PROBABILITY MEASURES ON POLISH SPACES

1. Introduction

2. Compact metric spaces

We start by some general property of metric space.

Fact 2.1. *Let (X, d) be a metric space and let $\epsilon > 0$ be a number. Then there exists a subset N such that*

$$\forall_{x_1, x_2 \in N} \left(x_1 \neq x_2 \Rightarrow 2 \cdot \epsilon < d(x_1, x_2) \right)$$

and X is the union of balls centered in points of N and with radius ϵ.

Proof. This is a consequence of Zorn's lemma applied to the family Consider the family of sets

$$\mathcal{N} = \left\{ N \subseteq X \,\middle|, \forall_{x_1, x_2 \in N} \left(x_1 \neq x_2 \Rightarrow 2 \cdot \epsilon < d(x_1, x_2) \right) \right\}$$

ordered by inclusion. The details are left for the reader. □

Definition 2.2. Let (X, d) be a metric space. Suppose that for each $\epsilon > 0$ there exists a finite family $\mathcal{B}$ of closed balls with respect to d such that each of them has radius equal to ϵ and

$$X = \bigcup_{B \in \mathcal{B}} \mathcal{B}$$

Then (X, d) is *a completely bounded metric space*.

Fact 2.3. *Let (X, d) be a completely bounded metric space. Then X is second countable.*

Proof. Left for the reader. □

Definition 2.4. Let X be a topological space. Suppose that for every sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of X there exists a convergent subsequence. Then X is *a sequentially compact space*.

Definition 2.5. Let (X, d) be a metric space and let $\mathcal{U}$ be its open cover. Assume that there exists $\lambda > 0$ such that for every subset A of X with $\mathrm{diam}(A) \leq \lambda$ there exists U in $\mathcal{U}$ such that $A \subseteq U$. Then λ is *a Lebesgue number of $\mathcal{U}$*.

Theorem 2.6. *Let (X, d) be a metric space. Then the following assertions are equivalent.*

- **(i)** *X is compact.*
- **(ii)** *(X, d) is complete and completely bounded.*
- **(iii)** *X is sequentially compact.*

Moreover, if these equivalent assertions hold, then every open cover of X admits Lebesgue number.

We prove partial result first.

Lemma 2.6.1. *Let (X, d) be a metric space. If X is sequentially compact, then every open cover of X admits a Lebesgue number.*

Proof of the lemma. Fix open cover $\mathcal{U}$ of X. Suppose that this cover does not admits a Lebesgue number. Pick a decreasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of elements in $\mathbb{R}_+$ which is convergent to zero. Since $\mathcal{U}$ does not admit a Lebesgue number, for each $n\in\mathbb{N}$ there exists a nonempty set A_n of diameter not greater than λ_n such that A_n is not contained in any element of $\mathcal{U}$. For each $n\in\mathbb{N}$ pick $x_n\in A_n$. By sequential compactness of X, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ which converges to some point x in X. Moreover, according to

$$X=\bigcup_{U\in\mathcal{U}}U$$

there exists $U\in\mathcal{U}$ such that $x\in\mathcal{U}$. Fix $\delta>0$ such that the open ball $B(x,2\cdot\delta)$ with respect to d is contained in U. Pick also k such that $d(x,x_{n_k})<\delta$ and $\lambda_{n_k}<\delta$. Then for every a in A_{n_k} we have

$$d(x,a)\leq d(x,x_{n_k})+d(x_{n_k},a)<\delta+\lambda_{n_k}<2\cdot\delta$$

Thus $a\in B(x,2\cdot\delta)$. Since this holds for every a in A_{n_k}, we infer that $A_{n_k}\subseteq B(x,2\cdot\delta)\subseteq U$. This is a contradiction. $\square$

Proof of the theorem. Suppose that X is compact. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to d. Define

$$F_n=\mathbf{cl}\left(\{x_n\,|\,n\geq k\}\right)$$

Clearly $\{F_n\}_{n\in\mathbb{N}}$ is a nondecreasing sequence of closed nonempty subsets of X. Thus by compactness of X it follows that $\{F_n\}_{n\in\mathbb{N}}$ has nonempty intersection. Since $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d, we derive that

$$\lim_{n\to+\infty}\mathrm{diam}(F_n)=0$$

Thus the intersection of $\{F_n\}_{n\in\mathbb{N}}$ consists of a single point say x. It follows that $\{x_n\}_{n\in\mathbb{N}}$ converges to x. Hence (X,d) is complete. The fact that X is completely bounded follows easily from compactness of X. Therefore, **(i)** $\Rightarrow$ **(ii)**.

