

# FILTERS IN TOPOLOGY

## 1. INTRODUCTION

In these short notes we study filters of subsets with their applications to topological spaces. Filters were introduced in [Cartan, 1937] as an effective tool in studying general topological spaces. Here we recapitulate some of Cartan's results. In particular, we give a concise proof of Tychonoff's theorem on compact spaces. For introduction to topological spaces we refer to [Monygham, 2024].

## 2. FILTERS

**Definition 2.1.** Let  $X$  be a set and let  $\mathcal{F}$  be a nonempty family of subsets of  $X$ . Assume that the following assertions hold.

- (1)  $\mathcal{F}$  is closed under finite intersections.
- (2) If  $F_1$  and  $F_2$  are subsets of  $X$  such that  $F_1 \in \mathcal{F}$  and  $F_1 \subseteq F_2$ , then  $F_2 \in \mathcal{F}$ .

Then  $\mathcal{F}$  is a *filter* on  $X$ .

We note the following fact.

**Fact 2.2.** Let  $X$  be a set and let  $\{\mathcal{F}_i\}_{i \in I}$  be a family of filters on  $X$ . Then

$$\bigcap_{i \in I} \mathcal{F}_i$$

is a filter on  $X$ .

*Proof.* Left for the reader as an exercise. □

**Definition 2.3.** Let  $X$  be a set and let  $\mathcal{F}$  be a filter on  $X$ . Assume that  $\emptyset \notin \mathcal{F}$ . Then  $\mathcal{F}$  is *proper*.

Filters are functorial as it is displayed in the following notion.

**Definition 2.4.** Let  $\mathcal{F}$  be a filter on a set  $X$  and let  $f : X \rightarrow Y$  be a map of sets. Then a filter

$$f(\mathcal{F}) = \{Z \subseteq Y \mid \text{there exists } F \in \mathcal{F} \text{ such that } f(F) \subseteq Z\}$$

on  $Y$  is the *image* of  $\mathcal{F}$  under  $f$ .

Let us note the following result.

**Fact 2.5.** Let  $\mathcal{F}$  be a filter on a set  $X$  and let  $f : X \rightarrow Y$  be a map of sets. If  $\mathcal{F}$  is proper, then  $f(\mathcal{F})$  is proper.

*Proof.* Left for the reader as an exercise. □

Now we introduce the notion of ultrafilter and prove its properties.

**Definition 2.6.** Let  $X$  be a set and let  $\mathcal{F}$  be a proper filter on  $X$ . Suppose that  $\mathcal{F}$  is maximal with respect to inclusion among proper filters on  $X$ . Then  $\mathcal{F}$  is an *ultrafilter* on  $X$ .

**Proposition 2.7.** Let  $X$  be a set and let  $\mathcal{F}$  be a proper filter on  $X$ . The following assertions are equivalent.

- (i)  $\mathcal{F}$  is an ultrafilter on  $X$ .
- (ii) For each subset  $F$  of  $X$  either  $F \in \mathcal{F}$  or  $X \setminus F \in \mathcal{F}$ .

*Proof.* Assume that  $\mathcal{F}$  is an ultrafilter and let  $F$  be a subset of  $X$ . Suppose that  $F \notin \mathcal{F}$ . Then the smallest filter containing  $\{F\} \cup \mathcal{F}$ , which exists according to Fact 2.2, is not a proper filter. This implies that there exists  $F' \in \mathcal{F}$  such that  $F \cap F' = \emptyset$ . Since  $F' \subseteq X \setminus F$  and  $\mathcal{F}$  is a filter, we derive that  $X \setminus F \in \mathcal{F}$ . This proves that (i)  $\Rightarrow$  (ii).

