RADON-NIKODYM THEOREM, HAHN-JORDAN DECOMPOSITION AND LEBESGUE DECOMPOSITION

1. Introduction

This notes are devoted to some more advanced notions in measure theory. Tools presented here are indispensable in probability theory and statistics. We refer to [Monygham, 2018] for extensive introduction to measure theory.

2. Signed and Complex Measures

In this section we define extension of the usual notion of measure. Denote by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ the topological space obtained as a two-point compactification of the line $\mathbb{R}$. Addition is partially defined operation on $\overline{\mathbb{R}}$ given by the following rules

$$(+\infty) + r = +\infty = r + (+\infty),\ (-\infty) + r = -\infty = r + (-\infty)$$

for every $r \in \mathbb{R}$

Definition 2.1. Let (X, Σ) be a measurable space. *A signed measure on* (X, Σ) is a function $\nu : \Sigma \to \overline{\mathbb{R}}$ such that $\nu(\varnothing) = 0$ and

$$\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \nu(A_n)$$

for every family $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint subsets of Σ. We also say that ν is *a real measure on* (X, Σ) if it is signed measure and takes values in $\mathbb{R}$.

Definition 2.2. Let (X, Σ) be a measurable space. *A complex measure* is a function $\nu : \Sigma \to \mathbb{C}$ such that $\nu(\varnothing) = 0$ and

$$\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \nu(A_n)$$

for every family $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint subsets of Σ.

Definition 2.3. Let (X, Σ) be a measurable space and let μ, ν be two measures (either complex or signed) on (X, Σ). Suppose that for every set A in Σ we have

$$\mu(A) = 0 \Rightarrow \nu(A) = 0$$

Then we write $\nu \ll \mu$ and say that ν is *absolutely continuous with respect to* μ.

Definition 2.4. Let (X, Σ) be a measurable space and let μ, ν be two measures (either complex or signed) on (X, Σ). Suppose that there exists a set $S \in \Sigma$ such that

$$\mu(A \cap S) = 0,\ \nu(A \setminus S) = 0$$

for every $A \in \Sigma$. Then we write $\nu \perp \mu$ and say that ν is *singular with respect to* μ.

3. Hahn-Jordan Decomposition

Theorem 3.1 (Hahn-Jordan decomposition)**.** *Let* (X, Σ) *be a measurable space and* $\nu : \Sigma \to \overline{\mathbb{R}}$ *be a signed measure. Then there exists the unique pair of measures* $\nu_+, \nu_- : \Sigma \to [0, +\infty]$ *such that*

$$\nu = \nu_+ - \nu_-$$

and $\nu_+ \perp \nu_-$.

For the proof we need the following notion.

Definition 3.2. Let (X,Σ,ν) be a space with signed measure. A set $A\in\Sigma$ is *positive* if for every subset B of A such that $B\in\Sigma$ we have inequality $\nu(B)\geq 0$.

Lemma 3.2.1. *Let $B\in\Sigma$ be a set such that $\nu(B)\in\mathbb{R}$ and $\nu(B)>0$. Then there exists a positive set $C\subseteq B$ such that $\nu(B)\leq\nu(C)$.*

Proof of the lemma. First note that all sets $A\in\Sigma$ contained in B have finite measure (we left the proof as an exercise for the reader). For every subset $A\in\Sigma$ contained in B we define

$$\delta(A)=\max\left\{\frac{1}{2}\inf\left\{\nu(D)\,\middle|\,D\text{ is a subset of }A\text{ in }\Sigma\right\},-1\right\}$$

Note that $\delta(A)\leq 0$. Now we define a sequence $\{D_n\}_{n\in\mathbb{N}}$ of disjoint subsets of B and members of Σ. This is done recursively as follows. If $D_0,...,D_n$ are defined, then we pick D_{n+1} as a subset of $B\setminus(D_0\cup...\cup D_n)$ in Σ such that

$$\nu(D_{n+1})\leq\delta\big(B\setminus(D_0\cup...\cup D_n)\big)$$

Let

$$C=B\setminus\bigcup_{n\in\mathbb{N}}D_n$$

be a subset of B. Clearly $C\in\Sigma$ and for every $n\in\mathbb{N}$ we have

$$\delta(C)\geq\delta\big(B\setminus(D_0\cup...\cup D_n)\big)$$

Thus

$$\nu(C)=\nu(B)-\sum_{n\in\mathbb{N}}\nu(D_n)\geq\nu(B)-\sum_{n\in\mathbb{N}}\delta\big(B\setminus(D_0\cup...\cup D_n)\big)=\nu(B)-\sum_{n\in\mathbb{N}}\delta(C)$$

