

FIBERED CATEGORIES AND EQUIVARIANT OBJECTS

1. INTRODUCTION

In these notes we often work with two distinct categories. In order to make our notation clear we denote by $h^{\mathcal{C}} : \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ the Yoneda embedding for category \mathcal{C} . In particular, if X is an object of \mathcal{C} , then $h_X^{\mathcal{C}}$ is a presheaf associated with X .

2. FIBERED CATEGORIES

We fix a functor $p : \mathcal{E} \rightarrow \mathcal{B}$. We introduce now some convenient notation that will help clarifying our definitions. Consider a morphism $\phi : \xi \rightarrow \eta$ of \mathcal{E} such that $p(\phi) = f$ and $f : X \rightarrow Y$. We depict this situation by the square diagram

$$\begin{array}{ccc} \xi & \xrightarrow{\phi} & \eta \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Note that to every such square there corresponds a commutative square

$$\begin{array}{ccc} h_{\xi}^{\mathcal{E}} & \xrightarrow{h_{\phi}^{\mathcal{E}}} & h_{\eta}^{\mathcal{E}} \\ p_{\text{hom}} \downarrow & & \downarrow p_{\text{hom}} \\ h_X^{\mathcal{B}} \cdot p & \xrightarrow{(h_f^{\mathcal{B}})_p} & h_Y^{\mathcal{B}} \cdot p \end{array}$$

of presheaves on \mathcal{E} , where p_{hom} denotes maps induced by p on sets of morphisms.

Definition 2.1. Consider a square

$$\begin{array}{ccc} \xi & \xrightarrow{\phi} & \eta \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

We call the square *cartesian* and ϕ a *cartesian morphism with respect to p* if the corresponding square of presheaves on \mathcal{E} is cartesian in the category of presheaves.

One can rephrase definition above in terms of presheaves as follows. Morphism $\phi : \xi \rightarrow \eta$ is cartesian with respect to p if the square

$$\begin{array}{ccc}
\mathrm{Mor}_{\mathcal{E}}(\zeta, \zeta) & \xrightarrow{\mathrm{Mor}_{\mathcal{E}}(1_{\zeta}, \phi)} & \mathrm{Mor}_{\mathcal{E}}(\zeta, \eta) \\
\downarrow p_{\mathrm{hom}} & & \downarrow p_{\mathrm{hom}} \\
\mathrm{Mor}_{\mathcal{B}}(p(\zeta), p(\zeta)) & \xrightarrow{\mathrm{Mor}_{\mathcal{B}}(1_{p(\zeta)}, p(\phi))} & \mathrm{Mor}_{\mathcal{B}}(p(\zeta), p(\eta))
\end{array}$$

of sets is cartesian for every object ζ of \mathcal{E} .

Fact 2.2. Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a functor, let $f : X \rightarrow Y$ be a morphism of \mathcal{B} and let η be an object of \mathcal{E} . Suppose that $\phi_1 : \zeta_1 \rightarrow \eta, \phi_2 : \zeta_2 \rightarrow \eta$ are morphisms of \mathcal{E} that are cartesian with respect to p and assume that $p(\phi_1) = p(\phi_2)$. Then there exists a unique morphism $\theta : \zeta_1 \rightarrow \zeta_2$ such that $\phi_1 = \phi_2 \cdot \theta$. Moreover, θ is an isomorphism.

Proof. We use the presheaf reformulation of a definition of cartesian morphisms of p . It implies that there exists a unique natural transformation $\sigma : h_{\zeta_1}^{\mathcal{E}} \rightarrow h_{\zeta_2}^{\mathcal{E}}$ such that $h_{\phi_1}^{\mathcal{E}} = h_{\phi_2}^{\mathcal{E}} \cdot \sigma$. Moreover, σ is a natural isomorphism. Since $h^{\mathcal{E}} : \mathcal{E} \rightarrow \widehat{\mathcal{E}}$ is full and faithful, we derive that there exists a unique morphism $\theta : \zeta_1 \rightarrow \zeta_2$ such that $h_{\theta}^{\mathcal{E}} = \sigma$. Then θ satisfies the assertion. \square

Definition 2.3. Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a functor, let $f : X \rightarrow Y$ be a morphism of \mathcal{B} and let η be an object of \mathcal{E} such that $p(\eta) = Y$. A pair (ζ, ϕ) such that ζ is an object of \mathcal{E} and $\phi : \zeta \rightarrow \eta$ is a morphism of \mathcal{E} is called a *pullback of η along f* if the following conditions are satisfied.

- (1) $p(\phi) = f$
- (2) ϕ is cartesian morphism of p .

Note that Fact 2.2 implies that pullbacks are unique up to a unique isomorphism.

Definition 2.4. Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a functor. Then p is a *fibred category* if and only if for every morphism $f : X \rightarrow Y$ of \mathcal{B} and every object η of \mathcal{E} such that $p(\eta) = Y$ there exists a pullback of η along f . If $p : \mathcal{E} \rightarrow \mathcal{B}$ is a fibred category, then we say that \mathcal{E} is *fibred over \mathcal{B} with respect to p* .

Now we give some examples of fibred categories. The first is a prototypical for the notion of a cartesian category. It shows that any category \mathcal{B} with fiber products gives rise in a canonical way to a fibred category over \mathcal{B} with cartesian arrows as cartesian squares in \mathcal{B} .

Example 2.5 (the fibred category of arrows). Let \mathcal{B} be a category. We define the category $\mathrm{Arr}(\mathcal{B})$ of arrows of \mathcal{B} as follows. Objects of $\mathrm{Arr}(\mathcal{B})$ are morphisms $\pi : \tilde{X} \rightarrow X$ of \mathcal{B} . Now if $\pi : \tilde{X} \rightarrow X$ and $\psi : \tilde{Y} \rightarrow Y$ are objects of $\mathrm{Arr}(\mathcal{B})$, then a morphism $\pi \rightarrow \psi$ is a pair (f, ϕ) such that $f : X \rightarrow Y$ and $\phi : \tilde{X} \rightarrow \tilde{Y}$ are morphisms in \mathcal{B} making the square

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\phi} & \tilde{Y} \\
\pi \downarrow & & \downarrow \psi \\
X & \xrightarrow{f} & Y
\end{array}$$

commutative. There exists a functor $p_{\mathrm{Arr}(\mathcal{B})} : \mathrm{Arr}(\mathcal{B}) \rightarrow \mathcal{B}$ given by formula $p_{\mathrm{Arr}(\mathcal{B})}((f, \phi)) = f$. Suppose now that $f : X \rightarrow Y$ and $\psi : \tilde{Y} \rightarrow Y$ are morphisms of \mathcal{B} and there exists a commutative square

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\phi} & \tilde{Y} \\
\pi \downarrow & & \downarrow \psi \\
X & \xrightarrow{f} & Y
\end{array}$$

It is a direct consequence of the definition that (f, ϕ) is a cartesian morphism of $p_{\text{Arr}(\mathcal{B})}$ if and only if the square above is cartesian. Thus $p_{\text{Arr}(\mathcal{B})}$ is a fibered category provided that \mathcal{B} admits fiber products.

Definition 2.6. Suppose that $p_1 : \mathcal{E}_1 \rightarrow \mathcal{B}$ and $p_2 : \mathcal{E}_2 \rightarrow \mathcal{B}$ are fibered categories. Then a functor $F : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is a *morphism of fibered categories* if the following two assertions are satisfied.

- (1) $p_1 = F \cdot p_2$ or in other words F is a functor over \mathcal{B} .
- (2) Image under F of a cartesian morphism of p_1 is a cartesian morphism of p_2 .

Next example is closely related to the previous one, but is of more topological flavour.

Example 2.7 (the fibered category vector bundles). Let **Top** be the category of topological spaces. We define a category **VectBund** $_{\mathbb{R}}$ of real vector bundles as follows. Objects of **VectBund** $_{\mathbb{R}}$ are topological \mathbb{R} -vector bundles $\pi : \mathcal{V} \rightarrow X$. Now if $\pi : \mathcal{V} \rightarrow X$ and $\psi : \mathcal{W} \rightarrow Y$ are topological \mathbb{R} -vector bundles, then a morphism $\pi \rightarrow \psi$ is a pair (f, ϕ) such that $f : X \rightarrow Y$ and $\phi : \mathcal{V} \rightarrow \mathcal{W}$ are continuous maps making the square

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\phi} & \mathcal{W} \\ \pi \downarrow & & \downarrow \psi \\ X & \xrightarrow{f} & Y \end{array}$$

commutative and moreover, ϕ induces an \mathbb{R} -linear map on fibers i.e. for each point x in X map ϕ induces an \mathbb{R} -linear map $\pi^{-1}(x) \rightarrow \psi^{-1}(f(x))$. We have the functor **VectBund** $_{\mathbb{R}} \rightarrow \text{Arr}(\mathbf{Top})$ that forgets about \mathbb{R} -vector bundle structure. Since topological vector bundles are stable under continuous change of base, we deduce (according to description of cartesian squares in Example 2.5) that the composition of this forgetful functor with $p_{\text{Arr}(\mathbf{Top})} : \text{Arr}(\mathbf{Top}) \rightarrow \mathbf{Top}$ is the fibered category. Thus we have a commutative triangle

$$\begin{array}{ccc} \mathbf{VectBund}_{\mathbb{R}} & \xrightarrow{\quad} & \text{Arr}(\mathbf{Top}) \\ & \searrow & \swarrow p_{\text{Arr}(\mathbf{Top})} \\ & \mathbf{Top} & \end{array}$$

and the functor **VectBund** $_{\mathbb{R}} \rightarrow \text{Arr}(\mathbf{Top})$ is a morphism of fibered categories.

3. PSEUDO-FUNCTORS AND FIBERED CATEGORIES OF ELEMENTS

Pseudo-functors are certain non-strict 2-functors. In this section we introduce a procedure that enables to construct a fibered category out of a pseudo-functor. We start by defining this notion.

Definition 3.1. Let \mathcal{B} be a category. Consider the tuple of collections

$$F = (\{F(X)\}_{X \in \text{Ob}(\mathcal{B})}, \{F(f)\}_{f \in \text{Mor}(\mathcal{B})}, \{\Theta^{f,g}\}_{f,g \in \text{Mor}(\mathcal{B}), \text{cod}(f)=\text{dom}(g)}, \{\epsilon^X\}_{X \in \text{Ob}(\mathcal{B})})$$

of the following data.

- (1) For each object X of \mathcal{B} a category $F(X)$.
- (2) For each arrow $f : X \rightarrow Y$ a functor $F(f) : F(Y) \rightarrow F(X)$.
- (3) For each object X of \mathcal{B} a natural isomorphism $\epsilon^X : 1_{F(X)} \rightarrow F(1_X)$.
- (4) For any two composable morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ of \mathcal{B} a natural isomorphism $\Theta^{g,f} : F(f) \cdot F(g) \rightarrow F(g \cdot f)$

Suppose that these data are subject to the following conditions.

(1) For every arrow $f : X \rightarrow Y$ in \mathcal{B} we have

$$1_{F(f)} = \Theta^{f, 1_X} \cdot \epsilon_{F(f)}^X, \quad 1_{F(f)} = \Theta^{1_Y, f} \cdot F(f) (\epsilon^Y)$$

(2) For any three morphisms $f : X \rightarrow Y, g : Y \rightarrow Z, h : Z \rightarrow W$ of \mathcal{B} the square of functors and natural isomorphisms

$$\begin{array}{ccc} F(f) \cdot F(g) \cdot F(h) & \xrightarrow{F(f)(\Theta^{h,g})} & F(f) \cdot F(h \cdot g) \\ \Theta_{F(h)}^{g,f} \downarrow & & \downarrow \Theta^{h,g,f} \\ F(g \cdot f) \cdot F(h) & \xrightarrow{\Theta^{h,g,f}} & F(h \cdot g \cdot f) \end{array}$$

is commutative.

Then F is called a *pseudo-functor* on \mathcal{B}

Now we show how to construct a fibered category from a pseudo-functor. Suppose that \mathcal{B} is a category and

$$F = (\{F(X)\}_{X \in \text{Ob}(\mathcal{B})}, \{F(f)\}_{f \in \text{Mor}(\mathcal{B})}, \{\Theta^{f,g}\}_{f,g \in \text{Mor}(\mathcal{B}), \text{cod}(f)=\text{dom}(g)}, \{\epsilon^X\}_{X \in \text{Ob}(\mathcal{B})})$$

is a pseudo-functor on \mathcal{B} . We define a category $\int_{\mathcal{B}} F$. Its objects are pairs (X, ξ) such that X is an object of \mathcal{B} and ξ is an object of $F(X)$. If (X, ξ) and (Y, η) are objects of $\int_{\mathcal{B}} F$, then a morphism between these objects is a pair (f, σ) such that $f : X \rightarrow Y$ is a morphism of \mathcal{B} and $\sigma : \xi \rightarrow F(f)(\eta)$ is a morphism of $F(X)$. Now suppose that $(f, \sigma) : (X, \xi) \rightarrow (Y, \eta)$ and $(g, \tau) : (Y, \eta) \rightarrow (Z, \zeta)$ are morphisms of $\int_{\mathcal{B}} F$. Then we define their composition by formula

$$(g, \tau) \cdot (f, \sigma) = (g \cdot f, \Theta_{\xi}^{g,f} \cdot F(f)(\tau) \cdot \sigma)$$

Fact 3.2. $\int_{\mathcal{B}} F$ is a well defined category.

Proof. We first verify that the composition of morphisms in $\int_{\mathcal{B}} F$ is associative. Suppose that $(f, \sigma) : (X, \xi) \rightarrow (Y, \eta), (g, \tau) : (Y, \eta) \rightarrow (Z, \zeta), (h, \rho) : (Z, \zeta) \rightarrow (W, \omega)$ are morphisms of $\int_{\mathcal{B}} F$. Then

$$\begin{aligned} ((h, \rho) \cdot (g, \tau)) \cdot (f, \sigma) &= (h \cdot g, \Theta_{\omega}^{h,g} \cdot F(g)(\rho) \cdot \tau) \cdot (f, \sigma) = \\ &= (h \cdot g \cdot f, \Theta_{\omega}^{h,g,f} \cdot F(f)(\Theta_{\omega}^{h,g} \cdot F(g)(\rho) \cdot \tau) \cdot \sigma) = (h \cdot g \cdot f, \Theta_{\omega}^{h,g,f} \cdot F(f)(\Theta_{\omega}^{h,g} \cdot F(f)(F(g)(\rho)) \cdot F(f)(\tau) \cdot \sigma) \end{aligned}$$

and

$$\begin{aligned} (h, \rho) \cdot ((g, \tau) \cdot (f, \sigma)) &= (h, \rho) \cdot (g \cdot f, \Theta_{\xi}^{g,f} \cdot F(f)(\tau) \cdot \sigma) = \\ &= (h \cdot g \cdot f, \Theta_{\omega}^{h,g,f} \cdot F(g \cdot f)(\rho) \cdot \Theta_{\xi}^{g,f} \cdot F(f)(\tau) \cdot \sigma) = (h \cdot g \cdot f, \Theta_{\omega}^{h,g,f} \cdot \Theta_{F(h)(\omega)}^{g,f} \cdot F(f)(F(g)(\rho)) \cdot F(f)(\tau) \cdot \sigma) \end{aligned}$$

Since $\Theta_{\omega}^{h,g,f} \cdot F(f)(\Theta_{\omega}^{h,g}) = \Theta_{\omega}^{h,g,f} \cdot \Theta_{F(h)(\omega)}^{g,f}$, we deduce that

$$((h, \rho) \cdot (g, \tau)) \cdot (f, \sigma) = (h, \rho) \cdot ((g, \tau) \cdot (f, \sigma))$$

and hence the composition in $\int_{\mathcal{B}} F$ is associative. Next we prove that for each object (X, ξ) of $\int_{\mathcal{B}} F$ there exists an identity morphism. We claim that $(1_X, \epsilon_{\xi}^X) : (X, \xi) \rightarrow (X, \xi)$ is the identity. Indeed, for morphisms $(f, \sigma) : (X, \xi) \rightarrow (Y, \eta)$ and $(g, \tau) : (Z, \zeta) \rightarrow (X, \xi)$ we have

$$(f, \sigma) \cdot (1_X, \epsilon_{\xi}^X) = (f, \Theta_{\eta}^{f, 1_X} \cdot F(1_X)(\sigma) \cdot \epsilon_{\xi}^X) = (f, \Theta_{\eta}^{f, 1_X} \cdot \epsilon_{F(f)(\eta)}^X \cdot \sigma) = (f, \sigma)$$

and

$$(1_X, \epsilon_{\xi}^X) \cdot (g, \tau) = (g, \Theta_{\xi}^{1_X, g} \cdot F(g)(\epsilon_{\xi}^X) \cdot \tau) = (g, \tau)$$

Therefore, $\int_{\mathcal{B}} F$ is a category. \square

Next we define a functor $p_F : \int_{\mathcal{B}} F \rightarrow \mathcal{B}$ by formula

$$p_F \left((f, \sigma) : (X, \xi) \rightarrow (Y, \tau) \right) = f : X \rightarrow Y$$

This is clearly a well defined functor. Now we prove the following statement.

