

CONDITIONAL EXPECTATIONS

1. INTRODUCTION

These notes introduce notion of conditional expectation of a random variable and discuss its properties. Aside basic measure-theoretic and probabilistic tools we use here Radon-Nikodym theorem [Monygham, 2018, Theorem 5.1].

2. EXISTENCE OF CONDITIONAL EXPECTATIONS

Fix a probability space (Ω, \mathcal{F}, P) .

Theorem 2.1. *Let $X : \Omega \rightarrow \mathbb{C}$ be an integrable random variable and \mathcal{G} be a σ -subalgebra of \mathcal{F} . Then there exists \mathcal{G} -measurable and integrable function $f : \Omega \rightarrow \mathbb{C}$ such that*

$$\int_G X dP = \int_G f dP$$

for every G in \mathcal{G} . Moreover, the set of all \mathcal{G} -measurable functions having the property described by the system of equations above is

$$\{g : \Omega \rightarrow \mathbb{C} \mid g \text{ is } \mathcal{G}\text{-measurable and } f(\omega) = g(\omega) \text{ almost surely}\}$$

Proof. We define a complex measure $\nu : \mathcal{G} \rightarrow \mathbb{C}$ by formula

$$\nu(G) = \int_G X dP$$

for $G \in \mathcal{G}$. Since $\nu \ll P|_{\mathcal{G}}$ and by Radon-Nikodym theorem, we derive that there exists a \mathcal{G} -measurable function $f : \Omega \rightarrow \mathbb{C}$ such that

$$\nu(G) = \int_G f dP$$

The last statement is clear and is left for the reader as an exercise. \square

Definition 2.2. Let $X : \Omega \rightarrow \mathbb{C}$ be an integrable random variable and \mathcal{G} be a σ -subalgebra of \mathcal{F} . Suppose that $f : \Omega \rightarrow \mathbb{C}$ is a \mathcal{G} -measurable and integrable function $f : \Omega \rightarrow \mathbb{C}$ such that

$$\int_G X dP = \int_G f dP$$

for every G in \mathcal{G} . Then f is called a *version of the conditional expectation of X with respect to \mathcal{G}* .

No we define important special case.

Definition 2.3. Let \mathcal{G} be a σ -subalgebra of \mathcal{F} . Let $f : \Omega \rightarrow \mathbb{C}$ be a \mathcal{G} -measurable, integrable function such that

$$P(A \cap G) = \int_G f dP$$

for every $G \in \mathcal{G}$. Then f is called a *version of conditional probability of A with respect to \mathcal{G}* .

Now that we discuss basic existence and uniqueness results concerning conditional expectation let us introduce some notation. Let (Ω, \mathcal{F}, P) be a probability space, $X : \Omega \rightarrow \mathbb{C}$ be an integrable random variable and \mathcal{G} be a σ -subalgebra of \mathcal{F} . We denote any version of the conditional expectation of X with respect to \mathcal{G} by a symbol

$$\mathbb{E}[X | \mathcal{G}]$$

and for every set $A \in \mathcal{F}$ we denote by

$$P[A | \mathcal{G}]$$

any version of conditional probability of A with respect to \mathcal{G} . We also often omit the word version and speak about conditional expectation and conditional probabilities. Nevertheless one should always keep in mind that these are \mathcal{G} -measurable and integrable functions defined up to sets in \mathcal{G} of probability zero.

3. PROPERTIES OF CONDITIONAL EXPECTATION

Let (Ω, \mathcal{F}, P) be a probability space and \mathcal{G} be a σ -subalgebra of \mathcal{F} .

