BIAŁYNICKI-BIRULA FUNCTORS

1. Introduction

In this notes we study Białynicki-Birula functors. In the first section we prove some results concerning the forgetful functor $Rep(M) \to Rep(G)$, where M is an affine monoid k-scheme and G is its group of units (we assume that G is open and schematically dense in M). These results will be used in the following sections.

We assume that *k* is a field. In these notes we use the following notational convention.

Remark 1.1. Since the Yoneda embedding $\mathbf{Sch}_k \hookrightarrow \widehat{\mathbf{Sch}_k}$ is full and faithful, we identify \mathbf{Sch}_k with the subcategory of $\widehat{\mathbf{Sch}_k}$ consisting of representable presheaves on \mathbf{Sch}_k . In particular, if X is a k-scheme, then we denote by the same symbol the presheaf representable by X.

2. RELATIONS BETWEEN REPRESENTATIONS OF A MONOID AND ITS GROUP OF UNITS

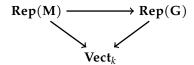
In this section we study the relation between the category $\mathbf{Rep}(\mathbf{M})$ of representations of an affine monoid k-scheme \mathbf{M} and the category $\mathbf{Rep}(\mathbf{G})$ of representations of its group of units \mathbf{G} . Let $i:k[\mathbf{M}] \to k[\mathbf{G}]$ be the morphism of k-bialgebras induced by $\mathbf{G} \to \mathbf{M}$. Let us first note the following elementary result.

Fact 2.1. Assume that G is open and schematically dense in M. Then i is an injective morphism of k-algebras.

Proof. This follows from [Görtz and Wedhorn, 2010, Proposition 9.19].

Fact 2.2. The forgetful functor $Rep(M) \rightarrow Rep(G)$ creates colimits and finite limits.

Proof. This follows from [Monygham, 2020b, Theorem 14.3, Theorem 14.4] and the commutative triangle



of functors. \Box

The theorem below characterizes representations of G which are contained in the image of the forgetful functor $Rep(M) \to Rep(G)$.

Theorem 2.3. Assume that G is open and schematically dense in M. Let V be a G-representation. Then the following are equivalent.

- (i) V is in the image of the forgetful functor $Rep(M) \rightarrow Rep(G)$.
- (ii) The coaction $d: V \to k[\mathbf{G}] \otimes_k V$ factors through $i \otimes_k 1_V : k[\mathbf{M}] \otimes_k V \hookrightarrow k[\mathbf{G}] \otimes_k V$.

Proof. In the proof we denote by $\Delta_{\mathbf{M}}$ and $\Delta_{\mathbf{G}}$ comultiplications of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. We also denote by $\xi_{\mathbf{M}}$ and $\xi_{\mathbf{G}}$ counits of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. According to Fact 2.1 i is an injective morphism of k-algebras.

Clearly (i) \Rightarrow (ii). We prove the converse. Suppose that (ii) holds. Let $c: V \to k[\mathbf{M}] \otimes_k V$ be a unique morphism such that $d = (i \otimes_k 1_V) \cdot c$. It suffices to prove that c is the coaction of the bialgebra $k[\mathbf{M}]$ on V. Observe that

$$(i \otimes_k i \otimes_k 1_V) \cdot (1_{k[\mathbf{M}]} \otimes_k c) \cdot c = (i \otimes_k d) \cdot c = (1_{k[\mathbf{G}]} \otimes_k d) \cdot d = (\Delta_{\mathbf{G}} \otimes_k 1_V) \cdot d = (\Delta_{\mathbf{$$

$$= (\Delta_{\mathbf{G}} \otimes_k 1_V) \cdot \left((i \otimes_k 1_V) \cdot c \right) = \left((\Delta_{\mathbf{G}} \cdot i) \otimes_k 1_V \right) \cdot c = (i \otimes_k i \otimes_k 1_V) \cdot (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot c$$

Since $i \otimes_k i \otimes_k 1_V$ is a monomorphism, we deduce that $(1_{k[\mathbf{M}]} \otimes_k c) \cdot c = (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot c$. Moreover, we have

