MONOID k-FUNCTORS AND THEIR REPRESENTATIONS

1. Introduction and notation

In these notes we study algebraic structures in the category of k-functors with special emphasis on monoid objects.
If R is a ring, then we denote by $R^{\times}$ its multplicative monoid.

2. Algebraic structures in the category of k-functors

In the sequel we assume that the reader is familiar with notions of a monoid, group etc. in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 2-5].

Definition 2.1. *A monoid (group, abelian group, ring) k-functor* is a monoid (group, abelian group, ring) object in the category of k-functors.

Example 2.2. Let $\mathfrak{X}$ be a k-functor such that $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{X})$ exists. Then $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{X})$ is a monoid k-functor with respect to composition of morphisms.

Example 2.3. Basic example of a ring k-functor is a k-functor $\mathfrak{K}$ given by

$$\mathfrak{K}(A) = k,\ \mathfrak{K}(f) = 1_k$$

for any k-algebra A and morphism f of k-algebras. It can be described as a constant k-functor ([ML98, page 67]) corresponding to k.

Definition 2.4. Let $\mathfrak{R}$ be a ring k-functor. Then we denote by $\mathfrak{R}^{\times}$ the k-subfunctor of $\mathfrak{R}$ defined by

$$\mathfrak{R}^{\times}(A) = \mathfrak{R}(A)^{\times}$$

for every k-algebra A. We call $\mathfrak{R}^{\times}$ *the multiplicative monoid k-functor of* $\mathfrak{R}$.

Definition 2.5. Let $\mathfrak{A}$ be a commutative ring k-functor. *An* $\mathfrak{A}$*-algebra* is an $\mathfrak{A}$-algebra object in the category of k-functors.

3. Global regular functions on a k-functor

Recall the ring k-functor $\mathfrak{K}$ from Example 2.3. Note that a $\mathfrak{K}$-algebra $\mathfrak{A}$ can be viewed as a functor $\mathfrak{A} : \mathbf{Alg}_k \to \mathbf{Alg}_k$.

Definition 3.1. The $\mathfrak{K}$-algebra $\mathfrak{O}_k$ represented by the identity functor on $\mathbf{Alg}_k$ is called *the structure* $\mathfrak{K}$*-algebra*.

Let $|-| : \mathbf{Alg}_k \to \mathbf{Set}$ be the forgetful k-functor. Note that $|-|$ is the underlying k-functor of $\mathfrak{K}$-algebra $\mathfrak{O}_k$. Recall that the affine line $\mathbb{A}^1_k$ is an affine k-scheme having k-algebra of polynomials with one variable as a k-algebra of regular functions.

Fact 3.2. *Let* $|-| : \mathbf{Alg}_k \to \mathbf{Set}$ *be the forgetful k-functor. Then we have natural isomorphism*

$$\mathfrak{P}_{\mathbb{A}^1_k} \cong |-|$$

Proof. Let B be a k-algebra. We have the following chain of identifications

$$\mathfrak{P}_{\mathbb{A}^1_k}(B) = \mathrm{Mor}_k\left(\mathrm{Spec}\,B, \mathbb{A}^1_k\right) = \mathrm{Mor}_k\left(\mathrm{Spec}\,B, \mathrm{Spec}\,k[x]\right) = \mathrm{Mor}_k\left(k[x], B\right) = |B|$$

natural in B. □

In particular, since $|-|$ carries the structure $\mathfrak{K}$-algebra $\mathfrak{O}_k$, we derive that $\mathfrak{P}_{\mathbb{A}^1_k}$ admits a structure of $\mathfrak{K}$-algebra isomorphic to $\mathfrak{O}_k$.
No we introduce regular functions on k-functors.

Definition 3.3. Let $\mathfrak{X}$ be a k-functor and assume that $\mathcal{M}\mathrm{or}_k\left(\mathfrak{X},\mathfrak{O}_k\right)$ is a set. Then $\mathrm{Mor}_k\left(\mathfrak{X},\mathfrak{O}_k\right)$ is a k-algebra with respect to the structure induced by $\mathfrak{O}_k$. We call this k-algebra *the k-algebra of global regular functions on $\mathfrak{X}$*. Its elements are called *global regular functions on $\mathfrak{X}$*.

Definition 3.4. Let $\mathfrak{X}$ be a k-functor. Suppose that A is a k-algebra, $x \in \mathfrak{X}(A)$ and $f \in \mathrm{Mor}_k\left(\mathfrak{X},\mathfrak{O}_k\right)$. The element $f(x) \in A$ is called *the value of f on a point x*.

For given k-functor $\mathfrak{X}$ we describe k-algebra operations on $\mathrm{Mor}_k\left(\mathfrak{X},\mathfrak{O}_k\right)$ in terms of values of its elements on points of $\mathfrak{X}$. For this consider $\alpha \in k$ and $f, g \in \mathrm{Mor}_k\left(\mathfrak{X},\mathfrak{O}_k\right)$. We have formulas

$$(f+g)(x) = f(x)+g(x),\ (f\cdot g)(x) = f(x)\cdot g(x),\ (\alpha\cdot f)(x) = \alpha\cdot f(x)$$

in which right hand side are k-algebra operations in A.

Example 3.5. Let $\mathfrak{X}$ be a k-functor and assume that $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{O}_k)$ exists. Fix k-algebra A. Note that $\mathrm{Mor}_A(\mathfrak{X}_A,\mathfrak{O}_A)$ is an A-algebra of global regular functions on $\mathfrak{X}_A$. Moreover, if B is an A-algebra, then

$$\mathrm{Mor}_A(\mathfrak{X}_A,\mathfrak{O}_A) \ni f \mapsto f_B \in \mathrm{Mor}_B(\mathfrak{X}_B,\mathfrak{O}_B)$$

is a morphism of A-algebras. This implies that $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{O}_k)$ admits a canonical structure of an $\mathfrak{O}_k$-algebra k-functor.

4. Internal hom and product of k-functors

We denote by $\mathbf{1}$ a k-functor that assigns to every k-algebra a set with one element. Then for every k-algebra A the restriction $\mathbf{1}_A$ is a terminal object in the category of A-functors.

