**MONOID $k$-FUNCTORS AND THEIR REPRESENTATIONS**

## 1. Introduction and notation

In these notes we study algebraic structures in the category of $k$-functors with special emphasis on monoid objects.
Throughout these notes $k$ is a fixed commutative ring. If $M$ is a monoid, then we denote by $M^*$ the group of units of $M$. If $R$ is a ring, then we denote by $R^{\times}$ its multplicative monoid.

## 2. Algebraic structures in the category of $k$-functors

In the sequel we assume that the reader is familiar with notions of a monoid, group etc. in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 2-5].

**Definition 2.1.** *A monoid (group, abelian group, ring) $k$-functor* is a monoid (group, abelian group, ring) object in the category of $k$-functors.

**Example 2.2.** Let $\mathfrak{X}$ be a $k$-functor such that $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{X})$ exists. Then $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{X})$ is a monoid $k$-functor with respect to composition of morphisms.

**Example 2.3.** Let $\mathfrak{G}$ be a monoid $k$-functor. Then we denote by $\mathfrak{G}^*$ the $k$-subfunctor of $\mathfrak{G}$ defined by

$$\mathfrak{G}^*(A) = \mathfrak{G}(A)^*$$

for every $k$-algebra $A$. We call $\mathfrak{G}^*$ *the unit group $k$-functor of* $\mathfrak{G}$.

**Example 2.4.** Basic example of a ring $k$-functor is a $k$-functor $\mathfrak{K}$ given by

$$\mathfrak{K}(A) = k,\ \mathfrak{K}(f) = 1_k$$

for any $k$-algebra $A$ and morphism $f$ of $k$-algebras. It can be described as a constant $k$-functor ([ML98, page 67]) corresponding to $k$.

**Definition 2.5.** Let $\mathfrak{R}$ be a ring $k$-functor. Then we denote by $\mathfrak{R}^{\times}$ the $k$-subfunctor of $\mathfrak{R}$ defined by

$$\mathfrak{R}^{\times}(A) = \mathfrak{R}(A)^{\times}$$

for every $k$-algebra $A$. We call $\mathfrak{R}^{\times}$ *the multiplicative monoid $k$-functor of* $\mathfrak{R}$.

**Definition 2.6.** Let $\mathfrak{A}$ be a commutative ring $k$-functor. *An $\mathfrak{A}$-algebra* is an $\mathfrak{A}$-algebra object in the category of $k$-functors.

## 3. Global regular functions on a $k$-functor

Recall the ring $k$-functor $\mathfrak{K}$ from Example 2.4. Note that a $\mathfrak{K}$-algebra $\mathfrak{A}$ can be viewed as a functor $\mathfrak{A} : \mathbf{Alg}_k \rightarrow \mathbf{Alg}_k$.

**Definition 3.1.** The $\mathfrak{K}$-algebra $\mathfrak{O}_k$ represented by the identity functor on $\mathbf{Alg}_k$ is called *the structure $\mathfrak{K}$-algebra*.

Let $|-| : \mathbf{Alg}_k \rightarrow \mathbf{Set}$ be the forgetful $k$-functor. Note that $|-|$ is the underlying $k$-functor of $\mathfrak{K}$-algebra $\mathfrak{O}_k$. Recall that the affine line $\mathbb{A}^1_k$ is an affine $k$-scheme having $k$-algebra of polynomials with one variable as a $k$-algebra of regular functions.

**Fact 3.2.** *Let $|-| : \mathbf{Alg}_k \to \mathbf{Set}$ be the forgetful k-functor. Then we have natural isomorphism*

$$\mathfrak{P}_{\mathbb{A}^1_k} \cong |-|$$

*Proof.* Let $B$ be a $k$-algebra. We have the following chain of identifications

$$\mathfrak{P}_{\mathbb{A}^1_k}(B) = \mathrm{Mor}_k\left(\mathrm{Spec}\, B, \mathbb{A}^1_k\right) = \mathrm{Mor}_k\left(\mathrm{Spec}\, B, \mathrm{Spec}\, k[x]\right) = \mathrm{Mor}_k\left(k[x], B\right) = |B|$$

natural in $B$. □

In particular, since $|-|$ carries the structure $\mathfrak{K}$-algebra $\mathfrak{O}_k$, we derive that $\mathfrak{P}_{\mathbb{A}^1_k}$ admits a structure of $\mathfrak{K}$-algebra isomorphic to $\mathfrak{O}_k$.

No we introduce regular functions on $k$-functors.

**Definition 3.3.** Let $\mathfrak{X}$ be a $k$-functor and assume that $\mathcal{M}\mathrm{or}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$ is a set. Then $\mathrm{Mor}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$ is a $k$-algebra with respect to the structure induced by $\mathfrak{O}_k$. We call this $k$-algebra *the k-algebra of global regular functions on* $\mathfrak{X}$. Its elements are called *global regular functions on* $\mathfrak{X}$.

**Definition 3.4.** Let $\mathfrak{X}$ be a $k$-functor. Suppose that $A$ is a $k$-algebra, $x \in \mathfrak{X}(A)$ and $f \in \mathrm{Mor}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$. The element $f(x) \in A$ is called *the value of $f$ on a point $x$*.

For given $k$-functor $\mathfrak{X}$ we describe $k$-algebra operations on $\mathrm{Mor}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$ in terms of values of its elements on points of $\mathfrak{X}$. For this consider $\alpha \in k$ and $f, g_1 \in \mathrm{Mor}_k\left(\mathfrak{X}, \mathfrak{O}_k\right)$. We have formulas

$$(f+g)(x) = f(x) + g(x),\ (f \cdot g)(x) = f(x)\cdot g(x),\ (\alpha \cdot f)(x) = \alpha \cdot f(x)$$

in which right hand side are $k$-algebra operations in $A$.

