FIBERED CATEGORIES AND EQUIVARIANT OBJECTS

1. Introduction

In these notes we often work with two distinct categories. In order to make our notation clear we denote by $h^{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ the Yoneda embedding for category $\mathcal{C}$. In particular, if X is an object of $\mathcal{C}$, then $h^{\mathcal{C}}_X$ is a presheaf associated with X.

2. Fibered categories

We fix a functor $p : \mathcal{E} \to \mathcal{B}$. We introduce now some convenient notation that will help clarifying our definitions. Consider a morphism $\phi : \xi \to \eta$ of $\mathcal{E}$ such that $p(\phi) = f$ and $f : X \to Y$. We depict this situation by the square diagram

$$\begin{tikzcd}
\xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Note that to every such square there corresponds a commutative square

$$\begin{tikzcd}
h^{\mathcal{E}}_{\xi} \arrow[r, "h^{\mathcal{E}}_{\phi}"] \arrow[d, "p_{\mathrm{hom}}"'] & h^{\mathcal{E}}_{\eta} \arrow[d, "p_{\mathrm{hom}}"] \\
h^{\mathcal{B}}_X \cdot p \arrow[r, "\left(h^{\mathcal{B}}_f\right)_p"'] & h^{\mathcal{B}}_Y \cdot p
\end{tikzcd}$$

of presheaves on $\mathcal{E}$.

Definition 2.1. Consider a square

$$\begin{tikzcd}
\xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

We call the square *cartesian* and ϕ *a cartesian morphism with respect to p* if the corresponding square of presheaves on $\mathcal{E}$ is cartesian in the category of presheaves.

One can rephrase definition above in terms of presheaves as follows. Morphism $\phi : \xi \to \eta$ is cartesian with respect to p if the square

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{E}}(\zeta,\xi) \arrow[r, "\mathrm{Mor}_{\mathcal{E}}(1_{\zeta},\phi)"] \arrow[d, "p_{\mathrm{hom}}"'] & \mathrm{Mor}_{\mathcal{E}}(\zeta,\eta) \arrow[d, "p_{\mathrm{hom}}"] \\ \mathrm{Mor}_{\mathcal{B}}\big(p(\zeta),p(\xi)\big) \arrow[r, "\mathrm{Mor}_{\mathcal{B}}(1_{p(\zeta)},p(\phi))"'] & \mathrm{Mor}_{\mathcal{B}}\big(p(\zeta),p(\eta)\big) \end{tikzcd}$$

of classes is cartesian for every object ζ of $\mathcal{E}$.

Fact 2.2. *Let $p:\mathcal{E}\to\mathcal{B}$ be a functor, let $f:X\to Y$ be a morphism of $\mathcal{B}$ and let η be an object of $\mathcal{E}$. Suppose that $\phi_1:\xi_1\to\eta,\phi_2:\xi_2\to\eta$ are morphisms of $\mathcal{E}$ that are cartesian with respect to p and assume that $p(\phi_1)=p(\phi_2)$. Then there exists a unique morphism $\theta:\xi_1\to\xi_2$ such that $\phi_1=\phi_2\cdot\theta$. Moreover, θ is an isomorphism.*

Proof. We use the presheaf reformulation of a definition of cartesian morphisms of p. It implies that there exists a unique natural transformation $\sigma:h^{\mathcal{E}}_{\xi_1}\to h^{\mathcal{E}}_{\xi_2}$ such that $h^{\mathcal{E}}_{\phi_1}=h^{\mathcal{E}}_{\phi_2}\cdot\sigma$. Moreover, σ is a natural isomorphism. Since $h^{\mathcal{E}}:\mathcal{E}\to\widehat{\mathcal{E}}$ is full and faithful, we derive that there exists a unique morphism $\theta:\xi_1\to\xi_2$ such that $h^{\mathcal{E}}_{\theta}=\sigma$. Then θ satisfies the assertion. $\square$

Definition 2.3. Let $p:\mathcal{E}\to\mathcal{B}$ be a functor, let $f:X\to Y$ be a morphism of $\mathcal{B}$ and let η be an object of $\mathcal{E}$ such that $p(\eta)=Y$. A pair (ξ,ϕ) such that ξ is an object of $\mathcal{E}$ and $\phi:\xi\to\eta$ is a morphism of $\mathcal{E}$ is called *a pullback of η along f* if the following conditions are satisfied.

(1) $p(\phi)=f$

(2) ϕ is cartesian morphism of p.

Note that Fact 2.2 implies that pullbacks are unique up to a unique isomorphism.

Definition 2.4. Let $p:\mathcal{E}\to\mathcal{B}$ be a functor. Then p is *a fibered category* if and only if for every morphism $f:X\to Y$ of $\mathcal{B}$ and every object η of $\mathcal{E}$ such that $p(\eta)=Y$ there exists a pullback of η along f. If $p:\mathcal{E}\to\mathcal{B}$ is a fibered category, then we say that *$\mathcal{E}$ is fibered over $\mathcal{B}$ with respect to p.*

Now we give some examples of fibered categories. The first is a prototypical for the notion of a cartesian category. It shows that any category $\mathcal{B}$ with fiber products gives rise in a canonical way to a fibered category over $\mathcal{B}$ with cartesian arrows as cartesian squares in $\mathcal{B}$.

Example 2.5 (the fibered category of arrows)**.** Let $\mathcal{B}$ be a category. We define the category $\mathrm{Arr}(\mathcal{B})$ of arrows of $\mathcal{B}$ as follows. Objects of $\mathrm{Arr}(\mathcal{B})$ are morphisms $\pi:\tilde{X}\to X$ of $\mathcal{B}$. Now if $\pi:\tilde{X}\to X$ and $\psi:\tilde{Y}\to Y$ are objects of $\mathrm{Arr}(\mathcal{B})$, then a morphism $\pi\to\psi$ is a pair (f,ϕ) such that $f:X\to Y$ and $\phi:\tilde{X}\to\tilde{Y}$ are morphisms in $\mathcal{B}$ making the square

$$\begin{tikzcd} \tilde{X} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \tilde{Y} \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutative. There exists a functor $p_{\mathrm{Arr}}:\mathrm{Arr}(\mathcal{B})\to\mathcal{B}$ given by formula $p_{\mathrm{Arr}}\big((f,\phi)\big)=f$. Suppose now that $f:X\to Y$ and $\psi:\tilde{Y}\to Y$ are morphisms of $\mathcal{B}$ and there exists a commutative square

$$\begin{tikzcd} \tilde{X} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \tilde{Y} \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

It is a direct consequence of the definition that (f,ϕ) is a cartesian morphisms of p_{Arr} if and only if the square above is cartesian. Thus p_{Arr} is a fibered category provided that $\mathcal{B}$ admits fiber products.

Definition 2.6. Suppose that $p_1 : \mathcal{E}_1 \to \mathcal{B}$ and $p_2 : \mathcal{E}_2 \to \mathcal{B}$ are fibered categories. Then a functor $F : \mathcal{E}_1 \to \mathcal{E}_2$ is *a morphism of fibered categories* if the following two assertions are satisfied.

(1) $p_1 = F \cdot p_2$ or in other words F is a functor over $\mathcal{B}$.

(2) Image under F of a cartesian morphism of p_1 is a cartesian morphism of p_2.

Next example is closely related to the previous one, but is of more topological flavour.

Example 2.7 (the fibered category vector bundles)**.** Let **Top** be the category of topological spaces. We define a subcategory $\mathbf{VectBund}_{\mathbb{R}}$ of $\mathrm{Arr}(\mathbf{Top})$ of vector bundles as follows. Objects of $\mathbf{VectBund}_{\mathbb{R}}$ are topological $\mathbb{R}$-vector bundles $\pi : \mathcal{V} \to X$. Now if $\pi : \mathcal{V} \to X$ and $\psi : \mathcal{W} \to Y$ are topological $\mathbb{R}$-vector bundles, then a morphism $\pi \to \psi$ is a pair (f,ϕ) such that $f : X \to Y$ is a continuous map and $\phi : \mathcal{V} \to \mathcal{W}$ is a continuous making the square

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \mathcal{W} \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutative and moreover, ϕ induces an $\mathbb{R}$-linear map on fibers i.e. for each point x in X map ϕ induces an $\mathbb{R}$-linear map $\pi^{-1}(x) \to \psi^{-1}(f(x))$. Since topological vector bundles are stable under continuous change of base, we obtain a fibered category $\mathbf{VectBund}_{\mathbb{R}} \to \mathbf{Top}$ as the restriction of $p_{\mathrm{Arr}} : \mathrm{Arr}(\mathbf{Top}) \to \mathbf{Top}$. Thus we have a commutative triangle

$$\begin{tikzcd} \mathbf{VectBund}_{\mathbb{R}} \arrow[rr, hook] \arrow[dr] & & \mathrm{Arr}(\mathbf{Top}) \arrow[dl, "p_{\mathrm{Arr}}"] \\ & \mathbf{Top} & \end{tikzcd}$$

According to Example 2.5 the inclusion $\mathbf{VectBund}_{\mathbb{R}} \hookrightarrow \mathrm{Arr}(\mathbf{Top})$ is a morphism of fibered categories.

3. Example: Principal Bundles

We devote this section to another important example of a fibered category. We fix a category with finite limits $\mathcal{B}$ and a monoid object $\mathbf{M}$ of $\mathcal{B}$. We denote by $\mu : \mathbf{M} \times \mathbf{M} \to \mathbf{M}$ and $e : \mathbf{1} \to \mathbf{M}$ the multiplication and unit of $\mathbf{M}$, respectively.

Definition 3.1. Let $\mathcal{P}$ be an object of $\mathcal{B}$ equipped with an action of $\mathbf{M}$, let T be an object of $\mathcal{B}$ with trivial action of $\mathbf{M}$ and let $\pi : \mathcal{P} \to T$ be an $\mathbf{M}$-equivariant morphism with respect to these $\mathbf{M}$-actions. We say that $\mathbf{M}$-equivariant morphism π is *a trivial principal* $\mathbf{M}$*-bundle on* T if there exists an $\mathbf{M}$-equivariant isomorphism $\phi : \mathcal{P} \to \mathbf{M} \times T$ such that $\mathbf{M} \times T$ is equipped with an action of $\mathbf{M}$ given by $\mu \times 1_T$ and the triangle

$$\begin{tikzcd} \mathcal{P} \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \mathbf{M} \times T \arrow[dl, "\mathrm{pr}_T"] \\ & T & \end{tikzcd}$$

is commutative.

Definition 3.2. Let $\mathcal{P}$ be an object of $\mathcal{B}$ equipped with an action of $\mathbf{M}$, let T be an object of $\mathcal{B}$ with trivial action of $\mathbf{M}$ and let $\pi : \mathcal{P} \to T$ be a $\mathbf{M}$-equivariant morphism with respect to these $\mathbf{M}$-actions. Consider a sieve S on T. For every arrow $g : \widetilde{T} \to T$ in S we construct a cartesian square

$$\begin{tikzcd} g^*\mathcal{P} \arrow[r] \arrow[d, "\pi_g"'] & \mathcal{P} \arrow[d, "\pi"] \\ \widetilde{T} \arrow[r, "g"'] & T \end{tikzcd}$$

in $\mathcal{B}$. We consider g as an $\mathbf{M}$-equivariant morphism with respect to trivial $\mathbf{M}$-actions on T and $\widetilde{T}$. Then there exists a unique action of $\mathbf{M}$ on $g^*\mathcal{P}$ which makes π_g into an $\mathbf{M}$-equivariant morphism in such a way that the square consists of objects of $\mathcal{B}$ with $\mathbf{M}$-actions and $\mathbf{M}$-equivariant morphisms. Suppose that $\mathbf{M}$-equivariant morphism π_g is a trivial principal $\mathbf{M}$-bundle on $\widetilde{T}$ for every g in S. Then we say that *S trivializes π*.

