MONOID k-FUNCTORS AND THEIR REPRESENTATIONS

1. Introduction and notation

In these notes we study algebraic structures in the category of k-functors with special emphasis on monoid objects.
Throughout these notes k is a fixed commutative ring and $\mathbf{Alg}_k$ denote the category of commutative k-algebras. If A, B are k-algebras, then we denote by $\mathrm{Mor}_k(A, B)$ the set of all morphisms $A \to B$ of k-algebras. Similarly if X, Y are k-schemes (i.e. schemes together with morphism to $\mathrm{Spec}\, k$), then we denote by $\mathrm{Mor}_k(X, Y)$ the set of all morphisms $X \to Y$ of k-schemes (morphisms of schemes that preserve structure morphisms to $\mathrm{Spec}\, k$). If M is a monoid, then we denote by M^* the group of units of M. If R is a ring, then we denote by $R^\times$ its multplicative monoid. Let A be a k-algebra and let V be an A-module and v be an element of V. Then for A-algebra B we denote by v_B the element $1 \otimes v$ of $B \otimes_A V$. If V is an A-module, then we denote $\mathrm{Hom}_A(V, A)$ by $V^\vee$. Thus we have a contravariant functor

$$(-)^\vee : \mathbf{Mod}(A)^{\mathrm{op}} \to \mathbf{Mod}(A)$$

2. Algebraic structures in the category of k-functors

In the sequel we assume that the reader is familiar with notions of a monoid, group etc. in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 2-5].

Definition 2.1. *A monoid (group, abelian group, ring) k-functor* is a monoid (group, abelian group, ring) object in the category of k-functors.

Example 2.2. Let $\mathfrak{X}$ be a k-functor such that $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ exists. Then $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ is a monoid k-functor with respect to composition of morphisms.

Example 2.3. Let $\mathfrak{G}$ be a monoid k-functor. Then we denote by $\mathfrak{G}^*$ the k-subfunctor of $\mathfrak{G}$ defined by

$$\mathfrak{G}^*(A) = \mathfrak{G}(A)^*$$

for every k-algebra A. We call $\mathfrak{G}^*$ *the unit group k-functor of* $\mathfrak{G}$.

Example 2.4. Basic example of a ring k-functor is a k-functor $\mathfrak{K}$ given by

$$\mathfrak{K}(A) = k,\ \mathfrak{K}(f) = 1_k$$

for any k-algebra A and morphism f of k-algebras. It can be described as a constant k-functor ([ML98, page 67]) corresponding to k.

Definition 2.5. Let $\mathfrak{R}$ be a ring k-functor. Then we denote by $\mathfrak{R}^\times$ the k-subfunctor of $\mathfrak{R}$ defined by

$$\mathfrak{R}^\times(A) = \mathfrak{R}(A)^\times$$

for every k-algebra A. We call $\mathfrak{R}^\times$ *the multiplicative monoid k-functor of* $\mathfrak{R}$.

Definition 2.6. Let $\mathfrak{A}$ be a commutative ring k-functor. *An $\mathfrak{A}$-algebra* is an $\mathfrak{A}$-algebra object in the category of k-functors.

3. Global regular functions on a k-functor

Recall the ring k-functor $\mathfrak{K}$ from Example 2.4. Note that a $\mathfrak{K}$-algebra $\mathfrak{A}$ can be viewed as a functor $\mathfrak{A} : \mathbf{Alg}_k \to \mathbf{Alg}_k$.

Definition 3.1. The $\mathfrak{K}$-algebra $\mathfrak{O}_k$ represented by the identity functor on $\mathbf{Alg}_k$ is called *the structure $\mathfrak{K}$-algebra*.

Let $|-| : \mathbf{Alg}_k \to \mathbf{Set}$ be the forgetful k-functor. Note that $|-|$ is the underlying k-functor of $\mathfrak{K}$-algebra $\mathfrak{O}_k$. Recall that the affine line $\mathbb{A}^1_k$ is an affine k-scheme having k-algebra of polynomials with one variable as a k-algebra of regular functions.

Fact 3.2. *Let $|-| : \mathbf{Alg}_k \to \mathbf{Set}$ be the forgetful k-functor. Then we have natural isomorphism*

$$\mathfrak{P}_{\mathbb{A}^1_k} \cong |-|$$

Proof. Let B be a k-algebra. We have the following chain of identifications

$$\mathfrak{P}_{\mathbb{A}^1_k}(B) = \mathrm{Mor}_k\left(\mathrm{Spec}\, B, \mathbb{A}^1_k\right) = \mathrm{Mor}_k\left(\mathrm{Spec}\, B, \mathrm{Spec}\, k[x]\right) = \mathrm{Mor}_k\left(k[x], B\right) = |B|$$

natural in B. □

In particular, since $|-|$ carries the structure $\mathfrak{K}$-algebra $\mathfrak{O}_k$, we derive that $\mathfrak{P}_{\mathbb{A}^1_k}$ admits a structure of $\mathfrak{K}$-algebra isomorphic to $\mathfrak{O}_k$.
No we introduce regular functions on k-functors.

Definition 3.3. Let $\mathfrak{X}$ be a k-functor and assume that $\mathcal{M}\mathrm{or}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$ is a set. Then $\mathrm{Mor}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$ is a k-algebra with respect to the structure induced by $\mathfrak{O}_k$. We call this k-algebra *the k-algebra of global regular functions on $\mathfrak{X}$*. Its elements are called *global regular functions on $\mathfrak{X}$*.

Definition 3.4. Let $\mathfrak{X}$ be a k-functor. Suppose that A is a k-algebra, $x \in \mathfrak{X}(A)$ and $f \in \mathrm{Mor}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$. The element $f(x) \in A$ is called *the value of f on a point x*.

For given k-functor $\mathfrak{X}$ we describe k-algebra operations on $\mathrm{Mor}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$ in terms of values of its elements on points of $\mathfrak{X}$. For this consider $\alpha \in k$ and $f, g_1 \in \mathrm{Mor}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$. We have formulas

$$(f+g)(x) = f(x) + g(x),\ (f \cdot g)(x) = f(x) \cdot g(x),\ (\alpha \cdot f)(x) = \alpha \cdot f(x)$$

in which right hand side are k-algebra operations in A.

Example 3.5. Let $\mathfrak{X}$ be a k-functor and assume that $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{O}_k)$ exists. Fix k-algebra A. Note that $\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{O}_A)$ is an A-algebra of global regular functions on $\mathfrak{X}_A$. Moreover, if B is an A-algebra, then

$$\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{O}_A) \ni f \mapsto f_B \in \mathrm{Mor}_B(\mathfrak{X}_B, \mathfrak{O}_B)$$

is a morphism of A-algebras. This implies that $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{O}_k)$ admits a canonical structure of an $\mathfrak{O}_k$-algebra k-functor.

Proposition 3.6. *Let X be a k-scheme. Then $\mathcal{M}\mathrm{or}_k\left(\mathfrak{P}_X, \mathfrak{O}_k\right)$ exists and*

$$\mathcal{M}\mathrm{or}_k\left(\mathfrak{P}_X, \mathfrak{O}_k\right)(A) = \Gamma\left(\mathrm{Spec}\, A \times X, \mathcal{O}_{\mathrm{Spec}\, A \times X}\right)$$

for every k-algebra A. In particular, if X is affine k-scheme, then we have an isomorphism

$$\mathcal{M}\mathrm{or}_k\left(\mathfrak{P}_X, \mathfrak{O}_k\right) = A \otimes_k \Gamma(X, \mathcal{O}_X)$$

natural in k-algebra A.

Proof. Fact 3.2 and the fact that morphisms $\mathrm{Spec}\, A \times X \to \mathbb{A}^1_A$ of A-schemes corresponds to regular functions on X_A imply that we have an identification

$$\mathrm{Mor}_A\left(\left(\mathfrak{P}_X\right)_A, \mathfrak{O}_A\right) = \Gamma\left(\mathrm{Spec}\, A \times X, \mathcal{O}_{\mathrm{Spec}\, A \times X}\right)$$

natural in k-algebra A. This proves the first part of the statement. Moreover, if X is affine over k, then there is another identification

$$\mathrm{Mor}_A\left(\operatorname{Spec} A \times X, \mathbb{A}^1_A\right) = A \otimes_k \Gamma(X, \mathcal{O}_X)$$

natural in A. Combining two isomorphisms above, we derive that the second assertion. □

4. Internal hom and product of k-functors

We denote by $\mathbf{1}$ a k-functor that assigns to every k-algebra a set with one element. Then for every k-algebra A the restriction $\mathbf{1}_A$ is a terminal object in the category of A-functors.

Fact 4.1. *Let $\mathfrak{X}$ be a k-functor. Suppose A is a k-algebra and $x \in \mathfrak{X}(A)$. Then x determines a morphism $\mathbf{1}_A \to \mathfrak{X}_A$ that for every A-algebra B with structural morphism $f : A \to B$ sends a unique element of $\mathbf{1}_A(B)$ to $\mathfrak{X}(f)(x) \in \mathfrak{X}_A(B)$. This gives rise to a bijection*

$$\mathfrak{X}(A) \cong \mathrm{Mor}_A\left(\mathbf{1}_A, \mathfrak{X}_A\right)$$

Proof. Left to the reader as an exercise. □

The discussion below is partially an application of the main result in [Mon19a, section 6]. For reader's convenience we make our presentation self-contained.

Definition 4.2. Let $\mathfrak{U}, \mathfrak{X}, \mathfrak{Y}$ be k-functors and let $\sigma : \mathfrak{U} \times \mathfrak{X} \to \mathfrak{Y}$ be a morphism of k-functors. Fix $z \in \mathfrak{U}(A)$ for some k-algebra A. We denote by $i_z : \mathbf{1}_A \to \mathfrak{U}_A$ the morphism of A-functors corresponding to z by Fact 4.1. Since $\mathbf{1}_A$ is terminal A-functor, a morphism $\sigma_A \cdot \left(i_z \times 1_{\mathfrak{X}_A}\right)$ is isomorphic to a morphism $\sigma_z : \mathfrak{X}_A \to \mathfrak{Y}_A$ of A-functors. We call σ_z *the slice of σ along z*.

Definition 4.3. Let $\mathfrak{X}, \mathfrak{Y}$ be k-functors. Let $\mathfrak{J}$ be a k-functor such that $\mathfrak{J}(A)$ is a subset of a class $\mathrm{Mor}_A\left(\mathfrak{X}_A, \mathfrak{Y}_A\right)$ for every k-algebra A. Assume that for every morphism $f : A \to B$ of k-algebras and every $\sigma \in \mathfrak{J}(A)$ we have

$$\mathfrak{J}(f)(\sigma) = \sigma_B$$

where $\sigma_B \in \mathrm{Mor}_B\left(\mathfrak{X}_B, \mathfrak{Y}_B\right)$ is the restriction of σ along f. Then we call $\mathfrak{J}$ *a k-subfunctor of internal hom of $\mathfrak{X}$ and $\mathfrak{Y}$*.

