UNIFORM SPACES

1. INTRODUCTION

These notes are devoted to general theory of uniform spaces, which are mathematical objects rigorously encapsulating the intuitive notion of uniformity. In the first section we introduce uniform spaces and their category **Unif**. We describe uniform structures induced by the family of maps, small limits in **Unif** and introduce embeddings of uniform spaces. Next we construct the canonical functor **Unif** $\rightarrow$ **Top** and show that it preserves small limits and embeddings. In the forth section we define complete uniform spaces and prove results concerning their properties with respect to products, embeddings and extensions of uniform morphisms. We also devote a separate section to study completion of a uniform space by means of minimal Cauchy filters. The sixth section is devoted to uniform Urysohn's lemma, which implies that the object-image of the canonical functor **Unif** $\rightarrow$ **Top** consists of completely regular spaces.

2. UNIFORM STRUCTURES AND UNIFORM SPACES

We start by recalling some elementary notions.

Definition 2.1. Let X be a set. The set

$$\Delta_X = \left\{(x,x) \in X \times X \,\middle|\, x \in X\right\}$$

is *the diagonal of* $X \times X$.

Definition 2.2. Let X, Y, Z be sets and let $U \subseteq X \times Y$ and $W \subseteq Y \times Z$ be relations. Consider

$$W \cdot U = \left\{(x,z) \in X \times Z \,\middle|\, (x,y) \in U \text{ and } (y,z) \in W \text{ for some } y \in Y\right\}$$

Then the relation $W \cdot U$ is *the composition of* U *and* W.

The following notion is the main object of our study in these notes.

Definition 2.3. Let X be a set. Suppose that $\mathfrak{U}$ is a collection of reflexive and symmetric relations on X which satisfies the following two assertions.

(1) If $U \in \mathfrak{U}$ and W is a reflexive and symmetric relation on X such that $U \subseteq W$, then $W \in \mathfrak{U}$.

(2) If $U, W \in \mathfrak{U}$, then $U \cap W \in \mathfrak{U}$.

(3) If $U \in \mathfrak{U}$, then there exists $W \in \mathfrak{U}$ such that $W \cdot W \subseteq U$.

Then $\mathfrak{U}$ is *a uniform structure on* X.

Example 2.4. Let X be a set. Then the family $\mathfrak{D}_X$ of all reflexive and symmetric relations on X is a uniform structure on X. It is called *the discrete uniform structure on* X.

Definition 2.5. A pair $(X, \mathfrak{U})$ consisting of a set X and a uniform structure $\mathfrak{U}$ on X is *a uniform space*.

Definition 2.6. Let $(X, \mathfrak{U})$ be a uniform space. Suppose that

$$\Delta_X = \bigcap_{U \in \mathfrak{U}} U$$

Then $(X, \mathfrak{U})$ is *a Hausdorff uniform space*.

Definition 2.7. Let $(X,\mathfrak{U}),(Y,\mathfrak{Y})$ be uniform spaces and let $f : X \to Y$ be a map. Suppose that $(f\times f)^{-1}(V) \in \mathfrak{U}$ for every $V \in \mathfrak{Y}$. Then f is *a morphism of uniform spaces*.

Remark 2.8. We denote by **Unif** the category which consists of uniform spaces and uniform morphisms with respect to the usual composition of maps.

Proposition 2.9. *Let X be a set and let $\{(X_i,\mathfrak{U}_i)\}_{i\in I}$ be a family of uniform spaces. Consider a family $\{f_i : X \to X_i\}_{i\in I}$ of maps. Then the following assertions hold.*

(1) *There exists the smallest (with respect to inclusion) uniform structure $\mathfrak{U}$ on X such that $f_i : (X,\mathfrak{U}) \to (X_i,\mathfrak{U}_i)$ is a uniform morphism for every $i \in I$.*

(2) *Let U be a reflexive and symmetric relation on X. Then $U \in \mathfrak{U}$ if and only if there exist $n \in \mathbb{N}_+$, $i_1,...,i_n \in I$ and $W_1 \in \mathfrak{U}_{i_1},...,W_n \in \mathfrak{U}_{i_n}$ such that*

$$\bigcap_{k=1}^{n}\left(f_{i_k}\times f_{i_k}\right)^{-1}(W_k) \subseteq U$$

Proof. Consider the family $\mathfrak{U}$ of all $U \in \mathfrak{D}_X$ such that there exist $n \in \mathbb{N}_+$, $i_1,...,i_n \in I$ and $W_1 \in \mathfrak{U}_{i_1},...,W_n \in \mathfrak{U}_{i_n}$ satisfying

$$\bigcap_{k=1}^{n}\left(f_{i_k}\times f_{i_k}\right)^{-1}(W_k) \subseteq U$$

It is easy to verify (we left it for the reader) that $\mathfrak{U}$ is a uniform structure on X. By definition of $\mathfrak{U}$ map f_i is a uniform morphism $(X,\mathfrak{U}) \to (X_i,\mathfrak{U}_i)$ for every $i \in I$. Moreover, if $\mathfrak{F}$ is a uniform structure on X which makes f_i into a uniform morphism $(X,\mathfrak{F}) \to (X_i,\mathfrak{U}_i)$ for every $i \in I$, then clearly $\mathfrak{U} \subseteq \mathfrak{F}$. □

Definition 2.10. Let X be a set, let $\{(X_i,\mathfrak{U}_i)\}_{i\in I}$ be a family of uniform spaces and let $\{f_i : X \to X_i\}_{i\in I}$ be a family of maps. Then the smallest (with respect to inclusion) uniform structure $\mathfrak{U}$ on X such that $f_i : (X,\mathfrak{U}) \to (X_i,\mathfrak{U}_i)$ is a uniform morphism for every $i \in I$ is called *the uniform structure induced by families $\{f_i\}_{i\in I}$ and $\{(X_i,\mathfrak{U}_i)\}_{i\in I}$*.

Now we describe small limits in **Unif**.