In the remaining part of this section we fix a Grothendieck topology $\mathcal{J}$ on $\mathcal{B}$.

Definition 3.3. Let $\mathcal{P}$ be an object of $\mathcal{B}$ equipped with an action of $\mathbf{M}$, let T be an object of $\mathcal{B}$ with trivial action of $\mathbf{M}$ and let $\pi : \mathcal{P} \to T$ be a $\mathbf{M}$-equivariant morphism with respect to these $\mathbf{M}$-actions. Suppose that there exists a covering sieve S in $\mathcal{J}(T)$ that trivializes π. Then π is called *a principal* $\mathbf{M}$*-bundle with respect to* $\mathcal{J}$.

Now we define a category $\mathbb{B}\mathbf{M}$ that depends on the site $(\mathcal{B}, \mathcal{J})$. Its objects are principal $\mathbf{M}$-bundles with respect to $\mathcal{J}$ and if $\pi : \mathcal{P} \to T$ and $\psi : \mathcal{Q} \to Z$ are principal $\mathbf{M}$-bundles with respect to $\mathcal{J}$, then a morphism $\pi \to \psi$ is a pair (f, ϕ) such that $f : T \to Z$ and $\phi : \mathcal{P} \to \mathcal{Q}$ are morphisms in $\mathcal{B}$ such that ϕ is $\mathbf{M}$-equivariant and the square

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \mathcal{Q} \arrow[d, "\psi"] \\ T \arrow[r, "f"'] & Z \end{tikzcd}$$

is commutative. We have a functor $p_{\mathbf{M},\mathcal{J}} : \mathbb{B}\mathbf{M} \to \mathcal{B}$ given by $p_{\mathbf{M},\mathcal{J}}\big((f, \phi)\big) = f$. Let $\psi : \mathcal{Q} \to Z$ be a principal $\mathbf{M}$-bundle with respect to $\mathcal{J}$ and let $f : T \to Z$ be a morphism. Consider the cartesian square

$$\begin{tikzcd} f^*\mathcal{Q} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \mathcal{Q} \arrow[d, "\psi"] \\ T \arrow[r, "f"'] & Z \end{tikzcd}$$

in $\mathcal{B}$. Then by the universal property there exists a unique action of $\mathbf{M}$ on $f^*\mathcal{Q}$ such that the square above consists of $\mathbf{M}$-equivariant morphisms (T, Z are equipped with trivial $\mathbf{M}$-actions). Moreover, with respect to this action $\psi : f^*\mathcal{Q} \to T$ becomes a principal $\mathbf{M}$-bundle with respect to $\mathcal{J}$. Indeed, if S is in $\mathcal{J}(Z)$ and S trivializes ψ, then its pullback f^*S trivializes π and is an element of $\mathcal{J}(T)$ (by definition of a Grothendieck topology). This shows that $p_{\mathbf{M},\mathcal{J}} : \mathbb{B}\mathbf{M} \to \mathcal{B}$ is a fibered category. Moreover, we have a functor $\mathbb{B}\mathbf{M} \to \mathrm{Arr}(\mathcal{B})$ that forgets about $\mathbf{M}$-actions. Hence there exists commutative triangle

$$\begin{tikzcd} \mathbb{B}\mathbf{M} \arrow[rr] \arrow[dr, "p_{\mathbf{M},\mathcal{J}}"'] & & \mathrm{Arr}(\mathcal{B}) \arrow[dl, "p_{\mathrm{Arr}}"] \\ & \mathcal{B} & \end{tikzcd}$$

According to Example 2.5 and description of cartesian morphisms of $p_{\mathbf{M},\mathcal{J}}$ the functor $\mathbb{B}\mathbf{M} \to \mathrm{Arr}(\mathcal{B})$ described above is a morphism of fibered categories.

Definition 3.4. $p_{\mathbf{M},\mathcal{J}} : \mathbb{B}\mathbf{M} \to \mathcal{B}$ is called *the fibered category of principal* $\mathbf{M}$*-bundles on* $(\mathcal{B},\mathcal{J})$.

Suppose that X is an object of $\mathcal{B}$ equipped with an action $a : \mathbf{M} \times X \to X$ of $\mathbf{M}$. We define a category $[X/\mathbf{M}]$ depending on a and the site $(\mathcal{B},\mathcal{J})$ as follows. Its objects are pairs (π,α) such that π is a principal $\mathbf{M}$-bundle with respect to $\mathcal{J}$ and α is an $\mathbf{M}$-equivariant morphism. We depict them by diagrams

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\alpha"] \arrow[d, "\pi"'] & X \\ T & \end{tikzcd}$$

Suppose that $(\pi : \mathcal{P} \to T, \alpha : \mathcal{P} \to X)$ and $(\psi : \mathcal{Q} \to Z, \beta : \mathcal{Q} \to X)$ are two such objects. Then a morphism $(\pi,\alpha) \to (\psi,\beta)$ is a morphism $(f,\phi) : \pi \to \psi$ in $\mathbb{B}\mathbf{M}$ such that $\alpha = \beta \cdot \phi$. We have a functor $[X/\mathbf{M}] \to \mathbb{B}\mathbf{M}$ which sends (π,α) to π. We denote by $p_{a,\mathcal{J}} : [X/\mathbf{M}] \to \mathcal{B}$ the composition of this functor $[X/\mathbf{M}] \to \mathbb{B}\mathbf{M}$ with $p_{\mathbf{M},\mathcal{J}} : \mathbb{B}\mathbf{M} \to \mathcal{B}$. By description of cartesian morphisms of $p_{\mathbf{M},\mathcal{J}}$ we deduce that $p_{a,\mathcal{J}}$ is a fibered category. We have a commutative triangle

$$\begin{tikzcd} {[X/\mathbf{M}]} \arrow[rr] \arrow[dr, "p_{a,\mathcal{J}}"'] & & \mathbb{B}\mathbf{M} \arrow[dl, "p_{\mathbf{M},\mathcal{J}}"] \\ & \mathcal{B} & \end{tikzcd}$$

and the functor $[X/\mathbf{M}] \to \mathbb{B}\mathbf{M}$ described above is a morphism of fibered categories. Note that if $\mathbf{1}$ is a terminal object of $\mathcal{B}$ equipped with trivial action of $\mathbf{M}$, then we have a canonical isomorphism $[\mathbf{1}/\mathbf{M}] \cong \mathbb{B}\mathbf{M}$ of categories over $\mathcal{B}$.

Definition 3.5. $p_{a,\mathcal{J}} : \mathbb{B}\mathbf{M} \to \mathcal{B}$ is called *the quotient fibered category of* $\mathbf{M}$*-object* X *on* $(\mathcal{B},\mathcal{J})$.

Results below show that up to some mild assumptions on Grothendieck topology $\mathcal{J}$ fibered category $p_{a,\mathcal{J}} : [X/\mathbf{M}] \to \mathcal{B}$ encapsulates all essential information concerning action of $\mathbf{M}$ on X. We start with the following observation.

Fact 3.6. *Let* X, Y *be objects of* $\mathcal{B}$ *equipped with actions* $a : \mathbf{M} \times X \to X$ *and* $b : \mathbf{M} \times Y \to Y$ *of* $\mathbf{M}$*. Consider a functor* $F : [X/\mathbf{M}] \to [Y/\mathbf{M}]$ *such that the triangle*

$$\begin{tikzcd} {[X/\mathbf{M}]} \arrow[rr, "F"] \arrow[dr] & & {[Y/\mathbf{M}]} \arrow[dl] \\ & \mathbb{B}\mathbf{M} & \end{tikzcd}$$

is commutative, where two other sides are canonical functors. Then F *is a morphism of fibered categories* $p_{a,\mathcal{J}}$ *and* $p_{b,\mathcal{J}}$*.*

Proof. The commutativity of the triangle implies that $F \cdot p_{b,\mathcal{J}} = p_{a,\mathcal{J}}$. Since a morphism in $[X/\mathbf{M}]$ is cartesian with respect to $p_{a,\mathcal{J}}$ if and only if its image under the canonical functor $[X/\mathbf{M}] \to \mathbb{B}\mathbf{M}$ is cartesian with respect to $p_{\mathbf{M},\mathcal{J}}$ and the same holds for $p_{b,\mathcal{J}}$, we derive that F sends cartesian morphisms of $p_{a,\mathcal{J}}$ to cartesian morphisms of $p_{b,\mathcal{J}}$. □

Let X, Y be objects of $\mathcal{B}$ equipped with actions $a : \mathbf{M} \times X \to X$ and $b : \mathbf{M} \times Y \to Y$ of $\mathbf{M}$. We denote the class of functors in Fact 3.6 by $\mathrm{Mor}_{\mathbb{B}\mathbf{M}}([X/\mathbf{M}], [Y/\mathbf{M}])$. We also denote (by abuse of notation) the class of $\mathbf{M}$-equivariant morphism $(X, a) \to (Y, b)$ by $\mathrm{Mor}_{\mathbf{M}}(X, Y)$.

Theorem 3.7. *Let $(\mathcal{B}, \mathcal{J})$ be a Grothendieck site and assume that representable presheaves on $\mathcal{B}$ are separated with respect to $\mathcal{J}$. Let X, Y be objects of $\mathcal{B}$ equipped with $\mathbf{M}$-actions $a : \mathbf{M} \times X \to X$ and $b : \mathbf{M} \times Y \to Y$, respectively. Then there exists a bijection*

$$\mathrm{Mor}_{\mathbf{M}}(X, Y) \cong \mathrm{Mor}_{\mathbb{B}\mathbf{M}}\big([X/\mathbf{M}], [Y/\mathbf{M}]\big)$$

that sends an $\mathbf{M}$-equivariant morphism f to a functor $F : [X/\mathbf{M}] \to [Y/\mathbf{M}]$ given by

$$\begin{tikzcd} \mathcal{P} \arrow[r, "d"] \arrow[d, "\pi"'] & X \\ T & \end{tikzcd} \quad \overset{F}{\longmapsto} \quad \begin{tikzcd} \mathcal{P} \arrow[r, "f \cdot d"] \arrow[d, "\pi"'] & Y \\ T & \end{tikzcd}$$

