GEOMETRY OF k-FUNCTORS

1. Introduction

In these notes we provide functorial approach to algebraic geometry. Our aim is to show that functorial and geometrical techniques are interrelated in a very efficient way.
Throughout these notes k is a fixed commutative ring and $\mathbf{Alg}_k$ denote the category of commutative k-algebras. If A, B are k-algebras, then we denote by $\mathrm{Mor}_k(A,B)$ the set of all morphisms $A \to B$ of k-algebras. Similarly if X, Y are k-schemes (i.e. schemes together with morphism to $\mathrm{Spec}\, k$), then we denote by $\mathrm{Mor}_k(X,Y)$ the set of all morphisms $X \to Y$ of k-schemes (morphisms of schemes that preserve structure morphisms to $\mathrm{Spec}\, k$).

2. k-functors

Definition 2.1. The category $\mathbf{Fun}(\mathbf{Alg}_k, \mathbf{Set})$ of copresheaves on $\mathbf{Alg}_k$ is called *the category of k-functors*.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are k-functors, then we denote by $\mathrm{Mor}_k(\mathfrak{X},\mathfrak{Y})$ the class of morphisms $\mathfrak{X} \to \mathfrak{Y}$ of k-functors. If $\sigma : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of k-functors, then for every k-algebra A we denote by σ^A the corresponding component of σ.
Let $\mathfrak{X}$ and $\mathfrak{Y}$ be A-functors for some k-algebra A. Then we denote by $\mathrm{Mor}_A(\mathfrak{X},\mathfrak{Y})$ the class of morphisms of A-functors $\mathfrak{X} \to \mathfrak{Y}$. For every A-algebra B and a morphism $\sigma : \mathfrak{X} \to \mathfrak{Y}$ of A-functors we denote by $\mathfrak{X}_B$, $\mathfrak{Y}_B$, σ_B the restrictions $\mathfrak{X}_{|\mathbf{Alg}_B}$, $\mathfrak{Y}_{|\mathbf{Alg}_B}$, $\sigma_{|\mathbf{Alg}_B}$ of these entities to the category of B-algebras.

Fact 2.2. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be k-functors. Assume that A is a k-algebra, B is an A-algebra, C is an B-algebra. Then the composition of maps of classes*

$$\mathrm{Mor}_A(\mathfrak{X}_A,\mathfrak{Y}_A) \xrightarrow{\sigma \mapsto \sigma_B} \mathrm{Mor}_B(\mathfrak{X}_B,\mathfrak{Y}_B) \xrightarrow{\sigma \mapsto \sigma_C} \mathrm{Mor}_C(\mathfrak{X}_C,\mathfrak{Y}_C)$$

equals

$$\mathrm{Mor}_A(\mathfrak{X}_A,\mathfrak{Y}_A) \xrightarrow{\sigma \mapsto \sigma_C} \mathrm{Mor}_C(\mathfrak{X}_C,\mathfrak{Y}_C)$$

Proof. Left to the reader. □

Definition 2.3. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be k-functors and suppose that for every k-algebra A the class $\mathrm{Mor}_A(\mathfrak{X}_A,\mathfrak{Y}_A)$ is a set. We define

$$\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{Y})(A) = \mathrm{Mor}_A(\mathfrak{X}_A,\mathfrak{Y}_A)$$

for every k-algebra A. This is a k-functor. Indeed, for every k-algebra A and A-algebra B we can compose a morphism $\sigma : \mathfrak{X}_A \to \mathfrak{Y}_A$ of k-functors with the forgetful functor $\mathbf{Alg}_B \to \mathbf{Alg}_A$. This induces a map

$$\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{Y})(A) \ni \sigma \mapsto \sigma_B \in \mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{Y})(B)$$

and according to Fact 2.2 these maps make $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{Y})$ a k-functor. The k-functor $\mathcal{M}\mathrm{or}_C(\mathfrak{X},\mathfrak{Y})$ is called *a hom k-functor of $\mathfrak{X}$ and $\mathfrak{Y}$*.

3. Zariski local k-functors and Zariski sheaves

In this part we use the notion of a Grothendieck topology on a category. For this notion we refer the reader to [Mon19b].

Definition 3.1. Let $\{f_i : X_i \to X\}_{i\in I}$ be a family of morphisms of k-schemes. We say that $\{f_i\}_{i\in I}$ is *a Zariski covering of* X if the following conditions are satisfied.

(1) For every $i \in I$ morphism f_i is an open immersion of schemes.

(2) Morphism $\coprod_{i\in I} X_i \to X$ induced by $\{f_i\}_{i\in I}$ is surjective.

The collection of all Zariski coverings on $\mathbf{Sch}_k$ is a Grothendieck pretopology.

Definition 3.2. We call the Grothendieck topology generated by the pretopology consisting of Zariski coverings on $\mathbf{Sch}_k$ *the Zariski topology on* $\mathbf{Sch}_k$. A presheaf on $\mathbf{Sch}_k$ that is a sheaf with respect to Zariski topology on $\mathbf{Sch}_k$ is called *a Zariski sheaf.*

Let $\mathfrak{X}$ be a presheaf on the category of k-schemes. Recall that by [Mon19b, Theorem 3.5] $\mathfrak{X}$ is a Zariski sheaf if and only if for every k-scheme X and for every Zariski covering $\{f_i : X_i \to X\}$ of X the diagram

$$\begin{tikzcd} \mathfrak{X}(X) \arrow[r, "\langle \mathfrak{X}(f_i)\rangle_{i\in I}"] & \prod_{i\in I}\mathfrak{X}(X_i) \arrow[r, shift left, "\langle \mathfrak{X}(f'_{ij})\cdot pr_i\rangle_{(i,j)}"] \arrow[r, shift right, "\langle \mathfrak{X}(f''_{ij})\cdot pr_j\rangle_{(i,j)}"'] & \prod_{(i,j)\in I\times I}\mathfrak{X}(X_i\times_X X_j) \end{tikzcd}$$

is a kernel of a pair of arrows, where for every $(i,j) \in I\times I$ morphisms f'_{ij} and f''_{ij} form a cartesian square

$$\begin{tikzcd} X_i\times_X X_j \arrow[r, "f''_{ij}"] \arrow[d, "f'_{ij}"'] & X_j \arrow[d, "f_j"] \\ X_i \arrow[r, "f_i"'] & X \end{tikzcd}$$

Now we repeat this definitions for k-algebras and k-functors.

Definition 3.3. Let $\{f_i : A \to A_i\}_{i\in I}$ be a family of morphisms of k-algebras. We say that $\{f_i\}_{i\in I}$ is *a Zariski covering of* A if the following conditions are satisfied.

(1) For every $i \in I$ morphism $\operatorname{Spec} f_i$ is an open immersion of schemes.

(2) Morphism $\coprod_{i\in I}\operatorname{Spec} A_i \to \operatorname{Spec} A$ induced by $\{\operatorname{Spec} f_i\}_{i\in I}$ is surjective.

The collection of all Zariski coverings on $\mathbf{Alg}_k$ induces on its opposite category $\mathbf{Aff}_k$ of affine k-schemes a Grothendieck pretopology.

Definition 3.4. We call the Grothendieck topology generated by the pretopology consisting of Zariski coverings on $\mathbf{Aff}_k$ *the Zariski topology on* $\mathbf{Aff}_k$. A k-functor that is a sheaf with respect to Zariski topology on $\mathbf{Aff}_k$ is called *a Zariski local k-functor*.