Fact 3.3. *The functor $p_F : \int_{\mathcal{B}} F \rightarrow \mathcal{B}$ is a fibered category.*

Proof. Let $f : X \rightarrow Y$ be a morphism in \mathcal{B} and η be an object of $F(Y)$. Thus (Y, η) is an object of $\int_{\mathcal{B}} F$. It suffices to show that (Y, η) admits a pullback along f . We claim that

$$(f, 1_{F(f)(\eta)}) : (X, F(f)(\eta)) \rightarrow (Y, \eta)$$

is a cartesian morphism of p_F that yields a pullback of η along f . To prove the claim consider an object (Z, ζ) of $\int_{\mathcal{B}} F$ and suppose that $(g, \tau) : (Z, \zeta) \rightarrow (Y, \eta)$ is a morphism of $\int_{\mathcal{B}} F$ such that g factors through f . Then there exists $h : Z \rightarrow X$ such that $f \cdot h = g$. Note that $\tau : \zeta \rightarrow F(g)(\eta)$. Since $g = f \cdot h$, we have

$$\tau = \Theta_{\eta}^{f, h} \cdot \left(\Theta_{\eta}^{f, h} \right)^{-1} \cdot \tau = \Theta_{\eta}^{f, h} \cdot F(h)(1_{F(f)(\eta)}) \cdot \left(\Theta_{\eta}^{f, h} \right)^{-1} \cdot \tau$$

and hence

$$(g, \tau) = (f, 1_{F(f)(\eta)}) \cdot \left(h, \left(\Theta_{\eta}^{f, h} \right)^{-1} \cdot \tau \right)$$

Thus (g, τ) factors through $(f, 1_{F(f)(\eta)})$ and the formula above shows that this factorization is unique. Hence $(f, 1_{F(f)(\eta)})$ is a cartesian morphism of p_F . \square

Definition 3.4. Let \mathcal{B} be a category and let F be a pseudo-functor on \mathcal{B} . A fibered category $p_F : \int_{\mathcal{B}} F \rightarrow \mathcal{B}$ constructed above is called *the fibered category of elements of the pseudo-functor F* .

It is possible to construct a pseudo-functor out of a fibered category. We will give a brief outline of this construction. For this we introduce notation that will be also used in other considerations.

Definition 3.5. Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a fibered category. For every object X of \mathcal{B} we denote by $p^{-1}(X)$ a subcategory of \mathcal{E} consisting of all morphisms $\phi : \xi \rightarrow \eta$ such that $p(\phi) = 1_X$. We call this category *the fiber of p over X* .

Suppose now that $p : \mathcal{E} \rightarrow \mathcal{B}$ is a fibered category. Let $f : X \rightarrow Y$ be a morphism. For every object η in $p^{-1}(Y)$ we pick its pullback $\tilde{f}_{\eta} : f^* \eta \rightarrow \eta$ along f . By universal property of cartesian morphisms we deduce that this induces a functor $f^* : p^{-1}(Y) \rightarrow p^{-1}(X)$. Universal property of cartesian morphisms implies also the following assertions.

- (1) For each object X of \mathcal{B} we may choose $(1_X)^* = 1_{p^{-1}(X)}$.
- (2) For any two composable morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ of \mathcal{B} there exists a unique natural isomorphism $\Theta^{g,f}_\zeta : f^*g^*\zeta \rightarrow (g \cdot f)^*\zeta$ of functors such that for every ζ in $p^{-1}(Z)$ we have commutative diagram

$$\begin{array}{ccccc}
 f^*g^*\zeta & \xrightarrow{\tilde{f}_{g^*\zeta}} & g^*\zeta & \xrightarrow{\tilde{g}_\zeta} & \zeta \\
 \Theta^{g,f}_\zeta \downarrow & & & & \downarrow 1_\zeta \\
 (g \cdot f)^*\zeta & \xrightarrow{\tilde{g \cdot f}_\zeta} & & & \zeta
 \end{array}$$

From (1), (2) and Fact 2.2 one can deduce that the collection

$$(\{p^{-1}(X)\}_{X \in \text{Ob}(\mathcal{B})}, \{f^*\}_{f \in \text{Mor}(\mathcal{B})}, \{\Theta^{f,g}\}_{f,g \in \text{Mor}(\mathcal{B}), \text{cod}(f)=\text{dom}(g)}, \{1_{p^{-1}(X)}\}_{X \in \text{Ob}(\mathcal{B})})$$

is a pseudo-functor.

Remark 3.6. The construction of the fibered category of elements is a part of 2-equivalence between appropriately defined category of pseudo-functors on \mathcal{B} and the category of fibered categories over \mathcal{B} .

4. EXAMPLE: QUASI-COHERENT SHEAVES

Note that all examples of fibered categories given so far were fibered subcategories of the fibered category of arrows $p_{\text{Arr}(\mathcal{B})} : \text{Arr}(\mathcal{B}) \rightarrow \mathcal{B}$ for a given category \mathcal{B} with fibered-products. In this section we employ the procedure that produces a fibered category out of a pseudo-functor to obtain an important example of a category fibered over \mathbf{Sch}_k (the category of schemes over a ring k), which is not of this type.

Let $f : X \rightarrow Y$ be a morphism of k -schemes. We have an adjunction

$$\begin{array}{ccc}
 & f_* & \\
 \Omega\text{coh}(X) & \xrightleftharpoons{\quad \perp \quad} & \Omega\text{coh}(Y) \\
 & f^* &
 \end{array}$$

It is determined by the bijection

$$\text{Hom}_{\mathcal{O}_Y}(f^*\mathcal{G}, \mathcal{F}) \xrightarrow{\Phi_{\mathcal{G}, \mathcal{F}}^f} \text{Hom}_{\mathcal{O}_X}(\mathcal{G}, f_*\mathcal{F})$$

Suppose now that $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are morphisms of k -schemes. Since $(g \cdot f)_* = g_* \cdot f_*$, there exists a unique natural isomorphism $\Theta^{g,f} : f^*g^* \rightarrow (g \cdot f)^*$ such that for every quasi-coherent sheaf \mathcal{F} in $\Omega\text{coh}(X)$ and every quasi-coherent sheaf \mathcal{H} in $\Omega\text{coh}(Z)$ we have

$$\Phi_{\mathcal{H}, \mathcal{F}}^{g \cdot f} = \Phi_{\mathcal{H}, f_*\mathcal{F}}^g \cdot \Phi_{g^*\mathcal{H}, \mathcal{F}}^f \cdot \text{Hom}_{\mathcal{O}_X}(\Theta_{\mathcal{H}}^{g,f}, 1_{\mathcal{F}})$$

Now we have the following result.

Fact 4.1. Suppose that $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and $h : Z \rightarrow W$ are morphism of k -schemes. Then the square

$$\begin{array}{ccc}
f^* g^* h^* & \xrightarrow{f^* \Theta^{h,g}} & f^* (h \cdot g)^* \\
\Theta_{h^*}^{g,f} \downarrow & & \downarrow \Theta^{h,g,f} \\
(g \cdot f)^* h^* & \xrightarrow{\Theta^{h,g,f}} & (h \cdot g \cdot f)^*
\end{array}$$

of functors and natural isomorphisms is commutative.

Proof. Suppose that \mathcal{F} is an object of $\mathcal{Q}\text{coh}(X)$ and \mathcal{K} is an object of $\mathcal{Q}\text{coh}(W)$. Then

$$\begin{aligned}
\Phi_{\mathcal{K}, g_* f_* \mathcal{F}}^h \cdot \Phi_{h^* \mathcal{K}, f_* \mathcal{F}}^g \cdot \Phi_{g^* h^* \mathcal{K}, \mathcal{F}}^f \cdot \text{Hom}_{\mathcal{O}_X}(\Theta_{h^* \mathcal{K}}^{g,f}, 1_{\mathcal{F}}) \cdot \text{Hom}_{\mathcal{O}_X}(\Theta_{\mathcal{K}}^{h,g,f}, 1_{\mathcal{F}}) &= \\
= \Phi_{\mathcal{K}, g_* f_* \mathcal{F}}^h \cdot \Phi_{h^* \mathcal{K}, f_* \mathcal{F}}^{g \cdot f} \cdot \text{Hom}_{\mathcal{O}_X}(\Theta_{h^* \mathcal{K}}^{g,f}, 1_{\mathcal{F}}) &= \Phi_{\mathcal{K}, \mathcal{F}}^{h \cdot g \cdot f}
\end{aligned}$$

and

$$\begin{aligned}
\Phi_{\mathcal{K}, g_* f_* \mathcal{F}}^h \cdot \Phi_{h^* \mathcal{K}, f_* \mathcal{F}}^g \cdot \Phi_{g^* h^* \mathcal{K}, \mathcal{F}}^f \cdot \text{Hom}_{\mathcal{O}_X}(f^* \Theta_{\mathcal{K}}^{h,g}, 1_{\mathcal{F}}) \cdot \text{Hom}_{\mathcal{O}_X}(\Theta_{\mathcal{K}}^{h,g,f}, 1_{\mathcal{F}}) &= \\
= \Phi_{\mathcal{K}, g_* f_* \mathcal{F}}^h \cdot \Phi_{h^* \mathcal{K}, f_* \mathcal{F}}^g \cdot \text{Hom}_{\mathcal{O}_X}(\Theta_{\mathcal{K}}^{h,g}, 1_{f_* \mathcal{F}}) \cdot \Phi_{(h \cdot g)^* \mathcal{K}, \mathcal{F}}^f \cdot \text{Hom}_{\mathcal{O}_X}(\Theta_{\mathcal{K}}^{h,g,f}, 1_{\mathcal{F}}) &= \\
= \Phi_{\mathcal{K}, f_* \mathcal{F}}^{h \cdot g} \cdot \Phi_{(h \cdot g)^* \mathcal{K}, \mathcal{F}}^f \cdot \text{Hom}_{\mathcal{O}_X}(\Theta_{\mathcal{K}}^{h,g,f}, 1_{\mathcal{F}}) &= \Phi_{\mathcal{K}, \mathcal{F}}^{h \cdot g \cdot f}
\end{aligned}$$

Therefore, we derive that

$$\begin{aligned}
\Phi_{\mathcal{K}, g_* f_* \mathcal{F}}^h \cdot \Phi_{h^* \mathcal{K}, f_* \mathcal{F}}^g \cdot \Phi_{g^* h^* \mathcal{K}, \mathcal{F}}^f \cdot \text{Hom}_{\mathcal{O}_X}(\Theta_{h^* \mathcal{K}}^{g,f}, 1_{\mathcal{F}}) \cdot \text{Hom}_{\mathcal{O}_X}(\Theta_{\mathcal{K}}^{h,g,f}, 1_{\mathcal{F}}) &= \\
= \Phi_{\mathcal{K}, g_* f_* \mathcal{F}}^h \cdot \Phi_{h^* \mathcal{K}, f_* \mathcal{F}}^g \cdot \Phi_{g^* h^* \mathcal{K}, \mathcal{F}}^f \cdot \text{Hom}_{\mathcal{O}_X}(f^* \Theta_{\mathcal{K}}^{h,g}, 1_{\mathcal{F}}) \cdot \text{Hom}_{\mathcal{O}_X}(\Theta_{\mathcal{K}}^{h,g,f}, 1_{\mathcal{F}}) &
\end{aligned}$$

and hence

$$\begin{aligned}
\text{Hom}_{\mathcal{O}_X}(\Theta_{\mathcal{K}}^{h,g,f} \cdot \Theta_{h^* \mathcal{K}}^{g,f}, 1_{\mathcal{F}}) &= \text{Hom}_{\mathcal{O}_X}(\Theta_{h^* \mathcal{K}}^{g,f}, 1_{\mathcal{F}}) \cdot \text{Hom}_{\mathcal{O}_X}(\Theta_{\mathcal{K}}^{h,g,f}, 1_{\mathcal{F}}) = \\
= \text{Hom}_{\mathcal{O}_X}(f^* \Theta_{\mathcal{K}}^{h,g}, 1_{\mathcal{F}}) \cdot \text{Hom}_{\mathcal{O}_X}(\Theta_{\mathcal{K}}^{h,g,f}, 1_{\mathcal{F}}) &= \text{Hom}_{\mathcal{O}_X}(\Theta_{\mathcal{K}}^{h \cdot g \cdot f}, f^* \Theta_{\mathcal{K}}^{h,g}, 1_{\mathcal{F}})
\end{aligned}$$

Since this equality holds for every quasi-coherent sheaf \mathcal{F} on X , we deduce that

$$\Theta_{\mathcal{K}}^{h,g,f} \cdot \Theta_{h^* \mathcal{K}}^{g,f} = \Theta_{\mathcal{K}}^{h \cdot g \cdot f} \cdot f^* \Theta_{\mathcal{K}}^{h,g}$$

for every quasi-coherent sheaf \mathcal{K} . This proves the assertion. \square

Note that for every k -scheme X we may assume that $(1_X)_* = 1_{\mathcal{Q}\text{coh}(X)} = (1_X)^*$ and $\Phi_{\mathcal{G}, \mathcal{F}}^{1_X} = \text{Hom}_{\mathcal{O}_X}(1_{\mathcal{F}}, 1_{\mathcal{G}})$.

Fact 4.2. *Let $f : X \rightarrow Y$ and $g : Z \rightarrow X$ be morphisms of k -schemes. Then*

$$\Theta^{f, 1_X} = 1_{f^*}, \quad \Theta^{1_X, g} = 1_{g^*}$$

Proof. Suppose that \mathcal{F} is an object of $\mathcal{Q}\text{coh}(X)$ and \mathcal{G} is an object of $\mathcal{Q}\text{coh}(Y)$. Then

$$\Phi_{\mathcal{G}, \mathcal{F}}^f = \Phi_{\mathcal{G}, \mathcal{F}}^{f \cdot 1_X} = \Phi_{\mathcal{G}, \mathcal{F}}^f \cdot \Phi_{f^* \mathcal{G}, \mathcal{F}}^{1_X} \cdot \text{Hom}_{\mathcal{O}_X}(\Theta_{\mathcal{G}}^{f, 1_X}, 1_{\mathcal{F}}) = \Phi_{\mathcal{G}, \mathcal{F}}^f \cdot \text{Hom}_{\mathcal{O}_X}(\Theta_{\mathcal{G}}^{f, 1_X}, 1_{\mathcal{F}})$$

and thus $\text{Hom}_{\mathcal{O}_X}(\Theta_{\mathcal{G}}^{f, 1_X}, 1_{\mathcal{F}}) = \text{Hom}_{\mathcal{O}_X}(1_{f^* \mathcal{G}}, 1_{\mathcal{F}})$. Since this holds for every quasi-coherent sheaf \mathcal{F} on X , we derive that $\Theta_{\mathcal{G}}^{f, 1_X} = 1_{f^* \mathcal{G}}$. Thus $\Theta^{f, 1_X} = 1_{f^*}$.

Suppose that \mathcal{H} is an object of $\mathcal{Q}\text{coh}(X)$ and \mathcal{F} is an object of $\mathcal{Q}\text{coh}(Z)$. Then

$$\Phi_{\mathcal{H}, \mathcal{F}}^g = \Phi_{\mathcal{H}, \mathcal{F}}^{1_X \cdot g} = \Phi_{\mathcal{H}, \mathcal{F}}^{1_X} \cdot \Phi_{\mathcal{H}, \mathcal{F}}^g \cdot \text{Hom}_{\mathcal{O}_Z}(\Theta_{\mathcal{H}}^{1_X, g}, 1_{\mathcal{F}}) = \Phi_{\mathcal{H}, \mathcal{F}}^g \cdot \text{Hom}_{\mathcal{O}_Z}(\Theta_{\mathcal{H}}^{1_X, g}, 1_{\mathcal{F}})$$

and thus $\text{Hom}_{\mathcal{O}_Z}(\Theta_{\mathcal{H}}^{1_X, g}, 1_{\mathcal{F}}) = \text{Hom}_{\mathcal{O}_Z}(1_{g^* \mathcal{H}}, 1_{\mathcal{F}})$. Since this holds for every quasi-coherent sheaf \mathcal{F} on Z , we derive that $\Theta_{\mathcal{H}}^{1_X, g} = 1_{g^* \mathcal{H}}$. Thus $\Theta^{1_X, g} = 1_{g^*}$. \square

Now Facts 4.1 and 4.2 imply that the collection

$$\left(\{ \mathcal{Q}\mathrm{coh}(X) \}_{X \in \mathbf{Sch}_k}, \{ f^* \}_{f \in \mathrm{Mor}(\mathbf{Sch}_k)}, \{ \Theta^{f,g} \}_{f,g \in \mathrm{Mor}(\mathbf{Sch}_k), \mathrm{cod}(f)=\mathrm{dom}(g)}, \{ 1_{\mathcal{Q}\mathrm{coh}(X)} \}_{X \in \mathbf{Sch}_k} \right)$$

forms a pseudo-functor on \mathbf{Sch}_k .

