SMOOTH, UNRAMIFIED AND ÉTALE MORPHISMS

1. Smooth morphisms

Definition 1.1. Let $f : X \to Y$ be a morphism of schemes and let x be a point in X. Suppose that there is an open affine neighborhood U of x in X and an open affine subscheme V of Y such that $f(U) \subseteq V$ and there is an open immersion

$$U \xhookrightarrow{i} \operatorname{Spec}\Gamma(V, \mathcal{O}_Y)[x_1, ..., x_{n+r}]/(f_1, ..., f_r)$$

where $f_1, ..., f_r \in \Gamma(V, \mathcal{O}_Y)[x_1, ..., x_{n+1}]$ are polynomials. In addition assume that *the Jacobian matrix*

$$\frac{\partial(f_1, ..., f_r)}{\partial(x_1, ..., x_{n+r})} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & ... & \frac{\partial f_1}{\partial x_{n+r}} \\ ... & ... & ... \\ \frac{\partial f_r}{\partial x_1} & ... & \frac{\partial f_r}{\partial x_{n+r}} \end{pmatrix}$$

of rank r at x. Then we say that f is *smooth of relative dimension n at x*.

Definition 1.2. Let $f : X \to Y$ be a morphism of schemes. For every $n \in \mathbb{N}$ we define

$$\operatorname{SmoothLocus}_n(f) = \left\{ x \in X \mid f \text{ is smooth of relative dimension } n \text{ at } x \right\}$$

We also define

$$\operatorname{SmoothLocus}(f) = \bigcup_{n \in \mathbb{N}} \operatorname{SmoothLocus}_n(f)$$

and call it *the smooth locus of f*.

Theorem 1.3. *Let $f : X \to Y$ be a morphism of schemes smooth of relative dimension n at some point x of X. Then there exist open affine neighbourhoods U of x in X and V of $f(x)$ in Y such that the following assertions hold.*

(1) *$f(U)$ is a subset of V.*

(2) *The restriction of f to the morphism $U \to V$ is formally smooth and of finite presentation.*

(3) *$\Omega_{X/Y_{|U}}$ is locally free of rank n.*

Proof. Assume that f is smooth of relative dimension n at x. By definition there exists an open affine neighborhood W of x and an open affine subset V of Y such that $f(W) \subseteq V$ and locally on W the morphism f factors as an open immersion

$$i : W \hookrightarrow \operatorname{Spec}\Gamma(V, \mathcal{O}_Y)[x_1, ..., x_{n+r}]/(f_1, ..., f_r)$$

composed with the structural morphism

$$\operatorname{Spec}\Gamma(V, \mathcal{O}_Y)[x_1, ..., x_{n+r}]/(f_1, ..., f_r) \to V$$

in such a way that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & ... & \frac{\partial f_1}{\partial x_{n+r}} \\ ... & ... & ... \\ \frac{\partial f_r}{\partial x_1} & ... & \frac{\partial f_r}{\partial x_{n+r}} \end{pmatrix}$$

is of rank r at x. Now suppose that $j_1,...,j_r$ are indices of columns of the Jacobian matrix evaluated at x that are linearly independent over $k(x)$. Let

$$\delta = \det\left(\left[\frac{\partial f_i}{\partial x_{j_k}}\right]_{1\leq i\leq r, 1\leq k\leq r}\right) \in \Gamma(V,\mathcal{O}_Y)[x_1,...,x_{n+r}]$$

By assumption $\delta(x) \neq 0$ and hence there exists an open affine neighbourhood U of x in W such that $\delta(z) \neq 0$ for every $z \in U$. Let Q be an open subset of $\operatorname{Spec}\Gamma(V,\mathcal{O}_Y)[x_1,...,x_{n+r}]$ contained in the nonvanishing set of δ such that $U = Q \cap \operatorname{Spec}\Gamma(V,\mathcal{O}_Y)[x_1,...,x_{n+r}]/(f_1,...,f_r)$. Clearly Q is formally smooth over V and a morphism $j : U \hookrightarrow Q$ induced by i is a closed immersion. We have the conormal sequence

