

# HAAR MEASURE

## 1. INTRODUCTION

In this notes we introduce Haar measure, which is a fundamental technical tool in representation theory of locally compact topological groups. We send the interested reader to [DSS14] for excellent exposition of aspects and applications of this notion beyond our rudimentary presentation.

## 2. EXISTENCE OF HAAR MEASURE

**Definition 2.1.** Let  $G$  be a topological group and let  $\mu$  be a Borel measure. Then  $\mu$  is *left-invariant* if  $\mu(xA) = \mu(A)$  for every  $A$  in  $\mathcal{B}(G)$ . Similarly  $\mu$  is *right-invariant* if  $\mu(Ax) = \mu(A)$  for every  $A$  in  $\mathcal{B}(G)$ .

**Definition 2.2.** Let  $G$  be a locally compact group and  $\mu$  be a Borel measure. If  $\mu$  is a nonzero, left-invariant, regular Borel measure on  $G$ , then we say that  $\mu$  is a *left Haar measure* on  $G$ . Similarly if  $\mu$  is a nonzero, right-invariant, regular Borel measure on  $G$ , then we say that  $\mu$  is a *right Haar measure* on  $G$ .

**Theorem 2.3.** Let  $G$  be a locally compact topological group. Then there exists a left (right) Haar measure  $\mu$  on  $G$ . If in addition  $G$  is  $\sigma$ -compact, then  $\mu$  is inner regular.

We denote by  $\mathcal{K}$  the set of all compact subsets of  $G$  and by  $\mathcal{U}$  the set of all open neighborhoods of identity in  $G$ . Let  $U$  be an open nonempty subset of  $G$  and  $K$  be a compact subset of  $G$ . We define

$$(K : U) = \inf \left\{ n \in \mathbb{N} \mid \text{there exist } x_1, \dots, x_n \in G \text{ such that } K \subseteq \bigcup_{i=1}^n x_i U \right\}$$

Throughout the proof we fix a compact subset  $Q$  of  $G$  such that  $\text{int}(Q) \neq \emptyset$ .

**Lemma 2.3.1.** Fix  $U \in \mathcal{U}$ . There exists a real valued function  $h_U$  on  $\mathcal{K}$  such that the following assertions hold.

- (1) For every compact subset  $K$  in  $\mathcal{K}$  we have  $h_U(K) \geq 0$ ,  $h_U(\emptyset) = 0$  and  $h_U(Q) = 1$ .
- (2) For every compact subset  $K$  in  $\mathcal{K}$  and for every element  $x$  in  $G$  we have  $h_U(xK) = h_U(K)$ .
- (3) If  $K \subseteq L$  are compact subsets in  $\mathcal{K}$ , then  $h_U(K) \leq h_U(L)$ .
- (4) For every compact subset  $K$  in  $\mathcal{K}$  we have  $h_U(K) \leq (K : \text{int}(Q))$ .
- (5) If  $K, L$  are compact subsets in  $\mathcal{K}$ , then

$$h_U(K \cup L) \leq h_U(K) + h_U(L)$$

and if  $K \cdot U^{-1} \cap L \cdot U^{-1} = \emptyset$ , then the equality holds.

*Proof of the lemma.* For every compact subset  $K$  of  $G$  we define

$$h_U(K) = \frac{(K : U)}{(Q : U)}$$

Now we check that  $h_U$  admits the properties above. Properties (1), (2) and (3) are clear. For (4) note that

$$(K : U) \leq (Q : U) \cdot (K : \text{int}(Q))$$

Indeed, if  $K \subseteq \bigcup_{i=1}^n y_i \cdot \text{int}(Q)$  and  $Q \subseteq \bigcup_{j=1}^m z_j U$ , then  $K \subseteq \bigcup_{i=1}^n \bigcup_{j=1}^m y_i z_j U$  and this implies the inequality above. Observe that  $xU \cap K \neq \emptyset$  implies that  $x \in K \cdot U^{-1}$  and similarly  $xU \cap L \neq \emptyset$

implies that  $x \in L \cdot U^{-1}$ . Assuming that for compact subsets  $K, L$  in  $G$  we have  $K \cdot U^{-1} \cap L \cdot U^{-1} = \emptyset$  we derive from this that for every  $x \in G$  we have  $xU \cap (K \cap L) = \emptyset$ . Thus if  $K \cdot U^{-1} \cap L \cdot U^{-1} = \emptyset$ , then we have  $(K \cup L : U) = (K : U) + (L : U)$  and hence  $h_U(K \cup L) = h_U(K) + h_U(L)$ . Note that in general case we have  $(K \cup L : U) \leq (K : U) + (L : U)$  and hence also (5) holds for  $h_U$ .  $\square$