Definition 4.3. The fibered category of quasi-coherent sheaves on \mathbf{Sch}_k is the fibered category of elements of the pseudo-functor

$$\left(\{ \mathcal{Q}\mathrm{coh}(X) \}_{X \in \mathbf{Sch}_k}, \{ f^* \}_{f \in \mathrm{Mor}(\mathbf{Sch}_k)}, \{ \Theta^{f,g} \}_{f,g \in \mathrm{Mor}(\mathbf{Sch}_k), \mathrm{cod}(f)=\mathrm{dom}(g)}, \{ 1_{\mathcal{Q}\mathrm{coh}(X)} \}_{X \in \mathbf{Sch}_k} \right)$$

We denote it by $\mathcal{Q}\mathrm{coh} \rightarrow \mathbf{Sch}_k$.

For every k -scheme X we have a category $\mathbf{Alg}(\mathcal{Q}\mathrm{coh}(X))$ of quasi-coherent \mathcal{O}_X -algebras. If $f : X \rightarrow Y$ is a morphism of k -schemes, then we have an adjunction

$$\begin{array}{ccc} & f_* & \\ \mathbf{Alg}(\mathcal{Q}\mathrm{coh}(X)) & \perp & \mathbf{Alg}(\mathcal{Q}\mathrm{coh}(Y)) \\ & f^* & \end{array}$$

Using similar argument as above one can show that there exists a canonical structure of a pseudo-functor on the collection

$$\left(\{ \mathbf{Alg}(\mathcal{Q}\mathrm{coh}(X)) \}_{X \in \mathbf{Sch}_k}, \{ f^* \}_{f \in \mathrm{Mor}(\mathbf{Sch}_k)} \right)$$

Definition 4.4. The fibered category of quasi-coherent algebras on \mathbf{Sch}_k is the fibered category of elements of the canonical pseudo-functor determined by the collection

$$\left(\{ \mathbf{Alg}(\mathcal{Q}\mathrm{coh}(X)) \}_{X \in \mathbf{Sch}_k}, \{ f^* \}_{f \in \mathrm{Mor}(\mathbf{Sch}_k)} \right)$$

We denote it by $\mathbf{Alg}(\mathcal{Q}\mathrm{coh}) \rightarrow \mathbf{Sch}_k$.

Remark 4.5. We also have a canonical functor $|-| : \mathbf{Alg}(\mathcal{Q}\mathrm{coh}) \rightarrow \mathcal{Q}\mathrm{coh}$ that forgets about an algebra structure. This gives rise to a morphism of fibered categories

$$\begin{array}{ccc} \mathbf{Alg}(\mathcal{Q}\mathrm{coh}) & \xrightarrow{|-|} & \mathcal{Q}\mathrm{coh} \\ & \searrow & \swarrow \\ & \mathbf{Sch}_k & \end{array}$$

Example 4.6 (The fibered category of relatively affine schemes). Note that $p_{\mathrm{Arr}(\mathbf{Sch}_k)} : \mathrm{Arr}(\mathbf{Sch}_k) \rightarrow \mathbf{Sch}_k$ admits a fibered subcategory that consists of affine morphisms $\pi : \tilde{X} \rightarrow X$ of k -schemes. We denote this fibered category by $\mathbf{Aff}(\mathbf{Sch}_k) \rightarrow \mathbf{Sch}_k$.

Remark 4.7. Let $f : X \rightarrow Y$ be a morphism of k -schemes and \mathcal{A} is a quasi-coherent \mathcal{O}_Y -algebra. Then there exists a morphism $\tilde{f} : \mathrm{Spec}_X f^* \mathcal{A} \rightarrow \mathrm{Spec}_Y \mathcal{A}$ such that the square

$$\begin{array}{ccc} \mathrm{Spec}_X f^* \mathcal{A} & \xrightarrow{\tilde{f}} & \mathrm{Spec}_Y \mathcal{A} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is cartesian in \mathbf{Sch}_k .

Example 4.8. For every k -scheme X we have the relative affine spectrum functor $\mathrm{Spec}_X : \mathbf{Alg}(\mathcal{Q}\mathrm{coh}(X)) \rightarrow \mathbf{Aff}_X$, where \mathbf{Aff}_X is the category of schemes affine over X . It is an equivalence of categories and note that \mathbf{Aff}_X is the fiber of $\mathbf{Aff}(\mathbf{Sch}_k) \rightarrow \mathbf{Sch}_k$ over X . This observation together with Remark

4.7 shows that the collection of functors Spec_X for all k -schemes X gives rise to a morphism of fibered categories

$$\begin{array}{ccc} \mathbf{Alg}(\mathfrak{Qcoh}) & \xrightarrow{\mathrm{Spec}} & \mathbf{Aff}(\mathbf{Sch}_k) \\ & \searrow & \swarrow \\ & \mathbf{Sch}_k & \end{array}$$

Moreover, $\mathrm{Spec} : \mathbf{Alg}(\mathfrak{Qcoh}) \rightarrow \mathbf{Aff}(\mathbf{Sch}_k)$ is an equivalence of categories.

5. EQUIVARIANT OBJECTS IN FIBERED CATEGORIES

Let k be a commutative ring. We fix a monoid k -scheme \mathbf{M} . The following notion is useful for studying actions of algebraic groups and monoids.

Definition 5.1. Let X be a k -scheme and let \mathbf{M} be a monoid k -scheme with an action $a : \mathbf{M} \times_k X \rightarrow X$ on X . We denote by $\pi : \mathbf{M} \times_k X \rightarrow X$ the projection. Consider a pair (\mathcal{F}, τ) consisting of a quasi-coherent sheaf \mathcal{F} on X and a morphism $\tau : \pi^* \mathcal{F} \rightarrow a^* \mathcal{F}$. We call it a *quasi-coherent \mathbf{M} -sheaf* on (X, a) if the following equalities

$$(\mu \times 1_X)^* \tau^\sigma = (1_{\mathbf{M}} \times a)^* \tau^\sigma \cdot \pi_{2,3}^* \tau^\sigma, \langle e, 1_X \rangle^* \tau^\sigma = 1_{\mathcal{F}}$$

hold, where $\mu : \mathbf{M} \times_k \mathbf{M} \rightarrow \mathbf{M}$ is the multiplication on \mathbf{M} , $\pi_{2,3} : \mathbf{M} \times_k \mathbf{M} \times_k X \rightarrow \mathbf{M} \times_k X$ is the projection on the last two factors and $e : 1 \rightarrow \mathbf{M}$ is the unit of \mathbf{M} .

Definition 5.2. Let X be a k -scheme and let \mathbf{M} be a monoid k -scheme with an action $a : \mathbf{M} \times_k X \rightarrow X$ on X . We denote by $\pi : \mathbf{M} \times_k X \rightarrow X$ the projection. Let (\mathcal{F}_1, τ_1) and (\mathcal{F}_2, τ_2) be quasi-coherent \mathbf{M} -sheaves on (X, a) . Suppose that $\phi : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is a morphism of quasi-coherent sheaves on X such that the square

$$\begin{array}{ccc} \pi^* \mathcal{F}_1 & \xrightarrow{\tau_1} & a^* \mathcal{F}_1 \\ \pi^* \phi \downarrow & & \downarrow a^* \phi \\ \pi^* \mathcal{F}_2 & \xrightarrow{\tau_2} & a^* \mathcal{F}_2 \end{array}$$

is commutative. Then ϕ is a *morphism of quasi-coherent \mathbf{M} -sheaves* on (X, a) . We denote by $\mathfrak{Qcoh}_{\mathbf{M}}(X)$ the category of quasi-coherent \mathbf{M} -sheaves and call it *the category of quasi-coherent \mathbf{M} -sheaves* on (X, a) .

Our goal in this section is to explain somewhat nonintuitive notion of quasi-coherent \mathbf{M} -sheaf on a k -scheme X equipped with action of \mathbf{M} . For this we use the machinery of fibered categories. We fix a fibered category $p : \mathcal{E} \rightarrow \mathcal{B}$. If $f : X \rightarrow Y$ and η is an object of $p^{-1}(Y)$, then we denote by $\tilde{f}_\eta : f^* \eta \rightarrow \eta$ a pullback of η . That is the square

$$\begin{array}{ccc} f^* \eta & \xrightarrow{\tilde{f}_\eta} & \eta \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is cartesian. Using some choice of pullback we obtain a functor $f^* : p^{-1}(Y) \rightarrow p^{-1}(X)$. We start with the following observation.

Remark 5.3. Consider morphisms f_1, f_2, g_1, g_2 in \mathcal{B} such that $g_1 \cdot f_1 = g_2 \cdot f_2$ with $\text{cod}(g_1) = Y = \text{cod}(g_2)$. For every object η in $p^{-1}(Y)$ we have a unique identification $f_1^* g_1^* \eta \cong f_2^* g_2^* \eta$ such that the square

$$\begin{array}{ccc} f_2^* g_2^* \eta \cong f_1^* g_1^* \eta & \xrightarrow{\tilde{f}_{1, g_1^* \eta}} & g_1^* \eta \\ \tilde{f}_{2, g_2^* \eta} \downarrow & & \downarrow \tilde{g}_{1, \eta} \\ g_2^* \eta & \xrightarrow{\tilde{g}_{2, \eta}} & \eta \end{array}$$

is commutative.

Now we have the following result.

Fact 5.4. Let X, \mathbf{M} be objects of \mathcal{B} and let ζ be an object of \mathcal{E} in $p^{-1}(X)$. Assume that the cartesian product of X and \mathbf{M} exists in \mathcal{B} and denote by $\pi : \mathbf{M} \times X \rightarrow X$ the projection. Then there exists a unique morphism (depicted by dotted arrow) such that the diagram

$$\begin{array}{ccccc} h_{\pi^* \zeta}^{\mathcal{E}} & & & & h_{\pi \zeta}^{\mathcal{E}} \\ \downarrow p_{\text{hom}} & \searrow \text{dotted} & & \xrightarrow{pr_{h_{\zeta}^{\mathcal{E}}}} & \downarrow p_{\text{hom}} \\ & h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\zeta}^{\mathcal{E}} & & & h_{\zeta}^{\mathcal{E}} \\ & \downarrow 1_{h_{\mathbf{M}}^{\mathcal{B}} \cdot p} \times p_{\text{hom}} & & & \downarrow p_{\text{hom}} \\ h_{\mathbf{M} \times X}^{\mathcal{B}} \cdot p & \xrightarrow{=} & h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_X^{\mathcal{B}} \cdot p & \xrightarrow{pr_{h_X^{\mathcal{B}} \cdot p}} & h_X^{\mathcal{B}} \cdot p \\ & & \text{curved arrow} & & \\ & & (h_{\pi}^{\mathcal{B}})_p & & \end{array}$$

is commutative, where $pr_{h_{\zeta}^{\mathcal{E}}}$ and $pr_{h_X^{\mathcal{B}} \cdot p}$ are projections. Moreover, this morphism is an isomorphism.

Proof. This is a consequence of the fact that both squares

$$\begin{array}{ccc} h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\zeta}^{\mathcal{E}} & \xrightarrow{pr_{h_{\zeta}^{\mathcal{E}}}} & h_{\zeta}^{\mathcal{E}} \\ \downarrow 1_{h_{\mathbf{M}}^{\mathcal{B}} \cdot p} \times p_{\text{hom}} & & \downarrow p_{\text{hom}} \\ h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_X^{\mathcal{B}} \cdot p & \xrightarrow{pr_{h_X^{\mathcal{B}} \cdot p}} & h_X^{\mathcal{B}} \cdot p \end{array} \quad \begin{array}{ccc} h_{\pi^* \zeta}^{\mathcal{E}} & \xrightarrow{h_{\pi \zeta}^{\mathcal{E}}} & h_{\zeta}^{\mathcal{E}} \\ \downarrow p_{\text{hom}} & & \downarrow p_{\text{hom}} \\ h_{\mathbf{M} \times X}^{\mathcal{B}} \cdot p & \xrightarrow{(h_{\pi}^{\mathcal{B}})_p} & h_X^{\mathcal{B}} \cdot p \end{array}$$

are cartesian. □

Fix now two objects \mathbf{M} and X of \mathcal{B} such that the product of \mathbf{M} and X exists. Denote by $\pi : \mathbf{M} \times X \rightarrow X$ the projection on X . Let $a : \mathbf{M} \times X \rightarrow X$ be a morphism in \mathcal{B} , let ζ be an object in $p^{-1}(X)$ and let $\sigma : h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\zeta}^{\mathcal{E}} \rightarrow h_{\zeta}^{\mathcal{E}}$ be a morphism of presheaves on \mathcal{E} . Suppose that the square

$$\begin{array}{ccc}
h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\xi}^{\mathcal{E}} & \xrightarrow{\sigma} & h_{\xi}^{\mathcal{E}} \\
p_{\text{hom}} \downarrow & & \downarrow p_{\text{hom}} \\
h_{\mathbf{M} \times X}^{\mathcal{B}} \cdot p & \xrightarrow{(h_a^{\mathcal{B}})_p} & h_X^{\mathcal{B}} \cdot p
\end{array}$$

is commutative. According to Fact 5.4 we deduce that σ is representable by some morphism $\alpha^{\sigma} : \pi^* \xi \rightarrow \xi$ of \mathcal{E} . By universal property of cartesian square

$$\begin{array}{ccc}
a^* \xi & \xrightarrow{\tilde{a}_{\xi}} & \xi \\
\downarrow & & \downarrow \\
\mathbf{M} \times X & \xrightarrow{a} & X
\end{array}$$

we deduce that there exists a unique morphism $\tau^{\sigma} : \pi^* \xi \rightarrow a^* \xi$ in $p^{-1}(\mathbf{M} \times X)$ such that $\alpha^{\sigma} = \tilde{a}_{\xi} \cdot \tau^{\sigma}$. Using this notation and Fact 5.4 we can now state the following result.

Proposition 5.5. *Let \mathbf{M} be a monoid object in \mathcal{B} and let X be an object of \mathcal{B} equipped with an action $a : \mathbf{M} \times X \rightarrow X$ of \mathbf{M} on X . Denote by $\pi : \mathbf{M} \times X \rightarrow X$ the projection on X . Consider an object ξ in $p^{-1}(X)$ and let $\sigma : h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\xi}^{\mathcal{E}} \rightarrow h_{\xi}^{\mathcal{E}}$ be a morphism of presheaves on \mathcal{E} . Suppose that the square*

$$\begin{array}{ccc}
h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\xi}^{\mathcal{E}} & \xrightarrow{\sigma} & h_{\xi}^{\mathcal{E}} \\
p_{\text{hom}} \downarrow & & \downarrow p_{\text{hom}} \\
h_{\mathbf{M} \times X}^{\mathcal{B}} \cdot p & \xrightarrow{(h_a^{\mathcal{B}})_p} & h_X^{\mathcal{B}} \cdot p
\end{array}$$

is commutative. Consider the following assertions.

- (i) σ is an action of a monoid presheaf $h_{\mathbf{M}}^{\mathcal{B}} \cdot p$ on a presheaf $h_{\xi}^{\mathcal{E}}$.
- (ii) Morphism τ^{σ} satisfies (up to identifications described in Remark 5.3) the identities

$$(\mu \times 1_X)^* \tau^{\sigma} = (1_{\mathbf{M}} \times a)^* \tau^{\sigma} \cdot \pi_{2,3}^* \tau^{\sigma}, \quad \langle e, 1_X \rangle^* \tau^{\sigma} = 1_{\xi}$$

where $\mu : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{M}$ is the multiplication on \mathbf{M} , $\pi_{2,3} : \mathbf{M} \times \mathbf{M} \times X \rightarrow \mathbf{M} \times X$ is the projection on the last two factors and $e : \mathbf{1} \rightarrow \mathbf{M}$ is the unit of \mathbf{M} .