Fact 4.1. *Let $\mathfrak{X}$ be a k-functor. Suppose A is a k-algebra and $x \in \mathfrak{X}(A)$. Then x determines a morphism $\mathbf{1}_A \rightarrow \mathfrak{X}_A$ that for every A-algebra B with structural morphism $f : A \rightarrow B$ sends a unique element of $\mathbf{1}_A(B)$ to $\mathfrak{X}(f)(x) \in \mathfrak{X}_A(B)$. This gives rise to a bijection*

$$\mathfrak{X}(A) \cong \mathrm{Mor}_A\left(\mathbf{1}_A,\mathfrak{X}_A\right)$$

Proof. Left to the reader as an exercise. □

The discussion below is partially an application of the main result in [Mon19, section 6]. For reader's convenience we make our presentation self-contained.

Definition 4.2. Let $\mathfrak{U},\mathfrak{X},\mathfrak{Y}$ be k-functors and let $\sigma : \mathfrak{U}\times\mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of k-functors. Fix $z \in \mathfrak{U}(A)$ for some k-algebra A. We denote by $i_z : \mathbf{1}_A \rightarrow \mathfrak{U}_A$ the morphism of A-functors corresponding to z by Fact 4.1. Since $\mathbf{1}_A$ is terminal A-functor, a morphism $\sigma_A\cdot\left(i_z\times 1_{\mathfrak{X}_A}\right)$ is isomorphic to a morphism $\sigma_z : \mathfrak{X}_A \rightarrow \mathfrak{Y}_A$ of A-functors. We call σ_z *the slice of σ over z*.

Definition 4.3. Let $\mathfrak{X},\mathfrak{Y}$ be k-functors. Let $\mathfrak{I}$ be a k-functor such that $\mathfrak{I}(A)$ is a subset of a class $\mathrm{Mor}_A\left(\mathfrak{X}_A,\mathfrak{Y}_A\right)$ for every k-algebra A. Assume that for every morphism $f : A \rightarrow B$ of k-algebras and every $\sigma \in \mathfrak{I}(A)$ we have

$$\mathfrak{I}(f)(\sigma) = \sigma_B$$

where $\sigma_B \in \mathrm{Mor}_B\left(\mathfrak{X}_B,\mathfrak{Y}_B\right)$ is the restriction of σ along f. Then we call $\mathfrak{I}$ *a k-subfunctor of internal hom of $\mathfrak{X}$ and $\mathfrak{Y}$*.

Definition 4.4. Let $\mathfrak{X},\mathfrak{Y},\mathfrak{U}$ be k-functors and let $\sigma : \mathfrak{U}\times\mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of k-functors. Suppose that $\mathfrak{I}$ is a k-subfunctor of internal hom of $\mathfrak{X}$ and $\mathfrak{Y}$. Assume that $\sigma_z : \mathfrak{X}_A \rightarrow \mathfrak{Y}_A$ is contained in $\mathfrak{I}(A)$ for every k-algebra A and $z \in \mathfrak{U}(A)$. Then we call σ *a family of $\mathfrak{I}$-morphisms parametrized by $\mathfrak{U}$*.

Let $\mathfrak{J}$ be a k-subfunctor of internal hom of $\mathfrak{X}$ and $\mathfrak{Y}$. Assume that $\sigma : \mathfrak{U}\times\mathfrak{X}\to\mathfrak{Y}$ is a $\mathfrak{J}$-family of morphism parametrized by $\mathfrak{U}$. Then the family of maps

$$\mathfrak{U}(A)\ni z\mapsto \sigma_z\in\mathfrak{J}(A)$$

gives rise to a morphism $\tau : \mathfrak{U}\to\mathfrak{J}$ of k-functors. Indeed, for a morphism $f : A\to B$ of k-algebras and $z\in\mathfrak{U}(A)$ we have

$$\sigma_B\cdot\left(i_{\mathfrak{U}(f)(z)}\times 1_{\mathfrak{X}_B}\right)=\left(\sigma_A\cdot\left(i_z\times 1_{\mathfrak{X}_A}\right)\right)_B$$

and hence $\sigma_{\mathfrak{U}(f)(z)}=(\sigma_z)_B$. This gives rise to a map Φ of classes

$$\left\{\text{families }\mathfrak{U}\times\mathfrak{X}\to\mathfrak{Y}\text{ of }\mathfrak{J}\text{-morphisms parametrized by }\mathfrak{U}\right\}\ni\sigma\mapsto\tau\in\mathrm{Mor}_k\left(\mathfrak{U},\mathfrak{J}\right)$$

Consider next a morphism $\tau : \mathfrak{U}\to\mathfrak{J}$ of k-functors and define $\sigma : \mathfrak{U}\times\mathfrak{X}\to\mathfrak{Y}$ by formula $\sigma^A(z,x)=\left(\tau^A(z)\right)^A(x)$ for every k-algebra A and points $z\in\mathfrak{U}(A)$, $x\in\mathfrak{X}(A)$. Let $f : A\to B$ be a morphism of k-algebras. Then

$$\sigma^B\left(\mathfrak{U}(f)(z),\mathfrak{X}(f)(x)\right)=\left(\tau^B\left(\mathfrak{U}(f)(z)\right)\right)^B\left(\mathfrak{X}(f)(x)\right)=\left(\left(\tau^A(z)\right)_B\right)^B\left(\mathfrak{X}(f)(x)\right)=$$

$$=\left(\tau^A(z)\right)^B\left(\mathfrak{X}(f)(x)\right)=\mathfrak{Y}(f)\left(\left(\tau^A(z)\right)^A(x)\right)=\mathfrak{Y}(f)\left(\sigma^A(z,x)\right)$$

Thus $\sigma : \mathfrak{U}\times\mathfrak{X}\to\mathfrak{Y}$ is a morphism of k-functors. For every k-algebra A and $z\in\mathfrak{U}(A)$ we have $\sigma_z=\tau^A(z)$. Indeed, let $f : A\to B$ be a morphism of k-algebras and x be an element in $\mathfrak{X}(B)$ then we have

$$(\sigma_z)^B(x)=\sigma^B\left(\mathfrak{U}(f)(z),x\right)=\left(\tau^B\left(\mathfrak{U}(f)(z)\right)\right)^B(x)=\left(\left(\tau^A(z)\right)_B\right)^B(x)=\left(\tau^A(z)\right)^B(x)$$

Hence σ is a family of $\mathfrak{J}$-morphisms parametrized by $\mathfrak{U}$. This gives rise to a map Ψ of classes

$$\mathrm{Mor}_k\left(\mathfrak{U},\mathfrak{J}\right)\ni\tau\mapsto\sigma\in\left\{\text{families }\mathfrak{U}\times\mathfrak{X}\to\mathfrak{Y}\text{ of }\mathfrak{J}\text{-morphisms parametrized by }\mathfrak{U}\right\}$$

Now we have the following result, which is an instance [Mon19, Theorem 6.3]. To make presentation self-contained we give a complete proof.