**Example 3.5.** Let $\mathfrak{X}$ be a $k$-functor and assume that $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{O}_k)$ exists. Fix $k$-algebra $A$. Note that $\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{O}_A)$ is an $A$-algebra of global regular functions on $\mathfrak{X}_A$. Moreover, if $B$ is an $A$-algebra, then

$$\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{O}_A) \ni f \mapsto f_B \in \mathrm{Mor}_B(\mathfrak{X}_B, \mathfrak{O}_B)$$

is a morphism of $A$-algebras. This implies that $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{O}_k)$ admits a canonical structure of an $\mathfrak{O}_k$-algebra $k$-functor.

## 4. Internal hom and product of $k$-functors

We denote by $\mathbf{1}$ a $k$-functor that assigns to every $k$-algebra a set with one element. Then for every $k$-algebra $A$ the restriction $\mathbf{1}_A$ is a terminal object in the category of $A$-functors.

**Fact 4.1.** *Let $\mathfrak{X}$ be a k-functor. Suppose $A$ is a k-algebra and $x \in \mathfrak{X}(A)$. Then $x$ determines a morphism $\mathbf{1}_A \to \mathfrak{X}_A$ that for every $A$-algebra $B$ with structural morphism $f : A \to B$ sends a unique element of $\mathbf{1}_A(B)$ to $\mathfrak{X}(f)(x) \in \mathfrak{X}_A(B)$. This gives rise to a bijection*

$$\mathfrak{X}(A) \cong \mathrm{Mor}_A\left(\mathbf{1}_A, \mathfrak{X}_A\right)$$

*Proof.* Left to the reader as an exercise. □

The discussion below is partially an application of the main result in [Mon19a, section 6]. For reader's convenience we make our presentation self-contained.

**Definition 4.2.** Let $\mathfrak{U}, \mathfrak{X}, \mathfrak{Y}$ be $k$-functors and let $\sigma : \mathfrak{U} \times \mathfrak{X} \to \mathfrak{Y}$ be a morphism of $k$-functors. Fix $z \in \mathfrak{U}(A)$ for some $k$-algebra $A$. We denote by $i_z : \mathbf{1}_A \to \mathfrak{U}_A$ the morphism of $A$-functors corresponding to $z$ by Fact 4.1. Since $\mathbf{1}_A$ is terminal $A$-functor, a morphism $\sigma_A \cdot \left(i_z \times 1_{\mathfrak{X}_A}\right)$ is isomorphic to a morphism $\sigma_z : \mathfrak{X}_A \to \mathfrak{Y}_A$ of $A$-functors. We call $\sigma_z$ *the slice of $\sigma$ along $z$*.

**Definition 4.3.** Let $\mathfrak{X},\mathfrak{Y}$ be $k$-functors. Let $\mathfrak{I}$ be a $k$-functor such that $\mathfrak{I}(A)$ is a subset of a class $\mathrm{Mor}_A\left(\mathfrak{X}_A,\mathfrak{Y}_A\right)$ for every $k$-algebra $A$. Assume that for every morphism $f : A \to B$ of $k$-algebras and every $\sigma \in \mathfrak{I}(A)$ we have

$$\mathfrak{I}(f)(\sigma) = \sigma_B$$

where $\sigma_B \in \mathrm{Mor}_B\left(\mathfrak{X}_B,\mathfrak{Y}_B\right)$ is the restriction of $\sigma$ along $f$. Then we call $\mathfrak{I}$ *a $k$-subfunctor of internal hom of $\mathfrak{X}$ and $\mathfrak{Y}$*.

**Definition 4.4.** Let $\mathfrak{X},\mathfrak{Y},\mathfrak{U}$ be $k$-functors and let $\sigma : \mathfrak{U}\times\mathfrak{X} \to \mathfrak{Y}$ be a morphism of $k$-functors. Suppose that $\mathfrak{I}$ is a $k$-subfunctor of internal hom of $\mathfrak{X}$ and $\mathfrak{Y}$. Assume that $\sigma_z : \mathfrak{X}_A \to \mathfrak{Y}_A$ is contained in $\mathfrak{I}(A)$ for every $k$-algebra $A$ and $z \in \mathfrak{U}(A)$. Then we call $\sigma$ *a family of $\mathfrak{I}$-morphisms parametrized by $\mathfrak{U}$*.

Let $\mathfrak{I}$ be a $k$-subfunctor of internal hom of $\mathfrak{X}$ and $\mathfrak{Y}$. Assume that $\sigma : \mathfrak{U}\times\mathfrak{X} \to \mathfrak{Y}$ is a $\mathfrak{I}$-family of morphism parametrized by $\mathfrak{U}$. Then the family of maps

$$\mathfrak{U}(A) \ni z \mapsto \sigma_z \in \mathfrak{I}(A)$$

gives rise to a morphism $\tau : \mathfrak{U} \to \mathfrak{I}$ of $k$-functors. Indeed, for a morphism $f : A \to B$ of $k$-algebras and $z \in \mathfrak{U}(A)$ we have

$$\sigma_B \cdot \left(i_{\mathfrak{U}(f)(z)} \times 1_{\mathfrak{X}_B}\right) = \left(\sigma_A \cdot \left(i_z \times 1_{\mathfrak{X}_A}\right)\right)_B$$

and hence $\sigma_{\mathfrak{U}(f)(z)} = (\sigma_z)_B$. This gives rise to a map $\Phi$ of classes

$$\left\{\text{families } \mathfrak{U}\times\mathfrak{X} \to \mathfrak{Y} \text{ of } \mathfrak{I}\text{-morphisms parametrized by } \mathfrak{U}\right\} \ni \sigma \mapsto \tau \in \mathrm{Mor}_k\left(\mathfrak{U},\mathfrak{I}\right)$$

