SMOOTH, UNRAMIFIED AND ÉTALE MORPHISMS

1. Smooth morphisms

Definition 1.1. Let $f : X \to Y$ be a morphism of schemes and let x be a point in X. Suppose that there is an open affine neighborhood U of x in X and an open affine subscheme V of Y such that $f(U) \subseteq V$ and there is an open immersion

$$U \xhookrightarrow{i} \operatorname{Spec} \Gamma(V, \mathcal{O}_Y)[x_1, ..., x_{n+r}]/(f_1, ..., f_r)$$

where $f_1, ..., f_r \in \Gamma(V, \mathcal{O}_Y)[x_1, ..., x_{n+1}]$ are polynomials. In addition assume that *the Jacobian matrix*

$$\frac{\partial(f_1, ..., f_r)}{\partial(x_1, ..., x_{n+r})} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & ... & \frac{\partial f_1}{\partial x_{n+r}} \\ ... & ... & ... \\ \frac{\partial f_r}{\partial x_1} & ... & \frac{\partial f_r}{\partial x_{n+r}} \end{pmatrix}$$

of rank r at x. Then we say that f is *smooth of relative dimension n at x*.

Definition 1.2. Let $f : X \to Y$ be a morphism of schemes. For every $n \in \mathbb{N}$ we define

$$\mathrm{SmoothLocus}_n(f) = \left\{ x \in X \mid f \text{ is smooth of relative dimension } n \text{ at } x \right\}$$

We also define

$$\mathrm{SmoothLocus}(f) = \bigcup_{n \in \mathbb{N}} \mathrm{SmoothLocus}_n(f)$$

and call it *the smooth locus of f*.

Theorem 1.3. *Let $f : X \to Y$ be a morphism of schemes smooth of relative dimension n at some point x of X. Then there exist open affine neighbourhoods U of x in X and V of $f(x)$ in Y such that the following assertions hold.*

(1) *$f(U)$ is a subset of V.*

(2) *The restriction of f to the morphism $U \to V$ is formally smooth and of finite presentation.*

(3) *$\Omega_{X/Y_{|U}}$ is locally free of rank n.*

Proof. Assume that f is smooth of relative dimension n at x. By definition there exists an open affine neighborhood W of x and an open affine subset V of Y such that $f(W) \subseteq V$ and locally on W the morphism f factors as an open immersion

$$i : W \hookrightarrow \operatorname{Spec} \Gamma(V, \mathcal{O}_Y)[x_1, ..., x_{n+r}]/(f_1, ..., f_r)$$

composed with the structural morphism

$$\operatorname{Spec} \Gamma(V, \mathcal{O}_Y)[x_1, ..., x_{n+r}]/(f_1, ..., f_r) \to V$$

in such a way that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & ... & \frac{\partial f_1}{\partial x_{n+r}} \\ ... & ... & ... \\ \frac{\partial f_r}{\partial x_1} & ... & \frac{\partial f_r}{\partial x_{n+r}} \end{pmatrix}$$

is of rank r at x. Now suppose that $j_1,...,j_r$ are indices of columns of the Jacobian matrix evaluated at x that are linearly independent over $k(x)$. Let

$$\delta = \det\left(\left[\frac{\partial f_i}{\partial x_{j_k}}\right]_{1\leq i\leq r, 1\leq k\leq r}\right) \in \Gamma(V, \mathcal{O}_Y)[x_1, ..., x_{n+r}]$$

By assumption $\delta(x) \neq 0$ and hence there exists an open affine neighbourhood U of x in W such that $\delta(z) \neq 0$ for every $z \in U$. Let Q be an open subset of $\operatorname{Spec}\Gamma(V, \mathcal{O}_Y)[x_1, ..., x_{n+r}]$ contained in the nonvanishing set of δ such that $U = Q \cap \operatorname{Spec}\Gamma(V, \mathcal{O}_Y)[x_1, ..., x_{n+r}]/(f_1, ..., f_r)$. Clearly Q is formally smooth over V and a morphism $j : U \hookrightarrow Q$ induced by i is a closed immersion. We have the conormal sequence

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\sigma} j^*\Omega_{Q/V} \longrightarrow \Omega_{U/V} \longrightarrow 0$$

of j. Here $\mathcal{I}$ is a quasi-coherent ideal in $\mathcal{O}_Q$ determining j. Now $j^*\Omega_{Q/V}$ is a locally free sheaf of $\mathcal{O}_U$-modules with basis

$$j^*d(x_{1|Q}), ..., j^*d(x_{n+r|Q})$$

and $\mathcal{I}/\mathcal{I}^2$ is a sheaf of $\mathcal{O}_U$-modules generated by

$$j^*(f_{1|Q}), ..., j^*(f_{r|Q})$$

Let $p : \mathcal{O}_U^r \twoheadrightarrow \mathcal{I}/\mathcal{I}^2$ be the epimorphism of sheaves of $\mathcal{O}_U$-modules determined by sections $j^*(f_{1|Q}),...,j^*(f_{r|Q})$ of $\mathcal{I}/\mathcal{I}^2$. The composition of p with the morphism $\sigma : \mathcal{I}/\mathcal{I}^2 \to j^*\Omega_{Q/V}$ coming from the conormal sequence is given by the transpose of a matrix

$$\mathbf{J} = \left[j^*\left(\frac{\partial f_i}{\partial x_j}_{|Q}\right)\right]_{1\leq i\leq r, 1\leq j\leq n+r}$$

Next we define a matrix $\mathbf{S} = [g_{ij}]_{1\leq i\leq r, 1\leq j\leq n+r}$ of regular functions on U as follows. We set $g_{ij} = 0$ if $j \neq j_k$ for $k = 1,...,r$ and we define the matrix $[g_{ij_k}]_{1\leq i\leq r, 1\leq k\leq r}$ to be the inverse of the matrix

$$\left[j^*\left(\frac{\partial f_i}{\partial x_{j_k}}_{|Q}\right)\right]_{1\leq k\leq r, 1\leq i\leq r}$$