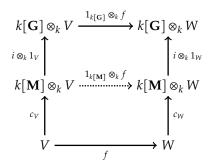
$$(\xi_{\mathbf{G}} \otimes_k 1_V) \cdot d = (\xi_{\mathbf{G}} \otimes_k 1_V) \cdot ((i \otimes_k 1_V) \cdot c) = (\xi_{\mathbf{M}} \otimes_k 1_V) \cdot c$$

and hence $(\xi_{\mathbf{M}} \otimes_k 1_V) \cdot c$ is the canonical isomorphism $V \cong k \otimes_k V$. Thus c is the coaction of $k[\mathbf{M}]$ and $d = (i \otimes_k 1_V) \cdot c$. Therefore, V is in the image of $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$.

Theorem 2.4. Assume that G is open and schematically dense in M. Then Rep(M) is a full subcategory of Rep(G) closed under subobjects and quotients.

Proof. In the proof we denote by $\Delta_{\mathbf{M}}$ and $\Delta_{\mathbf{G}}$ comultiplications of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. We also denote by $\xi_{\mathbf{M}}$ and $\xi_{\mathbf{G}}$ counits of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. According to Fact 2.1 i is an injective morphism of k-algebras.

We first prove that $\mathbf{Rep}(\mathbf{M})$ is a full subcategory of $\mathbf{Rep}(\mathbf{G})$. For this consider \mathbf{M} -representations V,W and a their morphism $f:V\to W$ as \mathbf{G} -representations. Let c_V and c_W be coactions of $k[\mathbf{M}]$ on V and W, respectively. Our goal is to prove that f is a morphism of \mathbf{M} -representations. Consider the diagram

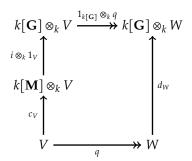


in which the outer square is commutative. Our goal is to prove that the bottom square is commutative. We have

$$(i \otimes_k 1_W) \cdot c_W \cdot f = (1_{k \lceil \mathbf{G} \rceil} \otimes_k f) \cdot (i \otimes_k 1_V) \cdot c_V = (i \otimes_k 1_W) \cdot (1_{k \lceil \mathbf{M} \rceil} \otimes_k f) \cdot c_V$$

Since $i \otimes_k 1_W$ is a monomorphism, we deduce that $c_W \cdot f = (1_{k[\mathbf{M}]} \otimes_k f) \cdot c_V$. Hence f is a morphism of \mathbf{M} -representations.

Next we prove that $\mathbf{Rep}(\mathbf{M})$ is a subcategory of $\mathbf{Rep}(\mathbf{G})$ that is closed under subquotients. Consider an \mathbf{M} -representation V and its quotient \mathbf{G} -representations $q:V \twoheadrightarrow W$. We show that W is a quotient \mathbf{M} -representation of V. Let c_V be the coaction of \mathbf{M} on V and let d_W be the coaction of \mathbf{G} on W. We have a commutative diagram



and hence $d_W(W) \subseteq k[\mathbf{M}] \otimes_k W$. Thus Theorem 2.3 implies that W is a representation of \mathbf{M} and q is a morphism of \mathbf{M} -representations. This shows that $\mathbf{Rep}(\mathbf{M})$ is a subcategory of $\mathbf{Rep}(\mathbf{G})$ closed under quotients. Next let $j: U \to V$ be a \mathbf{G} -subrepresentation of a \mathbf{M} -representation V. By what we proved above the cokernel $q: V \to W$ of j in $\mathbf{Rep}(\mathbf{G})$ is contained in $\mathbf{Rep}(\mathbf{M})$. Since both $\mathbf{Rep}(\mathbf{M})$ and $\mathbf{Rep}(\mathbf{G})$ are abelian and the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$ is exact, we derive that the kernel of q in $\mathbf{Rep}(\mathbf{M})$ coincides with its kernel in $\mathbf{Rep}(\mathbf{G})$. Thus U is a \mathbf{M} -representation and $j: U \to V$ is a morphism of \mathbf{M} -representations. Hence $\mathbf{Rep}(\mathbf{M})$ is the category of $\mathbf{Rep}(\mathbf{G})$ closed under subobjects.