- (iii) Morphism τ^{σ} is an isomorphism and satisfies (up to identifications described in Remark 5.3) the identity

$$(\mu \times 1_X)^* \tau^{\sigma} = (1_{\mathbf{M}} \times a)^* \tau^{\sigma} \cdot \pi_{2,3}^* \tau^{\sigma}$$

where $\mu : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{M}$ is the multiplication on \mathbf{M} , $\pi_{2,3} : \mathbf{M} \times \mathbf{M} \times X \rightarrow \mathbf{M} \times X$ is the projection on the last two factors.

Then (i) \Leftrightarrow (ii) and (iii) implies both these assertions. Moreover, if \mathbf{M} is a group object, then all three assertions are equivalent.

Proof. We prove that (i) \Leftrightarrow (ii). Our first goal is to prove that

$$\sigma \cdot (1_{h_{\mathbf{M}}^{\mathcal{B}} \cdot p} \times \sigma) = \sigma \cdot (1_{h_{\mu}^{\mathcal{B}} \cdot p} \times 1_{h_{\xi}^{\mathcal{E}}})$$

if and only if

$$(1_{\mathbf{M}} \times a)^* \tau^{\sigma} \cdot \pi_{2,3}^* \tau^{\sigma} = (\mu \times 1_X)^* \tau^{\sigma}$$

First note that the commutative square of presheaves

$$\begin{array}{ccc}
h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{a^* \zeta}^{\mathcal{E}} & \xrightarrow{1_{h_{\mathbf{M}}^{\mathcal{B}} \cdot p} \times h_{a^* \zeta}^{\mathcal{E}}} & h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\zeta}^{\mathcal{E}} \\
\downarrow p_{\text{hom}} & & \downarrow p_{\text{hom}} \\
h_{\mathbf{M} \times \mathbf{M} \times X}^{\mathcal{B}} \cdot p & \xrightarrow{(h_{\mathbf{M}^* a}^{\mathcal{B}})_p} & h_{\mathbf{M} \times X}^{\mathcal{B}} \cdot p
\end{array}$$

on \mathcal{E} is cartesian. Next according to Fact 5.4 we infer that projections

$$pr_{h_{a^* \zeta}^{\mathcal{E}}} : h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{a^* \zeta}^{\mathcal{E}} \rightarrow h_{a^* \zeta}^{\mathcal{E}}, \quad pr_{h_{\zeta}^{\mathcal{E}}} : h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\zeta}^{\mathcal{E}} \rightarrow h_{\zeta}^{\mathcal{E}}$$

are representable by morphisms $\widetilde{\pi_{23} a^* \zeta} : \pi_{23}^* a^* \zeta \rightarrow a^* \zeta$, $\widetilde{\pi_{\zeta}} : \pi^* \zeta \rightarrow \zeta$ in \mathcal{E} , respectively. Thus $1_{h_{\mathbf{M}}^{\mathcal{B}} \cdot p} \times h_{a^* \zeta}^{\mathcal{E}}$ is representable by a cartesian morphism

$$\pi_{23}^* a^* \zeta \xrightarrow{\cong} (1_{\mathbf{M}} \times a)^* \pi^* \zeta \xrightarrow{(\widetilde{1_{\mathbf{M}} \times a})_{\pi^* \zeta}} \pi^* \zeta$$

where \cong is the identification described in Remark 5.3. Since we have equality

$$\sigma \cdot (1_{h_{\mathbf{M}}^{\mathcal{B}} \cdot p} \times \sigma) = h_{a^* \zeta}^{\mathcal{E}} \cdot h_{\tau^{\sigma}}^{\mathcal{E}} \cdot (1_{h_{\mathbf{M}}^{\mathcal{B}} \cdot p} \times h_{a^* \zeta}^{\mathcal{E}}) \cdot (1_{h_{\mathbf{M}}^{\mathcal{B}} \cdot p} \times h_{\tau^{\sigma}}^{\mathcal{E}})$$

we derive that $\sigma \cdot (1_{h_{\mathbf{M}}^{\mathcal{B}} \cdot p} \times \sigma)$ is representable (again up to identifications of Remark 5.3) by a morphism

$$\widetilde{a_{\zeta}} \cdot \tau^{\sigma} \cdot (\widetilde{1_{\mathbf{M}} \times a})_{\pi^* \zeta} \cdot \pi_{23}^* \tau^{\sigma} = \widetilde{a_{\zeta}} \cdot (\widetilde{1_{\mathbf{M}} \times a})_{a^* \zeta} \cdot (1_{\mathbf{M}} \times a)^* \tau^{\sigma} \cdot \pi_{23}^* \tau^{\sigma}$$

in \mathcal{E} . Next note that the square of presheaves on \mathcal{E}

$$\begin{array}{ccc}
h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\zeta}^{\mathcal{E}} & \xrightarrow{h_{\mu}^{\mathcal{B}} \cdot p \times 1_{h_{\zeta}^{\mathcal{E}}}} & h_{\mathbf{M} \cdot p}^{\mathcal{B}} \times h_{\zeta}^{\mathcal{E}} \\
\downarrow \text{can} \times p_{\text{hom}} & & \downarrow p_{\text{hom}} \\
h_{\mathbf{M} \times \mathbf{M} \times X}^{\mathcal{B}} \cdot p & \xrightarrow{(h_{\mu \times 1_X}^{\mathcal{B}})_p} & h_{\mathbf{M} \times X}^{\mathcal{B}} \cdot p
\end{array}$$

is cartesian. According to Fact 5.4 we infer that projections

$$pr_{h_{\mathbf{M} \cdot p}^{\mathcal{B}} \times h_{\zeta}^{\mathcal{E}}} : h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\zeta}^{\mathcal{E}} \rightarrow h_{\mathbf{M} \cdot p}^{\mathcal{B}} \times h_{\zeta}^{\mathcal{E}}, \quad pr_{h_{\zeta}^{\mathcal{E}}} : h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\zeta}^{\mathcal{E}} \rightarrow h_{\zeta}^{\mathcal{E}}$$

are representable by morphisms $\widetilde{\pi_{23} \pi^* \zeta} : \pi_{23}^* \pi^* \zeta \rightarrow \pi^* \zeta$, $\widetilde{\pi_{\zeta}} : \pi^* \zeta \rightarrow \zeta$ in \mathcal{E} , respectively. Thus $h_{\mu}^{\mathcal{B}} \cdot p \times 1_{h_{\zeta}^{\mathcal{E}}}$ is representable by a cartesian morphism

$$\pi_{23}^* \pi^* \zeta \xrightarrow{\cong} (\mu \times 1_X)^* \pi^* \zeta \xrightarrow{(\widetilde{\mu \times 1_X})_{\pi^* \zeta}} \pi^* \zeta$$

where \cong is the identification described in Remark 5.3. Since we have equality

$$\sigma \cdot (1_{h_{\mu}^{\mathcal{B}} \cdot p} \times 1_{h_{\zeta}^{\mathcal{E}}}) = h_{a^* \zeta}^{\mathcal{E}} \cdot h_{\tau^{\sigma}}^{\mathcal{E}} \cdot (1_{h_{\mu}^{\mathcal{B}} \cdot p} \times 1_{h_{\zeta}^{\mathcal{E}}})$$

we derive that $\sigma \cdot \left(1_{h_{\mu}^{\mathcal{B}} \cdot p} \times 1_{h_{\xi}^{\mathcal{E}}}\right)$ is representable (again up to identifications of Remark 5.3) by a morphism

$$\widetilde{a}_{\xi} \cdot \tau^{\sigma} \cdot (\widetilde{\mu \times 1_X})_{\pi^* \xi} = \widetilde{a}_{\xi} \cdot (\widetilde{\mu \times 1_X})_{a^* \xi} \cdot (\mu \times 1_X)^* \tau^{\sigma}$$

We deduce that

$$\sigma \cdot \left(1_{h_{\mathbf{M}}^{\mathcal{B}} \cdot p} \times \sigma\right) = \sigma \cdot \left(1_{h_{\mu}^{\mathcal{B}} \cdot p} \times 1_{h_{\xi}^{\mathcal{E}}}\right)$$

if and only if

$$\widetilde{a}_{\xi} \cdot (\widetilde{1_{\mathbf{M}} \times a})_{a^* \xi} \cdot (1_{\mathbf{M}} \times a)^* \tau^{\sigma} \cdot \pi_{23}^* \tau^{\sigma} = \widetilde{a}_{\xi} \cdot (\widetilde{\mu \times 1_X})_{a^* \xi} \cdot (\mu \times 1_X)^* \tau^{\sigma}$$

Since $a \cdot (1_{\mathbf{M}} \times a) = a \cdot (\mu \times 1_X)$ and according to Remark 5.3, we have canonical identification $\widetilde{a}_{\xi} \cdot (\widetilde{1_{\mathbf{M}} \times a})_{a^* \xi} = \widetilde{a}_{\xi} \cdot (\widetilde{\mu \times 1_X})_{a^* \xi}$ of these cartesian morphisms. Therefore, we deduce that the formula above holds if and only if

$$(1_{\mathbf{M}} \times a)^* \tau^{\sigma} \cdot \pi_{23}^* \tau^{\sigma} = (\mu \times 1_X)^* \tau^{\sigma}$$

This proves our first claim. Now it suffices to prove that

$$\sigma \cdot \langle h_e^{\mathcal{B}} \cdot p, 1_{h_{\xi}^{\mathcal{E}}} \rangle = 1_{h_{\xi}^{\mathcal{E}}}$$

if and only if $\langle e, 1_X \rangle^* \tau^{\sigma} = 1_{\xi}$. Note that the square of presheaves on \mathcal{E}

$$\begin{array}{ccc} h_{\xi}^{\mathcal{E}} & \xrightarrow{\langle h_e^{\mathcal{B}} \cdot p, 1_{h_{\xi}^{\mathcal{E}}} \rangle} & h_{\mathbf{M} \cdot p}^{\mathcal{B}} \times h_{\xi}^{\mathcal{E}} \\ p_{\text{hom}} \downarrow & & \downarrow p_{\text{hom}} \\ h_X^{\mathcal{B}} \cdot p & \xrightarrow{(h_{(e, 1_X)})_p} & h_{\mathbf{M} \times X \cdot p}^{\mathcal{B}} \end{array}$$

is cartesian. Thus according to Fact 5.4 we derive that $\langle h_e^{\mathcal{B}} \cdot p, 1_{h_{\xi}^{\mathcal{E}}} \rangle$ is representable by a morphism

$$\xi \xrightarrow{\cong} \langle e, 1_X \rangle^* \pi^* \xi \xrightarrow{\widetilde{\langle e, 1_X \rangle}_{\pi^* \xi}} \pi^* \xi$$

where \cong is the identification described in Remark 5.3. Therefore, the morphism $\sigma \cdot \langle h_e^{\mathcal{B}} \cdot p, 1_{h_{\xi}^{\mathcal{E}}} \rangle$ is representable (up to identifications of Remark 5.3) by

$$\widetilde{a}_{\xi} \cdot \tau^{\sigma} \cdot \widetilde{\langle e, 1_X \rangle}_{\pi^* \xi} = \widetilde{a}_{\xi} \cdot \widetilde{\langle e, 1_X \rangle}_{a^* \xi} \cdot \langle e, 1_X \rangle^* \tau^{\sigma} = \langle e, 1_X \rangle^* \tau^{\sigma}$$

Thus

$$\sigma \cdot \langle h_e^{\mathcal{B}} \cdot p, 1_{h_{\xi}^{\mathcal{E}}} \rangle = 1_{h_{\xi}^{\mathcal{E}}}$$

if and only if

$$\langle e, 1_X \rangle^* \tau^{\sigma} = 1_{\xi}$$

This finishes the proof of (i) \Leftrightarrow (ii). Now we prove that (iii) implies (ii). For this note that up to identification of Remark 5.3 we have

$$\langle e, 1_X \rangle^* \tau^{\sigma} = \langle e, e, 1_X \rangle^* (\mu \times 1_X)^* \tau^{\sigma} = \langle e, e, 1_X \rangle^* \left((1_{\mathbf{M}} \times a)^* \tau^{\sigma} \cdot \pi_{23}^* \tau^{\sigma} \right) = \langle e, 1_X \rangle^* \tau^{\sigma} \cdot \langle e, 1_X \rangle^* \tau^{\sigma}$$

Since τ^{σ} is an isomorphism, we derive that $\langle e, 1_X \rangle^* \tau^{\sigma} = 1_{\xi}$ and hence (ii) holds.

Finally we show that if \mathbf{M} is a group object, then (ii) implies (iii). For this note that

$$\begin{array}{ccc}
h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\xi}^{\mathcal{E}} & \xrightarrow{\sigma} & h_{\xi}^{\mathcal{E}} \\
p_{\text{hom}} \downarrow & & \downarrow p_{\text{hom}} \\
h_{\mathbf{M} \times X}^{\mathcal{B}} \cdot p & \xrightarrow{(h_a^{\mathcal{B}})_p} & h_X^{\mathcal{B}} \cdot p
\end{array}$$

is a cartesian square in this case. Therefore, α^{σ} is a cartesian morphism of p . Since $\alpha^{\sigma} = \tilde{a}_{\xi} \cdot \tau^{\sigma}$ and according to Fact 2.2, we derive that τ^{σ} is an isomorphism. Thus (iii) holds. \square

Fact 5.6. Let \mathbf{M}, X be objects of \mathcal{B} such that the cartesian product of \mathbf{M} and X exist. Let $a : \mathbf{M} \times X \rightarrow X$ be a morphism. Denote by $\pi : \mathbf{M} \times X \rightarrow X$ the projection on X . Consider objects ξ_1, ξ_2 in $p^{-1}(X)$ and let $\sigma_1 : h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\xi_1}^{\mathcal{E}} \rightarrow h_{\xi_1}^{\mathcal{E}}, \sigma_2 : h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\xi_2}^{\mathcal{E}} \rightarrow h_{\xi_2}^{\mathcal{E}}$ be morphisms of presheaves on \mathcal{E} . Suppose that squares

$$\begin{array}{ccc}
h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\xi_1}^{\mathcal{E}} & \xrightarrow{\sigma_1} & h_{\xi_1}^{\mathcal{E}} \\
p_{\text{hom}} \downarrow & & \downarrow p_{\text{hom}} \\
h_{\mathbf{M} \times X}^{\mathcal{B}} \cdot p & \xrightarrow{(h_a^{\mathcal{B}})_p} & h_X^{\mathcal{B}} \cdot p
\end{array}
\quad
\begin{array}{ccc}
h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\xi_2}^{\mathcal{E}} & \xrightarrow{\sigma_2} & h_{\xi_2}^{\mathcal{E}} \\
p_{\text{hom}} \downarrow & & \downarrow p_{\text{hom}} \\
h_{\mathbf{M} \times X}^{\mathcal{B}} \cdot p & \xrightarrow{(h_a^{\mathcal{B}})_p} & h_X^{\mathcal{B}} \cdot p
\end{array}$$

are commutative. Let $\phi : \xi_1 \rightarrow \xi_2$ be a morphism in \mathcal{E} . Then the following assertions are equivalent.

(i) The square

$$\begin{array}{ccc}
h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\xi_1}^{\mathcal{E}} & \xrightarrow{\sigma_1} & h_{\xi_1}^{\mathcal{E}} \\
1_{h_{\mathbf{M}}^{\mathcal{B}} \cdot p} \times h_{\phi}^{\mathcal{E}} \downarrow & & \downarrow h_{\phi}^{\mathcal{E}} \\
h_{\mathbf{M}}^{\mathcal{B}} \cdot p \times h_{\xi_2}^{\mathcal{E}} & \xrightarrow{\sigma_2} & h_{\xi_2}^{\mathcal{E}}
\end{array}$$

is commutative.

(ii) The square

$$\begin{array}{ccc}
\pi^* \xi_1 & \xrightarrow{\tau^{\sigma_1}} & a^* \xi_1 \\
\pi^* \phi \downarrow & & \downarrow a^* \phi \\
\pi^* \xi_2 & \xrightarrow{\tau^{\sigma_2}} & a^* \xi_2
\end{array}$$

is commutative.

Proof. Note that up to identifications of Remark 5.3 and according to Fact 5.4 morphism $h_{\phi}^{\mathcal{E}} \cdot \sigma_1$ is representable by

$$\phi \cdot \alpha^{\sigma_1} = \phi \cdot \tilde{a}_{\xi_1} \cdot \tau^{\sigma_1} = \tilde{a}_{\xi_2} \cdot a^* \phi \cdot \tau^{\sigma_1}$$

and on the other hand morphism $\sigma_2 \cdot (1_{h_{\mathbf{M}}^{\mathcal{B}} \cdot p} \times h_{\phi}^{\mathcal{E}})$ is representable by

$$\alpha^{\sigma_2} \cdot \pi^* \phi = \tilde{a}_{\xi_2} \cdot \tau^{\sigma_2} \cdot \pi^* \phi$$

Since \tilde{a}_{ξ_2} is cartesian with respect to p , we derive that

$$h_{\phi}^{\mathcal{E}} \cdot \sigma_1 = \sigma_2 \cdot (1_{h_{\mathbf{M}}^{\mathcal{B}} \cdot p} \times h_{\phi}^{\mathcal{E}})$$

if and only if

$$a^* \phi \cdot \tau^{\sigma_1} = \tau^{\sigma_2} \cdot \pi^* \phi$$

This proves the assertion. \square

Guided by these two results we formulate a general notion of equivariant object in a fibered category.