Such an inverse exists according to the fact that $j^*(\delta_{|Q})$ is invertible on U. Note that $\mathbf{S}\cdot\mathbf{J}^T$ is the $r\times r$ identity matrix. This implies that $\sigma\cdot p$ admits a section $s : j^*\Omega_{Q/V} \to \mathcal{O}_U^r$. In particular, p is a monomorphism. Since it is an epimorphism, we derive that p is an isomorphism and thus σ is a monorphism having a section. Therefore, the conormal sequence for j is split exact. Thus $\Omega_{X/Y|U} = \Omega_{U/V}$ is free of rank n and $U \to V$ is formally smooth. Obviously $U \to V$ is of finite presentation. □

Proposition 1.4. *Let A be a commutative ring. Suppose that $f_1, ..., f_s \in A[x_1, ..., x_m]$ are polynomials and $\mathfrak{q}$ is a prime ideal in $A[x_1, ..., x_m]$ such that $f_1, ..., f_s \in \mathfrak{q}$. Let $\mathfrak{p} = A \cap \mathfrak{q}$ be a prime ideal of A. Assume that the rank of the Jacobian matrix*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_m} \end{pmatrix}$$

is r at $\mathfrak{q}$. Let $i_1, ..., i_r$ be numbers of rows of that matrix that are linearly independent at $\mathfrak{q}$ and consider the surjective morphism

$$\pi : A[x_1, ..., x_m]/(f_{i_1}, ..., f_{i_r}) \twoheadrightarrow A[x_1, ..., x_m]/(f_1, ..., f_s)$$

Then the following assertions hold.

(1) *If $\operatorname{Spec} A[x_1, ..., x_m]_{\mathfrak{q}}/(f_1, ..., f_s)_{\mathfrak{q}} \to \operatorname{Spec} A_{\mathfrak{p}}$ is formally smooth, then $\pi_{\mathfrak{q}}$ is an isomorphism.*

(2) *If $\pi_{\mathfrak{q}}$ is an isomorphism, then* $\operatorname{Spec} A[x_1,...,x_m]/(f_1,...,f_s) \to \operatorname{Spec} A$ *is smooth of relative dimension $m-r$ at $\mathfrak{q}$ reduced modulo $f_1,...,f_s$.*

Proof. For convenience in the proof we write $B = A[x_1,...,x_m]/(f_1,...,f_s)$, $C = A[x_1,...,x_m]/(f_{i_1},...,f_{i_r})$, $\mathfrak{b} = (f_1,...,f_s)$, $\mathfrak{c} = (f_{i_1},...,f_{i_r})$.
Assume that $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is formally smooth. Note that we have a commutative diagram

$$\begin{tikzcd}
\mathfrak{b}_{\mathfrak{q}}/\mathfrak{b}_{\mathfrak{q}}^2 \arrow[r] & B_{\mathfrak{q}} \otimes_{A[x_1,...,x_m]_{\mathfrak{q}}} \Omega_{A[x_1,...,x_m]_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] & \Omega_{B_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] & 0 \\
\mathfrak{c}_{\mathfrak{q}}/\mathfrak{c}_{\mathfrak{q}}^2 \arrow[r] \arrow[u] & C_{\mathfrak{q}} \otimes_{A[x_1,...,x_m]_{\mathfrak{q}}} \Omega_{A[x_1,...,x_m]_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] \arrow[u] & \Omega_{C_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

in which rows are conormal sequences of $A[x_1,...,x_m]_{\mathfrak{q}} \twoheadrightarrow B_{\mathfrak{q}}$ and $A[x_1,...,x_m]_{\mathfrak{q}} \twoheadrightarrow C_{\mathfrak{q}}$. Observe that the bottom row is split exact as $\operatorname{Spec} C \to \operatorname{Spec} A$ is smooth of relative dimension $m-r$ at $\mathfrak{q}$. Similarly by formal smoothness of $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ the top row is split exact. Hence after tensoring with $k(\mathfrak{q})$ we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & k(\mathfrak{q}) \otimes_{A[x_1,...,x_m]_{\mathfrak{q}}} \mathfrak{b}_{\mathfrak{q}}/\mathfrak{b}_{\mathfrak{q}}^2 \arrow[r] & k(\mathfrak{q}) \otimes_{A[x_1,...,x_m]_{\mathfrak{q}}} \Omega_{A[x_1,...,x_m]_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] & k(\mathfrak{q}) \otimes_{B_{\mathfrak{q}}} \Omega_{B_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] & 0 \\
0 \arrow[r] & k(\mathfrak{q}) \otimes_{A[x_1,...,x_m]_{\mathfrak{q}}} \mathfrak{c}_{\mathfrak{q}}/\mathfrak{c}_{\mathfrak{q}}^2 \arrow[r] \arrow[u] & k(\mathfrak{q}) \otimes_{A[x_1,...,x_m]_{\mathfrak{q}}} \Omega_{A[x_1,...,x_m]_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] \arrow[u, "="] & k(\mathfrak{q}) \otimes_{C_{\mathfrak{q}}} \Omega_{C_{\mathfrak{q}}/A_{\mathfrak{p}}} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

According to the choice of $f_{i_1},...,f_{i_r}$, we derive that

$$\dim_{k(\mathfrak{q})}\left(k(\mathfrak{q}) \otimes_{B_{\mathfrak{q}}} \Omega_{B_{\mathfrak{q}}/A_{\mathfrak{p}}}\right) = m - r = \dim_{k(\mathfrak{q})}\left(k(\mathfrak{q}) \otimes_{C_{\mathfrak{q}}} \Omega_{C_{\mathfrak{q}}/A_{\mathfrak{p}}}\right)$$