Let $\mathfrak{X}$ be a k-functor. Again by [Mon19b, Theorem 3.5] $\mathfrak{X}$ is a Zariski local k-functor if and only if for every k-algebra A and for every Zariski covering $\{f_i : A \to A_i\}$ of A the diagram

$$\begin{tikzcd} \mathfrak{X}(A) \arrow[r, "\langle \mathfrak{X}(f_i)\rangle_{i\in I}"] & \prod_{i\in I}\mathfrak{X}(A_i) \arrow[r, shift left, "\langle \mathfrak{X}(f'_{ij})\cdot pr_i\rangle_{(i,j)}"] \arrow[r, shift right, "\langle \mathfrak{X}(f''_{ij})\cdot pr_j\rangle_{(i,j)}"'] & \prod_{(i,j)\in I\times I}\mathfrak{X}(A_i\otimes_A A_j) \end{tikzcd}$$

is a kernel of a pair of arrows, where for every $(i,j) \in I\times I$ morphisms f'_{ij} and f''_{ij} form a cocartesian square

$$\begin{tikzcd} A \arrow[r, "f_j"] \arrow[d, "f_i"'] & A_j \arrow[d, "f'_{ji}"] \\ A_i \arrow[r, "f'_{ij}"'] & A_i \otimes_A A_j \end{tikzcd}$$

Now we state the main result of this section.

Theorem 3.5. *Let*

$$\widehat{\mathbf{Sch}_k} \longrightarrow \text{the category of } k\text{-functors}$$

be the restriction of presheaves on $\mathbf{Sch}_k$ *to copresheaves on* $\mathbf{Alg}_k$ *(k-functors) induced by the contravariant functor* $\mathrm{Spec} : \mathbf{Alg}_k \to \mathbf{Sch}_k$. *Then it induces an equivalence of categories between Zariski sheaves on* $\mathbf{Sch}_k$ *and Zariski local k-functors.*

Proof. Note that $\mathbf{Aff}_k$ with Zariski topology is a dense subsite ([Mon19b, definition 4.4]) of $\mathbf{Sch}_k$ with Zariski topology. Hence the result is a special case of a more general theorem [Mon19b, Theorem 4.6]. □

4. Schemes and their functors of points

Let X be a k-scheme. We define a k-functor $\mathfrak{P}_X$ by formula

$$\mathfrak{P}_X(A) = \mathrm{Mor}_k(\mathrm{Spec}\, A, X)$$

That is $\mathfrak{P}_X$ is the restriction of the presheaf on $\mathbf{Sch}_k$ represented by X to the category $\mathbf{Alg}_k$ along the functor $\mathrm{Spec} : \mathbf{Alg}_k \to \mathbf{Sch}_k$. Next if $f : X \to Y$ is a morphism of k-schemes, then $\mathfrak{P}_f$ is the restriction of a morphism of presheaves on $\mathbf{Sch}_k$ represented by f to the category of k-algebras along $\mathrm{Spec} : \mathbf{Alg}_k \to \mathbf{Sch}_k$. Thus we have a functor

$$\mathbf{Sch}_k \xrightarrow{\mathfrak{P}} \text{the category of } k\text{-functors}$$

v

Fact 4.1. *Functor*

$$\mathbf{Sch}_k \xrightarrow{\mathfrak{P}} \text{the category of } k\text{-functors}$$

is full, faithful and its image consists of Zariski local k-functors. Moreover, $\mathfrak{P}$ *preserves limits.*

Proof. Note that the presheaf h_X on $\mathbf{Sch}_k$ represented by X is a Zariski sheaf. Indeed, this just rephrases standard fact that morphism of schemes can be glued in Zariski topology. Next according to Theorem 3.5 the functor $\mathrm{Spec} : \mathbf{Alg}_k \to \mathbf{Sch}_k$ induces an equivalence between the category of Zariski sheaves and the category of local Zariski k-functors. Thus $\mathfrak{P}_X$ is a local Zariski k-functor and $\mathfrak{P}$ it is full and faithful. Note that Yoneda embedding $h : \mathbf{Sch}_k \to \widehat{\mathbf{Sch}_k}$ and the functor

$$\widehat{\mathbf{Sch}_k} \xrightarrow{\text{induced by } \mathrm{Spec}} \text{the category of } k\text{-functors}$$

preserve limits. Thus their composition $\mathfrak{P}$ also preserves limits. □

Definition 4.2. Let X be a k-scheme. Then $\mathfrak{P}_X$ is called *the k-functor of points of X*.

Finally note that for every k-algebra A we have an identification $\mathfrak{P}_{\operatorname{Spec} A} = \operatorname{Hom}_k(A, -)$ and this identification is natural with respect to A. In other words $\mathfrak{P} \cdot \operatorname{Spec}$ is the (co)Yoneda embedding of $\mathbf{Alg}_k$ into the category of k-functors.
Suppose now that A is a k-algebra and $\mathfrak{a} \subseteq A$ is an ideal. Then we define $V(\mathfrak{a}) = \operatorname{Spec} A/\mathfrak{a}$ as a closed subscheme $\operatorname{Spec} A$ induced by the quotient morphism $A \to A/\mathfrak{a}$. We define an open subscheme $D(\mathfrak{a}) = \operatorname{Spec} A \smallsetminus V(\mathfrak{a})$ of $\operatorname{Spec} A$.

Definition 4.3. Let $\sigma : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of k-functors. Assume that for every k-algebra A and every morphism $\tau : \mathfrak{P}_{\operatorname{Spec} A} \to \mathfrak{Y}$ of k-functors there exist an ideal $\mathfrak{a}$ in A and a morphism $\tau' : \mathfrak{P}_{D(\mathfrak{a})} \to \mathfrak{X}$ of k-functors such that the square

$$\begin{tikzcd}
\mathfrak{P}_{D(\mathfrak{a})} \arrow[r, "\tau'"] \arrow[d, hook] & \mathfrak{X} \arrow[d, "\sigma"] \\
\mathfrak{P}_{\operatorname{Spec} A} \arrow[r, "\tau"'] & \mathfrak{Y}
\end{tikzcd}$$

is cartesian. Then σ is *an open immersion of k-functors*.

Fact 4.4. *The class of open immersions of k-functors is closed under base change and composition.*

Proof. Left to the reader. $\square$

Definition 4.5. Let $\mathfrak{X}$ be a k-functor and $\{\sigma_i : \mathfrak{X}_i \to \mathfrak{X}\}_{i \in I}$ be a family of open immersions. Then for every k-algebra A and $x \in \mathfrak{X}(A)$ we have a family of ideals $\{\mathfrak{a}_i\}_{i \in I}$ defined by cartesian squares

$$\begin{tikzcd}
\mathfrak{P}_{D(\mathfrak{a}_i)} \arrow[r, "\tau'"] \arrow[d, hook] & \mathfrak{X}_i \arrow[d, "\sigma_i"] \\
\mathfrak{P}_{\operatorname{Spec} A} \arrow[r, "\tau"'] & \mathfrak{X}
\end{tikzcd}$$

in which bottom vertical morphism $\tau : \mathfrak{P}_{\operatorname{Spec} A} \to \mathfrak{X}$ corresponds to x. We say that $\{\sigma_i\}_{i \in I}$ is *an open cover of $\mathfrak{X}$* if for every k-algebra A and $x \in \mathfrak{X}(A)$ we have

$$\operatorname{Spec} A = \bigcup_{i \in I} D(\mathfrak{a}_i)$$

or in other words $A = \sum_{i \in I} \mathfrak{a}_i$.

