CONSTRUCTIBLE SETS

1. Constructible and locally constructible sets

Definition 1.1. Let X be a topological space. Suppose that Z is a subset of X such that the inclusion $Z \hookrightarrow X$ is quasi-compact. Then we say that Z is *retro-compact*.

Definition 1.2. Let X be a topological space. We define *the family of constructible subsets of* X as the smallest family of subsets of X that satisfy the following assertions.

(1) Each retro-compact open subset of X is constructible.

(2) If E is constructible subset of X, then $X \setminus E$ is constructible.

(3) If E_1, E_2,..., E_n are constructible subsets of X, then

$$\bigcup_{i=1}^{n} E_i$$

is constructible.

Rephrasing the definition above one can say that constructible subsets of a topological space X form an algebra of sets generated by retro-compact open subsets.

Fact 1.3. *Let $f : X \to Y$ be a morphism of schemes and E be a constructible subset of Y. Then $f^{-1}(E)$ is constructible subset of X.*

Proof. We set

$$\mathcal{F} = \left\{ E \subseteq Y \,\middle|\, f^{-1}(E) \text{ is constructible} \right\}$$

Obviously $\mathcal{F}$ is an algebra of subsets of Y. By base change for quasi-compact morphisms, we derive that $\mathcal{F}$ contains all retro-compact open subsets of Y. This implies that $\mathcal{F}$ contains all constructible subsets of Y. □

Definition 1.4. Let X be a topological space. A subset E of X is called *locally constructible in* X if for every point x in X there exists open neighbourhood U of x in X such that $U \cap E$ is constructible in U.

Theorem 1.5. *Let X be a quasi-compact and quasi-separated scheme and E be a locally constructible subset of X. Then E is constructible and there exists a morphism of schemes $f : Z \to X$ of finite presentation such that $E = f(Z)$. Moreover, one can choose Z to be an affine scheme.*

First we characterize constructible subsets of an affine schemes.

Lemma 1.5.1. *Let A be a ring and E be a subset of* $\operatorname{Spec} A$. *Then the following are equivalent.*

(i) *E is constructible subset of* $\operatorname{Spec} A$.

(ii) *There exists elements $a_1,...,a_n$ and finitely generated ideals $\mathfrak{a}_1,...,\mathfrak{a}_n$ such that*

$$E = \bigcup_{i=1}^{n} D(a_i) \cap V(\mathfrak{a}_i)$$

Proof of the lemma. Consider the family

$$\mathcal{F} = \left\{ \bigcup_{i=1}^{n} D(a_i) \cap V(\mathfrak{a}_i) \,\middle|\, a_1, ..., a_n \in A \text{ and } \mathfrak{a}_1, ..., \mathfrak{a}_n \text{ are finitely generated ideals of } A \right\}$$

Since every retro-compact open subset of $\operatorname{Spec} A$ is quasi-compact, it belongs to $\mathcal{F}$ because it is a finite union of distinguished open subsets. Moreover, subsets in $\mathcal{F}$ are closed under complements and finite unions. Therefore, $\mathcal{F}$ contains all constructible subsets of $\operatorname{Spec} A$. On the other hand each element of $\mathcal{F}$ is constructible in $\operatorname{Spec} A$. $\square$

Lemma 1.5.2. *Let X be a quasi-separated scheme and U be its open affine subset. Then every constructible subset E of U is constructible in X.*

Proof of the lemma. For every $f \in \Gamma(U, \mathcal{O}_X)$ nonvanishing set U_f of f is affine. Since X is quasi-separated, we derive that U_f is retro-compact in X and hence constructible. Suppose now that $\mathfrak{I} \subseteq \Gamma(U, \mathcal{O}_X)$ is an ideal generated by $f_1,...,f_n \in \Gamma(U, \mathcal{O}_X)$ and $V(\mathfrak{I}) \subseteq U$ is a vanishing set of this ideal in U. Then

$$V(\mathfrak{I}) = \left(X \setminus \bigcup_{i=1}^{n} U_{f_i}\right) \setminus (X \setminus U)$$