Proof. We first describe certain object of $[X/\mathbf{M}]$. Observe that $(\mathbf{M} \times X, \mu \times 1_X)$ is an object of $\mathcal{B}$ equipped with the action of $\mathbf{M}$. Next the projection $\mathrm{pr}_X : \mathbf{M} \times X \to X$ can be considered as an $\mathbf{M}$-equivariant morphism from this $\mathbf{M}$-object to X with the trivial action of $\mathbf{M}$. Since the square

$$\begin{tikzcd} \mathbf{M} \times \mathbf{M} \times X \arrow[r, "1_{\mathbf{M}} \times a"] \arrow[d, "\mu \times 1_X"'] & \mathbf{M} \times X \arrow[d, "a"] \\ \mathbf{M} \times X \arrow[r, "a"'] & X \end{tikzcd}$$

is commutative, we derive that a is an $\mathbf{M}$-equivariant morphism $(\mathbf{M} \times X, \mu \times 1_X) \to (X, a)$. This gives (pr_X, a) the structure of an object of $[X/\mathbf{M}]$. Fix a functor F in $\mathrm{Mor}_{\mathbb{B}\mathbf{M}}\big([X/\mathbf{M}], [Y/\mathbf{M}]\big)$. The functor F sends $\big(\mathrm{pr}_X, a\big)$ to some object of $[Y/\mathbf{M}]$. This object is necessarily of the form (pr_X, α) for some $\mathbf{M}$-equivariant morphism $\alpha : (\mathbf{M} \times X, \mu \times 1_X) \to (Y, b)$. Indeed, this follows from the fact that F is over $\mathbb{B}\mathbf{M}$. Now if F is determined by some $\mathbf{M}$-equivariant morphism f as it is described in the statement, then $\alpha = f \cdot a$ and hence $f = \alpha \cdot \langle e, 1_X \rangle$. This proves that the map $\mathrm{Mor}_{\mathbf{M}}\big(X, Y\big) \to \mathrm{Mor}_{\mathbb{B}\mathbf{M}}\big([X/\mathbf{M}], [Y/\mathbf{M}]\big)$ described in the statement is injective. Our goal is to show that it is surjective. That is our goal is to show that for the functor F in $\mathrm{Mor}_{\mathbb{B}\mathbf{M}}\big([X/\mathbf{M}], [Y/\mathbf{M}]\big)$ a morphism $f = \alpha \cdot \langle e, 1_X \rangle$ is $\mathbf{M}$-equivariant and determines F as it is described in the statement. First we fix some object T of $\mathcal{B}$ and the projection $\mathrm{pr}_T : \mathbf{M} \times T \to T$ considered as a trivial principal $\mathbf{M}$-bundle. Let (pr_T, c) be an object of $[X/\mathbf{M}]$. Then c is an $\mathbf{M}$-equivariant morphism $c : (\mathbf{M} \times T, \mu \times 1_T) \to (X, a)$. Functor F sends (pr_T, c) to some object (pr_T, γ). We claim that $\gamma = f \cdot c$. Let $\mathrm{pr}_{23} : \mathbf{M} \times \mathbf{M} \times T \to \mathbf{M} \times T$ be the projection on the last two factors. There are diagrams

$$\begin{tikzcd} \mathbf{M} \times \mathbf{M} \times T \arrow[r, "\mu \times 1_T"] \arrow[d, "\mathrm{pr}_{23}"'] & \mathbf{M} \times T \arrow[r, "c"] \arrow[d, "\mathrm{pr}_T"] & X \\ \mathbf{M} \times T \arrow[r, "\mathrm{pr}_T"'] & T & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{M} \times \mathbf{M} \times T \arrow[r, "1_{\mathbf{M}} \times c"] \arrow[d, "\mathrm{pr}_{23}"'] & \mathbf{M} \times X \arrow[r, "a"] \arrow[d, "\mathrm{pr}_X"] & X \\ \mathbf{M} \times T \arrow[r, "c"'] & X & \end{tikzcd}$$

representing morphisms

$$\big(\mathrm{pr}_T, \mu \times 1_T\big) : \big(\mathrm{pr}_{23}, c \cdot (\mu \times 1_T)\big) \to \big(\mathrm{pr}_T, c\big),\ (c, 1_{\mathbf{M}} \times c) : \big(\mathrm{pr}_{23}, a \cdot (1_{\mathbf{M}} \times c)\big) \to \big(\mathrm{pr}_X, a\big)$$

in $[X/\mathbf{M}]$. Moreover, c is $\mathbf{M}$-equivariant $(\mathbf{M}\times T, \mu\times 1_T) \to (X, a)$ and hence we derive that $c\cdot(\mu\times 1_T) = a\cdot(c\times 1_{\mathbf{M}})$. Thus the morphisms in $[X/\mathbf{M}]$ described above have common domain. Since F is over $\mathbb{B}\mathbf{M}$, we derive that their images under F are

$$\begin{tikzcd}
\mathbf{M}\times\mathbf{M}\times T \arrow[r, "\mu\times 1_T"] \arrow[d, "\mathrm{pr}_{23}"'] & \mathbf{M}\times T \arrow[r, "\gamma"] \arrow[d, "\mathrm{pr}_T"] & X \\
\mathbf{M}\times T \arrow[r, "\mathrm{pr}_T"'] & T &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{M}\times\mathbf{M}\times T \arrow[r, "1_{\mathbf{M}}\times c"] \arrow[d, "\mathrm{pr}_{23}"'] & \mathbf{M}\times X \arrow[r, "\alpha"] \arrow[d, "\mathrm{pr}_X"] & X \\
\mathbf{M}\times T \arrow[r, "c"'] & X &
\end{tikzcd}$$

This implies that $\gamma\cdot(\mu\times 1_T) = \alpha\cdot(1_{\mathbf{M}}\times c)$. We deduce that

$$\gamma = \gamma\cdot(\mu\times 1_T)\cdot\langle e, 1_{\mathbf{M}\times X}\rangle = \alpha\cdot(1_{\mathbf{M}}\times c)\cdot\langle e, 1_{\mathbf{M}\times X}\rangle = \alpha\cdot\langle e, 1_X\rangle\cdot c = f\cdot c$$

and the claim is proved. We apply this to α to derive that $\alpha = f\cdot a$. Next recall that $\alpha\cdot(\mu\times 1_X) = b\cdot(1_{\mathbf{M}}\times\alpha)$ because α is an $\mathbf{M}$-equivariant morphism $(\mathbf{M}\times X, \mu\times 1_X)\to(Y, b)$. Thus

$$b\cdot(1_{\mathbf{M}}\times f) = b\cdot(1_{\mathbf{M}}\times\alpha)\cdot(1_{\mathbf{M}}\times\langle e, 1_X\rangle) = \alpha\cdot(\mu\times 1_X)\cdot(1_{\mathbf{M}}\times\langle e, 1_X\rangle) = \alpha$$

Hence $f\cdot a = \alpha = b\cdot(1_{\mathbf{M}}\times f)$. Thus f is $\mathbf{M}$-equivariant and F is given as in the statement on the subcategory of $[X/\mathbf{M}]$ consisting of trivial principal $\mathbf{M}$-bundles. Now consider any principal $\mathbf{M}$-bundle $\pi:\mathcal{P}\to T$ with respect to $\mathcal{J}$ and let $d:\mathcal{P}\to X$ be a $\mathbf{M}$-equivariant morphism to (X, a). We know that F sends (π, d) to some object of $[Y/\mathbf{M}]$ of the form (π, δ). It suffices to prove that $\delta = f\cdot d$. For this consider a sieve S in $\mathcal{J}(T)$ such that S trivializes π. Pick $g:\widetilde{T}\to T$ in S and a cartesian square

$$\begin{tikzcd}
g^*\mathcal{P} \arrow[r, "g'"] \arrow[d, "\pi_g"'] & \mathcal{P} \arrow[d, "\pi"] \\
\widetilde{T} \arrow[r, "g"'] & T
\end{tikzcd}$$

Then $(\pi_g, d\cdot g')$ is an object of $[X/\mathbf{M}]$. Since F is over $\mathbb{B}\mathbf{M}$, we derive that $F(\pi_g, d\cdot g') = (\pi_g, \delta\cdot g')$. By definition π_g is trivial $\mathbf{M}$-bundle. Thus (from what we proved above) we have

$$\delta\cdot g' = f\cdot d\cdot g'$$

This holds for pullback g' of every g in S along π. These pullbacks $\{g'\}_{g\in S}$ generate some sieve S' on $\mathcal{P}$ and the formula

$$\delta\cdot h = f\cdot d\cdot h$$

holds for every h in S'. Moreover, S' is a covering sieve on a site $(\mathcal{B}, \mathcal{J})$ i.e. $S'\in\mathcal{J}(\mathcal{P})$. According to the assumption on $\mathcal{J}$ we infer that $h^{\mathcal{B}}_{\mathcal{P}} = \mathrm{Mor}_{\mathcal{B}}(-, \mathcal{P}) : \mathcal{B}^{\mathrm{op}}\to\mathbf{Set}$ is a separated presheaf with respect to $\mathcal{J}$. Thus the formula

$$\delta\cdot h = f\cdot d\cdot h$$

which holds for every h in S' implies that $\delta = f\cdot d$. □

4. Pseudo-functors and fibered categories of elements

Pseudo-functors are certain non-strict 2-functors. In this section we introduce a procedure that enables to construct a fibered category out of a pseudo-functor. We start by defining this notion.

Definition 4.1. Let $\mathcal{B}$ be a category. Consider the tuple of collections

$$F = \left(\{F(X)\}_{X\in\mathrm{Ob}(\mathcal{B})}, \{F(f)\}_{f\in\mathrm{Mor}(\mathcal{B})}, \{\Theta^{f,g}\}_{f,g\in\mathrm{Mor}(\mathcal{B}),\,\mathrm{cod}(f)=\mathrm{dom}(g)}, \{\epsilon^X\}_{X\in\mathrm{Ob}(\mathcal{B})}\right)$$

of the following data.

(1) For each object X of $\mathcal{B}$ a category $F(X)$.

(2) For each arrow $f : X \to Y$ a functor $F(f) : F(Y) \to F(X)$.

(3) For each object X of $\mathcal{B}$ a natural isomorphism $\epsilon^X : 1_{F(X)} \to F(1_X)$.

(4) For any two composable morphisms $f : X \to Y$ and $g : Y \to Z$ of $\mathcal{B}$ a natural isomorphism $\Theta^{g,f} : F(f) \cdot F(g) \to F(g \cdot f)$

Suppose that these data are subject to the following conditions.

(1) For every arrow $f : X \to Y$ in $\mathcal{B}$ we have
$$1_{F(f)} = \Theta^{f,1_X} \cdot \epsilon^X_{F(f)},\ 1_{F(f)} = \Theta^{1_Y,f} \cdot F(f)\left(\epsilon^Y\right)$$

(2) For any three morphisms $f : X \to Y, g : Y \to Z, h : Z \to W$ of $\mathcal{B}$ the square of functors and natural isomorphisms

$$\begin{tikzcd} F(f)\cdot F(g)\cdot F(h) \arrow[r, "F(f)(\Theta^{h,g})"] \arrow[d, "\Theta^{g,f}_{F(h)}"'] & F(f)\cdot F(h\cdot g) \arrow[d, "\Theta^{h\cdot g,f}"] \\ F(g\cdot f)\cdot F(h) \arrow[r, "\Theta^{h,g\cdot f}"'] & F(h\cdot g\cdot f) \end{tikzcd}$$

is commutative.