Suppose that for each subset  $F$  of  $X$  either  $F \in \mathcal{F}$  or  $X \setminus F \in \mathcal{F}$ . Consider a filter  $\tilde{\mathcal{F}}$  such that  $\mathcal{F} \subsetneq \tilde{\mathcal{F}}$ . If  $F \in \tilde{\mathcal{F}} \setminus \mathcal{F}$ , then  $X \setminus F \in \mathcal{F}$  and hence  $\emptyset = F \cap (X \setminus F) \in \tilde{\mathcal{F}}$ . This implies that  $\tilde{\mathcal{F}}$  is not a proper filter. Thus  $\mathcal{F}$  is an ultrafilter on  $X$ . This completes the proof of (ii)  $\Rightarrow$  (i).  $\square$

**Corollary 2.8.** *Let  $f : X \rightarrow Y$  be a map of sets and let  $\mathcal{F}$  be an ultrafilter of subsets of  $X$ . Then  $f(\mathcal{F})$  is an ultrafilter.*

*Proof.* Filter  $f(\mathcal{F})$  is proper according to Fact 2.5. Fix a subset  $F$  of  $Y$ . By Proposition 2.7 either  $f^{-1}(F) \in \mathcal{F}$  or  $f^{-1}(Y \setminus F) \in \mathcal{F}$ . Thus either  $F \in f(\mathcal{F})$  or  $Y \setminus F \in f(\mathcal{F})$ . Proposition 2.7 implies that  $f(\mathcal{F})$  is an ultrafilter.  $\square$

The following result uses axiom of choice.

**Proposition 2.9.** *Let  $X$  be a set and let  $\mathcal{F}$  be a proper filter on  $X$ . Then there exists an ultrafilter  $\tilde{\mathcal{F}}$  on  $X$  such that  $\mathcal{F} \subseteq \tilde{\mathcal{F}}$ .*

*Proof.* Consider the family

$$\mathbf{F} = \{\mathcal{G} \mid \mathcal{G} \text{ is a proper filter on } X \text{ and } \mathcal{F} \subseteq \mathcal{G}\}$$

Note that  $\mathbf{F}$  is nonempty because  $\mathcal{F} \in \mathbf{F}$ . The inclusion of filters introduces partial order on  $\mathbf{F}$  and if  $L \subseteq \mathbf{F}$  is a linearly ordered subset, then

$$\bigcup L$$

is a proper filter. Hence each chain in  $(\mathbf{F}, \subseteq)$  admits an upper bound. Zorn's lemma implies that  $(\mathbf{F}, \subseteq)$  has a maximal element  $\tilde{\mathcal{F}}$ . Clearly  $\tilde{\mathcal{F}}$  is an ultrafilter on  $X$  and  $\mathcal{F} \subseteq \tilde{\mathcal{F}}$ .  $\square$

### 3. FILTERS AND CONVERGENCE IN TOPOLOGICAL SPACES

**Definition 3.1.** Let  $X$  be a topological space and let  $\mathcal{F}$  be a proper filter on  $X$ . Consider a point  $x$  in  $X$ . Suppose that for every open neighborhood  $U$  of  $x$  we have  $U \in \mathcal{F}$ . Then  $\mathcal{F}$  converges to  $x$  in  $X$ .

**Proposition 3.2.** *Let  $X, Y$  be topological spaces and let  $f : X \rightarrow Y$  be a map of sets. Let  $x$  be a point in  $X$ . Then the following assertions are equivalent.*

- (i)  $f$  is continuous at  $x$ .
- (ii) If  $\mathcal{F}$  is a proper filter on  $X$  convergent to  $x$ , then  $f(\mathcal{F})$  converges to  $f(x)$ .
- (iii) If  $\mathcal{F}$  is an ultrafilter on  $X$  convergent to  $x$ , then  $f(\mathcal{F})$  converges to  $f(x)$ .

*Proof.* Suppose that  $f$  is continuous at  $x$ . Fix a proper filter  $\mathcal{F}$  on  $X$  convergent to  $x$ . Fix an open neighborhood  $V$  of  $f(x)$  in  $Y$ . Since  $f$  is continuous at  $x$ , there exists an open neighborhood  $U$  of  $x$  such that  $f(U) \subseteq V$ . Note that  $U \in \mathcal{F}$  and hence  $V \in f(\mathcal{F})$ . Since  $V$  is arbitrary open neighborhood of  $f(x)$  in  $Y$ , we derive that  $f(\mathcal{F})$  converges to  $f(x)$  in  $Y$ . This proves the implication (i)  $\Rightarrow$  (ii).

