

## LINEARLY REDUCTIVE GROUPS

### 1. MOTIVATION – LINEAR REPRESENTATIONS OF COMPACT TOPOLOGICAL GROUPS

In this section we fix a compact topological group  $\mathbf{G}$ . Assume that  $\rho : \mathbf{G} \rightarrow \mathrm{GL}_n(\mathbb{C})$  is a continuous homomorphism i.e. a complex,  $n$ -dimensional linear representation of  $\mathbf{G}$ . For every  $g \in \mathbf{G}$  we get a matrix

$$\rho(g) = [c_{ij}(g)]_{1 \leq i, j \leq n}$$

For  $i, j$  function  $c_{ij} : \mathbf{G} \rightarrow \mathbb{C}$  is a continuous complex valued function. Alternatively suppose that  $\{e_1, e_2, \dots, e_n\}$  is the standard basis of  $\mathbb{C}^n$  on which  $\mathrm{GL}_n(\mathbb{C})$  act. Then  $c_{ij}$  is equal to a function

$$\mathbf{G} \ni g \mapsto \langle g \cdot e_i, e_j \rangle \in \mathbb{C}$$

Fix now  $g_1, g_2 \in \mathbf{G}$  and note that

$$[c_{ij}(g_2 \cdot g_1)]_{1 \leq i, j \leq n} = \rho(g_2 \cdot g_1) = \rho(g_2) \cdot \rho(g_1) = \left[ \sum_{k=1}^n c_{ik}(g_2) \cdot c_{kj}(g_1) \right]_{1 \leq i, j \leq n}$$

Hence

$$c_{ij}(g_2 \cdot g_1) = \sum_{k=1}^n c_{ik}(g_2) \cdot c_{kj}(g_1)$$

for every  $1 \leq i, j \leq n$ . This implies that  $\sum_{1 \leq i, j \leq n} \mathbb{C} \cdot c_{ij} \subseteq \mathcal{L}^2(\mathbf{G}, \mathbb{C})$  is a linear  $\mathbf{G} \times \mathbf{G}^{\mathrm{op}}$ -subrepresentation of the regular representation  $\mathcal{L}^2(\mathbf{G}, \mathbb{C})$ . We call it *the matrix coefficients of  $\rho$* .

### 2. MATRIX COEFFICIENTS OF A REPRESENTATION

**Proposition 2.1.** *Let  $\mathfrak{X}$  be a monoid  $k$ -functor and let  $V$  be a finitely generated, projective  $k$ -module. Fix a morphism of monoids  $\rho : \mathfrak{X} \rightarrow \mathcal{L}_V$ . Fix  $k$ -algebra  $A$  and elements  $v \in A \otimes_k V$ ,  $w \in A \otimes_k V^\vee$ . For every  $A$ -algebra  $B$  and  $x \in \mathfrak{X}_A(B)$  we consider the formula*

$$c_{v,w}(x) = \langle \rho_A(x) \cdot v_B, w_B \rangle$$

*Then  $c_{v,w}$  defines a regular function on  $\mathfrak{X}_A$  for every  $k$ -algebra  $A$ .*

*Proof.* Suppose that  $f : B \rightarrow C$  is a morphism of  $A$ -algebras and pick  $x \in \mathfrak{X}_A(B)$ . Since  $\rho_A$  is natural and  $w : A \otimes_k V \rightarrow A$  is a morphism of  $A$ -modules, we derive that the diagram

$$\begin{array}{ccccc} V_B & \xrightarrow{\rho_A(x)} & V_B & \xrightarrow{w_B} & B \\ 1_{V_A} \otimes_A f \downarrow & & \downarrow 1_{V_A} \otimes_A f & & \downarrow f \\ V_C & \xrightarrow{\rho_A(\mathfrak{X}_A(f)(x))} & V_C & \xrightarrow{w_C} & C \end{array}$$

is commutative. Hence

$$c_{v,w}(\mathfrak{X}_A(f)(x)) = \langle \rho_A(\mathfrak{X}_A(f)(x)) \cdot v_C, w_C \rangle = f(\langle \rho_A(x) \cdot v_B, w_B \rangle) = f(c_{v,w}(x))$$

and this implies that  $c_{v,w} : \mathfrak{X}_A \rightarrow \mathbb{A}_A^1$  is natural. □

**Definition 2.2.** Let  $\mathfrak{X}$  be a monoid  $k$ -functor and let  $(V, \rho)$  be its representation with finitely generated, projective underlying  $k$ -module  $V$ . Fix  $k$ -algebra  $A$  and elements  $v \in A \otimes_k V$ ,  $w \in A \otimes_k V^\vee$ . Then the regular function  $c_{v,w}$  on  $\mathfrak{X}_A$  is called *the matrix coefficient of  $v$  and  $w$* .

**Proposition 2.3.** Let  $\mathfrak{X}$  be a monoid  $k$ -functor and let  $(V, \rho)$  be its representation with finitely generated projective underlying  $k$ -module  $V$ . Then the following assertions holds.

(1) For every  $k$ -algebra  $A$  map

$$(A \otimes_k V) \times (A \otimes_k V^\vee) \ni (v, w) \mapsto c_{v,w} \in \text{Mor}_A(\mathfrak{X}_A, \mathbb{A}_A^1)$$

is  $A$ -bilinear.

(2) The collection of maps

$$\{(A \otimes_k V) \times (A \otimes_k V^\vee) \ni (v, w) \mapsto c_{v,w} \in \text{Mor}_A(\mathfrak{X}_A, \mathbb{A}_A^1)\}_{A \in \mathbf{Alg}_k}$$

gives rise to a morphism of  $k$ -functors

$$V_a \times V_a^\vee \longrightarrow \text{Mor}_k(\mathfrak{X}, \mathbb{A}_k^1)$$

*Proof.* We left the proof of (1) to the reader.

We prove (2). Consider  $k$ -algebra  $A$  and an  $A$ -algebra  $B$  with structural morphism  $f : A \rightarrow B$ . Fix  $v \in A \otimes_k V$ ,  $w \in A \otimes_k V^\vee$ . We prove that restriction of  $c_{v,w} : \mathfrak{X}_A \rightarrow \mathbb{A}_A^1$  to the category  $\mathbf{Alg}_B$  is  $c_{v_B, w_B}$ . For this pick a  $B$ -algebra  $C$  and an element  $x \in \mathfrak{X}_A(C) = \mathfrak{X}_B(C)$ . Note that

$$c_{v,w}(x) = \langle \rho_A(x) \cdot v_C, w_C \rangle = \langle \rho_B(x) \cdot v_C, w_C \rangle = \langle \rho_B(x) \cdot (v_B)_C, (w_B)_C \rangle = c_{v_B, w_B}(x)$$

and hence  $c_{v,w}|_{\mathbf{Alg}_B} = c_{v_B, w_B}$ . Consider the square

$$\begin{array}{ccc} V_a(A) \times V_a^\vee(A) & \longrightarrow & \text{Mor}_k(\mathfrak{X}, \mathbb{A}^1)(A) \\ \downarrow V_a(f) \times V_a^\vee(f) & & \downarrow \text{Mor}_k(\mathfrak{X}, \mathbb{A}^1)(f) \\ V_a(B) \times V_a^\vee(B) & \longrightarrow & \text{Mor}_k(\mathfrak{X}, \mathbb{A}^1)(B) \end{array}$$

in which both horizontal arrows are given by formula  $(v, w) \mapsto c_{v,w}$ . We proved that the square commutes. Since  $f$  is an arbitrary morphism of  $k$ -algebras, we conclude the assertion.  $\square$

**Corollary 2.4.** Let  $\mathfrak{X}$  be a monoid  $k$ -functor and let  $(V, \rho)$  be its representation with finitely generated projective underlying  $k$ -module  $V$ . Then there exists a morphism of  $k$ -functors

$$(V \otimes_k V^\vee)_a \xrightarrow{c} \text{Mor}_k(\mathfrak{X}, \mathbb{A}_k^1)$$

given by formula

$$(A \otimes_k V) \otimes_A (A \otimes_k V^\vee) \ni (v, w) \mapsto c_{v,w} \in \text{Mor}_A(\mathfrak{X}_A, \mathbb{A}_A^1)$$

Moreover,  $c$  is a morphism of  $k$ -functors equipped with  $\mathfrak{X} \times \mathfrak{X}^{\text{op}}$ -actions.

