UNIFORM SPACES

1. Introduction

This notes are devoted to uniform spaces. In the first section we prove important result on existence of pseudometrics originally due to Weil. This result is crucial for further developments.

2. Existence of pseudometrics

Let X be a set. We start by introducing some set-theoretic notions concerning subsets of $X \times X$.

Definition 2.1. Let X be a set. A subset V of $X \times X$ such that

$$\forall_{x \in X}\,(x,x) \in V,\ \forall_{x,y \in X}\,(x,y) \in V \Leftrightarrow (y,x) \in V$$

is *a surrounding of the diagonal in* $X \times X$.

Definition 2.2. Let X be a set and let V, W be subsets of $X \times X$. We define a subset $W \cdot V$ of $X \times X$ called *a composition of* V *with* W such that

$$(x,z) \in W \cdot V \Leftrightarrow \exists_{y \in X}\,(x,y) \in V \text{ and } (y,z) \in W$$

for each $x, z \in X$.

Finally we recall the notion of pseudometric.

Definition 2.3. Let X be a set. Suppose that a function $\rho : X \times X \to [0, +\infty)$ satisfies the following assertions.

(1) $\rho(x,x) = 0$ for all $x \in X$.

(2) $\rho(x,y) = \rho(y,x)$ for all $x, y \in X$.

(3) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all $x, y, z \in X$.

Then ρ is *a pseudometric on* X.

Now we state and prove the fundamental result on the existence of pseudometrics.

Theorem 2.4. *Let X be a set and let $\{V_n\}_{n \in \mathbb{N}}$ be a sequence of surroundings of the diagonal in $X \times X$ such that*

$$V_{n+1} \cdot V_{n+1} \cdot V_{n+1} \subseteq V_n$$

for every $n \in \mathbb{N}$. Then there exists a pseudometric ρ on X bounded by 1 such that

$$\left\{(x,y) \in X \times X \,\middle|\, \rho(x,y) < \frac{1}{2^n}\right\} \subseteq V_n \subseteq \left\{(x,y) \in X \times X \,\middle|\, \rho(x,y) \leq \frac{1}{2^n}\right\}$$

for every $n \in \mathbb{N}$.

For the proof consider a function f defined on $X \times X$ given by formula

$$\begin{cases} 0 & \text{if } (x,y) \in V_n \text{ for each } n \in \mathbb{N} \\ \frac{1}{2^n} & \text{if } (x,y) \in V_n \setminus V_{n+1} \\ 1 & \text{if } (x,y) \notin V_0 \end{cases}$$

The proof relies on the following result.

Lemma 2.4.1. *For each $n \in \mathbb{N}$ and every finite sequence $x_0, ..., x_m$ the inequality*

$$\sum_{i=1}^{m} f(x_{i-1}, x_i) < \frac{1}{2^n}$$

implies that $(x_0, x_m) \in V_n$.

Proof of the lemma. The proof goes by induction on m. For $m = 0$ and $m = 1$ the claim is trivial. Assume that m is greater than one and suppose that the assertion holds for all numbers smaller than m. Suppose that

$$\sum_{i=1}^{m} f(x_{i-1}, x_i) < \frac{1}{2^n}$$

for some sequence $x_0, ..., x_m$ of elements in X. We have

$$\text{either } f(x_0, x_1) < \frac{1}{2^{n+1}} \text{ or } f(x_{m-1}, x_m) < \frac{1}{2^{n+1}}$$

Without loss of generality we may assume that the first inequality holds. Let k be the greatest number in $\{1, ..., m-1\}$ such that

$$\sum_{i=1}^{k} f(x_{i-1}, x_i) < \frac{1}{2^{n+1}}$$

Next we consider two cases.

