MONOID k-SCHEMES AND THEIR REPRESENTATIONS

1. Equivalence of representations and comodules

Let k be a commutative ring.

Theorem 1.1. *Let $\mathbf{M}$ be an affine monoid scheme over k. Suppose that $\rho : \mathbf{M} \to \mathcal{L}(V)$ is a morphism of functors of sets. Yoneda lemma implies that ρ is determined by some element*

$$p_\rho \in \mathrm{Hom}_{\Gamma(\mathbf{M},\mathcal{O}_\mathbf{M})}\left(\Gamma(\mathbf{M},\mathcal{O}_\mathbf{M}) \otimes_k V, \Gamma(\mathbf{M},\mathcal{O}_\mathbf{M}) \otimes_k V\right)$$

Next under the natural isomorphism

$$\mathrm{Hom}_k(V, \Gamma(\mathbf{M},\mathcal{O}_\mathbf{M}) \otimes_k V) \to \mathrm{Hom}_{\Gamma(\mathbf{M},\mathcal{O}_\mathbf{M})}\left(\Gamma(\mathbf{M},\mathcal{O}_\mathbf{M}) \otimes_k V, \Gamma(\mathbf{M},\mathcal{O}_\mathbf{M}) \otimes_k V\right)$$

we deduce that p_ρ corresponds to a unique k-linear morphism $d_\rho : V \to \Gamma(\mathbf{M},\mathcal{O}_\mathbf{M}) \otimes_k V$.
Then (V,ρ) is a representation of $\mathbf{M}$ if and only if (V, d_ρ) is a comodule over $\Gamma(\mathbf{M},\mathcal{O}_\mathbf{M})$.
Moreover, assume that (V,ρ_V), (W,ρ_W) are representations and (V, d_{ρ_V}), (W, d_{ρ_W}) are associated comodules. Then a morphism of k-modules $f : V \to W$ is a morphism of the representations if and only if it is a morphism of the comodules.

In order to give a proof we will fix some notation. For every affine scheme S over k we denote by O_S its k-algebra of global regular functions. We also denote by $\delta_S : O_S \otimes_k O_S \to O_S$ the multiplication on O_S and by $\eta_S : k \to O_S$ the structural morphism. In particular, $O_\mathbf{M} = \Gamma(\mathbf{M},\mathcal{O}_\mathbf{M})$ is a bialgebra of global regular functions on $\mathbf{M}$. We denote its counit and comultiplication by $\xi_\mathbf{M}$ and $\Delta_\mathbf{M}$, respectively. Finally for every morphism $f : S \to T$ of affine k-schemes denote by $f^\# : O_T \to O_S$ the corresponding morphism of k-algebras.
We identify freely $\mathbf{M}$ with its functor of points. Let V be a module over k. Note that every morphism $\rho : \mathbf{M} \to \mathcal{L}(V)$ of functors of sets gives rise by the rule described in the statement to a unique morphism of k-modules $d_\rho : V \to O_\mathbf{M} \otimes_k V$. This correspondence is one to one by Yoneda lemma.
Fix a morphism $\rho : \mathbf{M} \to \mathcal{L}(V)$ of functors and associated morphism of k-modules $d_\rho : V \to O_\mathbf{M} \otimes_k V$. Let S be an affine k-scheme and pick an S-point $m \in \mathbf{M}(S)$. Then m gives rise to a morphism $m^\# : O_\mathbf{M} \to O_S$ of k-algebras and

$$\rho(m) = (\delta_S \otimes_k 1_V) \cdot \left(1_{O_S} \otimes_k m^\# \otimes_k 1_V\right) \cdot \left(1_{O_S} \otimes_k d_\rho\right)$$

Now we need some additional results.

Lemma 1.1.1. *Let V be a module over k, $\rho : \mathbf{M} \to \mathcal{L}(V)$ be a morphism of functors and d_ρ be an associated morphism of k-modules. Then ρ is a morphism of semigroups if and only if d_ρ satisfies*

$$(\Delta_\mathbf{M} \otimes_k 1_V) \cdot d_\rho = \left(1_{O_\mathbf{M}} \otimes_k d_\rho\right) \cdot d_\rho$$