Since $\nu(C)\in\mathbb{R}$, we derive that $\delta(C)=0$. This implies that C is a positive set and $\nu(C)\geq\nu(B)$. □

Proof of the theorem. Assume that for every $A\in\Sigma$ we have $\nu(A)\in\mathbb{R}\cup\{-\infty\}$. Now consider

$$\alpha=\sup\left\{\nu(C)\,\middle|\,C\text{ is positive}\right\}$$

We can find a nondecreasing sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of nonnegative real numbers that converges to α and such that for every $n\in\mathbb{N}$ there exists a positive set C_n with $\nu(C_n)=\alpha_n$. Now pick $P=\bigcup_{n\in\mathbb{N}}C_n$. Obviously P is positive and $\nu(P)=\alpha$. In particular, $\alpha\in\mathbb{R}$. Assume that there exists $B\in\Sigma$ such that $B\subseteq X\setminus P$ and $\nu(B)>0$. According to Lemma 3.2.1 we deduce that there exists a positive set C inside B such that $\nu(B)\leq\nu(C)$. Then we get

$$\alpha=\nu(P)<\nu(P)+\nu(C)=\nu(P\cup C)$$

and $P\cup C$ is positive. This contradicts the definition of α. Hence for every $B\subseteq X\setminus P$ such that $B\in\Sigma$ we have $\nu(B)\leq 0$. Thus measures

$$\nu_+(A)=\nu(A\cap P),\ \nu_-(A)=-\nu(A\setminus P)$$

defined for $A\in\Sigma$ satisfy the assertion of the theorem. This finishes the proof of the Hahn-Jordan decomposition under the assumption that $\nu(A)\in\mathbb{R}\cup\{-\infty\}$ for all $A\in\Sigma$.
If $\nu(A)\in\mathbb{R}\cup\{+\infty\}$ for every $A\in\Sigma$, then we apply the result above for $-\nu$. Finally the case $\nu(A_1)=-\infty$ and $\nu(A_2)=+\infty$ for some $A_1,A_2\in\Sigma$ yields to the contradiction. Hence Hahn-Jordan decomposition is proved. □

Corollary 3.3. *Let (X,Σ) be a measurable space and $\nu:\Sigma\to\overline{\mathbb{R}}$ be a signed measure. Then either ν_+ or ν_- is finite.*

Proof. According to Theorem 3.1 we have $\nu=\nu_+-\nu_-$ and both ν_+,ν_- are nonnegative measures such that $\nu_+\perp\nu_-$. We cannot have $\nu_+(X)=\nu_-(X)=+\infty$, because then $\nu(X)$ would be undefined in $\overline{\mathbb{R}}$. This implies that either $\nu_+(X)\in\mathbb{R}$ or $\nu_-(X)\in\mathbb{R}$. □

4. Lebesgue decomposition

Definition 4.1. Let (X,Σ) be a measurable space and $\mu : \Sigma \to \overline{\mathbb{R}}$ be a signed measure. We say that μ is σ-finite if there exists a decomposition

$$X = \bigcup_{n\in\mathbb{N}} X_n$$

onto pairwise disjoint elements of Σ such that $\mu(X_n) \in \mathbb{R}$ for every $n \in \mathbb{N}$.

Theorem 4.2 (Lebesgue decomposition)**.** *Let (X,Σ) be a measurable space and let μ be a σ-finite, measure on (X,Σ). Suppose that ν is either a signed and σ-finite measure or a complex measure on (X,Σ). Then there exists a unique decomposition*

$$\nu = \nu_s + \nu_a$$

of measure ν such that $\nu_s \perp \mu$ and $\nu_a \ll \mu$.

Proof. Suppose first that ν is finite measure. Consider

$$\alpha = \sup_{A\in\Sigma,\, \mu(A)=0} \nu(A)$$

Since ν is finite, we derive that $\alpha \in \mathbb{R}$. Consider a sequence $\{A_n\}_{n\in\mathbb{N}}$ such that $A_n \in \Sigma$, $\mu(A_n) = 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} \nu(A_n) = \alpha$. Define $S = \bigcup_{n\in\mathbb{N}} A_n$. Then $\mu(S) = 0$ and $\nu(S) = \alpha$. Moreover, if $A \in \Sigma$ and $A \cap S = \emptyset$, then $\mu(A) = 0$ implies $\nu(A) = 0$. Now we define $\nu_s(A) = \nu(A\cap S)$ and $\nu_a(A) = \nu(A \setminus S)$ for every $A \in \Sigma$. Then $\nu = \nu_s + \nu_a$ and $\nu_s \perp \mu$, $\nu_a \ll \mu$.
Now assume that ν is σ-finite measure. There exists a decomposition