Suppose now that (X,d) is complete and completely bounded. For every $k\in\mathbb{N}$ let $B_{k,1},...B_{k,m_k}$ be a family of closed balls in X such that each of them has radius equal to $\frac{1}{2^k}$ and

$$X=B_{k,1}\cup...\cup B_{k,m_k}$$

Pick a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X. For every $k\in\mathbb{N}$ we will construct a sequence $\{x_n^k\}_{n\in\mathbb{N}}$ such that $\{x_n^{k+1}\}_{n\in\mathbb{N}}$ is a subsequence of $\{x_n^k\}_{n\in\mathbb{N}}$. We set $\{x_n^0\}_{n\in\mathbb{N}}$ to be $\{x_n\}_{n\in\mathbb{N}}$. Next if $\{x_n^k\}_{n\in\mathbb{N}}$ is constructed, then at least one of the balls

$$B_{k+1,1},...,B_{k+1,m_{k+1}}$$

contains infinitely many elements of $\{x_n^k\}_{n\in\mathbb{N}}$. We define $\{x_n^{k+1}\}_{n\in\mathbb{N}}$ to be a subsequence of $\{x_n^k\}_{n\in\mathbb{N}}$ which consists of these elements. It follows from the construction that all elements of $\{x_n^k\}_{n\in\mathbb{N}}$ are contained in some closed ball D_k of X having radius $\frac{1}{2^k}$. Now we define a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ by $x_{n_k}=x_k^k$ for every $k\in\mathbb{N}$. Then x_{m_k} is contained in a closed ball D_k of X having radius $\frac{1}{2^k}$ for every $m\geq k$ and $k\in\mathbb{N}$. It follows that $\{x_{n_k}\}_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to d. Thus it is convergent to some point x in X. This completes the proof of **(ii)** $\Rightarrow$ **(iii)**.

Suppose that X is sequentially compact. Consider an open cover $\mathcal{U}$ of X. By Lemma 2.6.1 there exists a Lebesgue number $\lambda>0$ of $\mathcal{U}$. According to Fact 2.1 there exists a set $N\subseteq X$ such that

$$X=\bigcup_{x\in N}B\left(x,\frac{\lambda}{2}\right)$$

and for every pair of points x_1,x_2 in N we have $\lambda<d(x_1,x_2)$. Clearly N is discrete subspace of X. Since X is sequentially compact, we infer that N is finite say $N=\{x_1,...,x_n\}$ for some $n\in\mathbb{N}$. For each $i\in\{1,...,n\}$ let $U_i\in\mathcal{U}$ be an open subset such that

$$B\left(x_i,\frac{\lambda}{2}\right)\subseteq U_i$$

Thus
$$X = \bigcup_{i=1}^{n} B\left(x_i, \frac{\lambda}{2}\right) = \bigcup_{i=1}^{n} U_i$$
and hence $\mathcal{U}$ has finite subcover. Therefore, X is compact and the implication **(iii)** $\Rightarrow$ **(i)** is proved. Now the additional assertion follows from Lemma 2.6.1. □

Corollary 2.7. *Each compact metrizable space is second countable.*

Proof. A consequence of Fact 2.3 and Theorem 2.6. □

3. Completely metrizable topological spaces

Definition 3.1. Let X be a topological space. If there exists a metric d on X which induces the topology of X, then X is *a metrizable space*. In addition if d is complete, then X is *a completely metrizable space*.

We start by some basic results.

Proposition 3.2. *Let X be a metrizable space. Then there exists a metric δ which induces topology of X and*
$$\delta(x_1, x_2) < 1$$
for every pair $x_1, x_2 \in X$.