Proof of the lemma. Let S be an affine k-scheme and suppose that $m_1, m_2 \in \mathbf{M}(S)$. Then

$$\rho(m_1) \cdot \rho(m_2) = \left(\left(\delta_S \cdot \left(1_{O_S} \otimes_k \delta_S\right)\right) \otimes_k 1_V\right) \cdot \left(1_{O_S} \otimes_k m_1^\# \otimes_k m_2^\# \otimes_k 1_V\right) \cdot \left(1_{O_S} \otimes_k \left(\left(1_{O_\mathbf{M}} \otimes_k d_\rho\right) \cdot d_\rho\right)\right)$$

and if $m_1 \cdot m_2$ denotes product of these elements in $\mathbf{M}(S)$, then

$$\rho(m_1 \cdot m_2) = \left(\left(\delta_S \cdot \left(1_{O_S} \otimes_k \delta_S\right)\right) \otimes_k 1_V\right) \cdot \left(1_{O_S} \otimes_k m_1^\# \otimes_k m_2^\# \otimes_k 1_V\right) \cdot \left(1_{O_S} \otimes_k \left((\Delta_\mathbf{M} \otimes_k 1_V) \cdot d_\rho\right)\right)$$

These formulas imply that if

$$(\Delta_\mathbf{M} \otimes_k 1_V) \cdot d_\rho = \left(1_{O_\mathbf{M}} \otimes_k d_\rho\right) \cdot d_\rho$$

then ρ is a morphism of functors of semigroups.
Conversely consider two canonical projections $\pi_1, \pi_2 : \mathbf{M} \times_{\operatorname{Spec} k} \mathbf{M} \to \mathbf{M}$ so that $\pi_1, \pi_2 \in \mathbf{M}\left(\mathbf{M} \times_{\operatorname{Spec} k} \mathbf{M}\right)$. Then formulas above together with the fact that $1_{O_{\mathbf{M}\times_{\operatorname{Spec} k}\mathbf{M}}} = \delta_{\mathbf{M}\times_{\operatorname{Spec} k}\mathbf{M}} \cdot \left(\pi_1^{\#} \otimes_k \pi_2^{\#}\right)$ imply

$$\rho(\pi_1 \cdot \pi_2) = \left(\delta_{\mathbf{M}\times_{\operatorname{Spec} k}\mathbf{M}} \otimes_k 1_V\right) \cdot \left(1_{O_{\mathbf{M}\times_{\operatorname{Spec} k}\mathbf{M}}} \otimes_k \left((\Delta_{\mathbf{M}} \otimes_k 1_M) \cdot d_\rho\right)\right)$$

and

$$\rho(\pi_1) \cdot \rho(\pi_2) = \left(\delta_{\mathbf{M}\times_{\operatorname{Spec} k}\mathbf{M}} \otimes_k 1_V\right) \cdot \left(1_{O_{\mathbf{M}\times_{\operatorname{Spec} k}\mathbf{M}}} \otimes_k \left(\left(1_{O_{\mathbf{M}}} \otimes_k d_\rho\right) \cdot d_\rho\right)\right)$$

Now if ρ is a morphism of functors of semigroups, then $\rho(\pi_1 \cdot \pi_2) = \rho(\pi_1) \cdot \rho(\pi_2)$ and hence

$$\left(1_{O_{\mathbf{M}}} \otimes_k d_\rho\right) \cdot d_\rho = (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho$$

□

Lemma 1.1.2. *Let V be a module over k, $\rho : \mathbf{M} \to \mathcal{L}(V)$ be a morphism of functors and d_ρ be an associated morphism of k-modules. Then ρ preserves identity elements if and only if $(\xi_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho$ coincides with the canonical morphism $V \to k \otimes_k V$.*

Proof of the lemma. For every affine k-scheme S define $e_S \in \mathbf{M}(S)$ as a structural morphism $S \to \operatorname{Spec} k$ composed with the neutral element $e : \operatorname{Spec} k \to \mathbf{M}$. This is an identity element of monoid $\mathbf{M}(S)$. We have

$$\rho(e_S) = (\delta_S \otimes_k 1_V) \cdot \left(1_{O_S} \otimes_k \eta_S \otimes_k 1_V\right) \cdot \left(1_{O_S} \otimes_k \left((\xi_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho\right)\right)$$

Therefore, if

$$\left((\xi_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho\right)$$

is equal to the canonical isomorphism $V \to k \otimes_k V$, then $\rho(e_S) = 1_{O_S \otimes_k V}$.
On the other hand if $\rho(e_S) = 1_{O_S \otimes_k V}$ for every affine k-scheme S, then setting $S = \operatorname{Spec} k$ we derive

$$1_{k\otimes_k V} = \rho(e_{\operatorname{Spec} k}) = \left(\delta_{\operatorname{Spec} k} \otimes_k 1_V\right) \cdot \left(1_k \otimes_k \left((\xi_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho\right)\right)$$

and thus $(\xi_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho$ is equal to the canonical morphism $V \to k \otimes_k V$. □