$$X = \bigcup_{n\in\mathbb{N}} X_n$$

onto pairwise disjoint elements of Σ such that $\mu(X_n) \in \mathbb{R}$ for every $n \in \mathbb{N}$. We define $\nu_n(A) = \nu(A \cap X_n)$ for each $n \in \mathbb{N}$ and $A \in \Sigma$. Then ν_n is a finite measure. By the case above we find $\nu_n = \nu_{ns} + \nu_{na}$ and $\nu_{ns} \perp \mu$, $\nu_{na} \ll \mu$ for some measures on Σ. Now we define

$$\nu_s = \sum_{n\in\mathbb{N}} \nu_{ns},\ \nu_a = \sum_{n\in\mathbb{N}} \nu_{an}$$

Then $\nu = \nu_s + \nu_a$ and $\nu_s \perp \mu$, $\nu_a \ll \mu$.
Now consider the case when ν is σ-finite and signed measure. According to Theorem 3.1 we write $\nu = \nu_+ - \nu_-$ for measures ν_+, ν_- such that $\nu_+ \perp \nu_-$. Then ν_+, ν_- are σ-finite measures. According to previous case we can write $\nu_+ = \nu_{+s} + \nu_{+a}$, $\nu_- = \nu_{-s} + \nu_{-a}$ for measures such that $\nu_{+s} \perp \mu, \nu_{-s} \perp \mu, \nu_{+a} \ll \mu, \nu_{-a} \ll \mu$. Then $\nu_s = \nu_{+s} - \nu_{-s}, \nu_a = \nu_{+a} - \nu_{-a}$ are signed measures and $\nu_s \perp \mu, \nu_a \ll \mu$.
Finally assume that ν is complex. Then $\nu = \nu^r + i\cdot\nu^i$, where ν^r and ν^i are finite, signed measures. Form the case above we have decompositions

$$\nu^r = \nu^r_s + \nu^r_a,\ \nu^i = \nu^i_s + \nu^i_s$$

and $\nu^r_s \perp \mu$, $\nu^i_s \perp \mu$, $\nu^r_a \ll \mu$, $\nu^i_a \ll \mu$. Then complex measures

$$\nu_s = \nu^r_s + i\cdot\nu^i_s,\ \nu_a = \nu^r_a + i\cdot\nu^i_a$$

satisfy $\nu_s \perp \mu$, $\nu_a \ll \mu$. □

5. Radon-Nikodym theorem

In this section we prove famous result of Radon and Nikodym.

Theorem 5.1 (Radon-Nikodym)**.** *Let (X,Σ) be a measurable space and let μ be a σ-finite measure on (X,Σ). Suppose that $\nu \ll \mu$ for ν that is either complex measure or σ-finite, signed measure. Then there exists a measurable function $f : X \to \mathbb{C}$ such that*

$$\nu(A) = \int_A f d\mu$$

for every $A \in \Sigma$.

Proof for finite measures μ, ν. Fix $n \in \mathbb{N}$ and $k \in \mathbb{N}$. According to Theorem 3.1 there exists a set $P_{n,k} \in \Sigma$ such that

$$\left(\nu - \frac{k}{2^n} \cdot \mu\right)\left(A \cap P_{n,k}\right) \geq 0, \left(\nu - \frac{k}{2^n} \cdot \mu\right)\left(A \smallsetminus P_{n,k}\right) \leq 0$$

for every $A \in \Sigma$. We may assume that $P_{n,0} = X$, $P_{n,k+1} \subseteq P_{n,k}$ and $P_{n,k} = P_{n+1,2k}$ for every $n, k \in \mathbb{N}$. Since $\nu \ll \mu$ and ν is finite, we derive that

$$\mu\left(\bigcap_{k \in \mathbb{N}} P_{n,k}\right) = \nu\left(\bigcap_{k \in \mathbb{N}} P_{n,k}\right) = 0$$

and may assume that this set is empty for every $n \in \mathbb{N}$. Pick $n \in \mathbb{N}$. We define a function $f_n : X \to \mathbb{C}$ by formula

$$f_n(x) = \sum_{k \in \mathbb{N}} \frac{k}{2^n} \cdot \chi_{P_{n,k} \smallsetminus P_{n,k+1}}(x)$$