Theorem 2.11. *Let $\mathcal{I}$ be a small category and let $F : \mathcal{I} \to$ **Unif** be a functor given by $F(i) = (X_i,\mathfrak{U}_i)$ for $i \in \mathcal{I}$. Let $\{f_i : X \to X_i\}_{i\in\mathcal{I}}$ be a limiting cone of the composition of F with the functor **Unif** $\to$ **Set** which sends each uniform space to its underlying set. Consider the uniform structure $\mathfrak{U}$ induced by $\{f_i\}_{i\in\mathcal{I}}$ on X. Then $(X,\mathfrak{U})$ together with family $\{f_i\}_{i\in\mathcal{I}}$ is a limiting cone of F.*

Proof. We may equivalently describe $\mathfrak{U}$ as the smallest uniform structure on X such that

$$(f_i\times f_i)^{-1}(W) \in \mathfrak{U}$$

for every $i \in \mathcal{I}$ and every $W \in \mathfrak{U}_i$. Suppose that $\{g_i : (Z,\mathfrak{O}) \to (X_i,\mathfrak{U}_i)\}_{i\in\mathcal{I}}$ is some cone over F. Then there exists a unique map $g : Z \to X$ such that $f_i \cdot g = g_i$ for every $i \in \mathcal{I}$. It is easy to verify that

$$\left\{U \in \mathfrak{U} \,\middle|\, (g\times g)^{-1}(U) \in \mathfrak{O}\right\}$$

is a uniform structure on X. Moreover, it contains $(f_i\times f_i)^{-1}(W)$ for every $i \in \mathcal{I}$ and every $W \in \mathfrak{U}_i$. Since $\mathfrak{U}$ is the smallest such uniform structure, we derive that $\mathfrak{U}$ and

$$\left\{U \in \mathfrak{U} \,\middle|\, (g\times g)^{-1}(U) \in \mathfrak{O}\right\}$$

coincide and hence g is a morphism of uniform spaces $(Z,\mathfrak{O}) \to (X,\mathfrak{U})$. This shows that $(X,\mathfrak{U})$ together with $\{f_i : (X,\mathfrak{U}) \to (X_i,\mathfrak{U}_i)\}_{i\in\mathcal{I}}$ is a limiting cone of F. □

Definition 2.12. Let $j : (Z, \mathfrak{O}) \to (X, \mathfrak{U})$ be a morphism of uniform spaces. If j is injective and $\mathfrak{O}$ is the uniform structure induced by j, then j is *an embedding of uniform spaces*.

3. Topology induced by uniform structure

Definition 3.1. Let X be a set and let U be a symmetric and reflexive relation on X. Let Z be a subset of X. A set

$$U(Z) = \left\{x \in X \,\middle|\, (z, x) \in U \text{ for some } z \in Z\right\}$$

is *the U-neighborhood of Z*.

Remark 3.2. Let X be a set. For every x in X and every symmetric and reflexive relation U on X we denote by $U(x)$ the set $U(\{x\})$.

Fact 3.3. *Let X be a set and let $\mathfrak{U}$ be a uniform structure on X. The family*

$$\tau_{\mathfrak{U}} = \left\{\mathcal{O} \subseteq X \,\middle|\, \text{for each } x \in \mathcal{O} \text{ there exists } U \in \mathfrak{U} \text{ such that } U(x) \subseteq \mathcal{O}\right\}$$

is a topology on X.

Proof. We left the proof for the reader as an exercise. □

Definition 3.4. Let X be a set and let $\mathfrak{U}$ be a uniform structure on X. Then the topology $\tau_{\mathfrak{U}}$ is *the topology on X induced by $\mathfrak{U}$*.

Proposition 3.5. *Let $(X, \mathfrak{U})$ be a uniform space and let Z be a subset of X. Then $x \in Z$ is an interior point of Z with respect to $\tau_{\mathfrak{U}}$ if and only if there exists $U \in \mathfrak{U}$ such that $U(x) \subseteq Z$.*

Proof. The "only if" part is clear. For the proof of "if" part consider the set

$$\tilde{Z} = \left\{x \in Z \,\middle|\, U(x) \text{ is a subset of } Z \text{ for some } U \in \mathfrak{U}\right\}$$

It suffices to prove that $\tilde{Z}$ is open with respect to $\tau_{\mathfrak{U}}$. Fix x in $\tilde{Z}$. Then there exists $U \in \mathfrak{U}$ such that $U(x)$ is a subset of Z. Since $\mathfrak{U}$ is a uniform structure, there exists W in $\mathfrak{U}$ such that $W \cdot W \subseteq U$. Then for every $z \in W(x)$ we have $W(z) \subseteq U(x) \subseteq Z$. Hence if $z \in W(x)$, then $z \in \tilde{Z}$ and thus $\tilde{Z}$ is open with respect to $\tau_{\mathfrak{U}}$. □

The following consequence of the previous result is very useful.

Corollary 3.6. *Let $(X, \mathfrak{U})$ be a uniform space. For every $x \in X$ and $U \in \mathfrak{U}$ set $U(x)$ contains an open neighborhood of x with respect to $\tau_{\mathfrak{U}}$.*

Proof. By Proposition 3.5 point x is an interior point of $U(x)$ with respect to $\tau_{\mathfrak{U}}$. □

Fact 3.7. *Let $(X, \mathfrak{U})$ and $(Y, \mathfrak{Y})$ be uniform spaces and let $f : X \to Y$ be a morphism of uniform spaces. Then f is a continuous map $(X, \tau_{\mathfrak{U}}) \to \left(Y, \tau_{\mathfrak{Y}}\right)$.*

Proof. Pick an open subset $\mathcal{O}$ with respect to the topology induced by $\mathfrak{Y}$ on Y. Suppose that $f(x) \in \mathcal{O}$ for some x in X. Then there exists $V_x \in \mathfrak{Y}$ such that $V_x(f(x)) \subseteq \mathcal{O}$. Note that the image of $\left((f \times f)^{-1}(V_x)\right)(x)$ under f is contained in $\mathcal{O}$. Therefore,

$$f^{-1}(\mathcal{O}) = \bigcup_{x \in f^{-1}(\mathcal{O})} \left((f \times f)^{-1}(V_x)\right)(x)$$

is open in the topology induced by $\mathfrak{U}$. □

Remark 3.8. According to Fact 3.7 topology induced by uniformity determines a functor **Unif** $\to$ **Top**.

