MINIMAX THEOREM AND LINEAR PROGRAMMING

1. SION'S MINIMAX THEOREM

Definition 1.1. Let X be a topological space and let $f : X \to \mathbb{R}$ be a function. Then f is *lower-semicontinuous* if for every $r \in \mathbb{R}$ the set

$$\{x \in X \,|\, f(x) \leq r\}$$

is closed. We say that f is *upper-semicontinuous* if $-f$ is lower-semicontinuous.

Definition 1.2. Let X be a convex subset of a linear space over $\mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function. Then f is *quasiconvex* if for every $x_1, x_2 \in X$ and $t \in [0,1]$ we have

$$f\left(tx_1 + (1-t)x_2\right) \leq \max\left\{f(x_1), f(x_2)\right\}$$

We say that f is *quasiconcave* if $-f$ is quasiconvex.

Proposition 1.3. *Let X be a convex subset of a linear space over $\mathbb{R}$. Suppose that $f : X \to \mathbb{R}$ is a function. Then the following are equivalent.*

(1) *f is quasiconvex.*

(2) *For every $r \in \mathbb{R}$ the set $\{x \in X \,|\, f(x) \leq r\}$ is convex.*

Proof. We prove **(1)** $\Rightarrow$ **(2)**. Pick $r \in \mathbb{R}$, $x_1, x_2 \in X$ and assume that $f(x_1), f(x_2)$ are both less or equal to r. Then

$$f\left(tx_1 + (1-t)x_2\right) \leq \max\left\{f(x_1), f(x_2)\right\} \leq r$$

for every $t \in [0,1]$ by **(1)**. Thus the set $\{x \in X \,|\, f(x) \leq r\}$ contains line segment joining x_1 with x_2 and hence it is convex. This is **(2)**.
We prove **(2)** $\Rightarrow$ **(1)**. Pick $x_1, x_2 \in X$ and $t \in [0,1]$. Let $r = \max\left\{f(x_1), f(x_2)\right\}$. Then by **(2)** we deduce that the set $\{x \in X \,|\, f(x) \leq r\}$ is convex. Hence $f\left(tx_1 + (1-t)x_2\right) \leq r$. This shows **(1)**. $\square$

Theorem 1.4 (Sion's theorem)**.** *Let X be a convex, compact subset of a topological vector space over $\mathbb{R}$ and let Y be a convex subset of a topological vector space over $\mathbb{R}$. Suppose that $f : X \times Y \to \mathbb{R}$ is a function such that the following assertions hold.*

(1) *f_x is lower-semicontinuous and quasiconvex for every x in X.*

(2) *f_y is upper-semicontinuous and quasiconcave for every y in Y.*

Then we have

$$\sup_{x \in X} \inf_{y \in Y} f(x,y) = \inf_{y \in Y} \sup_{x \in X} f(x,y)$$

The proof relies on series of lemmas.

Lemma 1.4.1. *Let X be a compact topological space and let $f_1, f_2 : X \to \mathbb{R}$ be upper-semicontinuous functions. Suppose that*

$$\left\{x \in X \,\middle|\, f_1(x) \geq r\right\} \cap \left\{x \in X \,\middle|\, f_2(x) \geq r\right\} = \emptyset$$

for some $r \in \mathbb{R}$. Then there exists $s < r$ such that

$$\left\{x \in X \,\middle|\, f_1(x) \geq s\right\} \cap \left\{x \in X \,\middle|\, f_2(x) \geq s\right\} = \emptyset$$

Proof of the lemma. Pick an increasing sequence $\{s_n\}_{n\in\mathbb{N}}$ of real numbers convergent to r. Define

$$F_n = \{x \in X \,|\, f_1(x) \geq s_n\} \cap \{x \in X \,|\, f_2(x) \geq s_n\}$$

for every $n \in \mathbb{N}$. Since f_1 and f_2 are upper-semicontinuous, we derive that the family of sets $\{F_n\}_{n\in\mathbb{N}}$ consists of closed sets. Moreover, this family is nonincreasing. If $F_n \neq \varnothing$ for all $n \in \mathbb{N}$, then $\{F_n\}_{n\in\mathbb{N}}$ admits finite intersection property. Hence by compactness of X there exists x in X such that

$$x \in \bigcap_{n\in\mathbb{N}} F_n = \{x \in X \,|\, \forall_{n\in\mathbb{N}}\, f_1(x) \geq s_n \text{ and } \forall_{n\in\mathbb{N}}\, f_2(x) \geq s_n\} =$$
$$= \{x \in X \,|\, f_1(x) \geq r\} \cap \{x \in X \,|\, f_2(x) \geq r\}$$

