RADON-NIKODYM THEOREM, HAHN-JORDAN DECOMPOSITION AND LEBESGUE DECOMPOSITION

1. Introduction

This notes are devoted to some more advanced notions in measure theory. Tools presented here are indispensable in probability theory and statistics. We refer to [Monygham, 2018] for extensive introduction to measure theory.

2. Signed and Complex Measures

In this section we define extension of the usual notion of nonnegative measure. Denote by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ the topological space obtained as a two-point compactification of the line $\mathbb{R}$. Addition is partially defined operation on $\overline{\mathbb{R}}$ given by the following rules

$$(+\infty) + r = +\infty = r + (+\infty),\ (-\infty) + r = -\infty = r + (-\infty)$$

for every $r \in \mathbb{R}$

Definition 2.1. Let (X, Σ) be a measurable space. *A signed measure on* (X, Σ) is a function $\nu : \Sigma \to \overline{\mathbb{R}}$ such that $\nu(\varnothing) = 0$ and

$$\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \nu(A_n)$$

for every family $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint subsets of Σ. We also say that ν is *a real measure on* (X, Σ) if it is signed measure and takes values in $\mathbb{R}$.

Definition 2.2. Let (X, Σ) be a measurable space. *A complex measure* is a function $\nu : \Sigma \to \mathbb{C}$ such that $\nu(\varnothing) = 0$ and

$$\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \nu(A_n)$$

for every family $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint subsets of Σ.

Definition 2.3. Let (X, Σ) be a measurable space and let μ, ν be two measures (either complex or signed) on (X, Σ). Suppose that for every set A in Σ we have

$$\mu(A) = 0 \Rightarrow \nu(A) = 0$$

Then we write $\nu \ll \mu$ and say that ν is *absolutely continuous with respect to* μ.

Definition 2.4. Let (X, Σ) be a measurable space and let μ, ν be two measures (either complex or signed) on (X, Σ). Suppose that there exists a set $S \in \Sigma$ such that

$$\mu(A \cap S) = 0,\ \nu(A \setminus S) = 0$$

for every $A \in \Sigma$. Then we write $\nu \perp \mu$ and say that ν is *singular with respect to* μ.

3. Functional analytic proof of Lebesgue decomposition and Radon-Nikodym theorem

Theorem 3.1. *Let* (X, Σ) *be a measurable space and let* μ, ν *be a finite, nonnegative measures. Then the following assertions hold.*

(1) *There exists a unique decomposition*

$$\nu = \nu_s + \nu_a$$

of measure ν such that $\nu_s \perp \mu$ and $\nu_a \ll \mu$.

(2) *There exists a unique up to a set of measure μ zero nonnegative and measurable function $f : X \to \mathbb{R}$ such that*

$$\nu_a(A) = \int_A f d\mu$$

for every $A \in \Sigma$.

The proof presented here is due to John von Neumman and it uses theory of Hilbert spaces. There are also measure-theoretic proofs available.

Proof. Consider a measure $\xi = \mu + \nu$. Then for every set A in Σ we have $\nu(A) \leq \xi(A)$ and $\mu(A) \leq \xi(A)$. We refer to these facts by notation $\nu \leq \xi$, $\mu \leq \xi$. Define $\mathbb{C}$-linear functional

$$L^2(\xi, \mathbb{C}) \ni f \mapsto \int_X f d\nu \in \mathbb{C}$$

From $\nu \leq \xi$ it follows that the functional is continuous. By representation of continuous functionals on Hilbert spaces, we derive that there exists $g \in L^2(\xi, \mathbb{C})$ such that

$$\int f d\nu = \int f \cdot g \, d\xi$$

for every $f \in L^2(\xi, \mathbb{C})$. We may assume that g is measurable by modifying it on a set of measure ξ (and hence also μ according to $\mu \leq \nu$) equal to zero. Pick $k \in \mathbb{N}$ and $A \in \Sigma$ and in the above equation set $f = \chi_A \cdot g^k$. Then

$$\int_A g^k \, d\nu = \int_A g^{k+1} \, d\xi \tag{*}$$

and hence we have

$$\int_A \left(g^k - g^{k+1}\right) d\nu = \int_A g^{k+1} d\mu$$

for every $A \in \Sigma$. Summing these equalities for $0 \leq k \leq n$ we derive that

$$\int_A \left(1 - g^{n+1}\right) d\nu = \int_A \left(g + g^2 + \ldots + g^{n+1}\right) d\mu \tag{**}$$