Theorem 4.6. *Let $\mathfrak{X}$ be a k-functor. Then the following are equivalent.*

- **(i)** *$\mathfrak{X}$ is isomorphic with functor of points of some k-scheme.*
- **(ii)** *$\mathfrak{X}$ is a Zariski local k-functor and there exists an open cover $\{\sigma_i : \mathfrak{P}_{X_i} \to \mathfrak{X}\}_{i \in I}$ of k-functors for some family $\{X_i\}_{i \in I}$ of k-schemes.*
- **(iii)** *$\mathfrak{X}$ is a Zariski local k-functor and there exists an open cover $\{\sigma_i : \mathfrak{P}_{\operatorname{Spec} A_i} \to \mathfrak{X}\}_{i \in I}$ of k-functors for some family $\{A_i\}_{i \in I}$ of k-algebras.*

The proof depends on two lemmas. Check [Mon19b, Definition 7.1] for the notion of a locally surjective morphism.

Lemma 4.6.1. *Let $f : X \to Y$ be a morphism of k-schemes. Suppose that f is surjective morphism and an open immersion locally on X. Then $\mathfrak{P}_f$ is a locally surjective morphism of Zariski local k-functors.*

Proof of the lemma. Let A be a k-algebra and $g : \operatorname{Spec} A \to Y$ be a morphism of k-schemes. Since f is surjective and an open immersion locally on X, there exist a Zariski cover $\{f_i : A \to A_i\}_{i \in I}$ and a family $\{g_i : \operatorname{Spec} A_i \to X\}_{i \in I}$ of morphisms of k-schemes such that $f \cdot g_i = g \cdot \operatorname{Spec} f_i$ for every $i \in I$.

This implies that $\mathfrak{P}_f(g_i) = \mathfrak{P}_Y(f_i)(g)$ for every $i \in I$. Thus $\mathfrak{P}_f$ is a locally surjective morphism of Zariski local k-functors. □

Lemma 4.6.2. *Let $X = \coprod_{i \in I} X_i, R = \coprod_{i,j \in I} R_{ij}$ be disjoint sums of k-schemes and let $p, q : R \to X$ be morphisms of k-schemes such that the following conditions are satisfied.*

(1) *For any $i, j \in I$ morphism $p_{|R_{ij}}$ induces an open immersion $R_{ij} \hookrightarrow X_i$ and morphism $q_{|R_{ij}}$ induces an open immersion $R_{ij} \hookrightarrow X_j$.*

(2) *For every $i \in I$ morphisms $p_{|R_{ii}}$ and $q_{|R_{ii}}$ are equal and induce an isomorphisms $R_{ii} \to X_i$.*

(3) *Triple (R, p, q) is an equivalence relation on X in the category of k-schemes.*

Then there exist a k-scheme Y and a morphism $f : X \to Y$ of k-schemes such that

$$\begin{tikzcd} \mathfrak{P}_R \arrow[r, shift left, "\mathfrak{P}_p"] \arrow[r, shift right, "\mathfrak{P}_q"'] & \mathfrak{P}_X \arrow[r, "\mathfrak{P}_f"] & \mathfrak{P}_Y \end{tikzcd}$$

is a cokernel of a pair $(\mathfrak{P}_p, \mathfrak{P}_q)$ in the category of Zariski local k-functors.

Proof of the lemma. Let

$$\begin{tikzcd} R \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

be a cokernel in the category of ringed spaces. It exists according to [Mon19c, Remark 2.3]. Moreover, [Mon19c, Theorem 3.2] states that for every $i \in I$ subset $f(X_i)$ is open in Y and we have an isomorphism of ringed spaces $X_i \cong f(X_i)$ induced by f. Therefore, Y is a k-scheme and $f : X \to Y$ is a morphism of k-schemes.

Now we verify that $\mathfrak{P}_f$ is the quotient in the category of Zariski local k-functors. For this note that we proved above that f is open immersion of k-schemes locally on X and it is surjective. Thus by Lemma 4.6.1 we derive that $\mathfrak{P}_f$ is a locally surjective morphism of Zariski local k-functors. Therefore ([Mon19b, Theorem 7.3]), it suffices to show that the square

$$\begin{tikzcd} \mathfrak{P}_R \arrow[r, "\mathfrak{P}_q"] \arrow[d, "\mathfrak{P}_p"'] & \mathfrak{P}_X \arrow[d, "\mathfrak{P}_f"] \\ \mathfrak{P}_X \arrow[r, "\mathfrak{P}_f"'] & \mathfrak{P}_Y \end{tikzcd}$$

is cartesian. Since $\mathfrak{P}$ preserves limits (Fact 4.1), we derive that it suffices to check that

$$\begin{tikzcd} R \arrow[r, "q"] \arrow[d, "p"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is cartesian square of k-schemes. By [Mon19c, Remark 2.3] we have $R_{ij} = X_i \times_Y X_j$ for every $i, j \in I$ and hence

$$X \times_Y X = \left(\coprod_{i \in I} X_i\right) \times_Y \left(\coprod_{i \in I} X_i\right) = \coprod_{i,j \in I} \left(X_i \times_Y X_j\right) = \coprod_{i,j \in I} R_{ij} = R$$

Thus the result follows. □

Proof of the theorem. If **(i)** holds, then we may assume that $\mathfrak{X} = \mathfrak{P}_Y$ for some k-scheme Y. Fact 4.1 states that $\mathfrak{P}_Y$ is a Zariski local k-functor and clearly $1_{\mathfrak{P}_Y} : \mathfrak{P}_Y \to \mathfrak{P}_Y$ is an open cover. Thus **(i)** $\Rightarrow$ **(ii)**.

Every functor of points of a k-scheme admits open cover by functors of points of affine k-schemes. Indeed, it suffices to take open affine subschemes that cover given k-scheme and apply $\mathfrak{P}$. This implies that every open cover of a k-functor $\mathfrak{X}$ by functors of points of k-schemes admits refinement by open cover of functors of points of affine k-schemes. Therefore, implication **(ii)** $\Rightarrow$ **(iii)** holds.

Suppose that a k-functor $\mathfrak{X}$ is Zariski local and $\left\{\sigma_i : \mathfrak{P}_{\operatorname{Spec} A_i} \to \mathfrak{X}\right\}_{i \in I}$ is an open cover of $\mathfrak{X}$. Note that for every $i, j \in I$ there exist a k-scheme R_{ij} and open immersions $p_{ij} : R_{ij} \hookrightarrow \operatorname{Spec} A_i$, $q_{ij} : R_{ij} \hookrightarrow \operatorname{Spec} A_j$ such that the square

$$\begin{tikzcd} \mathfrak{P}_{R_{ij}} \arrow[r, "\mathfrak{P}_{q_{ij}}"] \arrow[d, "\mathfrak{P}_{p_{ij}}"'] & \mathfrak{P}_{\operatorname{Spec} A_j} \arrow[d, "\sigma_j"] \\ \mathfrak{P}_{\operatorname{Spec} A_i} \arrow[r, "\sigma_i"'] & \mathfrak{X} \end{tikzcd}$$

is cartesian. Consider k-scheme $X = \coprod_{i \in I} \operatorname{Spec} A_i$ and morphism $\sigma : \mathfrak{P}_X \to \mathfrak{X}$ induced by $\{\sigma_i\}_{i \in I}$. Moreover, consider k-scheme $R = \coprod_{i,j \in I} R_{ij}$ and morphisms $p, q : R \to X$ induced by $\{p_{ij}\}_{i,j \in I}$ and $\{q_{ij}\}_{i,j \in I}$, respectively. Note that the square

$$\begin{tikzcd} \mathfrak{P}_R \arrow[r, "\mathfrak{P}_q"] \arrow[d, "\mathfrak{P}_p"'] & \mathfrak{P}_X \arrow[d, "\sigma"] \\ \mathfrak{P}_X \arrow[r, "\sigma"'] & \mathfrak{X} \end{tikzcd}$$

is cartesian and hence $\left(\mathfrak{P}_R, \mathfrak{P}_p, \mathfrak{P}_q\right)$ is an equivalence relation. By Lemma 4.6.2 there exist a k-scheme Y and a morphism $f : X \to Y$ such that

$$\begin{tikzcd} \mathfrak{P}_R \arrow[r, shift left, "\mathfrak{P}_p"] \arrow[r, shift right, "\mathfrak{P}_q"'] & \mathfrak{P}_X \arrow[r, "\mathfrak{P}_f"] & \mathfrak{P}_Y \end{tikzcd}$$

is a cokernel of $\left(\mathfrak{P}_p, \mathfrak{P}_q\right)$. Moreover, σ is locally surjective morphism of Zariski local k-functors and hence also

$$\begin{tikzcd} \mathfrak{P}_R \arrow[r, shift left, "\mathfrak{P}_p"] \arrow[r, shift right, "\mathfrak{P}_q"'] & \mathfrak{P}_X \arrow[r, "\sigma"] & \mathfrak{X} \end{tikzcd}$$

is a cokernel of $\left(\mathfrak{P}_p, \mathfrak{P}_q\right)$. Thus $\mathfrak{P}_Y$ is isomorphic with $\mathfrak{X}$. This proves **(iii)** $\Rightarrow$ **(i)**. □