Then F is called *a pseudo-functor on $\mathcal{B}$*

Now we show how to construct a fibered category from a pseudo-functor. Suppose that $\mathcal{B}$ is a category and
$$F = \left(\{F(X)\}_{X\in \mathrm{Ob}(\mathcal{B})}, \{F(f)\}_{f\in \mathrm{Mor}(\mathcal{B})}, \{\Theta^{f,g}\}_{f,g\in \mathrm{Mor}(\mathcal{B}),\, \mathrm{cod}(f)=\mathrm{dom}(g)}, \{\epsilon^X\}_{X\in \mathrm{Ob}(\mathcal{B})}\right)$$
is a pseudo-functor on $\mathcal{B}$. We define a category $\int_{\mathcal{B}} F$. Its objects are pairs (X, ξ) such that X is an object of $\mathcal{B}$ and ξ is an object of $F(X)$. If (X, ξ) and (Y, η) are objects of $\int_{\mathcal{B}} F$, then a morphism between these objects is a pair (f, σ) such that $f : X \to Y$ is a morphism of $\mathcal{B}$ and $\sigma : \xi \to F(f)(\eta)$ is a morphism of $F(X)$. Now suppose that $(f, \sigma) : (X, \xi) \to (Y, \eta)$ and $(g, \tau) : (Y, \eta) \to (Z, \zeta)$ are morphisms of $\int_{\mathcal{B}} F$. Then we define their composition by formula
$$(g, \tau) \cdot (f, \sigma) = \left(g \cdot f, \Theta^{g,f}_{\zeta} \cdot F(f)\,(\tau) \cdot \sigma\right)$$

Fact 4.2. *$\int_{\mathcal{B}} F$ is a well defined category.*

Proof. We first verify that the composition of morphisms in $\int_{\mathcal{B}} F$ is associative. Suppose that $(f, \sigma) : (X, \xi) \to (Y, \eta), (g, \tau) : (Y, \eta) \to (Z, \zeta), (h, \rho) : (Z, \zeta) \to (W, \omega)$ are morphisms of $\int_{\mathcal{B}} F$. Then
$$\left((h, \rho) \cdot (g, \tau)\right) \cdot (f, \sigma) = \left(h \cdot g, \Theta^{h,g}_{\omega} \cdot F(g)\,(\rho) \cdot \tau\right) \cdot (f, \sigma) =$$
$$= \left(h\cdot g\cdot f, \Theta^{h\cdot g,f}_{\omega} \cdot F(f)\big(\Theta^{h,g}_{\omega} \cdot F(g)\,(\rho)\cdot\tau\big)\cdot\sigma\right) = \left(h\cdot g\cdot f, \Theta^{h\cdot g,f}_{\omega} \cdot F(f)\big(\Theta^{h,g}_{\omega}\big)\cdot F(f)\big(F(g)\,(\rho)\,\big)\cdot F(f)\big(\tau\big)\cdot\sigma\right)$$
and
$$(h, \rho) \cdot \left((g, \tau) \cdot (f, \sigma)\right) = (h, \rho) \cdot \left(g \cdot f, \Theta^{g,f}_{\zeta} \cdot F(f)\big(\tau\big) \cdot \sigma\right) =$$

$$= \big(h\cdot g\cdot f, \Theta_{\omega}^{h,g\cdot f}\cdot F(g\cdot f)\big(\rho\big)\cdot \Theta_{\zeta}^{g,f}\cdot F(f)\big(\tau\big)\cdot\sigma\big) = \big(h\cdot g\cdot f, \Theta_{\omega}^{h,g\cdot f}\cdot \Theta_{F(h)(\omega)}^{g,f}\cdot F(f)\big(F(g)\,(\rho)\,\big)\cdot F(f)\big(\tau\big)\cdot\sigma\big)$$

Since $\Theta_{\omega}^{h\cdot g,f}\cdot F(f)\big(\Theta_{\omega}^{h,g}\big) = \Theta_{\omega}^{h,g\cdot f}\cdot\Theta_{F(h)(\omega)}^{g,f}$, we deduce that

$$\big((h,\rho)\cdot(g,\tau)\big)\cdot(f,\sigma) = (h,\rho)\cdot\big((g,\tau)\cdot(f,\sigma)\big)$$

and hence the composition in $\int_{\mathcal{B}} F$ is associative. Next we prove that for each object (X,ξ) of $\int_{\mathcal{B}} F$ there exists an identity morphism. We claim that $(1_X, \epsilon_{\xi}^X) : (X,\xi) \to (X,\xi)$ is the identity. Indeed, for morphisms $(f,\sigma) : (X,\xi) \to (Y,\eta)$ and $(g,\tau) : (Z,\zeta) \to (X,\xi)$ we have

$$(f,\sigma)\cdot(1_X,\epsilon_{\xi}^X) = \big(f, \Theta_{\eta}^{f,1_X}\cdot F(1_X)\,(\sigma)\cdot\epsilon_{\xi}^X\big) = \big(f,\Theta_{\eta}^{f,1_X}\cdot\epsilon_{F(f)(\eta)}^X\cdot\sigma\big) = (f,\sigma)$$

and

$$(1_X,\epsilon_{\xi}^X)\cdot(g,\tau) = \big(g,\Theta_{\zeta}^{1_X,g}\cdot F(g)\left(\epsilon_{\xi}^X\right)\cdot\tau\big) = (g,\tau)$$

Therefore, $\int_{\mathcal{B}} F$ is a category. □

Next we define a functor $p_F : \int_{\mathcal{B}} F \to \mathcal{B}$ by formula

$$p_F\Big((f,\sigma) : (X,\xi) \to (Y,\tau)\Big) = f : X \to Y$$

This is clearly a well defined functor. Now we prove the following statement.

The functor $p_F : \int_{\mathcal{B}} F \to \mathcal{B}$ is a fibered category.

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{B}$ and η be an object of $F(Y)$. Thus (Y,η) is an object of $\int_{\mathcal{B}} F$. It suffices to show that (Y,η) admits a pullback along f. We claim that

$$\big(f,1_{F(f)(\eta)}\big) : \big(X,F(f)(\eta)\big) \to (Y,\eta)$$

is a cartesian morphism of p_F that yields a pullback of η along f. To prove the claim consider an object (Z,ζ) of $\int_{\mathcal{B}} F$ and suppose that $(g,\tau) : (Z,\zeta) \to (Y,\eta)$ is a morphism of $\int_{\mathcal{B}} F$ such that g factors through f. Then there exists $h : Z \to X$ such that $f\cdot h = g$. Note that $\tau : \zeta \to F(g)(\eta)$. Since $g = f\cdot h$, we have

$$\tau = \Theta_{\eta}^{f,h}\cdot\left(\Theta_{\eta}^{f,h}\right)^{-1}\cdot\tau = \Theta_{\eta}^{f,h}\cdot F(h)\big(1_{F(f)(\eta)}\big)\cdot\left(\Theta_{\eta}^{f,h}\right)^{-1}\cdot\tau$$

and hence

$$(g,\tau) = \big(f,1_{F(f)(\eta)}\big)\cdot\left(h,\left(\Theta_{\eta}^{f,h}\right)^{-1}\cdot\tau\right)$$

Thus (g,τ) factors through $\big(f,1_{F(f)(\eta)}\big)$ and the formula above shows that this factorization is unique. Hence $\big(f,1_{F(f)(\eta)}\big)$ is a cartesian morphism of p_F. □

Definition 4.3. Let $\mathcal{B}$ be a category and let F be a pseudo-functor on $\mathcal{B}$. A fibered category $p_F : \int_{\mathcal{B}} F \to \mathcal{B}$ constructed above is called *the fibered category of elements of the pseudo-functor F*.

It is possible to construct a pseudo-functor out of a fibered category. We will give a brief outline of this construction. For this we introduce notation that will be also used in other considerations.

Definition 4.4. Let $p : \mathcal{E} \to \mathcal{B}$ be a fibered category. For every object X of $\mathcal{B}$ we denote by $p^{-1}(X)$ a subcategory of $\mathcal{E}$ consisting of all morphisms $\phi : \xi \to \eta$ such that $p(\phi) = 1_X$. We call this category *the fiber of p over X*.

Suppose now that $p : \mathcal{E} \to \mathcal{B}$ is a fibered category. Let $f : X \to Y$ be a morphism. For every object η in $p^{-1}(Y)$ we pick its pullback $\tilde{f}_\eta : f^*\eta \to \eta$ along f. By universal property of cartesian morphisms we deduce that this induces a functor $f^* : p^{-1}(Y) \to p^{-1}(X)$. Universal property of cartesian morphisms implies also the following assertions.

(1) For each object X of $\mathcal{B}$ we may choose $(1_X)^* = 1_{p^{-1}(X)}$.

(2) For any two composable morphisms $f : X \to Y$ and $g : Y \to Z$ of $\mathcal{B}$ there exists a unique natural isomorphism $\Theta^{g,f} : f^*g^* \to (g \cdot f)^*$ of functors such that for every ζ in $p^{-1}(Z)$ we have commutative diagram

$$\begin{tikzcd} f^*g^*\zeta \arrow[r, "\tilde{f}_{g^*\zeta}"] \arrow[d, "\Theta^{g,f}_\zeta"'] & g^*\zeta \arrow[r, "\tilde{g}_\zeta"] & \zeta \arrow[d, "1_\zeta"] \\ (g\cdot f)^*\zeta \arrow[rr, "\widetilde{g\cdot f}_\zeta"'] & & \zeta \end{tikzcd}$$

From **(1)**, **(2)** and Fact 2.2 one can deduce that the collection

$$\left(\{p^{-1}(X)\}_{X\in\mathrm{Ob}(\mathcal{B})}, \{f^*\}_{f\in\mathrm{Mor}(\mathcal{B})}, \{\Theta^{f,g}\}_{f,g\in\mathrm{Mor}(\mathcal{B}),\,\mathrm{cod}(f)=\mathrm{dom}(g)}, \{1_{p^{-1}(X)}\}_{X\in\mathrm{Ob}(\mathcal{B})}\right)$$

is a pseudo-functor.

Remark 4.5. The construction of the fibered category of elements is a part of 2-equivalence between appropriately defined category of pseudo-functors on $\mathcal{B}$ and the category of fibered categories over $\mathcal{B}$.

5. Example: Quasi-coherent sheaves

Note that all examples of fibered categories given so far were fibered subcategories of the fibered category of arrows $p_{\mathrm{Arr}} : \mathrm{Arr}(\mathcal{B}) \to \mathcal{B}$ for a given category $\mathcal{B}$ with fibered-products. In this section we employ the procedure that produces a fibered category out of a pseudo-functor to obtain an important example of a category fibered over $\mathbf{Sch}_k$ (the category of schemes over a ring k), which is not of this type.

Let $f : X \to Y$ be a morphism of k-schemes. We have an adjunction

$$\begin{tikzcd} \mathfrak{Qcoh}(X) \arrow[r, bend left, "f_*"] \arrow[r, phantom, "\perp"] & \mathfrak{Qcoh}(Y) \arrow[l, bend left, "f^*"] \end{tikzcd}$$

It is determined by the bijection

$$\mathrm{Hom}_{\mathcal{O}_Y}(f^*\mathcal{G}, \mathcal{F}) \xrightarrow{\Phi^f_{\mathcal{G},\mathcal{F}}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, f_*\mathcal{F})$$

Suppose now that $f : X \to Y$ and $g : Y \to Z$ are morphisms of k-schemes. Since $(g\cdot f)_* = g_* \cdot f_*$, there exists a unique natural isomorphism $\Theta^{g,f} : f^*g^* \to (g\cdot f)^*$ such that for every quasi-coherent sheaf $\mathcal{F}$ in $\mathfrak{Qcoh}(X)$ and every quasi-coherent sheaf $\mathcal{H}$ in $\mathfrak{Qcoh}(Z)$ we have

$$\Phi^{g\cdot f}_{\mathcal{H},\mathcal{F}} = \Phi^g_{\mathcal{H},f_*\mathcal{F}} \cdot \Phi^f_{g^*\mathcal{H},\mathcal{F}} \cdot \mathrm{Hom}_{\mathcal{O}_X}\left(\Theta^{g,f}_{\mathcal{H}}, 1_{\mathcal{F}}\right)$$

Now we have the following result.

Fact 5.1. *Suppose that $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$ are morphism of k-schemes. Then the square*

$$\begin{tikzcd}
f^*g^*h^* \arrow[r, "f^*\Theta^{h,g}"] \arrow[d, "\Theta^{g,f}_{h^*}"'] & f^*(h\cdot g)^* \arrow[d, "\Theta^{h\cdot g,f}"] \\
(g\cdot f)^*h^* \arrow[r, "\Theta^{h,g\cdot f}"'] & (h\cdot g\cdot f)^*
\end{tikzcd}$$

of functors and natural isomorphisms is commutative.