Lemma 1.1.3. *Suppose that V and W are k-modules and $f : V \to W$ be a morphism of k-modules. Let $\rho_V : \mathbf{M} \to \mathcal{L}(V)$, $\rho_W : \mathbf{M} \to \mathcal{L}(W)$ be morphisms of functors of sets and d_{ρ_V}, d_{ρ_W} be associated morphism of k-modules. Then the following assertions are equivalent.*

(i) *The formula*

$$\left(1_{\Gamma(\mathbf{M},\mathcal{O}_{\mathbf{M}})} \otimes_k f\right) \cdot d_{\rho_V} = d_{\rho_W} \cdot f$$

holds.

(ii) *The formula*

$$\rho_W(m) \cdot \left(1_{\Gamma(S,\mathcal{O}_S)} \otimes_k f\right) = \left(1_{\Gamma(S,\mathcal{O}_S)} \otimes_k f\right) \cdot \rho_V(m)$$

holds for every affine scheme S over k and $m \in \mathbf{M}(S)$.

Proof of the lemma. Let $m \in \mathbf{M}(S)$ be an S-point for some affine scheme S. We have

$$\left(1_{O_S} \otimes_k f\right) \cdot \rho_V(m) = (\delta_S \otimes_k 1_V) \cdot \left(1_{O_S} \otimes_k m^{\#} \otimes_k 1_V\right) \cdot \left(1_{O_S} \otimes_k \left(\left(1_{O_{\mathbf{M}}} \otimes_k f\right) \cdot d_{\rho_V}\right)\right)$$

and

$$\rho_W(m) \cdot (1_{O_S} \otimes_k f) = (\delta_S \otimes_k 1_V) \cdot \left(1_{O_S} \otimes_k m^{\#} \otimes_k 1_V\right) \cdot \left(1_{O_S} \otimes_k \left(d_{\rho_W} \cdot f\right)\right)$$

Hence clearly **(i)** ⇒ **(ii)**. Now suppose that **(ii)** holds. In particular

$$(\delta_{\mathbf{M}} \otimes_k 1_V) \cdot \left(1_{O_{\mathbf{M}}} \otimes_k \left(\left(1_{O_{\mathbf{M}}} \otimes_k f\right) \cdot d_{\rho_V}\right)\right) = \left(1_{O_{\mathbf{M}}} \otimes_k f\right) \cdot \rho_V(1_{\mathbf{M}}) =$$

$$= \rho_W(1_{\mathbf{M}}) \cdot (1_{O_{\mathbf{M}}} \otimes_k f) = (\delta_{\mathbf{M}} \otimes_k 1_V) \cdot \left(1_{O_{\mathbf{M}}} \otimes_k \left(d_{\rho_W} \cdot f\right)\right)$$

This implies that

$$\left(1_{O_{\mathbf{M}}} \otimes_k f\right) \cdot d_{\rho_V} = d_{\rho_W} \cdot f$$

□

Proof of the theorem. According to Lemmas 1.1.1 and 1.1.2 we deduce that ρ is a morphism of functors of monoids if and only if (M, d_ρ) is a comodule over the bialgebra $\Gamma(\mathbf{V}, \mathcal{O}_{\mathbf{M}})$. This proves that the correspondence $(V, \rho) \mapsto (V, d_\rho)$ between representations of $\mathbf{M}$ and comodules over $\Gamma(\mathbf{M}, \mathcal{O}_{\mathbf{M}})$ is bijective.
Now suppose that $f : V \to W$ is a morphism of k-modules and (V, ρ_V), (W, ρ_W) are representations. Lemma 1.1.3 shows that f is a morphism of representations if and only if f is a morphism of comodules over $\Gamma(\mathbf{M}, \mathcal{O}_{\mathbf{M}})$. $\square$

Corollary 1.2. *Let* $\mathbf{M}$ *be an affine monoid k-scheme. Then correspondence described in Theorem 1.1 gives rise to an isomorphism of categories*

$$\mathbf{Rep}_{\mathbf{M}} \to \mathbf{coMod}\left(\Gamma(\mathbf{M}, \mathcal{O}_{\mathbf{M}})\right)$$

Proof. This is just a reformulation of Theorem 1.1. $\square$