Now we let $k = 0$ in (*) and obtain

$$\nu(A) = \int_A g \, d\xi$$

Together with $\nu \leq \xi$ this implies that the inequality $0 \leq g(x) \leq 1$ holds ξ-almost everywhere and thus by simple modification of g we may assume that it holds everywhere. Define a set $Q = \left\{x \in X \middle| g(x) = 1\right\}$. Then we have $\nu(Q) = \xi(Q)$ and hence $\mu(Q) = 0$. Now for every $A \in \Sigma$ we define

$$\nu_s(A) = \nu(A \cap Q), \, \nu_a(A) = \nu(A \setminus Q)$$

Then we have $\nu_s \perp \mu$ and $0 \leq g(x) < 1$ for $x \notin Q$. Next by (**) and monotone convergence theorem we deduce that

$$\nu_a(A) = \nu(A \setminus Q) = \lim_{n \to +\infty} \int_{A \setminus Q} \left(1 - g^{n+1}\right) d\nu =$$
$$= \lim_{n \to +\infty} \int_{A \setminus Q} \left(g + g^2 + \ldots + g^{n+1}\right) d\mu = \int_A \chi_{X \setminus Q} \cdot \sum_{n \in \mathbb{N}} g^{n+1} \, d\mu$$

for every $A \in \Sigma$. Thus $\nu = \nu_s + \nu_a$, $\nu_s \perp \mu$ and for

$$f = \chi_{X \setminus Q} \cdot \sum_{n \in \mathbb{N}} g^{n+1}$$

we have

$$\nu_a(A) = \int_A f d\mu$$

for every $A \in \Sigma$. We proved **(1)** and **(2)** without uniqueness statements. We left them for the reader. $\square$

4. Hahn-Jordan Decomposition

Theorem 4.1 (Hahn-Jordan decomposition). *Let (X,Σ) be a measurable space and $\nu : \Sigma \to \overline{\mathbb{R}}$ be a signed measure. Then there exists the unique pair of measures $\nu_+, \nu_- : \Sigma \to [0,+\infty]$ such that*

$$\nu = \nu_+ - \nu_-$$

and $\nu_+ \perp \nu_-$.

For the proof we need the following notion.

Definition 4.2. Let (X,Σ,ν) be a space with signed measure. A set $A \in \Sigma$ is *positive* if for every subset B of A such that $B \in \Sigma$ we have inequality $\nu(B) \geq 0$.

Lemma 4.2.1. *Let $B \in \Sigma$ be a set such that $\nu(B) \in [0,+\infty)$. Then there exists a positive set $C \subseteq B$ such that $\nu(B) \leq \nu(C)$.*

Proof of the lemma. First note that all sets $A \in \Sigma$ contained in B have finite measure (we left the proof as an exercise for the reader). We define partial order $\preceq$ on the set $\mathcal{P}(B) \cap \Sigma$ by declaring $A_1 \preceq A_2$ if and only if $A_2 \subseteq A_1$ and $\nu(A_1) \leq \nu(A_2)$. Suppose now that $\mathfrak{I}$ is a chain (linearly ordered subset) in $(\mathcal{P}(B) \cap \Sigma, \preceq)$. Then the function $\mathfrak{I} \ni A \mapsto \nu(A) \in \mathbb{R}$ is strictly increasing. Hence there exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ of elements in $\mathfrak{I}$ such that $A_n \preceq A_{n+1}$ for every $n \in \mathbb{N}$ and

$$\lim_{n\to+\infty} \nu(A_n) = \sup_{A\in\mathfrak{I}} \nu(A)$$

Then

$$\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{P}(B) \cap \Sigma$$

is the least upper bound of $\mathfrak{I}$. Thus every chain in $(\mathcal{P}(B) \cap \Sigma, \preceq)$ has the least upper bound. Zorn's lemma implies that the ordered subset

$$\left\{A \in \mathcal{P}(B) \cap \Sigma \,\middle|\, B \preceq A\right\}$$

of $(\mathcal{P}(B) \cap \Sigma, \preceq)$ admits a maximal element C. We deduce that C is a positive subset of B and since $B \preceq C$ we have $\nu(B) \leq \nu(C)$. □