Theorem 2.5. Assume that G is open and schematically dense in M. Let V be a G-representation of G. There exists an M-representation W and a surjective morpism $q:V \twoheadrightarrow W$ of G-representations such that for every M-representation U and a morphism $f:V \rightarrow U$ of G-representations there exists a unique morphism $\tilde{f}:W \rightarrow U$ of M-representations making the triangle



commutative.

Proof. Assume first that V is finite dimensional. Let \mathcal{K} be a set of **G**-subrepresentations of V that consists of all $K \subseteq V$ such that V/K carries a structure of **M**-representation. Clearly $\mathcal{K} = \emptyset$ because $\{0\} \in \mathcal{K}$. Since V is finite dimensional, there exists a finite subset $\{K_1, ..., K_n\} \subseteq \mathcal{K}$ such that

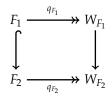
$$\bigcap_{i=1}^{n} K_i = \bigcap_{K \in \mathcal{K}} K$$

Then a morphism

$$V/\left(\bigcap_{K\in\mathcal{K}}K\right)\ni v\mapsto \left(v\bmod K_i\right)_{1\leq i\leq n}\in\bigoplus_{i=1}^nV/K_i$$

is a monomorphism and hence by Theorem 2.4 the quotient $W = V/(\bigcap_{K \in \mathcal{K}} K)$ is an **M**-representation. Let $q: V \twoheadrightarrow W$ be the canonical epimorphism. Consider now a morphism $f: V \to U$ of **G**-representations, where U is an **M**-representation. Then $\operatorname{im}(f)$ is a **G**-subrepresentation of U and by Theorem 2.4 we derive that $\operatorname{im}(f)$ is an **M**-representation. This implies that $\ker(f)$ is in \mathcal{K} . Hence f factors through g. Thus there exists a unique morphism $\tilde{f}: W \to U$ of **G**-representations such that $\tilde{f} \cdot g = f$. This completes the proof in case when V is finite dimensional.

Now consider the general V. Let \mathcal{F} be the set of all finite dimensional G-representations of V. According to [Monygham, 2020b, Corollary 15.2] we deduce that $V = \operatorname{colim}_{F \in \mathcal{F}} F$. By the case considered above we deduce that for every F in \mathcal{F} there exists a universal morphism $q_F : F \to W_F$ of G-representations into an G-representation G-representation G-representation into an G-representation G-representation G-representation into an G-representation G-represe



Thus $\{W_F\}_{F\in\mathcal{F}}$ together with morphisms $W_{F_1} \to W_{F_2}$ for $F_1 \subseteq F_2$ in \mathcal{F} form a diagram parametrized by the poset \mathcal{F} . The category $\mathbf{Rep}(\mathbf{M})$ has small colimits ([Monygham, 2020b, Corollary 14.5]) and we define $W = \mathrm{colim}_{F\in\mathcal{F}}W_F$. This is also a colimit of this diagram in the category $\mathbf{Rep}(\mathbf{G})$ by Fact 2.2. We also define $q = \mathrm{colim}_{F\in\mathcal{F}}q_F : V = \mathrm{colim}_{F\in\mathcal{F}}F \to W$. Since a colimit of a family of epimorphisms is an epimorphism, we derive that q is an epimorphism of \mathbf{G} -representations. Suppose now that $f: V \to U$ is a morphism of \mathbf{G} -representations and U is an \mathbf{M} -representation. Then $f_{|F}$ uniquely factors through q_F for every F in \mathcal{F} . Hence by universal property of colimits we derive that f factors through q in a unique way. This completes the proof.