The implication (ii)  $\Rightarrow$  (iii) follows from definition of an ultrafilter.

Suppose now that (iii) holds. Consider an open neighborhood  $V$  of  $f(x)$  in  $Y$ . Assume that for every open neighborhood  $U$  of  $x$  in  $X$  the set  $U \setminus f^{-1}(V)$  is nonempty. Let  $\mathcal{F}$  be a filter generated by all sets of the form  $U \setminus f^{-1}(V)$  where  $U$  is an open neighborhood of  $x$ . Then  $\mathcal{F}$  is a proper filter on  $X$ . Next by Proposition 2.9 there exists an ultrafilter  $\tilde{\mathcal{F}}$  on  $X$  which contains  $\mathcal{F}$ . Since  $\mathcal{F}$

converges to  $x$  in  $X$ , we derive that  $\tilde{\mathcal{F}}$  converges to  $x$  in  $X$ . Thus  $f(\tilde{\mathcal{F}})$  converges to  $f(x)$  in  $Y$ . Note that

$$f(X \setminus f^{-1}(V)) \in f(\tilde{\mathcal{F}})$$

This implies that  $Y \setminus V \in f(\tilde{\mathcal{F}})$  and hence  $V \notin f(\tilde{\mathcal{F}})$ . It follows that the filter  $f(\tilde{\mathcal{F}})$  cannot converge to  $f(x)$  in  $Y$ . We arrive at contradiction. This means that there exists an open neighborhood  $U$  of  $x$  in  $X$  such that  $U \subseteq f^{-1}(V)$ . This proves that  $f$  is continuous at  $x$ . We infer (iii)  $\Rightarrow$  (i).  $\square$

**Definition 3.3.** Let  $X$  be a topological space and let  $\mathcal{F}$  be a proper filter on  $X$ . Consider a point  $x$  in  $X$ . Suppose that for every open neighborhood  $U$  of  $x$  and every  $F \in \mathcal{F}$  we have  $F \cap U \neq \emptyset$ . Then  $x$  is an *accumulation point* of  $\mathcal{F}$ .

**Theorem 3.4.** Let  $X$  be a topological space. Then the following assertions are equivalent.

- (i) Each ultrafilter on  $X$  is convergent to some point of  $X$ .
- (ii) Each proper filter on  $X$  admits an accumulation point.
- (iii)  $X$  is a quasi-compact topological space.

*Proof.* Suppose that (i) holds. Let  $\mathcal{F}$  be a proper filter on  $X$ . By Proposition 2.9 there exists an ultrafilter  $\tilde{\mathcal{F}}$  that contains  $\mathcal{F}$ . By assumption  $\tilde{\mathcal{F}}$  is convergent to some point  $x$  of  $X$ . Then  $x$  is an accumulation point of  $\mathcal{F}$ . Hence (i)  $\Rightarrow$  (ii).

Assume that (ii). Pick a centered family  $\mathcal{F}$  of closed subsets of  $X$ . Define

$$\tilde{\mathcal{F}} = \{F \mid F \text{ contains some finite intersection of sets in } \mathcal{F}\}$$

Then  $\tilde{\mathcal{F}}$  is a proper filter on  $X$ . By assumption  $\tilde{\mathcal{F}}$  admits an accumulation point – say  $x \in X$ . Pick now  $F \in \mathcal{F}$ . Then for every open neighborhood  $U$  of  $x$  we have  $U \cap F \neq \emptyset$ . Since  $F$  is closed in  $X$ , we derive that  $x \in F$ . Thus  $x$  is contained in the intersection of  $\mathcal{F}$ . This implies that  $X$  is quasi-compact. This completes the proof of (ii)  $\Rightarrow$  (iii).

