ALGEBRAIC MONOIDS

1. The unit group of an algebraic monoid

Theorem 1.1. *Let $f : X \to Y$ be a dominant morphism of finite type between irreducible schemes. Suppose that η is a generic point and assume that the generic fiber $f^{-1}(\eta)$ is finite. Then there exists an open and nonempty subset V of Y such that the restriction $f^{-1}(V) \to V$ of V is finite.*

For the proof we need the following local version of the theorem.

Lemma 1.1.1. *Let A be a ring such that* $\operatorname{Spec} A$ *is irreducible and let B be an A-algebra of finite type. Suppose that a unique minimal prime ideal $\mathfrak{p}$ of A is nilpotent and $k(\mathfrak{p}) \otimes_A B$ is finite over $k(\mathfrak{p})$, where $k(\mathfrak{p})$ denotes the residue field of $\mathfrak{p}$ in A. Then there exists nonzero s in A such that B_s is a finite A_s-module.*

Proof of the lemma. Let $b_1, ..., b_n$ be generators of B as an A-algebra. Then

$$\overline{b_i} = b_i \operatorname{mod} \mathfrak{p}B$$

for $1 \leq i \leq n$ are generators of $B/\mathfrak{p}B$ as an $A/\mathfrak{p}$ algebra. Since $k(\mathfrak{p}) \otimes_A B$ is finite over $k(\mathfrak{p})$ for each i there exists positive integer m_i and a polynomial

$$f_i(x) = s_{m_i} x^{m_i} + s_{m_i - 1} x^{m_i - 1} + ... + s_0 \in (A/\mathfrak{p})\,[x]$$

such that $s_{m_i} \neq 0$ and $f_i(\overline{b_i}) = 0$. Let $s \in A$ be an element such that

$$\overline{s} = s \operatorname{mod} \mathfrak{p} = s_{m_1} \cdot s_{m_2} \cdot ... \cdot s_{m_n}$$

Clearly s is nonzero and $B_s / (\mathfrak{p}B)_s = (B/\mathfrak{p}B)_s$ is a finite A_s-algebra. Hence there exists a finite A_s-submodule M of B_s such that

$$B_s = M + (\mathfrak{p}B)_s = M + \mathfrak{p}B_s$$

Since $\mathfrak{p}$ is nilpotent, there exists $N \in \mathbb{N}$ such that $\mathfrak{p}^N = 0$. Thus

$$B_s = M + \mathfrak{p}B_s = M + \mathfrak{p}M + \mathfrak{p}^2 B_s = ... = M + \mathfrak{p}M + ... + \mathfrak{p}^{N-1}M + \mathfrak{p}^N B_s = M + \mathfrak{p}M + ... + \mathfrak{p}^{N-1}M$$

is a finite A_s-module. □

Proof of the theorem. Pick an open, nonempty, affine neighborhood W of η. Since f is of finite type, we derive that

$$f^{-1}(W) = \bigcup_{i=1}^{n} U_i$$

where each U_i is nonempty open affine subscheme of X and moreover, the morphism $U_i \to V$ induced by f is of finite type. According to Lemma 1.1.1 for each i there exists an open, affine and nonempty subscheme $W_i \subseteq W$ such that the morphism $f^{-1}(W_i) \cap U_i \to W_i$ induced by f is finite. Thus replacing W by the intersection of $W_1, ..., W_n$ we may assume that each $U_i \to W$ is finite. Consider

$$F = f^{-1}(W) \setminus \left(\bigcap_{i=1}^{n} U_i \right)$$

Then F is a closed subset of $f^{-1}(W)$ and it does not contain the generic point ξ of X. Since each restriction $U_i \to W$ of f is finite, we derive that $f\big(U_i \cap F\big)$ is closed in W for every $1 \leq i \leq n$ and does not contain $\eta = f(\xi)$ (f is dominant). Thus $f(F)$ is a closed subset of W and $\eta \notin f(F)$. Hence $V = W \setminus f(F)$ is an open neighborhood of η and $f^{-1}(V) \subseteq \bigcap_{i=1}^{n} U_i$. Thus the restriction $f^{-1}(V) \to V$ of f is finite. □

Theorem 1.2. *Let $\mathbf{M}$ be a geometrically integral algebraic monoid k-scheme. Suppose that $\mathbf{G}$ is a group of units of $\mathbf{M}$ and $i : \mathbf{G} \hookrightarrow \mathbf{M}$ is the canonical monomorphism. Then i is an open immersion.*

