CONCENTRATION INEQUALITIES

1. Introduction

Concentration inequalities estimate deviation of random variable from its mean value or variance. In this short notes we prove Azuma-Heffding inequality.

2. Azuma-Hoeffding inequality

Theorem 2.1 (Azuma-Hoeffding inequality)**.** *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of independent random variables on a probability space $(\Omega, \mathcal{F}, P)$. Assume that for every $n \in \mathbb{N}$ there exists real number c_n such that*

$$|X_n| \leq c_n$$

almost surely. Then

$$P\left(X_0 + X_1 + ... + X_n \geq \lambda\right) \leq \exp\left(\frac{-\lambda^2}{2\cdot\left(c_0^2 + c_1^2 + ... + c_n^2\right)}\right)$$

for every $\lambda \geq 0$.

Lemma 2.1.1. *Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that $\mathcal{G} \subseteq \mathcal{F}$ is a σ-subalgebra and c is a positive real number. Assume that $\mathbb{E}[X\,|\,\mathcal{G}] = 0$ and $|X| \leq c$ almost surely. Then for every convex function $\phi : \mathbb{R} \to \mathbb{R}$ we have*

$$\mathbb{E}[\phi(X)\,|\,\mathcal{G}] \leq \frac{\phi(-c) + \phi(c)}{2}$$

Proof of the lemma. Since ϕ is convex, we derive that

$$\phi(x) \leq \frac{c - x}{2c}\cdot\phi(-c) + \frac{c + x}{2c}\cdot\phi(c)$$

for every $x \in [-c, c]$. Hence the inequality

$$\phi(X) \leq \frac{c - X}{2c}\cdot\phi(-c) + \frac{c + X}{2c}\cdot\phi(c)$$

holds almost surely. Applying conditional expectation and using the fact that it is a monotone operator, we deduce that

$$\mathbb{E}[\phi(X)\,|\,\mathcal{G}] \leq \frac{c - \mathbb{E}[X\,|\,\mathcal{G}]}{2c}\cdot\phi(-c) + \frac{c + \mathbb{E}[X\,|\,\mathcal{G}]}{2c}\cdot\phi(c) = \frac{\phi(-c) + \phi(c)}{2}$$

$\square$

Lemma 2.1.2. *Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that $\mathcal{G} \subseteq \mathcal{F}$ is a σ-subalgebra and c is a positive real number. Assume that $\mathbb{E}[X\,|\,\mathcal{G}] = 0$ and $|X| \leq c$ almost surely. Then for every $\theta > 0$ we have*

$$\mathbb{E}[e^{\theta X}\,|\,\mathcal{G}] \leq \exp\left(\frac{\theta^2\cdot c^2}{2}\right)$$

Proof of the lemma. Note that

$$\mathbb{E}[e^{\theta X} \,|\, \mathcal{G}] \leq \frac{e^{-\theta \cdot c} + e^{\theta \cdot c}}{2} = \cosh(\theta \cdot c)$$

by Lemma 2.1.1. Next observe that

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{1}{2} \cdot \left(\sum_{n=0}^{+\infty} \frac{x^n}{n!} + \sum_{n=0}^{+\infty} \frac{(-1)^n x^n}{n!} \right) = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \leq \sum_{n=0}^{+\infty} \frac{x^{2n}}{2^n \cdot n!} = \exp\left(\frac{x^2}{2} \right)$$

for every $x \in \mathbb{R}$. Hence

$$\mathbb{E}[e^{\theta X} \,|\, \mathcal{G}] \leq \exp\left(\frac{\theta^2 \cdot c^2}{2} \right)$$

□

Proof of the theorem. Suppose that $\lambda \geq 0$ and $\theta > 0$. We have

$$P\left(X_0 + X_1 + \ldots + X_n \geq \lambda\right) = P\left(e^{\theta \cdot (X_0 + X_1 + \ldots + X_n)} \geq e^{\theta \cdot \lambda} \right)$$

Now applying Markov inequality, we derive that

$$P\left(e^{\theta \cdot (X_0 + X_1 + \ldots + X_n)} \geq e^{\theta \cdot \lambda} \right) \leq e^{-\theta \cdot \lambda} \cdot \mathbb{E}\left[e^{\theta \cdot (X_0 + X_1 + \ldots + X_n)} \right] = e^{-\theta \cdot \lambda} \cdot \mathbb{E}\left[e^{\theta \cdot X_0 + \theta \cdot X_1 + \ldots + \theta \cdot X_n} \right]$$

Now let $\mathcal{F}_{n-1}$ be a σ-algebra generated by random variables $X_0, \ldots, X_{n-1}$. According to the standard properties of conditional expectation we have

$$e^{-\theta \cdot \lambda} \cdot \mathbb{E}\left[e^{\theta \cdot X_0 + \theta \cdot X_1 + \ldots + \theta \cdot X_n} \right] = e^{-\theta \cdot \lambda} \cdot \mathbb{E}\left[\mathbb{E}\left[e^{\theta \cdot X_0 + \theta \cdot X_1 + \ldots + \theta \cdot X_{n-1}} \cdot e^{\theta \cdot X_n} \,|\, \mathcal{F}_{n-1} \right] \right] =$$

$$= e^{-\theta \cdot \lambda} \cdot \mathbb{E}\left[\mathbb{E}\left[e^{\theta \cdot X_0 + \theta \cdot X_1 + \ldots + \theta \cdot X_{n-1}} \,|\, \mathcal{F}_{n-1} \right] \cdot \mathbb{E}\left[e^{\theta \cdot X_n} \,|\, \mathcal{F}_{n-1} \right] \right]$$