Definition 5.7. Let $M : \mathcal{B}^{\text{op}} \rightarrow \mathbf{Mon}$ be a presheaf of monoids on \mathcal{B} and assume that for some object X of \mathcal{B} the presheaf $h_X^{\mathcal{B}}$ admits an action of M given by the morphism $\alpha : M \times h_X^{\mathcal{B}} \rightarrow h_X^{\mathcal{B}}$. Consider an object ζ in $p^{-1}(X)$. Suppose that there is an action $\sigma : M \cdot p \times h_{\zeta}^{\mathcal{E}} \rightarrow h_{\zeta}^{\mathcal{E}}$ of a monoid presheaf $M \cdot p$ on $h_{\zeta}^{\mathcal{E}}$ such that the square

$$\begin{array}{ccc} M \cdot p \times h_{\zeta}^{\mathcal{E}} & \xrightarrow{\sigma} & h_{\zeta}^{\mathcal{E}} \\ 1_{M \cdot p} \times p_{\text{hom}} \downarrow & & \downarrow p_{\text{hom}} \\ M \cdot p \times h_X^{\mathcal{B}} \cdot p & \xrightarrow{\alpha_p} & h_X^{\mathcal{B}} \cdot p \end{array}$$

is commutative. Then a pair (ζ, σ) is called an *M-equivariant object over α* .

Definition 5.8. Let $M : \mathcal{B}^{\text{op}} \rightarrow \mathbf{Mon}$ be a presheaf of monoids on \mathcal{B} and assume that for some object X of \mathcal{B} the presheaf $h_X^{\mathcal{B}}$ admits an action of M given by the morphism $\alpha : M \times h_X^{\mathcal{B}} \rightarrow h_X^{\mathcal{B}}$. Suppose that (ζ_1, σ_1) and (ζ_2, σ_2) are objects over X with M -equivariant structures. Then a morphism $\phi : \zeta_1 \rightarrow \zeta_2$ in \mathcal{E} is *M-equivariant* if the square

$$\begin{array}{ccc} M \cdot p \times h_{\zeta_1}^{\mathcal{E}} & \xrightarrow{\sigma_1} & h_{\zeta_1}^{\mathcal{E}} \\ 1_{M \cdot p} \times h_{\phi}^{\mathcal{E}} \downarrow & & \downarrow \phi \\ M \cdot p \times h_{\zeta_2}^{\mathcal{E}} & \xrightarrow{\sigma_2} & h_{\zeta_2}^{\mathcal{E}} \end{array}$$

is commutative. By abuse of notation we denote by $p_{\mathbf{M}}^{-1}(X)$ the category of M -equivariant objects over α with respect to the fibered category p .

Now we apply Proposition 5.5 and Fact 5.6 to the fibered category $\mathcal{Q}\text{coh} \rightarrow \mathbf{Sch}_k$. We deduce the following.

Corollary 5.9. Suppose that \mathbf{M} is a monoid k -scheme that acts on a k -scheme X through morphism $a : \mathbf{M} \times_k X \rightarrow X$ of k -schemes. Then the category $\mathcal{Q}\text{coh}_{\mathbf{M}}(X)$ is isomorphic to the category of $h_{\mathbf{M}}^{\mathbf{Sch}_k}$ -objects over $h_a^{\mathbf{Sch}_k}$ with respect to the fibered category $\mathcal{Q}\text{coh} \rightarrow \mathbf{Sch}_k$.

6. EXAMPLE: THE FIBERED CATEGORY OF EQUIVARIANT OBJECTS OVER THE CATEGORY OF ACTIONS

Fix a category \mathcal{B} and a monoid presheaf M on \mathcal{B} . We define a category $\mathbf{Act}_M(\mathcal{B})$ as follows. An object of $\mathbf{Act}_M(\mathcal{B})$ is a pair (X, α) such that X is an object of \mathcal{B} and $\alpha : M \times h_X^{\mathcal{B}} \rightarrow h_X^{\mathcal{B}}$ is an action of M on $h_X^{\mathcal{B}}$. Next if (X, α) and (Y, β) are objects of $\mathbf{Act}_M(\mathcal{B})$, then a morphism in $\mathbf{Act}_M(\mathcal{B})$ between them consists of a morphism $f : X \rightarrow Y$ in \mathcal{B} such that $h_f^{\mathcal{E}}$ is M -equivariant with respect to M -actions α and β .

Definition 6.1. Let \mathcal{B} be a category and let M be a monoid presheaf on \mathcal{B} . We call $\mathbf{Act}_M(\mathcal{B})$ the *category of actions of M on \mathcal{B}* .

Let \mathcal{B} be a category with finite products and let \mathbf{M} be a monoid object in \mathcal{B} . Then (for future reference) we denote by $\mathbf{Act}_{\mathbf{M}}(\mathcal{B})$ the category of actions of the presheaf $h_{\mathbf{M}}^{\mathcal{B}}$ of monoids representable by \mathbf{M} on \mathcal{B} . Let us note the following result.

Proposition 6.2. *Let \mathcal{B} be a category and let M be a monoid presheaf on \mathcal{B} . Consider the canonical functor $\mathbf{Act}_M(\mathcal{B}) \rightarrow \mathcal{B}$ that forgets about action of M . This functor creates limits.*

Proof. Note that limits in the category of presheaves are determined levelwise, for an abstract monoid M (a monoid object in **Set**) the forgetful functor from M -sets to **Set** creates all limits and the Yoneda embedding preserves limits. The assertion is a consequence of these facts. \square

We are ready to start the main discussion of this section. Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a fibered category and let M be a presheaf of monoids on \mathcal{B} . We construct a fibered category $p_M : \mathcal{E}_M \rightarrow \mathbf{Act}_M(\mathcal{B})$ out of $p : \mathcal{E} \rightarrow \mathcal{B}$ and M . First we construct a category \mathcal{E}_M . The class of objects of \mathcal{E}_M is the union of all M -equivariant objects over objects of $\mathbf{Act}_M(\mathcal{B})$. Now suppose that (ζ, σ) and (η, τ) are M -equivariant object over objects (X, α) and (Y, β) in $\mathbf{Act}_M(\mathcal{B})$, respectively. A morphism in \mathcal{E}_M between these objects consists of a morphism $\phi : \zeta \rightarrow \eta$ in \mathcal{E} such that $h_\phi^\mathcal{E}$ is $M \cdot p$ -equivariant with respect to σ and τ . We also have a functor $p_M : \mathcal{E}_M \rightarrow \mathbf{Act}_M(\mathcal{B})$ that sends ϕ to $p(\phi)$, which is M -equivariant with respect to α and β . This means that we have a square diagram

$$\begin{array}{ccc} (\zeta, \sigma) & \xrightarrow{\phi} & (\eta, \tau) \\ \downarrow & & \downarrow \\ (X, \alpha) & \xrightarrow{p(\phi)} & (Y, \beta) \end{array}$$

Next let us assume that $f : X \rightarrow Y$ is a morphism of $\mathbf{Act}_M(\mathcal{B})$ between (X, α) and (Y, β) . Suppose that (η, τ) is object with M -equivariant structure over β and consider a cartesian square

$$\begin{array}{ccc} f^* \eta & \xrightarrow{\tilde{f}_\eta} & \eta \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

with respect to p . Then by Proposition 6.2 there exists a unique M -equivariant structure $(f^* \eta, \sigma)$ that gives rise to a square diagram

$$\begin{array}{ccc} (f^* \eta, \sigma) & \xrightarrow{\tilde{f}_\eta} & (\eta, \tau) \\ \downarrow & & \downarrow \\ (X, \alpha) & \xrightarrow{f} & (Y, \beta) \end{array}$$

with respect to p_M . This square is cartesian with respect to p_M .

Definition 6.3. Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a fibered category and let M be a presheaf of monoids on \mathcal{B} . The fibered category $p_M : \mathcal{E}_M \rightarrow \mathbf{Act}_M(\mathcal{B})$ is called *the fibered category of M -equivariant objects*.

Suppose that $p : \mathcal{E} \rightarrow \mathcal{B}$ be a fibered category and assume that \mathbf{M} is a monoid object of \mathcal{B} (in particular, we assume that \mathcal{B} admits terminal object). Then we denote the fibered category of $h_{\mathbf{M}}^\mathcal{B}$ equivariant objects by $p_{\mathbf{M}} : \mathcal{E}_{\mathbf{M}} \rightarrow \mathbf{Act}_{\mathbf{M}}(\mathcal{B})$.

Example 6.4. Let \mathcal{B} be a category with finite limits and let \mathbf{M} be a monoid object of \mathcal{B} . Consider the fibered category of arrows $p_{\mathbf{Arr}(\mathcal{B})} : \mathbf{Arr}(\mathcal{B}) \rightarrow \mathcal{B}$. By Proposition 6.2 the category $\mathbf{Act}_{\mathbf{M}}(\mathcal{B})$ admits finite limits and there exists an isomorphism

$$\begin{array}{ccc}
\mathrm{Arr}(\mathcal{B})_{\mathbf{M}} & \xrightarrow{\cong} & \mathrm{Arr}(\mathrm{Act}_{\mathbf{M}}(\mathcal{B})) \\
\downarrow (p_{\mathrm{Arr}(\mathcal{B})})_{\mathbf{M}} & & \downarrow p_{\mathrm{Arr}(\mathrm{Act}_{\mathbf{M}}(\mathcal{B}))} \\
& \mathrm{Act}_{\mathbf{M}}(\mathcal{B}) &
\end{array}$$

of fibered categories.

7. APPLICATIONS TO EQUIVARIANT QUASI-COHERENT SHEAVES

This section is devoted to applications in algebraic geometry. We fix a commutative ring k and a monoid k -scheme \mathbf{M} . We are interested in a fibered category $\mathbf{Alg}(\Omega\mathrm{coh})_{\mathbf{M}} \rightarrow \mathbf{Act}_{\mathbf{M}}(\mathbf{Sch}_k)$ constructed from the fibered category $\mathbf{Alg}(\Omega\mathrm{coh}) \rightarrow \mathbf{Sch}_k$ by means of procedure described in Section 6.

Definition 7.1. Let X be a k -scheme equipped with \mathbf{M} -action $a : \mathbf{M} \times_k X \rightarrow X$. By abuse of notation we denote the fiber of $\mathbf{Alg}(\Omega\mathrm{coh})_{\mathbf{M}} \rightarrow \mathbf{Sch}_k$ over (X, a) by $\mathbf{Alg}(\Omega\mathrm{coh})_{\mathbf{M}}(X)$. We call it *the category of quasi-coherent of \mathbf{M} -algebras on (X, a)* .

Remark 7.2. We describe $\mathbf{Alg}(\Omega\mathrm{coh})_{\mathbf{M}}(X)$ for X with \mathbf{M} -action a in terms that are used in algebraic geometry. We denote by $\pi : \mathbf{M} \times_k X \rightarrow X$ the projection. By Proposition 5.5 an object of $\mathbf{Alg}(\Omega\mathrm{coh})_{\mathbf{M}}(X)$ (that is a quasi-coherent \mathbf{M} -algebra) is a pair (\mathcal{A}, τ) consisting of a quasi-coherent algebra \mathcal{A} on X and a morphism $\tau : \pi^* \mathcal{A} \rightarrow a^* \mathcal{A}$ such that the following equalities

$$(\mu \times 1_X)^* \tau^\sigma = (1_{\mathbf{M}} \times a)^* \tau^\sigma \cdot \pi_{2,3}^* \tau^\sigma, \langle e, 1_X \rangle^* \tau^\sigma = 1_{\mathcal{A}}$$

hold, where $\mu : \mathbf{M} \times_k \mathbf{M} \rightarrow \mathbf{M}$ is the multiplication on \mathbf{M} , $\pi_{2,3} : \mathbf{M} \times_k \mathbf{M} \times_k X \rightarrow \mathbf{M} \times_k X$ is the projection on the last two factors and $e : \mathbf{1} \rightarrow \mathbf{M}$ is the unit of \mathbf{M} . Next according to Fact 5.6 a morphism in $\mathbf{Alg}(\Omega\mathrm{coh})_{\mathbf{M}}(X)$ between (\mathcal{A}_1, τ_1) and (\mathcal{A}_2, τ_2) is a morphism $\phi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ of quasi-coherent sheaves on X such that the square

$$\begin{array}{ccc}
\pi^* \mathcal{A}_1 & \xrightarrow{\tau_1} & a^* \mathcal{A}_1 \\
\downarrow \pi^* \phi & & \downarrow a^* \phi \\
\pi^* \mathcal{A}_2 & \xrightarrow{\tau_2} & a^* \mathcal{A}_2
\end{array}$$

is commutative.

Remark 7.3. We describe the category $\mathbf{Aff}(\mathbf{Sch}_k)_{\mathbf{M}} \rightarrow \mathbf{Act}_{\mathbf{M}}(\mathbf{Sch}_k)$ obtained from the fibered category $\mathbf{Aff}(\mathbf{Sch}_k) \rightarrow \mathbf{Sch}_k$ (Example 4.6) by means of construction described in Section 6. $\mathbf{Aff}(\mathbf{Sch}_k)_{\mathbf{M}}$ is a full fibered subcategory of $\mathbf{Act}_{\mathbf{M}}(\mathrm{Arr}(\mathbf{Sch}_k))$. Indeed, this follows from Examples 4.6 and 6.4. More concretely, if X is a k -scheme and $a : \mathbf{M} \times_k X \rightarrow X$ is an \mathbf{M} -action, then object of $\mathbf{Aff}(\mathbf{Sch}_k)_{\mathbf{M}}$ over (X, a) consists of an affine morphism of k -schemes $\psi : \tilde{X} \rightarrow X$ and an action $b : \mathbf{M} \times \tilde{X} \rightarrow \tilde{X}$ such that the square

$$\begin{array}{ccc}
\mathbf{M} \times_k \tilde{X} & \xrightarrow{b} & \tilde{X} \\
\downarrow 1_{\mathbf{M}} \times_k \psi & & \downarrow \psi \\
\mathbf{M} \times_k X & \xrightarrow{a} & X
\end{array}$$

is commutative.

Let us note the following result.

Corollary 7.4. *The equivalence of fibered categories (Example 4.8)*

$$\begin{array}{ccc} \mathbf{Alg}(\Omega\mathbf{coh}) & \xrightarrow{\text{Spec}} & \mathbf{Aff}(\mathbf{Sch}_k) \\ & \searrow & \swarrow \\ & \mathbf{Sch}_k & \end{array}$$

induces an equivalence of fibered categories

$$\begin{array}{ccc} \mathbf{Alg}(\Omega\mathbf{coh})_{\mathbf{M}} & \xrightarrow{\quad} & \mathbf{Aff}(\mathbf{Sch}_k)_{\mathbf{M}} \\ & \searrow & \swarrow \\ & \mathbf{Act}_{\mathbf{M}}(\mathbf{Sch}_k) & \end{array}$$

of \mathbf{M} -equivariant objects.

Remark 7.5. Let X be a k -scheme and denote by $\pi : \mathbf{M} \times_k X \rightarrow X$ the projection. We fix a quasi-coherent algebra \mathcal{A} on X . By Remark 4.7 we choose a cartesian square

$$\begin{array}{ccc} \text{Spec}_{\mathbf{M} \times_k X} \pi^* \mathcal{A} & \xrightarrow{\tilde{\pi}} & \text{Spec}_X \mathcal{A} \\ \downarrow & & \downarrow \\ \mathbf{M} \times_k X & \xrightarrow{\pi} & X \end{array}$$

in \mathbf{Sch}_k . Note that there is a unique isomorphism $\mathbf{M} \times_k \text{Spec}_X \mathcal{A} \cong \text{Spec}_{\mathbf{M} \times_k X} \pi^* \mathcal{A}$ such that its composition with $\tilde{\pi}$ is the projection $\mathbf{M} \times_k \text{Spec}_X \mathcal{A} \rightarrow \text{Spec}_X \mathcal{A}$ on the second factor.

Corollary 7.4 can be stated in a more concrete form by means of language described in Remark 7.2. We realize this translation in the next two corollaries.