Definition 4.4. Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{U}$ be k-functors and let $\sigma : \mathfrak{U} \times \mathfrak{X} \to \mathfrak{Y}$ be a morphism of k-functors. Suppose that $\mathfrak{J}$ is a k-subfunctor of internal hom of $\mathfrak{X}$ and $\mathfrak{Y}$. Assume that $\sigma_z : \mathfrak{X}_A \to \mathfrak{Y}_A$ is contained in $\mathfrak{J}(A)$ for every k-algebra A and $z \in \mathfrak{U}(A)$. Then we call σ *a family of $\mathfrak{J}$-morphisms parametrized by $\mathfrak{U}$*.

Let $\mathfrak{J}$ be a k-subfunctor of internal hom of $\mathfrak{X}$ and $\mathfrak{Y}$. Assume that $\sigma : \mathfrak{U} \times \mathfrak{X} \to \mathfrak{Y}$ is a $\mathfrak{J}$-family of morphism parametrized by $\mathfrak{U}$. Then the family of maps

$$\mathfrak{U}(A) \ni z \mapsto \sigma_z \in \mathfrak{J}(A)$$

gives rise to a morphism $\tau : \mathfrak{U} \to \mathfrak{J}$ of k-functors. Indeed, for a morphism $f : A \to B$ of k-algebras and $z \in \mathfrak{U}(A)$ we have

$$\sigma_B \cdot \left(i_{\mathfrak{U}(f)(z)} \times 1_{\mathfrak{X}_B}\right) = \left(\sigma_A \cdot \left(i_z \times 1_{\mathfrak{X}_A}\right)\right)_B$$

and hence $\sigma_{\mathfrak{U}(f)(z)} = (\sigma_z)_B$. This gives rise to a map Φ of classes

$$\left\{\text{families } \mathfrak{U} \times \mathfrak{X} \to \mathfrak{Y} \text{ of } \mathfrak{J}\text{-morphisms parametrized by } \mathfrak{U}\right\} \ni \sigma \mapsto \tau \in \mathrm{Mor}_k\left(\mathfrak{U}, \mathfrak{J}\right)$$

Consider next a morphism $\tau : \mathfrak{U} \to \mathfrak{J}$ of k-functors and define $\sigma : \mathfrak{U} \times \mathfrak{X} \to \mathfrak{Y}$ by formula $\sigma^A(z, x) = \left(\tau^A(z)\right)^A(x)$ for every k-algebra A and points $z \in \mathfrak{U}(A)$, $x \in \mathfrak{X}(A)$. Let $f : A \to B$ be a morphism of k-algebras. Then

$$\sigma^B\left(\mathfrak{U}(f)(z), \mathfrak{X}(f)(x)\right) = \left(\tau^B\left(\mathfrak{U}(f)(z)\right)\right)^B\left(\mathfrak{X}(f)(x)\right) = \left(\left(\tau^A(z)\right)_B\right)^B\left(\mathfrak{X}(f)(x)\right) =$$

$$= \left(\tau^A(z)\right)^B (\mathfrak{X}(f)(x)) = \mathfrak{Y}(f)\left(\left(\tau^A(z)\right)^A(x)\right) = \mathfrak{Y}(f)\left(\sigma^A(z,x)\right)$$

Thus $\sigma : \mathfrak{U} \times \mathfrak{X} \to \mathfrak{Y}$ is a morphism of k-functors. For every k-algebra A and $z \in \mathfrak{U}(A)$ we have $\sigma_z = \tau^A(z)$. Indeed, let $f : A \to B$ be a morphism of k-algebras and x be an element in $\mathfrak{X}(B)$ then we have

$$(\sigma_z)^B(x) = \sigma^B(\mathfrak{U}(f)(z), x) = \left(\tau^B(\mathfrak{U}(f)(z))\right)^B(x) = \left(\left(\tau^A(z)\right)_B\right)^B(x) = \left(\tau^A(z)\right)^B(x)$$

Hence σ is a family of $\mathfrak{J}$-morphisms parametrized by $\mathfrak{U}$. This gives rise to a map Ψ of classes

$$\mathrm{Mor}_k(\mathfrak{U},\mathfrak{J}) \ni \tau \mapsto \sigma \in \left\{\text{families } \mathfrak{U}\times\mathfrak{X}\to\mathfrak{Y} \text{ of } \mathfrak{J}\text{-morphisms parametrized by } \mathfrak{U}\right\}$$

Now we have the following result, which is an instance [Mon19a, Theorem 6.3]. To make presentation self-contained we give a complete proof.

Theorem 4.5. *Let $\mathfrak{X}$, $\mathfrak{Y}$ be k-functors and let $\mathfrak{J}$ be a k-subfunctor of internal hom of $\mathfrak{X}$ and $\mathfrak{Y}$. Then maps*

$$\Phi : \left\{\textit{families } \mathfrak{U}\times\mathfrak{X}\to\mathfrak{Y} \textit{ of } \mathfrak{J}\textit{-morphisms parametrized by } \mathfrak{U}\right\} \to \mathrm{Mor}_k(\mathfrak{U},\mathfrak{J})$$

and

$$\Psi : \mathrm{Mor}_k(\mathfrak{U},\mathfrak{J}) \to \left\{\textit{families } \mathfrak{U}\times\mathfrak{X}\to\mathfrak{Y} \textit{ of } \mathfrak{J}\textit{-morphisms parametrized by } \mathfrak{U}\right\}$$

are mutually inverse bijections.

Proof. Pick a morphism $\tau : \mathfrak{U} \to \mathfrak{J}$ of k-functors. Let A be a k-algebra and $z \in \mathfrak{U}(A)$. In the discussion preceding the statement we showed that $\Psi(\tau)_z = \tau^A(z)$. Thus

$$(\Phi(\Psi(\tau)))^A(z) = \Psi(\tau)_z = \tau^A(z)$$

and hence $\Phi \cdot \Psi$ is the identity.
Pick a family of $\mathfrak{J}$-morphism $\sigma : \mathfrak{U}\times\mathfrak{X}\to\mathfrak{Y}$ parametrized by $\mathfrak{U}$. Let A be a k-algebra and $z \in \mathfrak{U}(A)$, $x \in \mathfrak{X}(A)$ be points. Then

$$(\Psi(\Phi(\sigma)))^A(z,x) = \left(\Phi(\sigma)^A(z)\right)^A(x) = \sigma_z^A(x) = \sigma^A(z,x)$$

Thus $\Psi \cdot \Phi$ is the identity map. □

Now we formulate some consequences of Theorem 4.5.

Corollary 4.6. *Let $\mathfrak{X}$, $\mathfrak{Y}$ be k-functors. Assume that for every k-algebra A the class $\mathrm{Mor}_A(\mathfrak{X}_A,\mathfrak{Y}_A)$ is a set. Then there is a bijection*

$$\mathrm{Mor}_k(\mathfrak{U}\times\mathfrak{X},\mathfrak{Y}) \to \mathrm{Mor}_k(\mathfrak{U},\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{Y}))$$

of classes.

Definition 4.7. Let $\mathfrak{X},\mathfrak{Y}$ be k-functors. If $\mathrm{Iso}_A(\mathfrak{X}_A,\mathfrak{Y}_A)$ is a set for every k-algebra A, then we define a k-subfunctor $\mathcal{I}\mathrm{so}_k(\mathfrak{X},\mathfrak{Y})$ of $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{Y})$ by

$$\mathcal{I}\mathrm{so}_k(\mathfrak{X},\mathfrak{Y})(A) = \mathrm{Iso}_A(\mathfrak{X}_A,\mathfrak{Y}_A)$$

for every k-algebra A. We call $\mathcal{I}\mathrm{so}_k(\mathfrak{X},\mathfrak{Y})$ *the k-functor of isomorphism.*

Definition 4.8. Let $\mathfrak{X},\mathfrak{Y},\mathfrak{U}$ be k-functors and let $\sigma : \mathfrak{U}\times\mathfrak{X}\to\mathfrak{Y}$ be a morphism of k-functors. Assume that $\sigma_z : \mathfrak{X}_A \to \mathfrak{Y}_A$ is an isomorphism of A-functors for every k-algebra A. Then we call *σ a family of isomorphisms parametrized by $\mathfrak{U}$.*

Corollary 4.9. *Let $\mathfrak{U},\mathfrak{X},\mathfrak{Y}$ be k-functors and suppose that for every k-algebra A the class $\mathrm{Iso}_A(\mathfrak{X}_A,\mathfrak{Y}_A)$ is a set. The the following map*

$$\left\{\textit{families } \mathfrak{U}\times\mathfrak{X}\to\mathfrak{Y} \textit{ of isomorphism parametrized by } \mathfrak{U}\right\} \to \mathrm{Mor}_k(\mathfrak{U},\mathcal{I}\mathrm{so}_k(\mathfrak{X},\mathfrak{Y}))$$

is a bijection of classes.

5. ACTIONS OF MONOID k-FUNCTORS

In this section we assume that the reader is familiar with notion of an action of a monoid object in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 5].
Let $\mathfrak{G}$ be a monoid k-functor and $\mathfrak{X}$ be a k-functor together with an action $\alpha : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X}$. Next assume that k-functor $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ exists. By Example 2.2 it is a monoid k-functor. We define a morphism $\rho : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ of k-functors by formula $\rho(g) = \alpha_g$. Note that by discussion preceding Theorem 4.5, we deduce that ρ is a well defined morphism of k-functors. We show now that ρ is a morphism of monoids. For this pick k-algebra A and $g_1, g_2 \in \mathfrak{G}(A)$. Since α is an action, we deduce that $\alpha_{g_1 \cdot g_2} = \alpha_{g_1} \cdot \alpha_{g_2}$ and hence also

$$\rho(g_1 \cdot g_2) = \alpha_{g_1 \cdot g_2} = \alpha_{g_1} \cdot \alpha_{g_2} = \rho(g_1) \cdot \rho(g_2)$$

Therefore, ρ is a morphism of monoid k-functors. This shows how to construct a morphism of monoid k-functors ρ from an action α of $\mathfrak{G}$.

Theorem 5.1. *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{X}$ be a k-functor such that $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ exists. Suppose that*

$$\left\{ \textit{actions of } \mathfrak{G} \textit{ on } \mathfrak{X} \right\} \longrightarrow \left\{ \textit{Morphisms } \rho : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X}) \textit{ of monoid } k\textit{-functors} \right\}$$

is a map of classes described above. Then it is bijection.