Since $\mathbb{E}\left[X_n \,|\, \mathcal{F}_{n-1}\right] = 0$ and $|X_n| \leq c_n$ almost surely, by Lemma 2.1.2 we have

$$\mathbb{E}\left[e^{\theta \cdot X_n} \,|\, \mathcal{F}_{n-1} \right] \leq \exp\left(\frac{\theta^2 \cdot c_n^2}{2} \right)$$

and thus we deduce that

$$e^{-\theta \cdot \lambda} \cdot \mathbb{E}\left[e^{\theta \cdot X_0 + \theta \cdot X_1 + \ldots + \theta \cdot X_n} \right] = e^{-\theta \cdot \lambda} \cdot \mathbb{E}\left[\mathbb{E}\left[e^{\theta \cdot X_0 + \theta \cdot X_1 + \ldots + \theta \cdot X_{n-1}} \,|\, \mathcal{F}_{n-1} \right] \cdot \mathbb{E}\left[e^{\theta \cdot X_n} \,|\, \mathcal{F}_{n-1} \right] \right] \leq$$

$$\leq e^{-\theta \cdot \lambda} \cdot \mathbb{E}\left[\mathbb{E}\left[e^{\theta \cdot X_0 + \theta \cdot X_1 + \ldots + \theta \cdot X_{n-1}} \,|\, \mathcal{F}_{n-1} \right] \cdot \exp\left(\frac{\theta^2 \cdot c_n^2}{2} \right) \right] = e^{-\theta \cdot \lambda} \cdot \mathbb{E}\left[e^{\theta \cdot X_0 + \theta \cdot X_1 + \ldots + \theta \cdot X_{n-1}} \right] \cdot \exp\left(\frac{\theta^2 \cdot c_n^2}{2} \right)$$

for every $n \in \mathbb{N}$. Hence by easy induction

$$e^{-\theta \cdot \lambda} \cdot \mathbb{E}\left[e^{\theta \cdot X_0 + \theta \cdot X_1 + \ldots + \theta \cdot X_n} \right] \leq \exp\left(-\theta \cdot \lambda\right) \cdot \exp\left(\frac{\theta^2 \cdot (c_0^2 + c_1^2 + \ldots + c_n^2)}{2} \right)$$

Therefore, we deduce that inequality

$$P\left(X_0 + X_1 + \ldots + X_n \geq \lambda\right) \leq \exp\left(\frac{c_0^2 + c_1^2 + \ldots + c_n^2}{2} \cdot \theta \cdot \left(\theta - \frac{2 \cdot \lambda}{c_0^2 + c_1^2 + \ldots + c_n^2} \right) \right)$$

holds for every $\theta > 0$. The right hand side of the inequality is continuous for every $\theta \in [0, +\infty)$ and attains global minimum for

$$\theta = \frac{\lambda}{c_0^2 + c_1^2 + \ldots + c_n^2} \in [0, +\infty)$$

Hence finally

$$P\left(X_0 + X_1 + \ldots + X_n \geq \lambda\right) \leq \exp\left(\frac{-\lambda^2}{2 \cdot \left(c_0^2 + c_1^2 + \ldots + c_n^2\right)} \right)$$

□

Corollary 2.2. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of independent random variables in probability space $(\Omega, \mathcal{F}, P)$. Assume that for every $n \in \mathbb{N}$ there exists real number c_n such that*

$$|X_n| \leq c_n$$

almost surely. Then

$$P\left(|X_0 + X_1 + ... + X_n| \geq \lambda\right) \leq 2 \cdot \exp\left(\frac{-\lambda^2}{2\cdot\left(c_0^2 + c_1^2 + ... + c_n^2\right)}\right)$$

for every $\lambda \geq 0$.

Proof. Fix $\lambda \geq 0$. According to Theorem 2.1 we have

$$P\left(X_0 + X_1 + ... + X_n \geq \lambda\right) \leq \exp\left(\frac{-\lambda^2}{2\cdot\left(c_0^2 + c_1^2 + ... + c_n^2\right)}\right)$$

Applying Theorem 2.1 to a sequence $\{-X_n\}_{n\in\mathbb{N}}$ we derive

$$P\left(X_0 + X_1 + ... + X_n \leq -\lambda\right) \leq \exp\left(\frac{-\lambda^2}{2\cdot\left(c_0^2 + c_1^2 + ... + c_n^2\right)}\right)$$

Merging these two inequalities we obtain the assertion. □

Corollary 2.3 (Hoeffding inequality)**.** *Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables in probability space $(\Omega, \mathcal{F}, P)$. Assume that there exists a positive real number c such that*

$$|X_1| \leq c$$

almost surely and let m be the expected value of X_1. Then

$$P\left(\left|\frac{X_1 + ... + X_n}{n} - m\right| \geq \lambda\right) \leq 2 \cdot \exp\left(\frac{-\lambda^2 \cdot n}{2\cdot(c + |m|)^2}\right)$$

for every $\lambda \geq 0$.

Proof. Write $Z_n = X_n - \mathbb{E}[X_n] = X_n - m$. Then $\{Z_n\}_{n\geq 1}$ are independent and $|Z_n| \leq c + |m|$. Fix $\lambda \geq 0$. Then applying Corollary 2.2 we derive that

$$P\left(\left|\frac{X_1 + ... + X_n}{n} - m\right| \geq \lambda\right) = P\left(|Z_1 + ... + Z_n| \geq n\cdot\lambda\right) \leq 2\cdot\exp\left(\frac{-\lambda^2 \cdot n^2}{2\cdot n \cdot (c + |m|)^2}\right) = 2\cdot\exp\left(\frac{-\lambda^2 \cdot n}{2\cdot(c + |m|)^2}\right)$$

□