Proof of the theorem. Assume that for every $A \in \Sigma$ we have $\nu(A) \in \mathbb{R} \cup \{-\infty\}$. Now consider

$$\alpha = \sup\left\{\nu(C) \,\middle|\, C \text{ is positive}\right\}$$

We can find a nondecreasing sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of nonnegative real numbers that converges to α and such that for every $n \in \mathbb{N}$ there exists a positive set C_n with $\nu(C_n) = \alpha_n$. Now pick $P = \bigcup_{n\in\mathbb{N}} C_n$. Obviously P is positive and $\nu(P) = \alpha$. In particular, $\alpha \in \mathbb{R}$. Assume that there exists $B \in \Sigma$ such that $B \subseteq X \setminus P$ and $\nu(B) > 0$. According to Lemma 4.2.1 we deduce that there exists a positive set C inside B such that $\nu(B) \leq \nu(C)$. Then we get

$$\alpha = \nu(P) < \nu(P) + \nu(C) = \nu(P \cup C)$$

and $P \cup C$ is positive. This contradicts the definition of α. Hence for every $B \subseteq X \setminus P$ such that $B \in \Sigma$ we have $\nu(B) \leq 0$. Thus measures

$$\nu_+(A) = \nu(A \cap P),\ \nu_-(A) = -\nu(A \setminus P)$$

defined for $A \in \Sigma$ satisfy the assertion of the theorem. This finishes the proof of the Hahn-Jordan decomposition under the assumption that $\nu(A) \in \mathbb{R} \cup \{-\infty\}$ for all $A \in \Sigma$.
If $\nu(A) \in \mathbb{R} \cup \{+\infty\}$ for every $A \in \Sigma$, then we apply the result above for $-\nu$. Finally the case $\nu(A_1) = -\infty$ and $\nu(A_2) = +\infty$ for some $A_1, A_2 \in \Sigma$ yields to the contradiction. Hence Hahn-Jordan decomposition is proved. □

Corollary 4.3. *Let (X,Σ) be a measurable space and $\nu : \Sigma \to \overline{\mathbb{R}}$ be a signed measure. Then either ν_+ or ν_- is finite.*

Proof. According to Theorem 4.1 we have $\nu = \nu_+ - \nu_-$ and both ν_+, ν_- are nonnegative measures such that $\nu_+ \perp \nu_-$. We cannot have $\nu_+(X) = \nu_-(X) = +\infty$, because then $\nu(X)$ would be undefined in $\overline{\mathbb{R}}$. This implies that either $\nu_+(X) \in \mathbb{R}$ or $\nu_-(X) \in \mathbb{R}$. □

5. Lebesgue decomposition and general form of Radon-Nikodym theorem

Definition 5.1. Let (X,Σ) be a measurable space and $\mu : \Sigma \to \overline{\mathbb{R}}$ be a signed measure. We say that μ is σ-finite if there exists a decomposition

$$X = \bigcup_{n\in\mathbb{N}} X_n$$

onto pairwise disjoint elements of Σ such that $\mu(X_n) \in \mathbb{R}$ for every $n \in \mathbb{N}$.

Theorem 5.2 (Lebesgue decomposition)**.** *Let (X,Σ) be a measurable space and let μ be an σ-finite, nonnegative measure on (X,Σ). Suppose that ν is either a signed and σ-finite measure or a complex measure on (X,Σ). Then there exists a unique decomposition*

$$\nu = \nu_s + \nu_a$$

of measure ν such that $\nu_s \perp \mu$ and $\nu_a \ll \mu$.

The uniqueness is left for the reader. The existence is a consequence of Theorem 3.1 and the following elementary observation.

Lemma 5.2.1. *Let (X,Σ) be a measurable space and let ν_1, ν_2, μ be measures (either signed or complex) on (X,Σ). Assume that $\nu_1 + \nu_2$ exists. Then the following assertions hold.*

(1) *If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, then $(\nu_1 + \nu_2) \ll \mu$.*

(2) *If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$, then $(\nu_1 + \nu_2) \perp \mu$.*