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\sigma} j^*\Omega_{Q/V} \longrightarrow \Omega_{U/V} \longrightarrow 0$$

of j. Here $\mathcal{I}$ is a quasi-coherent ideal in $\mathcal{O}_Q$ determining j. Now $j^*\Omega_{Q/V}$ is a locally free sheaf of $\mathcal{O}_U$-modules with basis

$$j^*d(x_{1|Q}), ..., j^*d(x_{n+r|Q})$$

and $\mathcal{I}/\mathcal{I}^2$ is a sheaf of $\mathcal{O}_U$-modules generated by

$$j^*(f_{1|Q}), ..., j^*(f_{r|Q})$$

Let $p : \mathcal{O}_U^r \twoheadrightarrow \mathcal{I}/\mathcal{I}^2$ be the epimorphism of sheaves of $\mathcal{O}_U$-modules determined by sections $j^*(f_{1|Q})$,...,$j^*(f_{r|Q})$ of $\mathcal{I}/\mathcal{I}^2$. The composition of p with the morphism $\sigma : \mathcal{I}/\mathcal{I}^2 \to j^*\Omega_{Q/V}$ coming from the conormal sequence is given by the transpose of a matrix

$$\mathbf{J} = \left[j^*\left(\frac{\partial f_i}{\partial x_j}_{|Q}\right)\right]_{1\leq i\leq r, 1\leq j\leq n+r}$$

Next we define a matrix $\mathbf{S} = [g_{ij}]_{1\leq i\leq r, 1\leq j\leq n+r}$ of regular functions on U as follows. We set $g_{ij} = 0$ if $j \neq j_k$ for $k = 1,...,r$ and we define the matrix $[g_{ij_k}]_{1\leq i\leq r, 1\leq k\leq r}$ to be the inverse of the matrix

$$\left[j^*\left(\frac{\partial f_i}{\partial x_{j_k}}_{|Q}\right)\right]_{1\leq k\leq r, 1\leq i\leq r}$$

Such an inverse exists according to the fact that $j^*(\delta_{|Q})$ is invertible on U. Note that $\mathbf{S}\cdot\mathbf{J}^T$ is the $r\times r$ identity matrix. This implies that $\sigma\cdot p$ admits a section $s : j^*\Omega_{Q/V} \to \mathcal{O}_U^r$. In particular, p is a monomorphism. Since it is an epimorphism, we derive that p is an isomorphism and thus σ is a monorphism having a section. Therefore, the conormal sequence for j is split exact. Thus $\Omega_{X/Y_{|U}} = \Omega_{U/V}$ is free of rank n and $U \to V$ is formally smooth. Obviously $U \to V$ is of finite presentation. □

Proposition 1.4. *Let A be a commutative ring. Suppose that $f_1,...,f_s \in A[x_1,...,x_m]$ are polynomials and $\mathfrak{q}$ is a prime ideal in $A[x_1,...,x_m]$ such that $f_1,...,f_s \in \mathfrak{q}$. Let $\mathfrak{p} = A\cap\mathfrak{q}$ be a prime ideal of A. Assume that the rank of the Jacobian matrix*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_m} \end{pmatrix}$$

is r at $\mathfrak{q}$. Let $j_1,...,j_r$ be numbers of rows of that matrix that are linearly independent at $\mathfrak{q}$ and consider the surjective morphism

$$\pi : A[x_1,...,x_m]/(f_{j_1},...,f_{j_r}) \twoheadrightarrow A[x_1,...,x_m]/(f_1,...,f_s)$$

Then the following assertions hold.

(1) *If $\operatorname{Spec} A[x_1,...,x_m]_{\mathfrak{q}}/(f_1,...,f_s)_{\mathfrak{q}} \to \operatorname{Spec} A_{\mathfrak{p}}$ is formally smooth, then $\pi_{\mathfrak{q}}$ is an isomorphism.*

(2) *If $\pi_{\mathfrak{q}}$ is an isomorphism, then* $\operatorname{Spec} A[x_1, ..., x_m]/(f_1, ..., f_s) \to \operatorname{Spec} A$ *is smooth of relative dimension $m - r$ at $\mathfrak{q}$.*