*Proof.* The first part is an immediate consequence of Proposition 2.3. We prove that  $c$  is a morphism of  $k$ -functors equipped with  $\mathfrak{X} \times \mathfrak{X}^{\text{op}}$ -actions. For this we fix a  $k$ -algebra  $k$  and elements  $v \in A \otimes_k V$ ,  $w \in A \otimes_k V^\vee$ . Pick a morphism of  $k$ -algebras  $f : A \rightarrow B$ ,  $(y, z) \in \mathfrak{X}(A) \times \mathfrak{X}(A)^{\text{op}}$  and  $x \in \mathfrak{X}_A(B)$ . Then we have

$$\begin{aligned} c_{\rho(y) \cdot v, w \cdot \rho(z)}(x) &= \langle \rho_A(x) \cdot (\rho(y) \cdot v)_B, (w \cdot \rho(z))_B \rangle = \\ &= \langle \rho_A(x) \cdot \rho_A((\mathfrak{X}_A(f)(y))) \cdot v_B, w_B \cdot \rho_A(\mathfrak{X}_A(f)(z)) \rangle = w_B(\rho_A(\mathfrak{X}_A(f)(z)) \cdot \rho_A(x) \cdot \rho_A(\mathfrak{X}_A(f)(y))) \cdot v_B = \\ &= w_B(\rho_A(\mathfrak{X}_A(f)(z)) \cdot x \cdot \mathfrak{X}_A(f)(y)) \cdot v_B = \langle \rho_A(\mathfrak{X}_A(f)(z)) \cdot x \cdot \mathfrak{X}_A(f)(y) \cdot v_B, w_B \rangle = \end{aligned}$$

$$= c_{v,w}(\mathfrak{X}_A(f)(z) \cdot x \cdot \mathfrak{X}_A(f)(y))$$

and hence  $c$  is a morphism of  $k$ -functors equipped with actions of  $\mathfrak{X} \times \mathfrak{X}^{\text{op}}$ .  $\square$

The discussion below is partially an application of the main result in [Mon19a, section 6] (Remark 5.6 shows that  $\mathcal{L}_V$  is a subcopresheaf of internal endomorphisms of  $V_a$  and hence the machinery developed in the citation above can be applied), but for the reader's convenience we decide to include all essential details even if this requires repetition.

Let  $\mathfrak{G}$  be a monoid  $k$ -functor and let  $V$  be a  $k$ -module. Suppose that  $\alpha : \mathfrak{G} \times V_a \rightarrow V_a$  is an action of  $\mathfrak{G}$  on  $V_a$ . Assume that  $A$  is a  $k$ -algebra and  $x \in \mathfrak{G}(A)$ . We denote by  $i_x : \mathbf{1}_A \rightarrow \mathfrak{G}_A$  the morphism of  $A$ -functors corresponding to  $x$  by means of [Mon19b, Fact 2.4]. Since  $\mathbf{1}_A$  is terminal  $A$ -functor, a morphism  $\alpha_A \cdot (i_x \times \mathbf{1}_{(V_a)_A})$  is isomorphic to a morphism  $\alpha_x : (V_a)_A \rightarrow (V_a)_A$  of  $A$ -functors.

**Definition 2.5.** Let  $\mathfrak{G}$  be a monoid  $k$ -functor and let  $V$  be a  $k$ -module. An action  $\alpha : \mathfrak{G} \times V_a \rightarrow V_a$  of  $\mathfrak{G}$  such that for any  $k$ -algebra  $A$  and point  $x \in \mathfrak{G}(A)$  morphism  $\alpha_x$  is linear is called a *linear action* of  $\mathfrak{G}$  on  $V$ .

Theorem ?? states that for every monoid  $k$ -functor  $\mathfrak{G}$  and every  $k$ -module  $V$  linear actions  $\alpha : \mathfrak{G} \times V_a \rightarrow V_a$  and morphisms  $\rho : \mathfrak{G} \rightarrow \mathcal{L}_V$  of  $k$ -monoids are in bijective correspondence. This shows that the formal machinery developed so far works as expected. Now we introduce the following notion.

**Definition 2.6.** Let  $\mathfrak{G}$  be a monoid  $k$ -functor. A pair  $(V, \rho)$  consisting of a  $k$ -module  $V$  and a morphism  $\rho : \mathfrak{G} \rightarrow \mathcal{L}_V$  of  $k$ -monoids is called a *linear representation* of  $\mathfrak{G}$ .

**Proposition 2.7.** Let  $\mathfrak{G}$  be a monoid  $k$ -functor. Suppose that  $\alpha : \mathfrak{G} \times V_a \rightarrow V_a$ ,  $\beta : \mathfrak{G} \times W_a \rightarrow W_a$  are  $k$ -linear actions on  $k$ -modules  $V$  and  $W$ , respectively. Suppose that  $\sigma : V_a \rightarrow W_a$  is a linear morphism of  $k$ -functors and  $\phi = \sigma^k$  is the corresponding morphism of  $k$ -modules. Then the following assertions are equivalent.

(i) The square

$$\begin{array}{ccc} \mathfrak{G} \times V_a & \xrightarrow{1_{\mathfrak{G}} \times \sigma} & \mathfrak{G} \times W_a \\ \beta \downarrow & & \downarrow \alpha \\ V_a & \xrightarrow{\sigma} & W_a \end{array}$$

is commutative.

(ii) For every  $k$ -algebra  $A$  and  $x \in \mathfrak{G}(A)$  we have

$$(1_A \otimes_k \phi) \cdot \rho(x) = \delta(x) \cdot (1_A \otimes_k \phi)$$

where  $\rho : \mathfrak{G} \rightarrow \mathcal{L}_V$  and  $\delta : \mathfrak{G} \rightarrow \mathcal{L}_W$  are morphism of monoid  $k$ -functors corresponding to  $\alpha$  and  $\beta$ , respectively.

*Proof.* Indeed, conditions expressed in (i) and (ii) are directly translatable to each other by virtue of Fact 5.3 and the bijection in Theorem ??.  $\square$

**Definition 2.8.** Let  $\mathfrak{G}$  be a monoid  $k$ -functor and let  $(V, \rho)$ ,  $(W, \delta)$  be its linear representations. A morphism  $\phi : V \rightarrow W$  of  $k$ -modules such that for every  $k$ -algebra  $A$  and  $x \in \mathfrak{G}(A)$  we have

$$(1_A \otimes_k \phi) \cdot \rho(x) = \delta(x) \cdot (1_A \otimes_k \phi)$$

is called a *morphism of linear representations* of  $\mathfrak{G}$ .

Let  $\mathfrak{G}$  be a monoid  $k$ -functor. We denote by  $\mathbf{Rep}(\mathfrak{G})$  its category of linear representations.

## 3. THE CATEGORY OF LINEAR REPRESENTATIONS

In this section we fix a monoid  $k$ -functor  $\mathfrak{G}$ . Note that there exists the forgetful functor  $\mathbf{Rep}(\mathfrak{G}) \rightarrow \mathbf{Mod}(k)$  that sends each linear representation to its underlying  $k$ -module.