Assume that  $X$  is quasi-compact and suppose that  $\mathcal{F}$  is an ultrafilter on  $X$ . Suppose that  $\mathcal{F}$  is not convergent. Then for every  $x \in X$  there exists open neighborhood  $U_x$  of  $x$  in  $X$  such that  $U_x \notin \mathcal{F}$ . Since  $X$  is quasi-compact, we deduce that there exist  $n \in \mathbb{N}_+$  and  $x_1, \dots, x_n \in X$  such that

$$X = \bigcup_{i=1}^n U_{x_i}$$

According to Proposition 2.7 we derive that  $X \setminus U_x \in \mathcal{F}$  for every  $x \in X$ . Hence

$$\bigcap_{i=1}^n (X \setminus U_{x_i}) \in \mathcal{F}$$

On the other hand we have

$$\bigcap_{i=1}^n (X \setminus U_{x_i}) = X \setminus \bigcup_{i=1}^n U_{x_i} = \emptyset$$

This is contradiction. Thus the implication (iii)  $\Rightarrow$  (i) holds.  $\square$

#### 4. TYCHONOFF'S THEOREM

The following result is a celebrated theorem due to Tychonoff.

**Theorem 4.1.** Let  $\{X_i\}_{i \in I}$  be a family of quasi-compact topological spaces. Then the product

$$\prod_{i \in I} X_i$$

is quasi-compact.

*Proof.* We denote  $\prod_{i \in I} X_i$  by  $X$ . For each  $i$  in  $I$  we denote by  $pr_i : X \rightarrow X_i$  the canonical projection onto  $i$ -th factor. Suppose that  $X_i$  is quasi-compact for every  $i \in I$ . Pick an ultrafilter  $\mathcal{F}$  on  $X$ . Fix  $i$  in  $I$ . According to Corollary 2.8 the filter  $pr_i(\mathcal{F})$  is an ultrafilter. Since  $X_i$  is quasi-compact, we derive that  $pr_i(\mathcal{F})$  is convergent to some point  $x_i \in X_i$ . Let  $x$  be a point of  $X$  such that  $pr_i(x) = x_i$  for each  $i \in I$ . Fix finite subset  $\{i_1, \dots, i_n\} \subseteq I$ . Consider open neighborhood  $U_j$  of  $x_{i_j}$ . Then  $U_{i_j} \in pr_{i_j}(\mathcal{F})$  for each  $j$  and hence  $pr_{i_j}^{-1}(U_{i_j}) \in \mathcal{F}$  for each  $j$ . Since  $\mathcal{F}$  is a filter, we derive that

$$\prod_{j=1}^n U_{i_j} \times \prod_{i \in I \setminus \{i_1, \dots, i_n\}} X_i = \bigcap_{j=1}^n pr_{i_j}^{-1}(U_{i_j}) \in \mathcal{F}$$

This implies that  $\mathcal{F}$  is convergent to  $x$  in  $X$ . Thus every ultrafilter in  $X$  is convergent and hence Theorem 3.4 shows that  $X$  is quasi-compact.  $\square$

**Theorem 4.2.** *Let  $\{X_i\}_{i \in I}$  be a family of nonempty topological spaces. If the product*

$$\prod_{i \in I} X_i$$

*is quasi-compact, then  $X_i$  is quasi-compact for every  $i \in I$ .*

*Proof.* We denote  $\prod_{i \in I} X_i$  by  $X$ . For each  $i$  in  $I$  we denote by  $pr_i : X \rightarrow X_i$  the canonical projection onto  $i$ -th factor. Assume that  $X$  is quasi-compact. Since  $X_i \neq \emptyset$  for every  $i \in I$ , we derive that  $pr_i : X \rightarrow X_i$  is a continuous and surjective map for every  $i \in I$ . Hence for each  $i \in I$  space  $X_i$  is quasi-compact as an image of a quasi-compact space under continuous map.  $\square$

#### REFERENCES

- [Cartan, 1937] Cartan, H. (1937). Théorie des filtres. *CR Acad. Sci. Paris*, 205:595–598.  
[Monygham, 2024] Monygham (2024). Topological spaces. *github repository*: "Monygham/Pedo-mellon-a-minno".