In particular, the rightmost vertical morphism in the diagram is an epimorphism of vector spaces over $k(\mathfrak{q})$ of the same finite dimension. Thus it is an isomorphism. This implies that the leftmost morphism in the diagram is an isomorphism. Thus by virtue of Nakayama lemma $\mathfrak{c}_{\mathfrak{q}}/\mathfrak{c}_{\mathfrak{q}}^2 \to \mathfrak{b}_{\mathfrak{q}}/\mathfrak{b}_{\mathfrak{q}}^2$ is an epimorphism. Using Nakayama lemma once more we deduce that $\mathfrak{c}_{\mathfrak{q}} \hookrightarrow \mathfrak{b}_{\mathfrak{q}}$ is an epimorphism and thus $\mathfrak{c}_{\mathfrak{q}} = \mathfrak{b}_{\mathfrak{q}}$. Hence $\pi_{\mathfrak{q}}$ is an isomorphism and the proof of **(1)** is completed.
On the other hand if $\pi_{\mathfrak{q}}$ is an isomorphism, then $\ker(\pi)$ is a finitely generated ideal in C that vanishes at $\mathfrak{q}$ modulo $f_{i_1},...,f_{i_r}$. Hence it vanishes on some open neighborhood of $\mathfrak{q}$ modulo $f_{i_1},...,f_{i_r}$ in $\operatorname{Spec} C$ and thus $\operatorname{Spec} \pi$ is an isomorphism on that neighborhood. Thus $\operatorname{Spec} B \to \operatorname{Spec} A$ is smooth of relative dimension $m-r$ at $\mathfrak{q}$ reduced modulo $f_1,...,f_s$ and **(2)** is proved. □

Corollary 1.5. *Let $f : X \to Y$ be a morphism of schemes and let x be a point in X. Then the following conditions are equivalent.*

(i) *f is smooth at x.*

(ii) *There exists an open neighborhood U of x in X such that $f_{|U}$ is formally smooth and locally of finite presentation*

(iii) *There exists an open neighborhood U of x in X such that $f_{|U}$ is locally of finite presentation and the local morphism $f^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is formally smooth.*

Proof. The implication **(i)** ⇒ **(ii)** follows from Theorem 1.3.
Suppose that **(ii)** holds. We want to prove **(iii)**. Since this is a local assertion, we may assume that there exists a closed immersion $i : U \to \mathbb{A}^m_Y$. Let $\mathcal{I}$ be a quasi-coherent ideal determining i. Then the conormal sequence

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow i^*\Omega_{\mathbb{A}^m_Y/Y} \longrightarrow \Omega_{U/Y} \longrightarrow 0$$

is locally split exact. Thus after localizing at x we derive a split exact sequence

$$\mathcal{I}_x/\mathcal{I}_x^2 \longrightarrow \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{\mathbb{A}^m_Y,i(x)}} \Omega_{\mathcal{O}_{\mathbb{A}^m_Y,i(x)}/\mathcal{O}_{Y,f(x)}} \longrightarrow \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}} \longrightarrow 0$$

which implies that $f^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is formally smooth.
Finally **(iii)** $\Rightarrow$ **(i)** is an easy consequence of Proposition 1.4. □

Corollary 1.6. *Let $f : X \to Y$ be a morphism of schemes and let $n \in \mathbb{N}$ be a natural number. Then* $\mathrm{SmoothLocus}_n(f)$ *is open subset of X and the sheaf $\Omega_{X/Y}$ is locally free of rank n on* $\mathrm{SmoothLocus}_n(f)$.

Proof. It follows from Corollary 1.5 that $\mathrm{SmoothLocus}(f)$ is open subset of X. From Theorem 1.3 we derive that $\Omega_{X/Y|\mathrm{SmoothLocus}(f)}$ is locally free of finite rank and

$$\mathrm{SmoothLocus}_n(f) = \left\{x \in \mathrm{SmoothLocus}(f) \,\middle|\, \mathrm{rank}\left(\Omega_{X/Y_x}\right) = n\right\}$$

Thus $\mathrm{SmoothLocus}_n(f)$ is open subset of X and $\Omega_{X/Y}$ is locally free of rank n on $\mathrm{SmoothLocus}_n(f)$. □

Proposition 1.7. *The following assertions hold.*

(1) *Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of schemes. Consider the cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

If f is smooth of relative dimension n at $x \in X$ and $x' \in X'$ is a point such that $g'(x') = x$, then f' is smooth of relative dimension n at x'.

(2) *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If f is smooth of relative dimension n at $x \in X$ and g is smooth of relative dimension m at $f(x)$, then $g \cdot f$ is smooth of relative dimension $n + m$ at x.*

Proof. Observe that classes of formally smooth and locally of finite presentation morphisms are closed under base change and composition. Therefore, by Corollary 1.5 in order to prove assertions it is enough to check relative dimensions. For this we use Corollary 1.6 and compute ranks of sheaves of differentials. In case **(1)** observe that $g'^*\Omega_{X/Y} \cong \Omega_{X'/Y'}$ and hence if $\Omega_{X/Y}$ has rank n at point x, then $\Omega_{X'/Y'}$ has rank n at x'. For **(2)** consider an exact sequence

$$f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

which splits locally in neighborhood of x. Since $f^*\Omega_{Y/Z}$ has rank m at x and $\Omega_{X/Y}$ has rank n at x, we derive that $\Omega_{X/Z}$ has rank $n + m$ at x. □

Theorem 1.8. *Let $f : X \to Y$ be a morphism locally of finite presentation. Suppose that $g : Y' \to Y$ is a morphism of schemes. Assume that g is flat at some point y' of Y'. Consider the cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Let x be a point of X and x' be a point of X' lying over x. Then the following assertions are equivalent.

(i) *f is smooth at x.*

(ii) *f' is smooth at x'.*

Proof. By Proposition 1.7 it suffices to prove **(ii)** $\Rightarrow$ **(i)**. Let $y = f(x)$. Clearly both f and f' are locally of finite presentation. The question is local so we may assume that schemes X, Y, Y' are affine. Consider closed immersion $i : X \hookrightarrow \mathbb{A}^m_Y$ determined by the ideal $\mathcal{I}$. Let $i' : X' \hookrightarrow \mathbb{A}^m_{Y'}$ be the base change of i along $g : Y' \to Y$ and denote its ideal by $\mathcal{I}'$. Then the conormal sequence

$$(\mathcal{I}'/\mathcal{I}'^2)_{x'} \longrightarrow (i'^*\Omega_{\mathbb{A}^m_{Y'}/Y'})_{x'} \longrightarrow (\Omega_{X'/Y'})_{x'} \longrightarrow 0$$

of i' at x' is a split short exact sequence. Since it is canonically isomorphic to the sequence