Proposition 3.9. *Let X be a set and let $\{(X_i, \mathfrak{U}_i)\}_{i\in I}$ be a family of uniform spaces. Consider a family $\{f_i : X \to X_i\}_{i\in I}$ of maps. If $\mathfrak{U}$ is the uniform structure induced on X by $\{f_i\}_{i\in I}$, then $\tau_{\mathfrak{U}}$ is the topology induced by the family of maps $\{f_i\}_{i\in I}$ with codomains $\{(X_i, \tau_{\mathfrak{U}_i})\}_{i\in I}$.*

Proof. Let τ denote the topology on X induced by the family of maps $\{f_i\}_{i\in I}$ with codomains $\{(X_i, \tau_{\mathfrak{U}_i})\}_{i\in I}$. Our goal is to prove that $\tau_{\mathfrak{U}} = \tau$. Pick an open subset $\mathcal{O}$ of $\tau_{\mathfrak{U}}$. Fix x in $\mathcal{O}$. According to definition of $\tau_{\mathfrak{U}}$ and Proposition 2.9 there exist $n \in \mathbb{N}$, $i_1, ..., i_n \in I$ and $V_{i_1} \in \mathfrak{U}_{i_1}, ..., V_{i_n} \in \mathfrak{U}_{i_n}$ such that

$$\left(\bigcap_{k=1}^{n} \left(f_{i_k} \times f_{i_k}\right)^{-1}\left(V_{i_k}\right)\right)(x) \subseteq \mathcal{O}$$

We have

$$\bigcap_{k=1}^{n} f_{i_k}^{-1}\Big(V_{i_k}\big(f_{i_k}(x)\big)\Big) = \bigcap_{k=1}^{n}\left(\left(f_{i_k} \times f_{i_k}\right)^{-1}\left(V_{i_k}\right)\right)(x) = \left(\bigcap_{k=1}^{n} \left(f_{i_k} \times f_{i_k}\right)^{-1}\left(V_{i_k}\right)\right)(x)$$

By Corollary 3.6 for each k there exists $\mathcal{O}_k \in \tau_{\mathfrak{U}_{i_k}}$ such that

$$f_{i_k}(x) \in \mathcal{O}_k \subseteq V_{i_k}\left(f_{i_k}(x)\right)$$

Therefore, we deduce that

$$x \in \bigcap_{k=1}^{n} f_{i_k}^{-1}(\mathcal{O}_k) \subseteq \mathcal{O}$$

and hence $\mathcal{O}$ is open with respect to τ. This means that $\tau_{\mathfrak{U}}$ is coarser than τ. On the other hand $f_i : (X, \tau_{\mathfrak{U}}) \to \left(X_i, \tau_{\mathfrak{U}_i}\right)$ is continuous for every $i \in I$ and thus $\tau_{\mathfrak{U}}$ is stronger than τ. Therefore, we have $\tau_{\mathfrak{U}} = \tau$. □

Corollary 3.10. *The functor*

$$\mathbf{Unif} \ni (X, \mathfrak{U}) \mapsto (X, \tau_{\mathfrak{U}}) \in \mathbf{Top}$$

preserves small limits.

Proof. This follows from the combination of Theorem 2.11 and Proposition 3.9 □

Corollary 3.11. *If $j : (Z, \mathfrak{D}) \hookrightarrow (X, \mathfrak{U})$ is an embedding of uniform spaces, then $j : (Z, \tau_{\mathfrak{D}}) \to (X, \tau_{\mathfrak{U}})$ is an embedding of topological spaces.*

Proof. This follows from definition of embeddings of uniform spaces and Proposition 3.9. □

Fact 3.12. *Let $(X, \mathfrak{U})$ be a uniform space. Then $(X, \mathfrak{U})$ is a Hausdorff uniform space if and only if $(X, \tau_{\mathfrak{U}})$ is a Hausdorff topological space.*

Proof. Left for the reader as an exercise. □

4. Cauchy filters and complete uniform spaces

In this section we study very important notion of completeness of uniform spaces. For this we use the notion of filter of subsets defined in [Monygham, 2022].

Definition 4.1. Let $(X, \mathfrak{U})$ be a uniform space. Suppose that $\mathcal{F}$ is a proper filter of subsets of X. Assume that for every $U \in \mathfrak{U}$ there exists $F \in \mathcal{F}$ such that $F \times F \subseteq U$. Then $\mathcal{F}$ is *a Cauchy filter in* $(X, \mathfrak{U})$.

First we prove that the image of a Cauchy filter under a morphism of uniform spaces is a Cauchy filter.

Fact 4.2. *Let $f : (X, \mathfrak{U}) \to (Y, \mathfrak{Y})$ be a morphism of uniform spaces. If $\mathcal{F}$ is a Cauchy filter in $(X, \mathfrak{U})$, then $f(\mathcal{F})$ is a Cauchy filter in $(Y, \mathfrak{Y})$.*

Proof. Pick $V \in \mathfrak{Y}$. Since $\mathcal{F}$ is a Cauchy filter in $(X,\mathfrak{U})$ and f is a morphism of uniform spaces, there exists $F \in \mathcal{F}$ such that $F \times F \subseteq (f \times f)^{-1}(V)$. Then $f(F) \times f(F) \subseteq V$. Since $f(F) \in f(\mathcal{F})$ and V is an arbitrary element of $\mathfrak{Y}$, this proves the assertion. □

Definition 4.3. Let $(X,\mathfrak{U})$ be a uniform space. Suppose that every Cauchy filter in in $(X,\mathfrak{U})$ is convergent with respect to $\tau_{\mathfrak{U}}$. Then $(X,\mathfrak{U})$ is *a complete uniform space.*

The following theorem is analogical to famous Tychonoff's theorem for compact topological spaces.

Theorem 4.4. *Let $(X_i,\mathfrak{U}_i)$ be complete uniform spaces for $i \in I$. Then the product*

$$\prod_{i\in I}(X_i,\mathfrak{U}_i)$$

is a complete uniform space.