Corollary 7.6. Let X be a k -scheme equipped with an action $a : \mathbf{M} \times_k X \rightarrow X$ of \mathbf{M} and denote by $\pi : \mathbf{M} \times_k X \rightarrow X$ the projection. Suppose that \mathcal{A} is a quasi-coherent algebra on X . Consider the cartesian square

$$\begin{array}{ccc} \text{Spec}_{\mathbf{M} \times_k X} a^* \mathcal{A} & \xrightarrow{\tilde{a}} & \text{Spec}_X \mathcal{A} \\ \downarrow & & \downarrow \\ \mathbf{M} \times_k X & \xrightarrow{a} & X \end{array}$$

in \mathbf{Sch}_k . Then the map

$$\tau \mapsto \tilde{a} \cdot \text{Spec}_{\mathbf{M} \times_k X} \tau$$

together with the unique isomorphism $\mathbf{M} \times_k \text{Spec}_X \mathcal{A} \cong \text{Spec}_{\mathbf{M} \times_k X} \pi^* \mathcal{A}$ described above (Remark 7.5) gives rise to a bijection between the class of structures (\mathcal{A}, τ) of quasi-coherent \mathbf{M} -algebras on \mathcal{A} and the class of actions $b : \mathbf{M} \times_k \text{Spec}_X \mathcal{A} \rightarrow \text{Spec}_X \mathcal{A}$ of \mathbf{M} such that b make the square

$$\begin{array}{ccc} \mathbf{M} \times_k \text{Spec}_X \mathcal{A} & \xrightarrow{b} & \text{Spec}_X \mathcal{A} \\ \downarrow & & \downarrow \\ \mathbf{M} \times_k X & \xrightarrow{a} & X \end{array}$$

commutative.

Corollary 7.7. *Let X be a k -scheme equipped with an action $a : \mathbf{M} \times_k X \rightarrow X$ of \mathbf{M} and denote by $\pi : \mathbf{M} \times_k X \rightarrow X$ the projection. Suppose that (\mathcal{A}_1, τ_1) and (\mathcal{A}_2, τ_2) are quasi-coherent \mathbf{M} -algebras on X . Consider cartesian squares*

$$\begin{array}{ccc} \mathrm{Spec}_{\mathbf{M} \times_k X} a^* \mathcal{A}_1 & \xrightarrow{\tilde{a}_1} & \mathrm{Spec}_X \mathcal{A}_1 \\ \downarrow & & \downarrow \\ \mathbf{M} \times_k X & \xrightarrow{a} & X \end{array} \quad \begin{array}{ccc} \mathrm{Spec}_{\mathbf{M} \times_k X} a^* \mathcal{A}_2 & \xrightarrow{\tilde{a}_2} & \mathrm{Spec}_X \mathcal{A}_2 \\ \downarrow & & \downarrow \\ \mathbf{M} \times_k X & \xrightarrow{a} & X \end{array}$$

in \mathbf{Sch}_k . Then the map

$$\phi \mapsto \mathrm{Spec}_X \phi$$

induces a bijection between the class of morphisms $\phi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ of quasi-coherent \mathbf{M} -algebras (with respect to τ_1 and τ_2) and the class of morphism of k -schemes with \mathbf{M} -actions $(\mathrm{Spec}_X \mathcal{A}_1, \tilde{a}_1 \cdot \mathrm{Spec}_{\mathbf{M} \times_k X} \tau_1) \rightarrow (\mathrm{Spec}_X \mathcal{A}_2, \tilde{a}_2 \cdot \mathrm{Spec}_{\mathbf{M} \times_k X} \tau_2)$.

Proof of Corollaries 7.6 and 7.7. The two results above are consequences of Corollary 7.4 and Remarks 7.2, 7.3. \square

Corollary 7.6 has a consequence which makes rigorous the intuitive statement that if X is a k -scheme equipped with an action $a : \mathbf{M} \times_k X \rightarrow X$ of \mathbf{M} , then there is a canonical structure of quasi-coherent \mathbf{M} -sheaf on \mathcal{O}_X . Indeed, observe that $\mathrm{Spec}_X \mathcal{O}_X$ can be canonically identified with X and the square

$$\begin{array}{ccc} \mathbf{M} \times_k X & \xrightarrow{a} & X \\ \downarrow 1_{\mathbf{M}} \times_k 1_X & & \downarrow 1_X \\ \mathbf{M} \times_k X & \xrightarrow{a} & X \end{array}$$

is commutative. Then Corollary 7.6 implies that there exists a corresponding structure of quasi-coherent \mathbf{M} -sheaf on \mathcal{O}_X .

Definition 7.8. Let X be a k -scheme equipped with an action $a : \mathbf{M} \times_k X \rightarrow X$ of \mathbf{M} . Then the structure of \mathbf{M} -sheaf on \mathcal{O}_X described above is called *the canonical structure of \mathbf{M} -sheaf on \mathcal{O}_X* .

We now describe this canonical structure more concretely.

Fact 7.9. Let X be a k -scheme equipped with an action $a : \mathbf{M} \times_k X \rightarrow X$ of \mathbf{M} and denote by $\pi : \mathbf{M} \times_k X \rightarrow X$ the projection. Then the canonical structure of \mathbf{M} -sheaf on \mathcal{O}_X is determined by the morphism $a_{\#}^{-1} \cdot \pi_{\#}$.

Proof. Consider cartesian squares

$$\begin{array}{ccc} \mathrm{Spec}_{\mathbf{M} \times_k X} \pi^* \mathcal{O}_X & \xrightarrow{\tilde{\pi}} & \mathrm{Spec}_X \mathcal{O}_X \\ \downarrow & & \downarrow \\ \mathbf{M} \times_k X & \xrightarrow{\pi} & X \end{array} \quad \begin{array}{ccc} \mathrm{Spec}_{\mathbf{M} \times_k X} a^* \mathcal{O}_X & \xrightarrow{\tilde{a}} & \mathrm{Spec}_X \mathcal{O}_X \\ \downarrow & & \downarrow \\ \mathbf{M} \times_k X & \xrightarrow{a} & X \end{array}$$

in \mathbf{Sch}_k . We have canonical identifications $\mathrm{Spec}_X \mathcal{O}_X = X$ and $\mathrm{Spec}_{\mathbf{M} \times_k X} \mathcal{O}_{\mathbf{M} \times_k X} = \mathbf{M} \times_k X$. Under these identifications squares

$$\begin{array}{ccc}
\mathbf{M} \times_k X & \xrightarrow{\tilde{\pi} \cdot \text{Spec}_{\mathbf{M} \times_k X} \pi_{\#}^{-1}} & X \\
1_{\mathbf{M} \times_k X} \downarrow & & \downarrow 1_X \\
\mathbf{M} \times_k X & \xrightarrow{\pi} & X
\end{array}
\quad
\begin{array}{ccc}
\mathbf{M} \times_k X & \xrightarrow{\tilde{a} \cdot \text{Spec}_{\mathbf{M} \times_k X} a_{\#}^{-1}} & X \\
1_{\mathbf{M} \times_k X} \downarrow & & \downarrow 1_X \\
\mathbf{M} \times_k X & \xrightarrow{a} & X
\end{array}$$

are cartesian in \mathbf{Sch}_k . Thus

$$a = \tilde{a} \cdot \text{Spec}_{\mathbf{M} \times_k X} (a_{\#}^{-1})$$

Next let $\tau : \pi^* \mathcal{O}_X \rightarrow a^* \mathcal{O}_X$ be a morphism giving the canonical structure of \mathbf{M} -sheaf on \mathcal{O}_X . Then we have $a = \tilde{a} \cdot \text{Spec}_{\mathbf{M} \times_k X} (\tau) \cdot \text{Spec}_{\mathbf{M} \times_k X} (\pi_{\#}^{-1})$ by definition of τ and Corollary 7.6. Thus

$$\tilde{a} \cdot \text{Spec}_{\mathbf{M} \times_k X} (a_{\#}^{-1}) = \tilde{a} \cdot \text{Spec}_{\mathbf{M} \times_k X} (\tau \cdot \pi_{\#}^{-1})$$

and this implies that $a_{\#}^{-1} = \tau \cdot \pi_{\#}^{-1}$. Hence $\tau = a_{\#}^{-1} \cdot \pi_{\#}$ because \tilde{a} is cartesian with respect to $\mathbf{Aff}(\mathbf{Sch}_k) \rightarrow \mathbf{Sch}_k$ and the functor $\text{Spec}_{\mathbf{M} \times_k X} : \mathbf{Alg}(\mathfrak{Qcoh}(\mathbf{M} \times_k X)) \rightarrow \mathbf{Aff}_{\mathbf{M} \times_k X}$ is an equivalence of categories. \square

Our next goal is to explain that the category of quasi-coherent \mathbf{M} -sheaves carries a canonical symmetric monoidal structure.

Remark 7.10. Let X be a k -scheme equipped with an action $a : \mathbf{M} \times_k X \rightarrow X$ of \mathbf{M} and denote by $\pi : \mathbf{M} \times_k X \rightarrow X$ the projection. Let (\mathcal{F}_1, τ_1) and (\mathcal{F}_2, τ_2) be quasi-coherent \mathbf{M} -sheaves on X . Then

$$(\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2, \tau_1 \otimes_{\mathcal{O}_X} \tau_2)$$

is a quasi-coherent \mathbf{M} -sheaf on X . This shows that $(-) \otimes_{\mathcal{O}_X} (-) : \mathfrak{Qcoh}(X) \times \mathfrak{Qcoh}(X) \rightarrow \mathfrak{Qcoh}(X)$ admits a lift to a functor $\mathfrak{Qcoh}_{\mathbf{M}}(X) \times \mathfrak{Qcoh}_{\mathbf{M}}(X) \rightarrow \mathfrak{Qcoh}_{\mathbf{M}}(X)$. This bifunctor together with $(\mathcal{O}_X, a_{\#} \cdot \pi_{\#}^{-1})$ and lifts of the associator isomorphisms, left and right units isomorphism and the symmetry isomorphism for $\mathfrak{Qcoh}(X)$ give rise to a symmetric monoidal structure on $\mathfrak{Qcoh}_{\mathbf{M}}(X)$. Note that the forgetful functor $\mathfrak{Qcoh}_{\mathbf{M}}(X) \rightarrow \mathfrak{Qcoh}(X)$ is symmetric monoidal functor. Moreover, we have the category of (commutative) algebras $\mathbf{Alg}(\mathfrak{Qcoh}_{\mathbf{M}}(X))$ in $\mathfrak{Qcoh}_{\mathbf{M}}(X)$ with respect to this symmetric monoidal structure. By definition it follows that this category is isomorphic with $\mathbf{Alg}(\mathfrak{Qcoh})_{\mathbf{M}}(X)$.

It is interesting to note that under mild assumptions the category of quasi-coherent \mathbf{M} -sheaves on a k -scheme with \mathbf{M} -action is an abelian category.

Theorem 7.11. *Let X be a k -scheme equipped with an action $a : \mathbf{M} \times_k X \rightarrow X$ of \mathbf{M} . Consider the forgetful functor $\mathfrak{Qcoh}_{\mathbf{M}}(X) \rightarrow \mathfrak{Qcoh}(X)$. Then the following assertions hold.*

- (1) *The forgetful functor $\mathfrak{Qcoh}_{\mathbf{M}}(X) \rightarrow \mathfrak{Qcoh}(X)$ creates colimits.*
- (2) *If a is flat morphism, then the forgetful functor $\mathfrak{Qcoh}_{\mathbf{M}}(X) \rightarrow \mathfrak{Qcoh}(X)$ creates finite limits.*

Proof. In the proof we denote by $\pi : \mathbf{M} \times_k X \rightarrow X$ the projection and by $\pi_{23} : \mathbf{M} \times_k \mathbf{M} \times_k X \rightarrow \mathbf{M} \times_k X$ the projection on the last two factors.

We prove that (1) holds. Consider a category I and a functor $F : I \rightarrow \mathfrak{Qcoh}_{\mathbf{M}}(X)$. Let $F(i) = (\mathcal{F}_i, \tau_i)$ for every $i \in I$. Assume that $\{u_i : \mathcal{F}_i \rightarrow \mathcal{F}\}_{i \in I}$ is a colimiting cocone in $\mathfrak{Qcoh}(X)$ for the composition of F with the forgetful functor $\mathfrak{Qcoh}_{\mathbf{M}}(X) \rightarrow \mathfrak{Qcoh}(X)$. Pick a morphism $\tau : \pi^* \mathcal{F} \rightarrow a^* \mathcal{F}$ of quasi-coherent sheaves on X . We first show that the following assertions are equivalent.

- (i) (\mathcal{F}, τ) is a quasi-coherent \mathbf{M} -sheaf on X and $\{u_i\}_{i \in I}$ are morphisms of quasi-coherent \mathbf{M} -sheaves with codomain (\mathcal{F}, τ) .
- (ii) Squares

$$\begin{array}{ccc}
\pi^* \mathcal{F}_i & \xrightarrow{\pi^* u_i} & \pi^* \mathcal{F} \\
\downarrow \tau_i & & \downarrow \tau \\
a^* \mathcal{F}_i & \xrightarrow{a^* u_i} & a^* \mathcal{F}
\end{array}$$

are commutative for all i in I .

The implication **(i)** \Rightarrow **(ii)** is a consequence of the definition of quasi-coherent \mathbf{M} -sheaves. We prove the implication **(ii)** \Rightarrow **(i)**. Assume that **(ii)** holds. Observe that the functor $\pi_{23}^* \pi^*$ preserves colimits and hence $\{\pi_{23}^* \pi^* u_i : \pi_{23}^* \pi^* \mathcal{F}_i \rightarrow \pi_{23}^* \pi^* \mathcal{F}\}_{i \in I}$ is a colimiting cone for the composition of F , the forgetful functor $\mathcal{Q}\text{coh}_{\mathbf{M}}(X) \rightarrow \mathcal{Q}\text{coh}(X)$ and $\pi_{23}^* \pi^*$. Note that we have

$$\begin{aligned}
(\mu \times_k 1_X)^* \tau \cdot \pi_{23}^* \pi^* u_i &= (\mu \times_k 1_X)^* \tau \cdot (\mu \times_k 1_X)^* \pi^* u_i = (\mu \times_k 1_X)^* a^* u_i \cdot (\mu \times_k 1_X)^* \tau_i = \\
&= (\mu \times_k 1_X)^* a^* u_i \cdot (1_{\mathbf{M}} \times_k a)^* \tau_i \cdot \pi_{23}^* \tau_i = (1_{\mathbf{M}} \times_k a)^* a^* u_i \cdot (1_{\mathbf{M}} \times_k a)^* \tau_i \cdot \pi_{23}^* \tau_i = \\
&= (1_{\mathbf{M}} \times_k a)^* \tau \cdot (1_{\mathbf{M}} \times_k a)^* \pi^* u_i \cdot \pi_{23}^* \tau_i = (1_{\mathbf{M}} \times_k a)^* \tau \cdot \pi_{23}^* a^* u_i \cdot \pi_{23}^* \tau_i = (1_{\mathbf{M}} \times_k a)^* \tau \cdot \pi_{23}^* \tau \cdot \pi_{23}^* \pi^* u_i
\end{aligned}$$

for every i in I . This shows that $(\mu \times_k 1_X)^* \tau = (1_{\mathbf{M}} \times_k a)^* \tau \cdot \pi_{23}^* \tau$ and hence (\mathcal{F}, τ) is a quasi-coherent \mathbf{M} -sheaf. Now since **(ii)** holds, we derive that $\{u_i : \mathcal{F}_i \rightarrow \mathcal{F}\}_{i \in I}$ are morphisms of quasi-coherent \mathbf{M} -sheaves with codomain (\mathcal{F}, τ) . This proves **(i)** and hence the proposition **(ii)** \Rightarrow **(i)** holds.

We prove now that **(ii)** holds for a unique τ . Indeed, if τ exists, then it is a morphism between cocones $\{a^* u_i : a^* \mathcal{F}_i \rightarrow a^* \mathcal{F}\}_{i \in I}$ and $\{\pi^* u_i \cdot \tau_i : a^* \mathcal{F} \rightarrow \pi^* \mathcal{F}\}_{i \in I}$ for the composition of F , the forgetful functor $\mathcal{Q}\text{coh}_{\mathbf{M}}(X) \rightarrow \mathcal{Q}\text{coh}(X)$ and a^* . Since a^* preserves colimits, we deduce that the cocone $\{a^* u_i : a^* \mathcal{F}_i \rightarrow a^* \mathcal{F}\}_{i \in I}$ is colimiting. Hence such morphism τ of cocones exists and is unique.

