HAAR MEASURE

1. Introduction

In this notes we introduce Haar measure. Haar measure is a fundamental technical tool in representation theory of locally compact topological groups. There are many excellent sources concerning this topic. We send the interested reader to

2. Existence of Haar measure

Definition 2.1. Let G be a topological group and let μ be a Borel measure. Then μ is *left-invariant* if $\mu(xA) = \mu(A)$ for every A in $\mathcal{B}(G)$. Similarly μ is right-invariant if $\mu(Ax) = \mu(A)$ for every A in $\mathcal{B}(G)$.

Definition 2.2. Let G be a locally compact group and μ be a Borel measure. If μ is a nonzero, left-invariant, regular Borel measure on G, then we say that μ is *a left Haar measure on* G. Similarly if μ is a nonzero, right-invariant, regular Borel measure on G, then we say that μ is *a right Haar measure on* G

Theorem 2.3. *Let G be a locally compact topological group. Then there exists a left (right) Haar measure μ on G. If in addition G is σ-compact, then μ is inner regular.*

We denote by $\mathcal{K}$ the set of all compact subsets of G and by $\mathcal{U}$ the set of all open neighborhoods of identity in G. Let U be an open nonempty subset of G and K be a compact subset of G. We define

$$(K:U) = \inf\left\{n \in \mathbb{N} \,\middle|\, \text{there exist } x_1, ..., x_n \in G \text{ such that } K \subseteq \bigcup_{i=1}^{n} x_i U\right\}$$

Throughout the proof we fix a compact subset Q of G such that $\mathbf{int}(Q) \neq \varnothing$.

Lemma 2.3.1. *Fix $U \in \mathcal{U}$. There exists a real valued function h_U on $\mathcal{K}$ such that the following assertions hold.*

(1) *For every compact subset K in $\mathcal{K}$ we have $h_U(K) \geq 0$, $h_U(\varnothing) = 0$ and $h_U(Q) = 1$.*

(2) *For every compact subset K in $\mathcal{K}$ and for every element x in G we have $h_U(xK) = h_U(K)$.*

(3) *If $K \subseteq L$ are compact subsets in $\mathcal{K}$, then $h_U(K) \subseteq h_U(L)$.*

(4) *For every compact subset K in $\mathcal{K}$ we have $h_U(K) \leq (K : \mathbf{int}(Q))$.*

(5) *If K, L are compact subsets in $\mathcal{K}$, then*

$$h_U(K \cup L) \leq h_U(K) + h_U(L)$$

and if $K \cdot U^{-1} \cap L \cdot U^{-1} = \varnothing$, then the equality holds.

Proof of the lemma. For every compact subset K of G we define

$$h_U(K) = \frac{(K:U)}{(Q:U)}$$

Now we check that h_U admits the properties above. Properties **(1)**, **(2)** and **(3)** are clear. For **(4)** note that

$$(K:U) \leq (Q:U) \cdot (K : \mathbf{int}(Q))$$

Indeed, if $K \subseteq \bigcup_{i=1}^{n} y_i \cdot \mathbf{int}(Q)$ and $Q \subseteq \bigcup_{j=1}^{m} z_j U$, then $K \subseteq \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} y_i z_j U$ and this implies the inequality above. Observe that $xU \cap K \neq \varnothing$ implies that $x \in K \cdot U^{-1}$ and similarly $xU \cap L \neq \varnothing$

implies that $x \in L \cdot U^{-1}$. Assuming that for compact subsets K, L in G we have $K \cdot U^{-1} \cap L \cdot U^{-1} = \emptyset$ we derive from this that for every $x \in G$ we have $xU \cap (K \cap L) = \emptyset$. Thus if $K \cdot U^{-1} \cap L \cdot U^{-1} = \emptyset$, then we have $(K \cup L : U) = (K : U) + (K : L)$ and hence $h_U(K \cup L) = h_U(K) + h_U(L)$. Note that in general case we have $(K \cup L : U) \leq (K : U) + (K : L)$ and hence also **(5)** holds for h_U. □

Lemma 2.3.2. *Let K, L in $\mathcal{K}$ and suppose that $K \cap L = \emptyset$. Then there exists $U \in \mathcal{U}$ such that*

$$K \cdot U^{-1} \cap L \cdot U^{-1} = \emptyset$$

Proof of the lemma. Left as an exercise. □

Lemma 2.3.3. *There exists a real valued function h on $\mathcal{K}$ such that the following assertions hold.*