**Theorem 3.1.** *The forgetful functor*

$$\mathbf{Rep}(\mathfrak{G}) \longrightarrow \mathbf{Mod}(k)$$

*creates small colimits.*

*Proof.* Suppose that  $I \ni i \mapsto (V_i, \rho_i) \in \mathbf{Rep}(\mathfrak{G})$  is a diagram of linear representations of  $\mathfrak{G}$  indexed by some category  $I$ . Let  $V$  together with  $u_i : V_i \rightarrow V$  for  $i \in I$  be a colimit of the diagram  $I \ni i \mapsto V_i \in \mathbf{Mod}(k)$ .

Assume first that  $(V, \rho)$  is a structure of the linear representation of  $\mathfrak{G}$  on  $V$  such that  $u_i : V_i \rightarrow V$  for  $i \in I$  becomes a cocone over the diagram  $I \ni i \mapsto (V_i, \rho_i) \in \mathbf{Rep}(\mathfrak{G})$ . For every  $k$ -algebra  $A$  the functor  $A \otimes_k (-)$  preserves colimits and hence  $1_A \otimes_k u_i$  for  $i \in I$  is a colimit of the diagram  $I \ni i \mapsto 1_A \otimes_k V_i \in \mathbf{Mod}(A)$ . For each  $i \in I$  we have an action  $\rho_i^A : \mathfrak{G}(A) \rightarrow \mathrm{Hom}_A(A \otimes_k V_i, A \otimes_k V_i)$  of  $\mathfrak{G}(A)$  on  $A \otimes_k V_i$  and we may view the diagram  $I \ni i \mapsto 1_A \otimes_k V_i \in \mathbf{Mod}(A)$  as a diagram in the category of  $A$ -modules equipped with  $\mathfrak{G}(A)$ -actions given by  $A$ -module morphisms. We refer to this category as to category of  $A$ -linear  $\mathfrak{G}(A)$ -actions. Now the forgetful functor

$$\left\{ \text{the category of } A\text{-linear } \mathfrak{G}(A)\text{-actions} \right\} \longrightarrow \mathbf{Mod}(A)$$

creates small limits. Indeed, the category on the right hand side is isomorphic with the category  $\mathbf{Mod}(A[\mathfrak{G}(A)])$  of left modules over the monoid  $A$ -algebra  $A[\mathfrak{G}(A)]$  and the forgetful functor

$$\mathbf{Mod}(A[\mathfrak{G}(A)]) \longrightarrow \mathbf{Mod}(A)$$

creates small colimits. This implies that  $\rho^A : \mathfrak{G}(A) \rightarrow \mathrm{Hom}_A(A \otimes_k V, A \otimes_k V)$  must be a unique morphism of monoids such that  $1_A \otimes_k u_i$  for every  $i \in I$  is a morphism of  $A$ -modules with  $A$ -linear action of  $\mathfrak{G}(A)$ . This implies that  $\rho$  is unique and hence  $(V, \rho)$  is unique lift of  $(V, \{u_i\}_{i \in I})$  to  $\mathbf{Rep}(\mathfrak{G})$ . This shows the uniqueness of a lift.

For the existence assume for given  $k$ -algebra  $A$  that  $\rho^A : \mathfrak{G}(A) \rightarrow \mathrm{Hom}_A(A \otimes_k V, A \otimes_k V)$  is a unique morphism of monoids such that  $1_A \otimes_k u_i$  for every  $i \in I$  is a morphism of  $A$ -modules with  $A$ -linear action of  $\mathfrak{G}(A)$ . Note that  $\rho^A$  exists because the forgetful functor

$$\left\{ \text{the category of } A\text{-linear } \mathfrak{G}(A)\text{-actions} \right\} \longrightarrow \mathbf{Mod}(A)$$

creates small colimits. Denote  $\rho = \{\rho^A\}_{A \in \mathbf{Alg}_k}$ . We verify that  $\rho$  is a morphism of  $k$ -functors  $\rho : \mathfrak{G} \rightarrow \mathcal{L}_V$ . For this consider morphism  $f : A \rightarrow B$  of  $k$ -algebras and the commutative square

$$\begin{array}{ccc} A \otimes_k V_i & \xrightarrow{1_A \otimes_k u_i} & A \otimes_k V \\ f \otimes_k 1_{V_i} \downarrow & & \downarrow f \otimes_k 1_V \\ B \otimes_k V_i & \xrightarrow{1_B \otimes_k u_i} & B \otimes_k V \end{array}$$

defined for every  $i \in I$ . Note that the top row of the square is a morphism of  $A$ -modules with  $A$ -linear  $\mathfrak{G}(A)$ -actions. Similarly interpreting  $B \otimes_k V_i$  and  $B \otimes_k V$  as  $A$ -modules with  $A$ -linear actions of  $\mathfrak{G}(A)$  given by  $\rho_i^B \cdot \mathfrak{G}(f)$  and  $\rho^B \cdot \mathfrak{G}(f)$ , respectively, we derive that the square consists of  $A$ -modules with  $A$ -linear actions of  $\mathfrak{G}(A)$  and all maps preserve actions except possibly  $f \otimes_k 1_V$ . Since  $A \otimes_k V$  together with  $1_A \otimes_k u_i$  for  $i \in I$  is a colimit of  $I \ni i \mapsto 1_A \otimes_k V_i \in \mathbf{Mod}(A)$  in the category of  $A$ -modules, we deduce that  $f \otimes_k 1_V$  is the only morphism of  $A$ -modules making squares commutative for all  $i \in I$ . Since  $A \otimes_k V$  with  $\rho^A$  and  $1_A \otimes_k u_i$  for  $i \in I$  is a colimit of the same diagram, but interpreted as a diagram of  $A$ -modules with  $A$ -linear action of  $\mathfrak{G}(A)$ -modules, we derive from uniqueness of  $f \otimes_k 1_V$  that it must also preserve  $\mathfrak{G}(A)$ -action. Hence  $(f \otimes_k 1_V) \cdot \rho^A = \rho^B \cdot \mathfrak{G}(f)$ . Thus  $\rho$  is a morphism of  $k$ -functors. By definition of  $\rho^A$  for each  $k$ -algebra  $A$ , we derive that it is a morphism of monoid  $k$ -functors. Hence  $(V, \rho)$  is a linear representation of  $\mathfrak{G}$  and again by componentwise definition of  $\rho$  we deduce that  $(V, \rho)$  is a colimit of the diagram  $I \ni i \mapsto (V_i, \rho_i) \in \mathbf{Rep}(\mathfrak{G})$ .  $\square$

**Theorem 3.2.** *Let  $A$  be a commutative ring. The following assertions are equivalent.*

- (i)  *$\text{Spec } A$  is a Hausdorff space.*
- (ii) *Every prime ideal of  $A$  is maximal.*
- (iii) *Every  $A/\mathcal{N}$ -module is flat, where  $\mathcal{N}$  is a nilradical of  $A$ .*
- (iv) *Every finitely generated ideal of  $A$  is generated by an idempotent.*

**Lemma 3.2.1.** *Let  $A$  be a commutative ring and  $M$  be an  $A$ -module. Then  $M$  is flat if and only if  $M_{\mathfrak{p}}$  is flat for all  $\mathfrak{p} \in \text{Spec } A$ .*

*Proof of the lemma.* For every  $\mathfrak{p} \in \text{Spec } A$  we have a natural isomorphism

$$M_{\mathfrak{p}} \otimes_A (-) \cong (M \otimes_A (-))_{\mathfrak{p}}$$

Now the statement follows from the fact that a chain complex of  $A$ -modules is exact if and only if it is exact after localization in every prime ideal  $\mathfrak{p} \in \text{Spec } A$   $\square$