Proof. Our goal is to construct the inverse of the map. Substitute $\mathfrak{I} = \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ in Theorem 4.5. Consider maps

$$\Phi : \left\{ \text{families } \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X} \text{ of morphisms} \right\} \to \mathrm{Mor}_k\left(\mathfrak{G}, \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})\right)$$

and

$$\Psi : \mathrm{Mor}_k\left(\mathfrak{G}, \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})\right) \to \left\{ \text{families } \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X} \text{ of morphisms} \right\}$$

in that Theorem. Then the map in the statement above is the restriction of Φ to $\mathfrak{G}$-actions on $\mathfrak{X}$ on the right and morphisms $\mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ of monoid k-functors on the left. Since by Theorem 4.5 maps Φ and Ψ are mutually inverse, it suffices to check that Ψ sends a morphism $\rho : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ of monoids to an action of $\mathfrak{G}$ on $\mathfrak{X}$. For this denote $\Psi(\rho)$ by α. Consider k-algebra A and A-points $g_1, g_2 \in \mathfrak{G}(A)$, $x \in \mathfrak{X}(A)$. Then

$$\alpha\left(g_1, \alpha(g_2, x)\right) = \rho(g_1)\left(\rho(g_2)(x)\right) = \left(\rho(g_1) \cdot \rho(g_2)\right)(x) = \rho\left(g_1 \cdot g_2\right)(x) = \alpha\left(g_1 \cdot g_2, x\right)$$

Therefore, α is an action of $\mathfrak{G}$ on $\mathfrak{X}$. ☐

Proposition 5.2. *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{X}_1, \mathfrak{X}_2$ be k-functors such that $\mathcal{M}\mathrm{or}_k(\mathfrak{X}_1, \mathfrak{X}_1), \mathcal{M}\mathrm{or}_k(\mathfrak{X}_2, \mathfrak{X}_2)$ exist. Suppose that $\alpha_1 : \mathfrak{G} \times \mathfrak{X}_1 \to \mathfrak{X}_1$, $\alpha_2 : \mathfrak{G} \times \mathfrak{X}_2 \to \mathfrak{X}_2$ are actions of $\mathfrak{G}$, respectively. Suppose that $\sigma : \mathfrak{X}_1 \to \mathfrak{X}_2$ is a morphism of k-functors. Then the following assertions are equivalent.*

(i) *The square*

$$\begin{tikzcd} \mathfrak{G} \times \mathfrak{X}_1 \arrow[r, "1_{\mathfrak{G}} \times \sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{G} \times \mathfrak{X}_2 \arrow[d, "\alpha_2"] \\ \mathfrak{X}_1 \arrow[r, "\sigma"'] & \mathfrak{X}_2 \end{tikzcd}$$

is commutative.

(ii) *For every k-algebra A and $g \in \mathfrak{G}(A)$ we have*

$$\sigma_A \cdot \rho_1(g) = \rho_2(g) \cdot \sigma_A$$

where $\rho_1 : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}_1, \mathfrak{X}_1)$ and $\rho_2 : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}_2, \mathfrak{X}_2)$ are morphism of monoid k-functors corresponding to α_1 and α_2, respectively.

Proof. Conditions expressed in **(i)** and **(ii)** are directly translatable to each other by virtue of the bijection in Theorem 5.1. □

Definition 5.3. Let $\mathfrak{G}$ be a monoid k-functor and let $(\mathfrak{X}_1, \alpha_1)$, $(\mathfrak{X}_2, \alpha_2)$ be k-functors with actions of $\mathfrak{G}$. Suppose that $\sigma : \mathfrak{X}_1 \to \mathfrak{X}_2$ is a morphism k-functors such that the square

$$\begin{tikzcd}
\mathfrak{G} \times \mathfrak{X}_1 \arrow[r, "1_{\mathfrak{G}} \times \sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{G} \times \mathfrak{X}_2 \arrow[d, "\alpha_2"] \\
\mathfrak{X}_1 \arrow[r, "\sigma"'] & \mathfrak{X}_2
\end{tikzcd}$$

is commutative. Then σ is called *an $\mathfrak{G}$-equivariant morphism*.

6. Modules over ring k-functors

Definition 6.1. Let $\mathfrak{R}$ be a ring k-functor. Suppose that $\mathfrak{M}$ is an abelian group k-functor and there exists a morphism $\mathfrak{R} \times \mathfrak{M} \to \mathfrak{M}$ of k-functors that for each k-algebra A makes $\mathfrak{M}(A)$ into an $\mathfrak{R}(A)$-module. Then we say that $\mathfrak{M}$ is *a module k-functor over $\mathfrak{R}$*.

Definition 6.2. Let $\mathfrak{R}$ be an ring k-functor and let $\mathfrak{M}_1, \mathfrak{M}_2$ be module k-functors over $\mathfrak{R}$. Suppose that $\sigma : \mathfrak{M}_1 \to \mathfrak{M}_2$ is a morphism of abelian group k-functors such that the diagram

$$\begin{tikzcd}
\mathfrak{R} \times \mathfrak{M}_1 \arrow[r, "1_{\mathfrak{R}} \times \sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{R} \times \mathfrak{M}_2 \arrow[d, "\alpha_2"] \\
\mathfrak{M}_1 \arrow[r, "\sigma"'] & \mathfrak{M}_2
\end{tikzcd}$$

is commutative, where $\alpha_i : \mathfrak{R} \times \mathfrak{M}_i \to \mathfrak{M}_i$ define $\mathfrak{R}$-module structure on $\mathfrak{M}_i$ for $i = 1, 2$. Then σ is *a morphism of modules over $\mathfrak{R}$*.

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be module k-functors over $\mathfrak{R}$. We denote by

$$\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{M}_1, \mathfrak{M}_2)$$

the class of all morphisms of modules $\mathfrak{M}_1 \to \mathfrak{M}_2$ over $\mathfrak{R}$. We denote the category of $\mathfrak{R}$-modules by $\mathbf{Mod}(\mathfrak{R})$.

Definition 6.3. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be module k-functors over $\mathfrak{R}$. Assume that $\mathrm{Hom}_{\mathfrak{R}_A}((\mathfrak{M}_1)_A, (\mathfrak{M}_2)_A)$ is a set for every k-algebra A. Then we define a k-subfunctor $\mathcal{H}om_{\mathfrak{R}}(\mathfrak{M}_1, \mathfrak{M}_2)$ of internal hom of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ by formula

$$\mathbf{Alg}_k \ni A \mapsto \mathrm{Hom}_{\mathfrak{R}_A}((\mathfrak{M}_1)_A, (\mathfrak{M}_2)_A) \in \mathbf{Set}$$

We call $\mathcal{H}om_{\mathfrak{R}}(\mathfrak{M}_1, \mathfrak{M}_2)$ *a k-functor of module morphisms of $\mathfrak{M}_1$ and $\mathfrak{M}_2$.*

If $\mathfrak{M}$ is a module k-functor over some ring k-functor $\mathfrak{R}$, then we denote (if it exists) $\mathcal{H}om_{\mathfrak{R}}(\mathfrak{M}, \mathfrak{M})$ by $\mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$.

Example 6.4. Let $\mathfrak{M}$ be a module over a ring k-functor $\mathfrak{R}$. Assume that $\mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$ exists. Then $\mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$ is a ring k-functor with respect to composition of morphisms of modules as the multiplication and the usual addition of module morphisms. Moreover, if $\mathfrak{A}$ is a commutative ring k-functor, then $\mathcal{E}nd_{\mathfrak{A}}(\mathfrak{M})$ (if exists) admits additional structure of a $\mathfrak{A}$-algebra k-functor induced via a unique morphism $\mathfrak{A} \to \mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$ of ring k-functors that sends $1 \mapsto 1_{\mathfrak{M}}$.

Let $\mathfrak{A}$ be a commutative ring k-functor and let $\mathfrak{R}$ be a $\mathfrak{A}$-algebra k-functor. This means that there exists a morphism $\mathfrak{A} \to \mathfrak{R}$ of ring k-functors and for every k-algebra A induced morphism $\mathfrak{A}(A) \to \mathfrak{R}(A)$ sends $\mathfrak{A}(A)$ to the center of a ring $\mathfrak{R}(A)$. Fix a module $\mathfrak{M}$ over $\mathfrak{A}$. Next assume that k-functor $\mathcal{E}nd_{\mathfrak{A}}(\mathfrak{M})$ exists. By Example 6.4 it is a ring k-functor.

Definition 6.5. In the setting above suppose that $\alpha : \mathfrak{R} \times \mathfrak{M} \to \mathfrak{M}$ is a morphism of k-functors. Suppose that α makes $\mathfrak{M}$ into $\mathfrak{R}$-module and moreover, for every k-algebra A and for every point $x \in \mathfrak{R}(A)$ morphism α_x is a morphism of $\mathfrak{A}_A$-modules. Then α is called *a $\mathfrak{A}$-linear $\mathfrak{R}$-action on $\mathfrak{M}$*.

We continue the discussion. We assume that we are given an $\mathfrak{A}$-linear $\mathfrak{R}$-action $\alpha : \mathfrak{R} \times \mathfrak{M} \to \mathfrak{M}$ on $\mathfrak{M}$. We define a morphism $\rho : \mathfrak{R} \to \mathcal{E}nd_{\mathfrak{A}}(\mathfrak{M})$ of k-functors by formula $\rho(r) = \alpha_r$. As in Section 5 we can prove that ρ is a morphism of ring k-functors. Now we have the following result.

Theorem 6.6. *Let $\mathfrak{R}$ be an algebra k-functor over commutative ring $\mathfrak{A}$ k-functor and let $\mathfrak{M}$ be a $\mathfrak{A}$-module such that $\mathcal{E}nd_{\mathfrak{A}}(\mathfrak{M})$ exists. Suppose that*

$$\Big\{\mathfrak{A} \text{ linear actions of } \mathfrak{R} \text{ on } \mathfrak{M}\Big\} \longrightarrow \Big\{\text{Morphisms } \rho : \mathfrak{R} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}) \text{ of ring } k\text{-functors}\Big\}$$

is a map of classes described above. Then it is bijection.

Proof. The proof is similar to the proof of Theorem 5.1. □

7. Monoid algebra $\mathfrak{O}_k[\mathfrak{G}]$ and its modules

Definition 7.1. Let $\mathfrak{G}$ be a monoid k-functor. Then we construct an $\mathfrak{O}_k$-algebra $\mathfrak{O}_k[\mathfrak{G}]$ as follows. For every k-algebra A we define

$$\mathfrak{O}_k[\mathfrak{G}](A) = A\big[\mathfrak{G}(A)\big]$$

where the right hand side is monoid A-algebra for the abstract monoid $\mathfrak{G}(A)$. The structure of monoid k-functor on $\mathfrak{G}$ and $\mathfrak{K}$-algebra $\mathfrak{O}_k$ makes $\mathfrak{O}_k[\mathfrak{G}]$ into a ring k-functor. Moreover, we have a morphism $\mathfrak{O}_k \to \mathfrak{O}_k[\mathfrak{G}]$ which for every k-algebra A is given by the canonical inclusion

$$A \hookrightarrow A\big[\mathfrak{G}(A)\big]$$

Thus $\mathfrak{O}_k[\mathfrak{G}]$ is $\mathfrak{O}_k$-algebra. We call $\mathfrak{O}_k[\mathfrak{G}]$ *a monoid $\mathfrak{O}_k$-algebra over $\mathfrak{G}$*.