Proof. For convenience in the proof we write $B = A[x_1, ..., x_m]/(f_1, ..., f_s)$, $C = A[x_1, ..., x_m]/(f_{j_1}, ..., f_{j_r})$, $\mathfrak{b} = (f_1, ..., f_s)$, $\mathfrak{c} = (f_{j_1}, ..., f_{j_r})$.
Assume that $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is formally smooth. Note that we have a commutative diagram

$$\begin{tikzcd}
\mathfrak{b}_{\mathfrak{q}}/\mathfrak{b}_{\mathfrak{q}}^2 \arrow[r] & B_{\mathfrak{q}} \otimes_{A[x_1,...,x_m]_{\mathfrak{q}}} \Omega_{A[x_1,...,x_m]_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] & \Omega_{B_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] & 0 \\
\mathfrak{c}_{\mathfrak{q}}/\mathfrak{c}_{\mathfrak{q}}^2 \arrow[r] \arrow[u] & C_{\mathfrak{q}} \otimes_{A[x_1,...,x_m]_{\mathfrak{q}}} \Omega_{A[x_1,...,x_m]_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] \arrow[u] & \Omega_{C_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

in which rows are conormal sequences of $A[x_1, ..., x_m]_{\mathfrak{q}} \twoheadrightarrow B_{\mathfrak{q}}$ and $A[x_1, ..., x_m]_{\mathfrak{q}} \twoheadrightarrow C_{\mathfrak{q}}$. Observe that the bottom row is split exact as $\operatorname{Spec} C \to \operatorname{Spec} A$ is smooth of relative dimension $m - r$ at $\mathfrak{q}$. Similarly by formal smoothness of $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ the top row is split exact. Hence after tensoring with $k(\mathfrak{q})$ we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & k(\mathfrak{q}) \otimes_{A[x_1,...,x_m]_{\mathfrak{q}}} \mathfrak{b}_{\mathfrak{q}}/\mathfrak{b}_{\mathfrak{q}}^2 \arrow[r] & k(\mathfrak{q}) \otimes_{A[x_1,...,x_m]_{\mathfrak{q}}} \Omega_{A[x_1,...,x_m]_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] & k(\mathfrak{q}) \otimes_{B_{\mathfrak{q}}} \Omega_{B_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] & 0 \\
0 \arrow[r] & k(\mathfrak{q}) \otimes_{A[x_1,...,x_m]_{\mathfrak{q}}} \mathfrak{c}_{\mathfrak{q}}/\mathfrak{c}_{\mathfrak{q}}^2 \arrow[r] \arrow[u] & k(\mathfrak{q}) \otimes_{A[x_1,...,x_m]_{\mathfrak{q}}} \Omega_{A[x_1,...,x_m]_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] \arrow[u, "="'] & k(\mathfrak{q}) \otimes_{C_{\mathfrak{q}}} \Omega_{C_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

According to the choice of $f_{j_1}, ..., f_{j_r}$, we derive that

$$\dim_{k(\mathfrak{q})}\big(k(\mathfrak{q}) \otimes_{B_{\mathfrak{q}}} \Omega_{B_{\mathfrak{q}}/A_{\mathfrak{p}}}\big) = m - r = \dim_{k(\mathfrak{q})}\big(k(\mathfrak{q}) \otimes_{C_{\mathfrak{q}}} \Omega_{C_{\mathfrak{q}}/A_{\mathfrak{p}}}\big)$$

In particular, the rightmost vertical morphism in the diagram is an epimorphism of vector spaces over $k(\mathfrak{q})$ of the same finite dimension. Thus it is an isomorphism. This implies that the leftmost morphism in the diagram is an isomorphism. Thus by virtue of Nakayama lemma $\mathfrak{c}_{\mathfrak{q}}/\mathfrak{c}_{\mathfrak{q}}^2 \to \mathfrak{b}_{\mathfrak{q}}/\mathfrak{b}_{\mathfrak{q}}^2$ is an epimorphism. Using Nakayama lemma once more we deduce that $\mathfrak{c}_{\mathfrak{q}} \hookrightarrow \mathfrak{b}_{\mathfrak{q}}$ is an epimorphism and thus $\mathfrak{c}_{\mathfrak{q}} = \mathfrak{b}_{\mathfrak{q}}$. Hence $\pi_{\mathfrak{q}}$ is an isomorphism and the proof of **(1)** is completed.
On the other hand if $\pi_{\mathfrak{q}}$ is an isomorphism, then $\ker(\pi)$ is a finitely generated ideal in C that vanishes at $\mathfrak{q}$. Hence it vanishes on some open neighborhood of $\mathfrak{q}$ in $\operatorname{Spec} C$ and thus $\operatorname{Spec} \pi$ is an isomorphism on that neighborhood. Thus $\operatorname{Spec} B \to \operatorname{Spec} A$ is smooth of relative dimension $m - r$ at $\mathfrak{q}$ and **(2)** is proved. □