Theorem 3.1. *Let $Y, X, \{X_n\}_{n \in \mathbb{N}}$ be integrable random variables $\Omega \rightarrow \mathbb{C}$. Then the following results hold.*

(1) *If X, Y have real values and $X \leq Y$ almost surely, then $\mathbb{E}[X|\mathcal{G}] \leq \mathbb{E}[Y|\mathcal{G}]$ almost surely.*

(2) *$\mathbb{E}[a \cdot X + b \cdot Y|\mathcal{G}] = a \cdot \mathbb{E}[X|\mathcal{G}] + b \cdot \mathbb{E}[Y|\mathcal{G}]$ almost surely for $a, b \in \mathbb{C}$.*

(3) *$|\mathbb{E}[X|\mathcal{G}]| \leq \mathbb{E}[|X||\mathcal{G}]$ almost surely.*

(4) *If $\{X_n\}_{n \in \mathbb{N}}$ converges almost surely to X and*

$$|X_n| \leq Y, |X| \leq Y$$

almost surely for every $n \in \mathbb{N}$, then $\{\mathbb{E}[X_n|\mathcal{G}]\}_{n \in \mathbb{N}}$ converges almost surely to $\mathbb{E}[X|\mathcal{G}]$.

Proof. For the proof of (1). We have

$$\int_G \mathbb{E}[X|\mathcal{G}] dP = \int_G X dP \leq \int_G Y dP = \int_G \mathbb{E}[Y|\mathcal{G}] dP$$

Since conditional expectations with respect to \mathcal{G} are \mathcal{G} -measurable, we deduce that $\mathbb{E}[X|\mathcal{G}] \leq \mathbb{E}[Y|\mathcal{G}]$.

Next we prove (2). Pick $a, b \in \mathbb{C}$. We have

$$\begin{aligned} \int_G \mathbb{E}[a \cdot X + b \cdot Y|\mathcal{G}] dP &= \int_G (a \cdot X + b \cdot Y) dP = a \cdot \int_G X dP + b \cdot \int_G Y dP = \\ &= a \cdot \int_G \mathbb{E}[X|\mathcal{G}] dP + b \cdot \int_G \mathbb{E}[Y|\mathcal{G}] dP = \int_G (a \cdot \mathbb{E}[X|\mathcal{G}] + b \cdot \mathbb{E}[Y|\mathcal{G}]) dP \end{aligned}$$

for every $G \in \mathcal{G}$. Since conditional expectations with respect to \mathcal{G} is \mathcal{G} -measurable, we derive that $\mathbb{E}[a \cdot X + b \cdot Y|\mathcal{G}] = a \cdot \mathbb{E}[X|\mathcal{G}] + b \cdot \mathbb{E}[Y|\mathcal{G}]$.

For (3) assume pick $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and

$$\alpha \cdot \mathbb{E}[X|\mathcal{G}] = |\mathbb{E}[X|\mathcal{G}]|$$

Then

$$|\mathbb{E}[X|\mathcal{G}]| = \alpha \cdot \mathbb{E}[X|\mathcal{G}] = \mathbb{E}[\alpha \cdot X|\mathcal{G}] = \mathbb{E}[\operatorname{re}(\alpha \cdot X)|\mathcal{G}] \leq \mathbb{E}[|\alpha \cdot X||\mathcal{G}] = \mathbb{E}[|X||\mathcal{G}]$$

almost surely. Thus (3) holds.

Finally we prove that (4). Set $Z_n = \sup_{k \leq n} |X_k - X|$. Then Z_n is nonnegative measurable function and $\lim_{n \rightarrow +\infty} Z_n = 0$. Moreover, $\{Z_n\}_{n \in \mathbb{N}}$ is pointwise decreasing and dominated by $2 \cdot |Y|$. Thus by dominated convergence theorem

$$\lim_{n \rightarrow +\infty} \int_\Omega Z_n dP = 0$$

Next $\{\mathbb{E}[Z_n|\mathcal{G}]\}_{n \in \mathbb{N}}$ are measurable, almost surely pointwise decreasing and nonnegative functions. Moreover, we derive that

$$\lim_{n \rightarrow +\infty} \int_\Omega \mathbb{E}[Z_n|\mathcal{G}] dP = \lim_{n \rightarrow +\infty} \int_\Omega Z_n dP = 0$$

and hence

$$\int_\Omega \left(\lim_{n \rightarrow +\infty} \mathbb{E}[Z_n|\mathcal{G}] \right) dP = 0$$