Proposition 4.7. *Let $\sigma : \mathfrak{X} \to \mathfrak{Y}$ be a monomorphism of k-functors and $\mathfrak{Y}$ be a Zariski local k-functor. Assume that for every k-algebra A and every morphism $\tau : \mathfrak{P}_{\operatorname{Spec} A} \to \mathfrak{Y}$ of k-functors there exist a Zariski local k-functor $\mathfrak{Z}$ that fits into a cartesian square*

$$\begin{tikzcd}
\mathfrak{Z} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, "\sigma"] \\
\mathfrak{P}_{\operatorname{Spec} A} \arrow[r, "\tau"'] & \mathfrak{Y}
\end{tikzcd}$$

Then $\mathfrak{X}$ is a Zariski local k-functor.

Proof. Suppose that A is a k-algebra and S is a covering sieve on A with respect to Zariski topology. Recall that by [Mon19b, page 2] we may consider S as a subcopresheaf of $\mathfrak{P}_{\operatorname{Spec} A}$. Suppose that $\tau : \mathfrak{P}_{\operatorname{Spec} A} \to \mathfrak{Y}$ and $m : S \to \mathfrak{X}$ are morphisms of k-functors such that $\sigma \cdot m$ is equal to the composition of $S \hookrightarrow \mathfrak{P}_{\operatorname{Spec} A}$ with τ. Next there exists a Zariski local k-functor $\mathfrak{Z}$ that fits into a cartesian square

$$\begin{tikzcd}
\mathfrak{Z} \arrow[r, "\tau'"] \arrow[d, "\sigma'"'] & \mathfrak{X} \arrow[d, "\sigma"] \\
\mathfrak{P}_{\operatorname{Spec} A} \arrow[r, "\tau"'] & \mathfrak{Y}
\end{tikzcd}$$

of k-functors. By universal property of cartesian squares there exists a unique morphism $n : S \to \mathfrak{Z}$ of k-functors such that the diagram

$$\begin{tikzcd}
S \arrow[dr, "n"] \arrow[drr, bend left, "m"] \arrow[ddr, hook, bend right] & & \\
& \mathfrak{Z} \arrow[r, "\tau'"] \arrow[d, "\sigma'"'] & \mathfrak{X} \arrow[d, "\sigma"] \\
& \mathfrak{P}_{\operatorname{Spec} A} \arrow[r, "\tau"'] & \mathfrak{Y}
\end{tikzcd}$$

is commutative. Since $\mathfrak{Z}$ is Zariski local, there exists a morphism $\rho : \mathfrak{P}_{\operatorname{Spec} A} \to \mathfrak{Z}$ such that $\rho_{|S} = n$. Then $(\tau' \cdot \rho)_{|S} = \tau' \cdot n = m$ and hence matching family m admits an amalgamation. Since σ is a monomorphism, this suffices to prove that $\mathfrak{X}$ is a Zariski local k-functor. □

5. Representable morphisms of k-functors

Definition 5.1. Let $\sigma : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of k-functors. Assume that for every k-algebra A and every morphism $\tau : \mathfrak{P}_{\operatorname{Spec} A} \to \mathfrak{Y}$ of k-functors there exist a k-scheme X, a morphism $f : X \to \operatorname{Spec} A$ and a morphism $\tau' : \mathfrak{P}_X \to \mathfrak{X}$ of k-functors such that the square

$$\begin{tikzcd}
\mathfrak{P}_X \arrow[r, "\tau'"] \arrow[d, "\mathfrak{P}_f"'] & \mathfrak{X} \arrow[d, "\sigma"] \\
\mathfrak{P}_{\operatorname{Spec} A} \arrow[r, "\tau"'] & \mathfrak{Y}
\end{tikzcd}$$

is cartesian. Then σ is *a representable morphism of k-functors*.

Fact 5.2. *The class of representable morphisms of k-functors is closed under base change and composition.*

Proof. Left to the reader. □

Proposition 5.3. *Let $\sigma : \mathfrak{X} \to \mathfrak{Y}$ be a representable morphism of Zariski local k-functors. Fix a k-scheme Y and a morphism $\tau : \mathfrak{P}_Y \to \mathfrak{Y}$. Then there exist a k-scheme X, a morphism $f : X \to Y$ and a morphism $\tau' : \mathfrak{P}_X \to \mathfrak{X}$ such that the square*

$$\begin{tikzcd}
\mathfrak{P}_X \arrow[r, "\tau'"] \arrow[d, "\mathfrak{P}_f"'] & \mathfrak{X} \arrow[d, "\sigma"] \\
\mathfrak{P}_Y \arrow[r, "\tau"'] & \mathfrak{Y}
\end{tikzcd}$$

is cartesian.

Proof. Let

$$\begin{tikzcd}
\mathfrak{Z} \arrow[r, "\tau'"] \arrow[d, "\sigma'"'] & \mathfrak{X} \arrow[d, "\sigma"] \\
\mathfrak{P}_Y \arrow[r, "\tau"'] & \mathfrak{Y}
\end{tikzcd}$$

be a cartesian square. According to [Mon19b, Theorem 2.12] k-functor $\mathfrak{Z}$ is Zariski local. Suppose that $\{f_i : \operatorname{Spec} A_i \to Y\}_{i\in I}$ is an open cover of Y. Then $\{\mathfrak{P}_{f_i} : \mathfrak{P}_{\operatorname{Spec} A_i} \to \mathfrak{P}_Y\}_{i\in I}$ is an open cover of $\mathfrak{P}_Y$ and hence its base change $\{\tau_i : \mathfrak{Z}_i \to \mathfrak{Z}\}_{i\in I}$ is an open cover of $\mathfrak{Z}$. Since σ is representable, we deduce that $\mathfrak{Z}_i$ is a functor of points of some k-scheme for $i \in I$. Now by Theorem 4.6 we derive that there exists a k-scheme X such that $\mathfrak{Z}$ is isomorphic with $\mathfrak{P}_X$. This proves the result. □

Definition 5.4. Let $\sigma : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of k-functors. Assume that for every k-algebra A and every morphism $\tau : \mathfrak{P}_{\operatorname{Spec} A} \to \mathfrak{Y}$ of k-functors there exist an ideal $\mathfrak{a}$ in A and morphism $\tau' : \mathfrak{P}_{V(\mathfrak{a})} \to \mathfrak{X}$ such that the square

$$\begin{tikzcd}
\mathfrak{P}_{V(\mathfrak{a})} = \mathfrak{P}_{\operatorname{Spec} A/\mathfrak{a}} \arrow[r, "\tau'"] \arrow[d, "\mathfrak{P}_{\operatorname{Spec} q}"'] & \mathfrak{X} \arrow[d, "\sigma"] \\
\mathfrak{P}_{\operatorname{Spec} A} \arrow[r, "\tau"'] & \mathfrak{Y}
\end{tikzcd}$$

is cartesian, where $q : A \to A/\mathfrak{a}$ is the quotient map. Then σ is *a closed immersion of k-functors*.