- If $k < m - 1$, then we have

  $$\sum_{i=1}^{k} f(x_{i-1}, x_i) < \frac{1}{2^{n+1}},\ f(x_k, x_{k+1}) \leq \frac{1}{2^{n+1}},\ \sum_{i=k+1}^{m} f(x_{i-1}, x_i) < \frac{1}{2^{n+1}}$$

  By induction hypothesis we have $(x_0, x_k) \in V_{n+1}, (x_{k+1}, x_m) \in V_{n+1}$ and by definition of f we have $(x_k, x_{k+1}) \in V_{n+1}$. Hence

  $$(x_0, x_m) \in V_{n+1} \cdot V_{n+1} \cdot V_{n+1} \subseteq V_n$$

  and the assertion holds.
- If $k = m - 1$. Then

  $$\sum_{i=1}^{m-1} f(x_{i-1}, x_i) < \frac{1}{2^{n+1}},\ f(x_{m-1}, x_m) \leq \frac{1}{2^{n+1}}$$

  By induction hypothesis we have $(x_0, x_{m-1}) \in V_{n+1}$ and by definition of f we have $(x_{m-1}, x_m) \in V_{n+1}$. Hence

  $$(x_0, x_m) \in V_{n+1} \cdot V_{n+1} \subseteq V_{n+1} \cdot V_{n+1} \cdot V_{n+1} \subseteq V_n$$

  and the assertion holds.

Thus the result follows from induction. □

Proof of the theorem. For $x, y \in X$ we define

$$\rho(x, y) = \inf\left\{ \sum_{i=1}^{m} f(x_{i-1}, x_i) \,\middle|\, \text{for every } m \in \mathbb{N} \text{ an every finite sequence } x_0, ..., x_m \text{ such that } x_0 = x,\ x_m = y \right\}$$

It is easy to verify that the function ρ is a pseudometric on X. It remains to prove that

$$\left\{ (x, y) \in X \times X \,\middle|\, \rho(x, y) < \frac{1}{2^n} \right\} \subseteq V_n \subseteq \left\{ (x, y) \in X \times X \,\middle|\, \rho(x, y) \leq \frac{1}{2^n} \right\}$$

The first inclusion follows from Lemma 2.4.1 and the second follows from the fact that $\rho(x, y) \leq f(x, y)$ for every $x, y \in X$. □

3. Uniform structures and uniform spaces

In this section we introduce main object of our study.

Definition 3.1. Let X be a set. Suppose that $\mathfrak{U}$ is a collection of surroundings of the diagonal in $X \times X$ which satisfies the following two assertions.

(1) If $U \in \mathfrak{U}$ and W is a surrounding of the diagonal in $X \times X$ such that $V \subseteq W$, then $W \in \mathfrak{U}$.

(2) If $U, W \in \mathfrak{U}$, then $U \cap W \in \mathfrak{U}$.

(3) If $U \in \mathfrak{U}$, then there exists $W \in \mathfrak{U}$ such that $W \cdot W \subseteq U$.

Then $\mathfrak{U}$ is *a uniform structure on* X.

Example 3.2. Let X be a set. Then the family $\mathfrak{D}_X$ of all surrounding of the diagonal in $X \times X$ is a uniform structure on X. It is called *the discrete uniform structure on* X.

Fact 3.3. *Let X be a set and let $\{\mathfrak{U}_i\}_{i \in I}$ be a family of uniform structures on X. Then*

$$\bigcap_{i \in I} \mathfrak{U}_i$$

is a uniform structure on X.

Proof. Left for the reader. □

Corollary 3.4. *Let X be a set and let $\mathcal{F}$ be a family of surrounding of the diagonal in $X \times X$. Then there exists the smallest (with respect to inclusion) uniform structure $\mathfrak{U}$ on X which contain $\mathcal{F}$.*

Proof. Let $\{\mathfrak{U}_i\}_{i \in I}$ be a family of all uniform structures on X which contain $\mathcal{F}$. The family is nonempty, since it contains the discrete uniform structure on X. The intersection

$$\mathfrak{U} = \bigcap_{i \in I} \mathfrak{U}_i$$

is a uniform structure on X by Fact 3.3. Hence it is the smallest uniform structure on X which contain $\mathcal{F}$. □

Definition 3.5. A pair $(X, \mathfrak{U})$ consisting of a set X and a uniform structure $\mathfrak{U}$ on X is *a uniform space*.

Definition 3.6. Let $(X, \mathfrak{U})$ be a uniform space. A surrounding V in $\mathfrak{U}$ is called *an entourage of the diagonal in* $(X, \mathfrak{U})$.

Definition 3.7. Let $(X, \mathfrak{U}), (Y, \mathfrak{V})$ be uniform spaces and let $f : X \to Y$ be a map. Suppose that $(f \times f)^{-1}(V) \in \mathfrak{U}$ for every $V \in \mathfrak{V}$. Then f is *a morphism of uniform spaces*.