**Lemma 3.2.2.** *Let  $A$  be a local ring such that each  $A$ -module is flat. Then  $A$  is a field.*

*Proof of the lemma.* Let  $\mathfrak{m}$  be a maximal ideal of  $A$  and  $k$  be a residue field. Pick finitely generated ideal  $\mathfrak{a} \subseteq \mathfrak{m}$ . Consider the canonical exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow A \xrightarrow{a \mapsto a \bmod \mathfrak{a}} A/\mathfrak{a} \longrightarrow 0$$

Since  $k$  is a flat  $A$ -module, we derive that the sequence

$$0 \longrightarrow k \otimes_A \mathfrak{a} \longrightarrow k \xrightarrow{\alpha \mapsto \alpha \bmod \mathfrak{a}k} k/\mathfrak{a}k \longrightarrow 0$$

is exact. Since  $\mathfrak{a}k = 0$  because  $\mathfrak{a} \subseteq \mathfrak{m}$ , we deduce from the short exact sequence that  $k \otimes_A \mathfrak{a} = 0$ . By Nakayama lemma this implies that  $\mathfrak{a} = 0$  ( $\mathfrak{a}$  is finitely generated over  $A$ ). Thus every finitely generated  $A$ -submodule of  $\mathfrak{m}$  is trivial. Thus  $\mathfrak{m} = 0$  and hence  $A$  is a field.  $\square$

## 4.

5. LINEAR REPRESENTATIONS OF MONOID  $k$ -FUNCTORS

**Example 5.1.** Let  $V$  be a  $k$ -module. We define a  $k$ -functor  $V_a$ . We set

$$V_a(A) = A \otimes_k V, \quad V_a(f) = f \otimes_k 1_V$$

for every  $k$ -algebra  $A$  and every morphism  $f : A \rightarrow B$  of  $k$ -algebras. Moreover,  $V_a$  admits a structure of a commutative group  $k$ -functor. Indeed,  $V_a(A)$  is a commutative group with respect to addition induced by its structure of  $A$ -module and  $V_a(f) : V_a(A) \rightarrow V_a(B)$  preserves the addition.

Suppose now that  $V, W$  are  $k$ -modules and  $\sigma : (V_a)_A \rightarrow (W_a)_A$  is a morphism of  $A$ -functors for some  $k$ -algebra  $A$ . Then for every  $A$ -algebra  $B$  we denote by  $\sigma^B : B \otimes_k V \rightarrow B \otimes_k W$  the component of  $\sigma$  for  $B$ .

**Definition 5.2.** Let  $V, W$  be  $k$ -modules and let  $A$  be a  $k$ -algebra. A morphism  $\sigma : (V_a)_A \rightarrow (W_a)_A$  of  $A$ -functors is *linear* if for every  $A$ -algebra  $B$  the component  $\sigma^B : B \otimes_k V \rightarrow B \otimes_k W$  is a morphism of  $B$ -modules.

Next result characterizes linear morphism.

**Fact 5.3.** Let  $V, W$  be  $k$ -modules and let  $A$  be a  $k$ -algebra. Suppose that  $\phi : A \otimes_k V \rightarrow A \otimes_k W$  is a morphism of  $A$ -modules. Then there exists a unique linear morphism  $\sigma : (V_a)_A \rightarrow (W_a)_A$  of  $A$ -functors such that  $\sigma^A = \phi$ .

*Proof.* Note that if such  $\sigma$  exists, then by requirement  $\sigma^A = \phi$  for every morphism  $f : A \rightarrow B$  of  $k$ -algebras the following diagram

$$\begin{array}{ccc} A \otimes_k V & \xrightarrow{\phi} & A \otimes_k W \\ f \otimes_k 1_V \downarrow & & \downarrow f \otimes_k 1_W \\ B \otimes_k V & \xrightarrow{\sigma^B} & B \otimes_k W \end{array}$$

must commute. We make this into a definition of a morphism  $\sigma^B$  of  $B$ -modules. It is a matter of linear algebra that this diagram uniquely determines  $\sigma^B$  and also that  $\sigma^A = \phi$ . It remains to verify that  $\sigma = \{\sigma^B\}_{B \in \mathbf{Alg}_A}$  defined in such a way is a morphism of  $A$ -functors. For this suppose that  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are morphisms of  $k$ -algebras. Then we have

$$\begin{aligned} \sigma^C \cdot (g \otimes_k 1_V) \cdot (f \otimes_k 1_V) &= \sigma^C \cdot ((g \cdot f) \otimes_k 1_V) = ((g \cdot f) \otimes_k 1_W) \cdot \phi = \\ &= (g \otimes_k 1_W) \cdot (f \otimes_k 1_V) \cdot \phi = (g \otimes_k 1_W) \cdot \sigma^B \cdot (f \otimes_k 1_V) \end{aligned}$$

and hence  $\sigma^C \cdot (g \otimes_k 1_V) = (g \otimes_k 1_W) \cdot \sigma^B$ . Thus  $\sigma$  is a linear morphism of  $A$ -functors.  $\square$

We restate Fact 5.3 in the form of the following result.

**Corollary 5.4.** Let  $V, W$  be  $k$ -modules and  $A$  be a  $k$ -algebra. Consider the map

$$\mathrm{Hom}_A(A \otimes_k V, A \otimes_k W) \longrightarrow \mathrm{Mor}_A((V_a)_A, (W_a)_A)$$

that sends morphism  $\phi$  to a unique linear morphism  $\sigma : (V_a)_A \rightarrow (W_a)_A$  of  $A$ -functors such that  $\sigma^A = \phi$ . Then this map is injective and its image consists of all linear morphisms of  $A$ -functors.

**Example 5.5.** Let  $V$  be a  $k$ -module. We define a  $k$ -functor  $\mathcal{L}_V$ . We set

$$\mathcal{L}_V(A) = \text{Hom}_A(A \otimes_k V, A \otimes_k V)$$

for every  $k$ -algebra  $A$ . Next for every morphism  $f : A \rightarrow B$  of  $k$ -algebras and every morphism  $\phi : A \otimes_k V \rightarrow A \otimes_k V$  of  $A$ -modules we define  $\mathcal{L}_V(f)(\phi)$  as a unique morphism of  $B$ -modules such that the diagram

$$\begin{array}{ccc} A \otimes_k V & \xrightarrow{\phi} & A \otimes_k V \\ f \otimes_k 1_V \downarrow & & \downarrow f \otimes_k 1_V \\ B \otimes_k V & \xrightarrow{\mathcal{L}_V(\phi)} & B \otimes_k V \end{array}$$

is commutative. Note also that  $\mathcal{L}_V(A)$  is a monoid with respect to the usual composition of morphism of  $A$ -modules and  $\mathcal{L}_V(f) : \mathcal{L}_V(A) \rightarrow \mathcal{L}_V(B)$  preserves this composition. Hence  $\mathcal{L}_V$  is a monoid  $k$ -functor.

**Remark 5.6.** Corollary 5.4 implies that there are injective maps that make the square

$$\begin{array}{ccc} \mathcal{L}_V(A) & \hookrightarrow & \text{Mor}_A((V_a)_A, (V_a)_A) \\ \mathcal{L}_V(f) \downarrow & & \downarrow \sigma \mapsto \sigma_B \\ \mathcal{L}_V(B) & \hookrightarrow & \text{Mor}_B((V_a)_B, (V_a)_B) \end{array}$$

commutative for every morphism  $f : A \rightarrow B$  of  $k$ -algebras. It also shows that for every  $k$ -algebra  $A$  this identifies  $\mathcal{L}_V(A)$  with a subset of the class  $\text{Mor}_A((V_a)_A, (V_a)_A)$  consisting of all linear morphism of the  $A$ -functor  $(V_a)_A$ .