Fact 7.2. *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{R}$ be an $\mathfrak{O}_k$-algebra k-functor. Then every morphism*

$$\sigma : \mathfrak{G} \to \mathfrak{R}^{\times}$$

of monoid k-functors admits a unique extension

$$\tilde{\sigma} : \mathfrak{O}_k[\mathfrak{G}] \to \mathfrak{R}$$

to a morphism of $\mathfrak{O}_k$-algebras.

Proof. This follows from the analogical universal property of algebras over abstract monoids. □

Definition 7.3. Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{M}$ be a module over $\mathfrak{O}_k$. Suppose that $\alpha : \mathfrak{G} \times \mathfrak{M} \to \mathfrak{M}$ is an action of $\mathfrak{G}$ such that for any k-algebra A and point $g \in \mathfrak{G}(A)$ morphism $\alpha_g : \mathfrak{M}_A \to \mathfrak{M}_A$ is a morphism of $\mathfrak{O}_A$-modules. Then α is called *a linear $\mathfrak{G}$-action on $\mathfrak{M}$*.

Suppose now that $\mathfrak{G}$ is a monoid k-functor and $\mathfrak{M}$ is a module $\mathfrak{O}_k$. Note that every linear $\mathfrak{G}$-action $\alpha : \mathfrak{G} \times \mathfrak{M} \to \mathfrak{M}$ extends uniquely to a $\mathfrak{O}_k$-linear action $\mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}$ of monoid $\mathfrak{O}_k$-algebra. This gives a bijection

$$\left\{\text{Linear actions of } \mathfrak{G} \text{ on } \mathfrak{M}\right\} \longrightarrow \left\{\mathfrak{O}_k\text{-linear actions } \mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}\right\}$$

Next assume that k-functor $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ exists. By Example 6.4 it is an $\mathfrak{O}_k$-algebra k-functor. Next by Theorem 6.6 we have a bijection

$$\left\{\mathfrak{O}_k\text{-linear actions of } \mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}\right\} \longrightarrow \left\{\text{Morphisms } \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}) \text{ of } \mathfrak{O}_k\text{-algebras}\right\}$$

Finally Fact 7.2 implies that we have a bijection

$$\left\{\text{Morphisms } \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}) \text{ of } \mathfrak{O}_k\text{-algebras}\right\} \longrightarrow \left\{\text{Morphisms } \mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}) \text{ of monoids}\right\}$$

This chain of bijections sends a linear action $\alpha : \mathfrak{G} \times \mathfrak{M} \to \mathfrak{M}$ of $\mathfrak{G}$ to a morphism $\rho : \mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ of monoid k-functors given by $\rho(g) = \alpha_g$ for every $g \in \mathfrak{G}(A)$ and every k-algebra A. We proved the following result.

Proposition 7.4. *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{M}$ be a $\mathfrak{O}_k$-module such that $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ exists. Then the following classes are in canonical bijections described above.*

(1) *Linear actions of $\mathfrak{G}$ on $\mathfrak{M}$.*

(2) *$\mathfrak{O}_k$-linear actions $\mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}$. These are precisely $\mathfrak{O}_k[\mathfrak{G}]$-modules.*

(3) *Morphisms $\mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ of $\mathfrak{O}_k$-algebras.*

(4) *Morphisms $\mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ of monoids.*

Moreover, the bijection between class **(1)** *and* **(2)** *does not require the existence of $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$.*

Now in a similar manner we can describe morphisms.

Proposition 7.5. *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{M}_1$, $\mathfrak{M}_2$ be k-functors of $\mathfrak{O}_k$-modules such that $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1), \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$ exist. Suppose that $\alpha_1 : \mathfrak{G} \times \mathfrak{M}_1 \to \mathfrak{M}_1$, $\alpha_2 : \mathfrak{G} \times \mathfrak{M}_2 \to \mathfrak{M}_2$ are linear actions of $\mathfrak{G}$. Suppose that $\sigma : \mathfrak{M}_1 \to \mathfrak{M}_2$ is a morphism of modules over $\mathfrak{O}_k$. Then the following assertions are equivalent.*

(i) *The square*

$$\begin{tikzcd}
\mathfrak{G} \times \mathfrak{M}_1 \arrow[r, "1_{\mathfrak{G}} \times \sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{G} \times \mathfrak{M}_2 \arrow[d, "\alpha_2"] \\
\mathfrak{M}_1 \arrow[r, "\sigma"'] & \mathfrak{M}_2
\end{tikzcd}$$

is commutative.

(ii) *The square*

$$\begin{tikzcd}
\mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M}_1 \arrow[r, "1_{\mathfrak{O}_k[\mathfrak{G}]} \times \sigma"] \arrow[d, "\tilde{\alpha_1}"'] & \mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M}_2 \arrow[d, "\tilde{\alpha_2}"] \\
\mathfrak{M}_1 \arrow[r, "\sigma"'] & \mathfrak{M}_2
\end{tikzcd}$$

is commutative, where $\tilde{\alpha_1}$ and $\tilde{\alpha_2}$ are $\mathfrak{O}_k$-linear actions of $\mathfrak{O}_k[\mathfrak{G}]$ corresponding to α_1 and α_2, respectively. This states that σ is a morphism of $\mathfrak{O}_k[\mathfrak{G}]$-modules.

(iii) *For every k-algebra A and $g \in \mathfrak{G}(A)$ we have*

$$\sigma_A \cdot \tilde{\rho}_1(g) = \tilde{\rho}_2(g) \cdot \sigma_A$$

where $\tilde{\rho}_1 : \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1)$ and $\tilde{\rho}_2 : \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$ are morphism of $\mathfrak{O}_k$-algebras corresponding to $\tilde{\alpha_1}$ and $\tilde{\alpha_2}$, respectively.

(iv) *For every k-algebra A and $g \in \mathfrak{G}(A)$ we have*

$$\sigma_A \cdot \rho_1(g) = \rho_2(g) \cdot \sigma_A$$

where $\rho_1 : \mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1)$ and $\rho_2 : \mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$ are restrictions of $\tilde{\rho_1}$ and $\tilde{\rho_2}$, respectively.

The equivalence of **(i)** *and* **(ii)** *does not require the existence of $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1)$ and $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$.*

Proof. Conditions expressed in **(i)-(iv)** are directly translatable to each other by virtue of bijections in Proposition 7.4. □

Let $\mathfrak{G}$ be a monoid k-functor. We denote by $\mathbf{Mod}\left(\mathfrak{O}_k[\mathfrak{G}]\right)$ the category of $\mathfrak{O}_k[\mathfrak{G}]$-modules.

8. Linear representations of a monoid k-functors

We start the discussion with some results that relates categories $\mathbf{Mod}(k)$ and $\mathbf{Mod}\left(\mathfrak{O}_k\right)$.

Example 8.1. Let V be a k-module. We define a k-functor V_{a}. We set

$$V_{\mathrm{a}}(A) = A \otimes_k V,\ V_{\mathrm{a}}(f) = f \otimes_k 1_V$$

for every k-algebra A and every morphism $f : A \to B$ of k-algebras. Note that V_{a} is $\mathfrak{O}_k$-module. Suppose that $\phi : V \to W$ is a morphism of k-modules, then we define $\phi_{\mathrm{a}} : V_{\mathrm{a}} \to W_{\mathrm{a}}$ by formula

$$\phi_{\mathrm{a}}^A = 1_A \otimes_k \sigma$$

for every k-algebra. Then ϕ_{a} is a morphism of $\mathfrak{O}_k$-modules.

Remark 8.2. Let V be a finitely generated, projective k-module. Then for each k-algebra A we have an isomorphism

$$\mathfrak{P}_{\mathrm{Spec\,Sym}(V^{\vee})}(A) = \mathrm{Mor}_k\left(\mathrm{Sym}(V^{\vee}), A\right) = \mathrm{Hom}_k\left(V^{\vee}, A\right) \cong A \otimes_k V$$

Clearly this isomorphism is natural in A. Thus V_{a} is representable by a k-scheme $\mathrm{Spec\,Sym}(V^{\vee})$.

Proposition 8.3. *The functor $(-)_{\mathrm{a}} : \mathbf{Mod}\,(k) \to \mathbf{Mod}\left(\mathfrak{O}_k\right)$ is full and faithful.*

Proof. Fix k-modules V, W. Then

$$\mathrm{Hom}_{\mathfrak{O}_k}\left(V_{\mathrm{a}}, W_{\mathrm{a}}\right) \ni \sigma \mapsto \sigma^k \in \mathrm{Hom}_k\left(V, W\right)$$

and

$$\mathrm{Hom}_k\left(V, W\right) \ni \phi \mapsto \phi_{\mathrm{a}} \in \mathrm{Hom}_{\mathfrak{O}_k}\left(V_{\mathrm{a}}, W_{\mathrm{a}}\right)$$

are mutually inverse bijections. Hence the functor is full and faithful. □

Example 8.4. Let V be a k-module. We define a k-functor $\mathcal{L}_V$. We set

$$\mathcal{L}_V(A) = \mathrm{Hom}_A(A \otimes_k V, A \otimes_k V)$$

for every k-algebra A. Next for every morphism $f : A \to B$ of k-algebras and every morphism $\phi : A \otimes_k V \to A \otimes_k V$ of A-modules we define $\mathcal{L}_V(f)(\phi)$ as a unique morphism of B-modules such that the diagram

$$\begin{tikzcd} A \otimes_k V \arrow[r, "\phi"] \arrow[d, "f \otimes_k 1_V"'] & A \otimes_k V \arrow[d, "f \otimes_k 1_V"] \\ B \otimes_k V \arrow[r, "\mathcal{L}_V(\phi)"'] & B \otimes_k V \end{tikzcd}$$

is commutative. Note also that $\mathcal{L}_V(A)$ is an A-algebra for every k-algebra A. Hence $\mathcal{L}_V$ is a monoid $\mathfrak{O}_k$-algebra. Note that we have natural identification

$$\mathcal{L}_V(A) = \mathrm{Hom}_k(V, A \otimes_k V)$$

for every k-algebra. One can describe $\mathfrak{O}_k$-algebra structure on $\mathcal{L}_V$ in terms of this identification as follows. Since $\mathrm{Hom}_k(V, A \otimes_k V)$ carries canonical structure of A-module it suffices to describe the multiplication. For this suppose that $d_1, d_2 \in \mathrm{Hom}_k(V, A \otimes_k V)$. Then their product is given by

$$(\mu_A \otimes_k 1_V) \cdot (1_A \otimes d_2) \cdot d_1$$

where $\mu_A : A \otimes_k A \to A$ is the multiplication on A.