Remark 3.8. Uniform spaces and their morphisms form a category. We denote this category by **Unif**.

Now we study limits in **Unif**. For this we use the following result.

Theorem 3.9. *Let X be a set and let $\{(X_i, \mathfrak{U}_i)\}_{i \in I}$ be a family of uniform spaces. Consider a family $\{f_i : X \to X_i\}_{i \in I}$ of maps. Suppose that $\mathfrak{U}$ is the smallest uniform structure on X which makes $\{f_i\}_{i \in I}$ into a family of uniform morphisms. Then*

$$U \in \mathfrak{U}$$

if and only if there exist $n \in \mathbb{N}_+$, $i_1, ..., i_n \in I$ and $V_1 \in \mathfrak{U}_{i_1}, ..., V_n \in \mathfrak{U}_{i_n}$ such that

$$\bigcap_{k=1}^{n} \left(f_{i_k} \times f_{i_k}\right)^{-1}(V_k) \subseteq U$$

Proof. Consider the family cU of all surrounding U of the diagonal in $X \times X$ such that there exist $n \in \mathbb{N}_+$, $i_1, ..., i_n \in I$ and $V_1 \in \mathfrak{U}_{i_1}, ..., V_n \in \mathfrak{U}_{i_n}$ satisfying

$$\bigcap_{k=1}^{n} \left(f_{i_k} \times f_{i_k}\right)^{-1}(V_k) \subseteq U$$

It is easy to verify (we left for the reader) that $\mathcal{U}$ is a uniform structure on X. Moreover, for every $n \in \mathbb{N}_+$, $i_1, ..., i_n \in I$ and $V_1 \in \mathfrak{U}_{i_1}, ..., V_n \in \mathfrak{U}_{i_n}$ we have

$$\bigcap_{k=1}^{n} \left(f_{i_k} \times f_{i_k}\right)^{-1}(V_k) \in \mathfrak{U}$$

Hence $\mathcal{U} \subseteq \mathfrak{U}$. Note also that f_i is a uniform morphism $(X, \mathcal{U}) \to (X_i, \mathfrak{U}_i)$ for each $i \in I$. Thus $\mathfrak{U} \subseteq \mathcal{U}$. Therefore, $\mathcal{U} = \mathfrak{U}$ and this proves the theorem. □

Definition 3.10. Let $(X, \mathfrak{U})$ be a uniform space and let Z be a subset of X. Then Z together with the smallest uniform structure which makes the inclusion $Z \hookrightarrow X$ into a uniform morphism is *a uniform subspace of* $(X, \mathfrak{U})$ *with* Z *as the underlying set*.

4. Topology induced by uniform structure

We start by introducing the notion of a ball with respect to surrounding of the diagonal.

Definition 4.1. Let X be a set. For every x in X and U in $\mathfrak{D}_X$ the set

$$B(x, U) = \left\{y \in X \,\middle|\, (x, y) \in U\right\}$$

is *a ball with center* x *and radius* V.

Fact 4.2. *Let* X *be a set and let* $\mathfrak{U}$ *be a uniform structure on* X*. The family*

$$\left\{\mathcal{O} \subseteq X \,\middle|\, \textit{for each } x \in \mathcal{O} \textit{ there exists } U \in \mathfrak{U} \textit{ such that } B(x, U) \subseteq \mathcal{O}\right\}$$

is a topology on X*.*

Proof. We left the proof for the reader as an exercise. □

Example 4.3. The interval $[0, 1]$ admits a uniform structure given by

$$\left\{U \subseteq \mathfrak{D}_{[0,1]} \,\middle|\, \text{there exists } \epsilon > 0 \text{ such that } |x - y| < \epsilon \text{ for some } x, y \in [0, 1] \text{ implies } (x, y) \in U\right\}$$

Now the topology induced by this uniform structure coincides with the natural topology on $[0, 1]$.

Definition 4.4. Let X be a set and let $\mathfrak{U}$ be a uniform structure on X. Then the topology

$$\left\{\mathcal{O} \subseteq X \,\middle|\, \text{for each } x \in \mathcal{O} \text{ there exists } U \in \mathfrak{U} \text{ such that } B(x, U) \subseteq \mathcal{O}\right\}$$

on X is *the topology induced by* $\mathfrak{U}$.