Proof. Let $X = \prod_{i\in I} X_i$ and let $\mathfrak{U}$ be the product uniform structure of $\mathfrak{U}_i$ for $i \in I$. For each $i \in I$ we denote by $pr_i : X \to X_i$ the canonical projection. Suppose that $(X_i,\mathfrak{U}_i)$ is a complete uniform space for every $i \in I$. Fix a Cauchy filter $\mathcal{F}$ in $(X,\mathfrak{U})$. Then $\mathcal{F}_i = pr_i(\mathcal{F})$ is a Cauchy filter on $(X_i,\mathfrak{U}_i)$ for every $i \in I$ according to Fact 4.2. Since $(X_i,\mathfrak{U}_i)$ is a complete uniform space and $\mathcal{F}_i$ is a Cauchy filter, we derive that $\mathcal{F}_i$ is convergent to some point $x_i \in X_i$ with respect to $\tau_{\mathfrak{U}_i}$ for each $i \in I$. Let x be a point in X such that $pr_i(x) = x_i$ for every $i \in I$. Then $\mathcal{F}$ is convergent to x with respect to the product of topologies $\tau_{\mathfrak{U}_i}$ for $i \in I$. By Corollary 3.10 we infer that $\tau_{\mathfrak{U}}$ is the product of $\tau_{\mathfrak{U}_i}$ for $i \in I$. Hence $\mathcal{F}$ is convergent to x with respect to $\tau_{\mathfrak{U}}$. Thus $(X,\mathfrak{U})$ is a complete uniform space. □

Theorem 4.5. *Let $(X_i,\mathfrak{U}_i)$ be nonempty uniform spaces for $i \in I$. If*

$$\prod_{i\in I}(X_i,\mathfrak{U}_i)$$

is a complete uniform space, then $(X_i,\mathfrak{U}_i)_{i\in I}$ is complete for every $i \in I$.

Proof. Denote $X = \prod_{i\in I} X_i$ and let $\mathfrak{U}$ be the product uniform structure induced by $\mathfrak{U}_i$ for all $i \in I$. For each $i \in I$ we denote by $pr_i : X \to X_i$ the canonical projection. Assume that $(X,\mathfrak{U})$ is a complete uniform space. Fix $i_0 \in I$. Suppose that $\mathcal{F}$ is a Cauchy filter in $(X_{i_0},\mathfrak{U}_{i_0})$. Since $X_i \neq \varnothing$ for every $i \in I$, we may pick $z \in \prod_{i\neq i_0} X_i$. Consider a filter

$$\tilde{\mathcal{F}} = \left\{\tilde{F} \subseteq X \,\middle|\, F \times \{z\} \subseteq \tilde{F} \text{ for some } F \in \mathcal{F}\right\}$$

Then $\tilde{\mathcal{F}}$ is a Cauchy filter in $(X,\mathfrak{U})$ and $\mathcal{F} = pr_{i_0}\left(\tilde{\mathcal{F}}\right)$. Since $\tilde{\mathcal{F}}$ is a Cauchy filter in a complete uniform space $(X,\mathfrak{U})$, it is convergent with respect to $\tau_{\mathfrak{U}}$ to some $x \in X$. Since $\tau_{\mathfrak{U}}$ is a product of topologies $\tau_{\mathfrak{U}_i}$ for $i \in I$ according to Corollary 3.10, we derive that $pr_{i_0}\left(\tilde{\mathcal{F}}\right)$ is convergent to $pr_{i_0}(x)$ with respect to $\tau_{\mathfrak{U}_{i_0}}$. Finally according to $\mathcal{F} = pr_{i_0}\left(\tilde{\mathcal{F}}\right)$ we derive that $\mathcal{F}$ is convergent to $pr_{i_0}(x)$. This shows that $(X_{i_0},\mathfrak{U}_{i_0})$ is a complete uniform space. Since $i_0 \in I$ is arbitrary, we derive that $(X_i,\mathfrak{U}_i)$ is a complete uniform space for every $i \in I$. □

Now we study completeness under embeddings of uniform spaces.

Theorem 4.6. *Let $j : (Z,\mathfrak{D}) \hookrightarrow (X,\mathfrak{U})$ be an embedding of uniform spaces. If $(X,\mathfrak{U})$ is a complete uniform space and $j(Z)$ is closed with respect to $\tau_{\mathfrak{U}}$, then $(Z,\mathfrak{D})$ is a complete uniform space.*

Proof. Pick a Cauchy filter $\mathcal{F}$ in $(Z,\mathfrak{D})$. Then according to Fact 4.2 the filter $j(\mathcal{F})$ is a Cauchy filter in $(X,\mathfrak{U})$. Thus $j(\mathcal{F})$ converges to some point $x \in X$ with respect to $\tau_{\mathfrak{U}}$. Since $j(Z)$ is closed in $\tau_{\mathfrak{U}}$, we infer that $x \in j(Z)$. Thus $x = j(z)$ for some $z \in Z$. According to Corollary 3.11 the map $j : (Z,\tau_{\mathfrak{D}}) \hookrightarrow (X,\tau_{\mathfrak{U}})$ is a topological embedding. Hence every open neighborhood of z in $(Z,\tau_{\mathfrak{D}})$ is of the form $j^{-1}(\mathcal{O})$ for an open neighborhood $\mathcal{O}$ of $j(z)$ in $(X,\tau_{\mathfrak{U}})$. Since $j(\mathcal{F})$ is convergent to $j(z)$ in $\tau_{\mathfrak{U}}$, we have $\mathcal{O} \in j(\mathcal{F})$ and hence $j^{-1}(\mathcal{O}) \in \mathcal{F}$. This proves that $\mathcal{F}$ is convergent to z in $\tau_{\mathfrak{O}}$. □

Theorem 4.7. *Let $j : (Z,\mathfrak{O}) \hookrightarrow (X,\mathfrak{U})$ be an embedding of uniform spaces. If $(X,\mathfrak{U})$ is a Hausdorff uniform space and $(Z,\mathfrak{O})$ is a complete uniform space, then $j(Z)$ is closed with respect to $\tau_{\mathfrak{U}}$.*

