KAKUTANI'S FIXED POINT THEOREM AND ITS APPLICATIONS

1. Introduction

In these notes we study applications of Kakutani's fixed point theorem to theory of Nash equilibria.

2. Brouwer fixed point theorem

In this section we present a Milnor's analytic proof of Brouwer fixed point theorem. The proof is based on the excellent paper [Rog80], where the author vastly simplifies the original Milnor's approach.
Let $n \in \mathbb{N}$ be a natural number. We denote by $\mathbb{B}^n$ a closed unit euclidean ball in $\mathbb{R}^n$ and we denote by S^n a unit euclidean sphere in $\mathbb{R}^{n+1}$.

Theorem 2.1 (Brouwer's fixed point theorem)**.** *Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be a continuous map. Then there exists x in $\mathbb{B}^n$ such that $f(x) = x$.*

Lemma 2.1.1. *Let U be an open subset of $\mathbb{R}^{n+1}$ containing $\mathbb{B}^{n+1}$. Then there is no continuously differentiable map $f : U \to \mathbb{R}^{n+1}$ such that $f(\mathbb{B}^{n+1}) = S^n$ and $f(x) = x$ for $x \in S^n$.*

Proof of the lemma. Assume that such f exists. For $t \in [0,1]$ we define $f_t : U \to \mathbb{R}^{n+1}$ given by formula

$$f_t(x) = x - t(f(x) - x) = (1-t)x + tf(x)$$

for every $x \in U$. We have $f_t\left(\mathbb{B}^{n+1}\right) \subseteq \mathbb{B}^{n+1}$ for every $t \in [0,1]$. There exists $T > 0$ such that the following assertions hold for every $t \in [0,T]$.

(1) ${f_t}_{|\mathbb{B}^{n+1}}$ is injective.

(2) $Df_t(x)$ is invertible for every $x \in \mathbb{B}^{n+1}$.

We explain now how to choose suitable T. For this consider a function $g : U \to \mathbb{R}^{n+1}$ given by formula $g(x) = x - f(x)$ for every $x \in U$. Let $L = 1 + \sup_{x \in \mathbb{B}^{n+1}} ||Dg(x)||$. Then g is Lipschitz function on $\mathbb{B}^{n+1}$ with constant L. Fix $t < L^{-1}$. If $x_1, x_2 \in \mathbb{B}^{n+1}$ are distinct, then

$$||f_t(x_1) - f_t(x_2)|| \geq ||x_1 - x_2|| - t \cdot ||g(x_1) - g(x_2)|| \geq ||x_1 - x_2|| - tL \cdot ||x_1 - x_2|| = (1 - tL) \cdot ||x_1 - x_2|| > 0$$

Therefore, ${f_t}_{|\mathbb{B}^{n+1}}$ is injective. We have

$$t \cdot ||Dg(x)|| = tL < 1$$

for every $x \in \mathbb{B}^{n+1}$ and hence $Df_t(x)$ is invertible for such x. Thus it suffices to take $T = \min\left(L^{-1}, 1\right)$. Now we fix $t \in [0,T]$. Property **(2)** and $f_t\left(\mathbb{B}^{n+1}\right) = \mathbb{B}^{n+1}$ imply that

$$U_t = f_t\left(\mathbf{int}\left(\mathbb{B}^{n+1}\right)\right) \subseteq \mathbb{R}^{n+1}$$

is an open subset contained in $\mathbb{B}^{n+1}$. If $U_t \neq \mathbf{int}\left(\mathbb{B}^{n+1}\right)$, then there exists

$$y \in (\mathbf{cl}(U_t) \setminus U_t) \cap \mathbf{int}\left(\mathbb{B}^{n+1}\right)$$

Consider a sequence $\{x_m\}_{m \in \mathbb{N}}$ of elements in $\mathbf{int}\left(\mathbb{B}^{n+1}\right)$ such that

$$\lim_{m \to +\infty} f_t(x_m) = y$$

We may assume that the sequence $\{x_m\}_{n\in\mathbb{N}}$ converges to some x in $\mathbb{B}^{n+1}$. Then $y = f_t(x)$. Clearly $x \notin \mathbf{int}\left(\mathbb{B}^{n+1}\right)$ because otherwise $y \in V_t$. Hence $x \in S^n$. But then

$$\mathbf{int}\left(\mathbb{B}^{n+1}\right) \ni y = f_t(x) = (1-t)x + tf(x) = x \in S^n$$

Thus the only possibility is that $V_t = \mathbf{int}\left(\mathbb{B}^{n+1}\right)$. Therefore, we have $f_t(\mathbb{B}^{n+1}) = \mathbb{B}^{n+1}$. Now **(1)** and **(2)** imply that f_t induces a diffeomorphism $\mathbb{B}^{n+1} \to \mathbb{B}^{n+1}$. Define a polynomial $p : [0,1] \to \mathbb{R}$ given by formula

$$p(t) = \int_{\mathbb{B}^{n+1}} \left|\det\left(Df_t(x)\right)\right| dx = \int_{\mathbb{B}^{n+1}} \left|\det\left(1_{\mathbb{R}^{n+1}} + t \cdot Dg(x)\right)\right| dx$$

