gamma \in \mathbb{R}$. Then $\tilde{f}$ satisfies the assertion. □

Proof of the theorem. Consider the family $\mathcal{G}$ which consists of $\mathbb{R}$-linear maps $g : U \to \mathbb{R}$ such that U is a $\mathbb{R}$-subspace of V containing W, $g_{|W} = f$ and $g(u) \leq p(u)$ for every $u \in U$. For $g_1 : U_1 \to \mathbb{R}$ and $g_2 : U_2 \to \mathbb{R}$ in $\mathcal{G}$ we define $g_1 \preceq g_2$ if and only if $U_1 \subseteq U_2$ and $(g_2)_{|U_1} = g_1$. Clearly $\preceq$ is a partial order on $\mathcal{G}$. By Zorn's lemma there exists element $\tilde{f} : \tilde{V} \to \mathbb{R}$ in $\mathcal{G}$ maximal with respect to $\preceq$. If $\tilde{V} \subsetneq V$, then by Lemma 2.3.1 there exists element of $\mathcal{G}$ greater than $\tilde{f}$ with respect to $\preceq$. This is a contradiction. Hence $\tilde{V} = V$ and $\tilde{f}$ satisfies the assertion of the theorem. □

We note here an immediate consequence of Hahn-Banach theorem.

Corollary 2.4. *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a $\mathbb{K}$-vector space and let $||-||$ be a seminorm on V. Suppose that $f : W \to \mathbb{K}$ is a $\mathbb{K}$-linear functional defined on some $\mathbb{K}$-vector subspace W of V. Assume that there exists $c \in \mathbb{R}_+$ such that*
$$|f(w)| \leq c \cdot ||w||$$
for every $w \in W$. Then there exists a $\mathbb{K}$-linear map $\tilde{f} : V \to \mathbb{K}$ such that $\tilde{f}_{|W} = f$ and
$$|\tilde{f}(v)| \leq c \cdot ||v||$$
for every $v \in V$.

For the proof we need the following notation. Let V be a $\mathbb{C}$-vector space and let $f : V \to \mathbb{C}$ be a $\mathbb{C}$-linear map. For each v in V we define
$$\left(\mathrm{Re}f\right)(v) = \mathrm{Re}\left(f(v)\right)$$
Clearly $\mathrm{Re}f : V \to \mathbb{R}$ is an $\mathbb{R}$-linear map. The following result shows that f is determined by $\mathrm{Re}f$.

Lemma 2.4.1. *Let V be a $\mathbb{C}$-vector space and let $\|-\|$ be a seminorm on V. Suppose that $f : V \to \mathbb{C}$ is a $\mathbb{C}$-linear map which is continuous with respect to the topology induced by $\|-\|$. Then*

$$f(v) = (\mathrm{Re} f)(v) - i \cdot (\mathrm{Re} f)(i \cdot v)$$

and

$$\sup_{v \in V, \|v\| \leq 1} |f(v)| = \sup_{v \in V, \|v\| \leq 1} \|(\mathrm{Re} f)(v)\|$$

Proof of the lemma. For every v in V we have

$$(\mathrm{Re} f)(i \cdot v) = \mathrm{Re}(f(i \cdot v)) = \mathrm{Re}(i \cdot f(v)) = -\mathrm{Im}(f(v))$$

Thus

$$\mathrm{Im}(f(v)) = -(\mathrm{Re} f)(i \cdot v)$$

and hence

$$f(v) = (\mathrm{Re} f)(v) - i \cdot (\mathrm{Re} f)(i \cdot v)$$

This completes the proof of the first part of the assertion. In order to prove the second part for each $v \in V$ such that $\|v\| \leq 1$ define $\alpha_v \in \mathbb{C}$ such that $\alpha_v \cdot f(v) = |f(v)|$. Then

$$\alpha_v \in \left\{ z \in \mathbb{C} \,\middle|\, |z| = 1 \right\} \cup \{0\}$$

and $\alpha_v \cdot f(v) = |(\mathrm{Re} f)(\alpha_v \cdot v)|$ for each v. We have

$$\sup_{v \in V, \|v\| \leq 1} \left| (\mathrm{Re} f)(v) \right| \leq \sup_{v \in V, \|v\| \leq 1} \left| f(v) \right| = \sup_{v \in V, \|v\| \leq 1} \alpha_v \cdot f(v) =$$

$$= \sup_{v \in V, \|v\| \leq 1} f(\alpha_v \cdot v) = \sup_{v \in V, \|v\| \leq 1} |(\mathrm{Re} f)(\alpha_v \cdot v)| \leq \sup_{v \in V, \|v\| \leq 1} \left| (\mathrm{Re} f)(v) \right|$$

□

Proof of the theorem. The case $\mathbb{K} = \mathbb{R}$ follows directly from Theorem 2.3. If $\mathbb{K} = \mathbb{C}$, then we apply Theorem 2.3 in order to obtain $\mathbb{R}$-linear map $g : V \to \mathbb{R}$ such that $g_{|W} = \mathrm{Re} f$ and

$$\sup_{v \in V, \|v\| \leq 1} |g(v)| = \sup_{w \in W, \|w\| \leq 1} |(\mathrm{Re} f)(w)|$$