Proof. Fix a point x of X in the closure of $j(Z)$ with respect to $\tau_{\mathfrak{U}}$. Consider the filter

$$\mathcal{F} = \left\{F \subseteq Z \,\middle|\, j^{-1}\left(U(x)\right) \subseteq F \text{ for some } U \in \mathfrak{U}\right\}$$

According to Corollary 3.6 and the fact that x is in the closure of $j(Z)$ with respect to $\tau_{\mathfrak{U}}$, we derive that $\mathcal{F}$ is a proper filter of subsets of Z. Since j is an embedding of uniform spaces, every element of $\mathfrak{O}$ is of the form $(j\times j)^{-1}(U)$ for some $U \in \mathfrak{U}$. Fix some $U \in \mathfrak{U}$ and consider $W \in \mathfrak{U}$ such that $W \cdot W \subseteq U$. Note that

$$j^{-1}\left(U(x)\right) \times j^{-1}\left(U(x)\right) \subseteq (j\times j)^{-1}\left(W\cdot W\right) \subseteq (j\times j)^{-1}\left(U\right)$$

This shows that $\mathcal{F}$ is a Cauchy filter in $(Z,\mathfrak{O})$. Since $(Z,\mathfrak{O})$ is complete, filter $\mathcal{F}$ is convergent to some z in Z with respect to $\tau_{\mathfrak{O}}$. Hence $j(\mathcal{F})$ is convergent to $j(z)$ with respect to $\tau_{\mathfrak{U}}$. Clearly $U(x) \in j(\mathcal{F})$ for every $U \in \mathfrak{U}$. This implies that $j(\mathcal{F})$ is convergent to x with respect to $\tau_{\mathfrak{U}}$. Since $(X,\mathfrak{U})$ is Hausdorff, Fact 3.12 implies that $(X,\tau_{\mathfrak{U}})$ is Hausdorff and hence we deduce that $x = j(z)$. This completes the proof that $j(Z)$ is closed with respect to $\tau_{\mathfrak{U}}$. □

Finally we discuss extensions of uniform morphisms defined on dense uniform subspaces and with values in complete uniform spaces.

Theorem 4.8. *Let $j : (Z,\mathfrak{O}) \hookrightarrow (X,\mathfrak{U})$ be an embedding of uniform spaces and let $f : (Z,\mathfrak{O}) \to (Y,\mathfrak{Y})$ be a uniform morphism. If $(Y,\mathfrak{Y})$ is a complete uniform space and $j(Z)$ is dense in X with respect to $\tau_{\mathfrak{U}}$, then there exists a uniform morphism $\tilde{f} : (X,\mathfrak{U}) \to (Y,\mathfrak{Y})$ such that $\tilde{f}\cdot j = f$. Moreover, if $(Y,\mathfrak{Y})$ is Hausdorff, then $\tilde{f}$ is unique.*

Proof. For each point x in X define a filter of subsets of Z by formula

$$\mathcal{F}_x = \left\{F \subseteq Z \,\middle|\, j^{-1}\left(U(x)\right) \subseteq F \text{ for some } U \in \mathfrak{U}\right\}$$

Since $j(Z)$ is dense in X with respect to $\tau_{\mathfrak{U}}$ and by Corollary 3.6, the filter $\mathcal{F}_x$ is proper. Since j is an embedding of uniform spaces, every element of $\mathfrak{O}$ is of the form $(j\times j)^{-1}(U)$ for some $U \in \mathfrak{U}$. Fix some $U \in \mathfrak{U}$ and consider $W \in \mathfrak{U}$ such that $W\cdot W \subseteq U$. Note that

$$j^{-1}\left(U(x)\right) \times j^{-1}\left(U(x)\right) \subseteq (j\times j)^{-1}\left(W\cdot W\right) \subseteq (j\times j)^{-1}\left(U\right)$$

This shows that $\mathcal{F}_x$ is a Cauchy filter in $(Z,\mathfrak{O})$. By Fact 4.2 the filter $f(\mathcal{F}_x)$ is Cauchy in $(Y,\mathfrak{Y})$. Since $(Y,\mathfrak{Y})$ is a complete uniform space, we derive that $f(\mathcal{F}_x)$ is convergent with respect to $\tau_{\mathfrak{Y}}$. If $z \in Z$, then according to Corollary 3.6 and Corollary 3.11 filter $\mathcal{F}_{j(z)}$ contains all open neighborhoods of z with respect to $\tau_{\mathfrak{O}}$ and hence $\mathcal{F}_{j(z)}$ is convergent to z with respect to $\tau_{\mathfrak{O}}$. Hence $f(\mathcal{F}_{j(z)})$ is convergent to $f(z)$ with respect to $\tau_{\mathfrak{Y}}$. We define $\tilde{f} : X \to Y$ in such a way that $f(\mathcal{F}_x)$ converges to $\tilde{f}(x)$ with respect to $\tau_{\mathfrak{Y}}$ for every $x \in X$ and $\tilde{f}\cdot j = f$.
We claim that $\tilde{f}$ is a uniform morphism $(X,\mathfrak{U}) \to (Y,\mathfrak{Y})$. For this fix $V \in \mathfrak{Y}$ and consider $E \in \mathfrak{Y}$ such that $E\cdot E\cdot E \subseteq V$. Then $(f\times f)^{-1}(E) \in \mathfrak{O}$. Since j is an embedding of uniform spaces, there exists $U \in \mathfrak{U}$ such that

$$(j\times j)^{-1}\left(U\cdot U\cdot U\right) \subseteq (f\times f)^{-1}\left(E\right)$$