Theorem 4.5. *Let $\mathfrak{X}$, $\mathfrak{Y}$ be k-functors and let $\mathfrak{J}$ be a k-subfunctor of internal hom of $\mathfrak{X}$ and $\mathfrak{Y}$. Then maps*

$$\Phi : \left\{\textit{families }\mathfrak{U}\times\mathfrak{X}\to\mathfrak{Y}\textit{ of }\mathfrak{J}\textit{-morphisms parametrized by }\mathfrak{U}\right\}\to\mathrm{Mor}_k\left(\mathfrak{U},\mathfrak{J}\right)$$

and

$$\Psi : \mathrm{Mor}_k\left(\mathfrak{U},\mathfrak{J}\right)\to\left\{\textit{families }\mathfrak{U}\times\mathfrak{X}\to\mathfrak{Y}\textit{ of }\mathfrak{J}\textit{-morphisms parametrized by }\mathfrak{U}\right\}$$

are mutually inverse bijections.

Proof. Pick a morphism $\tau : \mathfrak{U}\to\mathfrak{J}$ of k-functors. Let A be a k-algebra and $z\in\mathfrak{U}(A)$. In the discussion preceding the statement we showed that $\Psi(\tau)_z=\tau^A(z)$. Thus

$$\left(\Phi(\Psi(\tau))\right)^A(z)=\Psi(\tau)_z=\tau^A(z)$$

and hence $\Phi\cdot\Psi$ is the identity.
Pick a family of $\mathfrak{J}$-morphism $\sigma : \mathfrak{U}\times\mathfrak{X}\to\mathfrak{Y}$ parametrized by $\mathfrak{U}$. Let A be a k-algebra and $z\in\mathfrak{U}(A)$, $x\in\mathfrak{X}(A)$ be points. Then

$$\left(\Psi\left(\Phi(\sigma)\right)\right)^A(z,x)=\left(\Phi(\sigma)^A(z)\right)^A(x)=\sigma_z^A(x)=\sigma^A(z,x)$$

Thus $\Psi\cdot\Phi$ is the identity map. $\square$

Now we formulate some consequences of Theorem 4.5.

Corollary 4.6. *Let $\mathfrak{X}$, $\mathfrak{Y}$ be k-functors. Assume that for every k-algebra A the class* $\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ *is a set. Then there is a bijection*

$$\mathrm{Mor}_k(\mathfrak{U}\times\mathfrak{X}, \mathfrak{Y}) \to \mathrm{Mor}_k(\mathfrak{U}, \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y}))$$

of classes.

Definition 4.7. Let $\mathfrak{X}, \mathfrak{Y}$ be k-functors. If $\mathrm{Iso}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set for every k-algebra A, then we define a k-subfunctor $\mathcal{I}\mathrm{so}_k(\mathfrak{X}, \mathfrak{Y})$ of $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})$ by

$$\mathcal{I}\mathrm{so}_k(\mathfrak{X}, \mathfrak{Y})(A) = \mathrm{Iso}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$$

for every k-algebra A. We call $\mathcal{I}\mathrm{so}_k(\mathfrak{X}, \mathfrak{Y})$ *the k-functor of isomorphism.*

Definition 4.8. Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{U}$ be k-functors and let $\sigma : \mathfrak{U}\times\mathfrak{X} \to \mathfrak{Y}$ be a morphism of k-functors. Assume that $\sigma_z : \mathfrak{X}_A \to \mathfrak{Y}_A$ is an isomorphism of A-functors for every k-algebra A. Then we call σ *a family of isomorphisms parametrized by* $\mathfrak{U}$.

Corollary 4.9. *Let $\mathfrak{U}, \mathfrak{X}, \mathfrak{Y}$ be k-functors and suppose that for every k-algebra A the class* $\mathrm{Iso}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ *is a set. The the following map*

$$\left\{\textit{families } \mathfrak{U}\times\mathfrak{X}\to\mathfrak{Y} \textit{ of isomorphism parametrized by } \mathfrak{U}\right\} \to \mathrm{Mor}_k(\mathfrak{U}, \mathcal{I}\mathrm{so}_k(\mathfrak{X}, \mathfrak{Y}))$$

is a bijection of classes.

5. Actions of monoid k-functors

In this section we assume that the reader is familiar with notion of an action of a monoid object in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 5].
Let $\mathfrak{G}$ be a monoid k-functor and $\mathfrak{X}$ be a k-functor together with an action $\alpha : \mathfrak{G}\times\mathfrak{X} \to \mathfrak{X}$. Next assume that k-functor $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ exists. By Example 2.2 it is a monoid k-functor. We define a morphism $\rho : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ of k-functors by formula $\rho(x) = \alpha_x$. Note that by discussion preceding Theorem 4.5, we deduce that ρ is a well defined morphism of k-functors. We show now that ρ is a morphism of monoids. For this pick k-algebra A and $x, y \in \mathfrak{G}(A)$. Since α is an action, we deduce that $\alpha_{x\cdot y} = \alpha_x \cdot \alpha_y$ and hence also

$$\rho(x\cdot y) = \alpha_{x\cdot y} = \alpha_x\cdot\alpha_y = \rho(x)\cdot\rho(y)$$

Therefore, ρ is a morphism of monoid k-functors. This shows how to construct a morphism of monoid k-functors ρ from an action α of $\mathfrak{G}$.