Consider next a morphism $\tau : \mathfrak{U} \to \mathfrak{I}$ of $k$-functors and define $\sigma : \mathfrak{U}\times\mathfrak{X} \to \mathfrak{Y}$ by formula $\sigma^A(z,x) = \left(\tau^A(z)\right)^A(x)$ for every $k$-algebra $A$ and points $z \in \mathfrak{U}(A)$, $x \in \mathfrak{X}(A)$. Let $f : A \to B$ be a morphism of $k$-algebras. Then

$$\sigma^B\left(\mathfrak{U}(f)(z),\mathfrak{X}(f)(x)\right) = \left(\tau^B\left(\mathfrak{U}(f)(z)\right)\right)^B\left(\mathfrak{X}(f)(x)\right) = \left(\left(\tau^A(z)\right)_B\right)^B\left(\mathfrak{X}(f)(x)\right) =$$

$$= \left(\tau^A(z)\right)^B\left(\mathfrak{X}(f)(x)\right) = \mathfrak{Y}(f)\left(\left(\tau^A(z)\right)^A(x)\right) = \mathfrak{Y}(f)\left(\sigma^A(z,x)\right)$$

Thus $\sigma : \mathfrak{U}\times\mathfrak{X} \to \mathfrak{Y}$ is a morphism of $k$-functors. For every $k$-algebra $A$ and $z \in \mathfrak{U}(A)$ we have $\sigma_z = \tau^A(z)$. Indeed, let $f : A \to B$ be a morphism of $k$-algebras and $x$ be an element in $\mathfrak{X}(B)$ then we have

$$(\sigma_z)^B(x) = \sigma^B\left(\mathfrak{U}(f)(z),x\right) = \left(\tau^B\left(\mathfrak{U}(f)(z)\right)\right)^B(x) = \left(\left(\tau^A(z)\right)_B\right)^B(x) = \left(\tau^A(z)\right)^B(x)$$

Hence $\sigma$ is a family of $\mathfrak{I}$-morphisms parametrized by $\mathfrak{U}$. This gives rise to a map $\Psi$ of classes

$$\mathrm{Mor}_k\left(\mathfrak{U},\mathfrak{I}\right) \ni \tau \mapsto \sigma \in \left\{\text{families } \mathfrak{U}\times\mathfrak{X} \to \mathfrak{Y} \text{ of } \mathfrak{I}\text{-morphisms parametrized by } \mathfrak{U}\right\}$$

Now we have the following result, which is an instance [Mon19a, Theorem 6.3]. To make presentation self-contained we give a complete proof.

**Theorem 4.5.** *Let $\mathfrak{X}$, $\mathfrak{Y}$ be $k$-functors and let $\mathfrak{I}$ be a $k$-subfunctor of internal hom of $\mathfrak{X}$ and $\mathfrak{Y}$. Then maps*

$$\Phi : \left\{\textit{families } \mathfrak{U}\times\mathfrak{X} \to \mathfrak{Y} \textit{ of } \mathfrak{I}\textit{-morphisms parametrized by } \mathfrak{U}\right\} \to \mathrm{Mor}_k\left(\mathfrak{U},\mathfrak{I}\right)$$

*and*

$$\Psi : \mathrm{Mor}_k\left(\mathfrak{U},\mathfrak{I}\right) \to \left\{\textit{families } \mathfrak{U}\times\mathfrak{X} \to \mathfrak{Y} \textit{ of } \mathfrak{I}\textit{-morphisms parametrized by } \mathfrak{U}\right\}$$

*are mutually inverse bijections.*

*Proof.* Pick a morphism $\tau : \mathfrak{U} \to \mathfrak{J}$ of $k$-functors. Let $A$ be a $k$-algebra and $z \in \mathfrak{U}(A)$. In the discussion preceding the statement we showed that $\Psi(\tau)_z = \tau^A(z)$. Thus

$$(\Phi(\Psi(\tau)))^A(z) = \Psi(\tau)_z = \tau^A(z)$$

and hence $\Phi \cdot \Psi$ is the identity.
Pick a family of $\mathfrak{J}$-morphism $\sigma : \mathfrak{U} \times \mathfrak{X} \to \mathfrak{Y}$ parametrized by $\mathfrak{U}$. Let $A$ be a $k$-algebra and $z \in \mathfrak{U}(A)$, $x \in \mathfrak{X}(A)$ be points. Then

$$(\Psi(\Phi(\sigma)))^A(z,x) = \left(\Phi(\sigma)^A(z)\right)^A(x) = \sigma_z^A(x) = \sigma^A(z,x)$$

Thus $\Psi \cdot \Phi$ is the identity map. □

Now we formulate some consequences of Theorem 4.5.

**Corollary 4.6.** *Let $\mathfrak{X}$, $\mathfrak{Y}$ be k-functors. Assume that for every k-algebra A the class* $\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ *is a set. Then there is a bijection*

$$\mathrm{Mor}_k(\mathfrak{U} \times \mathfrak{X}, \mathfrak{Y}) \to \mathrm{Mor}_k(\mathfrak{U}, \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y}))$$

*of classes.*

**Definition 4.7.** Let $\mathfrak{X}, \mathfrak{Y}$ be $k$-functors. If $\mathrm{Iso}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set for every $k$-algebra $A$, then we define a $k$-subfunctor $\mathcal{I}\mathrm{so}_k(\mathfrak{X}, \mathfrak{Y})$ of $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})$ by

$$\mathcal{I}\mathrm{so}_k(\mathfrak{X}, \mathfrak{Y})(A) = \mathrm{Iso}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$$

for every $k$-algebra $A$. We call $\mathcal{I}\mathrm{so}_k(\mathfrak{X}, \mathfrak{Y})$ *the k-functor of isomorphism.*

**Definition 4.8.** Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{U}$ be $k$-functors and let $\sigma : \mathfrak{U} \times \mathfrak{X} \to \mathfrak{Y}$ be a morphism of $k$-functors. Assume that $\sigma_z : \mathfrak{X}_A \to \mathfrak{Y}_A$ is an isomorphism of $A$-functors for every $k$-algebra $A$. Then we call $\sigma$ *a family of isomorphisms parametrized by* $\mathfrak{U}$.