Proof. Suppose that $\mathcal{F}$ is an object of $\mathfrak{Qcoh}(X)$ and $\mathcal{K}$ is an object of $\mathfrak{Qcoh}(W)$. Then

$$\Phi^h_{\mathcal{K},g_*f_*\mathcal{F}}\cdot\Phi^g_{h^*\mathcal{K},f_*\mathcal{F}}\cdot\Phi^f_{g^*h^*\mathcal{K},\mathcal{F}}\cdot\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{g,f}_{h^*\mathcal{K}},1_{\mathcal{F}}\big)\cdot\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h,g\cdot f}_{\mathcal{K}},1_{\mathcal{F}}\big)=$$
$$=\Phi^h_{\mathcal{K},g_*f_*\mathcal{F}}\cdot\Phi^{g\cdot f}_{h^*\mathcal{K},\mathcal{F}}\cdot\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{g,f}_{h^*\mathcal{K}},1_{\mathcal{F}}\big)=\Phi^{h\cdot g\cdot f}_{\mathcal{K},\mathcal{F}}$$

and

$$\Phi^h_{\mathcal{K},g_*f_*\mathcal{F}}\cdot\Phi^g_{h^*\mathcal{K},f_*\mathcal{F}}\cdot\Phi^f_{g^*h^*\mathcal{K},\mathcal{F}}\cdot\mathrm{Hom}_{\mathcal{O}_X}\big(f^*\Theta^{h,g}_{\mathcal{K}},1_{\mathcal{F}}\big)\cdot\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h\cdot g,f}_{\mathcal{K}},1_{\mathcal{F}}\big)=$$
$$=\Phi^h_{\mathcal{K},g_*f_*\mathcal{F}}\cdot\Phi^g_{h^*\mathcal{K},f_*\mathcal{F}}\cdot\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h,g}_{\mathcal{K}},1_{f_*\mathcal{F}}\big)\cdot\Phi^f_{(h\cdot g)^*\mathcal{K},\mathcal{F}}\cdot\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h\cdot g,f}_{\mathcal{K}},1_{\mathcal{F}}\big)=$$
$$=\Phi^{h\cdot g}_{\mathcal{K},f_*\mathcal{F}}\cdot\Phi^f_{(h\cdot g)^*\mathcal{K},\mathcal{F}}\cdot\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h\cdot g,f}_{\mathcal{K}},1_{\mathcal{F}}\big)=\Phi^{h\cdot g\cdot f}_{\mathcal{K},\mathcal{F}}$$

Therefore, we derive that

$$\Phi^h_{\mathcal{K},g_*f_*\mathcal{F}}\cdot\Phi^g_{h^*\mathcal{K},f_*\mathcal{F}}\cdot\Phi^f_{g^*h^*\mathcal{K},\mathcal{F}}\cdot\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{g,f}_{h^*\mathcal{K}},1_{\mathcal{F}}\big)\cdot\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h,g\cdot f}_{\mathcal{K}},1_{\mathcal{F}}\big)=$$
$$=\Phi^h_{\mathcal{K},g_*f_*\mathcal{F}}\cdot\Phi^g_{h^*\mathcal{K},f_*\mathcal{F}}\cdot\Phi^f_{g^*h^*\mathcal{K},\mathcal{F}}\cdot\mathrm{Hom}_{\mathcal{O}_X}\big(f^*\Theta^{h,g}_{\mathcal{K}},1_{\mathcal{F}}\big)\cdot\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h\cdot g,f}_{\mathcal{K}},1_{\mathcal{F}}\big)$$

and hence

$$\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h,g\cdot f}_{\mathcal{K}}\cdot\Theta^{g,f}_{h^*\mathcal{K}},1_{\mathcal{F}}\big)=\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{g,f}_{h^*\mathcal{K}},1_{\mathcal{F}}\big)\cdot\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h,g\cdot f}_{\mathcal{K}},1_{\mathcal{F}}\big)=$$
$$=\mathrm{Hom}_{\mathcal{O}_X}\big(f^*\Theta^{h,g}_{\mathcal{K}},1_{\mathcal{F}}\big)\cdot\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h\cdot g,f}_{\mathcal{K}},1_{\mathcal{F}}\big)=\mathrm{Hom}_{\mathcal{O}_X}\big(\Theta^{h\cdot g,f}_{\mathcal{K}}\cdot f^*\Theta^{h,g}_{\mathcal{K}},1_{\mathcal{F}}\big)$$

Since this equality holds for every quasi-coherent sheaf $\mathcal{F}$ on X, we deduce that

$$\Theta^{h,g\cdot f}_{\mathcal{K}}\cdot\Theta^{g,f}_{h^*\mathcal{K}}=\Theta^{h\cdot g,f}_{\mathcal{K}}\cdot f^*\Theta^{h,g}_{\mathcal{K}}$$

for every quasi-coherent sheaf $\mathcal{K}$. This proves the assertion. □

Note that for every k-scheme X we may assume that $(1_X)_* = 1_{\mathfrak{Qcoh}(X)} = (1_X)^*$ and $\Phi^{1_X}_{\mathcal{G},\mathcal{F}} = \mathrm{Hom}_{\mathcal{O}_X}(1_{\mathcal{F}},1_{\mathcal{G}})$.

Fact 5.2. *Let $f : X \to Y$ and $g : Z \to X$ be morphisms of k-schemes. Then*

$$\Theta^{f,1_X} = 1_{f^*},\ \Theta^{1_X,g} = 1_{g^*}$$

Proof. Suppose that $\mathcal{F}$ is an object of $\mathfrak{Qcoh}(X)$ and $\mathcal{G}$ is an object of $\mathfrak{Qcoh}(Y)$. Then

$$\Phi^f_{\mathcal{G},\mathcal{F}}=\Phi^{f\cdot 1_X}_{\mathcal{G},\mathcal{F}}=\Phi^f_{\mathcal{G},\mathcal{F}}\cdot\Phi^{1_X}_{f^*\mathcal{G},\mathcal{F}}\cdot\mathrm{Hom}_{\mathcal{O}_X}\left(\Theta^{f,1_X}_{\mathcal{G}},1_{\mathcal{F}}\right)=\Phi^f_{\mathcal{G},\mathcal{F}}\cdot\mathrm{Hom}_{\mathcal{O}_X}\left(\Theta^{f,1_X}_{\mathcal{G}},1_{\mathcal{F}}\right)$$

and thus $\mathrm{Hom}_{\mathcal{O}_X}\left(\Theta^{f,1_X}_{\mathcal{G}},1_{\mathcal{F}}\right) = \mathrm{Hom}_{\mathcal{O}_X}\left(1_{f^*\mathcal{G}},1_{\mathcal{F}}\right)$. Since this holds for every quasi-coherent sheaf $\mathcal{F}$ on X, we derive that $\Theta^{f,1_X}_{\mathcal{G}} = 1_{f^*\mathcal{G}}$. Thus $\Theta^{f,1_X} = 1_{f^*}$.
Suppose that $\mathcal{H}$ is an object of $\mathfrak{Qcoh}(X)$ and $\mathcal{F}$ is an object of $\mathfrak{Qcoh}(Z)$. Then

$$\Phi^g_{\mathcal{H},\mathcal{F}}=\Phi^{1_X\cdot g}_{\mathcal{H},\mathcal{F}}=\Phi^{1_X}_{\mathcal{H},g_*\mathcal{F}}\cdot\Phi^g_{\mathcal{H},\mathcal{F}}\cdot\mathrm{Hom}_{\mathcal{O}_Z}\left(\Theta^{1_X,g}_{\mathcal{H}},1_{\mathcal{F}}\right)=\Phi^g_{\mathcal{H},\mathcal{F}}\cdot\mathrm{Hom}_{\mathcal{O}_Z}\left(\Theta^{1_X,g}_{\mathcal{H}},1_{\mathcal{F}}\right)$$

and thus $\mathrm{Hom}_{\mathcal{O}_Z}\left(\Theta^{1_X,g}_{\mathcal{H}},1_{\mathcal{F}}\right) = \mathrm{Hom}_{\mathcal{O}_Z}\left(1_{g^*\mathcal{H}},1_{\mathcal{F}}\right)$. Since this holds for every quasi-coherent sheaf $\mathcal{F}$ on Z, we derive that $\Theta^{1_X,g}_{\mathcal{H}} = 1_{g^*\mathcal{H}}$. Thus $\Theta^{1_X,g} = 1_{g^*}$. □

Now Facts 5.1 and 5.2 imply that the collection

$$\left(\{\mathfrak{Qcoh}(X)\}_{X\in\mathbf{Sch}_k},\{f^*\}_{f\in\mathrm{Mor}(\mathbf{Sch}_k)},\{\Theta^{f,g}\}_{f,g\in\mathrm{Mor}(\mathbf{Sch}_k),\,\mathrm{cod}(f)=\mathrm{dom}(g)},\{1_{1_{\mathfrak{Qcoh}(X)}}\}_{X\in\mathbf{Sch}_k}\right)$$

forms a pseudo-functor on $\mathbf{Sch}_k$.

Definition 5.3. *The fibered category of quasi-coherent sheaves on* $\mathbf{Sch}_k$ is the fibered category of elements of the pseudo-functor

$$\left(\{\mathfrak{Qcoh}(X)\}_{X\in\mathbf{Sch}_k},\{f^*\}_{f\in\mathrm{Mor}(\mathbf{Sch}_k)},\{\Theta^{f,g}\}_{f,g\in\mathrm{Mor}(\mathbf{Sch}_k),\,\mathrm{cod}(f)=\mathrm{dom}(g)},\{1_{1_{\mathfrak{Qcoh}(X)}}\}_{X\in\mathbf{Sch}_k}\right)$$

We denote it by $\mathfrak{Qcoh}\to\mathbf{Sch}_k$.

For every k-scheme X we have a category $\mathbf{Alg}\left(\mathfrak{Qcoh}(X)\right)$ of quasi-coherent $\mathcal{O}_X$-algebras. If $f: X\to Y$ is a morphism of k-schemes, then we have an adjuntion

$$\begin{tikzcd} \mathbf{Alg}\left(\mathfrak{Qcoh}(X)\right) \arrow[r, bend left, "f_*"] \arrow[r, phantom, "\perp"] & \mathbf{Alg}\left(\mathfrak{Qcoh}(Y)\right) \arrow[l, bend left, "f^*"] \end{tikzcd}$$

Using similar argument as above one can show that there exists a canonical structure of a pseudo-functor on the collection

$$\left(\{\mathbf{Alg}\left(\mathfrak{Qcoh}(X)\right)\}_{X\in\mathbf{Sch}_k},\{f^*\}_{f\in\mathrm{Mor}(\mathbf{Sch}_k)}\right)$$

Definition 5.4. *The fibered category of quasi-coherent algebras on* $\mathbf{Sch}_k$ is the fibered category of elements of the canonical pseudo-functor determined by the collection

$$\left(\{\mathbf{Alg}\left(\mathfrak{Qcoh}(X)\right)\}_{X\in\mathbf{Sch}_k},\{f^*\}_{f\in\mathrm{Mor}(\mathbf{Sch}_k)}\right)$$

We denote it by $\mathbf{Alg}\left(\mathfrak{Qcoh}\right)\to\mathbf{Sch}_k$.