This is contradiction. Thus $F_{n_0} = \varnothing$ for some $n_0 \in \mathbb{N}$ and hence for $s = s_{n_0}$ we have

$$\{x \in X \,|\, f_1(x) \geq s\} \cap \{x \in X \,|\, f_2(x) \geq s\} = \varnothing$$

Thus the assertion is proved. □

Lemma 1.4.2. *Let X be a convex, compact subset of a topological vector space over $\mathbb{R}$ and let Y be a convex subset of a topological vector space over $\mathbb{R}$. Suppose that $f : X \times Y \to \mathbb{R}$ is a function such that the following assertions hold.*

(1) *f_x is lower-semicontinuous and quasiconvex for every x in X.*

(2) *f_y is upper-semicontinuous and quasiconcave for every y in Y.*

For every $y \in Y$ and $r \in \mathbb{R}$ we denote

$$L_{r,y} = \{x \in X \,|\, f_y(x) \geq r\}$$

If every set in the family $\{L_{r,y}\}_{y\in Y}$ is nonempty, then any two sets in this family have nonempty intersection.

Proof of the lemma. Suppose conversely that $L_{r,y_1} \cap L_{r,y_2} = \varnothing$ for some $y_1, y_2 \in Y$. Then by Lemma 1.4.1 (f_y are upper-semicontinuous for all y so it can be applied) we derive that there exists $s < r$ such that $L_{s,y_1} \cap L_{s,y_2} = \varnothing$. Note that $L_{r,y} \subseteq L_{s,y}$ for all $y \in Y$ and hence $\{L_{s,y}\}_{y\in Y}$ consists of nonempty sets. Let $[y_1, y_2]$ be an interval joining y_1 and y_2 in Y. Pick now $y \in [y_1, y_2]$. If $x \in L_{s,y}$, then

$$s \geq f_y(x) = f_x(y) \leq \max\{f_x(y_1), f_x(y_2)\} = \max\{f_{y_1}(x), f_{y_2}(x)\}$$

by the fact that f_x is quasiconvex. This implies that $x \in L_{s,y_1} \cup L_{s,y_2}$. Thus $L_{s,y} \subseteq L_{s,y_1} \cup L_{s,y_2}$. Since f_y is quasiconcave, we derive that $L_{s,y}$ is convex and hence connected. Sets L_{s,y_1}, L_{s,y_2} are closed (f_y is upper-semicontinous for each y) and disjoint. Therefore, we have either $L_{s,y} \subseteq L_{s,y_1}$ or $L_{s,y} \subseteq L_{s,y_2}$ for every $y \in [y_1, y_2]$. Now we define

$$F_i = \{y \in [y_1, y_2] \,|\, L_{s,y} \subseteq L_{s,y_i}\}$$

for $i = 1, 2$. Then we have $[y_1, y_2] = F_1 \cup F_2$ and $F_1 \cap F_2 = \varnothing$. Next since $L_{r,y} \subseteq L_{s,y}$ for every $y \in Y$, we deduce that

$$F_i = \{y \in [y_1, y_2] \,|\, L_{r,y} \subseteq L_{s,y_i}\}$$

for $i = 1, 2$. Fix now $i = 1, 2$. Consider a sequence $\{z_n\}_{n\in\mathbb{N}}$ of elements in F_i convergent to some $z \in [y_1, y_2]$. Pick $x \in L_{r,z}$. Since f_x is lower-semicontinous, we derive that

$$\liminf_{n\to+\infty} f_{z_n}(x) = \liminf_{n\to+\infty} f_x(z_n) \geq f_x(z) = f_z(x) \geq r$$