$$\mathcal{O}_{X',x'} \otimes_{\mathcal{O}_{X,x}} (\mathcal{I}/\mathcal{I}^2)_x \longrightarrow \mathcal{O}_{X',x'} \otimes_{\mathcal{O}_{X,x}} (i^*\Omega_{\mathbb{A}^m_Y/Y})_x \longrightarrow \mathcal{O}_{X',x'} \otimes_{\mathcal{O}_{X,x}} (\Omega_{X/Y})_x \longrightarrow 0$$

i.e the conormal sequence for i at x tensored with $\mathcal{O}_{X',x'}$ over $\mathcal{O}_{X,x}$. We utilize the assumption that $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is faithfully flat to deduce that the conormal sequence

$$(\mathcal{I}/\mathcal{I}^2)_x \longrightarrow (i^*\Omega_{\mathbb{A}^m_Y/Y})_x \longrightarrow (\Omega_{X/Y})_x \longrightarrow 0$$

is short exact and $\mathcal{O}_{X,x}$-module $(\Omega_{X/Y})_x$ is flat. Since $(\Omega_{X/Y})_x$ is finitely presented, we derive that it is free and hence the conormal sequence

$$(\mathcal{I}/\mathcal{I}^2)_x \longrightarrow (i^*\Omega_{\mathbb{A}^m_Y/Y})_x \longrightarrow (\Omega_{X/Y})_x \longrightarrow 0$$

is a split short exact sequence. Therefore, $f^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is formally smooth and, since f is locally of finite presentation, we derive by Corollary 1.5 that f is smooth at x. $\square$

2. Smoothness for schemes over a field

In this section we use the following notation. Let X be a scheme over a field k. For every field extension K of k we denote by X_K the K-scheme $\operatorname{Spec} K \times_{\operatorname{Spec} k} X$.

Definition 2.1. Let X be a scheme over a field k and let x be a point of X. Then X is *geometrically regular at x* if for every field extension K of k and every point $\overline{x}$ in X_K lying over x the scheme X_K is regular at $\overline{x}$.

Theorem 2.2. *Let X be a scheme locally of finite type over a field k and let x be a point of X. Then the inequality*

$$\dim_x(X) \leq \dim_{k(x)}\big(k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/k}\big)$$

holds and the following assertions are equivalent.

(i) *X is smooth at x.*

(ii) *X is geometrically regular over k at x.*

(iii) *For some perfect extension K of k and some point $\overline{x} \in X_K$ lying over x the local ring $\mathcal{O}_{X_K,\overline{x}}$ is regular.*

(iv) $\dim_x(X) = \dim_{k(x)}(k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/k})$

We prove the theorem in a series of lemmas.

Lemma 2.2.1. *Let K be a perfect field and let X be a scheme locally of finite type over K. Suppose that x is a point X. Then*

$$\dim_x(X) \leq \dim_{k(x)}\big(k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/K}\big)$$

and the equality holds if and only if $\mathcal{O}_{X,x}$ is regular.

Proof of the lemma. Consider the conormal sequence

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/k} \longrightarrow \Omega_{k(x)/K} \longrightarrow 0$$

of the closed immersion $\operatorname{Spec} k(x) \hookrightarrow \operatorname{Spec} \mathcal{O}_{X,x}$ of K-schemes. Since $k(x)$ is formally smooth over K by [Monygham, 2021, Corollary 6.4] and K is perfect, we derive that the sequence above is short exact. Hence

$$\dim_x(X) = \dim\big(\mathcal{O}_{X,x}\big) + \operatorname{tr}_K\big(k(x)\big) \leq \dim_{k(x)}\big(\mathfrak{m}_x/\mathfrak{m}_x^2\big) + \operatorname{tr}_K\big(k(x)\big) =$$

$$= \dim_{k(x)}\big(\mathfrak{m}_x/\mathfrak{m}_x^2\big) + \dim_{k(x)}\big(\Omega_{k(x)/K}\big) = \dim_{k(x)}\big(k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/K}\big)$$

and the equality holds if and only if $\dim\big(\mathcal{O}_{X,x}\big) = \dim_{k(x)}\big(\mathfrak{m}_x/\mathfrak{m}_x^2\big)$. This equality is equivalent to regularity of $\mathcal{O}_{X,x}$. □

Lemma 2.2.2. *Let K be a perfect field and let X be a scheme locally of finite type over K. Suppose that x is a point X. If $\mathcal{O}_{X,x}$ is regular, then X is smooth at x.*

Proof of the lemma. Since the assertions are local we may assume that there exists a closed immersion $i : X \to \mathbb{A}^m_K$ given by a quasi-coherent ideal $\mathcal{I}$ generated by polynomials $f_1, ..., f_s \in K[x_1, ..., x_m]$. Next suppose that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_s}{\partial x_1} & \cdots & \frac{\partial f_s}{\partial x_m} \end{pmatrix}$$

is of rank r at x. Assume that $i_1,...,i_r$ are numbers of rows that are linearly independent over $k(x)$. Let $j : Z \hookrightarrow \mathbb{A}^m_K$ be a closed subscheme of $\mathbb{A}^m_K$ determined by the ideal $\mathcal{J}$ generated by $f_{i_1},...,f_{i_r}$. Then there exists a closed immersion $u : X \hookrightarrow Z$ such that $i \cdot u = j$. Consider a commutative diagram

$$\begin{tikzcd}
\left(u^*\left(\mathcal{J}/\mathcal{J}^2\right)\right)_x \arrow[r] \arrow[d] & (u^* j^* \Omega_{\mathbb{A}^m_K/K})_x \arrow[r] \arrow[d, "="] & (u^*\Omega_{Z/K})_x \arrow[r] \arrow[d] & 0 \\
(\mathcal{I}/\mathcal{I}^2)_x \arrow[r] & (i^*\Omega_{\mathbb{A}^m_K/K})_x \arrow[r] & (\Omega_{X/K})_x \arrow[r] & 0
\end{tikzcd}$$