Corollary 1.5. *Let $f : X \to Y$ be a morphism of schemes and let x be a point in X. Then the following conditions are equivalent.*

- **(i)** *f is smooth at x.*
- **(ii)** *There exists an open neighborhood U of x in X such that $f_{|U}$ is formally smooth and locally of finite presentation*
- **(iii)** *There exists an open neighborhood U of x in X such that $f_{|U}$ is locally of finite presentation and the local morphism $f^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is formally smooth.*

Proof. The implication **(i)** ⇒ **(ii)** follows from Theorem 1.3.
Suppose that **(ii)** holds. We want to prove **(iii)**. Since this is a local assertion, we may assume that there exists a closed immersion $i : U \to \mathbb{A}^m_Y$. Let $\mathcal{I}$ be a quasi-coherent ideal determining i. Then the conormal sequence

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow i^*\Omega_{\mathbb{A}^m_Y/Y} \longrightarrow \Omega_{U/Y} \longrightarrow 0$$

is locally split exact. Thus after localizing at x we derive a split exact sequence

$$\mathcal{I}_x/\mathcal{I}_x^2 \longrightarrow \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{\mathbb{A}^m_Y,i(x)}} \Omega_{\mathcal{O}_{\mathbb{A}^m_Y,i(x)}/\mathcal{O}_{Y,f(x)}} \longrightarrow \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}} \longrightarrow 0$$

which implies that $f^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is formally smooth.
Finally **(iii)** $\Rightarrow$ **(i)** is an easy consequence of Proposition 1.4. □

Corollary 1.6. *Let $f : X \to Y$ be a morphism of schemes and let $n \in \mathbb{N}$ be a natural number. Then* $\mathrm{SmoothLocus}_n(f)$ *is open subset of X and the sheaf $\Omega_{X/Y}$ is locally free of rank n on* $\mathrm{SmoothLocus}_n(f)$.

Proof. It follows from Corollary 1.5 that $\mathrm{SmoothLocus}(f)$ is open subset of X. From Theorem 1.3 we derive that $\Omega_{X/Y|\mathrm{SmoothLocus}(f)}$ is locally free of finite rank and

$$\mathrm{SmoothLocus}_n(f) = \left\{x \in \mathrm{SmoothLocus}(f) \,\middle|\, \mathrm{rank}\left(\Omega_{X/Y_x}\right) = n\right\}$$

Thus $\mathrm{SmoothLocus}_n(f)$ is open subset of X and $\Omega_{X/Y}$ is locally free of rank n on $\mathrm{SmoothLocus}_n(f)$. □

Proposition 1.7. *The following assertions hold.*

(1) *Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of schemes. Consider the cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

If f is smooth of relative dimension n at $x \in X$ and $x' \in X'$ is a point such that $g'(x') = x$, then f' is smooth of relative dimension n at x'.

(2) *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If f is smooth of relative dimension n at $x \in X$ and g is smooth of relative dimension m at $f(x)$, then $g \cdot f$ is smooth of relative dimension $n + m$ at x.*

Proof. Observe that classes of formally smooth and locally of finite presentation morphisms are closed under base change and composition. Therefore, by Corollary 1.5 in order to prove assertions it is enough to check relative dimensions. For this we use Corollary 1.6 and compute ranks of sheaves of differentials. In case **(1)** observe that $g'^*\Omega_{X/Y} \cong \Omega_{X'/Y'}$ and hence if $\Omega_{X/Y}$ has rank n at point x, then $\Omega_{X'/Y'}$ has rank n at x'. For **(2)** consider an exact sequence

$$f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

which splits locally in neighborhood of x. Since $f^*\Omega_{Y/Z}$ has rank m at x and $\Omega_{X/Y}$ has rank n at x, we derive that $\Omega_{X/Z}$ has rank $n + m$ at x. □

Theorem 1.8. *Let $f : X \to Y$ be a morphism locally of finite presentation. Suppose that $g : Y' \to Y$ is a morphism of schemes. Assume that g is flat at some point y' of Y'. Consider the cartesian diagram*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Let x be a point of X and x' be a point of X' lying over x. Then the following assertions are equivalent.

