PRO-CONSTRUCTIBLE SETS

1. Introduction

This is a continuation of [Monygham, 2018].

2. Prime spectrum and colimits of commutative algebras

Proposition 2.1. *Let A be a ring and $\{B_i\}_{i\in I}$ be a filtered diagram of A-algebras. Then the image of*

$$\mathrm{Spec}\,(\mathrm{colim}_{i\in I} B_i) \to \mathrm{Spec}\,A$$

is equal to the intersection of images $\left\{\mathrm{Spec}\,B_i \to \mathrm{Spec}\,A\right\}_{i\in I}$.

Lemma 2.1.1. *Let A be a ring and $\{B_i\}_{i\in I}$ be a filtered diagram of A-algebras. Then $\mathrm{colim}_{i\in I} B_i = 0$ if and only if there exists i_0 in I such that $B_{i_0} = 0$.*

Proof of the lemma. For every $i \in I$ let $f_i : B_i \to \mathrm{colim}_{i\in I} B_i$ be the canonical morphism. If $\mathrm{colim}_{i\in I} B_i = 0$, then $f_i(1) = 0$ for every $i \in I$. Since I is filtered category, this implies that there exists $i_0 \in I$ such that $1 = 0$ in B_{i_0}. Hence $B_{i_0} = 0$. The converse holds, because if $B_{i_0} = 0$ for some $i_0 \in I$, then

$$0 = f_{i_0}(0) = f_{i_0}(1) = 1$$

in $\mathrm{colim}_{i\in I} B_i$. □

Proof of the proposition. Consider $\mathfrak{p} \in \mathrm{Spec}\,A$ and let $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ be its residue field. For every A-algebra B we denote $k(\mathfrak{p}) \otimes_A B$ by $B(\mathfrak{p})$. We have

$$k(\mathfrak{p}) \otimes_A (\mathrm{colim}_{i\in I} B_i) \cong \mathrm{colim}_{i\in I}\,(k(\mathfrak{p}) \otimes_A B_i) \cong \mathrm{colim}_{i\in I} B_i(\mathfrak{p})$$

According to Lemma 2.1.1 we have

$$k(\mathfrak{p}) \otimes_A (\mathrm{colim}_{i\in I} B_i) = 0 \Leftrightarrow \exists_{i\in I}\; B_i(\mathfrak{p}) = 0$$

This implies that

$$\mathrm{Spec}\Big(k(\mathfrak{p}) \otimes_A (\mathrm{colim}_{i\in I} B_i)\Big) = \varnothing \Leftrightarrow \exists_{i\in I}\; B_i(\mathfrak{p}) = 0$$

Since the prime spectrum on the left hand side is the fiber of $\mathfrak{p}$ under the morphism

$$\mathrm{Spec}\,(\mathrm{colim}_{i\in I} B_i) \to \mathrm{Spec}\,A$$

we deduce that $\mathfrak{p}$ is not in the image of this map if and only if there exists $i \in I$ such that $B_i(\mathfrak{p}) = 0$. Hence $\mathfrak{p}$ is not in the image of

$$\mathrm{Spec}\,(\mathrm{colim}_{i\in I} B_i) \to \mathrm{Spec}\,A$$

if and only if it is not in the image of some $\mathrm{Spec}\,B_i \to \mathrm{Spec}\,A$. This finishes the proof. □

Corollary 2.2. *Let A be a ring and $\{B_i\}_{i\in I}$ be a family of A-algebras. We set*

$$\bigotimes_{i\in I} B_i = \mathrm{colim}_{n\in\mathbb{N},\,\{i_1,\dots,i_n\}\subseteq I}\left(B_{i_1} \otimes_A \dots \otimes_A B_{i_n}\right)$$

Then the image of the map

$$\mathrm{Spec}\left(\bigotimes_{i\in I} B_i\right) \to \mathrm{Spec}\,A$$

is the intersection of images of maps $\left\{\mathrm{Spec}\,B_i \to \mathrm{Spec}\,A\right\}_{i\in I}$.

Proof. For $\{i_1, ..., i_n\} \subseteq I$ the image of the map

$$\mathrm{Spec}\left(B_{i_1} \otimes_A ... \otimes_A B_{i_n}\right) \to \mathrm{Spec}\, A$$

is the intersection of images of maps $\left\{\mathrm{Spec}\, B_i \to \mathrm{Spec}\, A\right\}_{i \in I}$. Hence the assertion is an immediate consequence of Proposition 2.1. □

Corollary 2.3. *Let X be a quasi-compact scheme and E be a subset of X. Suppose that E is an intersection of constructible subsets of X. Then there exists an affine scheme Z and a morphism $f : Z \to X$ such that $f(Z) = E$.*