Proof. Consider a metric d which induces topology on X. Define
$$\delta(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}$$
Let $f : [0, +\infty) \to \mathbb{R}$ be a function given by formula
$$f(t) = \frac{t}{1+t}$$
Then $f(t_1 + t_2) \le f(t_1) + f(t_2)$ for $t_1, t_2 \in [0, +\infty)$ and f is strictly increasing. We derive that
$$\delta(x_1, x_3) = f\left(d(x_1, x_3)\right) \le f\left(d(x_1, x_2) + d(x_2, x_3)\right) \le$$
$$\le f\left(d(x_1, x_2)\right) + f\left(d(x_2, x_3)\right) = \delta(x_1, x_2) + \delta(x_2, x_3)$$
for every $x_1, x_2, x_3 \in X$. Clearly $\delta(x_1, x_2) = 0$ is equivalent to $d(x_1, x_2) = 0$ and hence it is equivalent to $x_1 = x_2$ for all $x_1, x_2 \in X$. Moreover, δ is symmetric which follows from the fact that d is symmetric. Therefore, δ is a metric on X. We claim that δ induces the same topology on X as d. In order to prove this we fix a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of X and a point x in X. Since f is strictly increasing, continuous and
$$f(0) = 0, \lim_{n\to+\infty} f(t) = 1$$
we infer that f induces a homeomorphism of $[0, +\infty)$ and $[0, 1)$. Thus
$$\lim_{n\to+\infty} d(x_n, x) = 0 \Leftrightarrow \lim_{n\to+\infty} f\left(d(x_n, x)\right) = 0 \Leftrightarrow \lim_{n\to+\infty} \delta(x_n, x) = 0$$
This implies that the class of convergent sequences for d is equal to the class of convergent sequences for δ. The claim is proved. Hence δ induces the topology of X. Finally as we noted above $\delta(x_1, x_2) = f\left(d(x_1, x_2)\right) < 1$ for all x_1, x_2 in X. □

Proposition 3.3. *Let (X, d) be a complete metric space and let F be its subset. The restriction of d to F makes it into a complete metric space if and only if F is a closed subset of X.*

Proof. Suppose that F is closed. Consider a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ with respect to d and such that $x_n \in F$ for all $n \in \mathbb{N}$. Since d is complete, there exists a limit x of $\{x_n\}_{n\in\mathbb{N}}$ inside x. Since F is closed, we derive that $x \in F$. According to the fact that $\{x_n\}_{n\in\mathbb{N}}$ is arbitrary Cauchy sequence with respect to d with elements in F, we derive that the restriction of d makes F into a complete metric space.
Suppose now that the restriction of d to F makes it into a complete metric space. Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of F and suppose that it converges to some x in X. Then $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy with respect to d. Since F is a complete with respect to restriction of d, we derive that $\{x_n\}_{n\in\mathbb{N}}$ is convergent to some element of F. Therefore, x is an element of F. This shows that F is closed subset of X. □

Now we introduce notion which plays important role in the study of complete metrizability.

Definition 3.4. Let X be a topological space. Then a subset of X which is a countable intersection of open subsets of X is *a G_δ subset of X*.

Now we shall prove important result due to Alexandrov.

Theorem 3.5 (Alexandrov)**.** *Let X be a topological space. If X is completely metrizable, then every G_δ subset of X is completely metrizable.*

For the proof we need some lemmas.

Lemma 3.5.1. *Let (X, d) be a complete metric space and let U be its open subset. Then U is completely metrizable.*

Proof of the lemma. Define a function $f : U \to \mathbb{R}$ by formula $f(x) = d(x, X \smallsetminus U)$. Let Γ_f be the graph of f inside $X \times \mathbb{R}$. That is

$$\Gamma_f = \left\{(x, r) \in X \times \mathbb{R} \,\middle|\, x \in U \text{ and } f(x) = r\right\}$$

Suppose that $\{(x_n, r_n)\}_{n\in\mathbb{N}}$ is a sequence of elements of Γ_f which is convergent in $X \times \mathbb{R}$. Let (x, r) be its limit. Then $x_n \to x$ for $n \to +\infty$ and hence

$$\lim_{n\to+\infty} d(x_n, X \smallsetminus U) = d(x, X \smallsetminus U)$$

Note that the left hand side potentially can be equal to $+\infty$. We rule out this possibility as follows. We have

$$\lim_{n\to+\infty} d(x_n, X \smallsetminus U) = \lim_{n\to+\infty} r_n = r \in \mathbb{R}$$

and hence $d(x, X \smallsetminus U) = r \in \mathbb{R}$. Thus $x \in U$ and we infer that $(x, r) \in \Gamma_f$. This implies that Γ_f is a closed subset of $X \times \mathbb{R}$. Since $X \times \mathbb{R}$ is completely metrizable, we derive that Γ_f is completely metrizable by Proposition 3.3. On the other hand the map

$$U \ni x \mapsto (x, f(x)) \in \Gamma_f$$

is a homeomorphism and thus U is completely metrizable. □

Lemma 3.5.2. *Let X be a set and let $\{d_n\}_{n\in\mathbb{N}}$ be a sequence of metrics on X. Assume that d_n is bounded from above by 1 for every $n \in \mathbb{N}$. Consider*

$$d = \sum_{n\in\mathbb{N}} \frac{1}{2^n} \cdot d_n$$

Then d is a metric on X and the following assertions hold.