Clearly f_n is a measurable function with real nonnegative values. If $m, n \in \mathbb{N}$ and $n \leq m$, then we have

$$f_n(x) \leq f_m(x) \leq f_n(x) + \frac{1}{2^n}$$

Thus $\{f_n\}_{n \in \mathbb{N}}$ is a nondecreasing sequence of measurable functions convergent uniformly to a measurable function $f : X \to \mathbb{C}$. Moreover, for every $A \in \Sigma$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
\nu(A) - \frac{1}{2^n}\mu(A) &= \sum_{k \in \mathbb{N}} \nu\left(A \cap \left(P_{n,k} \smallsetminus P_{n,k+1}\right)\right) - \frac{1}{2^n}\mu(A) \leq \\
&\leq \sum_{k \in \mathbb{N}} \frac{k+1}{2^n} \mu\left(A \cap \left(P_{n,k} \smallsetminus P_{n,k+1}\right)\right) - \frac{1}{2^n} \sum_{k \in \mathbb{N}} \mu\left(A \cap \left(P_{n,k} \smallsetminus P_{n,k+1}\right)\right) \leq \\
&\leq \sum_{k \in \mathbb{N}} \frac{k}{2^n} \mu\left(A \cap \left(P_{n,k} \smallsetminus P_{n,k+1}\right)\right) \leq \sum_{k \in \mathbb{N}} \nu\left(A \cap \left(P_{n,k} \smallsetminus P_{n,k+1}\right)\right) = \nu(A)
\end{aligned}$$

and since

$$\int_A f_n \, d\mu = \sum_{k \in \mathbb{N}} \frac{k}{2^n} \mu\left(A \cap \left(P_{n,k} \smallsetminus P_{n,k+1}\right)\right)$$

we derive that

$$\nu(A) - \frac{1}{2^n}\mu(A) \leq \int_A f_n \, d\mu \leq \nu(A)$$

This inequality together with monotone convergence theorem imply that

$$\nu(A) = \lim_{n \to +\infty} \int_A f_n \, d\mu = \int_A f \, d\mu$$

This finishes the proof for finite measures ν, μ. □

Reduction to finite case. Assume now that ν and μ are σ-finite measures on (X, Σ). Then there exists a decomposition

$$X = \bigcup_{n \in \mathbb{N}} X_n$$

onto disjoint subsets in Σ such that $\nu(X_n) \in \mathbb{R}$ and $\mu(X_n) \in \mathbb{R}$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we define $\nu_n(A) = \nu(A \cap X_n)$ and $\mu_n(A) = \mu(A \cap X_n)$ for $A \in \Sigma$. Since $\nu \ll \mu$, we derive that $\nu_n \ll \mu_n$ for every $n \in \mathbb{N}$. Measures $\{\nu_n\}_{n \in \mathbb{N}}$ and $\{\mu_n\}_{n \in \mathbb{N}}$ are finite. By finite case of the theorem we deduce that for each $n \in \mathbb{N}$ there exists a measurable function $f_n : X \to \mathbb{C}$ such that

$$\nu_n(A) = \int_A f_n \, d\mu_n$$

for every $A \in \Sigma$. Note that f_n is real nonnegative and can be set equal to zero outside X_n. Thus

$$\nu_n(A) = \int_A f_n \, d\mu_n = \int_A f_n \, d\mu$$

for every $A \in \Sigma$. Therefore, we deduce that

$$\nu(A) = \sum_{n\in\mathbb{N}} \nu(A \cap X_n) = \sum_{n\in\mathbb{N}} \nu_n(A) = \sum_{n\in\mathbb{N}} \int_A f_n\, d\mu = \int_A \left(\sum_{n\in\mathbb{N}} f_n\right) d\mu$$

by monotone convergence theorem.
Assume now that both ν is σ-finite, signed measure. In this situation we may write $\nu = \nu_+ - \nu_-$ for measures ν_+, ν_- such that $\nu_+ \perp \nu_-$. There exists a set $P \in \Sigma$ such that $\nu_-(P) = \nu_+(X \smallsetminus P) = 0$. Then $\nu_+ \ll \mu$ and $\nu_- \ll \mu$. By the case considered previously there exist measurable functions $f_+ : X \to \mathbb{C}$, $f_- : X \to \mathbb{C}$ such that

$$\nu_+(A) = \int_A f_+\, d\mu,\ \nu_-(A) = \int_A f_-\, d\mu$$

for every $A \in \Sigma$. Moreover, we may assume that f_+ is equal to zero outside P and f_- is equal to zero outside $X \smallsetminus P$. From this we have