Theorem 5.1. *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{X}$ be a k-functor such that $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ exists. Suppose that*

$$\left\{\textit{actions of } \mathfrak{G} \textit{ on } \mathfrak{X}\right\} \longrightarrow \left\{\textit{Morphisms } \rho : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X}) \textit{ of monoid k-functors}\right\}$$

is a map of classes described above. Then it is bijection.

Proof. Our goal is to construct the inverse of the map. Substitute $\mathfrak{J} = \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ in Theorem 4.5. Consider maps

$$\Phi : \left\{\text{families } \mathfrak{G}\times\mathfrak{X}\to\mathfrak{X} \text{ of morphisms}\right\} \to \mathrm{Mor}_k(\mathfrak{G}, \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X}))$$

and

$$\Psi : \mathrm{Mor}_k(\mathfrak{G}, \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})) \to \left\{\text{families } \mathfrak{G}\times\mathfrak{X}\to\mathfrak{X} \text{ of morphisms}\right\}$$

in that Theorem. Then the map in the statement above is the restriction of Φ to $\mathfrak{G}$-actions on $\mathfrak{X}$ on the right and morphisms $\mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ of monoid k-functors on the left. Since by Theorem 4.5 maps Φ and Ψ are mutually inverse, it suffices to check that Ψ sends a morphism $\rho : \mathfrak{G} \to$

$\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{X})$ of monoids to an action of $\mathfrak{G}$ on $\mathfrak{X}$. For this denote $\Psi(\rho)$ by α. Consider k-algebra A and A-points $x,y \in \mathfrak{G}(A)$, $z \in \mathfrak{X}(A)$. Then

$$\alpha\left(y,\alpha(x,z)\right) = \rho(y)\left(\rho(x)(z)\right) = \left(\rho(y)\cdot\rho(x)\right)(z) = \rho\left(x\cdot y\right)(z) = \alpha\left(x\cdot y,z\right)$$

Therefore, α is an action of $\mathfrak{G}$ on $\mathfrak{X}$. □

Proposition 5.2. *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{X}_1$, $\mathfrak{X}_2$ be k-functors such that $\mathcal{M}\mathrm{or}_k(\mathfrak{X}_1,\mathfrak{X}_1)$, $\mathcal{M}\mathrm{or}_k(\mathfrak{X}_2,\mathfrak{X}_2)$ exist. Suppose that $\alpha_1 : \mathfrak{G}\times\mathfrak{X}_1 \to \mathfrak{X}_1$, $\alpha_2 : \mathfrak{G}\times\mathfrak{X}_2 \to \mathfrak{X}_2$ are actions of $\mathfrak{G}$, respectively. Suppose that $\sigma : \mathfrak{X}_1 \to \mathfrak{X}_2$ is a morphism of k-functors. Then the following assertions are equivalent.*

(i) *The square*

$$\begin{tikzcd}
\mathfrak{G}\times\mathfrak{X}_1 \arrow[r, "1_{\mathfrak{G}}\times\sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{G}\times\mathfrak{X}_2 \arrow[d, "\alpha_2"] \\
\mathfrak{X}_1 \arrow[r, "\sigma"'] & \mathfrak{X}_2
\end{tikzcd}$$

is commutative.

(ii) *For every k-algebra A and $x \in \mathfrak{G}(A)$ we have*

$$\sigma_A\cdot\rho_1(x) = \rho_2(x)\cdot\sigma_A$$

where $\rho_1 : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}_1,\mathfrak{X}_1)$ and $\rho_2 : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}_2,\mathfrak{X}_2)$ are morphism of monoid k-functors corresponding to α_1 and α_2, respectively.

Proof. Conditions expressed in **(i)** and **(ii)** are directly translatable to each other by virtue of the bijection in Theorem 5.1. □

Definition 5.3. Let $\mathfrak{G}$ be a monoid k-functor and let $(\mathfrak{X}_1,\alpha_1)$, $(\mathfrak{X}_2,\alpha_2)$ be k-functors with actions of $\mathfrak{G}$. Suppose that $\sigma : \mathfrak{X}_1 \to \mathfrak{X}_2$ is a morphism k-functors such that the square

$$\begin{tikzcd}
\mathfrak{G}\times\mathfrak{X}_1 \arrow[r, "1_{\mathfrak{G}}\times\sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{G}\times\mathfrak{X}_2 \arrow[d, "\alpha_2"] \\
\mathfrak{X}_1 \arrow[r, "\sigma"'] & \mathfrak{X}_2
\end{tikzcd}$$

is commutative. Then σ is called *an $\mathfrak{G}$-equivariant morphism*.

6. Modules over ring k-functors

Definition 6.1. Let $\mathfrak{R}$ be a ring k-functor. Suppose that $\mathfrak{M}$ is an abelian group k-functor and there exists a morphism $\mathfrak{R}\times\mathfrak{M} \to \mathfrak{M}$ of k-functors that for each k-algebra A makes $\mathfrak{M}(A)$ into an $\mathfrak{R}(A)$-module. Then we say that $\mathfrak{M}$ is *a module k-functor over* $\mathfrak{R}$.

Definition 6.2. Let $\mathfrak{R}$ be an ring k-functor and let $\mathfrak{M}_1,\mathfrak{M}_2$ be module k-functors over $\mathfrak{R}$. Suppose that $\sigma : \mathfrak{M}_1 \to \mathfrak{M}_2$ is a morphism of abelian group k-functors such that the diagram

$$\begin{tikzcd}
\mathfrak{R}\times\mathfrak{M}_1 \arrow[r, "1_{\mathfrak{R}}\times\sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{R}\times\mathfrak{M}_2 \arrow[d, "\alpha_2"] \\
\mathfrak{M}_1 \arrow[r, "\sigma"'] & \mathfrak{M}_2
\end{tikzcd}$$

is commutative, where $\alpha_i : \mathfrak{R} \times \mathfrak{M}_i \to \mathfrak{M}_i$ define $\mathfrak{R}$-module structure on $\mathfrak{M}_i$ for $i = 1, 2$. Then σ is *a morphism of modules over* $\mathfrak{R}$.