3. BIAŁYNICKI-BIRULA FUNCTORS

In this section we fix an affine group k-scheme G. Let M be an affine monoid k-scheme with zero o such that G is its group of units. Note that if Y is a k-scheme, then $M \times_k Y$ admits canonical action of M and

Definition 3.1. Let X be a k-scheme equipped with an action of G. For every k-scheme Y we define

$$\mathcal{D}_X(Y) = \{ \gamma : \mathbf{M} \times_k Y \to X \mid \gamma \text{ is } \mathbf{G}\text{-equivariant} \}$$

This gives gives rise to a subfunctor \mathcal{D}_X of $\operatorname{Mor}_k(\mathbf{M} \times_k (-), X) : \operatorname{\mathbf{Sch}}_k^{\operatorname{op}} \to \operatorname{\mathbf{Set}}$. We call it *the Białynicki-Birula functor of* X.

Remark 3.2. Let *X* be a *k*-scheme equipped with an action of **G**. Then there are canonical morphism of functors

$$\mathcal{D}_X \xrightarrow{i_X} X$$

$$\downarrow_{r_X} XG$$

which we define now. First let us explain that in the diagram X stands for the presheaf representable by the k-scheme X (Remark 1.1) and X^G denotes the functor of fixed points of X ([Monygham, 2020a, Definition 7.1]). Now fix k-scheme Y and $\gamma \in \mathcal{D}_X(Y)$, then we define

$$i_X(\gamma) = \gamma_{|\{e\} \times_k X} = \gamma \cdot \langle e, 1_X \rangle, \, r_X(\gamma) = \gamma_{|\{o\} \times_k X} = \gamma \cdot \langle o, 1_X \rangle$$

where $e: \operatorname{Spec} k \to \mathbf{M}$ is the unit of \mathbf{M} and $\mathbf{o}: \operatorname{Spec} k \to \mathbf{M}$ is the zero. Next if $f: Y \to X$ is a morphism in $X^{\mathbf{G}}(Y)$, then we define

$$s_X(f) = f \cdot pr_Y$$

where $pr_Y : \mathbf{M} \times_k Y \to Y$ is the projection. Finally note that $r_X \cdot s_X = 1_{X^G}$.

Remark 3.3. Let X be a k-scheme equipped with an action of G. Then M (actually the presheaf of monoids represented by M) acts on \mathcal{D}_X . Indeed, fix k-scheme Y, $\gamma \in \mathcal{D}_X(Y)$ and $m: Y \to M$. Then we define the product

$$m\gamma = \gamma \cdot \langle m, 1_{\gamma} \rangle$$

and this determines an action of **M** on \mathcal{D}_X . Moreover, with respect to this action i_X is **G**-equivariant and r_X , s_X are **M**-equivariant (X^G is equipped with trivial action of **M**).

Remark 3.4. Let X, Y be k-schemes equipped with actions of G and let $f: X \to Y$ be a G-equivariant morphism, then there exists a morphism of functors $\mathcal{D}_f: \mathcal{D}_X \to \mathcal{D}_Y$ given by

$$\mathcal{D}_f(\gamma) = f \cdot \gamma$$

for every element γ of the functor \mathcal{D}_X .

Let *X* be a *k*-scheme equipped with an action of **G**. It is useful to discuss subfunctors of \mathcal{D}_X defined by closed **G**-stable subschemes of *X*.

Theorem 3.5. Let X be a k-scheme equipped with an action of the group G. Suppose that G is open and schematically dense in M. If $j: Z \hookrightarrow X$ is a closed G-stable subscheme of X, then the square

$$\mathcal{D}_{Z} \xrightarrow{\mathcal{D}_{j}} \mathcal{D}_{X} \\
\downarrow^{i_{Z}} \downarrow \qquad \qquad \downarrow^{i_{X}} \\
Z \xrightarrow{j} X$$

is cartesian in the category of presheaves on \mathbf{Sch}_k .