(1) *For every compact subset K in $\mathcal{K}$ we have $h(K) \geq 0$, $h(\emptyset) = 0$ and $h(Q) = 1$.*

(2) *For every compact subset K in $\mathcal{K}$ and for every element x in G we have $h(xK) = h(K)$.*

(3) *If $K \subseteq L$ are compact subsets in $\mathcal{K}$, then $h(K) \subseteq h(L)$.*

(4) *For every compact subset K in $\mathcal{K}$ we have $h(K) \leq (K : \mathbf{int}(Q))$.*

(5) *If K, L are compact subsets in $\mathcal{K}$, then*

$$h(K \cup L) \leq h(K) + h(L)$$

and if $K \cap L = \emptyset$, then the equality holds.

Proof of the lemma. Consider a topological space

$$X = \prod_{K \in \mathcal{K}} \left[0, (K : \mathbf{int}(Q))\right]$$

By Tichonoff's theorem X is compact. For every $U \in \mathcal{U}$ we define a subset $F_U \subseteq X$ that consists of tuples $\{a_K\}_{K \in \mathcal{K}}$ such that $a_{\emptyset} = 0$, $a_Q = 1$, $a_{xK} = a_K$ for $x \in G$ and K in $\mathcal{K}$, $a_K \leq a_L$ for $K \subseteq L$ in $\mathcal{K}$, $a_{K \cup L} \leq a_K + a_L$ for K, L in $\mathcal{K}$ and the equality holds if $K \cdot U^{-1} \cap L \cdot U^{-1} = \emptyset$. Conditions imposed on tuples in F_U imply that F_U is a closed subset. Note that $\{h_U(K)\}_{K \in \mathcal{K}} \in F_U$ for every $U \in \mathcal{U}$. Moreover, we have

$$F_{U_1 \cap U_2 \cap \ldots \cap U_n} \subseteq F_{U_1} \cap F_{U_2} \cap \ldots \cap F_{U_n}$$

for $U_1, U_2, \ldots, U_n \in \mathcal{U}$. This implies that $\left\{F_U\right\}_{U \in \mathcal{U}}$ is a centered family of nonempty closed subsets of a compact space X. Thus

$$\bigcap_{U \in \mathcal{U}} F_U \neq \emptyset$$

by compactness of X. Hence there exists $\{c_K\}_{K \in \mathcal{K}}$ in the intersection. We define a real function h on $\mathcal{K}$ by $h(K) = c_K$ for K in $\mathcal{K}$. The fact that properties **(1)**, **(2)**, **(3)** and **(4)** hold for h follows by definition of F_U for $U \in \mathcal{U}$. Since $\{c_K\}_{K \in \mathcal{K}}$ is an element in F_U for every $U \in \mathcal{U}$ we derive that

$$c_{K \cup L} \leq c_K + c_L$$

for K, L in $\mathcal{K}$. This implies $h(K \cup L) \leq h(K) + h(L)$ for $K, L \in \mathcal{K}$. Moreover, $c_{K \cup L} = c_K + c_L$ if $K \cdot U^{-1} \cap L \cdot U^{-1} = \emptyset$ for some $U \in \mathcal{U}$. This implies that $c_{K \cup L} = c_K + c_L$ if $K \cap L = \emptyset$ by Lemma 2.3.2. Thus h admits **(4)**. □

Proof of the theorem. We fix h as in Lemma 2.3.3 and we define $\mu^* : \mathcal{P}(G) \to [0, +\infty]$. First if U is an open subset of G, then we define

$$\mu^*(U) = \sup_{K \in \mathcal{K}, K \subseteq U} h(K)$$

Note that if U, V are open subsets of G and $U \subseteq V$, then $\mu^*(U) \leq \mu^*(V)$. Thus it makes sense to define

$$\mu^*(A) = \inf\left\{\mu^*(U) \,\middle|\, U \text{ is an open subset of } G \text{ containing } A\right\}$$

for arbitrary subset $A \subseteq G$. Note that $\mu^*(xA) = \mu^*(A)$ by definition of μ^* and the corresponding property of h. By [Mon19a, Theorem 1.3] we have that Borel sets $\mathcal{B}(G)$ are μ^*-measurable,

$\mu^*_{|\mathcal{B}(G)} = \mu$ is a regular Borel measure on G. According to this result if G is σ-compact, then μ is inner regular. Clearly μ is left-invariant and since

$$1 = h(Q) \le \mu(Q)$$

we derive that it is nonzero measure. □

3. Uniqueness of Haar measure

Theorem 3.1. *Let G be a locally compact group. If μ_1 and μ_2 are left (right) Haar measures on G, then there exists positive constant $a \in \mathbb{R}$ such that*

$$\mu_1 = a \cdot \mu_2$$

For the proof we need the following result.