Next we define $\tilde{f}(v) = g(v) - i \cdot g(i \cdot v)$ for every $v \in V$. Then it is easy to see that $\tilde{f} : V \to \mathbb{C}$ is $\mathbb{C}$-linear. Moreover, by Lemma 2.4.1 we have $\tilde{f}_{|W} = f$ and

$$\sup_{v \in V, \|v\| \leq 1} |\tilde{f}(v)| = \sup_{v \in V, \|v\| \leq 1} |g(v)| = \sup_{w \in W, \|w\| \leq 1} |(\mathrm{Re} f)(w)| = \sup_{w \in W, \|w\| \leq 1} |f(w)| \leq c$$

Hence

$$|\tilde{f}(v)| \leq c \cdot \|v\|$$

for every $v \in V$. Thus $\tilde{f}$ satisfies the assertion. □

3. Hyperplane separation theorem

Definition 3.1. Let V be an $\mathbb{R}$-vector space and let K be its subset. Suppose that for every $v \in V$ there exists $r \in \mathbb{R}_+$ such that $v \in r \cdot K$. Then K is *absorbent subset of V*.

Definition 3.2. Let V be an $\mathbb{R}$-vector space and let K be its subset. For every v in V we define

$$p_K(v) = \inf \left\{ r \in \mathbb{R}_+ \,\middle|\, v \in r \cdot K \right\}$$

Then $p_K : V \to [0, +\infty]$ is *the Minkowski functional of K*.

Minkowski functionals are extensively studied in functional analysis. Here we limit our study to the following results.

Fact 3.3. *Let V be an $\mathbb{R}$-vector space and let K be an absorbent subset of V. Then $p_K(v)$ is finite for every v in V.*

Proof. Left for the reader as an exercise. □

Proposition 3.4. *Let V be an $\mathbb{R}$-vector space and let K be convex and absorbent subset of V. Then the Minkowski functional $p_K : V \to [0,+\infty)$ is subadditive and positive homogeneous.*

Proof. Pick $\alpha \in \mathbb{R}_+$ and $v \in V$. We have

$$\alpha \cdot \left\{r \in \mathbb{R}_+ \,\middle|\, v \in r \cdot K\right\} = \left\{r \in \mathbb{R}_+ \,\middle|\, \alpha \cdot v \in r \cdot K\right\}$$

This implies that $p_K(\alpha \cdot v) = \alpha \cdot p_K(v)$ and hence p_K is positive homogeneous.
Next fix $v, w \in V$ and consider $r, t \in \mathbb{R}_+$ such that $v \in r \cdot K$ and $w \in t \cdot K$. Thus there exist $x, y \in K$ such that $v = r \cdot x$ and $w = t \cdot y$. Then

$$(v + w) = r \cdot x + t \cdot y = (r + t) \cdot \left(\frac{r}{r+t} \cdot v + \frac{t}{r+t} \cdot w\right)$$

and

$$\frac{r}{r+t} \cdot v + \frac{t}{r+t} \cdot w \in K$$

since K is convex. Therefore, we have $v + w \in (r + t) \cdot K$. This implies that

$$p_K(v + w) \leq r + t$$

Since $r, t \in \mathbb{R}_+$ are arbitrary numbers such that $v \in r \cdot K$ and $w \in t \cdot K$, we infer that $p_K(v + w) \leq p_K(v) + p_K(w)$. Thus p_K is subadditive. □

4. Preliminaries on topological vector spaces

Definition 4.1. Let $\mathfrak{X}$ be a vector space over $\mathbb{K}$ equipped with some topology. Suppose that the multiplication by scalars $\mathbb{K} \times \mathfrak{X} \to \mathfrak{X}$ and the addition $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ are continuous. Then $\mathfrak{X}$ is *a topological vector space over* $\mathbb{K}$.

Example 4.2. $\mathbb{K}^n$ admits a canonical structure of a topological vector space over $\mathbb{K}$ determined by the structure of topological field $\mathbb{K}$.

Definition 4.3. Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. A subset B of $\mathfrak{X}$ is *balanced* if $\alpha \cdot B \subseteq B$ for every $\alpha \in \mathbb{K}$ such that $|\alpha| \leq 1$.

Fact 4.4. *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. Then $\mathfrak{X}$ admits a local topological base at zero which consists of balanced sets.*

Proof. Left for the reader as an exercise. □

Definition 4.5. Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. A subset B of $\mathfrak{X}$ is *bounded* if for every open neighborhood U of zero in $\mathfrak{X}$ there exists $r \in \mathbb{R}_+$ such that $B \subseteq r \cdot U$.

Definition 4.6. Let $\mathfrak{X}, \mathfrak{Y}$ are topological vector spaces over $\mathbb{K}$. A map $f : \mathfrak{X} \to \mathfrak{Y}$ which is both continuous and $\mathbb{K}$-linear is *a morphism of topological vector spaces over* $\mathbb{K}$.

5. Finite dimensional Hausdorff topological vector spaces

We prove the following elementary but important result.