Proof. The fact that the square is commutative follows by examination of definitions in Remarks 3.2 and 3.4. Pick k-scheme Y, $f: Y \to Z$ and $\gamma \in \mathcal{D}_X(Y)$ such that $j \cdot f = i_X(\gamma)$. This is depicted in the diagram

$$f \longmapsto_{j} j \cdot f = \gamma_{|\{e\} \times_{k} X}$$

Our goal is to show that there exists a unique **G**-equivariant morphism $\eta : \mathbf{M} \times_k Y \to U$ such that $\mathcal{D}_j(\eta) = \gamma$ and $i_Z(\eta) = f$. This is depicted by the diagram

$$\frac{\eta}{r_{u}} \xrightarrow{\mathcal{D}_{j}} \gamma = j \cdot \eta$$

$$f = \eta_{|\{e\} \times_{k} X}$$

In order to achieve this it suffices to prove that γ factors through j. First note that the assumption $\gamma_{|\{e\}\times_k Y} = j \cdot f$ implies that

$$\gamma_{|\mathbf{G} \times_k Y} = j \cdot f \cdot pr_Y$$

where $pr_Y: \mathbf{G} \times_k Y \to Y$ is the projection. This implies that $\gamma_{|\mathbf{G} \times_k}$ factors through j. Consider scheme-theoretic preimage $\gamma^{-1}(Z)$. Then $\gamma^{-1}(Z)$ is a closed \mathbf{G} -stable (as an inverse image of a \mathbf{G} -stable closed subscheme under the \mathbf{G} -equivariant morphism) subscheme of $\mathbf{M} \times_k Y$, which contains $\mathbf{G} \times_k Y$. Since \mathbf{G} is open, schematically dense in \mathbf{M} and k is a field, we derive that $\mathbf{G} \times_k Y$ is open and schematically dense in $\mathbf{M} \times_k Y$. Thus $\gamma^{-1}(Z) = \mathbf{M} \times_k Y$ and hence γ factors through j.

In order to prove interesting result in the spirit of Theorem 3.5 which concerns open G-stable subschemes, we need to assume that M is a Kempf monoid.

Theorem 3.6. Let X be a k-scheme equipped with an action of the group G of units of a Kempf monoid M. If $j:U \hookrightarrow X$ is an open G-stable subscheme of X, then the square

$$\mathcal{D}_{U} \xrightarrow{\mathcal{D}_{j}} \mathcal{D}_{X} \\
\downarrow^{r_{U}} \qquad \qquad \downarrow^{r_{X}} \\
U^{G} \xrightarrow{j^{G}} X^{G}$$

is cartesian in the category of presheaves on \mathbf{Sch}_k .

As we shall see this result follows from the following.

Lemma 3.6.1. Let K be an algebraicaly closed field over k. Suppose that

$$\mathbf{M}_K = \operatorname{Spec} K \times_k \mathbf{M}, \mathbf{G}_K = \operatorname{Spec} K \times_k \mathbf{G}$$

and let \mathbf{o}_K be the unique K-point of \mathbf{M}_K lying over \mathbf{o} . Let V be an open \mathbf{G}_K -stable subscheme of \mathbf{M}_K such that $\mathbf{o}_K \in V$. Then $V = \mathbf{M}_K$.

Proof of the lemma. Since \mathbf{M} is a Kempf monoid, there exists a closed embedding of monoids $v: \mathbb{A}^1_K \to \mathbf{M}_K$ preserving zeros such that $v_{|G_{m,K}} \subseteq \mathbf{G}_K$. Fix a point $p \in \mathbf{M}_K$ and let $u: \operatorname{Spec} k(p) \to \mathbf{M}_K$ be the associated morphism of K-schemes. Consider the composition

$$\mathbb{A}^{1}_{k(p)} = \mathbb{A}^{1}_{K} \times_{K} \operatorname{Spec} k(p) \xrightarrow{c^{v} \times_{K} u} \mathbf{M}_{K} \times_{K} \mathbf{M}_{K} \longleftrightarrow \mathbf{M}_{K}$$

where the second morphism is the multiplication. Clearly h is $\mathbf{G}_{m,k(p)}$ -equivariant. Hence $h^{-1}(V)$ is an open $\mathbf{G}_{m,k(p)}$ -stable subscheme of $\mathbb{A}^1_{k(p)}$ containing zero of this monoid k(p)-scheme (because $\mathbf{o}_K \in V$ by assumption). Since the only open $\mathbf{G}_{m,k(p)}$ -stable subscheme of $\mathbb{A}^1_{k(p)}$ containing zero is $\mathbb{A}^1_{k(p)}$, we derive that $h^{-1}(V) = \mathbb{A}^1_{k(p)}$. Thus $p \in V$. Since p is arbitrary point of \mathbf{M}_K , we derive that $V = \mathbf{M}_K$.