(1) *Sequence $\{x_m\}_{m\in\mathbb{N}}$ of elements of X is convergent to some x in X with respect to d if and only if it is convergent to x with respect to d_n for all $n \in \mathbb{N}$*

(2) *Sequence $\{x_m\}_{m\in\mathbb{N}}$ of elements of X is a Cauchy sequence with respect to d if and only if it is a Cauchy sequence with respect to d_n for all $n \in \mathbb{N}$*

Proof of the lemma. It is clear that d is a metric on X. For each $n \in \mathbb{N}$ we have

$$d_n \leq 2^n \cdot d$$

and

$$d \leq \frac{1}{2^N} + \sum_{n=0}^{N} \frac{1}{2^n} \cdot d_n$$

From this two inequalities it is easy to deduce **(1)** and **(2)**. The details are left to the reader. □

Proof of the theorem. Suppose that $\{U_n\}_{n\in\mathbb{N}}$ is a nondecreasing sequence of open subsets of X. By lemma 3.5.1 each U_n is completely metrizable. Hence by Proposition 3.2 we may pick a complete metric d_n on U_n which induces the topology on U_n. Define

$$d(x_1, x_2) = \sum_{n\in\mathbb{N}} \frac{1}{2^n} \cdot d_n(x_1, x_2)$$

for every $x_1, x_2 \in \bigcap_{n\in\mathbb{N}} U_n$. Lemma 3.5.2 implies that d is a metric. We also have

$$\left\{\{x_m\}_{m\in\mathbb{N}} \in X^{\mathbb{N}} \,\middle|\, \forall_{m\in\mathbb{N}}\, x_m \in \bigcap_{n\in\mathbb{N}} U_n \text{ and } \exists_{x\in\bigcap_{n\in\mathbb{N}} U_n} \lim_{m\to+\infty} d(x_m, x) = 0\right\} =$$

$$= \left\{\{x_m\}_{m\in\mathbb{N}} \in X^{\mathbb{N}} \,\middle|\, \forall_{m\in\mathbb{N}}\, x_m \in \bigcap_{n\in\mathbb{N}} U_n \text{ and } \exists_{x\in\bigcap_{n\in\mathbb{N}} U_n} \forall_{n\in\mathbb{N}} \lim_{m\to+\infty} d_n(x_m, x) = 0\right\} =$$

$$= \bigcap_{n\in\mathbb{N}} \left\{\{x_m\}_{m\in\mathbb{N}} \in X^{\mathbb{N}} \,\middle|\, \forall_{m\in\mathbb{N}}\, x_m \in \bigcap_{n\in\mathbb{N}} U_n \text{ and } \exists_{x\in\bigcap_{n\in\mathbb{N}} U_n} \lim_{m\to+\infty} d_n(x_m, x) = 0\right\} =$$

$$= \text{the class of convergent sequences in } \bigcap_{n\in\mathbb{N}} U_n \text{ for the subspace topology induced from } X$$

The first equality follows from Lemma 3.5.2. The second is a consequence of the fact that (restrictions of) $\{d_n\}_{n\in\mathbb{N}}$ induce the same topology on $\bigcap_{n\in\mathbb{N}} U_n$. Finally, the third equality follows the fact that the topology induced by (the restriction of) d_n on $\bigcap_{n\in\mathbb{N}} U_n$ coincides with the subspace topology induced from X for every $n \in \mathbb{N}$. Thus d induces on $\bigcap_{n\in\mathbb{N}} U_n$ the topology of the subspace of X. Next suppose that $\{x_m\}_{m\in\mathbb{N}}$ is a sequence of elements of $\bigcap_{n\in\mathbb{N}} U_n$ which is a Cauchy sequence with respect to d. Then according to Lemma 3.5.2 we derive that $\{x_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence with respect to d_n for every $n \in \mathbb{N}$. Since d_n is a complete metric on U_n for $n \in \mathbb{N}$, $\{x_m\}_{m\in\mathbb{N}}$ is convergent to some point in U_n for every $n \in \mathbb{N}$. Hence $\{x_m\}_{m\in\mathbb{N}}$ is convergent to some point x of X and $x \in U_n$ for every $n \in \mathbb{N}$. Thus x is a point of $\bigcap_{n\in\mathbb{N}} U_n$. Therefore, $\{x_m\}_{m\in\mathbb{N}}$ is converges to some point in $\bigcap_{n\in\mathbb{N}} U_n$. Hence d is a complete metric on $\bigcap_{n\in\mathbb{N}} U_n$ which induces the topology of subspace of X. This completes the proof of the theorem. □

Now we prove the converse of the Alexandrov's theorem.