## 6. RESULTS ON AFFINE MONOIDS

**Definition 6.1.** Let  $\mathfrak{G}$  be a monoid  $k$ -functor. We say that  $\mathfrak{G}$  is a *monoid  $k$ -functor with zero* if there exists a  $k$ -point  $\mathbf{o}$  of  $\mathfrak{G}$  such that the following two morphisms

$$\mathbf{1} \times \mathfrak{G} \xrightarrow{\mathbf{o} \times 1_{\mathfrak{G}}} \mathfrak{G} \times \mathfrak{G} \xrightarrow{\text{mul}} \mathfrak{G} \quad \mathfrak{G} \times \mathbf{1} \xrightarrow{1_{\mathfrak{G}} \times \mathbf{o}} \mathfrak{G} \times \mathfrak{G} \xrightarrow{\text{mul}} \mathfrak{G}$$

where  $\text{mul} : \mathfrak{G} \times \mathfrak{G} \rightarrow \mathfrak{G}$  is the multiplication on  $\mathfrak{G}$ , factor through  $\mathbf{o}$ . If this is the case, then  $\mathbf{o}$  is called *the zero of  $\mathfrak{G}$* .

**Definition 6.2.** Let  $\mathfrak{G}$  be a monoid  $k$ -functor. For each  $k$ -algebra  $A$  we denote by  $\mathfrak{G}^*(A)$  the group of units of  $\mathfrak{G}(A)$ . This gives rise to a subgroup  $k$ -functor  $\mathfrak{G}^*$  of  $\mathfrak{G}$ . We call  $\mathfrak{G}^*$  *the group of units of  $\mathfrak{G}$* .

Now we describe the universal property of the group of units. Let  $\mathfrak{G}$  be a monoid  $k$ -functor and let  $\mathfrak{G}$  be a group  $k$ -functor. Suppose that  $\sigma : \mathfrak{G} \rightarrow \mathfrak{G}$  is a morphism of monoid  $k$ -functors. Then  $\sigma$  factors through  $\mathfrak{G}^*$ .

**Proposition 6.3.** Let  $\mathbf{M}$  be an affine  $k$ -monoid scheme and denote by  $\mathfrak{G}$  the  $k$ -monoid functor that represents  $\mathbf{M}$ . Then  $\mathfrak{G}^*$  is representable by an affine  $k$ -group scheme. Moreover, if  $\mathbf{M}$  is an affine integral  $k$ -monoid scheme of finite type over  $k$ , then  $\mathfrak{G}^*$  is an open  $k$ -subfunctor of  $\mathfrak{G}$ .

7. DIAGONALISABLE MONOID  $k$ -SCHEMES

Consider an abstract commutative monoid  $\Gamma$ . Consider the monoid  $k$ -algebra  $k[\Gamma]$ . Recall that  $k[\Gamma]$  as a free  $k$ -vector space over  $k$  and its elements can be uniquely written as

$$\sum_{\gamma \in \Gamma} k_{\gamma} \cdot \gamma$$

where almost all  $k_{\gamma}$  are zero for  $\gamma \in \Gamma$ . Next the  $k$ -algebra  $k[\Gamma]$  admits a structure of a commutative bialgebra with a comultiplication given by

$$k[\Gamma] \ni \sum_{\gamma \in \Gamma} k_{\gamma} \cdot \gamma \rightarrow \sum_{\gamma \in \Gamma} k_{\gamma} \cdot (\gamma \otimes \gamma) \in k[\Gamma] \otimes_k k[\Gamma]$$

and a counit

$$k[\Gamma] \ni \sum_{\gamma \in \Gamma} k_{\gamma} \cdot \gamma \mapsto \sum_{\gamma \in \Gamma} k_{\gamma} \in k$$

This makes  $\text{Spec} k[\Gamma]$  into a monoid  $k$ -scheme. We denote this monoid  $k$ -scheme by  $\mathbf{D}_{\Gamma}$ . For an alternative description note that we have identifications

$$\mathfrak{P}_{\mathbf{D}_{\Gamma}}(A) \cong \text{Mor}_k(k[\Gamma], A) \cong \mathbf{Mon}(\Gamma, A^{\times})$$

natural in  $k$ -algebra  $A$ , where the right hand side denotes the set of morphisms of monoids from  $\Gamma$  to the multiplicative monoid  $A^{\times}$  of  $A$ . The  $k$ -functor

$$\mathbf{Alg}_k \ni A \mapsto \mathbf{Mon}(\Gamma, A^{\times}) \in \mathbf{Set}$$

is a monoid  $k$ -functor with respect to multiplication of monoid homomorphisms in  $\mathbf{Mon}(\Gamma, A^{\times})$  for every  $k$ -algebra  $A$ . Hence the identification above makes the functor of points  $\mathfrak{P}_{\mathbf{D}_{\Gamma}}$  into the monoid  $k$ -functor and induces precisely the bialgebra structure on  $k[\Gamma]$  described above.

Note that if  $g : \Gamma_1 \rightarrow \Gamma_2$  is a morphism of commutative monoids, then  $k[g] : k[\Gamma_1] \rightarrow k[\Gamma_2]$  is a morphism of bialgebras (with respect to the structure described above). We denote  $\text{Spec} k[g]$  by  $\mathbf{D}_g$ .

**Definition 7.1.** Let  $\mathbf{M}$  be a monoid  $k$ -scheme. We say that  $\mathbf{M}$  is *diagonalisable* if there exists an abstract commutative monoid  $\Gamma$  such that  $\mathbf{M}$  is isomorphic to  $\mathbf{D}_{\Gamma}$  as a monoid  $k$ -scheme.

Now we prove the following important result.

**Theorem 7.2.** Suppose that  $k$  is commutative ring such that  $\text{Spec} k$  is connected (i.e.  $k$  has no nontrivial idempotents). Consider the functor

$$\begin{array}{ccc} \Gamma_1 & & \mathbf{D}_{\Gamma_1} \\ g \downarrow & \xrightarrow{\quad} & \uparrow \mathbf{D}_g \\ \Gamma_2 & & \mathbf{D}_{\Gamma_2} \end{array}$$

defined on the category of commutative monoids and with values in the category of monoid schemes over  $k$ . This functor preserves finite products and induces an equivalence of categories between abstract commutative monoids and diagonalisable monoid schemes over  $k$ .

*Proof.* Suppose that  $\Gamma_1, \Gamma_2$  are commutative monoids and  $f : k[\Gamma_1] \rightarrow k[\Gamma_2]$  is a morphism of bialgebras over  $k$ . Let  $\Delta_1, \xi_1$  and  $\Delta_2, \xi_2$  be comultiplications and counits for  $k[\Gamma_1], k[\Gamma_2]$ , respectively. Fix  $\gamma \in \Gamma_1$  and suppose that  $f(\gamma) = \sum_{\gamma' \in \Gamma_2} k_{\gamma'} \cdot \gamma'$ . The fact that  $f$  is a morphism of bialgebras over  $k$  implies that

$$\Delta_2(f(\gamma)) = (f \otimes_k f)(\Delta_1(\gamma)) = (f \otimes_k f)(\gamma \otimes_k \gamma) = f(\gamma) \otimes_k f(\gamma)$$



Substituting  $\sum_{\gamma' \in \Gamma_2} k_{\gamma'} \cdot \gamma'$  for  $f(\gamma)$  we deduce that

$$\sum_{\gamma' \in \Gamma_2} k_{\gamma'} \cdot (\gamma' \otimes \gamma') = \sum_{\gamma' \in \Gamma_2} \sum_{\gamma'' \in \Gamma_2} k_{\gamma'} \cdot k_{\gamma''} \cdot (\gamma' \otimes \gamma'')$$

Thus we derive that

$$k_{\gamma'} \cdot k_{\gamma''} = \begin{cases} 0 & \text{if } \gamma' \neq \gamma'' \\ k_{\gamma'} & \text{if } \gamma' = \gamma'' \end{cases}$$

Since there are no nontrivial idempotents in  $k$ , this implies that  $k_{\gamma'} = 0, 1$  for each  $\gamma' \in \Gamma_2$ . Again by the fact that  $f$  is a morphism of  $k$ -bialgebras, we derive that

$$\xi_1(\gamma) = \xi_2(f(\gamma))$$

Substituting  $\sum_{\gamma' \in \Gamma_2} k_{\gamma'} \cdot \gamma'$  for  $f(\gamma)$  yields that

$$\sum_{\gamma' \in \Gamma_2} k_{\gamma'} = 1$$

Combining this with previously established fact that  $k_{\gamma'} = 0, 1$  for each  $\gamma' \in \Gamma_2$  we deduce that there exists precisely one  $\gamma' \in \Gamma_2$  such that  $f(\gamma) = \gamma'$ . This proves that  $f(\Gamma_1) \subseteq \Gamma_2$ . Since  $f$  preserves multiplication and unit, we deduce that  $f = k[g]$  for some homomorphism of abstract monoids  $g : \Gamma_1 \rightarrow \Gamma_2$ . Thus the functor described in the statement is full.