Proof of the lemma. We left the proof of **(1)** for the reader.
We prove **(2)**. For this assume that $S_1, S_2 \in \Sigma$ are sets such that

$$\mu(A \cap S_1) = 0,\ \nu_1(A \smallsetminus S_1) = 0,\ \mu(A \cap S_2) = 0,\ \nu_2(A \smallsetminus S_2) = 0$$

for every $A \in \Sigma$. Hence also

$$\mu\left(A \cap (S_1 \cup S_2)\right) = \mu(A \cap S_1) + \mu\left((A \smallsetminus S_1) \cap S_2\right) = 0$$

and this implies that $(\nu_1 + \nu_2) \perp \mu$. □

Proof of the theorem. Suppose first that ν is σ-finite and nonnegative. Since μ is a σ-finite and nonnegative by assumption of the theorem, there exist

$$X = \bigcup_{n\in\mathbb{N}} X_n$$

a decompositon onto a sum of parwise disjoint elements of Σ such that $\mu(X_n) \in \mathbb{R}$ and $\mu(X_n) \in \mathbb{R}$. We define measures $\nu_n : \Sigma \to \mathbb{R}$ and $\mu_n : \Sigma \to [0,+\infty)$ by formulas $\mu_n(A) = \mu(A \cap X_n), \nu_n(A) = \nu(A \cap X_n)$ for every $A \in \Sigma$ and $n \in \mathbb{N}$. By **(1)** of Theorem 3.1 for every $n \in \mathbb{N}$ we have a decomposition $\nu_n = \nu_{ns} + \nu_{na}$, where $\nu_{ns} \perp \mu_n$ and $\nu_{na} \ll \mu_n$. Since $X_n \cap X_m = \varnothing$ for $n \neq m$, we derive that

$$\nu_s = \sum_{n\in\mathbb{N}} \nu_{ns},\ \nu_a = \sum_{n\in\mathbb{N}} \nu_{na}$$

are well defined σ-finite, nonnegative measures on (X,Σ) and $\nu = \nu_s + \nu_a$. Moreover, $\nu_s \perp \mu$ and $\nu_a \ll \mu$.
Next assume that ν is a σ-finite, signed measure. By Theorem 4.1 we write $\nu = \nu_+ - \nu_-$, where measures ν_+ and ν_- are nonnegative and σ-finite. Moreover, by Corollary 4.3 at least one of ν_+, ν_- is finite. By the above considerations we can write

$$\nu_+ = \nu_{+s} + \nu_{+a},\ \nu_- = \nu_{-s} + \nu_{-a}$$

where $\nu_{+s} \perp \mu$, $\nu_{-s} \perp \mu$, $\nu_{+a} \ll \mu$, $\nu_{-a} \ll \mu$. Note that measures $\nu_{+s} - \nu_{-s}$ and $\nu_{+a} - \nu_{-a}$ exist, because at least one measure ν_+, ν_- is finite and hence either ν_{+s}, ν_{+a} or ν_{-s}, ν_{-a} are finite. By Lemma 5.2.1 we deduce that

$$\nu_{+s} - \nu_{-s} \perp \mu,\ \nu_{+a} - \nu_{-a} \ll \mu$$

and hence $\nu_s = \nu_{+s} - \nu_{-s}$, $\nu_a = \nu_{+a} - \nu_{-a}$ satisfy

$$\nu = \nu_s + \nu_a$$

with $\nu_s \perp \mu$, $\nu_a \ll \mu$.
Finally assume that ν is complex. Then $\nu = \nu^r + i \cdot \nu^i$, where ν^r and ν^i are finite, signed measures. Form the case above we have decompositions

$$\nu^r = \nu^r_s + \nu^r_a,\ \nu^i = \nu^i_s + \nu^i_s$$

and $\nu^r_s \perp \mu$, $\nu^i_s \perp \mu$, $\nu^r_a \ll \mu$, $\nu^i_a \ll \mu$. Now Lemma 5.2.1 implies that

$$\nu_s = \nu^r_s + i \cdot \nu^i_s,\ \nu_a = \nu^r_a + i \cdot \nu^i_a$$

satisfy $\nu_s \perp \mu$, $\nu_a \ll \mu$. □

Theorem 5.3 (Radon-Nikodym)**.** *Let (X, Σ) be a measurable space and let μ, ν be either signed or complex measures on (X, Σ). Suppose that $\nu \ll \mu$ and assume that every signed measure in the set $\{\nu, \mu\}$ is σ-finite. Then there exists a measurable function $f : X \to \mathbb{C}$ such that*

$$\nu(A) = \int_A f d\mu$$

for every $A \in \Sigma$.