This implies that $\lim_{n \rightarrow +\infty} \mathbb{E}[Z_n | \mathcal{G}] = 0$ almost surely. By (1) and (3) we have

$$\sup_{k \geq n} |\mathbb{E}[X_k | \mathcal{G}] - \mathbb{E}[X | \mathcal{G}]| = \sup_{k \geq n} \mathbb{E}[|X_k - X| | \mathcal{G}] \leq \mathbb{E}[Z_n | \mathcal{G}]$$

Therefore

$$\lim_{n \rightarrow +\infty} \sup_{k \geq n} |\mathbb{E}[X_k | \mathcal{G}] - \mathbb{E}[X | \mathcal{G}]| = 0$$

and hence $\lim_{n \rightarrow +\infty} \mathbb{E}[X_n | \mathcal{G}] = \mathbb{E}[X | \mathcal{G}]$. \square

Theorem 3.2. Let $X, Y : \Omega \rightarrow \mathbb{C}$ be random variables such that $X, Y \cdot X$ are integrable and Y is \mathcal{G} -measurable. Then

$$\mathbb{E}[Y \cdot X | \mathcal{G}] = Y \cdot \mathbb{E}[X | \mathcal{G}]$$

Proof. First note that the result is clear for $Y = \mathbb{1}_G$ where $G \in \mathcal{G}$ and also for $\mathbb{R}_{>0}$ -linear combination of such functions. Next suppose that $Y : \Omega \rightarrow \mathbb{C}$ is integrable \mathcal{G} -measurable function with nonnegative real values. Then there exists a nondecreasing sequence $\{Y_n\}_{n \in \mathbb{N}}$ of positive combinations of indicator functions of sets in \mathcal{G} that converges to Y . Note that $|Y_n \cdot X| \leq |Y_n| \cdot |X|$ and $|Y_n \cdot \mathbb{E}[X | \mathcal{G}]| \leq Y_n \cdot |\mathbb{E}[X | \mathcal{G}]|$ for $n \in \mathbb{N}$. Then by dominated convergence theorem

$$\int_G \mathbb{E}[Y \cdot X | \mathcal{G}] dP = \int_G Y \cdot X dP = \lim_{n \rightarrow +\infty} \int_G Y_n \cdot X dP = \lim_{n \rightarrow +\infty} \int_G Y_n \cdot \mathbb{E}[X | \mathcal{G}] dP = \int_G Y \cdot \mathbb{E}[X | \mathcal{G}] dP$$

for every $G \in \mathcal{G}$. This implies that $\mathbb{E}[Y \cdot X | \mathcal{G}] = Y \cdot \mathbb{E}[X | \mathcal{G}]$. Suppose now that $Y : \Omega \rightarrow \mathbb{C}$ is a \mathcal{G} -measurable and integrable random variable taking real values. We write $Y_+ = \max\{0, Y\}$ and $Y_- = \min\{0, Y\}$. Then

$$\mathbb{E}[Y \cdot X | \mathcal{G}] = \mathbb{E}[Y_+ \cdot X | \mathcal{G}] + \mathbb{E}[Y_- \cdot X | \mathcal{G}] = Y_+ \cdot \mathbb{E}[X | \mathcal{G}] + Y_- \cdot \mathbb{E}[X | \mathcal{G}] = Y \cdot \mathbb{E}[X | \mathcal{G}]$$

This proves the assertion for every real-valued, integrable and \mathcal{G} -measurable random variable Y . Finally suppose that Y is complex valued, \mathcal{G} -measurable and integrable. Write $Y = Y_r + i \cdot Y_i$ for real valued Y_r, Y_i random variables. Then Y_r, Y_i are \mathcal{G} -measurable and integrable. Hence

$$\mathbb{E}[Y \cdot X | \mathcal{G}] = \mathbb{E}[Y_r \cdot X | \mathcal{G}] + i \cdot \mathbb{E}[Y_i \cdot X | \mathcal{G}] = Y_r \cdot \mathbb{E}[X | \mathcal{G}] + i \cdot Y_i \cdot \mathbb{E}[X | \mathcal{G}] = Y \cdot \mathbb{E}[X | \mathcal{G}]$$