Theorem 3.6. *Let X be a metrizable space and let A be its subspace. If A is completely metrizable, then A is a G_δ subset of X.*

Proof. Consider a metric d on X compatible with its topology. Suppose that δ is a complete metric on A which induces the topology of the subspace of X. For each point a in A consider a sequence $\{r_n(a)\}_{n\in\mathbb{N}}$ of positive real numbers such that

$$\left\{x \in A \,\middle|\, d(a, x) < r_n(a)\right\} \subseteq \left\{x \in A \,\middle|\, \delta(a, x) \leq 2^{-n}\right\}$$

and $r_n(a) \leq 2^{-n}$ for $n \in \mathbb{N}$. Define

$$U_n = \bigcup_{a\in A} \left\{x \in X \,\middle|\, d(a, x) < r_n(a)\right\}$$

for $n \in \mathbb{N}$. Clearly U_n is an open subset of X and A is contained in U_n for every $n \in \mathbb{N}$. Suppose now that x is a point of U_n for every $n \in \mathbb{N}$. Then there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that

$$d(a_n, x) < r_n(a_n)$$

for every $n \in \mathbb{N}$. Since $r_n(a_n) \leq 2^{-n}$, we derive that $\{a_n\}_{n\in\mathbb{N}}$ converges to x with respect to d. Now fix $\epsilon > 0$ and consider $k \in \mathbb{N}$ such that $2^{-k} < \epsilon$. Note that $\{a_n\}_{n\in\mathbb{N}}$ converges to x with respect to

d and $d(a_{k+1}, x) < r_{k+1}(a_{k+1})$. Thus there exists $N \in \mathbb{N}$ such that $d(a_{k+1}, a_n) < r_{k+1}(a_{k+1})$ for every $n \geq N$. Fix $n, m \geq N$. Then $d(a_{k+1}, a_n)$ and $d(a_{k+1}, a_m)$ are both smaller than $r_{k+1}(a_{k+1})$. It follows that $\delta(a_{k+1}, a_n)$ and $\delta(a_{k+1}, a_m)$ are both smaller than 2^{-k-1}. Hence

$$\delta(a_n, a_m) \leq \delta(a_{k+1}, a_n) + \delta(a_{k+1}, a_m) \leq 2 \cdot 2^{-k-1} = 2^{-k} < \epsilon$$

This inequality holds for all $n, m \geq N$. According to the fact that ϵ is arbitrary, we infer that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to δ. Since δ is complete metric which induces the topology of the subspace of X on A, it follows that $\{a_n\}_{n\in\mathbb{N}}$ is convergent to some element of A with respect to the topology of X. On the other hand it converges to x with respect to d. However, $\{a_n\}_{n\in\mathbb{N}}$ converges to x with respect to the topology of X. Thus x is an element of A. This shows that

$$A = \bigcup_{n\in\mathbb{N}} U_n$$

□

4. Ulam's theorem on inner regularity

Definition 4.1. Let X be a Hausdorff topological space. Suppose that X is normal and for every open subset U of X there is a family $\{F_n\}_{n\in\mathbb{N}}$ of closed subsets of X such that

$$U = \bigcup_{n\in\mathbb{N}} F_n$$

Then X is *a perfectly normal space*.

Proposition 4.2. *Let X be a perfectly normal space and let $\mu : \mathcal{B}(X) \to [0,1]$ be a probability measure on X. Then*

$$\mu(A) = \sup\left\{\mu(F) \,\middle|\, F \textit{ is closed in } X \textit{ and } F \subseteq A\right\}$$

and

$$\mu(A) = \inf\left\{\mu(U) \,\middle|\, U \textit{ is open in } X \textit{ and } A \subseteq U\right\}$$

for every Borel set A in X.