Since X, U_{f_i} for $1 \leq i \leq n$ and U are constructible in X, we derive that $V(\mathfrak{I})$ is constructible in X. Now the assertion is a consequence of the Lemma 1.5.1 and the fact that constructible sets form an algebra of sets. $\square$

Proof of the theorem. Let E be a locally constructible subset of X. Since X is quasi-compact, there exists an open cover

$$X = \bigcup_{j=1}^{m} U_j$$

by open affines such that each $E \cap U_j$ is constructible in U_j. By Lemma 1.5.2 set $E \cap U_j$ is constructible in X. Hence

$$E = \bigcup_{j=1}^{m} (U_j \cap E)$$

is constructible in X. Denote $U_j = \operatorname{Spec} A_j$ for $1 \leq j \leq m$. Fix j. By Lemma 1.5.1 there exists $a_{ji} \in A$ and finitely generated ideals $\mathfrak{a}_{ji} \subseteq A_j$ for $1 \leq i \leq n_j$ such that

$$U_j \cap E = \bigcup_{i=1}^{n_j} D(a_j) \cap V(\mathfrak{a}_j)$$

Consider a scheme $Z_j = \coprod_{i=1}^{n_j} \operatorname{Spec} \left(A_j/\mathfrak{a}_{ij}\right)_{a_{ji}}$ together with a canonical morphism $f_j : Z_j \to U_j$. Next let Z be an affine scheme $\coprod_{j=1}^{m} Z_j$ with a morphism $f : Z \to X$ such that $f_{|Z_j}$ is defined as f_j composed with the inclusion $U_j \hookrightarrow X$ for every $1 \leq j \leq m$. Then f is a finitely presented morphism and $E = f(Z)$. $\square$

Finally we discuss constructibility for noetherian and locally noetherian topological spaces.

Fact 1.6. *Let X be a locally noetherian topological space. Then the algebra of constructible sets of X is generated by open subsets of X.*

Proof. Every open subset of a locally noetherian topological space is retro-compact. $\square$

Proposition 1.7. *Let X be a noetherian topological space. Suppose that E is a subset of X such that for every irreducible closed subset F of X either $E \cap F$ contains open nonempty subset of F or $E \cap F = \varnothing$. Then E is constructible.*

Proof. Note that by Fact 1.6 every closed subset of X is constructible. Assume that E is not constructible. We set

$$\mathcal{F} = \left\{F \subseteq X \,\middle|\, F \text{ is closed subset of } X \text{ and } F \cap E \text{ is not constructible in } X\right\}$$

First note that $X \in \mathcal{F}$. Since X is noetherian, there exists the minimal (with respect to inclusion) subset F in $\mathcal{F}$. If F is not irreducible, then $F = F' \cup F''$ for some nonempty closed proper subsets

F', F'' of F. Since F is minimal in $\mathcal{F}$, we deduce that both $E \cap F'$ and $E \cap F''$ are constructible and hence $E \cap F = (E \cap F') \cup (E \cap F'')$ is constructible. This is a contradiction. Hence F must be irreducible. Since $E \cap F$ is not constructible, it is nonempty. Hence there exists nonempty subset $U \subseteq E \cap F$ open in F. According to $F \setminus U \subset F$ we infer that $E \cap (F \setminus U)$ is constructible. Thus

$$E \cap F = U \cup (E \cap (F \setminus U))$$

is constructible. This is a contradiction. Therefore, E is constructible. □

2. Noether normalization lemma

In this section we prove important theorem on the structure of commutative and finitely generated k-algebras.

Theorem 2.1 (Noether normalization lemma)**.** *Let k be a field and A be a finitely generated k-algebra. Then there exist elements $z_1,...,z_n$ in A algebraically independent over k such that*

$$k[z_1,...,z_n] \subseteq A$$

is a finite extension of rings.