**Corollary 4.9.** *Let $\mathfrak{U}, \mathfrak{X}, \mathfrak{Y}$ be k-functors and suppose that for every k-algebra A the class* $\mathrm{Iso}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ *is a set. The the following map*

$$\left\{\textit{families } \mathfrak{U} \times \mathfrak{X} \to \mathfrak{Y} \textit{ of isomorphism parametrized by } \mathfrak{U}\right\} \to \mathrm{Mor}_k(\mathfrak{U}, \mathcal{I}\mathrm{so}_k(\mathfrak{X}, \mathfrak{Y}))$$

*is a bijection of classes.*

## 5. Actions of monoid $k$-functors

In this section we assume that the reader is familiar with notion of an action of a monoid object in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 5].
Let $\mathfrak{G}$ be a monoid $k$-functor and $\mathfrak{X}$ be a $k$-functor together with an action $\alpha : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X}$. Next assume that $k$-functor $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ exists. By Example 2.2 it is a monoid $k$-functor. We define a morphism $\rho : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ of $k$-functors by formula $\rho(g) = \alpha_g$. Note that by discussion preceding Theorem 4.5, we deduce that $\rho$ is a well defined morphism of $k$-functors. We show now that $\rho$ is a morphism of monoids. For this pick $k$-algebra $A$ and $g_1, g_2 \in \mathfrak{G}(A)$. Since $\alpha$ is an action, we deduce that $\alpha_{g_1 \cdot g_2} = \alpha_{g_1} \cdot \alpha_{g_2}$ and hence also

$$\rho(g_1 \cdot g_2) = \alpha_{g_1 \cdot g_2} = \alpha_{g_1} \cdot \alpha_{g_2} = \rho(g_1) \cdot \rho(g_2)$$

Therefore, $\rho$ is a morphism of monoid $k$-functors. This shows how to construct a morphism of monoid $k$-functors $\rho$ from an action $\alpha$ of $\mathfrak{G}$.

**Theorem 5.1.** *Let $\mathfrak{G}$ be a monoid k-functor and let $\mathfrak{X}$ be a k-functor such that $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ exists. Suppose that*

$$\left\{\textit{actions of } \mathfrak{G} \textit{ on } \mathfrak{X}\right\} \longrightarrow \left\{\textit{Morphisms } \rho : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X}) \textit{ of monoid k-functors}\right\}$$

*is a map of classes described above. Then it is bijection.*

*Proof.* Our goal is to construct the inverse of the map. Substitute $\mathfrak{J} = \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ in Theorem 4.5. Consider maps

$$\Phi : \left\{ \text{families } \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X} \text{ of morphisms} \right\} \to \mathrm{Mor}_k\left(\mathfrak{G}, \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})\right)$$

and

$$\Psi : \mathrm{Mor}_k\left(\mathfrak{G}, \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})\right) \to \left\{ \text{families } \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X} \text{ of morphisms} \right\}$$

in that Theorem. Then the map in the statement above is the restriction of $\Phi$ to $\mathfrak{G}$-actions on $\mathfrak{X}$ on the right and morphisms $\mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ of monoid $k$-functors on the left. Since by Theorem 4.5 maps $\Phi$ and $\Psi$ are mutually inverse, it suffices to check that $\Psi$ sends a morphism $\rho : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{X})$ of monoids to an action of $\mathfrak{G}$ on $\mathfrak{X}$. For this denote $\Psi(\rho)$ by $\alpha$. Consider $k$-algebra $A$ and $A$-points $g_1, g_2 \in \mathfrak{G}(A)$, $x \in \mathfrak{X}(A)$. Then

$$\alpha\left(g_1, \alpha(g_2, x)\right) = \rho(g_1)\left(\rho(g_2)(x)\right) = \left(\rho(g_1) \cdot \rho(g_2)\right)(x) = \rho\left(g_1 \cdot g_2\right)(x) = \alpha\left(g_1 \cdot g_2, x\right)$$

Therefore, $\alpha$ is an action of $\mathfrak{G}$ on $\mathfrak{X}$. □

**Proposition 5.2.** *Let $\mathfrak{G}$ be a monoid $k$-functor and let $\mathfrak{X}_1, \mathfrak{X}_2$ be $k$-functors such that $\mathcal{M}\mathrm{or}_k(\mathfrak{X}_1, \mathfrak{X}_1), \mathcal{M}\mathrm{or}_k(\mathfrak{X}_2, \mathfrak{X}_2)$ exist. Suppose that $\alpha_1 : \mathfrak{G} \times \mathfrak{X}_1 \to \mathfrak{X}_1$, $\alpha_2 : \mathfrak{G} \times \mathfrak{X}_2 \to \mathfrak{X}_2$ are actions of $\mathfrak{G}$, respectively. Suppose that $\sigma : \mathfrak{X}_1 \to \mathfrak{X}_2$ is a morphism of $k$-functors. Then the following assertions are equivalent.*

**(i)** *The square*

$$\begin{tikzcd} \mathfrak{G} \times \mathfrak{X}_1 \arrow[r, "1_{\mathfrak{G}} \times \sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{G} \times \mathfrak{X}_2 \arrow[d, "\alpha_2"] \\ \mathfrak{X}_1 \arrow[r, "\sigma"'] & \mathfrak{X}_2 \end{tikzcd}$$