- **(i)** *f is smooth at x.*
- **(ii)** *f' is smooth at x'.*

Proof. By Proposition 1.7 it suffices to prove **(ii)** $\Rightarrow$ **(i)**. Let $y = f(x)$. Clearly both f and f' are locally of finite presentation. The question is local so we may assume that schemes X, Y, Y' are affine. Consider closed immersion $i : X \hookrightarrow \mathbb{A}^m_Y$ determined by the ideal $\mathcal{I}$. Let $i' : X' \hookrightarrow \mathbb{A}^m_{Y'}$ be the base change of i along $g : Y' \to Y$ and denote its ideal by $\mathcal{I}'$. Then the conormal sequence

$$(\mathcal{I}'/\mathcal{I}'^2)_{x'} \longrightarrow (i'^*\Omega_{\mathbb{A}^m_{Y'}/Y'})_{x'} \longrightarrow (\Omega_{X'/Y'})_{x'} \longrightarrow 0$$

of i' at x' is a split short exact sequence. Since it is canonically isomorphic to the sequence

$$\mathcal{O}_{X',x'} \otimes_{\mathcal{O}_{X,x}} (\mathcal{I}/\mathcal{I}^2)_x \longrightarrow \mathcal{O}_{X',x'} \otimes_{\mathcal{O}_{X,x}} (i^*\Omega_{\mathbb{A}^m_Y/Y})_x \longrightarrow \mathcal{O}_{X',x'} \otimes_{\mathcal{O}_{X,x}} (\Omega_{X/Y})_x \longrightarrow 0$$

i.e the conormal sequence for i at x tensored with $\mathcal{O}_{X',x'}$ over $\mathcal{O}_{X,x}$. We utilize the assumption that $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is faithfully flat to deduce that the conormal sequence

$$(\mathcal{I}/\mathcal{I}^2)_x \longrightarrow (i^*\Omega_{\mathbb{A}^m_Y/Y})_x \longrightarrow (\Omega_{X/Y})_x \longrightarrow 0$$

is short exact and $\mathcal{O}_{X,x}$-module $(\Omega_{X/Y})_x$ is flat. Since $(\Omega_{X/Y})_x$ is finitely presented, we derive that it is free and hence the conormal sequence

$$(\mathcal{I}/\mathcal{I}^2)_x \longrightarrow (i^*\Omega_{\mathbb{A}^m_Y/Y})_x \longrightarrow (\Omega_{X/Y})_x \longrightarrow 0$$

is a split short exact sequence. Therefore, $f^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is formally smooth and, since f is locally of finite presentation, we derive by Corollary 1.5 that f is smooth at x. $\square$

2. Smoothness for schemes over a field

In this section we use the following notation. Let X be a scheme over a field k. For every field extension K of k we denote by X_K the K-scheme $\operatorname{Spec} K \times_{\operatorname{Spec} k} X$.

Definition 2.1. Let X be a scheme over a field k and let x be a point of X. Then X is *geometrically regular at x* if for every field extension K of k and every point $\overline{x}$ in X_K lying over x the scheme X_K is regular at $\overline{x}$.

Theorem 2.2. *Let X be a scheme locally of finite type over a field k and let x be a point of X. Then the inequality*

$$\dim_x(X) \leq \dim_{k(x)}\big(k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/k}\big)$$

holds and the following assertions are equivalent.

(i) *X is smooth at x.*

(ii) *X is geometrically regular over k at x.*

(iii) *For some perfect extension K of k and some point $\overline{x} \in X_K$ lying over x the local ring $\mathcal{O}_{X_K,\overline{x}}$ is regular.*

(iv) $\dim_x(X) = \dim_{k(x)}\big(k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/k}\big)$

We prove the theorem in a series of lemmas.

Lemma 2.2.1. *Let K be a perfect field and let X be a scheme locally of finite type over K. Suppose that x is a point X. Then*

$$\dim_x(X) \leq \dim_{k(x)}\big(k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/K}\big)$$

and the equality holds if and only if $\mathcal{O}_{X,x}$ is regular.