Pick now $x_1, x_2 \in X$ such that $(x_1,x_2) \in U$. Denote $j^{-1}\big(U(x_1)\cup U(x_2)\big)$ by F. First note that F is an element of both $\mathcal{F}_{x_1}$ and $\mathcal{F}_{x_2}$. Filter $f(\mathcal{F}_i)$ is convergent to $\tilde{f}(x_i)$ in $\tau_{\mathfrak{Y}}$ for $i = 1,2$. By Corollary 3.6 we deduce that $f(F)$ has nonempty intersection with $E(\tilde{f}(x_i))$ for $i = 1,2$. On the other hand

$$F \times F \subseteq (j\times j)^{-1}\left(U\cdot U\cdot U\right) \subseteq (f\times f)^{-1}\left(E\right)$$

and hence $f(F)\times f(F)\subseteq E$. Thus

$$\big(\tilde{f}(x_1),\tilde{f}(x_2)\big)\in E\cdot E\cdot E\subseteq V$$

and this implies that for every $x_1,x_2\in X$ if $(x_1,x_2)\in U$, then $\big(\tilde{f}(x_1),\tilde{f}(x_2)\big)\in V$. Thus $\tilde{f}$ is a morphism of uniform spaces.
Suppose that $(Y,\mathfrak{Y})$ is Hausdorff. Fact 3.12 implies that $(Y,\tau_{\mathfrak{Y}})$ is a Hausdorff topological space. Moreover, $\tilde{f}$ is a continuous map $(X,\tau_{\mathfrak{U}})\to\big(Y,\tau_{\mathfrak{Y}}\big)$ and by Corollary 3.11 we have dense embedding of topological spaces $j:(Z,\tau_{\mathfrak{D}})\hookrightarrow(X,\tau_{\mathfrak{U}})$. Since each continuous map with codomain in Hausdorff topological space is uniquely determined by its restriction to the dense subspace of its domain, we derive that the map $\tilde{f}$ such that $\tilde{f}\cdot j=f$ is unique. □

5. Completion of uniform spaces

Definition 5.1. Let $(X,\mathfrak{U})$ be a uniform space and let $\mathcal{F}_1,\mathcal{F}_2$ be Cauchy filters in $(X,\mathfrak{U})$. If for every $U\in\mathfrak{U}$ there exist $F_1\in\mathcal{F}_1$ and $F_2\in\mathcal{F}_2$ such that $F_1\times F_2\subseteq U$, then $\mathcal{F}_1$ *is equivalent to* $\mathcal{F}_2$ *in* $(X,\mathfrak{U})$.

Fact 5.2. *Let $(X,\mathfrak{U})$ be a uniform space. Then equivalence of Cauchy filters in $(X,\mathfrak{U})$ is reflexive, symmetric and transitive relation.*

Proof. Left for the reader as an exercise. □

Fact 5.3. *Let $(X,\mathfrak{U})$ be a uniform space. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are proper filters of subsets in X such that $\mathcal{F}_1$ is a Cauchy filter in $(X,\mathfrak{U})$ and $\mathcal{F}_1\subseteq\mathcal{F}_2$, then $\mathcal{F}_2$ is a Cauchy filter in $(X,\mathfrak{U})$ equivalent to $\mathcal{F}_2$.*

Proof. Left for the reader as an exercise. □

Definition 5.4. Let $(X,\mathfrak{U})$ be a uniform space. *A minimal Cauchy filter in* $(X,\mathfrak{U})$ is a minimal element of the set of all Cauchy filters in $(X,\mathfrak{U})$ ordered by inclusion.

The following result describes main properties of minimal Cauchy filters.

Theorem 5.5. *Let $(X,\mathfrak{U})$ be a uniform space and let $\mathbf{K}$ be an equivalence class of Cauchy filters in $(X,\mathfrak{U})$. Then the following assertions hold.*

(1)
$$\bigcap\mathbf{K}=\left\{Z\subseteq X\,\middle|\,U(F)\subseteq Z\text{ for some }\mathcal{F}\in\mathbf{K},\,F\in\mathcal{F}\text{ and }U\in\mathfrak{U}\right\}$$

(2) $\bigcap\mathbf{K}$ *is an element of* $\mathbf{K}$.

(3) $\bigcap\mathbf{K}$ *is a minimal Cauchy filter in* $(X,\mathfrak{U})$.

Proof. For fixed filter $\mathcal{F}\in\mathbf{K}$ we denote by $\mathfrak{U}(\mathcal{F})$ the family

$$\left\{Z\subseteq X\,\middle|\,U(F)\subseteq Z\text{ for some }F\in\mathcal{F}\text{ and }U\in\mathfrak{U}\right\}$$

Note that $\mathfrak{U}(\mathcal{F})$ is a Cauchy filter in $(X,\mathfrak{U})$ such that $\mathfrak{U}(\mathcal{F})\subseteq\mathcal{F}$. Thus by Fact 5.3 it follows that $\mathfrak{U}(\mathcal{F})\in\mathbf{K}$ for every $\mathcal{F}\in\mathbf{K}$. Thus we have

$$\bigcap\mathbf{K}=\bigcap_{\mathcal{F}\in\mathbf{K}}\mathfrak{U}(\mathcal{F})$$

Fix now $\mathcal{F}_1,\mathcal{F}_2\in\mathbf{K}$, $F\in\mathcal{F}_1$ and $U\in\mathfrak{U}$. Pick $W\in\mathfrak{U}$ such that $W\cdot W\subseteq U$. Then there exist $F_1\in\mathcal{F}_1$ and $F_2\in\mathcal{F}_2$ such that $F_1\times F_2\subseteq W$. In particular, we have $(F\cap F_1)\times F_2\subseteq W$. Hence

$$W(F_2)\subseteq U\left(F\cap F_1\right)\subseteq U(F)$$

It follows that $U(F) \in \mathfrak{U}(\mathcal{F}_2)$. This proves that $\mathfrak{U}(\mathcal{F}_1) \subseteq \mathfrak{U}(\mathcal{F}_2)$. By symmetry we deduce that $\mathfrak{U}(\mathcal{F}_1) = \mathfrak{U}(\mathcal{F}_2)$ and this holds for each pair $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{K}$. Combining this with the fact that $\bigcap \mathbf{K}$ is the intersection of all $\mathfrak{U}(\mathcal{F})$ for $\mathcal{F} \in \mathbf{K}$, we deduce that

$$\bigcap \mathbf{K} = \mathfrak{U}(\mathcal{F})$$

for every $\mathcal{F} \in \mathbf{K}$. This completes the proof of **(1)** and **(2)**. The assertion **(3)** follows from Fact 5.3. □

Theorem 5.6. *Let $(X, \mathfrak{U})$ be a uniform space. Then there exists a complete Hausdorff uniform space $(\hat{X}, \hat{\mathfrak{U}})$ and a morphism $j : (X, \mathfrak{U}) \to (\hat{X}, \hat{\mathfrak{U}})$ of uniform spaces such that the following assertions hold.*

(1) *Equivalence classes of relation*

$$\bigcap_{U \in \mathfrak{U}} U$$

coincide with fibers of j.