*is commutative.*

**(ii)** *For every $k$-algebra $A$ and $g \in \mathfrak{G}(A)$ we have*

$$\sigma_A \cdot \rho_1(g) = \rho_2(g) \cdot \sigma_A$$

*where $\rho_1 : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}_1, \mathfrak{X}_1)$ and $\rho_2 : \mathfrak{G} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}_2, \mathfrak{X}_2)$ are morphism of monoid $k$-functors corresponding to $\alpha_1$ and $\alpha_2$, respectively.*

*Proof.* Conditions expressed in **(i)** and **(ii)** are directly translatable to each other by virtue of the bijection in Theorem 5.1. □

**Definition 5.3.** Let $\mathfrak{G}$ be a monoid $k$-functor and let $(\mathfrak{X}_1, \alpha_1)$, $(\mathfrak{X}_2, \alpha_2)$ be $k$-functors with actions of $\mathfrak{G}$. Suppose that $\sigma : \mathfrak{X}_1 \to \mathfrak{X}_2$ is a morphism $k$-functors such that the square

$$\begin{tikzcd} \mathfrak{G} \times \mathfrak{X}_1 \arrow[r, "1_{\mathfrak{G}} \times \sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{G} \times \mathfrak{X}_2 \arrow[d, "\alpha_2"] \\ \mathfrak{X}_1 \arrow[r, "\sigma"'] & \mathfrak{X}_2 \end{tikzcd}$$

is commutative. Then $\sigma$ is called *an $\mathfrak{G}$-equivariant morphism*.

## 6. Modules over ring $k$-functors

**Definition 6.1.** Let $\mathfrak{R}$ be a ring $k$-functor. Suppose that $\mathfrak{M}$ is an abelian group $k$-functor and there exists a morphism $\mathfrak{R}\times\mathfrak{M}\to\mathfrak{M}$ of $k$-functors that for each $k$-algebra $A$ makes $\mathfrak{M}(A)$ into an $\mathfrak{R}(A)$-module. Then we say that $\mathfrak{M}$ is *a module $k$-functor over* $\mathfrak{R}$.

**Definition 6.2.** Let $\mathfrak{R}$ be an ring $k$-functor and let $\mathfrak{M}_1,\mathfrak{M}_2$ be module $k$-functors over $\mathfrak{R}$. Suppose that $\sigma:\mathfrak{M}_1\to\mathfrak{M}_2$ is a morphism of abelian group $k$-functors such that the diagram

$$\begin{tikzcd} \mathfrak{R}\times\mathfrak{M}_1 \arrow[r, "1_{\mathfrak{R}}\times\sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{R}\times\mathfrak{M}_2 \arrow[d, "\alpha_2"] \\ \mathfrak{M}_1 \arrow[r, "\sigma"'] & \mathfrak{M}_2 \end{tikzcd}$$

is commutative, where $\alpha_i:\mathfrak{R}\times\mathfrak{M}_i\to\mathfrak{M}_i$ define $\mathfrak{R}$-module structure on $\mathfrak{M}_i$ for $i=1,2$. Then $\sigma$ is *a morphism of modules over* $\mathfrak{R}$.

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be module $k$-functors over $\mathfrak{R}$. We denote by

$$\mathrm{Hom}_{\mathfrak{R}}\left(\mathfrak{M}_1,\mathfrak{M}_2\right)$$

the class of all morphisms of modules $\mathfrak{M}_1\to\mathfrak{M}_2$ over $\mathfrak{R}$. We denote the category of $\mathfrak{R}$-modules by $\mathbf{Mod}\left(\mathfrak{R}\right)$.

**Definition 6.3.** Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be module $k$-functors over $\mathfrak{R}$. Assume that $\mathrm{Hom}_{\mathfrak{R}_A}\left((\mathfrak{M}_1)_A,(\mathfrak{M}_2)_A\right)$ is a set for every $k$-algebra $A$. Then we define a $k$-subfunctor $\mathcal{H}om_{\mathfrak{R}}(\mathfrak{M}_1,\mathfrak{M}_2)$ of internal hom of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ by formula

$$\mathbf{Alg}_k\ni A\mapsto \mathrm{Hom}_{\mathfrak{R}_A}\left((\mathfrak{M}_1)_A,(\mathfrak{M}_2)_A\right)\in\mathbf{Set}$$

We call $\mathcal{H}om_{\mathfrak{R}}(\mathfrak{M}_1,\mathfrak{M}_2)$ *a $k$-functor of module morphisms of $\mathfrak{M}_1$ and $\mathfrak{M}_2$*.

If $\mathfrak{M}$ is a module $k$-functor over some ring $k$-functor $\mathfrak{R}$, then we denote (if it exists) $\mathcal{H}om_{\mathfrak{R}}(\mathfrak{M},\mathfrak{M})$ by $\mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$.

**Example 6.4.** Let $\mathfrak{M}$ be a module over a ring $k$-functor $\mathfrak{R}$. Assume that $\mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$ exists. Then $\mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$ is a ring $k$-functor with respect to composition of morphisms of modules as the multiplication and the usual addition of module morphisms. Moreover, if $\mathfrak{A}$ is a commutative ring $k$-functor, then $\mathcal{E}nd_{\mathfrak{A}}(\mathfrak{M})$ (if exists) admits additional structure of a $\mathfrak{A}$-algebra $k$-functor induced via a unique morphism $\mathfrak{A}\to\mathcal{E}nd_{\mathfrak{R}}(\mathfrak{M})$ of ring $k$-functors that sends $1\mapsto 1_{\mathfrak{M}}$.