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be module k-functors over $\mathfrak{R}$. We denote by

$$\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{M}_1, \mathfrak{M}_2)$$

the class of all morphisms of modules $\mathfrak{M}_1 \to \mathfrak{M}_2$ over $\mathfrak{R}$. We denote the category of $\mathfrak{R}$-modules by $\mathbf{Mod}(\mathfrak{R})$.

Definition 6.3. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be module k-functors over $\mathfrak{R}$. Assume that $\mathrm{Hom}_{\mathfrak{R}_A}((\mathfrak{M}_1)_A, (\mathfrak{M}_2)_A)$ is a set for every k-algebra A. Then we define a k-subfunctor $\mathcal{H}om_{\mathfrak{R}}(\mathfrak{M}_1, \mathfrak{M}_2)$ of internal hom of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ by formula

$$\mathbf{Alg}_k \ni A \mapsto \mathrm{Hom}_{\mathfrak{R}_A}((\mathfrak{M}_1)_A, (\mathfrak{M}_2)_A) \in \mathbf{Set}$$

We call $\mathcal{H}om_{\mathfrak{R}}(\mathfrak{M}_1, \mathfrak{M}_2)$ *a k-functor of module morphisms of* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$.

If $\mathfrak{M}$ is a module k-functor over some ring k-functor $\mathfrak{R}$, then we denote (if it exists) $\mathcal{H}om_{\mathfrak{R}}(\mathfrak{M}, \mathfrak{M})$ by $\mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$.

Example 6.4. Let $\mathfrak{M}$ be a module over a ring k-functor $\mathfrak{R}$. Assume that $\mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$ exists. Then $\mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$ is a ring k-functor with respect to composition of morphisms of modules as the multiplication and the usual addition of module morphisms. Moreover, if $\mathfrak{A}$ is a commutative ring k-functor, then $\mathcal{E}nd_{\mathfrak{A}}(\mathfrak{M})$ (if exists) admits additional structure of a $\mathfrak{A}$-algebra k-functor induced via a unique morphism $\mathfrak{A} \to \mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$ of ring k-functors that sends $1 \mapsto 1_{\mathfrak{M}}$.

Let $\mathfrak{A}$ be a commutative ring k-functor and let $\mathfrak{R}$ be a $\mathfrak{A}$-algebra k-functor. This means that there exists a morphism $\mathfrak{A} \to \mathfrak{R}$ of ring k-functors and for every k-algebra A induced morphism $\mathfrak{A}(A) \to \mathfrak{R}(A)$ sends $\mathfrak{A}(A)$ to the center of a ring $\mathfrak{R}(A)$. Fix a module $\mathfrak{M}$ over $\mathfrak{A}$. Next assume that k-functor $\mathcal{E}nd_{\mathfrak{A}}(\mathfrak{M})$ exists. By Example 6.4 it is a ring k-functor.

Definition 6.5. In the setting above suppose that $\alpha : \mathfrak{R} \times \mathfrak{M} \to \mathfrak{M}$ is a morphism of k-functors. Suppose that α makes $\mathfrak{M}$ into $\mathfrak{R}$-module and moreover, for every k-algebra A and for every point $x \in \mathfrak{R}(A)$ morphism α_x is a morphism of $\mathfrak{A}_A$-modules. Then α is called *a* $\mathfrak{A}$*-linear* $\mathfrak{R}$*-action on* $\mathfrak{M}$.

We continue the discussion. We assume that we are given an $\mathfrak{A}$-linear $\mathfrak{R}$-action $\alpha : \mathfrak{R} \times \mathfrak{M} \to \mathfrak{M}$ on $\mathfrak{M}$. We define a morphism $\rho : \mathfrak{R} \to \mathcal{E}nd_{\mathfrak{A}}(\mathfrak{M})$ of k-functors by formula $\rho(x) = \alpha_x$. As in Section 5 we can prove that ρ is a morphism of ring k-functors. Now we have the following result.

Theorem 6.6. *Let* $\mathfrak{R}$ *be an algebra k-functor over commutative ring* $\mathfrak{A}$ *k-functor and let* $\mathfrak{M}$ *be a* $\mathfrak{A}$*-module such that* $\mathcal{E}nd_{\mathfrak{A}}(\mathfrak{M})$ *exists. Suppose that*

$$\Big\{\mathfrak{A} \text{ linear actions of } \mathfrak{R} \text{ on } \mathfrak{M}\Big\} \longrightarrow \Big\{\text{Morphisms } \rho : \mathfrak{R} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}) \text{ of ring } k\text{-functors}\Big\}$$

is a map of classes described above. Then it is bijection.

Proof. The proof is similar to the proof of Theorem 5.1. □

7. MONOID ALGEBRA $\mathfrak{O}_k[\mathfrak{G}]$ AND ITS MODULES

Definition 7.1. Let $\mathfrak{G}$ be a monoid k-functor. Then we construct an $\mathfrak{O}_k$-algebra $\mathfrak{O}_k[\mathfrak{G}]$ as follows. For every k-algebra A we define

$$\mathfrak{O}_k[\mathfrak{G}](A) = A\big[\mathfrak{G}(A)\big]$$

where the right hand side is monoid A-algebra for the abstract monoid $\mathfrak{G}(A)$. The structure of monoid k-functor on $\mathfrak{G}$ and $\mathfrak{K}$-algebra $\mathfrak{O}_k$ makes $\mathfrak{O}_k[\mathfrak{G}]$ into a ring k-functor. Moreover, we have a morphism $\mathfrak{O}_k \to \mathfrak{O}_k[\mathfrak{G}]$ which for every k-algebra A is given by the canonical inclusion

$$A \hookrightarrow A\big[\mathfrak{G}(A)\big]$$

Thus $\mathfrak{O}_k[\mathfrak{G}]$ is $\mathfrak{O}_k$-algebra. We call $\mathfrak{O}_k[\mathfrak{G}]$ *a monoid* $\mathfrak{O}_k$*-algebra over* $\mathfrak{G}$.