It is also clearly faithful. Indeed, for two distinct morphisms of monoids  $g_1, g_2 : \Gamma_1 \rightarrow \Gamma_2$  we have  $k[g_1] \neq k[g_2]$  and hence  $\text{Spec } k[g_1] \neq \text{Spec } k[g_2]$ .

By definition of diagonalisable monoid the image of the functor is an essential subcategory of the category of diagonalisable  $k$ -schemes.

Finally, consider commutative monoids  $\Gamma_1, \Gamma_2$  and note that isomorphism

$$k[\Gamma_1 \times \Gamma_2] \ni \sum_{(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2} k_{(\gamma_1, \gamma_2)} \cdot (\gamma_1, \gamma_2) \mapsto \sum_{(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2} k_{(\gamma_1, \gamma_2)} \cdot \gamma_1 \otimes \gamma_2 \in k[\Gamma_1] \otimes_k k[\Gamma_2]$$

is a morphism of  $k$ -bialgebras. This implies that the functor described in the statement preserves binary products. The functor preserves terminal objects, since  $k$  is a monoid  $k$ -algebra for trivial (zero) commutative monoid.  $\square$

## 8. REPRESENTATIONS OF DIAGONALISABLE MONOID $k$ -SCHEMES

**Definition 8.1.** Let  $\Gamma$  be a commutative monoid and let  $\mathbf{D}_\Gamma$  be the corresponding monoid  $k$ -scheme. Suppose that  $V$  is a representation of  $\mathbf{D}_\Gamma$  with respect to a morphism of monoid  $k$ -functors given by

$$\mathfrak{P}_{\mathbf{D}_\Gamma}(A) = \mathbf{Mod}(\Gamma, A^\times) \ni f \mapsto f(\gamma) \cdot (-) \in \mathcal{L}_V(A)$$

where  $\gamma$  is a fixed element of  $\Gamma$ . Then  $V$  is called a *representation of  $\mathbf{D}_\Gamma$  of weight  $\gamma$* .

**Fact 8.2.** Let  $\Gamma$  be a commutative monoid and let  $\gamma$  be its element. Suppose that  $V$  is a representation of  $\mathbf{D}_\Gamma$  of weight  $\gamma$ . Then  $V$  can be equivalently described as a comodule over  $k[\Gamma]$  with respect to the following coaction

$$V_\gamma \ni v \mapsto \gamma \otimes v \in k[\Gamma] \otimes_k V_\gamma$$

*Proof.* Denote by  $\rho : \mathfrak{P}_{\mathbf{D}_\Gamma} \rightarrow \mathcal{L}_V$  the morphism of monoid  $k$ -functors that makes a  $V$  into a representation of  $\mathbf{D}_\Gamma$ . Then  $\rho(1_{\mathbf{D}_\Gamma})$  is a morphism of  $k[\Gamma]$ -modules

$$k[\Gamma] \otimes_k V \ni 1 \otimes v \mapsto \gamma \otimes v \in k[\Gamma] \otimes_k V$$

We obtain the coaction of  $k[\Gamma]$  on  $V$  corresponding to  $\rho$  by transforming morphism  $\rho(1_{\mathbf{D}_\Gamma})$  via the canonical isomorphism

$$\text{Hom}_{k[\Gamma]}(k[\Gamma] \otimes_k V, k[\Gamma] \otimes_k V) \cong \text{Hom}_k(V, k[\Gamma] \otimes_k V)$$

Thus this coaction is given by formula

$$V \ni v \mapsto \gamma \otimes v \in k[\Gamma] \otimes_k V$$

□

**Fact 8.3.** *Let  $\Gamma$  be a commutative monoid and let  $\mathbf{D}_\Gamma$  be the corresponding monoid  $k$ -scheme. Suppose that  $V_1, V_2$  are representations of  $\mathbf{D}_\Gamma$  and assume that  $V_1, V_2$  have weights  $\gamma_1, \gamma_2$  with  $\gamma_1 \neq \gamma_2$ . Then*

$$\mathrm{Hom}_{\mathbf{D}_\Gamma}(V_1, V_2) = 0$$

*Proof.* This follows from Fact 8.2. □

Let  $\Gamma$  be a commutative monoid and let  $\mathbf{D}_\Gamma$  be the corresponding monoid  $k$ -scheme. For every representation  $V$  of  $\mathbf{D}_\Gamma$  and fixed  $\gamma$  in  $\Gamma$  define

$$V[\gamma] = \{v \in V \mid d(v) = \gamma \otimes v\}$$

where  $d : V \rightarrow k[\Gamma] \otimes_k V$  is the coaction. Then  $V[\gamma]$  is a subrepresentation of  $V$ . Note that according to Fact 8.2  $V[\gamma]$  is a subrepresentation of  $V$  of weight  $\gamma$ .

**Proposition 8.4.** *Let  $\Gamma$  be a commutative monoid and let  $\mathbf{D}_\Gamma$  be the corresponding monoid  $k$ -scheme. For every representation  $V$  of  $\mathbf{D}_\Gamma$  we have a direct sum*

$$V = \bigoplus_{\gamma \in \Gamma} V[\gamma]$$

*Proof.* Let  $\Delta, \xi$  be the comultiplication and the counit of  $k[\Gamma]$ , respectively. Let  $d : V \rightarrow k[\Gamma] \otimes_k V$  be a coaction. Fix  $v \in V$ . Then we have a unique decomposition  $d(v) = \sum_{\gamma \in \Gamma} \gamma \otimes v_\gamma$ . Then

$$\sum_{\gamma \in \Gamma} \gamma \otimes \gamma \otimes v_\gamma = (\Delta \otimes 1_V)(d(v)) = (1_{k[\Gamma]} \otimes d)(d(v)) = \sum_{\gamma \in \Gamma} \gamma \otimes d(v_\gamma)$$

This implies that  $d(v_\gamma) = \gamma \otimes v_\gamma$  and hence  $v_\gamma \in V[\gamma]$ . On the other hand we have

$$v = \xi(d(v)) = \sum_{\gamma \in \Gamma} v_\gamma$$

Thus

$$v \in \sum_{\gamma \in \Gamma} V[\gamma]$$

Hence

$$V = \sum_{\gamma \in \Gamma} V[\gamma]$$

Moreover, suppose that  $\sum_{\gamma \in \Gamma} v_\gamma = \sum_{\gamma \in \Gamma} v'_\gamma$  for some  $v_\gamma, v'_\gamma \in V[\gamma]$ . Then

$$\sum_{\gamma \in \Gamma} \gamma \otimes v_\gamma = d\left(\sum_{\gamma \in \Gamma} v_\gamma\right) = d\left(\sum_{\gamma \in \Gamma} v'_\gamma\right) = \sum_{\gamma \in \Gamma} \gamma \otimes v'_\gamma$$

and hence  $v_\gamma = v'_\gamma$  for each  $\gamma \in \Gamma$ . This proves the direct decomposition of  $V$  as we claimed. □