Proof. Assume first that ν, μ are σ-finite nonegative measures. There exist

$$X = \bigcup_{n \in \mathbb{N}} X_n$$

a decompositon onto a sum of parwise disjoint elements of Σ such that $\mu(X_n) \in \mathbb{R}$ and $\mu(X_n) \in \mathbb{R}$. We define a measures $\nu_n : \Sigma \to \mathbb{R}$ and $\mu_n : \Sigma \to [0, +\infty)$ by formulas $\mu_n(A) = \mu(A \cap X_n), \nu_n(A) = \nu(A \cap X_n)$ for every $A \in \Sigma$ and $n \in \mathbb{N}$. Clearly $\nu_n \ll \mu_n$ for every $n \in \mathbb{N}$. By **(2)** of Theorem 3.1 for every $n \in \mathbb{N}$ there exists nonnegative, measurable function $f_n : X \to \mathbb{R}$ such that

$$\nu_n(A) = \int_A f_n d\mu_n = \int_A f_n d\mu$$

for every $A \in \Sigma$. Then $f = \sum_{n \in \mathbb{N}} f_n$ satisfies

$$\nu(A) = \int_A f d\mu$$

for every $A \in \Sigma$.
Now suppose that ν is nonnegative, σ-finite measure and μ is signed, σ-finite measure. Then by Theorem 4.1 we deduce that $\mu = \mu_+ - \mu_-$ where $\mu_+ \perp \mu_-$ and μ_+, μ_- are both σ-finite and nonnegative. There exists a subset $P \in \Sigma$ such that

$$\mu_+(A) = \mu(A \cap P),\ \mu_-(A) = \mu(A \setminus P)$$

for every $A \in \Sigma$. For every $A \in \Sigma$ we define

$$\nu_{|P}(A) = \nu(A \cap P),\ \nu_{|X \setminus P}(A) = \nu(A \setminus P)$$

We have $\nu_{|P} \ll \mu_+$ and $\nu_{|X \setminus P} \ll \mu_-$. By previous considerations there exist nonnegative, measurable functions $f_P, f_{X \setminus P} : X \to \mathbb{R}$ such that

$$\nu_{|P}(A) = \int_A f_P d\mu_+,\ \nu_{|X \setminus P}(A) = \int_A f_{X \setminus P} d\mu_-$$

Modifying f_P on a set of measure μ_+ zero and modifying $f_{X \setminus P}$ on a set of measure μ_- zero we may assume $f_P(x) = 0$ for $x \notin P$ and $f_{X \setminus P}(x) = 0$ for $x \in P$. We have

$$\nu(A) = \nu_{|P}(A) + \nu_{|X \setminus P}(A) = \int_A f_P d\mu_+ + \int_A f_{X \setminus P} d\mu_- = \int_A f_P d\mu + \int_A (-f_{X \setminus P}) d\mu = \int_A (f_P - f_{X \setminus P}) d\mu$$

Next we assume that ν is σ-finite, signed measure and μ is σ-finite, signed measure. By Theorem 4.1 we write $\nu = \nu_+ - \nu_-$ where $\nu_+ \perp \nu_-$ and ν_+, ν_- are nonnegative and σ-finite measures. We have $\nu_+ \ll \mu$ and $\nu_- \ll \mu$. So by the case considered previously there exist a measurable functions $f_+ : X \to \mathbb{R}$ and $f_- : X \to \mathbb{R}$ such that

$$\nu_+(A) = \int_A f_+ d\mu,\ \nu_-(A) = \int_A f_- d\mu$$

for every $A \in \Sigma$. In addition, since one of measures ν_+, ν_- is finite by Corollary 4.3, we deduce that one of functions f_+, f_- is μ-integrable. Hence $f = f_+ - f_-$ is a well defined measurable function and

$$\nu(A) = \nu_+(A) - \nu_-(A) = \int_A f_+ d\mu - \int_A f_- d\mu = \int_A (f_+ - f_-) d\mu$$

Finally it suffices to prove the result if one (or both) of measures ν, μ is complex. This follows easily from the proof for signed case by decomposing complex measure into real and imaginary parts. □

Definition 5.4. Let (X, Σ) be a measurable space and let μ, ν be either signed or complex measures on (X, Σ). Suppose that $\nu \ll \mu$. Then a measurable function $f : X \to \mathbb{C}$ that for every $A \in \Sigma$ satisfies

$$\nu(A) = \int_A f d\mu$$

is called *a Radon-Nikodym derivative of ν with respect to μ*. It is sometimes denoted by

$$\frac{d\nu}{d\mu}$$

References

[Monygham, 2018] Monygham (2018). Introduction to measure theory. *github repository: "Monygham/Pedo-mellon-a-minno"*.