Fact 4.5. *Let* $(X, \mathfrak{U})$ *and* $(Y, \mathfrak{V})$ *be uniform spaces and let* $f : X \to Y$ *be a morphism of uniform spaces. Then* f *is a continuous map with respect to topologies induced by* $\mathfrak{U}$ *and* $\mathfrak{V}$ *on* X *and* Y*, respectively.*

Proof. Pick open subset $\mathcal{O}$ with respect to the topology induced by $\mathfrak{Y}$ on Y. Suppose that $f(x) \in \mathcal{O}$ for some x in X. Then there exists $V_x \in \mathfrak{Y}$ such that $B(f(x), V_x) \subseteq \mathcal{O}$. Note that the image of $B\big(x, (f \times f)^{-1}(V_x)\big)$ under f is contained in $\mathcal{O}$. Therefore,

$$f^{-1}(\mathcal{O}) = \bigcup_{x \in f^{-1}(\mathcal{O})} B\big(x, (f \times f)^{-1}(V_x)\big)$$

is open in the topology induced by $\mathfrak{U}$. □

Facts 4.2 and 4.5 imply the existence of the canonical functor **Unif** → **Top**. In the remaining part of this section we shall investigate the properties of this functor. We start by describing the image of the functor.

Definition 4.6. Let X be a topological space. Suppose that for every closed subset F of X and for every point x in $X \smallsetminus F$ there exists a continuous function $f : X \to [0,1]$ such that $f(F) \subseteq \{1\}$ and $f(x) = 0$. Then X is *a completely regular space.*

Theorem 4.7. *The image of the object part of the canonical functor* **Unif** → **Top** *consists of class of completely regular spaces.*

Proof. Let $(X, \mathfrak{U})$ be a uniform space. Consider a closed set F with respect to the topology induced by $\mathfrak{U}$. Let x be a point in $X \smallsetminus F$. Since $x \notin F$ and F is closed, we derive that there exists $U \in \mathfrak{U}$ such that $B(x, U) \cap F = \emptyset$. We define a sequence $\{V_n\}_{n\in\mathbb{N}}$ of elements in $\mathfrak{U}$ by recursion. We set $V_0 = U$ and if $V_0, ..., V_n$ are defined for some $n \in \mathbb{N}$, then we pick an element V_{n+1} of $\mathfrak{U}$ such that

$$V_{n+1} \cdot V_{n+1} \cdot V_{n+1} \subseteq V_n$$

According to Theorem 2.4 there exists a pseudometric ρ on X bounded by 1 such that

$$\left\{(x,y) \in X \times X \,\middle|\, \rho(x,y) < \frac{1}{2^n}\right\} \subseteq V_n \subseteq \left\{(x,y) \in X \times X \,\middle|\, \rho(x,y) \leq \frac{1}{2^n}\right\}$$

for every $n \in \mathbb{N}$. Note that

$$|\rho(x,y_1) - \rho(x,y_2)| \leq \rho(y_1,y_2)$$

for any pair $y_1, y_2 \in X$. Indeed, this is the triangle inequality for ρ. Thus if $(y_1, y_2) \in V_n$ for some $n \in \mathbb{N}$, then

$$|\rho(x,y_1) - \rho(x,y_2)| \leq \rho(y_1,y_2) \leq \frac{1}{2^n}$$

Hence the map $f : X \to [0,1]$ given by formula $f(y) = \rho(x,y) \in [0,1]$ is a morphism of uniform spaces, where $[0,1]$ is considered with uniform structure described in Example 4.3. This implies that f is a continuous map, where X admits topology induced by $\mathfrak{U}$. Pick $y \in F$. Then $y \notin B(x,U)$ and hence $(x,y) \notin U$. Since $V_0 = U$ and ρ is bounded by 1, we derive that $f(y) = \rho(x,y) = 1$. On the other hand $f(x) = \rho(x,x) = 0$. Therefore, $f(F) \subseteq \{1\}$ and $f(x) = 0$. Thus X with topology induced by $\mathfrak{U}$ is a completely regular space.
Suppose now that X is a completely regular space. Consider the set $C(X, \mathbb{R})$ of all continuous functions on X. For $m \in \mathbb{N}_+$ and set of m continuous functions $f_1, ..., f_m \in C(X, \mathbb{R})$ define