(2) *$j(X)$ is dense in $\hat{X}$ with respect to topology induced by $\hat{\mathfrak{U}}$.*

(3) *$\mathfrak{U}$ is induced by j and $(\hat{X}, \hat{\mathfrak{U}})$.*

Proof. Let $\hat{X}$ be the set of all minimal Cauchy filters in $(X, \mathfrak{U})$. For $U \in \mathfrak{U}$ we set

$$\hat{U} = \left\{ (\mathbf{x}_1, \mathbf{x}_2) \in \hat{X} \times \hat{X} \,\middle|\, \text{there exist } F_1 \in \mathbf{x}_1, F_2 \in \mathbf{x}_2 \text{ such that } F_1 \times F_2 \subseteq U \right\}$$

Clearly $\hat{U} \in \mathfrak{D}_{\hat{X}}$. We define

$$\hat{\mathfrak{U}} = \left\{ \mathbf{U} \in \mathfrak{D}_{\hat{X}} \,\middle|\, \hat{U} \subseteq \mathbf{U} \text{ for some } U \in \mathfrak{U} \right\}$$

For each $x \in X$ consider the minimal Cauchy filter $j(x)$ in $(X, \mathfrak{U})$ given by formula

$$\left\{ F \subseteq X \,\middle|\, U(x) \subseteq F \text{ for some } U \in \mathfrak{U} \right\}$$

This gives rise to a map $j : X \to \hat{X}$. We are going to prove the theorem in a series of claims.
We claim that $(\hat{X}, \hat{\mathfrak{U}})$ is a uniform space. Note that

$$\hat{U}_1 \cdot \hat{U}_2 \subseteq \widehat{U_1 \cdot U_2},\ \hat{U}_1 \cap \hat{U}_2 = \widehat{U_1 \cap U_2}$$

for every $U_1, U_2 \in \mathfrak{U}$ and if in addition $U_1 \subseteq U_2$ for $U_1, U_2 \in \mathfrak{U}$, then also $\hat{U}_1 \subseteq \hat{U}_2$. These assertions imply that $\hat{\mathfrak{U}}$ is a uniform structure on $\hat{X}$ and the claim is proved.
Next we claim that j is a morphism of uniform spaces $(X, \mathfrak{U}) \to (\hat{X}, \hat{\mathfrak{U}})$ such that $\mathfrak{U}$ is induced by j and $(\hat{X}, \hat{\mathfrak{U}})$. This follows from the fact that we have inclusions

$$W \subseteq (j \times j)^{-1}(\hat{U}) \subseteq U$$

which hold for every $W, U \in \mathfrak{U}$ such that $W \cdot W \subseteq U$.
Now we show that if $\mathbf{x}$ is a minimal Cauchy filter in $(X, \mathfrak{U})$, then $\hat{U}(\mathbf{x}) \in j(\mathbf{x})$ for every $U \in \mathfrak{U}$. For this we again fix $U, W \in \mathfrak{U}$ such that $W \cdot W \subseteq U$. Since there exists $F \in \mathbf{x}$ such that $F \times F \subseteq W$, we derive that for every $x \in F$ we have $W(x) \times F \subseteq U$. Hence for every $x \in F$ we have $j(x) \in \hat{U}$. Thus $F \subseteq j^{-1}\left(\hat{U}(\mathbf{x})\right)$. Therefore, $j^{-1}\left(\hat{U}(\mathbf{x})\right) \in \mathbf{x}$ for every $U \in \mathfrak{U}$. Hence $\hat{U}(\mathbf{x}) \subseteq j(\mathbf{x})$ for each $U \in \mathfrak{U}$ and each $\mathbf{x} \in \hat{X}$. This proves the claim.
The claim above combined with Corollary 3.6 this implies that $j(X)$ is dense in $\hat{X}$ with respect to the topology induced by $\hat{\mathfrak{U}}$.
Next we claim that $(\hat{X}, \hat{\mathfrak{U}})$ is complete. Suppose that $\mathfrak{F}$ is a minimal Cauchy filter in $(\hat{X}, \hat{\mathfrak{U}})$. By Theorem 5.5 and Proposition 3.5 sets in $\mathfrak{F}$ have nonempty interiors with respect to the topology induced by $\hat{\mathfrak{U}}$. Hence, according to the fact that $j(X)$ is dense in $\hat{X}$ with respect to $\tau_{\hat{\mathfrak{U}}}$, family

$$j^{-1}(\mathfrak{F}) = \left\{ F \subseteq X \,\middle|\, j^{-1}(\hat{F}) \subseteq F \text{ for some } \hat{F} \in \mathfrak{F} \right\}$$