Remark 5.5. For every k-scheme X we also have a canonical functor $|-|:\mathbf{Alg}\left(\mathfrak{Qcoh}(X)\right)\to\mathfrak{Qcoh}(X)$ that forgets about an algebra structure. The collection of all these functors for all k-schemes gives rise to a morphism of fibered categories

$$\begin{tikzcd} \mathbf{Alg}\left(\mathfrak{Qcoh}\right) \arrow[rr, "|-|"] \arrow[dr] & & \mathfrak{Qcoh} \arrow[dl] \\ & \mathbf{Sch}_k & \end{tikzcd}$$

Remark 5.6. Note that $\mathrm{Arr}(\mathbf{Sch}_k)$ admits a fibered subcategory that consists of affine morphisms $\pi:\widetilde{X}\to X$ of k-schemes. We denote this fibered category by $\mathbf{Aff}\left(\mathbf{Sch}_k\right)\to\mathbf{Sch}_k$. For every k-scheme X we have the relative affine spectrum functor $\mathrm{Spec}_X:\mathbf{Alg}\left(\mathfrak{Qcoh}(X)\right)\to\mathbf{Aff}_X$, where $\mathbf{Aff}_X$ is the category of schemes affine over X. It is an equivalence of categories and note that $\mathbf{Aff}_X$ is the fiber of $\mathbf{Aff}\left(\mathbf{Sch}_k\right)\to\mathbf{Sch}_k$ over X. Moreover, if $f:X\to Y$ is a morphism of k-schemes and $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_Y$-algebra, then the canonical square

$$\begin{tikzcd} \mathrm{Spec}_X f^*\mathcal{A} \arrow[r] \arrow[d] & \mathrm{Spec}_Y\mathcal{A} \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is cartesian. Thus the collection of functors Spec_X for all k-schemes X gives rise to a morphism of fibered categories

$$\begin{tikzcd} \mathbf{Alg}\,(\mathfrak{Qcoh}) \arrow[rr, "\mathrm{Spec}"] \arrow[dr] & & \mathbf{Aff}(\mathbf{Sch}_k) \arrow[dl] \\ & \mathbf{Sch}_k & \end{tikzcd}$$

6. Equivariant objects in fibered categories

Let k be a commutative ring. We fix a monoid k-scheme $\mathbf{M}$ with multiplication morphism μ. The following notion is useful for studying actions of algebraic groups and monoids.

Definition 6.1. Let X be a k-scheme and let $\mathbf{M}$ be a monoid k-scheme with an action $a : \mathbf{M} \times_k X \to X$ on X. We denote by $\pi : \mathbf{M} \times_k X \to X$ the projection. Consider a pair $(\mathcal{F}, \tau)$ consisting of a quasi-coherent sheaf $\mathcal{F}$ on X and an isomorphism $\tau : \pi^*\mathcal{F} \to a^*\mathcal{F}$. We call it *a quasi-coherent* $\mathbf{M}$*-sheaf on* (X, a) if the following equality

$$(\mu \times_k 1_X)^*\phi = (1_{\mathbf{M}} \times_k a)^*\phi \cdot \pi_{2,3}^*\phi$$

holds, where $\mu : \mathbf{M} \times_k \mathbf{M} \to \mathbf{M}$ is the multiplication on $\mathbf{M}$ and $\pi_{2,3} : \mathbf{M} \times_k \mathbf{M} \times_k X \to \mathbf{M} \times_k X$ is the projection on the last two factors.

Definition 6.2. Let X be a k-scheme and let $\mathbf{M}$ be a monoid k-scheme with an action $a : \mathbf{M} \times_k X \to X$ on X. We denote by $\pi : \mathbf{M} \times_k X \to X$ the projection. Let $(\mathcal{F}_1, \tau_1)$ and $(\mathcal{F}_2, \tau_2)$ be quasi-coherent $\mathbf{M}$-sheaves on (X, a). Suppose that $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism of quasi-coherent sheaves on X such that the square

$$\begin{tikzcd} \pi^*\mathcal{F}_1 \arrow[r, "\tau_1"] \arrow[d, "\pi^*\phi"'] & a^*\mathcal{F}_1 \arrow[d, "a^*\phi"] \\ \pi^*\mathcal{F}_2 \arrow[r, "\tau_2"'] & a^*\mathcal{F}_2 \end{tikzcd}$$

is commutative. Then ϕ is *a morphism of quasi-coherent* $\mathbf{M}$*-sheaves on* (X, a). We denote by $\mathfrak{Qcoh}_{\mathbf{M}}(X)$ the category of quasi-coherent $\mathbf{M}$-sheaves and call it *the category of quasi-coherent* $\mathbf{M}$*-sheaves on* (X, a).

Our goal in this section is to explain somewhat nonintuitive notion of quasi-coherent $\mathbf{M}$-sheaf on a k-scheme X equipped with action of $\mathbf{M}$. For this we use the machinery of fibered categories. We fix a fibered category $p : \mathcal{E} \to \mathcal{B}$. If $f : X \to Y$ and η is an object of $p^{-1}(Y)$, then we denote by $\tilde{f}_\eta : f^*\eta \to \eta$ a pullback of η. That is the square

$$\begin{tikzcd} f^*\eta \arrow[r, "\widetilde{f}_\eta"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is cartesian. Using some choice of pullback we obtain a functor $f^* : p^{-1}(Y) \to p^{-1}(X)$. We start with the following observation.

Remark 6.3. Consider morphisms f_1, f_2, g_1, g_2 in $\mathcal{B}$ such that $g_1 \cdot f_1 = g_2 \cdot f_2$ with $\mathrm{cod}(g_1) = Y = \mathrm{cod}(g_2)$. For every object η in $p^{-1}(Y)$ we have a unique identification $f_1^* g_1^* \eta \cong f_2^* g_2^* \eta$ such that the square

$$\begin{tikzcd} f_2^*g_2^*\eta \cong f_1^*g_1^*\eta \arrow[r, "\widetilde{f_1}_{g_1^*\eta}"] \arrow[d, "\widetilde{f_2}_{g_2^*\eta}"'] & g_1^*\eta \arrow[d, "\widetilde{g_1}_\eta"] \\ g_2^*\eta \arrow[r, "\widetilde{g_2}_\eta"'] & \eta \end{tikzcd}$$

is commutative.

Now we have the following result.

Fact 6.4. *Let $X, \mathbf{M}$ be objects of $\mathcal{B}$ and let ξ be an object of $\mathcal{E}$ in $p^{-1}(X)$. Assume that the cartesian product of X and $\mathbf{M}$ exists in $\mathcal{B}$ and denote by $\pi : \mathbf{M} \times X \to X$ the projection. Then there exists a unique morphism (depicted by dotted arrow) such that the diagram*

$$\begin{tikzcd} h^{\mathcal{E}}_{\pi^*\xi} \arrow[rr, bend left, "h^{\mathcal{E}}_{\widetilde{\pi}_\xi}"] \arrow[dr, dotted] \arrow[dd, "p_{\mathrm{hom}}"'] & & \\ & h^{\mathcal{B}}_{\mathbf{M}} \cdot p \times h^{\mathcal{E}}_{\xi} \arrow[r, "pr_{h^{\mathcal{E}}_{\xi}}"] \arrow[d, "1_{h^{\mathcal{B}}_{\mathbf{M}}\cdot p} \times p_{\mathrm{hom}}"'] & h^{\mathcal{E}}_{\xi} \arrow[d, "p_{\mathrm{hom}}"] \\ h^{\mathcal{B}}_{\mathbf{M}\times X} \cdot p \arrow[r, "="'] \arrow[rr, bend right, "(h^{\mathcal{B}}_{\pi})_p"'] & h^{\mathcal{B}}_{\mathbf{M}} \cdot p \times h^{\mathcal{B}}_{X} \cdot p \arrow[r, "pr_{h^{\mathcal{B}}_{X}\cdot p}"'] & h^{\mathcal{B}}_{X} \cdot p \end{tikzcd}$$

is commutative, where $pr_{h^{\mathcal{E}}_{\xi}}$ and $pr_{h^{\mathcal{B}}_{X}\cdot p}$ are projections. Moreover, this morphism is an isomorphism.

Proof. This is a consequence of the fact that both squares

$$\begin{tikzcd} h^{\mathcal{B}}_{\mathbf{M}} \cdot p \times h^{\mathcal{E}}_{\xi} \arrow[r, "pr_{h^{\mathcal{E}}_{\xi}}"] \arrow[d, "1_{h^{\mathcal{B}}_{\mathbf{M}}\cdot p} \times p_{\mathrm{hom}}"'] & h^{\mathcal{E}}_{\xi} \arrow[d, "p_{\mathrm{hom}}"] \\ h^{\mathcal{B}}_{\mathbf{M}} \cdot p \times h^{\mathcal{B}}_{X} \cdot p \arrow[r, "pr_{h^{\mathcal{B}}_{X}\cdot p}"'] & h^{\mathcal{B}}_{X} \cdot p \end{tikzcd} \qquad \begin{tikzcd} h^{\mathcal{E}}_{\pi^*\xi} \arrow[r, "h^{\mathcal{E}}_{\widetilde{\pi}_\xi}"] \arrow[d, "p_{\mathrm{hom}}"'] & h^{\mathcal{E}}_{\xi} \arrow[d, "p_{\mathrm{hom}}"] \\ h^{\mathcal{B}}_{\mathbf{M}\times X} \cdot p \arrow[r, "(h^{\mathcal{B}}_{\pi})_p"'] & h^{\mathcal{B}}_{X} \cdot p \end{tikzcd}$$

are cartesian. ☐

Fix now two objects $\mathbf{M}$ and X of $\mathcal{B}$ such that the product of $\mathbf{M}$ and X exists. Denote by $\pi : \mathbf{M} \times X \to X$ the projection on X. Let $a : \mathbf{M} \times X \to X$ be a morphism in $\mathcal{B}$, let ξ be an object in $p^{-1}(X)$ and let $\sigma : h^{\mathcal{B}}_{\mathbf{M}} \cdot p \times h^{\mathcal{E}}_{\xi} \to h^{\mathcal{E}}_{\xi}$ be a morphism of presheaves on $\mathcal{E}$. Suppose that the square

$$\begin{tikzcd} h^{\mathcal{B}}_{\mathbf{M}} \cdot p \times h^{\mathcal{E}}_{\xi} \arrow[r, "\sigma"] \arrow[d, "p_{\mathrm{hom}}"'] & h^{\mathcal{E}}_{\xi} \arrow[d, "p_{\mathrm{hom}}"] \\ h^{\mathcal{B}}_{\mathbf{M}\times X} \cdot p \arrow[r, "(h^{\mathcal{B}}_{a})_p"'] & h^{\mathcal{B}}_{X} \cdot p \end{tikzcd}$$

is commutative. According to Fact 6.4 we deduce that σ is representable by some morphism $\alpha^{\sigma} : \pi^*\xi \to \xi$ of $\mathcal{E}$. By universal property of cartesian square

$$\begin{tikzcd} a^*\xi \arrow[r, "\widetilde{a}_{\xi}"] \arrow[d, mapsto] & \xi \arrow[d, mapsto] \\ \mathbf{M} \times X \arrow[r, "a"'] & X \end{tikzcd}$$

we deduce that there exists a unique morphism $\tau^{\sigma} : \pi^*\xi \to a^*\xi$ in $p^{-1}(\mathbf{M} \times X)$ such that $\alpha^{\sigma} = \widetilde{a}_{\xi} \cdot \tau^{\sigma}$. Using this notation and Fact 6.4 we can now state the following result.