$$\nu(A) = \nu_+(A) + \nu_-(A) = \int_A f_+\, d\mu + \int_A f_-\, d\mu = \int_A (f_+ - f_-)\, d\mu$$

for every $A \in \Sigma$.
Suppose that ν is complex measure. Write $\nu = \nu_r - i \cdot \nu_i$. Then both ν_r, ν_- are finite, signed measures. Moreover, we have $\nu_r \ll \mu, \nu_i \ll \mu$. There exist measurable functions $f_r : X \to \mathbb{C}$ and $f_i : X \to \mathbb{C}$ that are real valued and satisfy

$$\nu_r(A) = \int_A f_r\, d\mu,\ \nu_i(A) = \int_A f_i\, d\mu$$

for every $A \in \Sigma$. Thus

$$\nu(A) = \nu_r(A) + i \cdot \nu_i(A) = \int_A f_r\, d\mu + i \cdot \int_A f_i\, d\mu = \int_A (f_r + i \cdot f_i)\, d\mu$$

for every $A \in \Sigma$. □

6. Applications of Radon-Nikodyn theorem

Proposition 6.1. *Let μ be a measure on (X, Σ) and $f : X \to \mathbb{C}$ be a measurable function taking nonnegative values. We define*

$$\nu(A) = \int_A g\, d\mu$$

for every $A \in \Sigma$. Then ν is a measure on (X, Σ) and the equality

$$\int_X g\, d\nu = \int_X g \cdot f\, d\mu$$

holds for every measurable function $g : X \to \mathbb{C}$ that is either ν-integrable or takes nonnegative values.

Proof. Suppose that $A = \bigcup_{n\in\mathbb{N}} A_n$ for $A \in \Sigma$ and $A_n \in \Sigma$ for every $n \in \mathbb{N}$. Assume also that $\{A_n\}_{n\in\mathbb{N}}$ are pairwise disjoint. Then by monotone convergence theorem

$$\nu(A) = \int_A f\, d\mu = \int_X \chi_A \cdot f\, d\mu = \int_X \left(\sum_{n\in\mathbb{N}} \chi_{A_n} \cdot f\right) d\mu = \sum_{n\in\mathbb{N}} \int_X \chi_{A_n} \cdot f\, d\mu = \sum_{n\in\mathbb{N}} \int_{A_n} f\, d\mu = \sum_{n\in\mathbb{N}} \nu(A_n)$$

Moreover, we have $\nu(\varnothing) = 0$. Thus ν is a measure on (X, Σ).
For the second part of the statement note that the family of measurable functions $g : X \to \mathbb{C}$ satisfying equality

$$\int_X g\, d\nu = \int_X g \cdot f\, d\mu$$

contains $\{\chi_A\}_{A\in\Sigma}$, is closed under $\mathbb{R}_{\geq 0}$-linear combinations of measurable functions taking nonnegative values, if it contains nondecreasing sequence $\{g_n : X \to \mathbb{C}\}_{n\in\mathbb{N}}$ of measurable functions taking only nonnegative values, then it also contains its limit. Thus this family contains all measurable functions $g : X \to \mathbb{C}$ taking nonnegative values. Since every real valued, ν-integrable function $g : X \to \mathbb{C}$ is a difference of a two ν-integrable functions taking nonnegative values, we deduce that this family contains all real, ν-integrable functions. Finally, if it contains two ν-integrable, real valued functions, then it contains its $\mathbb{C}$-linear combination. Thus it contains all ν-integrable functions. □

Theorem 6.2. *Let μ be a complex measure on (X, Σ). There exists a measurable function $f : X \to \mathbb{C}$ such that*

$$\mu(A) = \int_A f\, d|\mu|$$

for every $A \in \Sigma$ and $|f(x)| = 1$ for every x in X.

For the proof we need the following result.

Lemma 6.2.1. *Let μ be a measure on (X, Σ). Suppose that $f : X \to \mathbb{C}$ is a measurable function and F is a closed subset of $\mathbb{C}$. Assume that for every $A \in \Sigma$ such that $\mu(A) > 0$, we have*

$$\frac{1}{\mu(A)} \int_A f\, d\mu \in F$$

Then $\mu\left(X \smallsetminus f^{-1}(F)\right) = 0$.