Proof. Let $\mathcal{A}$ be a family of finitely generated k-subalgebras of A such that for every $B \in \mathcal{A}$ extension $B \subseteq A$ is finite. Clearly $A \in \mathcal{A}$ so $\mathcal{A}$ is nonempty. Now suppose that $n \in \mathbb{N}$ is a minimal number of k-algebra generators of any element in $\mathcal{A}$. Then there exist $z_1,...,z_n \in A$ such that $k[z_1,...,z_n] \subseteq A$ is finite. We show now that $z_1,...,z_n$ are algebraically independent over k. Let $k[x_1,...,x_n]$ be a polynomial k-algebra and assume that there exists nonzero $f \in k[x_1,..,x_n]$ such that $f(z_1,...,z_n) = 0$. Write

$$f(x_1,...,x_n) = \sum_{(d_1,...,d_n)\in F} a_{d_1,...,d_n} \cdot x_1^{d_1} \cdot ... \cdot x_n^{d_n}$$

where $F \subseteq \mathbb{N}^n$ is a finite subset and $a_{d_1,...,d_n} \in k$ are nonzero. Since f is nonzero, we derive that F is nonempty. Define

$$m = 1 + \max_{(d_1,...,d_n)\in F} \max_{1\leq i\leq n} d_i$$

Next define $g \in k[z_2,...,z_n][x]$ by formula

$$g(x) = f(x, z_2 - z_1^m + x^m, z_3 - z_1^{m^2} + x^{m^2}, ..., z_n - z_1^{m^{n-1}} + x^{m^{n-1}})$$

Now we prove that g is a monic polynomial of variable x. Let $\leq$ be the lexographical order on $\mathbb{N}^n$ that is

$$(d_1,...,d_n) \leq (e_1,...,e_n) \text{ if } d_i \leq e_i \text{ for } i = \max\left\{j \,\middle|\, 1 \leq j \leq n \text{ and } d_j \neq e_j\right\}$$

Since $F \subseteq \mathbb{N}^n$ is finite, there exists $(M_1,...,M_n)$ in F that is the greatest with respect to lexographical order $\leq$ restricted to F. This implies that

$$d_1 + d_2 \cdot m + d_3 \cdot m^2 + ... + d_n \cdot m^{n-1} < M_1 + M_2 \cdot m + M_3 \cdot m^2 + ... + M_n \cdot m^{n-1}$$

for every $(d_1,...,d_n) \in \mathbb{N}^n$. This fact and more precise investigation of how coefficients of powers of x in g are calculated show that g is monic. Note also that $h(z_1) = f(z_1,z_2,...,z_n) = 0$. This implies that z_1 is integral over $k[z_2,...,z_n]$ and hence $k[z_2,...,z_n] \subseteq A$ is a finite extension of rings. This proves that $k[z_2,...,z_n] \in \mathcal{A}$ and contradicts the definition of n. Therefore, such f does not exists and this proves that $z_1,...,z_n$ are algebraically independent over k. □

3. Chevalley's theorem on images

Theorem 3.1 (Chevalley's theorem on images). *Let $f : X \to Y$ be a morphism of schemes of finite presentation. Then for every locally constructible subset E of X its image $f(E)$ is locally constructible in Y.*

We start by a sequence of reductions. Since the question is local on Y, one can assume that Y is affine. Then X is quasi-compact and quasi-separated. Thus by Theorem 1.5 we may assume that $E = X$ and hence it suffices to prove that $f(X)$ is constructible in Y. Since f is of finite presentation and Y is affine, there exists a cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "g"'] & Y' \end{tikzcd}$$

with Y' the spectrum of a finitely generated $\mathbb{Z}$-algebra and f' is of finite type. We have

$$f(X) = g^{-1}\left(f'(X')\right)$$

Since preimage of a constructible subset is constructible by Fact 1.3, it suffices to prove that $f'(X')$ is constructible. Thus we may assume that Y is a noetherian affine scheme and f is of finite type. For the proof of the Chevalley's theorem in this simplified form we need the following interesting application of Theorem 2.1