Since f_t induces a diffeomorphism $\mathbb{B}^{n+1} \to \mathbb{B}^{n+1}$ for $t \in [0,T]$, we deduce that $p(t) = \mathrm{vol}\left(\mathbb{B}^{n+1}\right)$ for $t \in [0,T]$. Next $p(t)$ is a polynomial and hence we deduce that $p(t) = \mathrm{vol}\left(\mathbb{B}^{n+1}\right)$ for every $t \in [0,1]$. On the other hand we have

$$p(1) = \int_{\mathbb{B}^{n+1}} \left|\det\left(Df(x)\right)\right| dx$$

Since $f(\mathbb{B}^{n+1}) = S^n$, we deduce that $\det\left(Df(x)\right) = 0$ for every $x \in \mathbf{int}\left(\mathbb{B}^{n+1}\right)$ and hence $p(1) = 0$. This is contradiction, because $\mathrm{vol}\left(\mathbb{B}^{n+1}\right) = p(1) \neq 0$. □

Lemma 2.1.2. *Let U be an open subset of $\mathbb{R}^{n+1}$ containing $\mathbb{B}^{n+1}$. Suppose that $f : U \to \mathbb{R}^{n+1}$ is a continuously differentiable map such that $f(\mathbb{B}^{n+1}) \subseteq \mathbb{B}^{n+1}$. Then there exists a fixed point of f in $\mathbb{B}^{n+1}$.*

Proof of the lemma. Assume that f does not have fixed point in $\mathbb{B}^{n+1}$. Consider an open subset W of U defined by $f(x) \neq x$. Then W contains $\mathbb{B}^{n+1}$. For every x in W we define a point $r(x) \in S^n$ as the intersection of a line

$$\left\{f(x) + t \cdot (x - f(x)) \in \mathbb{R}^{n+1} \,\middle|\, t \in \mathbb{R}_+\right\}$$

with S^n. Then $r : W \to \mathbb{R}^{n+1}$ is continously differentiable, $r(W) = S^n$ and $r(x) = x$ for every $x \in S^n$. This is a contradiction with Lemma 2.1.1. □

Proof of the theorem. Suppose that $f : \mathbb{B}^{n+1} \to \mathbb{B}^{n+1}$ is a continuous map without fixed points. We consider f as a map $\tilde{f} : \mathbb{B}^{n+1} \to \mathbb{R}^{n+1}$. By Stone-Weierstrass theorem there exists a sequence $\left\{p_m : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}\right\}_{m\in\mathbb{N}}$ of polynomials such that the sequence $\{p_{m|\mathbb{B}^{n+1}}\}_{m\in\mathbb{N}}$ is uniformly convergent to $\tilde{f}$. Let

$$\alpha_m = \sup_{x\in\mathbb{B}^{n+1}} \|\tilde{f}(x) - p_m(x)\|$$

and consider an open subset U_m such that $\|p_m(x)\| < 1 + \alpha_m$ for every $x \in U_m$. Clearly U_m is an open subset of $\mathbb{R}^{n+1}$ containing $\mathbb{B}^{n+1}$. Define a sequence $\left\{q_m : U_m \to \mathbb{R}^{n+1}\right\}_{m\in\mathbb{N}}$ by formula $q_m(x) = (1+\alpha_m)^{-1} \cdot p_m(x)$ for $x \in U_m$. Then $q_m(\mathbb{B}^{n+1}) \subseteq \mathbb{B}^{m+1}$ and q_m is continuously differentiable for every $m \in \mathbb{N}$. By Lemma 2.1.2 we derive that there exists $x_m \in \mathbb{B}^{m+1}$ such that $q_m(x_m) = x_m$. Since $\mathbb{B}^{n+1}$ is compact, we may assume that the sequence $\{x_m\}_{m\in\mathbb{N}}$ converges to some $x \in \mathbb{B}^{n+1}$. Note also that $\{q_{m|\mathbb{B}^{n+1}}\}_{m\in\mathbb{N}}$ is uniformly convergent to $\tilde{f}$. Thus we have

$$x = \lim_{m\to+\infty} x_m = \lim_{m\to+\infty} q_m(x_m) = \tilde{f}(x)$$

and hence $f(x) = x$. This proves the theorem in the case $n \geq 1$. For $n = 0$ the set $\mathbb{B}^n$ consists of a point and hence the theorem holds trivially. □

References

[Rog80] Claude Ambrose Rogers. A less strange version of milnor's proof of brouwer's fixed-point theorem. *The American Mathematical Monthly*, 87(7):525–527, 1980.