Proof of the theorem. The fact that the square is commutative follows by examination of definitions in Remarks 3.2 and 3.4. Pick k-scheme Y, $f \in U^G$ and $\gamma \in \mathcal{D}_X(Y)$ such that $j^G(f) = r_X(\gamma)$. This is depicted in the diagram

$$f \mapsto \int_{j^{\mathbf{G}}}^{\gamma} j \cdot f = \gamma_{|\{\mathbf{o}\} \times_{k} X}$$

Our goal is to show that there exists a unique **G**-equivariant morphism $\eta : \mathbf{M} \times_k Y \to U$ such that $\mathcal{D}_i(\eta) = \gamma$ and $r_U(\eta) = f$. This is depicted by the diagram

$$\frac{\eta}{r_u} \xrightarrow{\mathcal{D}_j} \gamma = j \cdot \eta$$

$$f = \eta_{|\{\mathbf{o}\} \times_k X}$$

In order to achieve this it suffices to prove that γ factors through j. Consider $W = \gamma^{-1}(U)$. Note that W is an open G-stable (as an inverse image of a G-stable open subscheme under the G-equivariant morphism) subscheme of $\mathbf{M} \times_k Y$, which contains $\{\mathbf{o}\} \times_k Y$. Lemma 3.6.1 asserts that for every geometric point \overline{y} of Y we have $W_{\overline{y}} = \mathbf{M}_{k(\overline{y})}$, where $W_{\overline{y}}$ is the fiber over \overline{y} of the projection $\mathbf{M} \times_k Y \to Y$ restricted to W. Since W is open subscheme of $\mathbf{M} \times_k Y$, this implies that $W = \mathbf{M} \times_k Y$ and hence γ factors through j.

As we shall see below both Theorems are extremely useful properties of Białynicki-Birula functors. Now we introduce a formal version of this functor.

Definition 3.7. Let M be an affine monoid k-scheme with zero o and let G be its group of units. For every $n \in \mathbb{N}$ let $M_n \to M$ be an n-th infinitesimal neighborhood of o in M. Let X be a k-scheme equipped with an action of G. For every k-scheme Y we define

$$\widehat{\mathcal{D}}_X(Y) = \left\{ \left\{ \gamma_n : \mathbf{M}_n \times_k Y \to X \right\}_{n \in \mathbb{N}} \middle| \forall_{n \in \mathbb{N}} \gamma_n \text{ is } \mathbf{G}\text{-equivariant and } \gamma_{n+1 \mid \mathbf{M}_n \times_k Y} = \gamma_n \right\}$$

This gives gives rise to a functor $\widehat{\mathcal{D}}_X$. We call it the formal Biatynicki-Birula functor of X.

Remark 3.8. Let **M** be an affine monoid k-scheme with zero **o** and let **G** be its group of units. Let X be a k-scheme equipped with an action of **G**. Then there exists a canonical morphism of functors $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ given by

$$\gamma \mapsto \{\gamma_{|\mathbf{M}_n \times_k Y}\}_{n \in \mathbb{N}}$$

for every $\gamma \in \mathcal{D}_X(Y)$ and every k-scheme Y.

4. Representability of Białynicki-Birula functor for locally linear schemes

We first investigate open covers of Białynicki-Birula functor

Remark 4.1. Since **G** is geometrically connected and locally algebraic it follows by [Monygham, 2020a, Theorem 7.2] that for every k-scheme X equipped with an action of **G** there exists closed subscheme $X^{\mathbf{G}}$ of X representing

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