Proof. Consider the family $\mathcal{F}$ all Borel sets A in X such that

$$\mu(A) = \sup\left\{\mu(F) \,\middle|\, F \text{ is closed in } X \text{ and } F \subseteq A\right\}$$

and

$$\mu(A) = \inf\left\{\mu(U) \,\middle|\, U \text{ is open in } X \text{ and } A \subseteq U\right\}$$

We claim that $\mathcal{F}$ is a λ-system. Consider a countable sequence $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{F}$. Pick $\epsilon > 0$ and for every $n \in \mathbb{N}$ consider a closed subset F_n of X and an open subset U_n of X such that

$$F_n \subseteq A_n \subseteq U_n$$

and

$$\mu(A_n) \leq \mu(F_n) + \frac{\epsilon}{2^{n+1}},\ \mu(U_n) \leq \mu(A_n) + \frac{\epsilon}{2^{n+1}}$$

Then

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n) \leq \sum_{n\in\mathbb{N}} \left(\mu(F_n) + \frac{\epsilon}{2^{n+1}}\right) = \epsilon + \sum_{n\in\mathbb{N}} \mu(F_n) = \mu\left(\bigcup_{n\in\mathbb{N}} F_n\right) + \epsilon$$

and

$$\mu\left(\bigcup_{n\in\mathbb{N}} U_n\right) \leq \sum_{n\in\mathbb{N}} \mu(U_n) \leq \sum_{n\in\mathbb{N}} \left(\mu(A_n) + \frac{\epsilon}{2^{n+1}}\right) = \epsilon + \sum_{n\in\mathbb{N}} \mu(A_n) = \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) + \epsilon$$

Pick $N \in \mathbb{N}$ such that

$$\mu\left(\bigcup_{n\in\mathbb{N}} F_n\right) \leq \mu\left(\bigcup_{n=0}^{N} F_n\right) + \epsilon$$

and set $F = \bigcup_{n=0}^{N} F_n$ and $U = \bigcup_{n\in\mathbb{N}} U_n$. Then we derive that F is a closed subset of X and U is an open subset of X such that

$$F \subset \bigcup_{n\in\mathbb{N}} A_n \subseteq U$$

and

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \mu(F) + 2\epsilon,\ \mu(U) \leq \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) + \epsilon$$

Since $\epsilon > 0$ was chosen arbitrarily, we derive that $\bigcup_{n\in\mathbb{N}} A_n$ is in $\mathcal{F}$. Thus $\mathcal{F}$ is closed under countable unions of pairwise disjoint elements. Next suppose that A in $\mathcal{F}$. Pick $\epsilon > 0$ and consider a closed subset F of X and an open subset U of X such that

$$F \subseteq A \subseteq U$$

and

$$\mu(A) \leq \mu(F) + \epsilon,\ \mu(U) \leq \mu(A) + \epsilon$$

Then we have

$$X \setminus U \subseteq X \setminus A \subseteq X \setminus F$$

and

$$\mu(X \setminus A) \leq \mu(X \setminus U) + \epsilon,\ \mu(X \setminus F) \leq \mu(X \setminus A) + \epsilon$$

Again since $\epsilon > 0$ was chosen arbitrarily, we derive that $X \setminus A$ is in $\mathcal{F}$. Thus $\mathcal{F}$ is closed under complements. Therefore, the claim is proved i.e. $\mathcal{F}$ is a λ-system. Since X is completely normal, we derive that the family τ of all open subsets of X is contained in $\mathcal{F}$. Hence $\mathcal{F}$ contains the smallest λ-system generated by τ. We denote this λ-system by $\lambda(\tau)$. Since τ is a π-system, we deduce by Dynkin's $\pi - \lambda$ lemma ([Monygham, 2018, Theorem 1.4]) that $\lambda(\tau) = \sigma(\tau) = \mathcal{B}(X)$. Thus $\mathcal{B}(X) \subseteq \mathcal{F}$ and hence all Borel subsets of X are in $\mathcal{F}$. □

We introduce important notion.

Definition 4.3. Let $(X, \mathcal{F}, \mu)$ be a space with measure. Suppose that τ is a Hausdorff topology on X such that $\tau \subseteq \mathcal{F}$ and for every $A \in \mathcal{F}$ we have

$$\mu(A) = \sup\left\{\mu(K) \,\middle|\, K \text{ is compact with respect to } \tau \text{ and } K \subseteq A\right\}$$

Then μ is *an inner regular measure with respect to* τ.

Theorem 4.4 (Ulam)**.** *Let X be a Polish space. Then every probability measure* $\mu : \mathcal{B}(X) \to [0,1]$ *is inner regular.*

We start by proving easy but useful result.