Fact 5.5. *The class of closed immersions of k-functors is closed under base change and composition.*

Proof. Left to the reader. □

Proposition 5.6. *Let $\sigma : \mathfrak{X} \to \mathfrak{Y}$ be a closed (open) immersion of k-functors. Fix a k-scheme Y and a morphism $\tau : \mathfrak{P}_Y \to \mathfrak{Y}$. Then there exist a k-scheme X, a closed (open) immersion $f : X \to Y$ of schemes and a morphism $\tau' : \mathfrak{P}_X \to \mathfrak{X}$ of k-functors such that the square*

$$\begin{tikzcd}
\mathfrak{P}_X \arrow[r, "\tau'"] \arrow[d, "\mathfrak{P}_f"'] & \mathfrak{X} \arrow[d, "\sigma"] \\
\mathfrak{P}_Y \arrow[r, "\tau"'] & \mathfrak{Y}
\end{tikzcd}$$

is cartesian.

Proof. According to Fact 5.5 (Fact 4.4) pullback $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{P}_Y \rightarrow \mathfrak{P}_Y$ of σ along τ is a closed (open) immersion of k-functors. Since $\mathfrak{P}_Y$ is a Zariski local k-functor by Fact 4.1 and closed (open) immersions are monomorphisms, we derive by Proposition 4.7 that a fiber-product $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{P}_Y$ of σ and τ is a Zariski local k-functor. Since closed (open) immersions of k-functors are representable, we deduce by Proposition 5.3 that there exists a k-scheme X, a morphism $f : X \rightarrow Y$ of k-schemes and a morphism $\tau' : \mathfrak{P}_X \rightarrow \mathfrak{X}$ of k-functors such that the square

$$\begin{tikzcd}
\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{P}_Y \cong \mathfrak{P}_X \arrow[r, "\tau'"] \arrow[d, "\mathfrak{P}_f"'] & \mathfrak{X} \arrow[d, "\sigma"] \\
\mathfrak{P}_Y \arrow[r, "\tau"'] & \mathfrak{Y}
\end{tikzcd}$$

is cartesian and $\mathfrak{P}_f$ is a closed (open) immersion of k-functors. Since the functor

$$\widehat{\mathbf{Sch}_k} \xrightarrow{\mathfrak{P}} \text{the category of } k\text{-functors}$$

preserves finite limits, it follows that for every open affine subset V of Y we have a cartesian square

$$\begin{tikzcd}
\mathfrak{P}_{f^{-1}(V)} \arrow[r, hook] \arrow[d, "\mathfrak{P}_{f_V}"'] & \mathfrak{P}_X \arrow[d, "\mathfrak{P}_f"] \\
\mathfrak{P}_V \arrow[r, hook] & \mathfrak{P}_Y
\end{tikzcd}$$

where $f_V : f^{-1}(V) \rightarrow V$ is the restriction of f. Next as $\mathfrak{P}_f$ is a closed (open) immersion and V is affine, we derive that f_V is a closed (open) immersion of schemes. Since this holds for every affine open subset V of Y, we deduce that f is a closed (open) immersion. $\square$

The next result is frequently used in the theory of *algebraic spaces*.

Proposition 5.7. *Let $\mathfrak{Y}$ be a k-functor such that the diagonal $\mathfrak{Y} \rightarrow \mathfrak{Y} \times \mathfrak{Y}$ is representable. Then every morphism $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$ of k-functors is representable.*

Proof. Fix a morphism of k-functors $\sigma : \mathfrak{X} \rightarrow \mathfrak{Y}$. Let Y be a k-scheme and let $\tau : \mathfrak{P}_Y \rightarrow \mathfrak{Y}$ be a morphism of k-functors. Consider the cartesian square

$$\begin{tikzcd}
\mathfrak{Z} \arrow[r, "\tau'"] \arrow[d, "\sigma'"'] & \mathfrak{X} \arrow[d, "\sigma"] \\
\mathfrak{P}_Y \arrow[r, "\tau"'] & \mathfrak{Y}
\end{tikzcd}$$

Then there exists a cartesian square

$$\begin{tikzcd} \mathfrak{Z} \arrow[r] \arrow[d] & \mathfrak{Y} \arrow[d, "\textbf{diagonal}"] \\ \mathfrak{P}_Y \times \mathfrak{Y} \arrow[r, "\tau \times \sigma"'] & \mathfrak{Y} \times \mathfrak{Y} \end{tikzcd}$$

Since the diagonal of $\mathfrak{Y}$ is representable, we derive that $\mathfrak{Z}$ is isomorphic with functor of points of some k-scheme. This finishes the proof. □

6. Closed immersions and hom k-functors

Definition 6.1. Let X be a k-scheme. Suppose that there exists an open affine cover $X = \bigcup_{i \in I} X_i$ such that k-algebra $\Gamma(X_i, \mathcal{O}_{X_i})$ is free as a k-module. Then we say that X is *a locally free k-scheme.*

Next theorem is the main result of this section.

Theorem 6.2. *Let $j : \mathfrak{Y}' \to \mathfrak{Y}$ be a closed immersion of k-functors and X be a locally free k-scheme. Suppose that classes* $\mathrm{Mor}_A\left((\mathfrak{P}_X)_A, \mathfrak{Y}_A\right)$ *are sets for every k-algebra A. Then classes* $\mathrm{Mor}_A\left((\mathfrak{P}_X)_A, \mathfrak{Y}'_A\right)$ *are sets for every k-algebra A and the morphism*

$$\mathcal{M}\mathrm{or}_k\left(1_{\mathfrak{P}_X}, j\right) : \mathcal{M}\mathrm{or}_k\left(\mathfrak{P}_X, \mathfrak{Y}'\right) \to \mathcal{M}\mathrm{or}_k\left(\mathfrak{P}_X, \mathfrak{Y}\right)$$

is a closed immersion of k-functors.

It is useful to isolate crucial steps in the argument. For this we proceed by proving some lemmas.

Lemma 6.2.1. *Suppose that A is a commutative ring. Let $j : \mathfrak{Y}' \to \mathfrak{Y}$ be a closed immersion of A-functors and X be an affine A-scheme such that $\Gamma(X, \mathcal{O}_X)$ is a free A-module. Assume that $\tau : \mathfrak{P}_X \to \mathfrak{Y}$ is a morphism of A-functors. Then there exists an ideal $\mathfrak{a} \subseteq A$ such that for every A-algebra B the restriction τ_B factors through j_B if and only if the structure morphism $f : A \to B$ of B satisfies* $\mathfrak{a} \subseteq \ker(f)$*.*