Thus assertion holds for any \mathcal{G} -measurable, integrable random variable $Y : \Omega \rightarrow \mathbb{C}$. Suppose now that Y is \mathcal{G} -measurable and $Y \cdot X, X$ are integrable. Define $W_n = \{\omega \in \Omega : |Y(\omega)| \leq n\}$ and $Y_n = \mathbb{1}_{W_n} \cdot Y$. Then $\{Y_n\}_{n \in \mathbb{N}}$ is a sequence of integrable \mathcal{G} -measurable random variables convergent to Y and $|Y_n \cdot X| \leq |Y \cdot X|$ for every $n \in \mathbb{N}$. Hence

$$Y \cdot \mathbb{E}[X | \mathcal{G}] = \lim_{n \rightarrow +\infty} Y_n \cdot \mathbb{E}[X | \mathcal{G}] = \lim_{n \rightarrow +\infty} \mathbb{E}[Y_n \cdot X | \mathcal{G}] = \mathbb{E}[Y \cdot X | \mathcal{G}]$$

and the last equality follow from (4) of Theorem 3.1 \square

Theorem 3.3 (Tower Property). Let $\mathcal{G}_2 \subseteq \mathcal{G}_1 \subseteq \mathcal{F}$ be σ -algebras and $X : \Omega \rightarrow \mathbb{C}$ be an integrable random variable. Then

$$\mathbb{E}[\mathbb{E}[X | \mathcal{G}_1] | \mathcal{G}_2] = \mathbb{E}[X | \mathcal{G}_2]$$

Proof. Fix $G \in \mathcal{G}_2$. Then also $G \in \mathcal{G}_1$ and

$$\int_G \mathbb{E}[\mathbb{E}[X | \mathcal{G}_1] | \mathcal{G}_2] dP = \int_G \mathbb{E}[X | \mathcal{G}_1] dP = \int_G X dP = \int_G \mathbb{E}[X | \mathcal{G}_2] dP$$

Therefore, we derive that $\mathbb{E}[\mathbb{E}[X | \mathcal{G}_1] | \mathcal{G}_2] = \mathbb{E}[X | \mathcal{G}_2]$. \square

Theorem 3.4. Let \mathcal{G} be a σ -subalgebra of \mathcal{F} , $X : \Omega \rightarrow \mathbb{R}$ be an integrable random variable and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. Suppose that $\phi(X)$ is integrable. Then

$$\phi(\mathbb{E}[X | \mathcal{G}]) \leq \mathbb{E}[\phi(X) | \mathcal{G}]$$

Proof. Let L_ϕ be a set of functions $\mathbb{R} \ni x \mapsto a \cdot x + b \in \mathbb{R}$ for $a, b \in \mathbb{R}$ such that $a \cdot x + b \leq \phi(x)$ for every $x \in \mathbb{R}$. Since ϕ is convex, we derive that for every $x \in \mathbb{R}$ we have $\phi(x) = \sup_{l \in L_\phi} l(x)$. Hence

$$\phi(\mathbb{E}[X|\mathcal{G}]) = \sup_{l \in L_\phi} l(\mathbb{E}[X|\mathcal{G}]) = \sup_{l \in L_\phi} \mathbb{E}[l(X)|\mathcal{G}] \leq \mathbb{E}[\phi(X)|\mathcal{G}]$$

□

REFERENCES

[Monygham, 2018] Monygham (2018). Radon-nikodym theorem, hahn-jordan decomposition and lebesgue decomposition. *github repository: "Monygham/Pedo-mellon-a-minno"*.