Lemma 4.4.1. *Let* (X, d) *be a separable metric space and let* $\mu : \mathcal{B}(X) \to [0,1]$ *be a probability measure. Fix a closed subset F of X. Then for every* $r > 0$ *and* $\epsilon > 0$*, there exists a closed subset* $F_{r,\epsilon}$ *of F such that*

$$\mu(F) \leq \mu(F_{r,\epsilon}) + \epsilon$$

and $F_{\epsilon,r}$ *admits a finite cover by closed balls in X each having radius r.*

Proof of the lemma. Let $\mathcal{B}$ be a family of all closed balls in X such that each of them has radius r. Then

$$F \subseteq \bigcup_{B\in\mathcal{B}} B$$

By separability of X there exists a countable subset $\{B_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}$ such that

$$F \subseteq \bigcup_{n\in\mathbb{N}} B_n$$

In particular, by continuity of measure it follows that

$$\mu(F) = \lim_{N\to+\infty} \mu\left(F \cap \bigcup_{n=0}^{N} B_n\right)$$

Hence for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$\mu(F) \leq \mu\left(F \cap \bigcup_{n=0}^{N_\epsilon} B_n\right) + \epsilon$$

It suffices to pick $F_{\epsilon,r} = F \cap \bigcup_{n=0}^{N_\epsilon} B_n$. □

Lemma 4.4.2. *Let X be a Polish space and let $\mu : \mathcal{B}(X) \to [0,1]$ be a probability measure. Then for every $\epsilon > 0$ there exists a compact subset K of X such that*

$$\mu(X) \leq \mu(K) + \epsilon$$

Proof of the lemma. Fix a complete and separable metric d on X. We construct a sequence $\{F_n\}_{n\in\mathbb{N}}$ of closed subsets of X as follows. We set $F_0 = X$ and if F_n is constructed, then we pick for F_{n+1} a closed subset of F_n such that

$$\mu(F_n) \leq \mu(F_{n+1}) + \frac{\epsilon}{2^{n+1}}$$

and F_{n+1} admits a finite cover by closed balls in X each having radius $\frac{1}{n+1}$. Such construction is possible according to Lemma 4.4.1. Next consider

$$K = \bigcap_{n\in\mathbb{N}} F_n$$

Then K is closed and for every $n \in \mathbb{N}$ it admits a finite cover by closed balls in X each having radius $\frac{1}{n+1}$. Since d is complete metric, it follows that K is a compact subset of X. Moreover, we have

$$\mu(X) \leq \mu(F_n) + \epsilon \cdot \left(\frac{1}{2} + \ldots + \frac{1}{2^n}\right)$$

for every $n \in \mathbb{N}$. Thus by continuity of μ we obtain

$$\mu(X) \leq \mu(K) + \epsilon$$

□

Proof of the theorem. Fix a Borel set A in X and fix $\epsilon > 0$. By Proposition 4.2 there exists a closed subset F of X such that $F \subseteq A$ and $\mu(A) \leq \mu(F) + \frac{\epsilon}{2}$. By Lemma 4.4.2 there exists a compact subset K of X such that $\mu(X) \leq \mu(K) + \frac{\epsilon}{2}$. Now we have

$$mu\,(A) \leq \mu(F) + \frac{\epsilon}{2} = \mu(F \cap K) + \mu(F \setminus K) + \frac{\epsilon}{2} \leq$$

$$\leq \mu(F \cap K) + \mu(X \setminus K) + \frac{\epsilon}{2} \leq \mu(F \cap K) + \frac{\epsilon}{2} + \frac{\epsilon}{2} = \mu(F \cap K) + \epsilon$$

Note that $F \cap K$ is a compact subset of X contained in A. Since A and $\epsilon > 0$ are arbitrary, we derive that μ is inner regular. □

5. Hilbert's cube

Definition 5.1. The topological product $[0,1]^{\mathbb{N}}$ is called *the Hilbert's cube*.

Fact 5.2. *Let (X_n, d_n) for $n \in \mathbb{N}$ are metric spaces and for every $n \in \mathbb{N}$ let τ_n be the topology on X_n induced by d_n. We define $d : \prod_{n\in\mathbb{N}} X_n \times \prod_{n\in\mathbb{N}} X_n \to [0, +\infty)$ by formula*

$$d\left((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\right) = \sup_{n\in\mathbb{N}} d_n(x_n, y_n)$$

Then d is a metric which induces product topology on $\prod_{n\in\mathbb{N}} X_n$.