Proof of the lemma. Since j is a closed immersion of A-functors and X is affine k-scheme there exists an affine A-scheme X', a closed immersion $j' : X' \to X$ of schemes and a cartesian square

$$\begin{tikzcd} \mathfrak{P}_{X'} \arrow[r] \arrow[d, "\mathfrak{P}_{j'}"'] & \mathfrak{Y}' \arrow[d, "j"] \\ \mathfrak{P}_X \arrow[r, "\tau"'] & \mathfrak{Y} \end{tikzcd}$$

of A-functors. Next let B be an A-algebra with the structure morphism $f : A \to B$. Then τ_B factors through j_B if and only if the projection $\mathrm{Spec}\, B \times_{\mathrm{Spec}\, A} X \to X$ induced by f factors through X'. Let $A[X]$ be the A-algebra of global regular functions on X and let $\mathfrak{J}$ be an ideal in $A[X]$ such that $A[X]/\mathfrak{J} = A[X']$ is the A-algebra of global regular functions of X'. With this notation we derive that the projection $\mathrm{Spec}\, B \times_{\mathrm{Spec}\, A} X \to X$ induced by f factors through X' if and only if the morphism $A[X] \to B \otimes_A A[X]$ induced by f sends every element of $\mathfrak{J}$ to zero. Since $A[X]$ is a free A-module, we write $A[X] = A^{\oplus I}$ for some index set I. Then the morphism $A[X] \to B \otimes_A A[X]$ induced by f is just $f^{\oplus I} : A^{\oplus I} \to B^{\oplus I}$. We have $f^{\oplus I}(\mathfrak{J}) = 0$ if and only if $\left(pr_i^B \cdot f^{\oplus I}\right)(\mathfrak{J}) =$ for every $i \in I$, where $pr_i^B : B^{\oplus I} \to B$ is the projection on i-th component. Pick $i \in I$ and consider the commutative diagram

$$\begin{tikzcd} A^{\oplus I} \arrow[r, "f^{\oplus I}"] \arrow[d, "pr_i^A"'] & B^{\oplus I} \arrow[d, "pr_i^B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

In the diagram pr_i^A is the projection on i-th component. Diagram implies that $\left(pr_i^B \cdot f^{\oplus I}\right)(\mathfrak{J}) =$ for every $i \in I$ if and only if $\left(f \cdot pr_i^A\right)(\mathfrak{J}) = 0$ for every $i \in I$. This is equivalent with the condition that $f(\mathfrak{a}) = 0$ for ideal $\mathfrak{a}$ in A generated by $\sum_{i \in I} pr_i^A(\mathfrak{J})$. Thus the lemma is proved. □

Lemma 6.2.2. *Suppose that A is a commutative ring. Let $j : \mathfrak{Y}' \to \mathfrak{Y}$ be a closed immersion of A-functors and X be an A-scheme with open cover*

$$X = \bigcup_{i \in I} X_i$$

Assume that $\tau : \mathfrak{P}_X \to \mathfrak{Y}$ is a morphism of A-functors. Fix an A-algebra B. Then τ_B factors through j_B if and only if $\left(\tau_{|\mathfrak{P}_{X_i}}\right)_B$ factors through j_B for every $i \in I$.

Proof of the lemma. If τ_B factors through j_B, then also $\left(\tau_{|\mathfrak{P}_{X_i}}\right)_B$ factors through j_B for every $i \in I$. It suffices to prove the converse. So suppose that $\left(\tau_{|\mathfrak{P}_{X_i}}\right)_B$ factors through j_B for every $i \in I$. Since j is a closed immersion of A-functors and X is an A-scheme, Proposition 5.6 implies that there exists a cartesian square

$$\begin{tikzcd} \mathfrak{P}_{X'} \arrow[r] \arrow[d, "\mathfrak{P}_{j'}"'] & \mathfrak{Y}' \arrow[d, "j"] \\ \mathfrak{P}_X \arrow[r, "\tau"'] & \mathfrak{Y} \end{tikzcd}$$

where $j' : X' \to X$ is a closed immersion of A-schemes. For each $i \in I$ let $j_i' : j'^{-1}(X_i) \to X_i$ be the restriction of j'. We have the induced cartesian square

$$\begin{tikzcd} \mathfrak{P}j'^{-1}(X_i) \arrow[r] \arrow[d, "\mathfrak{P}_{j_i'}"'] & \mathfrak{Y}' \arrow[d, "j"] \\ \mathfrak{P}_{X_i} \arrow[r, "\tau_{|\mathfrak{P}_{X_i}}"'] & \mathfrak{Y} \end{tikzcd}$$

Now $\left(\tau_{|\mathfrak{P}_{X_i}}\right)_B$ factors through j_B. This implies that $(\mathfrak{P}_{j_i'})_B$ admits a section for every $i \in I$. Then $(\mathfrak{P}_{j_i'})_B$ is an isomorphism for every $i \in I$. Thus $j_i' \times_{\operatorname{Spec} A} 1_{\operatorname{Spec} B}$ is an isomorphism for every $i \in I$ and hence $j' \times_{\operatorname{Spec} A} 1_{\operatorname{Spec} B}$ is an isomorphism of B-schemes. This means that τ_B factors through j_B. □

Proof of the theorem. Let A be a k-algebra. The restriction functor $(-)_{|\mathbf{Alg}_A} = (-)_A$ preserves all closed immersions. Thus j_A is a closed immersion of A-functors and hence we derive that $j_A : \mathfrak{Y}'_A \to \mathfrak{Y}_A$ is a monomorphism of A-functors. Thus we have an injective map of classes

$$\operatorname{Mor}_A\left(1_{(\mathfrak{P}_X)_A}, j_A\right) : \operatorname{Mor}_A\left((\mathfrak{P}_X)_A, \mathfrak{Y}'_A\right) \hookrightarrow \operatorname{Mor}_A\left((\mathfrak{P}_X)_A, \mathfrak{Y}_A\right)$$

Hence if $\mathrm{Mor}_A\left((\mathfrak{P}_X)_A, \mathfrak{Y}_A\right)$ is a set, then $\mathrm{Mor}_A\left((\mathfrak{P}_X)_A, \mathfrak{Y}'_A\right)$ is a set. All these facts imply that both internal homs

$$\mathcal{M}\mathrm{or}_k\left(\mathfrak{P}_X, \mathfrak{Y}'\right), \mathcal{M}\mathrm{or}_k\left(\mathfrak{P}_X, \mathfrak{Y}\right)$$

exist and morphism $\mathcal{M}\mathrm{or}_k(1_{\mathfrak{P}_X}, j)$ of k-functors is a monomorphism. Our task is to prove that it is a closed immersion. For this consider a k-algebra A and a morphism $\sigma : \mathfrak{P}_{\mathrm{Spec}\,A} \to \mathcal{M}\mathrm{or}_k\left(\mathfrak{P}_X, \mathfrak{Y}\right)$ of k-functors that sends 1_A to some morphism $\tau : (\mathfrak{P}_X)_A \to \mathfrak{Y}_A$ of A-functors. Consider a cartesian square

$$\begin{tikzcd}
\mathfrak{U} \arrow[r] \arrow[d] & \mathcal{M}\mathrm{or}_k\left(\mathfrak{P}_X, \mathfrak{Y}'\right) \arrow[d, "\mathcal{M}\mathrm{or}_k\left(1_{\mathfrak{P}_X}, j\right)"] \\
\mathfrak{P}_{\mathrm{Spec}\,A} \arrow[r, "\sigma"'] & \mathcal{M}\mathrm{or}_k\left(\mathfrak{P}_X, \mathfrak{Y}\right)
\end{tikzcd}$$