Lemma 3.1.1. *Let A be a domain and $f : A \to B$ be an injective morphism of finite type. Then there exists nonzero $s \in A$ such that the image of* $\operatorname{Spec} f : \operatorname{Spec} B \to \operatorname{Spec} A$ *contains the distinguished set $D(s)$ of* $\operatorname{Spec} A$.

Proof of the lemma. Let $S = A \setminus \{0\}$. Then $K = S^{-1}A$ is a field of fractions of A and $S^{-1}B$ is a finitely generated K-algebra. By Theorem 2.1 we derive that there exists $\frac{b_1}{s_1},\ldots,\frac{b_n}{s_n} \in S^{-1}B$ algebraically independent over K such that

$$K\left[\frac{b_1}{s_1},\ldots,\frac{b_n}{s_n}\right] \subseteq S^{-1}B$$

is a finite extension of rings. Here $b_1,\ldots,b_n \in B$ and $s_1,\ldots,s_n \in S$. It follows that

$$K\left[b_1,\ldots,b_n\right] \subseteq S^{-1}B$$

is a finite extension of rings and $b_1,\ldots,b_n$ are algebraically independent over K. There exists a finite set $c_1,\ldots,c_m$ that generates B as an $A\left[b_1,\ldots,b_n\right]$-algebra and all these elements are integral over $K\left[b_1,\ldots,b_n\right]$. This implies that for every $1 \leq i \leq m$ there exists a monic polynomial $f_i \in K\left[b_1,\ldots,b_n\right]\left[x\right]$ such that $f_i(c_i) = 0$. Now there are finitely many coefficients of each f_i and each of them is some algebraic expression in $b_1,\ldots,b_n$ having coefficients in $K = S^{-1}A$. This implies that there exists nonzero $s \in A$ such that f_i is a monic polynomial in $A_s[b_1,\ldots,b_n][x]$ for every $1 \leq i \leq n$. Hence the extension

$$A_s[b_1,\ldots,b_n] \subseteq B_s$$

is finite. We also know that $b_1,\ldots,b_n$ are algebraically independent over K. Thus $A_s \subseteq B_s$ can be decomposed as a polynomial extension followed by a finite extension

$$A_s \subseteq A_s[b_1,\ldots,b_n] \subseteq B_s$$

Both polynomial extension and finite extension induce surjective morphism on prime spectra. Thus the morphism $\operatorname{Spec} B_s \to \operatorname{Spec} A_s$ induced by $\operatorname{Spec} f$ is surjective. Hence $D(s) \subseteq \operatorname{Spec} A$ is in the image of $\operatorname{Spec} f$. $\square$

Proof of the theorem. Let $f : X \to Y$ be a finite type morphism with Y affine and noetherian. As we explained above it suffices to prove that $f(X)$ is constructible. Suppose that F is an irreducible closed subset of Y. We consider it as a subscheme of Y with integral structure. By Lemma 3.1.1 we deduce that either the image of a morphism $f^{-1}(F) \to F$ induced by f contains nonempty open subset of F or this image is empty. Thus for every irreducible closed subset F of Y either $f(X) \cap F$ contains nonempty open subset of F or $f(X) \cap F = \varnothing$. By Proposition 1.7 we derive that $f(X)$ is constructible in Y. □

Corollary 3.2 (characterization of constructible sets on qcqs schemes)**.** *Let X be a quasi-compact and quasi-separated scheme. Then the following are equivalent.*

- **(i)** *E is locally constructible.*
- **(ii)** *E is constructible.*
- **(iii)** *There exists an affine scheme Z and a morphism $f : Z \to X$ of finite presentation such that $E = f(Z)$.*

Proof. The implications **(i)** $\Rightarrow$ **(ii)** $\Rightarrow$ **(iii)** follow from Theorem 1.5 and **(iii)** $\Rightarrow$ **(i)** follows from Theorem 3.1. □

4. Closedness criterion

In this section we prove important criterion for closedness of subsets of schemes.