Fact 7.2. *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{R}$ be an $\mathfrak{O}_k$-algebra k-functor. Then every morphism*

$$\sigma : \mathfrak{G} \to \mathfrak{R}^{\times}$$

of monoid k-functors admits a unique extension

$$\tilde{\sigma} : \mathfrak{O}_k[\mathfrak{G}] \to \mathfrak{R}$$

to a morphism of $\mathfrak{O}_k$-algebras.

Proof. This follows from the analogical universal property of algebras over abstract monoids. □

Definition 7.3. Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{M}$ be a module over $\mathfrak{O}_k$. Suppose that $\alpha : \mathfrak{G} \times \mathfrak{M} \to \mathfrak{M}$ is an action of $\mathfrak{G}$ such that for any k-algebra A and point $x \in \mathfrak{G}(A)$ morphism $\alpha_x : \mathfrak{M}_A \to \mathfrak{M}_A$ is a morphism of $\mathfrak{O}_A$-modules. Then α is called *a linear $\mathfrak{G}$-action on $\mathfrak{M}$.*

Suppose now that $\mathfrak{G}$ is a monoid k-functor and $\mathfrak{M}$ is a module $\mathfrak{O}_k$. Note that every linear $\mathfrak{G}$-action $\alpha : \mathfrak{G} \times \mathfrak{M} \to \mathfrak{M}$ extends uniquely to a $\mathfrak{O}_k$-linear action $\mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}$ of monoid $\mathfrak{O}_k$-algebra. This gives a bijection

$$\left\{\text{Linear actions of } \mathfrak{G} \text{ on } \mathfrak{M}\right\} \longrightarrow \left\{\mathfrak{O}_k\text{-linear actions } \mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}\right\}$$

Next assume that k-functor $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ exists. By Example 6.4 it is an $\mathfrak{O}_k$-algebra k-functor. Next by Theorem 6.6 we have a bijection

$$\left\{\mathfrak{O}_k\text{-linear actions of } \mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}\right\} \longrightarrow \left\{\text{Morphisms } \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}) \text{ of } \mathfrak{O}_k\text{-algebras}\right\}$$

Finally Fact 7.2 implies that we have a bijection

$$\left\{\text{Morphisms } \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}) \text{ of } \mathfrak{O}_k\text{-algebras}\right\} \longrightarrow \left\{\text{Morphisms } \mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}) \text{ of monoids}\right\}$$

This chain of bijections sends a linear action $\alpha : \mathfrak{G} \times \mathfrak{M} \to \mathfrak{M}$ of $\mathfrak{G}$ to a morphism $\rho : \mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ of monoid k-functors given by $\rho(x) = \alpha_x$ for every $x \in \mathfrak{G}(A)$ and every k-algebra A. We proved the following result.

Proposition 7.4. *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{M}$ be a $\mathfrak{O}_k$-module such that $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ exists. Then the following classes are in canonical bijections described above.*

(1) *Linear actions of $\mathfrak{G}$ on $\mathfrak{M}$.*

(2) *$\mathfrak{O}_k$-linear actions $\mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}$. These are precisely $\mathfrak{O}_k[\mathfrak{G}]$-modules.*

(3) *Morphisms $\mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ of $\mathfrak{O}_k$-algebras.*

(4) *Morphisms $\mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ of monoids.*

Moreover, the bijection between class **(1)** *and* **(2)** *does not require the existence of $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$.*

Now in a similar manner we can describe morphisms.

Proposition 7.5. *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{M}_1$, $\mathfrak{M}_2$ be k-functors of $\mathfrak{O}_k$-modules such that $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1), \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$ exist. Suppose that $\alpha_1 : \mathfrak{G} \times \mathfrak{M}_1 \to \mathfrak{M}_1$, $\alpha_2 : \mathfrak{G} \times \mathfrak{M}_2 \to \mathfrak{M}_2$ are linear actions of $\mathfrak{G}$. Suppose that $\sigma : \mathfrak{M}_1 \to \mathfrak{M}_2$ is a morphism of modules over $\mathfrak{O}_k$. Then the following assertions are equivalent.*

(i) *The square*

$$\begin{tikzcd}
\mathfrak{G} \times \mathfrak{M}_1 \arrow[r, "1_{\mathfrak{G}} \times \sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{G} \times \mathfrak{M}_2 \arrow[d, "\alpha_2"] \\
\mathfrak{M}_1 \arrow[r, "\sigma"'] & \mathfrak{M}_2
\end{tikzcd}$$

is commutative.

(ii) *The square*

$$\begin{tikzcd}
\mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M}_1 \arrow[r, "1_{\mathfrak{O}_k[\mathfrak{G}]} \times \sigma"] \arrow[d, "\tilde{\alpha_1}"'] & \mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M}_2 \arrow[d, "\tilde{\alpha_2}"] \\
\mathfrak{M}_1 \arrow[r, "\sigma"'] & \mathfrak{M}_2
\end{tikzcd}$$

is commutative, where $\tilde{\alpha_1}$ and $\tilde{\alpha_2}$ are $\mathfrak{O}_k$-linear actions of $\mathfrak{O}_k[\mathfrak{G}]$ corresponding to α_1 and α_2, respectively. This states that σ is a morphism of $\mathfrak{O}_k[\mathfrak{G}]$-modules.

(iii) *For every k-algebra A and $x \in \mathfrak{G}(A)$ we have*

$$\sigma_A \cdot \tilde{\rho}_1(x) = \tilde{\rho}_2(x) \cdot \sigma_A$$

where $\tilde{\rho}_1 : \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1)$ and $\tilde{\rho}_2 : \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$ are morphism of $\mathfrak{O}_k$-algebras corresponding to $\tilde{\alpha_1}$ and $\tilde{\alpha_2}$, respectively.

(iv) *For every k-algebra A and $x \in \mathfrak{G}(A)$ we have*

$$\sigma_A \cdot \rho_1(x) = \rho_2(x) \cdot \sigma_A$$

where $\rho_1 : \mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1)$ and $\rho_2 : \mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$ are restrictions of $\tilde{\rho}_1$ and $\tilde{\rho}_2$, respectively.