Since $\mathcal{M}\mathrm{or}_k\left(1_{\mathfrak{P}_X}, j\right)$ is a monomorphism, we may consider $\mathfrak{U}$ as a k-subfunctor of $\mathfrak{P}_{\mathrm{Spec}\,A}$. For every k-algebra B subset $\mathfrak{U}(B) \subseteq \mathrm{Mor}_k(A, B) = \mathrm{Mor}_k\left(\mathrm{Spec}\,B, \mathrm{Spec}\,A\right)$ consists of A-algebras B with structure morphisms $f : A \to B$ such that τ_B factors through $j_B : \mathfrak{Y}'_B \to \mathfrak{Y}_B$. Since X is a locally free k-scheme, we deduce that $(\mathfrak{P}_X)_A$ is a functor of points of a locally free A-scheme

$$\mathrm{Spec}\,A \times_{\mathrm{Spec}\,k} X$$

Pick an open affine cover $\bigcup_{i \in I} X_i$ of this A-scheme such that $\Gamma(X_i, \mathcal{O}_X)$ is a free A-module. Now Lemma 6.2.2 implies that τ_B factors through j_B if and only if $\left(\tau_{|X_i}\right)_B$ factors through j_B for every $i \in I$. Next by Lemma 6.2.1 we deduce that $\left(\tau_{|X_i}\right)_B$ factors through j_B for given $i \in I$ if and only if $f(\mathfrak{a}_i) = 0$ for some ideal $\mathfrak{a}_i \subseteq A$ independent of f. Thus $\mathfrak{U}$ consists of all morphisms $f : A \to B$ of k-algebras such that $f(\mathfrak{a}) = 0$ where $\mathfrak{a} = \sum_{i \in I} \mathfrak{a}_i$. Therefore, $\mathfrak{U} \hookrightarrow \mathfrak{P}_{\mathrm{Spec}\,A}$ is isomorphic with $\mathfrak{P}_{V(\mathfrak{a})} = \mathfrak{P}_{\mathrm{Spec}\,A/\mathfrak{a}} \hookrightarrow \mathfrak{P}_{\mathrm{Spec}\,A}$ induced by the quotient map $A \to A/\mathfrak{a}$ and hence $\mathcal{M}\mathrm{or}_k(1_{\mathfrak{P}_X}, j)$ is a closed immersion of k-functors. □

7. Example: Grassmannians

In this section we use representability results to prove the existence of grassmannian k-scheme. We start by recalling the notion of quotient.

Definition 7.1. Let $\mathcal{C}$ be a category and let X be an object of $\mathcal{C}$. Suppose that $f_1 : X \twoheadrightarrow X_1$ and $f_2 : X \twoheadrightarrow X_2$ are epimorphisms in $\mathcal{C}$. We say that f_1 and f_2 are *equivalent* if there exists a commutative triangle

$$\begin{tikzcd}
X_1 \arrow[rr, "\cong"] & & X_2 \\
& X \arrow[ul, "f_1"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

in $\mathcal{C}$ in which horizontal arrow is an isomorphism. Class of epimorphisms with domain in X which are equivalent with respect to the relation above is called *a quotient of* X.

Definition 7.2. Let V be a k-module and let n be a positive integer. For k-algebra A we define

$$\mathrm{Grass}_{V,n}(A) = \left\{ \begin{matrix} \text{Quotients of } A \otimes_k V \text{ represented by epimorphisms} \\ \text{with codomains that are projective } A\text{-modules of rank } n \end{matrix} \right\}$$

Note that if $f : A \to B$ is a morphism of k-algebras (making B into an A-algebra), then the functor $B \otimes_A (-)$ induces the canonical map

$$\mathrm{Grass}_{V,n}(f) : \mathrm{Grass}_{V,n}(A) \to \mathrm{Grass}_{V,n}(B)$$

This makes $\mathrm{Grass}_{V,n}$ into a k-functor. We call it *the grassmannian k-functor of quotients of rank n of V.*

Theorem 7.3. *Let V be a k-module and let n be a positive integer. Then the k-functor $\mathrm{Grass}_{V,n}$ is representable.*

We start with the following general result.

Lemma 7.3.1. *Let X be a locally ringed space and $\phi : \mathcal{P} \to \mathcal{Q}$ be a morphism of $\mathcal{O}_X$-modules such that $\mathcal{P}$ is of finite type and $\mathcal{Q}$ is locally free of finite rank. Then for every point x in X the following assertions are equivalent.*

- **(i)** $1_{k(x)} \otimes_{\mathcal{O}_{X,x}} \phi_x$ *is an isomorphism of vector spaces over $k(x)$.*
- **(ii)** ϕ_x *is an isomorphism of $\mathcal{O}_{X,x}$-modules.*

Moreover, the subset

$$\left\{x \in X \,\middle|\, \phi_x \text{ is an isomorphism of } \mathcal{O}_{X,x}\text{-modules}\right\}$$

of X is open.

Proof of the lemma. Suppose that $\mathcal{K} = \ker(\phi)$, $\mathcal{L} = \mathrm{coker}(\phi)$. Note first that $\mathcal{L}$ is $\mathcal{O}_X$-module of finite type as the homomorphic image of $\mathcal{Q}$. Fix a point x in X such that $1_{k(x)} \otimes_{\mathcal{O}_{X,x}} \phi_x$ is an isomorphisms of $k(x)$ vector spaces. This implies that $k(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x = 0$ and hence by Nakayama lemma we derive that $\mathcal{L}_x = 0$. Thus we have a short exact sequence

$$0 \longrightarrow \mathcal{K}_x \longrightarrow \mathcal{P}_x \xrightarrow{\phi_x} \mathcal{Q}_x \longrightarrow 0$$

Facts that $\mathcal{Q}_x$ is finitely presented and $\mathcal{P}_x$ is finitely generated over $\mathcal{O}_{X,x}$ imply that $\mathcal{K}_x$ is finitely generated over $\mathcal{O}_{X,x}$. Since $\mathcal{Q}_x$ is free, we derive that the sequence above is split exact. Therefore, also the sequence

$$0 \longrightarrow k(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{K}_x \longrightarrow k(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{P}_x \xrightarrow{1_{k(x)} \otimes_{\mathcal{O}_{X,x}} \phi_x} k(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{Q}_x \longrightarrow 0$$

is exact and hence $k(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{K}_x = 0$. Nakayama lemma implies that $\mathcal{K}_x = 0$. Thus we derive that $1_{k(x)} \otimes_{\mathcal{O}_{X,x}} \phi_x$ is an isomorphisms of $k(x)$ vector spaces if and only if ϕ_x is an isomorphisms of $\mathcal{O}_{X,x}$-modules. In other words

$$\left\{x \in X \,\middle|\, 1_{k(x)} \otimes_{\mathcal{O}_{X,x}} \phi_x \text{ is an isomorphism of vector spaces over } k(x)\right\} =$$

$$= \left\{x \in X \,\middle|\, \phi_x \text{ is an isomorphism of } \mathcal{O}_{X,x}\text{-modules}\right\}$$

Note that

$$\left\{x \in X \,\middle|\, \phi_x \text{ is an isomorphism of } \mathcal{O}_{X,x}\text{-modules}\right\} \subseteq \left\{x \in X \,\middle|\, \phi_x \text{ is an epimorphism of } \mathcal{O}_{X,x}\text{-modules}\right\}$$

and

$$\left\{x \in X \,\middle|\, \phi_x \text{ is an epimorphism of } \mathcal{O}_{X,x}\text{-modules}\right\} = X \smallsetminus \mathrm{supp}(\mathcal{L})$$

Since $\mathcal{L}$ is finitely generated, we derive that $\mathrm{supp}(\mathcal{L})$ is closed and $X \smallsetminus \mathrm{supp}(\mathcal{L})$ is open. Now there is a short exact sequence

$$0 \longrightarrow \mathcal{K}_{|X \smallsetminus \mathrm{supp}(\mathcal{L})} \longrightarrow \mathcal{P}_{|X \smallsetminus \mathrm{supp}(\mathcal{L})} \xrightarrow{\phi_{|X \smallsetminus \mathrm{supp}(\mathcal{L})}} \mathcal{Q}_{|X \smallsetminus \mathrm{supp}(\mathcal{L})} \longrightarrow 0$$

It follows that $\mathcal{K}_{|X\smallsetminus \mathrm{supp}(\mathcal{L})}$ is $\mathcal{O}_X$-module of finite type. Thus

$$\left\{x \in X \,\middle|\, \phi_x \text{ is an isomorphism of } \mathcal{O}_{X,x}\text{-modules}\right\} = (X \smallsetminus \mathrm{supp}(\mathcal{L})) \smallsetminus \mathrm{supp}(\mathcal{K}_{|X\smallsetminus \mathrm{supp}(\mathcal{L})})$$

is an open subset of X. □

Let V be a free k-module and let n be a positive integer. Consider a morphism $u : k^{\oplus n} \to V$ of k-modules. Now we define a k-subfunctor Grass^u_V of $\mathrm{Grass}_{V,n}$ by formula

$$\mathrm{Grass}^u_V(A) = \left\{\begin{matrix}\text{Elements of } \mathrm{Grass}_{V,n}(A) \text{ which are represented by epimorphisms } \phi : A \otimes_k V \to U \\ \text{such that the composition } \phi \cdot (1_A \otimes_k u) \text{ is an isomorphism}\end{matrix}\right\}$$

for every k-algebra. Next we proceed in steps.