Remark 8.5. Let V be a k-module. Proposition 8.3 implies that there are bijective maps that make the square

$$\begin{tikzcd} \mathcal{L}_V(A) \arrow[r, "\cong"] \arrow[d, "\mathcal{L}_V(f)"'] & \mathcal{E}nd_{\mathfrak{O}_A}\big((V_{\mathrm{a}})_A, (V_{\mathrm{a}})_A\big) \arrow[d, "\sigma \mapsto \sigma_B"] \\ \mathcal{L}_V(B) \arrow[r, "\cong"'] & \mathcal{E}nd_{\mathfrak{O}_B}\big((V_{\mathrm{a}})_B, (V_{\mathrm{a}})_B\big) \end{tikzcd}$$

commutative for every morphism $f : A \to B$ of k-algebras. This induces an idenitification $\mathcal{L}_V = \mathcal{E}nd_{\mathfrak{O}_k}(V_{\mathrm{a}})$ of $\mathfrak{O}_k$-algebras.

Remark 8.6. Suppose that V is a finitely generated, projective k-module. Then for each k-algebra A we have an isomorphism

$$\mathcal{L}_V(A) = \mathrm{Hom}_A(V, A \otimes_k V) \cong A \otimes_k V^{\vee} \otimes_k V$$

Clearly this isomorphism is natural in A. Hence $\mathcal{L}_V$ is isomorphic with $(V^{\vee} \otimes_k V)_{\mathrm{a}}$ and thus (Remark 8.2) it is representable by a k-scheme $\mathrm{Spec}\,\mathrm{Sym}(V \otimes_k V^{\vee})$.

Definition 8.7. Let $\mathfrak{G}$ be a monoid k-functor. A pair (V, ρ) consisting of a k-module V and a morphism $\rho : \mathfrak{G} \to \mathcal{L}_V$ of k-monoids is called *a linear representation of* $\mathfrak{G}$.

Next result characterizes linear representations of monoid k-functors.

Corollary 8.8. *Let $\mathfrak{G}$ be a monoid k-functor and let V be a k-module. Then the following classes are in canonical bijections.*

(1) *Linear actions of $\mathfrak{G}$ on V_{a}.*

(2) *$\mathfrak{O}_k$-linear actions $\mathfrak{O}_k[\mathfrak{G}] \times V_{\mathrm{a}} \to V_{\mathrm{a}}$. These are precisely $\mathfrak{O}_k[\mathfrak{G}]$-modules.*

(3) *Morphisms $\mathfrak{O}_k[\mathfrak{G}] \to \mathcal{L}_V$ of $\mathfrak{O}_k$-algebras.*

(4) *Morphisms $\mathfrak{G} \to \mathcal{L}_V$ of monoids.*

Proof. This follows from Proposition 7.4. $\square$

Definition 8.9. Let $\mathfrak{G}$ be a monoid k-functor and let (V, ρ), (W, δ) be its linear representations. A morphism $\phi : V \to W$ of k-modules such that

$$\phi_{\mathrm{a}}^A \cdot \rho(g) = \delta(g) \cdot \phi_{\mathrm{a}}^A$$

for every k-algebra A and $g \in \mathfrak{G}(A)$ is called *a morphism of linear representations of* $\mathfrak{G}$.

Next result characterizes morphisms of linear representations of monoid k-functor.

Corollary 8.10. *Let $\mathfrak{G}$ be a monoid k-functor and let V, W be k-modules. Suppose that $\alpha_1 : \mathfrak{G} \times V_{\mathrm{a}} \to V_{\mathrm{a}}$, $\alpha_2 : \mathfrak{G} \times W_{\mathrm{a}} \to W_{\mathrm{a}}$ are linear actions of $\mathfrak{G}$. Suppose that $\phi : V \to W$ is a morphism of k-modules. Then the following assertions are equivalent.*

(i) *The square*

$$\begin{tikzcd}
\mathfrak{G} \times V_{\mathrm{a}} \arrow[r, "1_{\mathfrak{G}} \times \phi_{\mathrm{a}}"] \arrow[d, "\alpha_1"'] & \mathfrak{G} \times W_{\mathrm{a}} \arrow[d, "\alpha_2"] \\
V_{\mathrm{a}} \arrow[r, "\phi_{\mathrm{a}}"'] & W_{\mathrm{a}}
\end{tikzcd}$$

is commutative.

(ii) *The square*

$$\begin{tikzcd}
\mathfrak{O}_k[\mathfrak{G}] \times V_{\mathrm{a}} \arrow[r, "1_{\mathfrak{O}_k[\mathfrak{G}]} \times \phi_{\mathrm{a}}"] \arrow[d, "\tilde{\alpha_1}"'] & \mathfrak{O}_k[\mathfrak{G}] \times W_{\mathrm{a}} \arrow[d, "\tilde{\alpha_2}"] \\
V_{\mathrm{a}} \arrow[r, "\phi_{\mathrm{a}}"'] & W_{\mathrm{a}}
\end{tikzcd}$$

is commutative, where $\tilde{\alpha_1}$ and $\tilde{\alpha_2}$ are $\mathfrak{O}_k$-linear actions of $\mathfrak{O}_k[\mathfrak{G}]$ corresponding to α_1 and α_2, respectively.

(iii) *For every k-algebra A and $g \in \mathfrak{G}(A)$ we have*

$$\phi_{\mathrm{a}}^A \cdot \tilde{\rho}_1(g) = \tilde{\rho}_2(g) \cdot \phi_{\mathrm{a}}^A$$

where $\tilde{\rho}_1 : \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{L}_V$ and $\tilde{\rho}_2 : \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{L}_W$ are morphism of $\mathfrak{O}_k$-algebras corresponding to $\tilde{\alpha_1}$ and $\tilde{\alpha_2}$, respectively.

(iv) *For every k-algebra A and $g \in \mathfrak{G}(A)$ we have*

$$\phi_{\mathrm{a}}^A \cdot \rho_1(g) = \rho_2(g) \cdot \phi_{\mathrm{a}}^A$$

where $\rho_1 : \mathfrak{G} \to \mathcal{L}_V$ and $\rho_2 : \mathfrak{G} \to \mathcal{L}_W$ are restrictions of $\tilde{\rho}_1$ and $\tilde{\rho}_2$, respectively. This states that ϕ is a morphism of linear representations of $\mathfrak{G}$.

Proof. This follows from Proposition 7.5. □

Let $\mathfrak{G}$ be a monoid k-functor. We denote by $\mathbf{Rep}(\mathfrak{G})$ its category of linear representations. Note that $\mathbf{Rep}(\mathfrak{G})$ is a full subcategory of $\mathbf{Mod}(\mathfrak{O}_k[\mathfrak{G}])$.

9. Constructions of linear representations

Example 9.1 (Outer tensor product of representations)**.** Let (V_1, ρ_1) and (V_2, ρ_2) are linear representations of monoid k-functors $\mathfrak{G}_1$ and $\mathfrak{G}_2$, respectively. Then we define a linear representation of $\mathfrak{G}_1 \times \mathfrak{G}_2$ with $V_1 \otimes_k V_2$ as the underlying k-module that corresponds to a morphism $\rho : \mathfrak{G}_1 \times \mathfrak{G}_2 \to \mathcal{L}_{V_1 \otimes_k V_2}$ of monoid k-functors given by

$$\rho(g_1, g_2) = \rho_1(g_1) \otimes_A \rho_2(g_2) : A \otimes_k V_1 \otimes_k V_2 \to A \otimes_k V_1 \otimes_k V_2$$

for $(g_1, g_2) \in \mathfrak{G}_1(A) \times \mathfrak{G}_2(A)$, where A is a k-algebra.

Example 9.2 (Tensor product of representations)**.** Let (V_1,ρ_1) and (V_2,ρ_2) are linear representations of monoid k-functor $\mathfrak{G}$. Then we define a linear representation of $\mathfrak{G}$ with $V_1\otimes_k V_2$ as the underlying k-module given as the composition of the outer tensor product of (V_1,ρ_1) and (V_2,ρ_2) with the diagonal $\mathfrak{G}\hookrightarrow\mathfrak{G}\times\mathfrak{G}$.

Example 9.3 (Tensor operations)**.** Let $\mathfrak{G}$ be a monoid k-functor, let V be k-module and let $\rho:\mathfrak{G}\to\mathcal{L}_V$ be a morphism of monoid k-functors. Then both $\bigwedge^n V$ and $\mathrm{Sym}^n(V)$ for $n\in\mathbb{N}$ carry canonical structure of linear representation of $\mathfrak{G}$.

Note that if V is a finitely generated, projective k-module, then there is a canonical isomorphism of A-modules $(V^\vee)_{\mathrm{a}}(A)\cong(A\otimes_k V)^\vee$ natural in k-algebra A. Under these assumptions on V there exists an anti-isomorphism of A-algebras

$$\mathrm{Hom}_A(A\otimes_k V,A\otimes_k V)\ni\phi\mapsto\phi^\vee\in\mathrm{Hom}_A\left((A\otimes_k V)^\vee,(A\otimes_k V)^\vee\right)$$

natural in k-algebra A. This proves the following result.

Fact 9.4. *Let V be a finitely generated, projective k-module. Then we have an identification of k-functors of $\mathfrak{O}_k$-algebras*

$$\mathcal{L}_V^{\mathrm{op}}=\mathcal{L}_{V^\vee}$$

Example 9.5 (Dual representation)**.** Let $\mathfrak{G}$ be a monoid k-functor, let V be k-module and let $\rho:\mathfrak{G}\to\mathcal{L}_V$ be a morphism of monoid k-functors. Suppose that V is a projective and finitely generated k-module. Fact 9.4 implies that morphism of a monoid k-functors $\rho^{\mathrm{op}}:\mathfrak{G}^{\mathrm{op}}\to\mathcal{L}_V^{\mathrm{op}}$ can be identified with $\rho^\vee:\mathfrak{G}^{\mathrm{op}}\to\mathcal{L}_{V^\vee}$. Hence a pair $(V^\vee,\rho^\vee)$ is a linear representation of $\mathfrak{G}^{\mathrm{op}}$.

Example 9.6 (Hom representation)**.** Let (V_1,ρ_1) and (V_2,ρ_2) are linear representations of monoid k-functor $\mathfrak{G}$. Suppose that V_1,V_2 are finitely generated, projective k-module. Then we have an identification

$$\mathrm{Hom}_k(V_1,V_2)_{\mathrm{a}}=\left(V_1^\vee\otimes_k V_2\right)_{\mathrm{a}}$$

of $\mathfrak{O}_k$-modules. By Examples 9.1 and 9.5 this isomorphism makes $\mathrm{Hom}_k(V_1,V_2)$ into linear representation of $\mathfrak{G}\times\mathfrak{G}^{\mathrm{op}}$.

10. Example of $\mathfrak{G}$-action: Regular functions k-functor

First we need the following notion.

Definition 10.1. Let $(-)^{\mathrm{op}}:\mathbf{Mon}\to\mathbf{Mon}$ be the opposite monoid functor and let $\mathfrak{G}$ be a monoid k-functor. Then the composition $\mathfrak{G}^{\mathrm{op}}=(-)^{\mathrm{op}}\cdot\mathfrak{G}$ is called *the opposite monoid k-functor of $\mathfrak{G}$*.