Proof. Let $X = \bigcup_{j=1}^{m} U_j$ be an affine open cover. By [Monygham, 2018, Corollary 3.4] and Corollary 2.2 for every $1 \leq j \leq m$ there exists an affine scheme Z_j and a morphism $f_j : Z_j \to U_j$ such that $f_j(Z_j) = E \cap U_j$. Define affine scheme $Z = \coprod_{j=1}^{m} Z_j$ and let $f : Z \to X$ be a morphism such that $f_{|Z_j}$ is the composition of f_j with the inclusion $U_j \hookrightarrow Z$. Then

$$f(Z) = \bigcup_{j=1}^{m} f_j(Z_j) = \bigcup_{j=1}^{m} \left(E \cap U_j\right) = E$$

□

3. Pro-constructible sets

Definition 3.1. Let X be a topological space. A subset E of X is called *pro-constructible in X* if for every point x in X there exists an open neighbourhood U of x in X such that $U \cap E$ is an intersection of locally constructible subsets of U.

Fact 3.2. *Let $f : X \to Y$ be a morphism of schemes and E be a pro-constructible subset of Y. Then $f^{-1}(E)$ is a pro-constructible subset of X.*

Proof. This is an immediate consequence of [Monygham, 2018, Fact 3.5] and the definition of pro-constructible sets. □

Corollary 3.3. *Let X be a scheme and E be a subset of X. Then the following are equivalent.*

- **(i)** *E is pro-constructible.*
- **(ii)** *$E \cap U$ is an intersection of constructible sets in U for every open quasi-compact and quasi-separated subset U of X.*
- **(iii)** *$E \cap U$ is an intersection of constructible sets in U for every affine open subset U of X.*

Proof. This is a consequence of [Monygham, 2018, Theorem 3.2] and the fact that union of sets is distributive over (arbitrary) intersection. □

The next theorem is a version of Chevalley's theorem on images for pro-constructible sets.

Theorem 3.4. *Let $f : X \to Y$ be a quasi-compact morphism of schemes and E be a pro-constructible subset of X. Then $f(E)$ is pro-constructible in Y.*

Lemma 3.4.1. *Let A be a ring and B be an A-algebra. Then B is a filtered colimit of finitely presented A-algebras.*

Proof of the lemma. Left as an exercise. □

The next result is very simple but useful.

Lemma 3.4.2. *Let X be a quasi-compact scheme. Then there exists an affine scheme W and a surjective morphism $W \to X$.*

Proof of the lemma. Let $X = \bigcup_{j=1}^{m} U_j$ be an open affine cover of X. Pick $W = \coprod_{j=1}^{m} U_j$ with the canonical morphism $W \to X$. □

Proof of the theorem. According to Corollary 3.3, we may assume that Y is affine. Then X is quasi-compact. Lemma 3.4.2 yields affine scheme W and a surjective morphism $g : W \to X$. By Fact 3.2 we derive that $g^{-1}(E)$ is pro-constructible subset of W. Thus replacing f by $f \cdot g$ we may assume that X is affine. In this case E is an intersection of constructible subsets of X according to Corollary 3.3. Corollary 2.3 implies that we can further assume that $E = X$. Hence it suffices to show that the image of a morphism $f : X \to Y$ of affine schemes is an intersection of constructible sets. By Lemma 3.4.1 there exists a filtered diagram $\{f_i : X_i \to Y\}_{i \in I}$ of morphisms of finite presentation such that

$$\mathrm{colim}_{i \in I} \Gamma(X_i, \mathcal{O}_{X_i}) = \Gamma(X, \mathcal{O}_X)$$

in the category of $\Gamma(Y, \mathcal{O}_Y)$-algebras. By [Monygham, 2018, Corollary 3.4] we deduce that $f_i(X_i)$ is constructible in Y for each $i \in I$. Proposition 2.1 implies that

$$f(X) = \bigcap_{i \in I} f_i(X_i)$$

This finishes the proof. □

Corollary 3.5 (Characterization of pro-constructible sets on qcqs schemes)**.** *Let X be a quasi-compact and quasi-separated scheme. Then the following are equivalent.*

- **(i)** *E is pro-constructible.*
- **(ii)** *E is an intersection constructible subsets in X.*
- **(iii)** *There exists an affine scheme Z and a morphism $f : Z \to X$ such that $E = f(Z)$.*

Proof. Assume that E is pro-constructible subset of X. Corollary 3.3 implies E is an intersection of constructible subsets of X. Thus **(i)** ⇒ **(ii)** is true.
If **(i)** holds, then Corollary 2.3 gives an affine scheme Z and a morphism $f : Z \to X$ such that $E = f(Z)$. This implies that **(ii)** ⇒ **(iii)**.
For the proof of **(iii)** ⇒ **(i)** note that such f is quasi-compact (this follows because X is quasi-separated) and hence the implication follows from Theorem 3.4. □

4. Open and closed subsets of schemes

Definition 4.1. Let X be a topological space and let η be a point of X. Every point x in $\mathbf{cl}(\{\eta\})$ is called *a specialization of* η. If x is a specialization of η, then η is called *a generization of* x.