Theorem 5.3 (Tychonoff)**.** *Let X be a completely regular space with weight $\mathfrak{m}$. Then there exists an immersion $i : X \hookrightarrow [0,1]^{\mathfrak{m}}$ of topological spaces.*

Proof. Consider an open base $\mathcal{B}$ of X having cardinality $\mathfrak{m}$. Fix B in $\mathcal{B}$. For every z in B let $f_{B,z} : X \to [0,1]$ be a continuous function such that $f_{B,z}(z) = 0$ and $X \setminus B \subseteq f_{B,z}^{-1}(1)$. Clearly

$$B = \bigcup_{z \in B} f_{B,z}^{-1}\left([0,1)\right)$$

Since $\mathcal{B}$ is of cardinality $\mathfrak{m}$, there exists a set $Z_B \subseteq B$ of cardinality $\mathfrak{m}$ such that

$$B = \bigcup_{z \in Z_B} f_{B,z}^{-1}\left([0,1)\right)$$

Denote $\mathcal{P} = \bigcup_{B \in \mathcal{B}} \left(\{B\} \times Z_B\right)$. Next define a map $i : X \to [0,1]^{\mathcal{P}}$ by formula $i(x) = \langle f_{B,z}(x) \rangle_{(B,z) \in \mathcal{P}}$. By universal property of cartesian products it follows that this map is continuous. For every (B,z) in $\mathcal{P}$ let $\pi_{B,z} : [0,1]^{\mathcal{P}} \to [0,1]$ be the projection. Then

$$i^{-1}\left(\pi_{B,z}^{-1}\left([0,1)\right)\right) = \left(\pi_{B,z} \cdot i\right)^{-1}\left([0,1)\right) = f_{B,z}^{-1}\left([0,1)\right)$$

and hence

$$i^{-1}\left(\bigcup_{z \in Z_B} \pi_{B,z}^{-1}\left([0,1)\right)\right) = \bigcup_{z \in Z_B} f_{B,z}^{-1}\left([0,1)\right) = B$$

for every B in $\mathcal{B}$. Therefore, in order to prove that i is an immersion of topological spaces it suffices to prove that it is injective. For this pick two distinct points x_1, x_2 in X. Then there exists B in $\mathcal{B}$ such that $x_1 \in B$ and $x_2 \notin B$. Then

$$x_1 \in \bigcup_{z \in Z_B} f_{B,z}^{-1}\left([0,1)\right),\ x_2 \notin \bigcup_{z \in Z_B} f_{B,z}^{-1}\left([0,1)\right)$$

Hence there exists $z \in Z_B$ such that $f_{B,z}(x_1) < 1$ and $f_{B,z}(x_2) = 1$. Thus $i(x_1) \neq i(x_2)$ and this completes the proof of the injectivity of i. Note that $\mathcal{P}$ is of cardinality $\mathfrak{m}$. Thus $i : X \hookrightarrow [0,1]^{\mathfrak{m}}$ is an immersion of topological spaces. □

Corollary 5.4. *Let X be a topological space. Then the following assertions are equivalent.*

- **(i)** *X is second countable and completely regular space.*
- **(ii)** *There exists an immersion $i : X \hookrightarrow [0,1]^{\mathbb{N}}$ of topological spaces.*
- **(iii)** *X is second countable and metrizable space.*

Proof. The implication **(i)** ⇒ **(ii)** follows from Theorem 5.3.
Suppose that there exists an immersion $i : X \hookrightarrow [0,1]^{\mathbb{N}}$. Note that Hilbert's cube is metrizable. For example define

$$d\left(\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}\right) = \sum_{n \in \mathbb{N}} 2^{-n} \cdot |x_n - y_n|$$

for every $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \in [0,1]^{\mathbb{N}}$. Then d is a metric which induces the Hilbert's cube topology. Let D_n be the subset of $[0,1]^{\mathbb{N}}$ consisting of sequences which have first n-elements rational and the remaining elements equal to zero. Then

$$\bigcup_{n \in \mathbb{N}} D_n \subseteq [0,1]^{\mathbb{N}}$$

is dense and countable subset. Thus $[0,1]^{\mathbb{N}}$ is second countable. Moreover, the subspace of a metrizable second countable space is itself metrizable and second countable. Thus **(ii)** ⇒ **(iii)** holds.
Suppose that X is metrizable and let $d : X \times X \to [0, +\infty)$ be the metric compatible with topology on X. Fix a point x in X and a closed subset F in X such that $x \notin F$. Then

$$f(z) = 1 - d(z, F)$$

□

References

[Monygham, 2018] Monygham (2018). Introduction to measure theory. *github repository: "Monygham/Pedo-mellon-a-minno"*.