$$\rho_{f_1,...,f_m}(x,y) = \max\left\{|f_1(x) - f_1(y)|, ..., |f_m(x) - f_m(y)|\right\}$$

where $x, y \in X$. Clearly $\rho_{f_1,...,f_m}$ is a pseudometric on X. Next consider a family $\mathfrak{U}$ of all $U \in \mathfrak{D}_X$ such that there exist a finite subset $\{f_1, ..., f_m\} \subseteq C(X, \mathbb{R})$ for some $m \in \mathbb{N}_+$ and $\epsilon > 0$ such that

$$\left\{(x,y) \in X \times X \,\middle|\, \rho_{f_1,...,f_m}(x,y) < \epsilon\right\} \subseteq U$$

Clearly $\mathfrak{U}$ is a uniform structure on X. Suppose that $\mathcal{O}$ is a subset of X which is open in the topology induced by $\mathfrak{U}$. For each point z in $\mathcal{O}$ there exists U in $\mathfrak{U}$ such that $B(z,U) \subseteq \mathcal{O}$. By definition there exist a finite subset $\{f_1, ..., f_m\} \subseteq C(X, \mathbb{R})$ for some $m \in \mathbb{N}_+$ and $\epsilon > 0$ such that

$$\left\{(x,y) \in X \times X \,\middle|\, \rho_{f_1,...,f_m}(x,y) < \epsilon\right\} \subseteq U$$

Thus

$$\bigcap_{i=1}^{m} f_i^{-1}\Big(\big(f_i(z) - \epsilon, f_i(z) + \epsilon\big)\Big) = \left\{y \in X \,\middle|\, \rho_{f_1,...,f_m}(z,y) < \epsilon\right\} \subseteq B(z,U) \subseteq \mathcal{O}$$

Since $f_1, ..., f_m$ are continuous on X with respect to the original topology, we derive that $\mathcal{O}$ is an open subset in the original topology. This proves that the original topology is stronger than the topology induced by $\mathfrak{U}$. Now we prove the converse. For this assume that a subset $\mathcal{O}$ is open in the original topology of X. We claim that $\mathcal{O}$ is also open in the topology induced by $\mathfrak{U}$. For this pick $z \in \mathcal{O}$. Since X is completely regular, there exists a function $f_z : X \to \mathbb{R}$ such that

$f_z(X \setminus \mathcal{O}) \subseteq \{1\}$ and $f_z(z) = 0$. Let U_z be a subset of $X \times X$ consisting of all pairs for which ρ_{f_z} is smaller than 1. Then $U_z \in \mathfrak{U}$ and obviously $B(z, U_z) \subseteq \mathcal{O}$. Thus

$$\mathcal{O} = \bigcup_{z \in \mathcal{O}} B(z, U_z)$$

and this proves the claim that $\mathcal{O}$ is open in the topology induced by $\mathfrak{U}$. Concluding the topology induced by $\mathfrak{U}$ and the original topology on X coincide. □

Next we note the following important fact.

Theorem 4.8. *Let $(X, \mathfrak{U})$ be a uniform space and let Z be a subset of X. Let $\mathfrak{U}_Z$ be the subspace uniform structure on Z. Then the topology induced by $\mathfrak{U}_Z$ on Z and the subspace topology of Z with respect to the topology induced by $\mathfrak{U}$ on X coincide.*

For the proof we need the following easy result.

Lemma 4.8.1. *Let $(X, \mathfrak{U})$ be a uniform space and let $U \in \mathfrak{U}$. Then there exists an infinite sequence $\{U_n\}_{n \in \mathbb{N}_+}$ of entourages of the diagonal in $(X, \mathfrak{U})$ such that*

$$U_1 \cdot U_2 \cdot \ldots \cdot U_n \subseteq U$$

for every $n \in \mathbb{N}_+$.