Proof of the lemma. Consider the conormal sequence

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/k} \longrightarrow \Omega_{k(x)/K} \longrightarrow 0$$

of the closed immersion $\operatorname{Spec} k(x) \hookrightarrow \operatorname{Spec} \mathcal{O}_{X,x}$ of K-schemes. Since $k(x)$ is formally smooth over K by [Monygham, 2021, Corollary 6.4] and K is perfect, we derive that the sequence above is short exact. Hence

$$\dim_x(X) = \dim\big(\mathcal{O}_{X,x}\big) + \operatorname{tr}_K\big(k(x)\big) \leq \dim_{k(x)}\big(\mathfrak{m}_x/\mathfrak{m}_x^2\big) + \operatorname{tr}_K\big(k(x)\big) =$$

$$= \dim_{k(x)}\big(\mathfrak{m}_x/\mathfrak{m}_x^2\big) + \dim_{k(x)}\big(\Omega_{k(x)/K}\big) = \dim_{k(x)}\big(k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/K}\big)$$

and the equality holds if and only if $\dim\big(\mathcal{O}_{X,x}\big) = \dim_{k(x)}\big(\mathfrak{m}_x/\mathfrak{m}_x^2\big)$. This equality is equivalent to regularity of $\mathcal{O}_{X,x}$. □

Lemma 2.2.2. *Let K be a perfect field and let X be a scheme locally of finite type over K. Suppose that x is a point X. If $\mathcal{O}_{X,x}$ is regular, then X is smooth at x.*

Proof of the lemma. Since the assertions are local we may assume that there exists a closed immersion $i : X \to \mathbb{A}^m_K$ given by a quasi-coherent ideal $\mathcal{I}$ generated by polynomials $f_1,...,f_s \in K[x_1,...,x_m]$. Next suppose that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_m} \\ \dots & \dots & \dots \\ \frac{\partial f_s}{\partial x_1} & \dots & \frac{\partial f_s}{\partial x_m} \end{pmatrix}$$

is of rank r at x. Assume that $j_1,...,j_r$ are numbers of rows that are linearly independent over $k(x)$. Let $j : Z \hookrightarrow \mathbb{A}^m_K$ be a closed subscheme of $\mathbb{A}^m_K$ determined by the ideal $\mathcal{J}$ generated by $f_{j_1},...,f_{j_r}$. Then there exists a closed immersion $u : X \hookrightarrow Z$ such that $i \cdot u = j$. Consider a commutative diagram

$$\begin{tikzcd} \left(u^*\left(\mathcal{J}/\mathcal{J}^2\right)\right)_x \arrow[r] \arrow[d] & (u^* j^* \Omega_{\mathbb{A}^m_K/K})_x \arrow[r] \arrow[d, "="] & (u^*\Omega_{Z/K})_x \arrow[r] \arrow[d] & 0 \\ (\mathcal{I}/\mathcal{I}^2)_x \arrow[r] & (i^*\Omega_{\mathbb{A}^m_K/K})_x \arrow[r] & (\Omega_{X/K})_x \arrow[r] & 0 \end{tikzcd}$$

In the diagram rows are induced by conormal sequences for j and i. The leftmost vetical arrow is induced by the inclusion $\mathcal{J} \hookrightarrow \mathcal{I}$ and the rightmost vertical arrow is induced by the cotangent morphism $u^*\Omega_{Z/K} \to \Omega_{X/K}$. It follows from the choice of $j_1, ..., j_r$ that morphisms

$$\left(u^*\left(\mathcal{J}/\mathcal{J}^2\right)\right)_x \to (u^*j^*\Omega_{\mathbb{A}^m_K/K})_x,\ (\mathcal{I}/\mathcal{I}^2)_x \to (i^*\Omega_{\mathbb{A}^m_K/K})_x$$

have the same image in $(u^*j^*\Omega_{\mathbb{A}^m_K/K})_x = (i^*\Omega_{\mathbb{A}^m_K/K})_x$. Thus the morphism $\left(u^*\Omega_{Z/K}\right)_x \to \left(\Omega_{X/K}\right)_x$ induced by the cotangent morphism of u is an isomorphism. This implies that

$$\dim_{k(x)}\left(k(x) \otimes_{\mathcal{O}_{Z,x}} \Omega_{\mathcal{O}_{Z,x}/K}\right) = \dim_{k(x)}\left(k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/K}\right)$$

By definition Z is smooth at x. Hence by [Monygham, 2021, Theorem 6.3] and Corollary 1.5 we deduce that Z is regular at x. By assumption X is regular at x. Thus by Lemma 2.2.1 we derive that

$$\dim_x(Z) = \dim_{k(x)}\left(k(x) \otimes_{\mathcal{O}_{Z,x}} \Omega_{\mathcal{O}_{Z,x}/K}\right) = \dim_{k(x)}\left(k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/K}\right) = \dim_x(X)$$