Let $\mathfrak{A}$ be a commutative ring $k$-functor and let $\mathfrak{R}$ be a $\mathfrak{A}$-algebra $k$-functor. This means that there exists a morphism $\mathfrak{A}\to\mathfrak{R}$ of ring $k$-functors and for every $k$-algebra $A$ induced morphism $\mathfrak{A}(A)\to\mathfrak{R}(A)$ sends $\mathfrak{A}(A)$ to the center of a ring $\mathfrak{R}(A)$. Fix a module $\mathfrak{M}$ over $\mathfrak{A}$. Next assume that $k$-functor $\mathcal{E}nd_{\mathfrak{A}}(\mathfrak{M})$ exists. By Example 6.4 it is a ring $k$-functor.

**Definition 6.5.** In the setting above suppose that $\alpha:\mathfrak{R}\times\mathfrak{M}\to\mathfrak{M}$ is a morphism of $k$-functors. Suppose that $\alpha$ makes $\mathfrak{M}$ into $\mathfrak{R}$-module and moreover, for every $k$-algebra $A$ and for every point $x\in\mathfrak{R}(A)$ morphism $\alpha_x$ is a morphism of $\mathfrak{A}_A$-modules. Then $\alpha$ is called *a $\mathfrak{A}$-linear $\mathfrak{R}$-action on* $\mathfrak{M}$.

We continue the discussion. We assume that we are given an $\mathfrak{A}$-linear $\mathfrak{R}$-action $\alpha:\mathfrak{R}\times\mathfrak{M}\to\mathfrak{M}$ on $\mathfrak{M}$. We define a morphism $\rho:\mathfrak{R}\to\mathcal{E}nd_{\mathfrak{A}}(\mathfrak{M})$ of $k$-functors by formula $\rho(r)=\alpha_r$. As in Section 5 we can prove that $\rho$ is a morphism of ring $k$-functors. Now we have the following result.

**Theorem 6.6.** *Let $\mathfrak{R}$ be an algebra $k$-functor over commutative ring $\mathfrak{A}$ $k$-functor and let $\mathfrak{M}$ be a $\mathfrak{A}$-module such that $\mathcal{E}nd_{\mathfrak{A}}(\mathfrak{M})$ exists. Suppose that*

$$\left\{\mathfrak{A}\text{ linear actions of }\mathfrak{R}\text{ on }\mathfrak{M}\right\}\longrightarrow\left\{\text{Morphisms }\rho:\mathfrak{R}\to\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})\text{ of ring }k\text{-functors}\right\}$$

*is a map of classes described above. Then it is bijection.*

*Proof.* The proof is similar to the proof of Theorem 5.1. □

## 7. Monoid algebra $\mathfrak{O}_k[\mathfrak{G}]$ and its modules

**Definition 7.1.** Let $\mathfrak{G}$ be a monoid $k$-functor. Then we construct an $\mathfrak{O}_k$-algebra $\mathfrak{O}_k[\mathfrak{G}]$ as follows. For every $k$-algebra $A$ we define

$$\mathfrak{O}_k[\mathfrak{G}](A) = A\big[\mathfrak{G}(A)\big]$$

where the right hand side is monoid $A$-algebra for the abstract monoid $\mathfrak{G}(A)$. The structure of monoid $k$-functor on $\mathfrak{G}$ and $\mathfrak{K}$-algebra $\mathfrak{O}_k$ makes $\mathfrak{O}_k[\mathfrak{G}]$ into a ring $k$-functor. Moreover, we have a morphism $\mathfrak{O}_k \to \mathfrak{O}_k[\mathfrak{G}]$ which for every $k$-algebra $A$ is given by the canonical inclusion

$$A \hookrightarrow A\big[\mathfrak{G}(A)\big]$$

Thus $\mathfrak{O}_k[\mathfrak{G}]$ is $\mathfrak{O}_k$-algebra. We call $\mathfrak{O}_k[\mathfrak{G}]$ *a monoid $\mathfrak{O}_k$-algebra over $\mathfrak{G}$*.

**Fact 7.2.** *Let $\mathfrak{G}$ be a monoid $k$-functor and let $\mathfrak{R}$ be an $\mathfrak{O}_k$-algebra $k$-functor. Then every morphism*

$$\sigma : \mathfrak{G} \to \mathfrak{R}^{\times}$$

*of monoid $k$-functors admits a unique extension*

$$\tilde{\sigma} : \mathfrak{O}_k[\mathfrak{G}] \to \mathfrak{R}$$

*to a morphism of $\mathfrak{O}_k$-algebras.*

*Proof.* This follows from the analogical universal property of algebras over abstract monoids. □

**Definition 7.3.** Let $\mathfrak{G}$ be a monoid $k$-functor and let $\mathfrak{M}$ be a module over $\mathfrak{O}_k$. Suppose that $\alpha : \mathfrak{G} \times \mathfrak{M} \to \mathfrak{M}$ is an action of $\mathfrak{G}$ such that for any $k$-algebra $A$ and point $g \in \mathfrak{G}(A)$ morphism $\alpha_g : \mathfrak{M}_A \to \mathfrak{M}_A$ is a morphism of $\mathfrak{O}_A$-modules. Then $\alpha$ is called *a linear $\mathfrak{G}$-action on $\mathfrak{M}$*.