Let $\mathfrak{G}$ be a monoid k-functor. In this section we discuss important example of a $\mathfrak{O}_k[\mathfrak{G}]$-module. Fix a k-functor $\mathfrak{X}$ for which $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{O}_k)$ exists. Recall that by Example 3.5 $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{O}_k)$ is $\mathfrak{O}_k$-algebra k-functor. Let $\alpha:\mathfrak{G}\times\mathfrak{X}\to\mathfrak{X}$ be an action of $\mathfrak{G}$ on $\mathfrak{X}$. For every k-algebra A we have a map of sets

$$\mathrm{Mor}_A(\mathfrak{X}_A,(\mathfrak{O}_k)_A)\ni f\mapsto f\cdot\alpha_g\in\mathrm{Mor}_A(\mathfrak{X}_A,(\mathfrak{O}_k)_A)$$

where $g\in\mathfrak{G}(A)$. From this description it follows that the map $f\mapsto f\cdot\alpha_g$ is a morphism of A-algebras. Moreover, note that if $g_1,g_2\in\mathfrak{G}(A)$, then $\left(f\cdot\alpha_{g_1}\right)\cdot\alpha_{g_2}=f\cdot\alpha_{g_1\cdot g_2}$, where $g_1\cdot g_2\in\mathfrak{G}(A)$ is a product of g_1 and g_2. Thus the opposite monoid $\mathfrak{G}^{\mathrm{op}}(A)$ acts on the A-algebra $\mathrm{Mor}_A\left(\mathfrak{X}_A,(\mathfrak{O}_k)_A\right)$ by morphism of A-algebras. Next for every A-algebra B and every point $x\in\mathfrak{X}(B)$ we have

$$(f\cdot\alpha_g)(x)=f\left(\alpha_g(x)\right)$$

where $g\in\mathfrak{G}(A)$. This proves the following result.

Proposition 10.2. *Let $\mathfrak{X}$ be a k-functor and let $\alpha:\mathfrak{G}\times\mathfrak{X}\to\mathfrak{X}$ be an action of a monoid k-functor $\mathfrak{G}$. Suppose that $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{O}_k)$ exists. Then $\mathfrak{G}^{\mathrm{op}}$ acts canonically on $\mathfrak{O}_k$-algebra k-functor $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{O}_k)$ by morphisms of $\mathfrak{O}_k$-algebras.*

Let us note one important consequence of this result.

Corollary 10.3. *Let $\mathfrak{G}$ be a monoid k-functor. The action of $\mathfrak{G} \times \mathfrak{G}^{\mathrm{op}}$ on $\mathfrak{G}$ induces the action of $\mathfrak{G}^{\mathrm{op}} \times \mathfrak{G}$ on $\mathfrak{O}_k$-algebra k-functor $\mathcal{M}\mathrm{or}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$ by morphisms of $\mathfrak{O}_k$-algebras.*

11. Matrix coefficients of a representation

Proposition 11.1. *Let $\mathfrak{G}$ be a monoid k-functor and let V be a finitely generated, projective k-module. Fix a morphism $\rho : \mathfrak{G} \to \mathcal{L}_V$ of monoid k-functors. Fix k-algebra A and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. For every A-algebra B and $g \in \mathfrak{G}(B)$ we consider the formula*

$$c_{v,w}(g) = \langle \rho_A(g) \cdot v_B, w_B \rangle$$

Then $c_{v,w}$ defines a regular function on $\mathfrak{G}_A$ for every k-algebra A.

Proof. Suppose that $f : B \to C$ is a morphism of A-algebras and pick $g \in \mathfrak{G}(B)$. Since ρ_A is natural and $w : A \otimes_k V \to A$ is a morphism of A-modules, we derive that the diagram

$$\begin{tikzcd} B \otimes_k V \arrow[r, "\rho_A(g)"] \arrow[d, "f \otimes_A 1_{A \otimes_k V}"'] & B \otimes_k V \arrow[r, "w_B"] \arrow[d, "f \otimes_A 1_{A \otimes_k V}"] & B \arrow[d, "f"] \\ C \otimes_k V \arrow[r, "\rho_A\left(\mathfrak{G}_A(f)(g)\right)"'] & C \otimes_k V \arrow[r, "w_C"'] & C \end{tikzcd}$$

is commutative. Hence

$$c_{v,w}\big(\mathfrak{G}_A(f)(g)\big) = \langle \rho_A\big(\mathfrak{G}_A(f)(g)\big) \cdot v_C, w_C \rangle = f\big(\langle \rho_A(g) \cdot v_B, w_B \rangle\big) = f\big(c_{v,w}(g)\big)$$

and this implies that $c_{v,w} : \mathfrak{G}_A \to \mathfrak{O}_A$ is a morphism of A-functors. □

Definition 11.2. Let $\mathfrak{G}$ be a monoid k-functor and let (V, ρ) be its representation with finitely generated, projective underlying k-module V. Fix k-algebra A and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. Then the regular function $c_{v,w}$ on $\mathfrak{G}_A$ is called *the matrix coefficient of v and w*.

Proposition 11.3. *Let $\mathfrak{G}$ be a monoid k-functor and let (V, ρ) be its representation with finitely generated projective underlying k-module V. Then the following assertions holds.*

(1) *For every k-algebra A map*

$$(A \otimes_k V) \times \left(A \otimes_k V^{\vee}\right) \ni (v, w) \mapsto c_{v,w} \in \mathrm{Mor}_A\left(\mathfrak{G}_A, \mathfrak{O}_A\right)$$

is A-bilinear.

(2) *Suppose that $\mathcal{M}\mathrm{or}_k\left(\mathfrak{G}, \mathfrak{O}_k\right)$ exists. Then the collection of maps*

$$\left\{\left(A \otimes_k V\right) \times \left(A \otimes_k V^{\vee}\right) \ni (v, w) \mapsto c_{v,w} \in \mathrm{Mor}_A\big(\mathfrak{G}_A, \mathfrak{O}_A\big)\right\}_{A \in \mathbf{Alg}_k}$$

gives rise to a morphism of k-functors

$$V_{\mathrm{a}} \times V_{\mathrm{a}}^{\vee} \longrightarrow \mathcal{M}\mathrm{or}_k\left(\mathfrak{G}, \mathfrak{O}_k\right)$$

Proof. We left the proof of **(1)** to the reader.
We prove **(2)**. Consider k-algebra A and an A-algebra B with structural morphism $f : A \to B$. Fix $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. We prove that restriction of $c_{v,w} : \mathfrak{G}_A \to \mathfrak{O}_A$ to the category $\mathbf{Alg}_B$ is c_{v_B,w_B}. For this pick a B-algebra C and an element $g \in \mathfrak{G}(C)$. Note that

$$c_{v,w}(g) = \langle \rho_A(g) \cdot v_C, w_C \rangle = \langle \rho_B(g) \cdot v_C, w_C \rangle = \langle \rho_B(g) \cdot (v_B)_C, (w_B)_C \rangle = c_{v_B,w_B}(g)$$

and hence $c_{v,w|\mathbf{Alg}_B} = c_{v_B,w_B}$. Consider the square

$$\begin{tikzcd}
V_{\mathrm{a}}(A) \times V_{\mathrm{a}}^{\vee}(A) \arrow[r] \arrow[d, "V_a(f) \times V_a^{\vee}(f)"'] & \mathcal{M}\mathrm{or}_k(\mathfrak{G}, \mathfrak{O}_A)(A) \arrow[d, "\mathcal{M}\mathrm{or}_k(\mathfrak{G}, \mathfrak{O}_k)(f)"] \\
V_{\mathrm{a}}(B) \times V_{\mathrm{a}}^{\vee}(B) \arrow[r] & \mathcal{M}\mathrm{or}_k(\mathfrak{G}, \mathfrak{O}_B)(B)
\end{tikzcd}$$

in which both horizontal arrows are given by formula $(v, w) \mapsto c_{v,w}$. We proved that the square commutes. Since f is an arbitrary morphism of k-algebras, we conclude the assertion. □

Corollary 11.4. *Let $\mathfrak{G}$ be a monoid k-functor and let (V, ρ) be its representation with finitely generated projective underlying k-module V. Suppose that $\mathcal{M}\mathrm{or}_k(\mathfrak{G}, \mathfrak{O}_k)$ exists. Then there exists a morphism of k-functors*

$$(V \otimes_k V^{\vee})_{\mathrm{a}} \xrightarrow{c} \mathcal{M}\mathrm{or}_k(\mathfrak{G}, \mathfrak{O}_k)$$

given by formula

$$(A \otimes_k V) \otimes_A (A \otimes_k V^{\vee}) \ni (v, w) \mapsto c_{v,w} \in \mathrm{Mor}_A(\mathfrak{G}_A, \mathfrak{O}_A)$$

Moreover, c is a morphism of k-functors equipped with $\mathfrak{G} \times \mathfrak{G}^{\mathrm{op}}$-actions.

Proof. The first part is an immediate consequence of Proposition 11.3. We prove that c is a morphism of k-functors equipped with $\mathfrak{G} \times \mathfrak{G}^{\mathrm{op}}$-actions. For this we fix a k-algebra k and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. Pick a morphism of k-algebras $f : A \to B$, $(g_1, g_2) \in \mathfrak{G}(A) \times \mathfrak{G}(A)^{\mathrm{op}}$ and $g \in \mathfrak{G}(B)$. Then we have

$$c_{\rho(g_1)\cdot v, w \cdot \rho(g_2)}(g) = \big\langle \rho_A(g) \cdot (\rho(g_1) \cdot v)_B, (w \cdot \rho(g_2))_B \big\rangle =$$
$$= \big\langle \rho_A(g) \cdot \rho_A((\mathfrak{G}_A(f)(g_1))) \cdot v_B, w_B \cdot \rho_A(\mathfrak{G}_A(f)(g_2)) \big\rangle = w_B\big(\rho_A(\mathfrak{G}_A(f)(g_2)) \cdot \rho_A(g) \cdot \rho_A(\mathfrak{G}_A(f)(g_1)) \cdot v_B\big) =$$
$$= w_B\big(\rho_A(\mathfrak{G}_A(f)(g_2) \cdot g \cdot \mathfrak{G}_A(f)(g_1)) \cdot v_B\big) = \big\langle \rho_A(\mathfrak{G}_A(f)(g_2) \cdot g \cdot \mathfrak{G}_A(f)(g_1)) \cdot v_B, w_B \big\rangle =$$
$$= c_{v,w}\big(\mathfrak{G}_A(f)(g_2) \cdot g \cdot \mathfrak{G}_A(f)(g_1)\big)$$

and hence c is a morphism of k-functors equipped with actions of $\mathfrak{G} \times \mathfrak{G}^{\mathrm{op}}$. □

12. Monoid k-schemes

Definition 12.1. *A monoid k-scheme* $\mathbf{M}$ is a monoid object in the category of k-schemes. If $\mathbf{M}$ is affine, then we say that $\mathbf{M}$ is *an affine monoid k-scheme*.

Definition 12.2. *A group k-scheme* $\mathbf{G}$ is a group object in the category of k-schemes. If $\mathbf{G}$ is affine, then we say that $\mathbf{G}$ is *an affine group k-scheme*.