Concluding our arguments above we derive that there exists a unique $\tau : \pi^* \mathcal{F} \rightarrow a^* \mathcal{F}$ such that **(i)** holds. This means that there exists precisely one lift along the forgetful functor $\mathcal{Q}\text{coh}_{\mathbf{M}}(X) \rightarrow \mathcal{Q}\text{coh}(X)$ of the cocone $\{u_i : \mathcal{F}_i \rightarrow \mathcal{F}\}_{i \in I}$ for F composed with the forgetful functor $\mathcal{Q}\text{coh}_{\mathbf{M}}(X) \rightarrow \mathcal{Q}\text{coh}(X)$ to a cone for F . In order to finish the proof of **(1)** it suffices to prove that this lift is colimiting for F . Suppose that (\mathcal{G}, σ) is a quasi-coherent \mathbf{M} -sheaf and $\{v_i : \mathcal{F}_i \rightarrow \mathcal{G}\}_{i \in I}$ is a cocone for F . Then there exists a unique morphism $v : \mathcal{F} \rightarrow \mathcal{G}$ of quasi-coherent sheaves such that $v \cdot u_i = v_i$ for all i in I . It suffices to prove that v is a morphism between (\mathcal{F}, τ) and (\mathcal{G}, σ) , where τ satisfies **(ii)**. We have

$$\sigma \cdot \pi^* v \cdot \pi^* u_i = \sigma \cdot \pi^* v_i \cdot \tau_i = a^* v_i \cdot \tau_i = a^* v \cdot a^* u_i \cdot \tau_i = a^* v \cdot \tau \cdot \pi^* u_i$$

for every i in I . Since π^* preserves colimits, we derive that $\{\pi^* u_i : \pi^* \mathcal{F}_i \rightarrow \pi^* \mathcal{F}\}_{i \in I}$ is a colimiting cocone for F composed with the forgetful functor $\mathcal{Q}\text{coh}_{\mathbf{M}}(X) \rightarrow \mathcal{Q}\text{coh}(X)$ and π^* . Thus $\sigma \cdot \pi^* v = a^* v \cdot \tau$ and this finishes the proof of **(1)**.

The proof of **(2)** is analogical to **(1)**. As above consider a finite category I and a functor $F : I \rightarrow \mathcal{Q}\text{coh}_{\mathbf{M}}(X)$. Let $F(i) = (\mathcal{F}_i, \tau_i)$ for every $i \in I$. Assume that $\{p_i : \mathcal{F} \rightarrow \mathcal{F}_i\}_{i \in I}$ is a limiting cone in $\mathcal{Q}\text{coh}(X)$ for the composition of F with the forgetful functor $\mathcal{Q}\text{coh}_{\mathbf{M}}(X) \rightarrow \mathcal{Q}\text{coh}(X)$. Pick a morphism $\tau : \pi^* \mathcal{F} \rightarrow a^* \mathcal{F}$ of quasi-coherent sheaves on X . We show that the following assertions are equivalent.

(i) (\mathcal{F}, τ) is a quasi-coherent \mathbf{M} -sheaf on X and $\{p_i\}_{i \in I}$ are morphisms of quasi-coherent \mathbf{M} -sheaves with domain (\mathcal{F}, τ) .

(ii) Squares

$$\begin{array}{ccc}
\pi^* \mathcal{F} & \xrightarrow{\pi^* p_i} & \pi^* \mathcal{F}_i \\
\tau \downarrow & & \downarrow \tau_i \\
a^* \mathcal{F} & \xrightarrow{a^* p_i} & a^* \mathcal{F}_i
\end{array}$$

are commutative for all i in I .

As above the implication (i) \Rightarrow (ii) is a consequence of the definition of quasi-coherent \mathbf{M} -sheaf. We now prove (ii) \Rightarrow (i). We have

$$\begin{aligned}
(1_{\mathbf{M}} \times_k a)^* a^* p_i \cdot (\mu \times_k 1_X)^* \tau &= (\mu \times_k 1_X)^* a^* p_i \cdot (\mu \times_k 1_X)^* \tau = (\mu \times_k 1_X)^* \tau_i \cdot (\mu \times_k 1_X)^* \pi^* p_i = \\
&= (1_{\mathbf{M}} \times_k a)^* \tau_i \cdot \pi_{23}^* \tau_i \cdot (\mu \times_k 1_X)^* \pi^* p_i = (1_{\mathbf{M}} \times_k a)^* \tau_i \cdot \pi_{23}^* \tau_i \cdot \pi_{23}^* \pi^* p_i = (1_{\mathbf{M}} \times_k a)^* \tau_i \cdot \pi_{23}^* a^* p_i \cdot \pi_{23}^* \tau = \\
&= (1_{\mathbf{M}} \times_k a)^* \tau_i \cdot (1_{\mathbf{M}} \times_k a)^* \pi^* p_i \cdot \pi_{23}^* \tau = (1_{\mathbf{M}} \times_k a)^* a^* p_i \cdot (1_{\mathbf{M}} \times_k a)^* \tau \cdot \pi_{23}^* \tau
\end{aligned}$$

for every i in I and since both a^* , $(1_{\mathbf{M}} \times_k a)^*$ preserve finite limits (a is flat), we derive that $(\mu \times_k 1_X)^* \tau = (1_{\mathbf{M}} \times_k a)^* \tau \cdot \pi_{23}^* \tau$. Thus this implication holds.

Next due to the fact that a^* preserves finite limits (a is flat), we deduce that (ii) holds for a unique morphism $\tau : \pi^* \mathcal{F} \rightarrow a^* \mathcal{F}$. Thus we conclude that there exists a unique lift along the forgetful functor $\Omega\text{coh}_{\mathbf{M}}(X) \rightarrow \Omega\text{coh}(X)$ of a cone $\{p_i : \mathcal{F} \rightarrow \mathcal{F}_i\}_{i \in I}$ for the composition of F with $\Omega\text{coh}_{\mathbf{M}}(X) \rightarrow \Omega\text{coh}(X)$ to a cone for F . In order to finish the proof it suffices to observe that this unique lift is a limiting cone for F . Suppose that (\mathcal{G}, σ) is a quasi-coherent \mathbf{M} -sheaf and $\{q_i : \mathcal{G} \rightarrow \mathcal{F}_i\}_{i \in I}$ is a cone for F . Then there exists a unique morphism $q : \mathcal{G} \rightarrow \mathcal{F}$ of quasi-coherent sheaves such that $p_i \cdot q = q_i$ for all i in I . It suffices to prove that q is a morphism between (\mathcal{G}, σ) and (\mathcal{F}, τ) , where τ satisfies (ii). We have

$$a^* p_i \cdot a^* q \cdot \sigma = a^* q_i \cdot \sigma = \tau_i \cdot \pi^* q_i = \tau_i \cdot \pi^* p_i \cdot \pi^* q = a^* p_i \cdot \tau \cdot a^* q$$

for every i in I . Since a^* preserves finite limits (a is flat), we deduce $a^* q \cdot \sigma = \tau \cdot a^* q$ and this finishes the proof of (2). \square

Corollary 7.12. *Let X be a k -scheme equipped with an action $a : \mathbf{M} \times_k X \rightarrow X$ of \mathbf{M} . Suppose that a is flat. Then $\Omega\text{coh}_{\mathbf{M}}(X)$ is an **Ab5**-category.*

Proof. This follows from Theorem 7.11 and the fact that $\Omega\text{coh}(X)$ is **Ab5**. \square

Definition 7.13. Let X be a k -scheme equipped with an action $a : \mathbf{M} \times_k X \rightarrow X$ of \mathbf{M} . Consider a locally closed subscheme Z of X . Then Z is called \mathbf{M} -invariant if the restriction of a to $\mathbf{M} \times_k Z$ factors scheme-theoretically through Z .

Now we discuss relation between quasi-coherent \mathbf{M} -ideals and \mathbf{M} -invariant closed subschemes.

Proposition 7.14. *Let X be a k -scheme equipped with an action $a : \mathbf{M} \times_k X \rightarrow X$ of \mathbf{M} and denote by $\pi : \mathbf{M} \times_k X \rightarrow X$ the projection. Suppose that Z is a closed subscheme of X and recall that \mathcal{O}_X carries the canonical \mathbf{M} -sheaf structure. Consider the following assertions.*

- (i) Z is an \mathbf{M} -invariant closed subscheme of X .
- (ii) There exists a unique structure of quasi-coherent \mathbf{M} -algebra on \mathcal{O}_Z .
- (iii) There exists a structure of quasi-coherent \mathbf{M} -algebra on \mathcal{O}_Z and the canonical epimorphism $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$ is a morphism of \mathbf{M} -sheaves.
- (iv) Let \mathcal{I} be a quasi-coherent ideal of Z . There exists a unique structure of quasi-coherent \mathbf{M} -sheaf on \mathcal{I} such that $\mathcal{I} \hookrightarrow \mathcal{O}_X$ is a morphism of \mathbf{M} -sheaves.

Then (i), (ii), (iii) are equivalent and (iv) implies all these assertions. Moreover, if a is flat, then all assertions are equivalent.

Proof. Note that there exists at most one \mathbf{M} -action on Z such that $Z \hookrightarrow X$ is \mathbf{M} -equivariant. Thus Corollary 7.6 implies that (i) is equivalent with (ii).

Suppose that (i) holds. Then there exists a \mathbf{M} -action on Z such that $Z \hookrightarrow X$ is \mathbf{M} -equivariant. Thus $\mathrm{Spec}_X \mathcal{O}_Z \hookrightarrow \mathrm{Spec}_X \mathcal{O}_X$ induced by the epimorphism $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$ is \mathbf{M} -equivariant and hence by Corollaries 7.6, 7.7 we derive that there exists a structure of quasi-coherent \mathbf{M} -algebra on \mathcal{O}_Z such that $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$ is a morphism of \mathbf{M} -algebras. Hence (i) \Rightarrow (iii).

Next if (iii) holds, then by Corollary 7.6 we derive that there exists \mathbf{M} -action on Z such that $Z \hookrightarrow X$ is \mathbf{M} -equivariant. Therefore, the implication (iii) \Rightarrow (i) holds.

This completes the proof that (i), (ii), (iii) are equivalent.

Now suppose that (iv) holds. Then there exists a morphism $\tau : \pi^* \mathcal{I} \rightarrow a^* \mathcal{I}$ such that $\mathcal{I} \hookrightarrow \mathcal{O}_X$ is a morphism of quasi-coherent \mathbf{M} -sheaves with respect to (\mathcal{I}, τ) . Then there exists a unique morphism $\sigma : \pi^* \mathcal{O}_Z \rightarrow a^* \mathcal{O}_Z$ of quasi-coherent sheaves such that the diagram

$$\begin{array}{ccccc} \pi^* \mathcal{I} & \longrightarrow & \pi^* \mathcal{O}_X & \twoheadrightarrow & \pi^* \mathcal{O}_Z \\ \tau \downarrow & & a_{\#}^{-1} \cdot \pi_{\#} \downarrow & & \downarrow \sigma \\ a^* \mathcal{I} & \longrightarrow & a^* \mathcal{O}_X & \twoheadrightarrow & a^* \mathcal{O}_Z \end{array}$$

is commutative. Moreover, σ is a morphism of $\mathcal{O}_{\mathbf{M} \times_k X}$ -algebras. The epimorphism $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$ is a cokernel of $\mathcal{I} \hookrightarrow \mathcal{O}_X$ in $\mathcal{Q}\mathrm{coh}(X)$. By Theorem 7.11 there exists a unique lift of this cokernel to $\mathcal{Q}\mathrm{coh}_{\mathbf{M}}(X)$. Since σ is a unique morphism of quasi-coherent sheaves on $\mathbf{M} \times_k X$ that makes the diagram above commutative, we derive that $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$ is a cokernel of $\mathcal{I} \hookrightarrow \mathcal{O}_X$ in $\mathcal{Q}\mathrm{coh}_{\mathbf{M}}(X)$ with respect to the structure of \mathbf{M} -sheaf determined by σ . Thus (\mathcal{O}_Z, σ) is quasi-coherent \mathbf{M} -sheaf on X . Since σ is a morphism of $\mathcal{O}_{\mathbf{M} \times_k X}$ -algebras, we derive that (\mathcal{O}_Z, σ) is a quasi-coherent \mathbf{M} -algebra on X . Thus (iv) \Rightarrow (iii).

Assume that a is flat and (iii) holds. Since $\mathcal{I} \hookrightarrow \mathcal{O}_X$ is a kernel in $\mathcal{Q}\mathrm{coh}(X)$ of $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$, we derive by Theorem 7.11 that there exists a structure of \mathbf{M} -sheaf on \mathcal{I} such that $\mathcal{I} \hookrightarrow \mathcal{O}_X$ is a morphism in $\mathcal{Q}\mathrm{coh}_{\mathbf{M}}(X)$. This shows that if a is flat, then (iii) implies (iv). \square

8. EXAMPLE: PRINCIPAL BUNDLES

We devote this whole section to another important example of a fibered category. We fix a category with finite limits \mathcal{B} and a monoid object \mathbf{M} of \mathcal{B} . We denote by $\mu : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{M}$ and $e : \mathbf{1} \rightarrow \mathbf{M}$ the multiplication and unit of \mathbf{M} , respectively.

Definition 8.1. Let \mathcal{P} be an object of \mathcal{B} equipped with an action of \mathbf{M} , let T be an object of \mathcal{B} with trivial action of \mathbf{M} and let $\pi : \mathcal{P} \rightarrow T$ be an \mathbf{M} -equivariant morphism with respect to these \mathbf{M} -actions. We say that \mathbf{M} -equivariant morphism π is a *trivial principal \mathbf{M} -bundle on T* if there exists an \mathbf{M} -equivariant isomorphism $\phi : \mathcal{P} \rightarrow \mathbf{M} \times T$ such that $\mathbf{M} \times T$ is equipped with an action of \mathbf{M} given by $\mu \times 1_T$ and the triangle

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\phi} & \mathbf{M} \times T \\ \pi \searrow & & \swarrow \mathrm{pr}_T \\ & T & \end{array}$$

is commutative.

Definition 8.2. Let \mathcal{P} be an object of \mathcal{B} equipped with an action of \mathbf{M} , let T be an object of \mathcal{B} with trivial action of \mathbf{M} and let $\pi : \mathcal{P} \rightarrow T$ be a \mathbf{M} -equivariant morphism with respect to these \mathbf{M} -actions. Consider a sieve S on T . For every arrow $g : \tilde{T} \rightarrow T$ in S we construct a cartesian square

$$\begin{array}{ccc}
g^*\mathcal{P} & \longrightarrow & \mathcal{P} \\
\pi_g \downarrow & & \downarrow \pi \\
\tilde{T} & \xrightarrow{g} & T
\end{array}$$

in \mathcal{B} . We consider g as an \mathbf{M} -equivariant morphism with respect to trivial \mathbf{M} -actions on T and \tilde{T} . Then there exists a unique action of \mathbf{M} on $g^*\mathcal{P}$ which makes π_g into an \mathbf{M} -equivariant morphism in such a way that the square consists of objects of \mathcal{B} with \mathbf{M} -actions and \mathbf{M} -equivariant morphisms. Suppose that \mathbf{M} -equivariant morphism π_g is a trivial principal \mathbf{M} -bundle on \tilde{T} for every g in S . Then we say that S *trivializes* π .

In the remaining part of this section we fix a Grothendieck topology \mathcal{J} on \mathcal{B} .

Definition 8.3. Let \mathcal{P} be an object of \mathcal{B} equipped with an action of \mathbf{M} , let T be an object of \mathcal{B} with trivial action of \mathbf{M} and let $\pi : \mathcal{P} \rightarrow T$ be a \mathbf{M} -equivariant morphism with respect to these \mathbf{M} -actions. Suppose that there exists a covering sieve S in $\mathcal{J}(T)$ that trivializes π . Then π is called a *principal \mathbf{M} -bundle with respect to \mathcal{J}* .