Hence

$$\mathrm{tr}_K(k(x)) + \dim\left(\mathcal{O}_{Z,x}\right) = \dim_x(Z) = \dim_x(X) = \mathrm{tr}_K(k(x)) + \dim\left(\mathcal{O}_{X,x}\right)$$

We conclude that $\dim\left(\mathcal{O}_{Z,x}\right) = \dim\left(\mathcal{O}_{X,x}\right)$. This implies that $u^\# : \mathcal{O}_{Z,x} \twoheadrightarrow \mathcal{O}_{X,x}$ is a surjective morphism of regular rings of the same dimension. Hence it is an isomorphism. By Proposition 1.4 we deduce that X is smooth at x. □

Proof of the theorem. The implication **(i)** ⇒ **(ii)** follows from the fact that smoothness is closed under base change (Proposition 1.7), the fact that smoothness implies formal smoothness (Corollary 1.5) and [Monygham, 2021, Theorem 6.3], which states that formally smooth noetherian local k-algebras are regular. The implication **(ii)** ⇒ **(iii)** is obvious. Suppose now that **(iii)** holds. Then Lemma 2.2.2 implies that X_K is smooth at $\overline{x}$. By Theorem 1.8 we deduce that X is smooth at x. Hence also **(iii)** ⇒ **(i)**. This completes the proof that **(i)**, **(ii)**, **(iii)** are equivalent.
Now we prove the inequality

$$\dim_x(X) \leq \dim_{k(x)}\left(k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/K}\right)$$

and the fact that **(iv)** is equivalent with **(i)**, **(ii)**, **(iii)**. Suppose that $\mathrm{char}(k) = p$. If $p > 0$, then consider $K = k^{\frac{1}{p^\infty}}$ i.e. the perfect closure of k. If $p = 0$, then pick $K = k$. Let $\pi : X_K \to X$ be the canonical projection. The morphism $\mathrm{Spec}\, K \to \mathrm{Spec}\, k$ is surjective, universally injective and integral. Since π is a base change of $\mathrm{Spec}\, K \to \mathrm{Spec}\, k$, we derive that π is also surjective, universally injective and integral. Hence π is a homeomorphism. Thus there exists a unique point $\overline{x}$ in X_K lying over x and

$$\dim_{\overline{x}}\left(X_K\right) = \dim_x(X)$$

Moreover, we have

$$k(\overline{x}) \otimes_{\mathcal{O}_{X_K,\overline{x}}} \Omega_{\mathcal{O}_{X_K,\overline{x}}/K} \cong k(\overline{x}) \otimes_{\mathcal{O}_{X_K,\overline{x}}} \left(\Omega_{X_K/K}\right)_{\overline{x}} \cong k(\overline{x}) \otimes_{\mathcal{O}_{X_K,\overline{x}}} \left(\pi^*\Omega_{X/k}\right)_{\overline{x}} \cong$$
$$\cong k(\overline{x}) \otimes_{\mathcal{O}_{X_K,\overline{x}}} \mathcal{O}_{X_K,\overline{x}} \otimes_{\mathcal{O}_{X,x}} \left(\Omega_{X/k}\right)_x = k(\overline{x}) \otimes_{k(x)} \left(k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/k}\right)$$

Thus

$$\dim_{k(\overline{x})}\left(k(\overline{x}) \otimes_{\mathcal{O}_{X_K,\overline{x}}} \Omega_{\mathcal{O}_{X_K,\overline{x}}/K}\right) = \dim_{k(x)}\left(k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/k}\right)$$

Now according to Lemma 2.2.1 we have

$$\dim_x(X) = \dim_{\overline{x}}\left(X_K\right) \leq \dim_{k(\overline{x})}\left(k(\overline{x}) \otimes_{\mathcal{O}_{X_K,\overline{x}}} \Omega_{\mathcal{O}_{X_K,\overline{x}}/K}\right) = \dim_{k(x)}\left(k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/k}\right)$$

and the equality holds if and only if $\mathcal{O}_{X_K,\overline{x}}$ is regular. Thus by Lemma 2.2.2 the equality holds if and only if X_K is smooth at $\overline{x}$, but this according to Proposition 1.7 and Theorem 1.8 is equivalent with smoothness of X at x. The proof is complete. □

References

[Monygham, 2021] Monygham (2021). Formally smooth and formally unramified morphisms. *github repository: "Monygham/Pedo-mellon-a-minno"*.