Corollary 12.3. *The functor*

$$\mathbf{Sch}_k \xrightarrow{\mathfrak{P}} \text{the category of } k\text{-functors}$$

induces an equivalence of categories

$$\text{the category of monoid } k\text{-schemes} \cong \text{monoid } k\text{-functors representable by } k\text{-schemes}$$

Similarly for categories of groups.

Proof. Follows from [Mon19b, Fact 4.1]. □

Recall that by Example 2.3 each monoid k-functor $\mathfrak{G}$ has its group k-functor $\mathfrak{G}^*$ of units.

Proposition 12.4. *Let $\mathbf{M}$ be an affine monoid k-scheme. Then the k-functor of units $\mathfrak{P}^*_{\mathbf{M}}$ is representable (by an affine k-scheme).*

Proof. Note that $\mathfrak{P}^*_{\mathbf{M}}$ fits into a cartesian square

$$\begin{tikzcd} \mathfrak{P}^*_{\mathbf{M}} \arrow[r] \arrow[d] & \mathbf{1} \arrow[d, "\mathfrak{P}_e"] \\ \mathfrak{P}_{\mathbf{M}} \times \mathfrak{P}_{\mathbf{M}} \arrow[r, "\mathfrak{P}_m"'] & \mathfrak{P}_{\mathbf{M}} \end{tikzcd}$$

where $m : \mathbf{M} \times \mathbf{M} \to \mathbf{M}$ is the multiplication and $e : \operatorname{Spec} k \to \mathbf{M}$ is the unit. By [Mon19b, Fact 4.1] the functor $\mathfrak{P}$ preserves finite products and hence it preserves fiber-products. This implies that $\mathfrak{P}^*_{\mathbf{M}}$ is represented by a unique (up to an isomorphism) k-scheme $\mathbf{M}^*$ that fit into a cartesian square below.

$$\begin{tikzcd} \mathbf{M}^* \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d, "e"] \\ \mathbf{M} \times \mathbf{M} \arrow[r, "m"'] & \mathbf{M} \end{tikzcd}$$

Note that if $\mathbf{M}$ is affine, then also $\mathbf{M}^*$ is affine. □

Definition 12.5. Let $\mathbf{M}$ be a monoid k-scheme. Then the group k-scheme $\mathbf{M}^*$ representing $\mathfrak{P}^*_{\mathbf{M}}$ is called *the group of units of* $\mathbf{M}$.

Remark 12.6. Under the embedding given in Corollary 12.3 notions defined for monoid k-functors can be translated to monoid k-schemes.

We give two instances of the use of Remark 12.6 below.

Definition 12.7. Let $\mathbf{M}$ be a monoid k-scheme. Then *the category of linear representations of* $\mathbf{M}$ is the category of linear representations of the monoid k-functor $\mathfrak{P}_{\mathbf{M}}$. We denote this category by $\mathbf{Rep}\,(\mathbf{M})$.

Definition 12.8. Let $\mathbf{M}$ be a monoid k-functor and let $\alpha : \mathfrak{P}_{\mathbf{M}} \times \mathfrak{X} \to \mathfrak{X}$ be an action of $\mathfrak{P}_{\mathbf{M}}$ on a k-functor $\mathfrak{X}$. Then we say that α is *an action of* $\mathbf{M}$ *on* $\mathfrak{X}$.

13. Bialgebras and affine monoid k-schemes

We start here with a general notion of k-coalgebras.

Definition 13.1. Let (C, Δ, ξ) be a triple consisting of a module C over k and morphisms

$$\Delta : C \to C \otimes_k C, \xi : C \to k$$

of k-modules such that the following diagrams are commutative.

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes_k C \arrow[d, "1_C \otimes_k \Delta"] \\ C \otimes_k C \arrow[r, "\Delta \otimes_k 1_C"'] & C \otimes_k C \otimes_k C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[dr, "\cong"'] & C \otimes_k C \arrow[d, "1 \otimes_k \xi"] \\ & C \otimes_k k \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[dr, "\cong"'] & C \otimes_k C \arrow[d, "\xi \otimes_k 1_C"] \\ & k \otimes_k C \end{tikzcd}$$

Then (C, Δ, ξ) is called *a k-coalgebra*. Morphisms Δ, ξ are called *a comultiplication* and *a counit*, respectively.

Definition 13.2. Let (C_1, Δ_1, ξ_1) and (C_2, Δ_2, ξ_2) are k-coalgebras. Then a morphism $f : C_1 \to C_2$ of k-modules is *a morphism of k-coalgebras* if the following diagrams are commutative.

$$
\begin{tikzcd}
C_1 \otimes_k C_1 \arrow[r, "f \otimes_k f"] & C_2 \otimes_k C_2 \\
C_1 \arrow[u, "\Delta_1"] \arrow[r, "f"'] & C_2 \arrow[u, "\Delta_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[rr, "f"] \arrow[dr, "\xi_1"'] & & C_2 \arrow[dl, "\xi_2"] \\
& k &
\end{tikzcd}
$$

By k-algebra we mean commutative and unital k-algebra.

Definition 13.3. Let B be a k-module with structures of both k-algebra and k-coalgebra. Assume that the comultiplication and the counit of B are morphisms of k-algebras with respect to k-algebra structure of B. Then we say that B with these structures is *a k-bialgebra*.

Definition 13.4. Let B_1, B_2 be k-bialgebras and let $f : B_1 \to B_2$ be a morphism of k-modules. We say that f is *a morphism of k-bialgebras* if it is simultaneously morphism of k-algebras and k-coalgebras.

Theorem 13.5. *The functor* Spec : $\mathbf{Alg}_k \to \mathbf{Sch}_k$ *induces an equivalence of categories*

$$k\text{-bialgebras} \cong \text{the category of affine monoid } k\text{-schemes}$$

Proof. This is an exercise in translation. For details see [DG70, II, 1.6]. □

Let $\mathbf{M}$ be an affine monoid k-scheme. Then we denote by $k[\mathbf{M}]$ its coordinate k-bialgebra, by $\Delta_{\mathbf{M}}$ its comultiplication and by $\xi_{\mathbf{M}}$ its counit. This is a notation that we consistently use in these notes.

14. Comodules over k-coalgebras

Definition 14.1. Let C be a k-coalgebra with the comultiplication Δ and the counit ξ. A pair (V, d) consisting of a k-module V and a morphism $d : V \to C \otimes_k V$ of k-modules such that the following diagrams are commutative

$$
\begin{tikzcd}
V \arrow[r, "d"] \arrow[d, "d"'] & C \otimes_k V \arrow[d, "1_C \otimes_k d"] \\
C \otimes_k V \arrow[r, "\Delta \otimes_k 1_V"'] & C \otimes_k C \otimes_k V
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "d"] \arrow[dr, "\cong"'] & C \otimes_k V \arrow[d, "\xi \otimes_k 1_V"] \\
& k \otimes_k V
\end{tikzcd}
$$

is called *a C-comodule*. Morphism d is called *a coaction of C on V*.

Definition 14.2. Let C be a k-coalgebra and let $(V_1, d_1), (V_2, d_2)$ be two comodules over C. A morphism of k-modules $f : V_1 \to V_2$ is *a morphism of C-comodules* if the diagram

$$
\begin{tikzcd}
C \otimes_k V_1 \arrow[r, "1_C \otimes_k f"] & C \otimes_k V_2 \\
V_1 \arrow[u, "d_1"] \arrow[r, "f"'] & V_2 \arrow[u, "d_2"']
\end{tikzcd}
$$

is commutative.

We denote by $\mathbf{coMod}(C)$ the category of C-comodules for a k-coalgebra C.

Theorem 14.3. *Let C be a k-coalgebra. Then the forgetful functor* $\mathbf{coMod}(C) \to \mathbf{Mod}(k)$ *creates colimits.*

Proof. Let Δ, ξ be the comultiplication and the counit of C, respectively. Suppose that $I \ni i \mapsto (V_i, d_i) \in \mathbf{coMod}(C)$ is a diagram of C-comodules indexed by some category I. Let V together with $u_i : V_i \to V$ for $i \in I$ be a colimit of the diagram $I \ni i \mapsto V_i \in \mathbf{Mod}(k)$. By the universal property of colimit we deduce that there exists a unique morphism $d : V \to C \otimes_k V$ such that diagrams

$$\begin{tikzcd}
C \otimes_k V_i \arrow[r, "1_C \otimes_k u_i"] & C \otimes_k V \\
V_i \arrow[u, "d_i"] \arrow[r, "u_i"'] & V \arrow[u, "d"']
\end{tikzcd}$$

are commutative for every $i \in I$. In order to verify that diagrams

$$\begin{tikzcd}
V \arrow[r, "d"] \arrow[d, "d"'] & C \otimes_k V \arrow[d, "1_C \otimes_k d"] \\
C \otimes_k V \arrow[r, "\Delta \otimes_k 1_V"'] & C \otimes_k C \otimes_k V
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "d"] \arrow[dr, "\cong"'] & C \otimes_k V \arrow[d, "\xi \otimes_k 1_V"] \\
& k \otimes_k V
\end{tikzcd}$$

are commutative it suffices to note that for every $i \in I$ we have chains of equalities

$$(1_C \otimes_k d) \cdot d \cdot u_i = (1_C \otimes_k 1_C \otimes_k u_i) \cdot (1_C \otimes_k 1_C \otimes_k d_i) \cdot d_i = (1_C \otimes 1_C \otimes_k u_i) \cdot \left(\Delta \otimes_k 1_{V_i}\right) \cdot d_i = (\Delta \otimes_k 1_V) \cdot d \cdot u_i$$

and

$$(\xi \otimes_k 1_V) \cdot d \cdot u_i = (1_k \otimes_k u_i) \cdot \left(\xi \otimes_k 1_{V_i}\right) \cdot d_i = (1_k \otimes_k u_i) \cdot j_{V_i} = j_V \cdot u_i$$

where $j_W : W \to k \otimes_k W$ is the natural isomorphism for every k-module W. Hence (V, d) is a C-comodule. Suppose now that (W, e) is a C-comodule and $w_i : V_i \to W$ for $i \in I$ is a family of C-comodule morphisms compatible with the diagram $I \ni i \mapsto (V_i, d_i) \in \mathbf{coMod}(C)$. Since $\{u_i : V_i \to V\}_{i \in I}$ form a colimiting cocone for $I \ni i \mapsto V_i \in \mathbf{Mod}(k)$, there exists a unique morphism of k-modules $f : V \to W$ such that $f \cdot u_i = w_i$. Note that

$$e \cdot f \cdot u_i = e \cdot w_i = (1_C \otimes_k w_i) \cdot d_i = (1_C \otimes_k f) \cdot (1_C \otimes_k u_i) \cdot d_i = (1_C \otimes_k f) \cdot d \cdot u_i$$

for every $i \in I$. Hence $e \cdot f = (1_C \otimes_k f) \cdot d$. Thus f is a morphism of C-comodules. Thus (V, d) together with family $\left\{u_i : (V_i, d_i) \to (V, d)\right\}_{i \in I}$ is a colimit of the diagram $I \ni i \mapsto (V_i, d_i) \in \mathbf{coMod}(C)$ of C-comodules. This implies that the forgetful functor $\mathbf{coMod}(C) \to \mathbf{Mod}(k)$ creates colimits. □

Theorem 14.4. *Let C be a k-coalgebra such that C is a flat k-module. Then the forgetful functor* $\mathbf{coMod}(C) \to \mathbf{Mod}(k)$ *creates finite limits.*

Proof. The proof is similar to the proof of Theorem 14.3. □

Corollary 14.5. *Let C be a coalgebra over k and assume that C is flat as a k-module. Then* $\mathbf{coMod}(C)$ *is an abelian category with small colimits.*

Proof. This follows from Theorems 14.3 and 14.4. □

The next result is of fundamental importance.