In the diagram rows are induced by conormal sequences for j and i. The leftmost vetical arrow is induced by the inclusion $\mathcal{J} \hookrightarrow \mathcal{I}$ and the rightmost vertical arrow is induced by the cotangent morphism $u^*\Omega_{Z/K} \to \Omega_{X/K}$. It follows from the choice of $i_1,...,i_r$ that morphisms

$$\left(u^*\left(\mathcal{J}/\mathcal{J}^2\right)\right)_x \to (u^*j^*\Omega_{\mathbb{A}^m_K/K})_x,\ (\mathcal{I}/\mathcal{I}^2)_x \to (i^*\Omega_{\mathbb{A}^m_K/K})_x$$

have the same image in $(u^*j^*\Omega_{\mathbb{A}^m_K/K})_x = (i^*\Omega_{\mathbb{A}^m_K/K})_x$. Thus the morphism $\left(u^*\Omega_{Z/K}\right)_x \to \left(\Omega_{X/K}\right)_x$ induced by the cotangent morphism of u is an isomorphism. This implies that

$$\dim_{k(x)}\left(k(x)\otimes_{\mathcal{O}_{Z,x}}\Omega_{\mathcal{O}_{Z,x}/K}\right) = \dim_{k(x)}\left(k(x)\otimes_{\mathcal{O}_{X,x}}\Omega_{\mathcal{O}_{X,x}/K}\right)$$

By definition Z is smooth at x. Hence by [Monygham, 2021, Theorem 6.3] and Corollary 1.5 we deduce that Z is regular at x. By assumption X is regular at x. Thus by Lemma 2.2.1 we derive that

$$\dim_x(Z) = \dim_{k(x)}\left(k(x)\otimes_{\mathcal{O}_{Z,x}}\Omega_{\mathcal{O}_{Z,x}/K}\right) = \dim_{k(x)}\left(k(x)\otimes_{\mathcal{O}_{X,x}}\Omega_{\mathcal{O}_{X,x}/K}\right) = \dim_x(X)$$

Hence

$$\mathrm{tr}_K(k(x)) + \dim\left(\mathcal{O}_{Z,x}\right) = \dim_x(Z) = \dim_x(X) = \mathrm{tr}_K(k(x)) + \dim\left(\mathcal{O}_{X,x}\right)$$

We conclude that $\dim\left(\mathcal{O}_{Z,x}\right) = \dim\left(\mathcal{O}_{X,x}\right)$. This implies that $u^\# : \mathcal{O}_{Z,x} \twoheadrightarrow \mathcal{O}_{X,x}$ is a surjective morphism of regular rings of the same dimension. Hence it is an isomorphism. By Proposition 1.4 we deduce that X is smooth at x. □

Proof of the theorem. The implication **(i)** $\Rightarrow$ **(ii)** follows from the fact that smoothness is closed under base change (Proposition 1.7), the fact that smoothness implies formal smoothness (Corollary 1.5) and [Monygham, 2021, Theorem 6.3], which states that formally smooth noetherian local k-algebras are regular. The implication **(ii)** $\Rightarrow$ **(iii)** is obvious. Suppose now that **(iii)** holds. Then Lemma 2.2.2 implies that X_K is smooth at $\overline{x}$. By Theorem 1.8 we deduce that X is smooth at x. Hence also **(iii)** $\Rightarrow$ **(i)**. This completes the proof that **(i)**, **(ii)**, **(iii)** are equivalent.
Now we prove the inequality

$$\dim_x(X) \leq \dim_{k(x)}\left(k(x)\otimes_{\mathcal{O}_{X,x}}\Omega_{\mathcal{O}_{X,x}/K}\right)$$

and the fact that **(iv)** is equivalent with **(i)**, **(ii)**, **(iii)**. Suppose that $\mathrm{char}(k) = p$. If $p > 0$, then consider $K = k^{\frac{1}{p^\infty}}$ i.e. the perfect closure of k. If $p = 0$, then pick $K = k$. Let $\pi : X_K \to X$ be the canonical projection. The morphism $\mathrm{Spec}\,K \to \mathrm{Spec}\,k$ is surjective, universally injective and integral. Since π is a base change of $\mathrm{Spec}\,K \to \mathrm{Spec}\,k$, we derive that π is also surjective, universally injective and integral. Hence π is a homeomorphism. Thus there exists a unique point $\overline{x}$ in X_K lying over x and

$$\dim_{\overline{x}}\left(X_K\right) = \dim_x(X)$$

Moreover, we have

$$k(\overline{x})\otimes_{\mathcal{O}_{X_K,\overline{x}}}\Omega_{\mathcal{O}_{X_K,\overline{x}}/K} \cong k(\overline{x})\otimes_{\mathcal{O}_{X_K,\overline{x}}}\left(\Omega_{X_K/K}\right)_{\overline{x}} \cong k(\overline{x})\otimes_{\mathcal{O}_{X_K,\overline{x}}}\left(\pi^*\Omega_{X/k}\right)_{\overline{x}} \cong$$

$$\cong k(\overline{x})\otimes_{\mathcal{O}_{X_K,\overline{x}}}\mathcal{O}_{X_K,\overline{x}}\otimes_{\mathcal{O}_{X,x}}\left(\Omega_{X/k}\right)_x = k(\overline{x})\otimes_{k(x)}\left(k(x)\otimes_{\mathcal{O}_{X,x}}\Omega_{\mathcal{O}_{X,x}/k}\right)$$

Thus

$$\dim_{k(\overline{x})}\left(k(\overline{x})\otimes_{\mathcal{O}_{X_K,\overline{x}}}\Omega_{\mathcal{O}_{X_K,\overline{x}}/K}\right) = \dim_{k(x)}\left(k(x)\otimes_{\mathcal{O}_{X,x}}\Omega_{\mathcal{O}_{X,x}/k}\right)$$

Now according to Lemma 2.2.1 we have

$$\dim_x(X) = \dim_{\overline{x}}\left(X_K\right) \leq \dim_{k(\overline{x})}\left(k(\overline{x})\otimes_{\mathcal{O}_{X_K,\overline{x}}}\Omega_{\mathcal{O}_{X_K,\overline{x}}/K}\right) = \dim_{k(x)}\left(k(x)\otimes_{\mathcal{O}_{X,x}}\Omega_{\mathcal{O}_{X,x}/k}\right)$$

and the equality holds if and only if $\mathcal{O}_{X_K,\overline{x}}$ is regular. Thus by Lemma 2.2.2 the equality holds if and only if X_K is smooth at $\overline{x}$, but this according to Proposition 1.7 and Theorem 1.8 is equivalent with smoothness of X at x. The proof is complete. □