Proof of the lemma. Let D be a closed disc in $\mathbb{C}$ such that $D \cap F = \emptyset$. If $\mu\left(f^{-1}(D)\right) > 0$, then

$$\frac{1}{\mu\left(f^{-1}(D)\right)} \int_{f^{-1}(D)} f\, d\mu \in D$$

by convexity of D. This implies that for every closed disc in $\mathbb{C}$ disjoint from F we have $\mu\left(f^{-1}(D)\right) = 0$. Since $\mathbb{C} \smallsetminus F$ can be covered by such discs, we derive that $\mu\left(X \smallsetminus f^{-1}(F)\right) = 0$. □

Proof of the theorem. Consider Radon-Nikodym derivative $f : X \to \mathbb{C}$ of μ with respect to $|\mu|$. It exists according to Theorem 5.1. Since

$$\frac{1}{\mu(A)} \left| \int_A f\, d|\mu| \right| \le \frac{1}{\mu(A)} \int_A |f|\, d|\mu| = \frac{|\mu|(A)}{\mu(A)} \le 1$$

for every $A \in \Sigma$ such that $A \in \Sigma$, we derive by Lemma 6.2.1 that $f(x) \in D$ almost everywhere with respect to measure $|\mu|$, where D is a closed unit disc in $\mathbb{C}$. Changing values of f on a set of measure $|\mu|$ equal to zero, we may assume that $f(x) \in D$ for every x in X.
Suppose next that $0 < \alpha < 1$ and denote $A_\alpha = f^{-1}\left(\{z \in \mathbb{C} \mid |f(z)| \le \alpha\}\right)$. Let

$$A_\alpha = \bigcup_{n \in \mathbb{N}} A_n$$

be a decomposition on disjoint subsets in Σ. Then

$$\sum_{n \in \mathbb{N}} |\mu(A_n)| = \sum_{n \in \mathbb{N}} \left| \int_{A_n} f\, d|\mu| \right| \le \sum_{n \in \mathbb{N}} \int_{A_n} |f|\, d|\mu| \le \alpha \cdot \sum_{n \in \mathbb{N}} |\mu|(A_n) = \alpha \cdot |\mu|(A_\alpha)$$

Hence

$$|\mu|(A_\alpha) \le \alpha \cdot |\mu|(A_\alpha)$$

Therefore, $|\mu|(A_\alpha) = 0$. Since α is arbitrary number in $(0, 1)$, we deduce that

$$|\mu|\Big(\big\{z \in \mathbb{C} \,\big|\, |f(z)| < 1\big\}\Big) = 0$$

Thus changing values of f on a set of measure $|\mu|$ equal to zero, we may assume that $|f(x)| = 1$ for every x in X. □

Corollary 6.3. *Let μ be a measure on (X, Σ) and $f : X \to \mathbb{C}$ be a μ-integrable function. Define*

$$\nu(A) = \int_A f\, d\mu$$

for every $A \in \mathbb{C}$*. Then* ν *is a complex measure on* (X, Σ) *and*

$$|\nu|(A) = \int_A |f|\, d\mu$$

for every $A \in \Sigma$*.*

Proof. Clearly $\nu(A) \in \mathbb{C}$ for every $A \in \Sigma$. Suppose that $A = \bigcup_{n \in \mathbb{N}} A_n$ for $A \in \Sigma$ and $A_n \in \Sigma$ for every $n \in \mathbb{N}$. Assume also that $\{A_n\}_{n \in \mathbb{N}}$ are pairwise disjoint. Then by dominated convergence theorem

$$\nu(A) = \int_A f\, d\mu = \int_X \chi_A \cdot f\, d\mu = \int_X \left(\sum_{n \in \mathbb{N}} \chi_{A_n} \cdot f\right) d\mu = \sum_{n \in \mathbb{N}} \int_X \chi_{A_n} \cdot f\, d\mu = \sum_{n \in \mathbb{N}} \int_{A_n} f\, d\mu = \sum_{n \in \mathbb{N}} \nu(A_n)$$

Moreover, we have $\nu(\emptyset) = 0$. Thus ν is a complex measure on (X, Σ). Since f is μ-integrable, there exists a σ-finite subset $\Omega \in \Sigma$ such that $|f(x)| = 0$ for $x \notin \Omega$. We define $\tilde{\mu}(A) = \mu(A \cap \Omega)$ for every $A \in \Sigma$. Clearly

$$\nu(A) = \int_A f\, d\mu = \int_A f\, d\tilde{\mu}$$

for every $A \in \Sigma$. Hence we have $|\nu| \ll \tilde{\mu}$ by definition of ν and $|\nu|$. Note that $\tilde{\mu}$ is a σ-finite measure. By Theorem 5.1 there exists a measurable function $g : X \to \mathbb{C}$ equal to zero outside Ω such that