**Corollary 8.5.** *Let  $k$  be a field. Suppose that  $\Gamma$  is a commutative monoid and let  $\mathbf{D}_\Gamma$  be the corresponding monoid  $k$ -scheme. Then the category  $\mathbf{Rep}(\mathbf{D}_\Gamma)$  is semisimple. Moreover, each irreducible representation of  $\mathbf{D}_\Gamma$  is isomorphic to one-dimensional representation of weight  $\gamma$  for a unique  $\gamma \in \Gamma$ .*

*Proof.* This is a consequence of Fact 8.3 and Proposition 8.4. □

9. DIAGONALISABLE GROUP  $k$ -SCHEMES

Let  $\Gamma$  be an abstract commutative group. Then in addition to  $k$ -bialgebra structure the  $k$ -algebra  $k[\Gamma]$  admits an antipode map

$$k[\Gamma] \ni \sum_{\gamma \in \Gamma} k_\gamma \cdot \gamma \mapsto \sum_{\gamma \in \Gamma} k_\gamma \cdot \gamma^{-1} \in k[\Gamma]$$

That makes  $k[\Gamma]$  into a commutative Hopf  $k$ -algebra. Thus  $\mathbf{D}_\Gamma$  is a group  $k$ -scheme in this case. The forgetful functor  $|-| : \mathbf{Ab} \rightarrow \mathbf{CMon}$  sending commutative (abelian) group to its underlying commutative monoid admits left adjoint  $(-)_\mathbf{Grp} : \mathbf{CMon} \rightarrow \mathbf{Ab}$ . Hence for every commutative monoid  $\Gamma$  there exists a universal commutative group  $\Gamma_\mathbf{Grp}$  generated by  $\Gamma$ . This is used in the following result.

**Proposition 9.1.** *Let  $\Gamma$  be a commutative monoid. Then the canonical morphism  $\Gamma \rightarrow \Gamma_\mathbf{Grp}$  induces a monomorphism of monoid  $k$ -schemes*

$$\mathbf{D}_{\Gamma_\mathbf{Grp}} \hookrightarrow \mathbf{D}_\Gamma$$

that identifies  $\mathbf{D}_{\Gamma_\mathbf{Grp}}$  with  $(\mathbf{D}_\Gamma)^*$ .

*Proof.* For every  $k$ -algebra we have an isomorphism of groups

$$\mathbf{Mon}(\Gamma, A^\times)^* \cong \mathbf{Mon}(\Gamma, A^*) \cong \mathbf{Mon}(\Gamma_\mathbf{Grp}, A^*) \cong \mathbf{Mon}(\Gamma_\mathbf{Grp}, A^\times)$$

natural in  $A$ . Note that this natural isomorphisms identifies  $\mathfrak{P}_{\mathbf{D}_\Gamma}^*$  with  $\mathfrak{P}_{\mathbf{D}_{\Gamma_\mathbf{Grp}}}$  by morphism induced by the unit  $\Gamma \rightarrow \Gamma_\mathbf{Grp}$  of the adjunction  $|-| \vdash (-)_\mathbf{Grp}$ .  $\square$

**Corollary 9.2.** *Let  $\mathbf{G}$  be a group  $k$ -scheme. Suppose that  $\mathbf{G}$  is isomorphic to  $\mathbf{D}_\Gamma$  as a monoid  $k$ -scheme for some commutative monoid  $\Gamma$ . Then  $\Gamma$  is a group.*

*Proof.* Suppose that  $\mathbf{G} \cong \mathbf{D}_\Gamma$  as a monoid  $k$ -schemes. We derive that  $\mathbf{D}_\Gamma$  is a group  $k$ -scheme. Hence  $\mathbf{D}_{\Gamma_\mathbf{Grp}} \hookrightarrow \mathbf{D}_\Gamma$  is an isomorphism of monoid  $k$ -schemes. This implies that  $\Gamma = \Gamma_\mathbf{Grp}$  and thus  $\Gamma$  is an abstract group.  $\square$

**Definition 9.3.** Let  $\mathbf{G}$  be a group  $k$ -scheme. We say that  $\mathbf{G}$  is *diagonalisable group  $k$ -scheme* if it is diagonalisable as a monoid scheme over  $k$ .

**Example 9.4.** Let  $\mathbb{Z}$  be a commutative group of additive integers. We denote by  $\mathbf{G}_m$  the monoid  $k$ -scheme  $\mathbf{D}_\mathbb{Z}$ . Note that  $\mathbf{G}_m$  represents the group  $k$ -functor

$$\mathbf{Alg}_k \ni A \mapsto A^* \in \mathbf{Ab}$$

We call  $\mathbf{G}_m$  the *multiplicative group over  $k$* .

**Definition 9.5.** Let  $\mathfrak{G}$  be a monoid  $k$ -functor. Then the morphisms  $\mathfrak{G} \rightarrow \mathfrak{P}_{\mathbf{G}_m}$  of monoid  $k$ -functors are called *characters of  $\mathfrak{G}$* . They form a group  $\mathcal{X}(\mathfrak{G})$  called *the group of characters of  $\mathfrak{G}$* .

**Corollary 9.6.** *Suppose that  $k$  is commutative ring such that  $\mathrm{Spec} k$  is connected (i.e.  $k$  has no nontrivial idempotents). Functors*

$$\begin{array}{ccc} \Gamma_1 & & \mathbf{D}_{\Gamma_1} \\ \downarrow g & \dashrightarrow & \uparrow \mathbf{D}_g \\ \Gamma_2 & & \mathbf{D}_{\Gamma_2} \end{array} \qquad \begin{array}{ccc} \mathbf{G}_1 & & \mathcal{X}(\mathbf{G}_1) \\ \downarrow f & \dashrightarrow & \uparrow \mathcal{X}(f) \\ \mathbf{G}_2 & & \mathcal{X}(\mathbf{G}_2) \end{array}$$

induce an equivalence between categories of abstract commutative groups and diagonalisable group schemes over  $k$ .

*Proof.* This is a consequence of Theorem 7.2.  $\square$

### 9.1. Results on linear representations.

**Proposition 9.7.** *Let  $\mathbf{M}$  be an affine monoid  $k$ -scheme and let  $V$  be a representation of  $\mathbf{M}$ . Then for every  $k$ -algebra  $A$  the natural morphism of  $A$ -modules*

$$V^{\mathbf{M}} \otimes_k A \rightarrow (A \otimes_k V)^{\mathbf{M}_A}$$

*is an isomorphism.*

*Proof.* Note that we have a left exact sequence of  $k$ -vector spaces defining invariants

$$0 \longrightarrow V^{\mathbf{M}} \longrightarrow V \xrightarrow{\Delta - p} \Gamma(\mathbf{M}, \mathcal{O}_{\mathbf{M}}) \otimes_k V$$

where  $\Delta : V \rightarrow \Gamma(\mathbf{M}, \mathcal{O}_{\mathbf{M}}) \otimes_k V$  is the coaction and  $p : V \rightarrow \Gamma(\mathbf{M}, \mathcal{O}_{\mathbf{M}}) \otimes_k V$  is the trivial coaction defined by formula  $p(v) = 1 \otimes v$  for every  $v$  in  $V$ . Now tensoring the sequence with  $k$ -algebra  $A$  yields a left exact sequence

$$0 \longrightarrow V^{\mathbf{M}} \otimes_k A \longrightarrow A \otimes_k V \xrightarrow{\Delta_A - p_A} \Gamma(\mathbf{M}_A, \mathcal{O}_{\mathbf{M}_A}) \otimes_A (A \otimes_k V)$$

where  $\Delta_A$  is the coaction on  $A \otimes_k V$  induced by  $\Delta$  and  $p_A$  is the trivial coaction on  $A \otimes_k V$ . This shows that  $V^{\mathbf{M}} \otimes_k A \rightarrow (A \otimes_k V)^{\mathbf{M}_A}$  is an isomorphism.  $\square$