Definition 4.2. Let X be a topological space and Z be its subset. We say that Z is *closed under specialization (generization)* if Z contains all specializations (generizations) of its points.

Theorem 4.3. *Let X be a scheme and $f : Z \to X$ be a quasi-compact morphism of schemes. Then the following are equivalent.*

- **(i)** *$f(Z)$ is a closed subset of X.*
- **(ii)** *$f(Z)$ is closed under specialization.*

For the proof we need the following result.

Lemma 4.3.1. *Let $f : A \to B$ be a morphism of rings. If the image of* $\mathrm{Spec}\, f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ *is closed under specialization, then it is closed.*

Proof of the lemma. The image of $\mathrm{Spec}\, f$ is equal to the image of its factor $\mathrm{Spec}\, B \to \mathrm{Spec}\,(A/\ker(f))$. Therefore, we may additionally assume that f is injective. We prove that under this extra assumption $\mathrm{Spec}\, f$ is surjective. For this assume that $\mathfrak{p} \in \mathrm{Spec}\, A$ is a prime ideal. Then f induces an injective map $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$. Thus $B_{\mathfrak{p}}$ is nonzero. Hence $\operatorname{Spec} B_{\mathfrak{p}}$ is nonempty. We also have a commutative square

$$\begin{tikzcd} \varnothing \neq \operatorname{Spec} B_{\mathfrak{p}} \arrow[r] \arrow[d, hook] & \operatorname{Spec} A_{\mathfrak{p}} \arrow[d, hook] \\ \operatorname{Spec} B \arrow[r, "\operatorname{Spec} f"'] & \operatorname{Spec} A \end{tikzcd}$$

of topological spaces. This imply that there exists a prime ideal $\mathfrak{q} \in \operatorname{Spec} B$ such that $\mathfrak{p}$ is a specialization of $(\operatorname{Spec} f)(\mathfrak{q})$. Since the image of $\operatorname{Spec} f$ is closed under specialization, we derive that $\mathfrak{p}$ is contained in the image of $\operatorname{Spec} f$. $\square$

Proof. Closed subsets are closed under specialization. Hence **(i)** $\Rightarrow$ **(ii)** holds.
Now assume **(ii)** i.e. $f(Z)$ is closed under specialization. Fix open affine U in X. Since f is quasi-compact, we derive that $f^{-1}(U)$ is quasi-compact. Write $f^{-1}(U) = \bigcup_{j=1}^{m} W_j$ for open affine subsets W_j of $f^{-1}(U)$. Let $W = \coprod_{j=1}^{m} W_j$ and consider a morphism $g : W \to U$ given as the composition

$$\coprod_{j=1}^{m} W_j \longrightarrow f^{-1}(U) \longrightarrow U$$

where the first arrow is induced by inclusions $\{W_j \hookrightarrow f^{-1}(U)\}_{1 \leq j \leq m}$ and the second is the restriction of f. Note that $g(W) = f(Z) \cap U$ and hence $g(W)$ is closed under specialization in U. By Lemma 4.3.1 we deduce that $g(W)$ is closed in U and hence $f(X) \cap U$ is closed in U. Since this holds for every open affine U in X, we infer that $f(X)$ is closed in X. This proves **(i)**. $\square$

Corollary 4.4. *Let X be a scheme and E be its subset. Then the following are equivalent.*

- **(i)** *E is a closed subset of X.*
- **(ii)** *E is pro-constructible and closed under specialization.*

Proof. Suppose that E is closed subset of X and let U be an open affine subset of X. Then $E \cap U$ is the image of some closed affine subscheme of U. By Corollary 3.5 we deduce that $E \cap U$ is an intersection of constructible subsets of U. Thus E is pro-constructible. Since E is closed, it is also closed under specialization. Hence **(i)** $\Rightarrow$ **(ii)**.
Assume that **(ii)** holds. Then for every open affine subset U of X set $E \cap U$ is pro-constructible and closed under specialization in U. By Corollary 3.5 and Theorem 4.3 we derive that $E \cap U$ is closed subset of U. Since U is arbitrary, we derive that E is closed. This is **(i)**. $\square$

Definition 4.5. Let X be a topological space. A subset E of X is called *ind-constructible in X* if $X \setminus E$ is pro-constructible in X.

Corollary 4.6. *Let X be a scheme and E be its subset. Then the following are equivalent.*

- **(i)** *E is an open subset of X.*
- **(ii)** *E is ind-constructible and closed under generization.*

Proof. This is a consequence of Corollary 4.4. Details are left to the reader. $\square$

References

[Monygham, 2018] Monygham (2018). Constructible and locally constructible sets. *github repository: "Monygham/Pedo-mellon-a-minno"*.