$$|\nu|(A) = \int_A g\, d\tilde{\mu} = \int_A g\, d\mu$$

for every $A \in \Sigma$. We may assume that g takes only nonnegative values. By Theorem 6.2 there exists a function $h : X \to \mathbb{C}$ such that

$$\nu(A) = \int_A h\, d|\nu|$$

for every $A \in \Sigma$ and $|h(x)| = 1$ for all x in X. By Proposition 6.1 we deduce that

$$\int_A f\, d\mu = \nu(A) = \int_A h\, d|\nu| = \int_A h \cdot g\, d\mu$$

for every $A \in \Sigma$. Therefore, $f = h \cdot g$ almost everywhere with respect to μ. Thus

$$g(x) = |h(x)| \cdot g(x) = |f(x)|$$

almost everywhere with respect to μ. □

7. Banach spaces of complex and real measures

Proposition 7.1. *Let* μ *be a complex measure on* (X, Σ)*. For every* $A \in \Sigma$ *we define*

$$|\mu|(A) = \sup\left\{ \sum_{n \in \mathbb{N}} |\mu(A_n)| \;\middle|\; A = \bigcup_{n \in \mathbb{N}} A_n \text{ is a partition of } A \text{ onto subsets in } \Sigma \right\}$$

Then $|\mu|$ *is a finite measure on* (X, Σ)*.*

Proof. Let $\mu = \mu^r + i \cdot \mu^i$ be decomposition onto real and imaginary part. Then μ^r, μ^i are finite, signed measures. By Theorem 3.1 we derive that there exist decompositions $\mu^r = \mu^r_+ - \mu^r_-$, $\mu^i = \mu^i_+ - \mu^i_-$ such that $\mu^r_+, \mu^r_-, \mu^i_+, \mu^i_-$ are finite measures and $\mu^r_+ \perp \mu^r_-, \mu^i_+ \perp \mu^i_-$. Then for every partition

$$A = \bigcup_{n \in \mathbb{N}} A_n$$

of $A \in \Sigma$ onto sets in Σ we have

$$\sum_{n \in \mathbb{N}} |\mu(A_n)| = \sum_{n \in \mathbb{N}} \sqrt{(\mu^r(A_n))^2 + (\mu^i(A_n))^2} \le \sum_{n \in \mathbb{N}} \left(|\mu^r(A_n)| + |\mu^i(A_n)|\right) \le$$

$$\le \sum_{n \in \mathbb{N}} \left(\mu^r_+(A_n) + \mu^r_-(A_n) + \mu^i_+(A_n) + \mu^i_-(A_n)\right) = \mu^r_+(A) + \mu^r_-(A) + \mu^i_+(A) + \mu^i_-(A)$$

Left hand side of the inequality does not depend on the partition and hence

$$|\mu|(A) \leq \mu^r_+(A) + \mu^r_-(A) + \mu^i_+(A) + \mu^i_-(A)$$

This implies that $|\mu|(A) \in \mathbb{R}$ for every $A \in \Sigma$. Note also that $|\mu|(\varnothing) = 0$. Suppose now that $A \in \Sigma$ and we have partitions

$$A = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} C_n,\ A_n = \bigcup_{m\in\mathbb{N}} A_{n,m} \text{ for every } n \in \mathbb{N}$$

onto subsets in Σ. Then

$$\sum_{n\in\mathbb{N}} |\mu(C_n)| \leq \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} |\mu(A_n \cap C_m)| \leq \sum_{n\in\mathbb{N}} |\mu|(A_n)$$

and

$$\sum_{n\in\mathbb{N}} \left(\sum_{m\in\mathbb{N}} |\mu(A_{n,m})|\right) \leq |\mu|(A)$$

These inequalities imply that

$$|\mu|(A) \leq \sum_{n\in\mathbb{N}} |\mu|(A_n) \leq |\mu|(A)$$

Therefore, $|\mu|$ is a finite measure. □

Definition 7.2. Let μ be a complex measure on (X, Σ). Then we define

$$||\mu|| = |\mu|(X)$$

and call it *the total variation of* μ.

Theorem 7.3. *Let (X, Σ) be a measurable space and $\mathcal{M}(X, \Sigma)$ be a space of all complex measures on (X, Σ). Then the following assertions hold.*

(1) *$\mathcal{M}(X, \Sigma)$ is a $\mathbb{C}$-linear space.*

(2) *The mapping*

$$\mathcal{M}(X, \Sigma) \ni \mu \mapsto ||\mu|| \in [0, +\infty)$$

is a norm on that space.