**Proposition 9.8.** *Let  $\mathbf{G}$  be an affine group  $k$ -scheme and let  $V, W$  be representations of  $\mathbf{G}$ . If  $V$  is finite dimensional, then for every  $k$ -algebra  $A$  the canonical morphism*

$$A \otimes_k \mathrm{Hom}_{\mathbf{G}}(V, W) \longrightarrow \mathrm{Hom}_{\mathbf{G}_A}(A \otimes_k V, A \otimes_k W)$$

*is an isomorphism of  $A$ -modules.*

*Proof.* Fix a  $k$ -algebra  $A$ . Since  $V$  is finite dimensional, for every  $k$ -algebra  $B$  there exists an isomorphism  $B \otimes_k \mathrm{Hom}_k(V, W) \rightarrow \mathrm{Hom}_B(B \otimes_k V, B \otimes_k W)$  of  $B$ -modules natural in  $B$ . This implies that  $\mathrm{Hom}_k(V, W)$  is a representation of  $\mathbf{G}$  via the action given by formula

$$(g \cdot f)(v) = g \cdot f(g^{-1} \cdot v)$$

where  $f \in \mathrm{Hom}_B(B \otimes_k V, B \otimes_k W)$ ,  $v \in B \otimes_k V$  and  $g \in \mathfrak{P}_{\mathbf{G}}(B)$ . Similarly  $\mathrm{Hom}_A(A \otimes_k V, A \otimes_k W)$  is a representation of  $\mathbf{G}_A$  and the canonical isomorphism  $A \otimes_k \mathrm{Hom}_k(V, W) \rightarrow \mathrm{Hom}_A(A \otimes_k V, A \otimes_k W)$  of  $A$ -modules is  $\mathbf{G}_A$ -equivariant. Now we apply Proposition 9.7 to derive a chain of isomorphisms

$$\mathrm{Hom}_A(A \otimes_k V, A \otimes_k W)^{\mathbf{G}_A} \cong (A \otimes_k \mathrm{Hom}_k(V, W))^{\mathbf{G}_A} \cong A \otimes_k \mathrm{Hom}_k(V, W)^{\mathbf{G}}$$

of  $A$ -modules. Since we have identifications

$$\mathrm{Hom}_{\mathbf{G}_A}(A \otimes_k V, A \otimes_k W) \cong \mathrm{Hom}_A(A \otimes_k V, A \otimes_k W)^{\mathbf{G}_A}, \quad \mathrm{Hom}_{\mathbf{G}}(V, W) \cong \mathrm{Hom}_k(V, W)^{\mathbf{G}}$$

we deduce the statement.  $\square$

**Proposition 9.9.** *Let  $\mathbf{G}$  be an affine group scheme over  $k$  and let  $V, W$  be  $\mathbf{G}$ -representation such that  $\mathrm{Hom}_{\mathbf{G}}(U, W) = 0$  for every finite dimensional  $\mathbf{G}$ -subrepresentation of  $V$ . Then for every  $k$ -algebra  $A$  we have*

$$\mathrm{Hom}_{\mathbf{G}_A}(A \otimes_k V, A \otimes_k W) = 0$$

*Proof.* Let  $\mathcal{F}$  be a set of all finite dimensional  $\mathbf{G}$ -subrepresentations of  $V$ . Since  $V$  is a  $\mathbf{G}$ -representation and  $\mathbf{G}$  is an affine group  $k$ -scheme, we have

$$V = \operatorname{colim}_{U \in \mathcal{F}} U$$

Fix  $k$ -algebra  $A$  then we have identifications of  $A$ -modules

$$\begin{aligned} \operatorname{Hom}_{\mathbf{G}_A} (A \otimes_k V, A \otimes_k W) &= \operatorname{Hom}_{\mathbf{G}_A} (A \otimes_k \operatorname{colim}_{U \in \mathcal{F}} U, A \otimes_k W) = \\ &= \operatorname{Hom}_{\mathbf{G}_A} (\operatorname{colim}_{U \in \mathcal{F}} A \otimes_k U, A \otimes_k W) = \lim_{U \in \mathcal{F}} \operatorname{Hom}_{\mathbf{G}_A} (A \otimes_k U, A \otimes_k W) = \\ &= \lim_{U \in \mathcal{F}} (A \otimes_k \operatorname{Hom}_{\mathbf{G}} (U, W)) = 0 \end{aligned}$$

where we apply Proposition 9.8. □

## 9.2. Linear algebraic monoids.

**Proposition 9.10.** *Let  $\mathbf{M}$  be a monoid  $k$ -scheme. Then the  $k$ -functor of units  $\mathfrak{P}_{\mathbf{M}}^*$  of  $\mathfrak{P}_{\mathbf{M}}$  is representable by a group  $k$ -scheme  $\mathbf{M}^*$ . Moreover, if  $\mathbf{M}$  is affine and of finite type over  $k$ , then  $\mathbf{M}^*$  is an open subscheme of  $\mathbf{M}$ .*

*Proof.* Note that  $\mathfrak{P}_{\mathbf{M}}^*$  fits into a cartesian square

$$\begin{array}{ccc} \mathfrak{P}_{\mathbf{M}}^* & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \mathfrak{P}_e \\ \mathfrak{P}_{\mathbf{M}} \times \mathfrak{P}_{\mathbf{M}} & \xrightarrow{\mathfrak{P}_m} & \mathfrak{P}_{\mathbf{M}} \end{array}$$

where  $m : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{M}$  is the multiplication and  $e : \operatorname{Spec} k \rightarrow \mathbf{M}$  is the unit. Since the functor

$$\widehat{\operatorname{Sch}}_k \longrightarrow \text{the category of } k\text{-functors}$$

preserves fiber products, we derive that  $\mathfrak{P}_{\mathbf{M}}^*$  is isomorphic to  $\mathfrak{P}_{\mathbf{M}^*}$ , where  $\mathbf{M}^*$  is a  $k$ -scheme defined by the cartesian diagram

$$\begin{array}{ccc} \mathbf{M}^* & \longrightarrow & \operatorname{Spec} k \\ \downarrow & & \downarrow e \\ \mathbf{M} \times \mathbf{M} & \xrightarrow{m} & \mathbf{M} \end{array}$$

Since  $\mathfrak{P}_{\mathbf{M}^*} \cong \mathfrak{P}_{\mathbf{M}}^*$ , we deduce that  $\mathbf{M}^*$  admits a structure of a group  $k$ -scheme.

Now suppose that  $\mathbf{M}$  is affine monoid  $k$ -scheme of finite type over  $k$ . Then there exist a finite dimensional vector space  $V$  over  $k$  and a closed immersion  $i : \mathbf{M} \rightarrow L(V)$  of monoid  $k$ -schemes. □

**Definition 9.11.** Let  $\mathbf{M}$  be an affine monoid  $k$ -scheme. Suppose that the group  $\mathbf{G}$  of units of  $\mathbf{M}$  is an algebraic group over  $k$  and that the open immersion  $\mathbf{G} \hookrightarrow \mathbf{M}$  is schematically dense. Then  $\mathbf{M}$  is a *linear algebraic monoid over  $k$* .

**Definition 9.12.** Let  $\mathbf{M}$  be a linear algebraic monoid over  $k$ . Suppose that the group  $\mathbf{G}$  of units of  $\mathbf{M}$  is (linearly) reductive. Then  $\mathbf{M}$  is a (linearly) *reductive monoid over  $k$* .

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