Proposition 6.5. *Let* $\mathbf{M}$ *be a monoid object in* $\mathcal{B}$ *and let* X *be an object of* $\mathcal{B}$ *equipped with an action* $a : \mathbf{M} \times X \to X$ *of* $\mathbf{M}$ *on* X*. Denote by* $\pi : \mathbf{M} \times X \to X$ *the projection on* X*. Consider an object* ξ *in* $p^{-1}(X)$ *and let* $\sigma : h^{\mathcal{B}}_{\mathbf{M}} \cdot p \times h^{\mathcal{E}}_{\xi} \to h^{\mathcal{E}}_{\xi}$ *be a morphism of presheaves on* $\mathcal{E}$*. Suppose that the square*

$$\begin{tikzcd} h^{\mathcal{B}}_{\mathbf{M}} \cdot p \times h^{\mathcal{E}}_{\xi} \arrow[r, "\sigma"] \arrow[d, "p_{\mathrm{hom}}"'] & h^{\mathcal{E}}_{\xi} \arrow[d, "p_{\mathrm{hom}}"] \\ h^{\mathcal{B}}_{\mathbf{M} \times X} \cdot p \arrow[r, "(h^{\mathcal{B}}_{a})_p"'] & h^{\mathcal{B}}_{X} \cdot p \end{tikzcd}$$

is commutative. Then the following assertions are equivalent.

(i) *σ is an action of a monoid presheaf $h^{\mathcal{B}}_{\mathbf{M}} \cdot p$ on a presheaf $h^{\mathcal{E}}_{\xi}$.*

(ii) *Morphism τ^{σ} satisfies (up to identifications described in Remark 6.3) the identities*

$$(\mu \times 1_X)^*\tau^{\sigma} = (1_{\mathbf{M}} \times a)^*\tau^{\sigma} \cdot \pi^*_{2,3}\tau^{\sigma},\ \langle e, 1_X\rangle^*\tau^{\sigma} = 1_{\xi}$$

where $\mu : \mathbf{M} \times \mathbf{M} \to \mathbf{M}$ is the multiplication on $\mathbf{M}$, $\pi_{2,3} : \mathbf{M} \times \mathbf{M} \times X \to \mathbf{M} \times X$ is the projection on the last two factors and $e : \mathbf{1} \to \mathbf{M}$ is the unit of $\mathbf{M}$.

Proof. Our first goal is to prove that

$$\sigma \cdot \left(1_{h^{\mathcal{B}}_{\mathbf{M}} \cdot p} \times \sigma\right) = \sigma \cdot \left(1_{h^{\mathcal{B}}_{\mu} \cdot p} \times 1_{h^{\mathcal{E}}_{\xi}}\right)$$

if and only if

$$(1_{\mathbf{M}} \times a)^*\,\tau^{\sigma} \cdot \pi^*_{23}\tau^{\sigma} = (\mu \times 1_X)^*\,\tau^{\sigma}$$

First note that the commutative square of presheaves

$$\begin{tikzcd} h^{\mathcal{B}}_{\mathbf{M}} \cdot p \times h^{\mathcal{E}}_{a^*\xi} \arrow[r, "1_{h^{\mathcal{B}}_{\mathbf{M}} \cdot p} \times h^{\mathcal{E}}_{\widetilde{a}_{\xi}}"] \arrow[d, "p_{\mathrm{hom}}"'] & h^{\mathcal{B}}_{\mathbf{M}} \cdot p \times h^{\mathcal{E}}_{\xi} \arrow[d, "p_{\mathrm{hom}}"] \\ h^{\mathcal{B}}_{\mathbf{M} \times \mathbf{M} \times X} \cdot p \arrow[r, "(h^{\mathcal{B}}_{1_{\mathbf{M}} \times a})_p"'] & h^{\mathcal{B}}_{\mathbf{M} \times X} \cdot p \end{tikzcd}$$

on $\mathcal{E}$ is cartesian. Next according to Fact 6.4 we infer that projections

$$pr_{h^{\mathcal{E}}_{a^*\xi}} : h^{\mathcal{B}}_{\mathbf{M}} \cdot p \times h^{\mathcal{E}}_{a^*\xi} \to h^{\mathcal{E}}_{a^*\xi},\ pr_{h^{\mathcal{E}}_{\xi}} : h^{\mathcal{B}}_{\mathbf{M}} \cdot p \times h^{\mathcal{E}}_{\xi} \to h^{\mathcal{E}}_{\xi}$$

are representable by morphisms $\widetilde{\pi_{23}}_{a^*\xi} : \pi^*_{23}a^*\xi \to a^*\xi$, $\widetilde{\pi}_{\xi} : \pi^*\xi \to \xi$ in $\mathcal{E}$, respectively. Thus $1_{h^{\mathcal{B}}_{\mathbf{M}} \cdot p} \times h^{\mathcal{E}}_{\widetilde{a}_{\xi}}$ is representable by a cartesian morphism

$$\pi_{23}^{*}a^{*}\xi \xrightarrow{\cong} (1_{\mathbf{M}}\times a)^{*}\,\pi^{*}\xi \xrightarrow{\widetilde{(1_{\mathbf{M}}\times a)}_{\pi^{*}\xi}} \pi^{*}\xi$$

where $\cong$ is the identification described in Remark 6.3. Since we have equality

$$\sigma\cdot\left(1_{h^{\mathcal{B}}_{\mathbf{M}}\cdot p}\times\sigma\right)=h^{\mathcal{E}}_{\widetilde{a}_{\xi}}\cdot h^{\mathcal{E}}_{\tau^{\sigma}}\cdot\left(1_{h^{\mathcal{B}}_{\mathbf{M}}\cdot p}\times h^{\mathcal{E}}_{\widetilde{a}_{\xi}}\right)\cdot\left(1_{h^{\mathcal{B}}_{\mathbf{M}}\cdot p}\times h^{\mathcal{E}}_{\tau^{\sigma}}\right)$$

we derive that $\sigma\cdot\left(1_{h^{\mathcal{B}}_{\mathbf{M}}\cdot p}\times\sigma\right)$ is representable (again up to identifications of Remark 6.3) by a morphism

$$\widetilde{a}_{\xi}\cdot\tau^{\sigma}\cdot\widetilde{(1_{\mathbf{M}}\times a)}_{\pi^{*}\xi}\cdot\pi_{23}^{*}\tau^{\sigma}=\widetilde{a}_{\xi}\cdot\widetilde{(1_{\mathbf{M}}\times a)}_{a^{*}\xi}\cdot(1_{\mathbf{M}}\times a)^{*}\,\tau^{\sigma}\cdot\pi_{23}^{*}\tau^{\sigma}$$

in $\mathcal{E}$. Next note that the square of presheaves on $\mathcal{E}$

$$\begin{tikzcd}
h^{\mathcal{B}}_{\mathbf{M}}\cdot p\times h^{\mathcal{B}}_{\mathbf{M}}\cdot p\times h^{\mathcal{E}}_{\xi} \arrow[r, "h^{\mathcal{B}}_{\mu}\cdot p\times 1_{h^{\mathcal{E}}_{\xi}}"] \arrow[d, "\mathbf{can}\times p_{\mathrm{hom}}"'] & h^{\mathcal{B}}_{\mathbf{M}\cdot p}\times h^{\mathcal{E}}_{\xi} \arrow[d, "p_{\mathrm{hom}}"] \\
h^{\mathcal{B}}_{\mathbf{M}\times\mathbf{M}\times X}\cdot p \arrow[r, "\left(h^{\mathcal{B}}_{\mu\times 1_X}\right)_p"'] & h^{\mathcal{B}}_{\mathbf{M}\times X\cdot p}
\end{tikzcd}$$

is cartesian. According to Fact 6.4 we infer that projections

$$pr_{h^{\mathcal{B}}_{\mathbf{M}}\cdot p\times h^{\mathcal{E}}_{\xi}}:h^{\mathcal{B}}_{\mathbf{M}}\cdot p\times h^{\mathcal{B}}_{\mathbf{M}}\cdot p\times h^{\mathcal{E}}_{\xi}\to h^{\mathcal{B}}_{\mathbf{M}}\cdot p\times h^{\mathcal{E}}_{\xi},\ pr_{h^{\mathcal{E}}_{\xi}}:h^{\mathcal{B}}_{\mathbf{M}}\cdot p\times h^{\mathcal{E}}_{\xi}\to h^{\mathcal{E}}_{\xi}$$

are representable by morphisms $\widetilde{\pi_{23}}_{\pi^{*}\xi}:\pi_{23}^{*}\pi^{*}\xi\to\pi^{*}\xi$, $\widetilde{\pi}_{\xi}:\pi^{*}\xi\to\xi$ in $\mathcal{E}$, respectively. Thus $h^{\mathcal{B}}_{\mu}\cdot p\times 1_{h^{\mathcal{E}}_{\xi}}$ is representable by a cartesian morphism

$$\pi_{23}^{*}\pi^{*}\xi \xrightarrow{\cong} (\mu\times 1_{X})^{*}\pi^{*}\xi \xrightarrow{\widetilde{(\mu\times 1_{X})}_{\pi^{*}\xi}} \pi^{*}\xi$$

where $\cong$ is the identification described in Remark 6.3. Since we have equality

$$\sigma\cdot\left(1_{h^{\mathcal{B}}_{\mu}\cdot p}\times 1_{h^{\mathcal{E}}_{\xi}}\right)=h^{\mathcal{E}}_{\widetilde{a}_{\xi}}\cdot h^{\mathcal{E}}_{\tau^{\sigma}}\cdot\left(1_{h^{\mathcal{B}}_{\mu}\cdot p}\times 1_{h^{\mathcal{E}}_{\xi}}\right)$$

we derive that $\sigma\cdot\left(1_{h^{\mathcal{B}}_{\mu}\cdot p}\times 1_{h^{\mathcal{E}}_{\xi}}\right)$ is representable (again up to identifications of Remark 6.3) by a morphism

$$\widetilde{a}_{\xi}\cdot\tau^{\sigma}\cdot\widetilde{(\mu\times 1_{X})}_{\pi^{*}\xi}=\widetilde{a}_{\xi}\cdot\widetilde{(\mu\times 1_{X})}_{a^{*}\xi}\cdot(\mu\times 1_{X})^{*}\,\tau^{\sigma}$$

We deduce that

$$\sigma\cdot\left(1_{h^{\mathcal{B}}_{\mathbf{M}}\cdot p}\times\sigma\right)=\sigma\cdot\left(1_{h^{\mathcal{B}}_{\mu}\cdot p}\times 1_{h^{\mathcal{E}}_{\xi}}\right)$$

if and only if

$$\widetilde{a}_{\xi}\cdot\widetilde{(1_{\mathbf{M}}\times a)}_{a^{*}\xi}\cdot(1_{\mathbf{M}}\times a)^{*}\,\tau^{\sigma}\cdot\pi_{23}^{*}\tau^{\sigma}=\widetilde{a}_{\xi}\cdot\widetilde{(\mu\times 1_{X})}_{a^{*}\xi}\cdot(\mu\times 1_{X})^{*}\,\tau^{\sigma}$$

Since $a\cdot(1_{\mathbf{M}}\times a)=a\cdot(\mu\times 1_{X})$ and according to Remark 6.3, we have canonical identification $\widetilde{a}_{\xi}\cdot\widetilde{(1_{\mathbf{M}}\times a)}_{a^{*}\xi}=\widetilde{a}_{\xi}\cdot\widetilde{(\mu\times 1_{X})}_{a^{*}\xi}$ of these cartesian morphisms. Therefore, we deduce that the formula above holds if and only if

$$(1_{\mathbf{M}}\times a)^{*}\,\tau^{\sigma}\cdot\pi_{23}^{*}\tau^{\sigma}=(\mu\times 1_{X})^{*}\,\tau^{\sigma}$$