**Lemma 2.3.2.** *Let  $K, L$  in  $\mathcal{K}$  and suppose that  $K \cap L = \emptyset$ . Then there exists  $U \in \mathcal{U}$  such that*

$$K \cdot U^{-1} \cap L \cdot U^{-1} = \emptyset$$

*Proof of the lemma.* Left as an exercise.  $\square$

**Lemma 2.3.3.** *There exists a real valued function  $h$  on  $\mathcal{K}$  such that the following assertions hold.*

- (1) *For every compact subset  $K$  in  $\mathcal{K}$  we have  $h(K) \geq 0$ ,  $h(\emptyset) = 0$  and  $h(Q) = 1$ .*
- (2) *For every compact subset  $K$  in  $\mathcal{K}$  and for every element  $x$  in  $G$  we have  $h(xK) = h(K)$ .*
- (3) *If  $K \subseteq L$  are compact subsets in  $\mathcal{K}$ , then  $h(K) \leq h(L)$ .*
- (4) *For every compact subset  $K$  in  $\mathcal{K}$  we have  $h(K) \leq (K : \text{int}(Q))$ .*
- (5) *If  $K, L$  are compact subsets in  $\mathcal{K}$ , then*

$$h(K \cup L) \leq h(K) + h(L)$$

*and if  $K \cap L = \emptyset$ , then the equality holds.*

*Proof of the lemma.* Consider a topological space

$$X = \prod_{K \in \mathcal{K}} [0, (K : \text{int}(Q))]$$

By Tichonoff's theorem  $X$  is compact. For every  $U \in \mathcal{U}$  we define a subset  $F_U \subseteq X$  that consists of tuples  $\{a_K\}_{K \in \mathcal{K}}$  such that  $a_\emptyset = 0$ ,  $a_Q = 1$ ,  $a_{xK} = a_K$  for  $x \in G$  and  $K$  in  $\mathcal{K}$ ,  $a_K \leq a_L$  for  $K \subseteq L$  in  $\mathcal{K}$ ,  $a_{K \cup L} \leq a_K + a_L$  for  $K, L$  in  $\mathcal{K}$  and the equality holds if  $K \cdot U^{-1} \cap L \cdot U^{-1} = \emptyset$ . Conditions imposed on tuples in  $F_U$  imply that  $F_U$  is a closed subset. Note that  $\{h_U(K)\}_{K \in \mathcal{K}} \in F_U$  for every  $U \in \mathcal{U}$ . Moreover, we have

$$F_{U_1 \cap U_2 \cap \dots \cap U_n} \subseteq F_{U_1} \cap F_{U_2} \cap \dots \cap F_{U_n}$$

for  $U_1, U_2, \dots, U_n \in \mathcal{U}$ . This implies that  $\{F_U\}_{U \in \mathcal{U}}$  is a centered family of nonempty closed subsets of a compact space  $X$ . Thus

$$\bigcap_{U \in \mathcal{U}} F_U \neq \emptyset$$

by compactness of  $X$ . Hence there exists  $\{c_K\}_{K \in \mathcal{K}}$  in the intersection. We define a real function  $h$  on  $\mathcal{K}$  by  $h(K) = c_K$  for  $K$  in  $\mathcal{K}$ . The fact that properties (1), (2), (3) and (4) hold for  $h$  follows by definition of  $F_U$  for  $U \in \mathcal{U}$ . Since  $\{c_K\}_{K \in \mathcal{K}}$  is an element in  $F_U$  for every  $U \in \mathcal{U}$  we derive that

$$c_{K \cup L} \leq c_K + c_L$$

for  $K, L$  in  $\mathcal{K}$ . This implies  $h(K \cup L) \leq h(K) + h(L)$  for  $K, L \in \mathcal{K}$ . Moreover,  $c_{K \cup L} = c_K + c_L$  if  $K \cdot U^{-1} \cap L \cdot U^{-1} = \emptyset$  for some  $U \in \mathcal{U}$ . This implies that  $c_{K \cup L} = c_K + c_L$  if  $K \cap L = \emptyset$  by Lemma 2.3.2. Thus  $h$  admits (4).  $\square$

*Proof of the theorem.* We fix  $h$  as in Lemma 2.3.3 and we define  $\mu^* : \mathcal{P}(G) \rightarrow [0, +\infty]$ . First if  $U$  is an open subset of  $G$ , then we define

$$\mu^*(U) = \sup_{K \in \mathcal{K}, K \subseteq U} h(K)$$

Note that if  $U, V$  are open subsets of  $G$  and  $U \subseteq V$ , then  $\mu^*(U) \leq \mu^*(V)$ . Thus it makes sense to define

$$\mu^*(A) = \inf \{ \mu^*(U) \mid U \text{ is an open subset of } G \text{ containing } A \}$$

for arbitrary subset  $A \subseteq G$ . Note that  $\mu^*(xA) = \mu^*(A)$  by definition of  $\mu^*$  and the corresponding property of  $h$ . By [Mon18a, Theorem 1.3] we have that Borel sets  $\mathcal{B}(G)$  are  $\mu^*$ -measurable,

$\mu_{|\mathcal{B}(G)}^* = \mu$  is a regular Borel measure on  $G$ . According to this result if  $G$  is  $\sigma$ -compact, then  $\mu$  is inner regular. Clearly  $\mu$  is left-invariant and since

$$1 = h(Q) \leq \mu(Q)$$

we derive that it is nonzero measure.  $\square$

### 3. UNIQUENESS OF HAAR MEASURE

**Theorem 3.1.** *Let  $G$  be a locally compact group. If  $\mu_1$  and  $\mu_2$  are left (right) Haar measures on  $G$ , then there exists positive constant  $a \in \mathbb{R}$  such that*

$$\mu_1 = a \cdot \mu_2$$

For the proof we need the following result.