Lemma 7.3.2. *Let V be a free k-module and let n be a positive integer. Then*

$$\left\{\mathrm{Grass}^u_V \hookrightarrow \mathrm{Grass}_{V,n}\right\}_{u \in \mathrm{Hom}_k(k^{\oplus n}, V)}$$

is an open cover of $\mathrm{Grass}_{V,n}$.

Proof of the lemma. Let A be a k-algebra. Consider a morphism $\tau : \mathfrak{P}_{\mathrm{Spec}\,A} \to \mathrm{Grass}_{V,n}$ that corresponds to some quotient of $A \otimes_k V$ that is represented by an epimorphism $\phi : A \otimes_k V \to U$ of A-modules with projective A-module U of rank n. Let $u : k^{\oplus n} \to V$ be a morphism of k-modules. Consider a cartesian square

$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[d, hook] & \mathrm{Grass}^u_V \arrow[d, hook] \\ \mathfrak{P}_{\mathrm{Spec}\,A} \arrow[r, "\tau"'] & \mathrm{Grass}_{V,n} \end{tikzcd}$$

Pick a k-algebra B and a morphism $f : A \to B$ of k-algebras. Note that f makes B into an A-algebra. Then $f \in \mathfrak{X}(B)$ if and only if $(1_B \otimes_A \phi) \cdot (1_B \otimes_k u)$ is an isomorphism of B-modules. Thus by Lemma 7.3.1 we deduce that $f \in \mathfrak{X}(B)$ if and only if $\mathrm{Spec}\,f : \mathrm{Spec}\,B \to \mathrm{Spec}\,A$ factors through an open subscheme

$$W_u = \left\{\mathfrak{q} \in \mathrm{Spec}\,A \,\middle|\, \big(\phi \cdot (1_A \otimes_k u)\big)_{\mathfrak{q}} \text{ is an isomorphism of } A_{\mathfrak{q}}\text{-modules}\right\} =$$

$$= \left\{\mathfrak{q} \in \mathrm{Spec}\,A \,\middle|\, k(\mathfrak{q}) \otimes_{A_{\mathfrak{q}}} \big(\phi \cdot (1_A \otimes_k u)\big)_{\mathfrak{q}} \text{ is an isomorphism of } k(\mathfrak{q})\text{-vector spaces}\right\}$$

This implies that $\mathfrak{X} \hookrightarrow \mathfrak{P}_{\mathrm{Spec}\,A}$ is isomorphic to an open immersion $\mathfrak{P}_{W_u} \hookrightarrow \mathfrak{P}_{\mathrm{Spec}\,A}$.
Pick now $\mathfrak{q} \in \mathrm{Spec}\,A$ and an epimorphism $\theta : k^{\oplus I} \twoheadrightarrow V$ for some set I. Then there exist $J \subseteq I$ with $\mathbf{card}(J) = n$ such that the restriction to $k(\mathfrak{q})^{\oplus J}$ of the morphism

$$1_{k(\mathfrak{q})} \otimes_{A_{\mathfrak{q}}} \big(\phi \cdot (1_A \otimes \theta)\big)_{\mathfrak{q}} : k(\mathfrak{q})^{\oplus I} \to k(\mathfrak{q}) \otimes_{A_{\mathfrak{q}}} U_{\mathfrak{q}}$$

is an isomorphism of $k(\mathfrak{q})$-vector spaces. Let $u : k^{\oplus n} \to V$ be a morphism given as the composition of the canonical monomorphism $k^{\oplus n} = k^{\oplus J} \hookrightarrow k^{\oplus I}$ with θ. Then

$$\left(1_{k(\mathfrak{q})} \otimes_{A_{\mathfrak{q}}} \big(\phi \cdot (1_A \otimes u)\big)_{\mathfrak{q}}\right)$$

is an isomorphism of $k(\mathfrak{q})$-vector spaces. Note that module $U_{\mathfrak{q}}$ is a free $A_{\mathfrak{q}}$-module of rank n. Hence by Lemma 7.3.1 we derive that

$$\big(\phi \cdot (1_A \otimes u)\big)_{\mathfrak{q}}$$

is an isomorphism of $A_{\mathfrak{q}}$-modules. Thus $\mathfrak{q} \in W_u$. Since $\mathfrak{q}$ is arbitrary, we deduce that

$$\operatorname{Spec} A = \bigcup_{u \in \operatorname{Hom}_k(k^{\oplus n}, V)} W_u$$

This finishes the proof. □

Lemma 7.3.3. *Let $V = k^{\oplus I}$ be a free k-module, where I is a set, and let n be a positive integer. Fix a subset J of I such that* $\mathbf{card}(J) = n$. *Then* Grass_V^J *is representable by a scheme* $\operatorname{Spec} k\big[x_{ji} \,\big|\, i \in I \smallsetminus J, 1 \leq j \leq n\big]$.

Proof of the lemma. Let $\{e_i\}_{i \in I}$ be the canonical basis of $k^{\oplus I}$ and let $\{f_j\}_{j=1}^n$ be the canonical basis of $k^{\oplus n}$. Fix a k-algebra A. Suppose that $\phi : A^{\oplus I} \twoheadrightarrow A^{\oplus n}$ represents element of $\mathrm{Grass}_{V,n}(A)$. Then ϕ can be encoded as a matrix $M_\phi = [a_{ji}]_{1 \leq j \leq n,\, i \in I}$ with entries in A such that

$$\phi(e_i) = \sum_{j=1}^{n} a_{ji} f_j$$

Note that $\phi_1, \phi_2 : A^{\oplus I} \twoheadrightarrow A^{\oplus n}$ represent the same element of $\mathrm{Grass}_{V,n}(A)$ if and only if there exists $n \times n$ invertible matrix M with entries in A such that $M \cdot M_{\phi_1} = M_{\phi_2}$. Thus for every quotient in $\mathrm{Grass}_V^J(A)$ there exists a unique representative $\phi : A^{\oplus I} \twoheadrightarrow A^{\oplus n}$ such that $M_\phi = [a_{ji}]_{1 \leq j \leq n,\, i \in I}$ and $[a_{ji}]_{1 \leq j \leq n, i \in J}$ is the identity matrix. Therefore, we have an identification

$$\mathrm{Grass}_V^J(A) = \big\{[a_{ji}]_{1 \leq j \leq n,\, i \in I} \,\big|\, a_{ji} \in A \text{ and } [a_{ji}]_{1 \leq j \leq n, i \in J} \text{ is the identity matrix}\big\}$$

This identification is natural in A. Hence the k-functor Grass_V^J is representable by a k-scheme $\operatorname{Spec} k\big[x_{ji} \,\big|\, i \in I \smallsetminus J, 1 \leq j \leq n\big]$. □

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