This proves our first claim. Now it suffices to prove that

$$\sigma\cdot\langle h^{\mathcal{B}}_{e}\cdot p,1_{h^{\mathcal{E}}_{\xi}}\rangle=1_{h^{\mathcal{E}}_{\xi}}$$

if and only if $\langle e, 1_X\rangle^*\tau^\sigma = 1_\xi$. Note that the square of presheaves on $\mathcal{E}$

$$\begin{tikzcd} h^{\mathcal{E}}_{\xi} \arrow[r, "\langle h^{\mathcal{B}}_{e}\cdot p, 1_{h^{\mathcal{E}}_{\xi}}\rangle"] \arrow[d, "p_{\mathrm{hom}}"'] & h^{\mathcal{B}}_{\mathbf{M}\cdot p}\times h^{\mathcal{E}}_{\xi} \arrow[d, "p_{\mathrm{hom}}"] \\ h^{\mathcal{B}}_{X}\cdot p \arrow[r, "\left(h^{\mathcal{B}}_{\langle e,1_X\rangle}\right)_p"'] & h^{\mathcal{B}}_{\mathbf{M}\times X\cdot p} \end{tikzcd}$$

is cartesian. Thus according to Fact 6.4 we derive that $\langle h^{\mathcal{B}}_{e}\cdot p, 1_{h^{\mathcal{E}}_{\xi}}\rangle$ is representable by morphism

$$\xi \xrightarrow{\cong} \langle e, 1_X\rangle^*\pi^*\xi \xrightarrow{\widetilde{\langle e,1_X\rangle}_{\pi^*\xi}} \pi^*\xi$$

where $\cong$ is the identification described in Remark 6.3. Therefore, the morphism $\sigma\cdot\langle h^{\mathcal{B}}_{e}\cdot p, 1_{h^{\mathcal{E}}_{\xi}}\rangle$ is representable (up to identifications of Remark 6.3) by

$$\widetilde{a}_\xi\cdot\tau^\sigma\cdot\widetilde{\langle e,1_X\rangle}_{\pi^*\xi} = \widetilde{a}_\xi\cdot\widetilde{\langle e,1_X\rangle}_{a^*\xi}\cdot\langle e,1_X\rangle^*\tau^\sigma = \langle e,1_X\rangle^*\tau^\sigma$$

Thus

$$\sigma\cdot\langle h^{\mathcal{B}}_{e}\cdot p, 1_{h^{\mathcal{E}}_{\xi}}\rangle = 1_{h^{\mathcal{E}}_{\xi}}$$

if and only if

$$\langle e, 1_X\rangle^*\tau^\sigma = 1_\xi$$

This finishes the proof. □

Fact 6.6. *Let $\mathbf{M}, X$ be objects of $\mathcal{B}$ such that the cartesian product of $\mathbf{M}$ and X exist. Let $a : \mathbf{M}\times X \to X$ be a morphism. Denote by $\pi : \mathbf{M}\times X \to X$ the projection on X. Consider objects ξ_1, ξ_2 in $p^{-1}(X)$ and let $\sigma_1 : h^{\mathcal{B}}_{\mathbf{M}}\cdot p\times h^{\mathcal{E}}_{\xi_1} \to h^{\mathcal{E}}_{\xi_1}, \sigma_2 : h^{\mathcal{B}}_{\mathbf{M}}\cdot p\times h^{\mathcal{E}}_{\xi_2} \to h^{\mathcal{E}}_{\xi_2}$ be morphisms of presheaves on $\mathcal{E}$. Suppose that squares*

$$\begin{tikzcd} h^{\mathcal{B}}_{\mathbf{M}}\cdot p\times h^{\mathcal{E}}_{\xi_1} \arrow[r, "\sigma_1"] \arrow[d, "p_{\mathrm{hom}}"'] & h^{\mathcal{E}}_{\xi_1} \arrow[d, "p_{\mathrm{hom}}"] \\ h^{\mathcal{B}}_{\mathbf{M}\times X}\cdot p \arrow[r, "\left(h^{\mathcal{B}}_{a}\right)_p"'] & h^{\mathcal{B}}_{X}\cdot p \end{tikzcd} \qquad \begin{tikzcd} h^{\mathcal{B}}_{\mathbf{M}}\cdot p\times h^{\mathcal{E}}_{\xi_2} \arrow[r, "\sigma_2"] \arrow[d, "p_{\mathrm{hom}}"'] & h^{\mathcal{E}}_{\xi_2} \arrow[d, "p_{\mathrm{hom}}"] \\ h^{\mathcal{B}}_{\mathbf{M}\times X}\cdot p \arrow[r, "\left(h^{\mathcal{B}}_{a}\right)_p"'] & h^{\mathcal{B}}_{X}\cdot p \end{tikzcd}$$

are commutative. Let $\phi : \xi_1 \to \xi_2$ be a morphism in $\mathcal{E}$. Then the following assertions are equivalent.

(i) *The square*

$$\begin{tikzcd} h^{\mathcal{B}}_{\mathbf{M}}\cdot p\times h^{\mathcal{E}}_{\xi_1} \arrow[r, "\sigma_1"] \arrow[d, "1_{h^{\mathcal{B}}_{\mathbf{M}}}\times h^{\mathcal{E}}_{\phi}"'] & h^{\mathcal{E}}_{\xi_1} \arrow[d, "h^{\mathcal{E}}_{\xi}"] \\ h^{\mathcal{B}}_{\mathbf{M}}\cdot p\times h^{\mathcal{E}}_{\xi_2} \arrow[r, "\sigma_2"'] & h^{\mathcal{E}}_{\xi_2} \end{tikzcd}$$

is commutative.

(ii) *The square*

$$\begin{tikzcd} \pi^*\xi_1 \arrow[r, "\tau^{\sigma_1}"] \arrow[d, "\pi^*\phi"'] & a^*\xi_1 \arrow[d, "a^*\phi"] \\ \pi^*\xi_2 \arrow[r, "\tau^{\sigma_2}"'] & a^*\xi_2 \end{tikzcd}$$

is commutative.

Proof. Note that up to identifications of Remark 6.3 and according to Fact 6.4 morphism $h^{\mathcal{E}}_{\phi} \cdot \sigma_1$ is representable by

$$\phi \cdot \alpha^{\sigma_1} = \phi \cdot \widetilde{a}_{\xi_1} \cdot \tau^{\sigma_1} = \widetilde{a}_{\xi_2} \cdot a^*\phi \cdot \tau^{\sigma_1}$$

and on the other hand morphism $\sigma_2 \cdot \left(1_{h^{\mathcal{B}}_{\mathbf{M}} \cdot p} \times h^{\mathcal{E}}_{\phi}\right)$ is representable by

$$\alpha^{\sigma_2} \cdot \pi^*\phi = \widetilde{a}_{\xi_2} \cdot \tau^{\sigma_2} \cdot \pi^*\phi$$

Since $\widetilde{a}_{\xi_2}$ is cartesian with respect to p, we derive that

$$h^{\mathcal{E}}_{\phi} \cdot \sigma_1 = \sigma_2 \cdot \left(1_{h^{\mathcal{B}}_{\mathbf{M}} \cdot p} \times h^{\mathcal{E}}_{\phi}\right)$$

if and only if

$$a^*\phi \cdot \tau^{\sigma_1} = \tau^{\sigma_2} \cdot \pi^*\phi$$

This proves the assertion. □

Guided by these two results we formulate a general notion of equivariant object in a fibered category.

Definition 6.7. Let $M : \mathcal{B}^{\mathrm{op}} \to \mathbf{Mon}$ be a presheaf of monoids on $\mathcal{B}$ and assume that for some object X of $\mathcal{B}$ the presheaf $h^{\mathcal{B}}_X$ admits an action of M given by the morphism $\alpha : M \times h^{\mathcal{B}}_X \to h^{\mathcal{B}}_X$. Consider an object ξ in $p^{-1}(X)$. Suppose that there is an action $\sigma : M \cdot p \times h^{\mathcal{E}}_{\xi} \to h^{\mathcal{E}}_{\xi}$ of a monoid presheaf $M \cdot p$ on $h^{\mathcal{E}}_{\xi}$ such that the square

$$\begin{tikzcd} M \cdot p \times h^{\mathcal{E}}_{\xi} \arrow[r, "\sigma"] \arrow[d, "1_{M \cdot p} \times p_{\mathrm{hom}}"'] & h^{\mathcal{E}}_{\xi} \arrow[d, "p_{\mathrm{hom}}"] \\ M \cdot p \times h^{\mathcal{B}}_X \cdot p \arrow[r, "\alpha_p"'] & h^{\mathcal{B}}_X \cdot p \end{tikzcd}$$

is commutative. Then a pair (ξ, σ) is called *an M-equivariant object over α*.

Definition 6.8. Let $M : \mathcal{B}^{\mathrm{op}} \to \mathbf{Mon}$ be a presheaf of monoids on $\mathcal{B}$ and assume that for some object X of $\mathcal{B}$ the presheaf $h^{\mathcal{B}}_X$ admits an action of M given by the morphism $\alpha : M \times h^{\mathcal{B}}_X \to h^{\mathcal{B}}_X$. Suppose that (ξ_1, σ_1) and (ξ_2, σ_2) are objects over X with M-equivariant structures. Then a morphism $\phi : \xi_1 \to \xi_2$ in $\mathcal{E}$ is *M-equivariant* if the square

$$\begin{tikzcd} M \cdot p \times h^{\mathcal{E}}_{\xi_1} \arrow[r, "\sigma_1"] \arrow[d, "1_{M \cdot p} \times h^{\mathcal{E}}_{\phi}"'] & h^{\mathcal{E}}_{\xi_1} \arrow[d, "\phi"] \\ M \cdot p \times h^{\mathcal{E}}_{\xi_2} \arrow[r, "\sigma_2"'] & h^{\mathcal{E}}_{\xi_2} \end{tikzcd}$$

is commutative.

We denote the category of M-equivariant objects over α with respect to the fibered category $p : \mathcal{E} \to \mathcal{B}$ by $p^{-1}(X)_M$.

Now we can apply Proposition 6.5 and Fact 6.6 to the fibered category $\mathfrak{Qcoh} \to \mathbf{Sch}_k$.

Corollary 6.9. *Suppose that* $\mathbf{M}$ *is a monoid k-scheme that acts on a k-scheme X through morphism* $a : \mathbf{M} \times_k X \to X$ *of k-schemes. Then the category* $\mathfrak{Qcoh}(X)_{\mathbf{M}}$ *is isomorphic to the category of* $h_{\mathbf{M}}^{\mathbf{Sch}_k}$*-objects over* $h_a^{\mathbf{Sch}_k}$ *with respect to the fibered category* $\mathfrak{Qcoh} \to \mathbf{Sch}_k$.

Moreover, we have the following general result.

Corollary 6.10. *Let* $\mathcal{B}$ *be a category with all finite limits. Suppose that* $\mathbf{M}$ *is a monoid object in* $\mathcal{B}$ *that acts on an object X of* $\mathcal{B}$ *via* $a : \mathbf{M} \times X \to X$*. Then the category of* $h_{\mathbf{M}}^{\mathcal{B}}$*-objects over* $h_a^{\mathcal{B}}$ *with respect to the fibered category* $p_{\mathrm{Arr}} : \mathrm{Arr}(\mathcal{B}) \to \mathcal{B}$ *is isomorphic to the category of* $\mathbf{M}$*-equivariant morphisms* $\pi : \widetilde{X} \to X$ *as objects and with*

7. Equivariant sheaves of quasi-coherent algebras

In this section we fix a commutative ring k. Let $\mathbf{M}$ be a monoids scheme and let X be a k-scheme together with an action $a : \mathbf{M} \times_k X \to X$ of $\mathbf{M}$.