Suppose now that $\mathfrak{G}$ is a monoid $k$-functor and $\mathfrak{M}$ is a module $\mathfrak{O}_k$. Note that every linear $\mathfrak{G}$-action $\alpha : \mathfrak{G} \times \mathfrak{M} \to \mathfrak{M}$ extends uniquely to a $\mathfrak{O}_k$-linear action $\mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}$ of monoid $\mathfrak{O}_k$-algebra. This gives a bijection

$$\Big\{\text{Linear actions of } \mathfrak{G} \text{ on } \mathfrak{M}\Big\} \longrightarrow \Big\{\mathfrak{O}_k\text{-linear actions } \mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}\Big\}$$

Next assume that $k$-functor $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ exists. By Example 6.4 it is an $\mathfrak{O}_k$-algebra $k$-functor. Next by Theorem 6.6 we have a bijection

$$\Big\{\mathfrak{O}_k\text{-linear actions of } \mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}\Big\} \longrightarrow \Big\{\text{Morphisms } \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}) \text{ of } \mathfrak{O}_k\text{-algebras}\Big\}$$

Finally Fact 7.2 implies that we have a bijection

$$\Big\{\text{Morphisms } \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}) \text{ of } \mathfrak{O}_k\text{-algebras}\Big\} \longrightarrow \Big\{\text{Morphisms } \mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}) \text{ of monoids}\Big\}$$

This chain of bijections sends a linear action $\alpha : \mathfrak{G} \times \mathfrak{M} \to \mathfrak{M}$ of $\mathfrak{G}$ to a morphism $\rho : \mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ of monoid $k$-functors given by $\rho(g) = \alpha_g$ for every $g \in \mathfrak{G}(A)$ and every $k$-algebra $A$. We proved the following result.

**Proposition 7.4.** *Let $\mathfrak{G}$ be a monoid $k$-functor and let $\mathfrak{M}$ be a $\mathfrak{O}_k$-module such that $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ exists. Then the following classes are in canonical bijections described above.*

**(1)** *Linear actions of $\mathfrak{G}$ on $\mathfrak{M}$.*

**(2)** *$\mathfrak{O}_k$-linear actions $\mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M} \to \mathfrak{M}$. These are precisely $\mathfrak{O}_k[\mathfrak{G}]$-modules.*

**(3)** *Morphisms $\mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ of $\mathfrak{O}_k$-algebras.*

**(4)** *Morphisms* $\mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$ *of monoids.*

*Moreover, the bijection between class* **(1)** *and* **(2)** *does not require the existence of* $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M})$*.*

Now in a similar manner we can describe morphisms.

**Proposition 7.5.** *Let* $\mathfrak{G}$ *be a monoid k-functor and let* $\mathfrak{M}_1$*,* $\mathfrak{M}_2$ *be k-functors of* $\mathfrak{O}_k$*-modules such that* $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1), \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$ *exist. Suppose that* $\alpha_1 : \mathfrak{G} \times \mathfrak{M}_1 \to \mathfrak{M}_1$*,* $\alpha_2 : \mathfrak{G} \times \mathfrak{M}_2 \to \mathfrak{M}_2$ *are linear actions of* $\mathfrak{G}$*. Suppose that* $\sigma : \mathfrak{M}_1 \to \mathfrak{M}_2$ *is a morphism of modules over* $\mathfrak{O}_k$*. Then the following assertions are equivalent.*

**(i)** *The square*

$$\begin{tikzcd} \mathfrak{G} \times \mathfrak{M}_1 \arrow[r, "1_{\mathfrak{G}} \times \sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{G} \times \mathfrak{M}_2 \arrow[d, "\alpha_2"] \\ \mathfrak{M}_1 \arrow[r, "\sigma"'] & \mathfrak{M}_2 \end{tikzcd}$$

*is commutative.*

**(ii)** *The square*

$$\begin{tikzcd} \mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M}_1 \arrow[r, "1_{\mathfrak{O}_k[\mathfrak{G}]} \times \sigma"] \arrow[d, "\tilde{\alpha_1}"'] & \mathfrak{O}_k[\mathfrak{G}] \times \mathfrak{M}_2 \arrow[d, "\tilde{\alpha_2}"] \\ \mathfrak{M}_1 \arrow[r, "\sigma"'] & \mathfrak{M}_2 \end{tikzcd}$$

*is commutative, where* $\tilde{\alpha_1}$ *and* $\tilde{\alpha_2}$ *are* $\mathfrak{O}_k$*-linear actions of* $\mathfrak{O}_k[\mathfrak{G}]$ *corresponding to* $\alpha_1$ *and* $\alpha_2$*, respectively. This states that* $\sigma$ *is a morphism of* $\mathfrak{O}_k[\mathfrak{G}]$*-modules.*

**(iii)** *For every k-algebra* $A$ *and* $g \in \mathfrak{G}(A)$ *we have*

$$\sigma_A \cdot \tilde{\rho}_1(g) = \tilde{\rho}_2(g) \cdot \sigma_A$$

*where* $\tilde{\rho}_1 : \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1)$ *and* $\tilde{\rho}_2 : \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$ *are morphism of* $\mathfrak{O}_k$*-algebras corresponding to* $\tilde{\alpha_1}$ *and* $\tilde{\alpha_2}$*, respectively.*

**(iv)** *For every k-algebra* $A$ *and* $g \in \mathfrak{G}(A)$ *we have*

$$\sigma_A \cdot \rho_1(g) = \rho_2(g) \cdot \sigma_A$$

*where* $\rho_1 : \mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1)$ *and* $\rho_2 : \mathfrak{G} \to \mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$ *are restrictions of* $\tilde{\rho}_1$ *and* $\tilde{\rho}_2$*, respectively.*

*The equivalence of* **(i)** *and* **(ii)** *does not require the existence of* $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_1)$ *and* $\mathcal{E}nd_{\mathfrak{O}_k}(\mathfrak{M}_2)$*.*

*Proof.* Conditions expressed in **(i)-(iv)** are directly translatable to each other by virtue of bijections in Proposition 7.4. □

Let $\mathfrak{G}$ be a monoid $k$-functor. We denote by **Mod** $(\mathfrak{O}_k[\mathfrak{G}])$ the category of $\mathfrak{O}_k[\mathfrak{G}]$-modules.