(3) *Suppose that $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of complex measures on (X, Σ) that is a Cauchy sequence with respect to total variation. Then there exists a complex measure μ such that*

$$\lim_{n\to+\infty} \mu_n = \mu$$

Moreover, for every $A \in \Sigma$ we have

$$\lim_{n\to+\infty} \mu_n(A) = \mu(A)$$

Proof. We left **(1)** and **(2)** for the reader as an exercise. Fix $A \in \Sigma$. Then

$$|\mu_n(A) - \mu_m(A)| \leq |\mu_n - \mu_m|(A) \leq ||\mu_n - \mu_m||$$

for every $n, m \in \mathbb{N}$. Since $\{\mu_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to total variation, we derive that there exists the limit of $\{\mu_n(A)\}_{n\in\mathbb{N}}$. We denote

$$\mu(A) = \lim_{n\to+\infty} \mu_n(A)$$

Suppose that

$$A = \bigcup_{k\in\mathbb{N}} A_k$$

for $A \in \Sigma$ and $A_k \in \Sigma$ for $k \in \mathbb{N}$. Assume that sets $\{A_k\}_{k\in\mathbb{N}}$ are disjoint. Pick $N \in \mathbb{N}$. Then

$$\sum_{k=0}^{N} |\mu_n(A_k) - \mu(A_k)| = \lim_{m\to+\infty} \sum_{k=0}^{N} |\mu_n(A_k) - \mu_m(A_k)| \leq$$

$$\leq \limsup_{m\to+\infty} \sum_{k\in\mathbb{N}} |\mu_n(A_k) - \mu_m(A_k)| \leq \limsup_{m\to+\infty} |\mu_n - \mu_m|(A) = \limsup_{m\to+\infty} ||\mu_n - \mu_m||$$

This implies that

$$\sum_{k\in\mathbb{N}} |\mu_n(A_k) - \mu(A_k)| \leq \limsup_{m\to+\infty} ||\mu_n - \mu_m||$$

regardless of set A and partition $\{A_k\}_{k\in\mathbb{N}}$. Thus we deduce that there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers, convergent to zero such that

$$\sum_{k\in\mathbb{N}} |\mu_n(A_k) - \mu(A_k)| \leq a_n$$

for every $n \in \mathbb{N}$, $A \in \Sigma$ and partition $\{A_k\}_{k\in\mathbb{N}}$ as above. Therefore, for fixed $N \in \mathbb{N}$ we have

$$\left|\mu(A) - \sum_{k=0}^{N} \mu(A_k)\right| \leq |\mu(A) - \mu_n(A)| + \left|\mu_n(A) - \sum_{k=0}^{N} \mu_n(A_k)\right| + \sum_{k=0}^{N} |\mu_n(A_k) - \mu(A_k)| \leq$$

$$\leq |\mu(A) - \mu_n(A)| + \left|\mu_n(A) - \sum_{k=0}^{N} \mu_n(A_k)\right| + \sum_{k\in\mathbb{N}} |\mu_n(A_k) - \mu(A_k)| \leq 2a_n + \left|\mu_n(A) - \sum_{k=0}^{N} \mu_n(A_k)\right|$$

Hence we derive that

$$\mu(A) = \sum_{k\in\mathbb{N}} \mu(A_k)$$

thus μ is a complex measure and according to

$$\sum_{k\in\mathbb{N}} |\mu_n(A_k) - \mu(A_k)| \leq a_n$$

for every $n \in \mathbb{N}$ we deduce that

$$\lim_{n\to+\infty} |\mu_n - \mu|(A) = 0$$

for every $A \in \Sigma$. Hence also $\lim_{n\to+\infty} ||\mu_n - \mu|| = 0$. This finishes the proof of **(3)**. □

Corollary 7.4. *Let (X, Σ) be a measurable space and μ be a measure on Σ. Then there exists an isometrical embedding*

$$L^1(X, \mu) \ni f \mapsto \left(\Sigma \ni A \mapsto \int_A f\, d\mu \in \mathbb{C}\right) \in \mathcal{M}(X, \Sigma)$$

of Banach spaces. If in addition μ is σ-finite, then it is surjective map onto the subspace of $\mathcal{M}(X, \Sigma)$ consisting of complex measures which are absolutely continous with respect to μ.

Proof. The first assertion follows from Corollary 6.3 and Theorem 7.3. The second is a recapitulation of Theorem 5.1. □

References

[Monygham, 2018] Monygham (2018). Introduction to measure theory. *github repository: "Monygham/Pedo-mellon-a-minno"*.