Corollary 2.3. *Let X be a scheme locally of finite type over a field k. Suppose that X is smooth of relative dimension n at some point x. Then $\dim_x(X) = n$.*

Proof. According to Corollary 1.6 and Theorem 2.2 we have

$$n = \dim_{k(x)}\big(k(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/k}\big) = \dim_x(X)$$

□

3. Smooth morphisms are flat families of smooth schemes over fields

Theorem 3.1. *Let $f : X \to Y$ be a morphism of schemes and let x be a point in X. Suppose that there exists a neighborhood U of x such that $f_{|U}$ is locally of finite presentation. Then the following assertions are equivalent.*

- **(i)** *f is smooth at x.*
- **(ii)** *f is flat at x and the fiber $X_{f(x)}$ is smooth at x.*

Proof. We will prove that **(i)** ⇒ **(ii)**. Since smoothness is stable under base change, we derive that fiber of $X_{f(x)}$ is smooth. It suffices to prove that f is flat at x. This question is local on base and domain. Hence we may assume that Y is affine and X is a closed subscheme of $\mathbb{A}^{n+r}_Y$ determined by ideal generated by polynomials $f_1,...,f_r \in \Gamma(Y,\mathcal{O}_Y)[x_1,...,x_{n+r}]$ such that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_{n+r}} \\ \dots & \dots & \dots \\ \frac{\partial f_r}{\partial x_1} & \dots & \frac{\partial f_r}{\partial x_{n+r}} \end{pmatrix}$$

of rank r at x. Consider the set of all nonzero coefficients of polynomials $f_1,...,f_r$ and define $A \subseteq \Gamma(Y,\mathcal{O}_Y)$ as the $\mathbb{Z}$-subalgebra generated by all these coefficients. We define $Y = \operatorname{Spec} A$. Note that polynomials $f_1,...,f_r$ have coefficients in A and hence we can define X_0 as a closed subscheme of $\mathbb{A}^{n+r}_{Y_0}$ determined by the ideal generated by $f_1,...,f_r$. We have cartesian square

$$\begin{tikzcd} X \arrow[r, "g'"] \arrow[d, "f"'] & X_0 \arrow[d, "f_0"] \\ Y \arrow[r, "g"'] & Y_0 \end{tikzcd}$$

where $f_0 : X_0 \to Y_0$ is the canonical morphism. Next we define $x_0 = g'(x)$ in X_0. It is straightforward to verify that rank of the matrix

$$\frac{\partial(f_1,...,f_r)}{\partial(x_1,...,x_{n+r})} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_{n+r}} \\ \dots & \dots & \dots \\ \frac{\partial f_r}{\partial x_1} & \dots & \frac{\partial f_r}{\partial x_{n+r}} \end{pmatrix}$$

in x_0 is the same as its rank in x. Hence f_0 is smooth at x_0. It suffices to prove that f_0 is flat at x_0. Note that Y_0 is noetherian. Therefore, without loss of generality we may assume that Y is noetherian. Let $f(x) = y$ and $h : \mathbb{A}^{n+r}_{k(y)} \hookrightarrow \mathbb{A}^{n+r}_Y$ be the canonical monomorphism. Let $\overline{x} = h(x)$ and $\overline{f}_1 = h^{\#}(f_1),...,\overline{f}_r = h^{\#}(f_r)$. Note that the closed subscheme $V(f_1,...,f_i)$ of $\mathbb{A}^{n+r}_Y$ determined by the ideal generated by $f_1,...,f_i$ is smooth of relative dimension $n+r-i$ at x and consequently the closed subscheme $V(\overline{f}_1,...,\overline{f}_i)$ of $\mathbb{A}^{n+r}_{k(y)}$ determined by the ideal generated by $\overline{f}_1,...,\overline{f}_i$ is smooth of relative dimension $n+r-i$ at $\overline{x}$. According to Theorem 2.2 local ring $\mathcal{O}_{V(\overline{f}_1,...,\overline{f}_i),\overline{x}}$ is regular. This shows that $\overline{f}_1,...,\overline{f}_r$ is a regular sequence in $\mathcal{O}_{\mathbb{A}^{n+r}_{k(y)},\overline{x}}$. Define finitely generated $\mathcal{O}_{\mathbb{A}^{n+r}_Y,x}$-modules

$$C_i = \begin{cases} \mathcal{O}_{\mathbb{A}^{n+r}_Y,x} & \text{if } i = 0 \\ \mathcal{O}_{V(f_1,...,f_i),x} & \text{if } i > 0 \end{cases}$$

for $0 \leq i \leq r$. Then C_0 is flat $\mathcal{O}_{\mathbb{A}^{n+r}_Y,x}$-module and for every $0 \leq i \leq r-1$ multiplication by the germ of f_{i+1} on C_i is a monomorphism after reduction to $k(x)$. By [Monygham, 2018, Corollary 3.5] if M, N are finitely generated $\mathcal{O}_{\mathbb{A}^{n+r}_Y,x}$-modules, N is a flat $\mathcal{O}_{\mathbb{A}^{n+r}_Y,x}$-module and $\phi : M \to N$ is $\mathcal{O}_{\mathbb{A}^{n+r}_Y,x}$-module morphism such that $\phi \otimes_{\mathcal{O}_{\mathbb{A}^{n+r}_Y,x}} 1_{k(x)}$ is a monomorphism, then ϕ is a monomorphism and $N/\phi(M)$ is flat $\mathcal{O}_{\mathbb{A}^{n+r}_Y,x}$-module. Applying this fact we deduce that C_i are flat $\mathcal{O}_{\mathbb{A}^{n+r}_Y,x}$-modules for $0 \leq i \leq r$ and the germ of f_{i+1} is a nonzero divisor in C_i for $0 \leq i \leq r-1$. In particular, $C_r = \mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{\mathbb{A}^{n+r}_Y,x}$-module. Thus f is flat at x.