Now we define a category \mathbf{BM} that depends on the site $(\mathcal{B}, \mathcal{J})$. Its objects are principal \mathbf{M} -bundles with respect to \mathcal{J} and if $\pi : \mathcal{P} \rightarrow T$ and $\psi : Q \rightarrow Z$ are principal \mathbf{M} -bundles with respect to \mathcal{J} , then a morphism $\pi \rightarrow \psi$ is a pair (f, ϕ) such that $f : T \rightarrow Z$ and $\phi : \mathcal{P} \rightarrow Q$ are morphisms in \mathcal{B} such that ϕ is \mathbf{M} -equivariant and the square

$$\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\phi} & Q \\
\pi \downarrow & & \downarrow \psi \\
T & \xrightarrow{f} & Z
\end{array}$$

is commutative. We have a functor $p_{\mathbf{M}, \mathcal{J}} : \mathbf{BM} \rightarrow \mathcal{B}$ given by $p_{\mathbf{M}, \mathcal{J}}((f, \phi)) = f$. Let $\psi : Q \rightarrow Z$ be a principal \mathbf{M} -bundle with respect to \mathcal{J} and let $f : T \rightarrow Z$ be a morphism. Consider the cartesian square

$$\begin{array}{ccc}
f^*Q & \xrightarrow{\phi} & Q \\
\pi \downarrow & & \downarrow \psi \\
T & \xrightarrow{f} & Z
\end{array}$$

in \mathcal{B} . Then by the universal property there exists a unique action of \mathbf{M} on f^*Q such that the square above consists of \mathbf{M} -equivariant morphisms (T, Z are equipped with trivial \mathbf{M} -actions). Moreover, with respect to this action $\psi : f^*Q \rightarrow T$ becomes a principal \mathbf{M} -bundle with respect to \mathcal{J} . Indeed, if S is in $\mathcal{J}(Z)$ and S trivializes ψ , then its pullback f^*S trivializes π and is an element of $\mathcal{J}(T)$ (by definition of a Grothendieck topology). This shows that $p_{\mathbf{M}, \mathcal{J}} : \mathbf{BM} \rightarrow \mathcal{B}$ is a fibered category. Moreover, we have a functor $\mathbf{BM} \rightarrow \text{Arr}(\mathcal{B})$ that forgets about \mathbf{M} -actions. Hence there exists commutative triangle

$$\begin{array}{ccc}
\mathbf{BM} & \longrightarrow & \text{Arr}(\mathcal{B}) \\
\searrow p_{\mathbf{M}, \mathcal{J}} & & \swarrow p_{\text{Arr}(\mathcal{B})} \\
& \mathcal{B} &
\end{array}$$

According to Example 2.5 and description of cartesian morphisms of $p_{\mathbf{M}, \mathcal{J}}$ the functor $\mathbf{BM} \rightarrow \text{Arr}(\mathcal{B})$ described above is a morphism of fibered categories.

Definition 8.4. $p_{\mathbf{M}, \mathcal{J}} : \mathbf{BM} \rightarrow \mathcal{B}$ is called *the fibered category of principal \mathbf{M} -bundles on $(\mathcal{B}, \mathcal{J})$* .

Suppose that X is an object of \mathcal{B} equipped with an action $a : \mathbf{M} \times X \rightarrow X$ of \mathbf{M} . We define a category $[X/\mathbf{M}]$ depending on a and the site $(\mathcal{B}, \mathcal{J})$ as follows. Its objects are pairs (π, α) such that π is a principal \mathbf{M} -bundle with respect to \mathcal{J} and α is an \mathbf{M} -equivariant morphism. We depict them by diagrams

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\alpha} & X \\ \pi \downarrow & & \\ T & & \end{array}$$

Suppose that $(\pi : \mathcal{P} \rightarrow T, \alpha : \mathcal{P} \rightarrow X)$ and $(\psi : Q \rightarrow Z, \beta : Q \rightarrow X)$ are two such objects. Then a morphism $(\pi, \alpha) \rightarrow (\psi, \beta)$ is a morphism $(f, \phi) : \pi \rightarrow \psi$ in $\mathbb{B}\mathbf{M}$ such that $\alpha = \beta \cdot \phi$. We have a functor $[X/\mathbf{M}] \rightarrow \mathbb{B}\mathbf{M}$ which sends (π, α) to π . We denote by $p_{a, \mathcal{J}} : [X/\mathbf{M}] \rightarrow \mathcal{B}$ the composition of this functor $[X/\mathbf{M}] \rightarrow \mathbb{B}\mathbf{M}$ with $p_{\mathbf{M}, \mathcal{J}} : \mathbb{B}\mathbf{M} \rightarrow \mathcal{B}$. By description of cartesian morphisms of $p_{\mathbf{M}, \mathcal{J}}$ we deduce that $p_{a, \mathcal{J}}$ is a fibered category. We have a commutative triangle

$$\begin{array}{ccc} [X/\mathbf{M}] & \xrightarrow{\quad} & \mathbb{B}\mathbf{M} \\ & \searrow p_{a, \mathcal{J}} & \swarrow p_{\mathbf{M}, \mathcal{J}} \\ & \mathcal{B} & \end{array}$$

and the functor $[X/\mathbf{M}] \rightarrow \mathbb{B}\mathbf{M}$ described above is a morphism of fibered categories. Note that if $\mathbf{1}$ is a terminal object of \mathcal{B} equipped with trivial action of \mathbf{M} , then we have a canonical isomorphism $[\mathbf{1}/\mathbf{M}] \cong \mathbb{B}\mathbf{M}$ of categories over \mathcal{B} .

Definition 8.5. $p_{a, \mathcal{J}} : [X/\mathbf{M}] \rightarrow \mathcal{B}$ is called *the quotient fibered category of \mathbf{M} -object X on $(\mathcal{B}, \mathcal{J})$* .

Results below show that up to some mild assumptions on Grothendieck topology \mathcal{J} fibered category $p_{a, \mathcal{J}} : [X/\mathbf{M}] \rightarrow \mathcal{B}$ encapsulates all essential information concerning action of \mathbf{M} on X . We start with the following observation.

Fact 8.6. Let X, Y be objects of \mathcal{B} equipped with actions $a : \mathbf{M} \times X \rightarrow X$ and $b : \mathbf{M} \times Y \rightarrow Y$ of \mathbf{M} . Consider a functor $F : [X/\mathbf{M}] \rightarrow [Y/\mathbf{M}]$ such that the triangle

$$\begin{array}{ccc} [X/\mathbf{M}] & \xrightarrow{F} & [Y/\mathbf{M}] \\ & \searrow & \swarrow \\ & \mathbb{B}\mathbf{M} & \end{array}$$

is commutative, where two other sides are canonical functors. Then F is a morphism of fibered categories $p_{a, \mathcal{J}}$ and $p_{b, \mathcal{J}}$.

Proof. The commutativity of the triangle implies that $F \cdot p_{b, \mathcal{J}} = p_{a, \mathcal{J}}$. Since a morphism in $[X/\mathbf{M}]$ is cartesian with respect to $p_{a, \mathcal{J}}$ if and only if its image under the canonical functor $[X/\mathbf{M}] \rightarrow \mathbb{B}\mathbf{M}$ is cartesian with respect to $p_{\mathbf{M}, \mathcal{J}}$ and the same holds for $p_{b, \mathcal{J}}$, we derive that F sends cartesian morphisms of $p_{a, \mathcal{J}}$ to cartesian morphisms of $p_{b, \mathcal{J}}$. \square

Let X, Y be objects of \mathcal{B} equipped with actions $a : \mathbf{M} \times X \rightarrow X$ and $b : \mathbf{M} \times Y \rightarrow Y$ of \mathbf{M} . We denote the class of functors in Fact 8.6 by $\text{Mor}_{\mathbb{B}\mathbf{M}}([X/\mathbf{M}], [Y/\mathbf{M}])$. We also denote (by abuse of notation) the class of \mathbf{M} -equivariant morphism $(X, a) \rightarrow (Y, b)$ by $\text{Mor}_{\mathbf{M}}(X, Y)$.

Theorem 8.7. Let $(\mathcal{B}, \mathcal{J})$ be a Grothendieck site and assume that representable presheaves on \mathcal{B} are separated with respect to \mathcal{J} . Let X, Y be objects of \mathcal{B} equipped with \mathbf{M} -actions $a : \mathbf{M} \times X \rightarrow X$ and $b : \mathbf{M} \times Y \rightarrow Y$, respectively. Then there exists a bijection

$$\text{Mor}_{\mathbf{M}}(X, Y) \cong \text{Mor}_{\mathbb{B}\mathbf{M}}([X/\mathbf{M}], [Y/\mathbf{M}])$$

that sends an \mathbf{M} -equivariant morphism f to a functor $F : [X/\mathbf{M}] \rightarrow [Y/\mathbf{M}]$ given by

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{d} & X \\ \pi \downarrow & & \\ T & & \end{array} \quad \xrightarrow{F} \quad \begin{array}{ccc} \mathcal{P} & \xrightarrow{f \cdot d} & Y \\ \pi \downarrow & & \\ T & & \end{array}$$

Proof. We first describe certain object of $[X/\mathbf{M}]$. Observe that $(\mathbf{M} \times X, \mu \times 1_X)$ is an object of \mathcal{B} equipped with the action of \mathbf{M} . Next the projection $\text{pr}_X : \mathbf{M} \times X \rightarrow X$ can be considered as an \mathbf{M} -equivariant morphism from this \mathbf{M} -object to X with the trivial action of \mathbf{M} . Since the square

$$\begin{array}{ccc} \mathbf{M} \times \mathbf{M} \times X & \xrightarrow{1_{\mathbf{M}} \times a} & \mathbf{M} \times X \\ \mu \times 1_X \downarrow & & \downarrow a \\ \mathbf{M} \times X & \xrightarrow{a} & X \end{array}$$

is commutative, we derive that a is an \mathbf{M} -equivariant morphism $(\mathbf{M} \times X, \mu \times 1_X) \rightarrow (X, a)$. This gives (pr_X, a) the structure of an object of $[X/\mathbf{M}]$. Fix a functor F in $\text{Mor}_{\mathbf{BM}}([X/\mathbf{M}], [Y/\mathbf{M}])$. The functor F sends (pr_X, a) to some object of $[Y/\mathbf{M}]$. This object is necessarily of the form (pr_X, α) for some \mathbf{M} -equivariant morphism $\alpha : (\mathbf{M} \times X, \mu \times 1_X) \rightarrow (Y, b)$. Indeed, this follows from the fact that F is over \mathbf{BM} . Now if F is determined by some \mathbf{M} -equivariant morphism f as it is described in the statement, then $\alpha = f \cdot a$ and hence $f = \alpha \cdot \langle e, 1_X \rangle$. This proves that the map $\text{Mor}_{\mathbf{M}}(X, Y) \rightarrow \text{Mor}_{\mathbf{BM}}([X/\mathbf{M}], [Y/\mathbf{M}])$ described in the statement is injective. Our goal is to show that it is surjective. That is our goal is to show that for the functor F in $\text{Mor}_{\mathbf{BM}}([X/\mathbf{M}], [Y/\mathbf{M}])$ a morphism $f = \alpha \cdot \langle e, 1_X \rangle$ is \mathbf{M} -equivariant and determines F as it is described in the statement. First we fix some object T of \mathcal{B} and the projection $\text{pr}_T : \mathbf{M} \times T \rightarrow T$ considered as a trivial principal \mathbf{M} -bundle. Let (pr_T, c) be an object of $[X/\mathbf{M}]$. Then c is an \mathbf{M} -equivariant morphism $c : (\mathbf{M} \times T, \mu \times 1_T) \rightarrow (X, a)$. Functor F sends (pr_T, c) to some object (pr_T, γ) . We claim that $\gamma = f \cdot c$. Let $\text{pr}_{23} : \mathbf{M} \times \mathbf{M} \times T \rightarrow \mathbf{M} \times T$ be the projection on the last two factors. There are diagrams

$$\begin{array}{ccc} \mathbf{M} \times \mathbf{M} \times T & \xrightarrow{\mu \times 1_T} & \mathbf{M} \times T \xrightarrow{c} X \\ \text{pr}_{23} \downarrow & & \downarrow \text{pr}_T \\ \mathbf{M} \times T & \xrightarrow{\text{pr}_T} & T \end{array} \quad \begin{array}{ccc} \mathbf{M} \times \mathbf{M} \times T & \xrightarrow{1_{\mathbf{M}} \times c} & \mathbf{M} \times X \xrightarrow{a} X \\ \text{pr}_{23} \downarrow & & \downarrow \text{pr}_X \\ \mathbf{M} \times T & \xrightarrow{c} & X \end{array}$$

representing morphisms

$$(\text{pr}_T, \mu \times 1_T) : (\text{pr}_{23}, c \cdot (\mu \times 1_T)) \rightarrow (\text{pr}_T, c), (c, 1_{\mathbf{M}} \times c) : (\text{pr}_{23}, a \cdot (1_{\mathbf{M}} \times c)) \rightarrow (\text{pr}_X, a)$$

in $[X/\mathbf{M}]$. Moreover, c is \mathbf{M} -equivariant $(\mathbf{M} \times T, \mu \times 1_T) \rightarrow (X, a)$ and hence we derive that $c \cdot (\mu \times 1_T) = a \cdot (c \times 1_{\mathbf{M}})$. Thus the morphisms in $[X/\mathbf{M}]$ described above have common domain. Since F is over \mathbf{BM} , we derive that their images under F are

$$\begin{array}{ccc}
\mathbf{M} \times \mathbf{M} \times T & \xrightarrow{\mu \times 1_T} & \mathbf{M} \times T \xrightarrow{\gamma} X \\
\text{pr}_{23} \downarrow & & \downarrow \text{pr}_T \\
\mathbf{M} \times T & \xrightarrow{\text{pr}_T} & T
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{M} \times \mathbf{M} \times T & \xrightarrow{1_{\mathbf{M}} \times c} & \mathbf{M} \times X \xrightarrow{\alpha} X \\
\text{pr}_{23} \downarrow & & \downarrow \text{pr}_X \\
\mathbf{M} \times T & \xrightarrow{c} & X
\end{array}$$

This implies that $\gamma \cdot (\mu \times 1_T) = \alpha \cdot (1_{\mathbf{M}} \times c)$. We deduce that

$$\gamma = \gamma \cdot (\mu \times 1_T) \cdot \langle e, 1_{\mathbf{M} \times X} \rangle = \alpha \cdot (1_{\mathbf{M}} \times c) \cdot \langle e, 1_{\mathbf{M} \times X} \rangle = \alpha \cdot \langle e, 1_X \rangle \cdot c = f \cdot c$$

and the claim is proved. We apply this to α to derive that $\alpha = f \cdot a$. Next recall that $\alpha \cdot (\mu \times 1_X) = b \cdot (1_{\mathbf{M}} \times \alpha)$ because α is an \mathbf{M} -equivariant morphism $(\mathbf{M} \times X, \mu \times 1_X) \rightarrow (Y, b)$. Thus

$$b \cdot (1_{\mathbf{M}} \times f) = b \cdot (1_{\mathbf{M}} \times \alpha) \cdot (1_{\mathbf{M}} \times \langle e, 1_X \rangle) = \alpha \cdot (\mu \times 1_X) \cdot (1_{\mathbf{M}} \times \langle e, 1_X \rangle) = \alpha$$

Hence $f \cdot a = \alpha = b \cdot (1_{\mathbf{M}} \times f)$. Thus f is \mathbf{M} -equivariant and F is given as in the statement on the subcategory of $[X/\mathbf{M}]$ consisting of trivial principal \mathbf{M} -bundles. Now consider any principal \mathbf{M} -bundle $\pi : \mathcal{P} \rightarrow T$ with respect to \mathcal{J} and let $d : \mathcal{P} \rightarrow X$ be a \mathbf{M} -equivariant morphism to (X, a) . We know that F sends (π, d) to some object of $[Y/\mathbf{M}]$ of the form (π, δ) . It suffices to prove that $\delta = f \cdot d$. For this consider a sieve S in $\mathcal{J}(T)$ such that S trivializes π . Pick $g : \tilde{T} \rightarrow T$ in S and a cartesian square

$$\begin{array}{ccc}
g^* \mathcal{P} & \xrightarrow{g'} & \mathcal{P} \\
\pi_g \downarrow & & \downarrow \pi \\
\tilde{T} & \xrightarrow{g} & T
\end{array}$$

Then $(\pi_g, d \cdot g')$ is an object of $[X/\mathbf{M}]$. Since F is over \mathbf{BM} , we derive that $F(\pi_g, d \cdot g') = (\pi_g, \delta \cdot g')$. By definition π_g is trivial \mathbf{M} -bundle. Thus (from what we proved above) we have

$$\delta \cdot g' = f \cdot d \cdot g'$$

This holds for pullback g' of every g in S along π . These pullbacks $\{g'\}_{g \in S}$ generate some sieve S' on \mathcal{P} and the formula

$$\delta \cdot h = f \cdot d \cdot h$$

holds for every h in S' . Moreover, S' is a covering sieve on a site $(\mathcal{B}, \mathcal{J})$ i.e. $S' \in \mathcal{J}(\mathcal{P})$. According to the assumption on \mathcal{J} we infer that $h_{\mathcal{P}}^{\mathcal{B}} = \text{Mor}_{\mathcal{B}}(-, \mathcal{P}) : \mathcal{B}^{\text{op}} \rightarrow \mathbf{Set}$ is a separated presheaf with respect to \mathcal{J} . Thus the formula

$$\delta \cdot h = f \cdot d \cdot h$$

which holds for every h in S' implies that $\delta = f \cdot d$. □