The equivalence of **(i)** *and* **(ii)** *does not require the existence of $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1)$ and $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$.*

Proof. Conditions expressed in **(i)-(iv)** are directly translatable to each other by virtue of bijections in Proposition 7.4. □

Let $\mathfrak{G}$ be a monoid k-functor. We denote by **Mod** $(\mathfrak{O}_k[\mathfrak{G}])$ the category of $\mathfrak{O}_k[\mathfrak{G}]$-modules.

8. Example of $\mathfrak{G}$-action: Regular functions k-functor

First we need the following notion.

Definition 8.1. Let $(-)^{\mathrm{op}} : \mathbf{Mon} \to \mathbf{Mon}$ be the functor of opposite monoid and let $\mathfrak{G}$ be a monoid k-functor. Then the composition $\mathfrak{G}^{\mathrm{op}} = (-)^{\mathrm{op}} \cdot \mathfrak{G}$ is called *the opposite monoid k-functor of* $\mathfrak{G}$.

Let $\mathfrak{G}$ be a monoid k-functor. In this section we discuss important example of a $\mathfrak{O}_k[\mathfrak{G}]$-module. Fix a k-functor $\mathfrak{X}$ for which $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{O}_k)$ exists. Recall that by Example 3.5 $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{O}_k)$ is $\mathfrak{O}_k$-algebra k-functor. Let $\alpha : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X}$ be an action of $\mathfrak{G}$ on $\mathfrak{X}$. For every k-algebra A we have a map of sets

$$\mathrm{Mor}_A\left(\mathfrak{X}_A, (\mathfrak{O}_k)_A\right) \ni f \mapsto f \cdot \alpha_x \in \mathrm{Mor}_A\left(\mathfrak{X}_A, (\mathfrak{O}_k)_A\right)$$

where $x \in \mathfrak{G}(A)$. From this description it follows that the map $f \mapsto f \cdot \alpha_x$ is a morphism of A-algebras. Moreover, note that if $y \in \mathfrak{G}(A)$ is some other A-point, then $(f \cdot \alpha_x) \cdot \alpha_y = f \cdot \alpha_{x \cdot y}$, where $x \cdot y \in \mathfrak{G}(A)$ is a product of x and y. Thus the opposite monoid $\mathfrak{G}^{\mathrm{op}}(A)$ acts on the A-algebra $\mathrm{Mor}_A\left(\mathfrak{X}_A, (\mathfrak{O}_k)_A\right)$ by morphism of A-algebras. Next for every A-algebra B and every point $y \in \mathfrak{X}(B)$ we have

$$(f \cdot \alpha_x)(y) = f\left(\alpha_x(y)\right)$$

This proves the following result.

Proposition 8.2. *Let $\mathfrak{X}$ be a k-functor and let $\alpha : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X}$ be an action of a monoid k-functor $\mathfrak{G}$. Suppose that $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{O}_k)$ exists. Then $\mathfrak{G}^{\mathrm{op}}$ acts canonically on $\mathfrak{O}_k$-algebra k-functor $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{O}_k)$ by morphisms of $\mathfrak{O}_k$-algebras.*

Let us note one important consequence of this result.

Corollary 8.3. *Let $\mathfrak{G}$ be a monoid k-functor. The action of $\mathfrak{G} \times \mathfrak{G}^{\mathrm{op}}$ on $\mathfrak{G}$ induces the action of $\mathfrak{G}^{\mathrm{op}} \times \mathfrak{G}$ on $\mathfrak{O}_k$-algebra k-functor $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{O}_k)$ by morphisms of $\mathfrak{O}_k$-algebras.*

9. Linear representations of a monoid k-functors

We start the discussion with some results that relates categories $\mathbf{Mod}(k)$ and $\mathbf{Mod}(\mathfrak{O}_k)$.

Example 9.1. Let V be a k-module. We define a k-functor V_{a}. We set

$$V_{\mathrm{a}}(A) = A \otimes_k V,\ V_{\mathrm{a}}(f) = f \otimes_k 1_V$$

for every k-algebra A and every morphism $f : A \to B$ of k-algebras. Note that V_{a} is $\mathfrak{O}_k$-module. Suppose that $\phi : V \to W$ is a morphism of k-modules, then we define $\phi_{\mathrm{a}} : V_{\mathrm{a}} \to W_{\mathrm{a}}$ by formula

$$\phi_{\mathrm{a}}^A = 1_A \otimes_k \sigma$$

for every k-algebra. Then ϕ_{a} is a morphism of $\mathfrak{O}_k$-modules.

Proposition 9.2. *The functor $(-)_{\mathrm{a}} : \mathbf{Mod}(k) \to \mathbf{Mod}(\mathfrak{O}_k)$ is full and faithful.*

Proof. Fix k-modules V, W. Then

$$\mathrm{Hom}_{\mathfrak{O}_k}(V_{\mathrm{a}}, W_{\mathrm{a}}) \ni \sigma \mapsto \sigma^k \in \mathrm{Hom}_k(V, W)$$

and

$$\mathrm{Hom}_k(V, W) \ni \phi \mapsto \phi_{\mathrm{a}} \in \mathrm{Hom}_{\mathfrak{O}_k}(V_{\mathrm{a}}, W_{\mathrm{a}})$$

are mutually inverse bijections. Hence the functor is full and faithful. □

Example 9.3. Let V be a k-module. We define a k-functor $\mathcal{L}_V$. We set

$$\mathcal{L}_V(A) = \mathrm{Hom}_A(A \otimes_k V, A \otimes_k V)$$

for every k-algebra A. Next for every morphism $f : A \to B$ of k-algebras and every morphism $\phi : A \otimes_k V \to A \otimes_k V$ of A-modules we define $\mathcal{L}_V(f)(\phi)$ as a unique morphism of B-modules such that the diagram

$$\begin{tikzcd}
A \otimes_k V \arrow[r, "\phi"] \arrow[d, "f \otimes_k 1_V"'] & A \otimes_k V \arrow[d, "f \otimes_k 1_V"] \\
B \otimes_k V \arrow[r, "\mathcal{L}_V(\phi)"'] & B \otimes_k V
\end{tikzcd}$$

is commutative. Note also that $\mathcal{L}_V(A)$ is an A-algebra. Hence $\mathcal{L}_V$ is a monoid $k\mathfrak{O}_k$-algebra.