Proof. Assume that k is algebraically closed. Denote by $\mu : \mathbf{M} \times_k \mathbf{M} \to \mathbf{M}$ and $e : \operatorname{Spec} k \to \mathbf{M}$ the multiplication and the unit, respectively. Since $\mathbf{M}$ is integral and of finite type over k, we derive that $\mathbf{M} \times_k \mathbf{M}$ is integral and

$$\dim\left(\mathbf{M} \times_k \mathbf{M}\right) = 2 \cdot \dim\left(\mathbf{M}\right)$$

Moreover, μ is surjective (which can be checked on k-functors of points). Pick any irreducible component Z of $\mu^{-1}(e)$. By [Görtz and Wedhorn, 2010, Lemma 14.109] we deduce

$$\dim\left(Z\right) \geq \dim\left(\mu^{-1}(\eta)\right)$$

where η is the generic point of $\mathbf{M}$. Since

$$\dim\left(\mu^{-1}(\eta)\right) = \dim\left(\mathbf{M} \times_k \mathbf{M}\right) - \dim\left(\mathbf{M}\right) = 2 \cdot \dim\left(\mathbf{M}\right) - \dim\left(\mathbf{M}\right) = \dim\left(\mathbf{M}\right)$$

we deduce that $\dim\left(Z\right) \geq \dim(\mathbf{M})$. Moreover, we have $\mathbf{G} \cong \mu^{-1}(e)$ as k-schemes and this isomorphism is given by the restriction $\pi : \mu^{-1}(e) \to \mathbf{G}$ to $\mu^{-1}(e)$ of the projection $\mathrm{pr} : \mathbf{M} \times_k \mathbf{M} \to \mathbf{M}$ on the first factor (this can be checked on k-functors of points). Hence $\mathbf{G}$ is of finite type over k as it is isomorphic with a closed subscheme of $\mathbf{M} \times_k \mathbf{M}$ and each irreducible component Z of $\mathbf{G}$ is of dimension at least $\dim(\mathbf{M})$. Now we fix an irreducible component Z of $\mathbf{G}$ and consider it as a closed subscheme of $\mathbf{G}$ with reduced structure. Then the morphism $i_{|Z} : Z \hookrightarrow \mathbf{M}$ is a monomorphism of finite type and $\dim(Z) \geq \dim(\mathbf{M})$. Hence $i_{|Z}$ is dominant. Since i is a monomorphism, this implies that $\mathbf{G}$ has only one irreducible component and $i : \mathbf{G} \hookrightarrow \mathbf{M}$ is dominant. By Theorem 1.1 there exists an open and nonempty subset V of $\mathbf{M}$ such that the morphism $i^{-1}(V) \hookrightarrow V$ induced by i is finite. Finite monomorphisms are closed immersions and dominant, closed immersions with integral scheme as a codomain are isomorphisms. Thus $i^{-1}(V) \to V$ is an isomorphism. Now pick a k-point g of $\mathbf{G}$. Since $\mathbf{G}$ is a group k-scheme, we derive that $g \cdot (-) : \mathbf{M} \to \mathbf{M}$ is an automorphism of k-scheme $\mathbf{M}$. This implies that $i^{-1}(g \cdot V) \to g \cdot V$ is an isomorphism. This holds for every k-point of $\mathbf{G}$ and

$$i(\mathbf{G}) \subseteq \bigcup_{g \in \mathbf{G}(k)} g \cdot V$$

where $\mathbf{G}(k)$ is the set of k-points of $\mathbf{G}$. Therefore, i is an open immersion.
If k is not algebraically closed, then we pick an algebraically closed extension K of k and consider $1_{\operatorname{Spec} K} \times_k i$. This is an open immersion according to the case considered above. By faithfully flat descent i is an open immersion. □

The more general result for algebraically closed fields is [Brion, 2014, Theorem 1]. Let us also note the following theorems.

Theorem 1.3 ([Demazure and Gabriel, 1970, Chapitre 2, &2, Corollaire 3.6])**.** *Let $\mathbf{M}$ be an affine, algebraic monoid k-scheme. Suppose that $\mathbf{G}$ is a group of units of $\mathbf{M}$. Then there exists a regular function f on $\mathbf{M}$ such that canonical morphism $\mathbf{G} \hookrightarrow \mathbf{M}$ is the inclusion of open subscheme of $\mathbf{M}$ on which f is nonzero.*

The converse is also true.