Proof of the lemma. We pick $U_1 \in \mathfrak{U}$ such that $U_1 \cdot U_1 \subseteq U$. Next suppose that U_n is defined for some $n \in \mathbb{N}_+$. Then there exists $U_{n+1} \in \mathfrak{U}$ such that $U_{n+1} \cdot U_{n+1} \subseteq U_n$. This recursive method constructs a sequence $\{U_n\}_{n \in \mathbb{N}}$ of elements of $\mathfrak{U}$. By easy induction we have

$$U_1 \cdot U_2 \cdot \ldots \cdot U_n \subseteq U$$

and this proves the lemma. □

Proof of the theorem. Let $\mathcal{O}$ be a subset of X open with respect to the topology induced by $\mathfrak{U}$. Then for each x in $\mathcal{O}$ there exists $U_x \in \mathfrak{U}$ such that $B(x, U_x) \subseteq \mathcal{O}$. Thus

$$\mathcal{O} \cap Z = \bigcup_{z \in \mathcal{O} \cap Z} B\big(z, U_z \cap (Z \times Z)\big)$$

Since $U_z \cap (Z \times Z) \in \mathfrak{U}_Z$ for every $z \in \mathcal{O} \cap Z$, it follows that $\mathcal{O} \cap Z$ is open with respect to the topology induced by $\mathfrak{U}_Z$ on Z.
Suppose now that $\mathcal{O}_Z$ is a subset of Z which is open with respect to the topology induced by $\mathfrak{U}_Z$. Then for each $z \in \mathcal{O}_Z$ there exists $U_z \in \mathfrak{U}$ such that $B\big(z, U_z \cap (Z \times Z)\big) \subseteq \mathcal{O}_Z$. By Lemma 4.8.1 for each $z \in \mathcal{O}_Z$ there exists a sequence $\{U_{z,n}\}_{n \in \mathbb{N}_+}$ such that

$$U_{z,1} \cdot U_{z,2} \cdot \ldots \cdot U_{z,n} \subseteq U_z$$

It follows that

$$\bigcup_{n \in \mathbb{N}_+} B\big(z, U_{z,1} \cdot U_{z,2} \cdot \ldots \cdot U_{z,n}\big)$$

is an open subset of X with respect to topology induced by $\mathfrak{U}$. Moreover, it is a subset of $B(z, U_z)$. Thus

$$\mathcal{O} = \bigcup_{z \in \mathcal{O}_Z} \bigcup_{n \in \mathbb{N}_+} B\big(z, U_{z,1} \cdot U_{z,2} \cdot \ldots \cdot U_{z,n}\big)$$

is an open subset of X with respect to topology induced by $\mathfrak{U}$. Moreover, we have

$$\mathcal{O} \subseteq \bigcap_{z \in \mathcal{O}_Z} B(z, U_z)$$

and hence $Z \cap \mathcal{O} = \mathcal{O}_Z$. Therefore, $\mathcal{O}_Z$ is open subset in the subspace topology induced on Z by the topology induced by $\mathfrak{U}$ on X. This completes the proof. □

Theorem 4.9. *Let $|-| : \mathbf{Unif} \to \mathbf{Set}$ be a functor which sends each uniform space to its underlying set and let $D : \mathcal{I} \to \mathbf{Unif}$ be a functor such that I is a small category.*

Proof. Consider a small diagram $D : I \to \mathbf{Unif}$ of uniform spaces. Let $\{f_i : X \to |D(i)|\}_{i\in I}$ be a limiting cone of $|-|\cdot D$ in **Set**. Let $\mathfrak{U}$ be the smallest uniform structure on X such that $(f_i \times f_i)^{-1}(V) \in \mathfrak{U}$ for every $i \in I$ and every entourage V of the diagonal in $D(i)$. By Corollary 3.4 the structure $\mathfrak{U}$ exists. Then $\mathfrak{X} = (X, \mathfrak{U})$ is a uniform space such that map f_i is a morphism of uniform spaces $D(i) \to \mathfrak{X}$ for every i in I. Suppose now that $\{g_i : \mathfrak{Y} \to D(i)\}_{i\in I}$ is some cone over D. Then there exists a unique map $h : |\mathfrak{Y}| \to X$ such that $h \cdot f_i = g_i$ in **Set** for every $i \in I$. It is easy to verify that

$$\left\{U \in \mathfrak{U} \,\middle|\, (h\times h)^{-1}(U) \text{ is an entourage of the diagonal in } \mathfrak{Y}\right\}$$

is a uniform structure on X which contains $(f_i \times f_i)^{-1}(V)$ for every $i \in I$ and every entourage V of the diagonal in $D(i)$. Therefore, $\mathfrak{U}$ and

$$\left\{U \in \mathfrak{U} \,\middle|\, (h\times h)^{-1}(U) \text{ is an entourage of the diagonal in } \mathfrak{Y}\right\}$$

coincide and hence h is a morphism of uniform spaces $\mathfrak{Y} \to \mathfrak{X}$. This shows that $\mathfrak{X}$ together with $\{f_i : \mathfrak{X} \to D(i)\}_{i\in I}$ is a limiting cone of D. This completes the proof. □

5. Pseudometrizable uniform spaces and their products

In this section we use Theorem 2.4 to prove certain structure theorems concerning uniform spaces.