is a proper filter of subsets of X. Since $\mathfrak{U}$ is induced by j and $\left(\hat{X},\hat{\mathfrak{U}}\right)$, we derive that $j^{-1}(\mathfrak{F})$ is a Cauchy filter in $(X,\mathfrak{U})$. Theorem 5.5 shows that there exists a minimal Cauchy filter $\mathbf{x}$ such that $\mathbf{x} \subseteq j^{-1}(\mathfrak{F})$. Then $j(\mathbf{x}) \subseteq j\left(j^{-1}(\mathfrak{F})\right)$ and hence $\hat{U}(\mathbf{x}) \in j\left(j^{-1}(\mathfrak{F})\right)$ for each $U \in \mathfrak{U}$. This together with Corollary 3.6 shows that $j\left(j^{-1}(\mathfrak{F})\right)$ converges to $\mathbf{x}$ with respect to the topology induced by $\hat{\mathfrak{U}}$. Clearly we have $\mathfrak{F} \subseteq j\left(j^{-1}(\mathfrak{F})\right)$ and Fact 5.3 implies that $\mathfrak{F}$ and $j\left(j^{-1}(\mathfrak{F})\right)$ are equivalent Cauchy filters in $\left(\hat{X},\hat{U}\right)$. Hence $\mathfrak{F}$ is also convergent to $\mathbf{x}$ with respect to $\tau_{\hat{\mathfrak{U}}}$. This proves that every minimal Cauchy filter in $(\hat{X},\hat{\mathfrak{U}})$ is convergent with respect to $\tau_{\hat{\mathfrak{U}}}$. According to Theorem 5.5 it follows that $(\hat{X},\hat{\mathfrak{U}})$ is complete.

Now we prove that $\left(\hat{X},\hat{U}\right)$ is Hausdorff. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are distinct minimal Cauchy filters in $(X,\mathfrak{U})$, then they are not equivalent and hence there exists $U \in \mathfrak{U}$ such that for all $F_1 \in \mathbf{x}_1$ and $F_2 \in \mathbf{x}_2$ we have $(F_1 \times F_2) \smallsetminus U \neq \emptyset$. This shows that $(\mathbf{x}_1,\mathbf{x}_2) \notin \hat{U}$ and the claim on Hausdorffness of $\left(\hat{X},\hat{\mathfrak{U}}\right)$ follows.

Finally we prove that fibers of j coincide with equivalence classes of the relation given by the intersection of all elements of $\mathfrak{U}$. For this fix $x_1, x_2 \in X$. Then $j(x_1) = j(x_2)$ if and only if for every $U \in \mathfrak{U}$ there exists $W_1, W_2 \in \mathfrak{U}$ such that

$$W_1(x_1) \subseteq U(x_2),\ W_2(x_2) \subseteq U(x_1)$$

This last assertion is equivalent to

$$(x_1,x_2) \in \bigcap_{U \in \mathfrak{U}} U$$

The proof of the claim is completed and thus the theorem is proved. □

6. Uniform Urysohn's lemma

The following result is a uniform version of Urysohn lemma.

Theorem 6.1. *Let X be a set and let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of reflexive and symmetric relations on X such that*

$$U_{n+1} \cdot U_{n+1} \subseteq U_n$$

for every $n \in \mathbb{N}$. Let Z be a subset of X. Then there exists a map $f : X \to \mathbb{R}$ such that the following assertions hold.

(1) *The inequality*

$$0 \le f(x) \le 1$$

holds for each x in X.

(2) *$f_{|Z}$ is the constant zero function on Z.*

(3) *Fix $n \in \mathbb{N}$. If*

$$f(x) < \frac{1}{2^n}$$

for some x in X, then x in $U_n(Z)$.

(4) *If $x_1, x_2 \in X$ and $(x_1,x_2) \in U_{n+1}$ for some $n \in \mathbb{N}$, then the inequality*

$$|f(x_1) - f(x_2)| < \frac{1}{2^n}$$

holds.

Proof. For each $n \in \mathbb{N}$ and each integer $k \in \{0,1,...,2^n\}$ we construct a set $Z_{\frac{k}{2^n}}$. The construction goes by recursion on $n \in \mathbb{N}$. We set $Z_0 = Z$ and $Z_1 = U_0(Z)$. Next suppose that for $n \in \mathbb{N}$ and $k \in \{0,1,...,2^n\}$ sets $Z_{\frac{k}{2^n}}$ are defined. Then we define

$$Z_{\frac{2k+1}{2^{n+1}}} = U_{n+1}\left(Z_{\frac{k}{2^n}}\right)$$

for each $k \in \{0, 1, ..., 2^n\}$. By construction

$$Z_{\frac{k}{2^n}} \subseteq U_n\left(Z_{\frac{k}{2^n}}\right) \subseteq Z_{\frac{k+1}{2^n}}$$

for every $n \in \mathbb{N}$ and every $k \in \{0, 1, ..., 2^n\}$. If $x \in U_0(Z)$, then we define

$$f(x) = \inf\left\{\frac{k}{2^n}\,\middle|\, x \in Z_{\frac{k}{2^n}} \text{ for some } n \in \mathbb{N} \text{ and } k \in \{0, 1, ..., 2^n\}\right\}$$

For $x \notin U_0(Z)$ we set $f(x) = 1$. Then f is a real valued function on X such that $0 \leq f(x) \leq 1$ for every x in X. Moreover, $f_{|Z} \equiv 0$ and for $n \in \mathbb{N}$ and $x \in X$ the inequality

$$f(x) < \frac{1}{2^n}$$

implies that $x \in U_n(Z)$. Fix now $x_1, x_2 \in X$ such that $(x_1, x_2) \in U_{n+1}$. If $f(x_1) = 1$, then

$$f(x_2) \leq 1 < f(x_1) + \frac{1}{2^n}$$

If $f(x_1) < 1$, then there exists the smallest $k \in \{0, 1, ..., 2^{n+1}\}$ such that $x_1 \in Z_{\frac{k}{2^{n+1}}}$. In that case we have $x_2 \in U_{n+1}\left(Z_{\frac{k}{2^{n+1}}}\right) \subseteq Z_{\frac{k+1}{2^{n+1}}}$. Thus

$$f(x_2) \leq \frac{k+1}{2^{n+1}} = \frac{k-1}{2^{n+1}} + \frac{2}{2^{n+1}} < f(x_1) + \frac{1}{2^n}$$

We proved that

$$f(x_2) < f(x_1) + \frac{1}{2^n}$$

By symmetry we have

$$f(x_2) < f(x_1) + \frac{1}{2^n}$$

and hence the inequality

$$|f(x_1) - f(x_2)| < \frac{1}{2^n}$$

holds. □

References

[Monygham, 2022] Monygham (2022). Filters in topology. *github repository: "Monygham/Pedo-mellon-a-minno"*.