Therefore, the inequality $s < r$ implies that there exists $n \in \mathbb{N}$ such that $f_{z_n}(x) \geq s$. Hence $x \in L_{s,z_n}$. The point z_n is contained in F_i and hence $x \in L_{s,z_n} \subseteq L_{s,y_i}$. Thus $\varnothing = L_{s,y_i} \cap L_{r,z} \subseteq L_{s,y_i} \cap L_{s,z}$. Thus $L_{s,z} \subseteq L_{s,y_i}$ and $z \in F_i$. We deduce that set F_i is closed in $[y_1, y_2]$. Note that we obtain

$$[y_1, y_2] = F_1 \cup F_2, \varnothing = F_1 \cap F_2$$

for some closed subsets F_1, F_2 of $[y_1, y_2]$. This is a contradiction with connectedness. □

Let $\mathcal{F}$ be a class of all functions $f : X \times Y \to \mathbb{R}$ such that the following assertions hold.

(1) X is a convex, compact subset of a topological vector space over $\mathbb{R}$ and Y is a convex subset of a topological vector space over $\mathbb{R}$.

(2) f_x is lower-semicontinuous and quasiconvex for every x in X.

(3) f_y is upper-semicontinuous and quasiconcave for every y in Y.

For every $f : X \times Y \to \mathbb{R}$ and $r \in \mathbb{R}$ we denote by $\mathcal{L}_{f,r}$ the family of sets

$$L_{f,ry} = \left\{ x \in X \,\middle|\, f_y(x) \geq r \right\}$$

parametrized by $y \in Y$.

Lemma 1.4.3. *For every $f \in \mathcal{F}$ the following assertion holds. If for some $r \in \mathbb{R}$ the family $\mathcal{L}_{f,r}$ consists of nonempty sets, then it has finite intersection property.*

Proof of the lemma. Suppose that $y_1, ..., y_m$ are points in Y. We want to show that

$$\bigcap_{i=1}^{m} L_{f,r,y_i} \neq \varnothing$$

The case $m = 1$ holds by assumption so we may assume that $m \geq 2$. Consider $X' = L_{f,r,y_m}$. This is a convex, compact and nonempty subset of a topological vector space over $\mathbb{R}$. Next let $f' : X' \times Y \to \mathbb{R}$ be the restriction of f. Then $f' \in \mathcal{F}$. In addition

$$L_{f',r,y} = X' \cap L_{f,r,y} = L_{f,r,y_m} \cap L_{f,r,y}$$

and by Lemma 1.4.2 this is nonempty set. So it suffices to prove that

$$\bigcap_{i=1}^{m-1} L_{f',r,y_i} \neq \varnothing$$

Hence the proof goes on induction on m. □

Proof of the theorem. We always have

$$\sup_{x \in X} \inf_{y \in Y} f(x,y) \leq \inf_{y \in Y} \sup_{x \in X} f(x,y)$$

Next pick $r \in \mathbb{R}$ such that

$$r < \inf_{y \in Y} \sup_{x \in X} f(x,y)$$

Then $\mathcal{L}_{f,r}$ consists of nonempty and compact subsets of X. By Lemma 1.4.3 we deduce that $\mathcal{L}_{f,r}$ admits finite intersection property. Hence there exists x in X such that

$$r \leq f(x,y)$$

for every $y \in Y$. This implies that

$$r \leq \inf_{y \in Y} f(x,y) \leq \sup_{x \in X} \inf_{y \in Y} f(x,y)$$

This shows that

$$\sup_{x \in X} \inf_{y \in Y} f(x,y) \leq \inf_{y \in Y} \sup_{x \in X} f(x,y)$$

□

2. Convex and concave functions

Definition 2.1. Let X be a convex subset of a linear space over $\mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function. Then f is *convex* if for every $x_1, x_2 \in X$ and $t \in [0, 1]$ we have

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

We say that f is *concave* if $-f$ is convex.

Fact 2.2. *Every convex function is quasiconvex.*

Proof. We left the proof to the reader. $\square$