Theorem 1.4 ([Brion, 2014, Theorem 2])**.** *Let $\mathbf{M}$ be a geometrically integral algebraic monoid over a field k and let $\mathbf{G}$ be an group of units of $\mathbf{M}$. If $\mathbf{G}$ is affine, then $\mathbf{M}$ is affine.*

2. Kempf monoids

In this section we discuss an important class of monoid k-schemes.

Proposition 2.1. *Let* $\mathbf{M}$ *be a monoid* k*-scheme with zero* $\mathbf{o}$ *and with group* $\mathbf{G}$ *of units. Suppose that for some field* K *over* k *there exists a closed immersion*

$$i : \mathbb{A}^1_K \hookrightarrow \operatorname{Spec} K \times_k \mathbf{M}$$

of monoid K*-schemes sending the zero of* $\mathbb{A}^1_K$ *to the unique zero* $\mathbf{o}_K$ *of* $\operatorname{Spec} K \times_{\operatorname{Spec} k} \mathbf{M}$ *lying over* $\mathbf{o}$*. Let* U *be an open* $\mathbf{G}$*-stable subscheme of* $\mathbf{M}$*. Then the following are equivalent.*

- **(i)** $\mathbf{o}$ *is contained in* U
- **(ii)** $U = \mathbf{M}$

Proof. Suppose that **(i)** holds. Denote

$$\operatorname{Spec} K \times_k \mathbf{M}, \operatorname{Spec} K \times_k \mathbf{G}, \operatorname{Spec} K \times_k U$$

by $\mathbf{M}_K$, $\mathbf{G}_K$, U_K, respectively. Note that $i\left(\mathbb{G}_{m,K}\right) \subseteq \mathbf{G}_K$. Fix a field L over K and a morphism $j : \operatorname{Spec} L \hookrightarrow \mathbf{M}_K$. Next consider the composition

$$\begin{tikzcd} \mathbb{A}^1_L = \mathbb{A}^1_K \times_K \operatorname{Spec} L \arrow[r, hook, "i \times_K j"] \arrow[rr, bend left, "f"] & \mathbf{M}_K \times_k \mathbf{M}_K \arrow[r, hook, "\mu_K"'] & \mathbf{M}_K \end{tikzcd}$$

where the second morphism $\mu_K : \mathbf{M}_K \times_k \mathbf{M}_K \to \mathbf{M}_K$ is the multiplication. Clearly f is $\mathbb{G}_{m,L}$-equivariant. Hence $f^{-1}(U_K)$ is an open $\mathbb{G}_{m,L}$-stable subscheme of $\mathbb{A}^1_L$ containing zero of this monoid L-scheme because $\mathbf{o}_K \in U_K$ by **(i)**. Since the only open $\mathbb{G}_{m,L}$-stable subscheme of $\mathbb{A}^1_L$ containing zero is $\mathbb{A}^1_L$, we derive that $f^{-1}(U_K) = \mathbb{A}^1_L$. Thus the image of j is in U_K. Hence $U_K = \mathbf{M}_K$ because $j : \operatorname{Spec} L \to \mathbf{M}_K$ and L are arbitrary. By faithfuly flat descent, we derive that $U = \mathbf{M}$ i.e. we deduced **(ii)**.
The implication **(ii)** $\Rightarrow$ **(i)** is obvious. □

Definition 2.2. Let $\mathbf{M}$ be an affine, geometrically integral monoid of finite type over k. Assume that $\mathbf{M}$ admits a zero $\mathbf{o}$. Suppose that for some field K over k there exists a closed immersion

$$i : \mathbb{A}^1_K \hookrightarrow \operatorname{Spec} K \times_k \mathbf{M}$$

of monoid K-schemes sending the zero of $\mathbb{A}^1_K$ to the unique zero $\mathbf{o}_K$ of $\operatorname{Spec} K \times_{\operatorname{Spec} k} \mathbf{M}$ lying over $\mathbf{o}$. Then $\mathbf{M}$ is called *a Kempf monoid*.

References

[Brion, 2014] Brion, M. (2014). On algebraic semigroups and monoids. In *Algebraic monoids, group embeddings, and algebraic combinatorics*, pages 1–54. Springer.

[Demazure and Gabriel, 1970] Demazure, M. and Gabriel, P. (1970). *Groupes algébriques. Tome I. Géométrie algébrique généralités. Groupes commutatifs*. North-Holland.

[Görtz and Wedhorn, 2010] Görtz, U. and Wedhorn, T. (2010). *Algebraic Geometry: Part I: Schemes. With Examples and Exercises*. Advanced Lectures in Mathematics.