Remark 9.4. Let V be a k-module. Proposition 9.2 implies that there are bijective maps that make the square

$$\begin{tikzcd}
\mathcal{L}_V(A) \arrow[r, "\cong"] \arrow[d, "\mathcal{L}_V(f)"'] & \mathcal{E}nd_{\mathfrak{O}_A}\big((V_{\mathrm{a}})_A, (V_{\mathrm{a}})_A\big) \arrow[d, "\sigma \mapsto \sigma_B"] \\
\mathcal{L}_V(B) \arrow[r, "\cong"'] & \mathcal{E}nd_{\mathfrak{O}_B}\big((V_{\mathrm{a}})_B, (V_{\mathrm{a}})_B\big)
\end{tikzcd}$$

commutative for every morphism $f : A \to B$ of k-algebras. This induces an idenitification $\mathcal{L}_V = \mathcal{E}nd_{\mathfrak{O}_k}(V_{\mathrm{a}})$ of $\mathfrak{O}_k$-algebras.

Definition 9.5. Let $\mathfrak{G}$ be a monoid k-functor. A pair (V,ρ) consisting of a k-module V and a morphism $\rho : \mathfrak{G} \to \mathcal{L}_V$ of k-monoids is called *a linear representation of* $\mathfrak{G}$.

Next result characterizes linear representations of monoid k-functors.

Corollary 9.6. *Let $\mathfrak{G}$ be a monoid k-functor and let V be a k-module. Then the following classes are in canonical bijections.*

(1) *Linear actions of $\mathfrak{G}$ on V_{a}.*

(2) *$\mathfrak{O}_k$-linear actions $\mathfrak{O}_k[\mathfrak{G}] \times V_{\mathrm{a}} \to V_{\mathrm{a}}$. These are precisely $\mathfrak{O}_k[\mathfrak{G}]$-modules.*

(3) *Morphisms $\mathfrak{O}_k[\mathfrak{G}] \to \mathcal{L}_V$ of $\mathfrak{O}_k$-algebras.*

(4) *Morphisms $\mathfrak{G} \to \mathcal{L}_V$ of monoids.*

Proof. This follows from Proposition 7.4. □

Definition 9.7. Let $\mathfrak{G}$ be a monoid k-functor and let (V,ρ), (W,δ) be its linear representations. A morphism $\phi : V \to W$ of k-modules such that

$$\phi_{\mathrm{a}}^{A} \cdot \rho(x) = \delta(x) \cdot \phi_{\mathrm{a}}^{A}$$

for every k-algebra A and $x \in \mathfrak{G}(A)$ is called *a morphism of linear representations of* $\mathfrak{G}$.

Next result characterizes morphisms of linear representations of monoid k-functor.

Corollary 9.8. *Let $\mathfrak{G}$ be a monoid k-functor and let V, W be k-modules. Suppose that $\alpha_1 : \mathfrak{G} \times V_{\mathrm{a}} \to V_{\mathrm{a}}$, $\alpha_2 : \mathfrak{G} \times W_{\mathrm{a}} \to W_{\mathrm{a}}$ are linear actions of $\mathfrak{G}$. Suppose that $\phi : V \to W$ is a morphism of k-modules. Then the following assertions are equivalent.*

(i) *The square*

$$\begin{tikzcd} \mathfrak{G} \times V_{\mathrm{a}} \arrow[r, "1_{\mathfrak{G}} \times \phi_{\mathrm{a}}"] \arrow[d, "\alpha_1"'] & \mathfrak{G} \times W_{\mathrm{a}} \arrow[d, "\alpha_2"] \\ V_{\mathrm{a}} \arrow[r, "\phi_{\mathrm{a}}"'] & W_{\mathrm{a}} \end{tikzcd}$$

is commutative.

(ii) *The square*

$$\begin{tikzcd} \mathfrak{O}_k[\mathfrak{G}] \times V_{\mathrm{a}} \arrow[r, "1_{\mathfrak{O}_k[\mathfrak{G}]} \times \phi_{\mathrm{a}}"] \arrow[d, "\tilde{\alpha}_1"'] & \mathfrak{O}_k[\mathfrak{G}] \times W_{\mathrm{a}} \arrow[d, "\tilde{\alpha}_2"] \\ V_{\mathrm{a}} \arrow[r, "\phi_{\mathrm{a}}"'] & W_{\mathrm{a}} \end{tikzcd}$$

is commutative, where $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are $\mathfrak{O}_k$-linear actions of $\mathfrak{O}_k[\mathfrak{G}]$ corresponding to α_1 and α_2, respectively.

(iii) *For every k-algebra A and $x \in \mathfrak{G}(A)$ we have*

$$\phi_{\mathrm{a}}^{A} \cdot \tilde{\rho}_1(x) = \tilde{\rho}_2(x) \cdot \phi_{\mathrm{a}}^{A}$$

where $\tilde{\rho}_1 : \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{L}_V$ and $\tilde{\rho}_2 : \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{L}_W$ are morphism of $\mathfrak{O}_k$-algebras corresponding to $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, respectively.

(iv) *For every k-algebra A and $x \in \mathfrak{G}(A)$ we have*

$$\phi_{\mathrm{a}}^{A} \cdot \rho_1(x) = \rho_2(x) \cdot \phi_{\mathrm{a}}^{A}$$

where $\rho_1 : \mathfrak{G} \to \mathcal{L}_V$ and $\rho_2 : \mathfrak{G} \to \mathcal{L}_W$ are restrictions of $\tilde{\rho_1}$ and $\tilde{\rho_2}$, respectively. This states that ϕ is a morphism of linear representations of $\mathfrak{G}$.

Proof. This follows from Proposition 7.5. □

Let $\mathfrak{G}$ be a monoid k-functor. We denote by $\mathbf{Rep}(\mathfrak{G})$ its category of linear representations. Note that $\mathbf{Rep}(\mathfrak{G})$ is a full subcategory of $\mathbf{Mod}(\mathfrak{O}_k[\mathfrak{G}])$.

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