Now we prove that **(ii)** $\Rightarrow$ **(i)**. We assume **(ii)** and want to deduce **(i)**. Again the question is local both on target and domain. As above we may assume that Y is affine and X is a closed subscheme of $\mathbb{A}^m_Y$ determined by ideal generated by polynomials $f_1, ..., f_s \in \Gamma(Y, \mathcal{O}_Y)[x_1, ..., x_m]$. Denote $y = f(x)$. Next let r be the rank of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_m} \end{pmatrix}$$

at x. Let $i_1, ..., i_r$ be numbers of rows of that matrix which are linearly independent over $k(x)$. Next we define Z as a closed subscheme of $\mathbb{A}^m_Y$ determined by the ideal generated by $f_{i_1}, ..., f_{i_r}$. In particular, $Z \to Y$ is smooth at x. We also have the canonical closed immersion $u : X \hookrightarrow Z$. We show that $u^\# : \mathcal{O}_{Z,x} \twoheadrightarrow \mathcal{O}_{X,x}$ is an isomorphism. By assumption X_y is smooth at x. Thus Proposition 1.4 implies that $u^\# \otimes_{\mathcal{O}_{Y,y}} 1_{k(y)}$ is an isomorphism. Since $\mathcal{O}_{X,x}$ is flat $\mathcal{O}_{Y,y}$-module, the exact sequence

$$0 \longrightarrow \ker(u^\#) \longrightarrow \mathcal{O}_{Z,x} \xrightarrow{u^\#} \mathcal{O}_{X,x} \longrightarrow 0$$

remains exact after tensoring with $k(y)$ over $\mathcal{O}_{Y,y}$. Thus $\ker(u^\#) \otimes_{\mathcal{O}_{Y,y}} k(y) = 0$ which means that

$$\mathfrak{m}_y \cdot \ker(u^\#) = \ker(u^\#)$$

Note that $\ker(u^\#)$ is generated by the germs of f_j at x for $j \neq i_1, ..., i_r$. Hence $\ker(u^\#)$ is a finitely generated module over $\mathcal{O}_{Z,x}$. Moreover, $\mathfrak{m}_y \cdot \mathcal{O}_{Z,x}$ is contained in the Jacobson radical of $\mathcal{O}_{Z,x}$. We infer by Nakayama lemma that $\ker(u^\#) = 0$. Thus $u^\#$ is an isomorphism of $\mathcal{O}_{Y,y}$-algebras. By Proposition 1.4 we derive that $f : X \to Y$ is smooth at x. □

4. Formal criterion for smoothness

Let A be a commutative ring and let $A[[x_1, ..., x_n]]$ be a ring of formal power series with coefficients in A. For every power series $f \in A[[x_1, ..., x_n]]$ we denote by $f(o)$ its constant coefficient. The result below is a formal version of the inverse function theorem.

Theorem 4.1. *Let A be a ring and let $f_1, ..., f_n \in A[[x_1, ..., x_n]]$ be formal power series. Assume that*

(1) *$f_i(o)$ is zero for every $1 \leq i \leq n$.*

(2) *The determinant*

$$\det\left[\frac{\partial f_i}{\partial x_j}(o)\right]_{1 \leq i,j \leq n}$$

is invertible element of A.

Then the morphism $A[[x_1,...,x_n]] \to A[[x_1,...,x_n]]$ of A-algebras given by

$$x_i \mapsto f_i$$

for every $1 \leq i \leq n$ is an isomorphism.

Proof. Denote by $g : A[[x_1,...,x_n]] \to A[[x_1,...,x_n]]$ the morphism of A-algebras given by $g(x_i) = f_i$ for $1 \leq i \leq n$. Let $I \subseteq A[[x_1,...,x_n]]$ be an ideal generated by $x_1,...,x_n$. Note that there is a canonical isomorphism

$$\mathrm{gr}_I(A[[x_1,...,x_n]]) \cong A[X_1,...,X_n]$$

Hence

$$\mathrm{gr}_I(g) : \mathrm{gr}_I(A[[x_1,...,x_n]]) \to \mathrm{gr}_I(A[[x_1,...,x_n]])$$

is a graded endomorphism of a polynomial A-algebra $A[X_1,...,X_n]$ induced by an A-linear map given by

$$X_i \mapsto \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j}(o)X_j$$

for $1 \leq i \leq n$. Since this A-linear map is induced by matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(o) & ... & \frac{\partial f_1}{\partial x_n}(o) \\ ... & ... & ... \\ \frac{\partial f_n}{\partial x_1}(o) & ... & \frac{\partial f_n}{\partial x_n}(o) \end{pmatrix}$$

which is invertible, we deduce that $\mathrm{gr}_I(g)$ is an isomorphism. There are commutative diagrams with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I^{k-1}/I^k \arrow[r] \arrow[d, "\mathrm{gr}_I(g)_{k-1}"] & A[[x_1,...,x_n]]/I^k \arrow[r] \arrow[d, "g_k"] & A[[x_1,...,x_n]]/I^{k-1} \arrow[r] \arrow[d, "g_{k-1}"] & 0 \\ 0 \arrow[r] & I^{k-1}/I^k \arrow[r] & A[[x_1,...,x_n]]/I^k \arrow[r] & A[[x_1,...,x_n]]/I^{k-1} \arrow[r] & 0 \end{tikzcd}$$

in which g_k and g_{k-1} are induced by g for $k \geq 2$. Moreover, observe that $g_1 : A \to A$ is an isomorphism as g is morphism of A-algebras. Thus the fact that g is an isomorphism follows by induction on k and the equality $g = \lim_{k \in \mathbb{N}} g_k$. □

Theorem 4.2. *Let $f : X \to Y$ be a morphism and $x \in X$ be a point. Consider the following statements.*

- **(i)** *f induces an isomorphism $k(x) \cong k(f(x))$ and it is smooth of relative dimension n at x.*
- **(ii)** *The morphism $f^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induces an isomorphism of algebras*

$$\widehat{\mathcal{O}_{X,x}} \cong \widehat{\mathcal{O}_{Y,f(x)}}[[x_1,...,x_n]]$$

Then **(i)** $\Rightarrow$ **(ii)**. *If there exists a neighborhood U of x such that $f_{|U}$ is locally of finite type and Y is locally noetherian, then also* **(ii)** $\Rightarrow$ **(i)**.