**Lemma 3.1.1.** *Let  $G$  be a locally compact group. Then there exists a  $\sigma$ -compact, open subgroup  $H$  of  $G$ .*

*Proof of the lemma.* Let  $U$  be an open neighborhood of identity in  $G$  such that  $\text{cl}(U)$  is compact. Consider  $V = U \cap U^{-1}$ . Then  $V$  is open neighborhood of identity in  $G$  such that  $V = V^{-1}$  and  $\text{cl}(V)$  is compact. We define  $H = \bigcup_{n \in \mathbb{N}} V^n$ . Then  $H$  is an open subgroup of  $G$ . We have

$$H = G \setminus \left( \bigcup_{g \in G \setminus H} gH \right)$$

and hence  $H$  is also a closed subgroup of  $G$ . Moreover, for every  $n \in \mathbb{N}$  set  $\text{cl}(V^n)$  is compact in  $G$ . Since

$$H = \bigcup_{n \in \mathbb{N}} (H \cap \text{cl}(V^n))$$

we derive that  $H$  is  $\sigma$ -compact.  $\square$

*Proof of the theorem.* By Lemma 3.1.1 there exists an open subgroup  $H$  of  $G$  that is  $\sigma$ -compact. We prove now that there exists  $a \in \mathbb{R}$  such that

$$\mu_{1|\mathcal{B}(H)} = a \cdot \mu_{2|\mathcal{B}(H)}$$

For this consider  $\mu = \mu_{1|\mathcal{B}(H)} + \mu_{2|\mathcal{B}(H)}$  and denote  $\mu_{2|\mathcal{B}(H)}$  by  $\nu$ . Measures  $\mu, \nu$  are  $\sigma$ -finite as they are finite on compact subsets of  $H$  and  $H$  is  $\sigma$ -compact space. Moreover,  $\nu \ll \mu$  and hence by [Mon18b, Theorem 5.3] there exists a Borel function  $f : H \rightarrow \mathbb{C}$  such that

$$\nu(A) = \int_A f d\mu$$

for every Borel subset  $A$  in  $H$ . Since  $\mu$  and  $\nu$  are nonnegative measures, we derive that  $f$  is real and nonnegative  $\mu$ -almost everywhere. Hence we may assume that  $f$  takes only nonnegative real values. We define

$$E = \{(x, y) \in H \times H \mid f(xy) - f(y) \neq 0\}$$

Next as  $\nu, \mu$  are left-invariant, we deduce that

$$0 = \nu(l_x(A)) - \nu(A) = \int_{l_x(A)} f d\mu - \int_A f d\mu = \int_A (f \cdot l_x - f) d\mu$$

for every  $x \in H$  and  $A \in \mathcal{B}(H)$ , where  $l_x : H \rightarrow H$  is a continuous map given by left multiplication by  $x$ . This implies that for all  $x \in H$  we have

$$\mu(E_x) = 0$$

By [Mon19, Theorem 7.5] applied to measure  $\mu \otimes \mu$  on  $H \times H$ , we deduce that there exists  $y \in H$  such that the set

$$E_y = \{x \in H \mid f(xy) - f(y) \neq 0\}$$

has measure  $\mu$  zero. This implies that  $f$  is constant almost everywhere with respect to  $\mu$  and thus there exists nonzero  $b \in \mathbb{R}$  such that  $\nu = b \cdot \mu$ . Hence we have

$$\mu_1|_{\mathcal{B}(H)} = a \cdot \mu_2|_{\mathcal{B}(H)}$$

for  $a = (1 - b)b^{-1}$ . Let  $K$  be a compact subset of  $G$ . Since  $H$  is an open subgroup of  $G$ , there exists  $x_1, \dots, x_n \in G$  such that

$$K \subseteq x_1 H \cup \dots \cup x_n H$$

and the sum is disjoint. Therefore, we have

$$\mu_1(K) = \sum_{i=1}^n \mu_1(K \cap x_i H) = \sum_{i=1}^n \mu_1(x_i^{-1} K \cap H) = a \cdot \sum_{i=1}^n \mu_2(x_i^{-1} K \cap H) = a \cdot \sum_{i=1}^n \mu_2(K \cap x_i H) = a \cdot \mu_2(K)$$

This implies that  $\mu_1 = a \cdot \mu_2$  because  $\mu_1, \mu_2$  are regular Borel measures.  $\square$

#### 4. MODULAR FUNCTION AND INVARIANCE OF HAAR MEASURE ON COMPACT GROUPS

##### REFERENCES

- [DSS14] J. Diestel, A. Spalsbury, and American Mathematical Society. *The Joys of Haar Measure*. Graduate Studies in Mathematics. American Mathematical Society, 2014.
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