Definition 5.1. A uniform space $(X, \mathfrak{U})$ is *pseudometrizable* if there exists a pseudometric ρ on X such that

$$\left\{U \in \mathfrak{D}_X \,\middle|\, \text{there exists } \epsilon > 0 \text{ such that for all } x, y \in X \text{ if } \rho(x,y) \le \epsilon \text{ then } (x,y) \in U\right\}$$

coincides with $\mathfrak{U}$.

Theorem 5.2. *Let $(X, \mathfrak{U})$ be a uniform space. The following assertions are equivalent.*

- **(i)** *$(X, \mathfrak{U})$ is a pseudometrizable uniform space.*
- **(ii)** *There exists a sequence $\{U_n\}_{n\in\mathbb{N}}$ of elements in $\mathfrak{U}$ such that the family*

$$\left\{U \in \mathfrak{D}_X \,\middle|\, \exists_{n\in\mathbb{N}}\, U_n \subseteq U\right\}$$

coincides with $\mathfrak{U}$.

Proof. For **(i)** ⇒ **(ii)** observe that if ρ is a pseudometric on $(X, \mathfrak{U})$ such that

$$\left\{U \in \mathfrak{D}_X \,\middle|\, \text{there exists } \epsilon > 0 \text{ such that for all } x, y \in X \text{ if } \rho(x,y) \le \epsilon \text{ then } (x,y) \in U\right\}$$

coincides with $\mathfrak{U}$, then the sequence $\{U_n\}_{n\in\mathbb{N}}$ given by formula

$$U_n = \left\{(x,y) \in X \times X \,\middle|\, \rho(x,y) \le \frac{1}{2^n}\right\}$$

satisfies **(ii)**.

Suppose now that **(ii)** holds. We define a sequence $\{V_n\}_{n\in\mathbb{N}}$ of elements in $\mathfrak{U}$ by recursion. We set $V_0 = U_0$ and if $V_0, ..., V_n$ are defined for some $n \in \mathbb{N}$, then we pick an element W of $\mathfrak{U}$ such that

$$W \cdot W \cdot W \subseteq V_n$$

and set V_{n+1} as $W \cap U_{n+1}$. Note that $\{V_n\}_{n\in\mathbb{N}}$ satisfies

$$V_{n+1} \cdot V_{n+1} \cdot V_{n+1} \subseteq V_n$$

for each $n \in \mathbb{N}$. Moreover, we have

$$\left\{U \in \mathfrak{D}_X \,\middle|\, \exists_{n\in\mathbb{N}}\, V_n \subseteq U\right\}$$

By Theorem 2.4 there exists a pseudometric ρ on X such that

$$\left\{(x,y) \in X \times X \,\middle|\, \rho(x,y) < \frac{1}{2^n}\right\} \subseteq V_n \subseteq \left\{(x,y) \in X \times X \,\middle|\, \rho(x,y) \leq \frac{1}{2^n}\right\}$$

for every $n \in \mathbb{N}$. This implies that

$$\left\{U \in \mathfrak{D}_X \,\middle|\, \text{there exists } \epsilon > 0 \text{ such that for all } x, y \in X \text{ if } \rho(x,y) \leq \epsilon \text{ then } (x,y) \in U\right\}$$

coincides with $\mathfrak{U}$. Hence **(ii)** $\Rightarrow$ **(i)**. □

Corollary 5.3. *Let $(X, \mathfrak{U})$ be a uniform space. Then there exists a family $\{(X_i, \mathfrak{U}_i)\}_{i \in I}$ of pseudometrizable uniform spaces and a morphism*

$$(X, \mathfrak{U}) \xrightarrow{f} \prod_{i \in I} (X_i, \mathfrak{U}_i)$$

such that f is an injective map.