Lemma 3.1.1. *Let G be a locally compact group. Then there exists a σ-compact, open subgroup H of G.*

Proof of the lemma. Let U be an open neighborhood of identity in G such that $\mathbf{cl}(U)$ is compact. Consider $V = U \cap U^{-1}$. Then V is open neighborhood of identity in G such that $V = V^{-1}$ and $\mathbf{cl}(V)$ is compact. We define $H = \bigcup_{n\in\mathbb{N}} V^n$. Then H is an open subgroup of G. We have

$$H = G \setminus \left(\bigcup_{g\in G\setminus H} gH\right)$$

and hence H is also a closed subgroup of G. Moreover, for every $n \in \mathbb{N}$ set $\mathbf{cl}\left(V^n\right)$ is compact in G. Since

$$H = \bigcup_{n\in\mathbb{N}} \left(H \cap \mathbf{cl}\left(V^n\right)\right)$$

we derive that H is σ-compact. □

Proof of the theorem. By Lemma 3.1.1 there exists an open subgroup H of G that is σ-compact. We prove now that there exists $a \in \mathbb{R}$ such that

$$\mu_{1|\mathcal{B}(H)} = a \cdot \mu_{2|\mathcal{B}(H)}$$

For this consider $\mu = \mu_{1|\mathcal{B}(H)} + \mu_{2|\mathcal{B}(H)}$ and denote $\mu_{2|\mathcal{B}(H)}$ by ν. Measures μ, ν are σ-finite as they are finite on compact subsets of H and H is σ-compact space. Moreover, $\nu \ll \mu$ and hence by [Mon19b, Theorem 5.3] there exists a Borel function $f : H \to \mathbb{C}$ such that

$$\nu(A) = \int_A f d\mu$$

for every Borel subset A in H. Since μ and ν are nonnegative measures, we derive that f is real and nonnegative μ-almost everywhere. Hence we may assume that f takes only nonnegative real values. Next as ν, μ are left-invariant, we deduce that

$$0 = \int_{xA} f d\mu - \int_A f d\mu = \int_A (f \cdot l_x - f)\, d\mu$$

for every $x \in H$, where $l_x : H \to H$ is a continuous map given by left multiplication by x. This implies that for given $x \in H$ the set

$$A_x = \left\{y \in H \,\middle|\, f(xy) - f(y) = 0\right\}$$

has measure μ zero. By Fubini's theorem applied to measure $\mu \otimes \mu$ on $H \times H$, we deduce that there exists $y \in H$ such that the set

$$B = \left\{x \in H \,\middle|\, f(xy) - f(y) = 0\right\}$$

has measure μ zero. This implies that f is constant almost everywhere with respect to μ and thus there exists nonzero $b \in \mathbb{R}$ such that $\nu = b \cdot \mu$. Hence we have

$$\mu_{1|\mathcal{B}(H)} = a \cdot \mu_{2|\mathcal{B}(H)}$$

for $a = (1-b)b^{-1}$. Let K be a compact subset of G. Since H is an open subgroup of G, there exists $x_1, ..., x_n \in G$ such that

$$K \subseteq x_1 H \cup ... \cup x_n H$$

and the sum is disjoint. Therefore, we have

$$\mu_1(K) = \sum_{i=1}^{n} \mu_1\left(K \cap x_i H\right) = \sum_{i=1}^{n} \mu_1\left(x_i^{-1} K \cap H\right) = a \cdot \sum_{i=1}^{n} \mu_2\left(x_i^{-1} K \cap H\right) = a \cdot \sum_{i=1}^{n} \mu_2\left(K \cap x_i H\right) = a \cdot \mu_2(K)$$

This implies that $\mu_1 = a \cdot \mu_2$ because μ_1, μ_2 are regular Borel measures. □

4. Modular function and invariance of Haar measure on compact groups

References

[Mon19a] Monygham. Borel measures on locally compact spaces. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2019.

[Mon19b] Monygham. Radon-nikodym theorem, hahn-jordan decomposition and lebesgue decomposition. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2019.