Definition 4.1. Let X be a topological space and let η be a point of X. Every point $x \in \mathbf{cl}(\{\eta\})$ is called *a specialization of η*.

Definition 4.2. Let X be a topological space and let x be a point of X. Every point η such that x is a specialization of η is called *a generization of x*.

Definition 4.3. Let X be a topological space and Z be its subset. We say that Z is *closed under specialization (generization)* if Z contains all specializations (generizations) of its points.

Theorem 4.4. *Let X be a scheme and $f : Z \to X$ be a quasi-compact morphism of schemes. Then the following are equivalent.*

- **(i)** *$f(Z)$ is a closed subset of X.*
- **(ii)** *$f(Z)$ is closed under specialization.*

For the proof we need the following result.

Lemma 4.4.1. *Let $f : A \to B$ be a morphism of rings. If the image of* $\mathrm{Spec}\, f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ *is closed under specialization, then it is closed.*

Proof of the lemma. The image of $\mathrm{Spec}\, f$ is equal to the image of its factor $\mathrm{Spec}\, B \to \mathrm{Spec}\,(A/\ker(f))$. Therefore, we may additionally assume that f is injective. We prove that under this extra assumption $\mathrm{Spec}\, f$ is surjective. For this assume that $\mathfrak{p} \in \mathrm{Spec}\, A$ is a prime ideal. Then f induces an injective map $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$. Thus $B_{\mathfrak{p}}$ is nonzero. Hence $\mathrm{Spec}\, B_{\mathfrak{p}}$ is nonempty. We also have a commutative square

$$\begin{tikzcd} \varnothing \neq \mathrm{Spec}\, B_{\mathfrak{p}} \arrow[r] \arrow[d, hook] & \mathrm{Spec}\, A_{\mathfrak{p}} \arrow[d, hook] \\ \mathrm{Spec}\, B \arrow[r, "\mathrm{Spec}\, f"'] & \mathrm{Spec}\, A \end{tikzcd}$$

of topological spaces. This imply that there exists a prime ideal $\mathfrak{q} \in \operatorname{Spec} B$ such that $\mathfrak{p}$ is a specialization of $(\operatorname{Spec} f)(\mathfrak{q})$. Since the image of $\operatorname{Spec} f$ is closed under specialization, we derive that $\mathfrak{p}$ is contained in the image of $\operatorname{Spec} f$. $\square$

Proof. Closed subsets are closed under specialization. Hence **(i)** $\Rightarrow$ **(ii)** holds.
Now assume **(ii)** i.e. $f(Z)$ is closed under specialization. Fix open affine U in X. Since f is quasi-compact, we derive that $f^{-1}(U)$ is quasi-compact. Write $f^{-1}(U) = \bigcup_{j=1}^{m} W_j$ for open affine subsets W_j of $f^{-1}(U)$. Let $W = \coprod_{j=1}^{m} W_j$ and consider a morphism $g : W \to U$ given as the composition

$$\coprod_{j=1}^{m} W_j \longrightarrow f^{-1}(U) \longrightarrow U$$

where the first arrow is induced by inclusions $\{W_j \hookrightarrow f^{-1}(U)\}_{1 \leq j \leq m}$ and the second is the restriction of f. Note that $g(W) = f(Z) \cap U$ and hence $g(W)$ is closed under specialization in U. By Lemma 4.4.1 we deduce that $g(W)$ is closed in U and hence $f(X) \cap U$ is closed in U. Since this holds for any open affine U in X, we infer that $f(Z)$ is closed in X. This is **(i)**. $\square$