Proposition 5.1. *Let $f : \mathfrak{X} \to \mathbb{K}$ be a $\mathbb{K}$-linear map between topological vector spaces over $\mathbb{K}$. Then the following are equivalent.*

- **(i)** *f is continuous.*
- **(ii)** $\ker(f)$ *is a closed subspace of $\mathfrak{X}$.*
- **(iii)** *Either f is the zero map or* $\ker(f)$ *is not dense in $\mathfrak{X}$.*

(iv) *There exists open neighborhood U of zero in $\mathfrak{X}$ such that $f(U)$ is bounded subset of $\mathbb{K}$.*

(v) *f is continuous at zero.*

Proof. The implications **(i)** $\Rightarrow$ **(ii)** and **(ii)** $\Rightarrow$ **(iii)** are obvious.
If f is the zero map, then **(iv)** holds. Assume that $f(U)$ is unbounded for every open neighborhood U of zero in $\mathfrak{X}$. Let $\mathcal{U}$ be a local topological base of $\mathfrak{X}$ at zero which consists of balanced sets (Fact 4.4). For every $U \in \mathcal{U}$ the set $f(U)$ is balanced and unbounded in $\mathbb{K}$. Thus $f(U) = \mathbb{K}$ for every $U \in \mathcal{U}$. Consider now an open subset W of $\mathfrak{X}$ and pick a point x in W. Let U be a set in $\mathcal{U}$ such that $x + U \subseteq W$. There exists $y \in U$ such that $f(y) = f(x)$. Since U is balanced, we have $-y \in U$ and hence $x - y \in x + U$. Therefore, we have $x - y \in W$ and $f(x - y) = 0$. This implies that $\ker(f)$ is dense in $\mathfrak{X}$. By contraposition we infer that if $\ker(f)$ is not dense in $\mathfrak{X}$, then **(iv)** holds. This completes the proof of **(iii)** $\Rightarrow$ **(iv)**.
Suppose that $f(U)$ is bounded subset of $\mathbb{K}$, where U is some open neighborhood of zero in $\mathfrak{X}$. Let V be an open neighborhood of zero in $\mathbb{K}$. Then there exists $\alpha \in \mathbb{R}_+$ such that

$$f(\alpha \cdot U) = \alpha \cdot f(U) \subseteq V$$

This shows that f is continuous at zero and hence the implication **(iv)** $\Rightarrow$ **(v)** holds.
Finally suppose that f is continuous at zero. Since it is additive, we derive that it is continuous. Thus **(v)** $\Rightarrow$ **(i)**. $\square$

Fact 5.2. *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. Suppose that $f : \mathbb{K}^n \to \mathfrak{X}$ is a $\mathbb{K}$-linear map for some $n \in \mathbb{N}$. Then f is continuous.*

Proof. Let $\{e_1, ..., e_n\}$ be the canonical basis of $\mathbb{K}^n$. For every i let $pr_i : \mathbb{K}^n \to \mathbb{K}$ be the projection onto i-th axis and let $m_i : \mathbb{K} \to \mathfrak{X}$ be the composition of the multiplication of scalars $\mathbb{K} \times \mathfrak{X} \to \mathfrak{X}$ with the continuous embedding $\mathbb{K} \ni \alpha \mapsto (\alpha, f(e_i)) \in \mathbb{K} \times \mathfrak{X}$. Since pr_i and m_i are continuous for each i, we derive that their compositions $m_i \cdot pr_i$ are also continuous. According to the fact that the addition $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ is continuous, we infer that the sum

$$\sum_{i=1}^{n} m_i \cdot pr_i$$

is continuous. This sum is equal to f. Thus f is continuous. $\square$

Corollary 5.3. *Let $\mathfrak{X}$ be a topological vector space over $\mathbb{K}$. If $\mathfrak{X}$ is Hausdorff and of dimension n for some $n \in \mathbb{N}$, then $\mathfrak{X}$ is isomorphic with $\mathbb{K}^n$.*

Proof. There exists $\mathbb{K}$-linear isomorphism $f : \mathbb{K}^n \to \mathfrak{X}$. Fact 5.2 shows that f is continuous. For each $i \in \{1, ..., n\}$ let $pr_i : \mathbb{K}^n \to \mathbb{K}$ be the projection. According to Proposition 5.1 we derive that $pr_i \cdot f^{-1}$ $\square$

6. Completeness of topological groups

In this section we study some results concerning generalization of completeness for metric spaces to arbitrary topological groups.

Definition 6.1. Let G be a topological group and let Σ be a directed set. Suppose that $\{x_i\}_{i \in \Sigma}$ is a net in G. If for every open neighborhood V of unit in G there exists $k \in \Sigma$ such that

$$x_i \cdot x_j^{-1} \in V$$

for each $i, j \geq k$, then $\{x_i\}_{i \in \Sigma}$ is *a Cauchy's net in G*.

Definition 6.2. Let G be a topological group. Suppose that every Cauchy net in G is convergent. Then G is *a complete topological group*.

Fact 6.3. *Let G be a complete topological group. Then G is Hausdorff.*

Proof. □