Theorem 14.6. *Let C be a k-coalgebra that is free as a k-module. Suppose that V is a C-comodule over C. Then for every finitely generated k-submodule $U \subseteq V$ there exists a C-subcomodule W of V such that $U \subseteq W$ and W is a finitely generated k-module.*

The theorem follows from the following simple lemma.

Lemma 14.6.1. *Let C be a k-coalgebra over k that is free as a k-module. Suppose that V is a C-comodule over C and fix an element $v \in V$. Then there exists a C-subcomodule W of V such that $v \in W$ and W is a finitely generated k-module.*

Proof of the lemma. Let $\{e_j\}_{j\in J}$ be a free basis of C over k and let $d : V \to C \otimes_k V$ be a left coaction of C on V. Denote by $\Delta : C \to C \otimes_k C$ the comultiplication of C. Then we have

$$d(v) = \sum_{j\in J} e_j \otimes v_j$$

where $v_j \in V$ are zero for almost all $j \in J$. Next according to

$$(\Delta \otimes_k 1_V) \cdot d = (1_C \otimes_k d) \cdot d$$

we derive that equality

$$\sum_{j\in J} e_j \otimes d(v_j) = (1_C \otimes_k d)\big(d(v)\big) = (\Delta \otimes_k 1_V)\big(d(v)\big) = \sum_{j\in J} \Delta(e_j) \otimes v_j \subseteq \sum_{j\in J} C \otimes_k C \otimes_k k \cdot v_j$$

holds. This implies that $d(v_j) \subseteq C \otimes_k \left(\sum_{j\in J} k \cdot v_j\right)$. Hence k-submodule W of V generated by v and $\{v_j\}_{j\in J}$ is C-subcomodule of V. It is finitely generated as a k-module and $v \in W$. □

Proof of the theorem. Suppose that U is generated by $\{v_1, ..., v_n\}$ as a k-module. For each i pick C-subcomodule W_i of V such that W_i is finitely generated as a k-module and $v_i \in W_i$. This can be done by Lemma 14.6.1. Next

$$W = W_1 + ... + W_n$$

is a C-subcomodule of V that is finitely generated as a k-module and contains U. □

15. Linear representations and comodules

Let $\mathbf{M}$ be an affine monoid k-scheme and let $\rho : \mathfrak{P}_{\mathbf{M}} \to \mathcal{L}_V$ be a morphism of functors of sets, where V is a k-module. Yoneda Lemma implies that ρ is determined by some element (Example 8.4)

$$d_\rho \in \mathrm{Hom}_k\left(V, k[\mathbf{M}] \otimes_k V\right)$$

Theorem 15.1. *Let $\mathbf{M}$ be an affine monoid k-scheme. Then the correspondence*

$$\mathbf{Rep}(\mathbf{M}) \ni (V, \rho) \mapsto \left(V, d_\rho\right) \in \mathbf{coMod}\left(k[\mathbf{M}]\right)$$

is an isomorphism of categories over $\mathbf{Mod}(k)$.

Proof. We fix notation in the proof. We denote by $\mu_A : A \otimes_k A \to A$ the multiplication and by $\eta_A : k \to A$ the unit for every k-algebra A. If A is a k-algebra, then we denote by e_A the composition $\eta_A \cdot \xi_{\mathbf{M}} : k[\mathbf{M}] \to A$. Note that $e_A \in \mathfrak{P}_{\mathbf{M}}(A)$ is the neutral element.
We start the proof with some useful remarks. If V is a k-module, then

$$\mathcal{L}_V(A) = \mathrm{Hom}_k(V, A \otimes_k V)$$

for every k-algebra A with $\mathfrak{O}_k$-algebra structure discussed in Example 8.4. Moreover, if $\rho : \mathfrak{P}_{\mathbf{M}} \to \mathcal{L}_V$ is a morphism of k-functors corresponding to $d_\rho : V \to k[\mathbf{M}] \otimes_k V$, then for every k-algebra A and a morphism $f : k[\mathbf{M}] \to A$ of k-algebras we have

$$\rho(f) = (f \otimes_k 1_V) \cdot d_\rho$$

Our discussion in Example 8.4 and Yoneda Lemma show that the following assertions hold.

(1) For k-algebra A and $f_1, f_2 \in \mathrm{Hom}_k\left(k[\mathbf{M}], A\right) = \mathfrak{P}_{\mathbf{M}}(A)$ we have

$$\rho(f_1) \cdot \rho(f_2) = (\mu_A \otimes_k 1_V) \cdot (f_2 \otimes_k f_1 \otimes_k 1_V) \cdot \left(1_{k[\mathbf{M}]} \otimes_k d_\rho\right) \cdot d_\rho$$

and

$$\rho(f_1 \cdot f_2) = (\mu_A \otimes_k 1_V) \cdot (f_2 \otimes_k f_1 \otimes_k 1_V) \cdot (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho$$

(2) For k-algebra A we have

$$\rho(e_A) = (\eta_A \otimes_k 1_V) \cdot (\xi_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho$$

Now **(1)** imply that if $(\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho = \left(1_{O_{\mathbf{M}}} \otimes_k d_\rho\right) \cdot d_\rho$ then $\rho(f_1 \cdot f_2) = \rho(f_1) \cdot \rho(f_2)$. On the other hand suppose that $\rho(f_1 \cdot f_2) = \rho(f_1) \cdot \rho(f_2)$ for any two $f_1, f_2 : k[\mathbf{M}] \to A$ morphism of k-algebras and for every k-algebra A. Pick inclusions $f_1, f_2 : k[\mathbf{M}] \to k[\mathbf{M}] \otimes_k k[\mathbf{M}]$ onto first and second component, respectively. Then

$$\left(\mu_{k[\mathbf{M}]\otimes_k k[\mathbf{M}]} \otimes_k 1_V\right) \cdot (f_2 \otimes_k f_1 \otimes_k 1_V) = 1_{k[\mathbf{M}]} \otimes_k 1_{k[\mathbf{M}]} \otimes_k 1_V$$

and hence $(\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho = \left(1_{O_{\mathbf{M}}} \otimes_k d_\rho\right) \cdot d_\rho$ by **(1)**.
Now if $(\xi_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho$ is the canonical isomorphism $V \cong k \otimes_k V$. Then by **(2)** we derive that $\rho(e_A)$ is the canonical morphism $V \to A \otimes_k V$. On the other hand if $\rho(e_A)$ is $V \to A \otimes_k V$ for every k-algebra A, then substituting k for A we deduce by **(2)** that $\rho(e_k) = (\xi_{\mathbf{M}} \otimes_k 1_V) \cdot d_\rho$ is the canonical isomorphism $V \cong k \otimes_k V$.
These considerations prove that ρ is a morphism of monoid k-functors if and only if d_ρ is a coaction of $k[\mathbf{M}]$ on V.
Now suppose that V_1, V_2 are k-modules and $\rho_1 : \mathfrak{P}_{\mathbf{M}} \to \mathcal{L}_V$, $\rho_2 : \mathfrak{P}_{\mathbf{M}} \to \mathcal{L}_W$ are morphisms of k-functors. Suppose that $\phi : V_1 \to V_2$ is a morphism of k-modules. Pick a k-algebra A and a morphism $f : k[\mathbf{M}] \to A$ of k-algebras. Assume that the diagram

$$\begin{tikzcd} k[\mathbf{M}] \otimes_k V_1 \arrow[r, "1_{k[\mathbf{M}]} \otimes_k \phi"] & k[\mathbf{M}] \otimes_k V_2 \\ V_1 \arrow[u, "d_{\rho_1}"] \arrow[r, "\phi"'] & V_2 \arrow[u, "d_{\rho_2}"'] \end{tikzcd}$$

is commutative. Since the square

$$\begin{tikzcd} A \otimes_k V_1 \arrow[r, "1_A \otimes_k \phi"] & A \otimes_k V_2 \\ k[\mathbf{M}] \otimes_k V_1 \arrow[u, "f \otimes_k 1_V"] \arrow[r, "1_{k[\mathbf{M}]} \otimes_k \phi"'] & k[\mathbf{M}] \otimes_k V_2 \arrow[u, "f \otimes_k 1_W"'] \end{tikzcd}$$

is commutative, we derive that

$$\begin{tikzcd} A \otimes_k V_1 \arrow[r, "1_A \otimes_k \phi"] & A \otimes_k V_2 \\ V_1 \arrow[u, "\rho_1(f)"] \arrow[r, "\phi"'] & V_2 \arrow[u, "\rho_2(f)"'] \end{tikzcd}$$

Moreover, if the square above commutes for every k-algebra A, then it also commutes for $A = k[\mathbf{M}]$ and this recovers the commutativity of the first square. Suppose now that (V, ρ_1) and (W, ρ_2) are linear representations of $\mathbf{M}$, then the discussion above implies that ϕ is a morphism of linear representations if and only if ϕ is a morphism of $k[\mathbf{M}]$-comodules (V, d_{ρ_1}) and (W, d_{ρ_2}). $\square$

We obtain immediate consequence.

Corollary 15.2. *Let k be a field. Let (V,ρ) be a linear representation of an affine monoid k-scheme $\mathbf{M}$. Then for every finitely generated k-subspace $U \subseteq V$ there exists a subrepresentation W of (V,ρ) such that $U \subseteq W$ and W is a finitely generated k-module.*

Proof. This follows from Theorems 15.1 and 14.6. □

16. Regular representation and matrix coefficients for affine monoid k-schemes

Proposition 16.1. *Suppose that $\mathbf{M}$ is a monoid k-scheme. Then $\mathcal{M}\mathrm{or}_k(\mathfrak{P}_{\mathbf{M}}, \mathfrak{O}_k)$ exists and*

References

[DG70] Michel Demazure and Pierre Gabriel. *Groupes algébriques. Tome I. Géométrie algébrique généralités. Groupes commutatifs*. North-Holland, 1970.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[Mon19a] Monygham. Categories of presheaves. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2019.

[Mon19b] Monygham. Geometry of k-functors. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2019.

[Mon20] Monygham. Monoid k-functors and their representations. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2020.