Proof. Suppose that f is smooth of relative dimension n at x and induces an isomorphism $k(x) \cong k(f(x))$. By taking base change to $\mathrm{Spec}\,\mathcal{O}_{Y,f(x)}$ we are reduced to the statement over a local ring $(A, \mathfrak{m})$. Suppose that $\mathfrak{q}$ is a prime ideal in the polynomial ring $A[x_1,...,x_{n+r}]$ and there are polynomials f_1,...,f_r such that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & ... & \frac{\partial f_1}{\partial x_{n+r}} \\ ... & ... & ... \\ \frac{\partial f_r}{\partial x_1} & ... & \frac{\partial f_r}{\partial x_{n+r}} \end{pmatrix}$$

has rank r at point $\mathfrak{q}$. Moreover, assume that $\mathfrak{m} = \mathfrak{q} \cap A$ and the canonical morphism $k(\mathfrak{m}) \to k(\mathfrak{q})$ is an isomorphism. Then $\mathfrak{q}$ is maximal ideal and that there exist $a_1,...,a_{n+r} \in A$ such that

$$(x_1 - a_1), ..., (x_{n+r} - a_{n+r}) \in \mathfrak{q}$$

We derive that $\mathfrak{q} = \mathfrak{m}[x_1, ..., x_{n+r}] + (x_1 - a_1, ..., x_{n+r} - a_{n+r})$. Now by A-linear change of variables we may assume that $a_1 = ... = a_{n+r} = 0$. Thus $\mathfrak{q} = \mathfrak{m}[x_1, ..., x_{n+r}] + (x_1, ..., x_{n+r})$. Since completion is right exact we deduce that the completion of $A[x_1, ..., x_{n+r}]_{\mathfrak{q}}/(f_1, ..., f_r)_{\mathfrak{q}}$ in $\mathfrak{q}$-adic topology is isomorphic with

$$\widehat{A}[[x_1, ..., x_{n+r}]]/(f_1, ..., f_r)$$

as $\widehat{A}$-algebra. Here $\widehat{A}$ is the completion of A with respect to $\mathfrak{m}$-adic topology. Consider now the continuous morphism $g : \widehat{A}[[x_1, ..., x_{n+r}]] \to \widehat{A}[[x_1, ..., x_{n+r}]]$ of $\widehat{A}$-algebras given by

$$g(x_i) = \begin{cases} f_i & \text{if } 1 \leq i \leq r \\ x_{j_i} & \text{if } r+1 \leq i \leq n+r \end{cases}$$

where $x_{j_{r+1}},...,x_{j_{n+r}}$ are chosen in such a way that the matrix

$$\begin{pmatrix} \frac{\partial g(x_1)}{\partial x_1} & \dots & \frac{\partial g(x_1)}{\partial x_{n+r}} \\ \dots & \dots & \dots \\ \frac{\partial g(x_{n+r})}{\partial x_1} & \dots & \frac{\partial g(x_{n+r})}{\partial x_{n+r}} \end{pmatrix}$$

is invertible at a unique maximal ideal of $\widehat{A}[[x_1, ..., x_{n+r}]]$. Hence this matrix is invertible also modulo $(x_1..., x_{n+r})$. Now using Theorem 4.1 we deduce that g is an isomorphism of $\widehat{A}$-algebras. Therefore, we have following isomorphisms of $\widehat{A}$-algebras

$$\widehat{A}[[x_1, ...x_n]] \cong \widehat{A}[[x_1, ..., x_{n+r}]]/(x_1, ..., x_r) \cong \widehat{A}[[x_1, ..., x_{n+r}]]/(f_1, ..., f_r)$$

This shows that **(i)** $\Rightarrow$ **(ii)**.

Suppose now that there exists a neighbourhood U of x such that $f_{|U}$ is locally of finite type and Y is locally noetherian. Assume that **(ii)** holds. Clearly f induces an isomorphism $k(x) \cong k(f(x))$. Next formation of formal power series with noetherian coefficient ring is flat. Thus we deduce that $\widehat{\mathcal{O}_{X,x}} \cong \widehat{\mathcal{O}_{Y,f(x)}}[[x_1, ..., x_n]]$ is flat over $\widehat{\mathcal{O}_{Y,f(x)}}$. Since $\widehat{\mathcal{O}_{Y,f(x)}}$ is flat over $\mathcal{O}_{Y,f(x)}$, we infer that $\widehat{\mathcal{O}_{X,x}}$ is flat $\mathcal{O}_{Y,f(x)}$-algebra. Finally since $\widehat{\mathcal{O}_{X,x}}$ is faithfully flat over $\mathcal{O}_{X,x}$, we derive that $\mathcal{O}_{X,x}$ is flat $\mathcal{O}_{Y,f(x)}$-algebra. Hence f is flat at x. Let $X_{f(x)}$ be the fiber of f over $f(x)$. Since $k((f(x)) \to k(x)$ is separable algebraic, the conormal sequence for inclusion of the closed point into Spec $\mathcal{O}_{X_{f(x)},x}$ is exact and yields an isomorphism

$$\mathfrak{m}_{X_{f(x)},x}/\mathfrak{m}^2_{X_{f(x)},x} \cong k(x) \otimes_{\mathcal{O}_{X_{f(x)},x}} \Omega_{\mathcal{O}_{X_{f(x)},x}/k(f(x))}$$

One can easily check that $\widehat{\mathcal{O}_{X,f(x)}} \cong k(f(x))[[x_1, ..., x_n]]$ is a formal power series $k(f(x))$-algebra. Thus $\mathcal{O}_{X_{f(x)},x}$ is a regular local ring of dimension n. This shows that

$$\dim_x\left(X_{f(x)}\right) = n = \dim_{k(x)}\left(k(x) \otimes_{\mathcal{O}_{X_{f(x)},x}} \Omega_{\mathcal{O}_{X_{f(x)},x}/k(f(x))}\right)$$

By Theorem 2.2 scheme $X_{f(x)}$ is smooth of relative dimension n at x. We showed that f is flat at x and that $X_{f(x)}$ is smooth of relative dimension n at x. Since Y is locally noetherian, the restriction $f_{|U}$ is locally of finite presentation. Hence f is smooth of relative dimension n by Theorem 3.1. This shows that **(ii)** $\Rightarrow$ **(i)** holds under additional assumptions described in the statement. □

References

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