## 8. Example of $\mathfrak{G}$-action: Regular functions $k$-functor

First we need the following notion.

**Definition 8.1.** Let $(-)^{\mathrm{op}} : \mathbf{Mon} \to \mathbf{Mon}$ be the opposite monoid functor and let $\mathfrak{G}$ be a monoid $k$-functor. Then the composition $\mathfrak{G}^{\mathrm{op}} = (-)^{\mathrm{op}} \cdot \mathfrak{G}$ is called *the opposite monoid k-functor of* $\mathfrak{G}$.

Let $\mathfrak{G}$ be a monoid $k$-functor. In this section we discuss important example of a $\mathfrak{O}_k[\mathfrak{G}]$-module. Fix a $k$-functor $\mathfrak{X}$ for which $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{O}_k)$ exists. Recall that by Example 3.5 $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{O}_k)$ is $\mathfrak{O}_k$-algebra $k$-functor. Let $\alpha : \mathfrak{G}\times\mathfrak{X}\to\mathfrak{X}$ be an action of $\mathfrak{G}$ on $\mathfrak{X}$. For every $k$-algebra $A$ we have a map of sets

$$\mathrm{Mor}_A(\mathfrak{X}_A,(\mathfrak{O}_k)_A)\ni f\mapsto f\cdot\alpha_g\in\mathrm{Mor}_A(\mathfrak{X}_A,(\mathfrak{O}_k)_A)$$

where $g\in\mathfrak{G}(A)$. From this description it follows that the map $f\mapsto f\cdot\alpha_g$ is a morphism of $A$-algebras. Moreover, note that if $g_1,g_2\in\mathfrak{G}(A)$, then $\left(f\cdot\alpha_{g_1}\right)\cdot\alpha_{g_2}=f\cdot\alpha_{g_1\cdot g_2}$, where $g_1\cdot g_2\in\mathfrak{G}(A)$ is a product of $g_1$ and $g_2$. Thus the opposite monoid $\mathfrak{G}^{\mathrm{op}}(A)$ acts on the $A$-algebra $\mathrm{Mor}_A\left(\mathfrak{X}_A,(\mathfrak{O}_k)_A\right)$ by morphism of $A$-algebras. Next for every $A$-algebra $B$ and every point $x\in\mathfrak{X}(B)$ we have

$$(f\cdot\alpha_g)(x)=f\left(\alpha_g(x)\right)$$

where $g\in\mathfrak{G}(A)$. This proves the following result.

**Proposition 8.2.** *Let $\mathfrak{X}$ be a $k$-functor and let $\alpha:\mathfrak{G}\times\mathfrak{X}\to\mathfrak{X}$ be an action of a monoid $k$-functor $\mathfrak{G}$. Suppose that $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{O}_k)$ exists. Then $\mathfrak{G}^{\mathrm{op}}$ acts canonically on $\mathfrak{O}_k$-algebra $k$-functor $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{O}_k)$ by morphisms of $\mathfrak{O}_k$-algebras.*

Let us note one important consequence of this result.

**Corollary 8.3.** *Let $\mathfrak{G}$ be a monoid $k$-functor. The action of $\mathfrak{G}\times\mathfrak{G}^{\mathrm{op}}$ on $\mathfrak{G}$ induces the action of $\mathfrak{G}^{\mathrm{op}}\times\mathfrak{G}$ on $\mathfrak{O}_k$-algebra $k$-functor $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{O}_k)$ by morphisms of $\mathfrak{O}_k$-algebras.*

## 9. Linear representations of a monoid $k$-functors

We start the discussion with some results that relates categories $\mathbf{Mod}(k)$ and $\mathbf{Mod}(\mathfrak{O}_k)$.

**Example 9.1.** Let $V$ be a $k$-module. We define a $k$-functor $V_{\mathrm{a}}$. We set

$$V_{\mathrm{a}}(A)=A\otimes_k V,\ V_{\mathrm{a}}(f)=f\otimes_k 1_V$$

for every $k$-algebra $A$ and every morphism $f:A\to B$ of $k$-algebras. Note that $V_{\mathrm{a}}$ is $\mathfrak{O}_k$-module. Suppose that $\phi:V\to W$ is a morphism of $k$-modules, then we define $\phi_{\mathrm{a}}:V_{\mathrm{a}}\to W_{\mathrm{a}}$ by formula

$$\phi_{\mathrm{a}}^A=1_A\otimes_k\sigma$$

for every $k$-algebra. Then $\phi_{\mathrm{a}}$ is a morphism of $\mathfrak{O}_k$-modules.

**Proposition 9.2.** *The functor $(-)_{\mathrm{a}}:\mathbf{Mod}(k)\to\mathbf{Mod}(\mathfrak{O}_k)$ is full and faithful.*

*Proof.* Fix $k$-modules $V,W$. Then

$$\mathrm{Hom}_{\mathfrak{O}_k}(V_{\mathrm{a}},W_{\mathrm{a}})\ni\sigma\mapsto\sigma^k\in\mathrm{Hom}_k(V,W)$$

and

$$\mathrm{Hom}_k(V,W)\ni\phi\mapsto\phi_{\mathrm{a}}\in\mathrm{Hom}_{\mathfrak{O}_k}(V_{\mathrm{a}},W_{\mathrm{a}})$$

are mutually inverse bijections. Hence the functor is full and faithful. □

**Example 9.3.** Let $V$ be a $k$-module. We define a $k$-functor $\mathcal{L}_V$. We set

$$\mathcal{L}_V(A)=\mathrm{Hom}_A(A\otimes_k V,A\otimes_k V)$$

for every $k$-algebra $A$. Next for every morphism $f:A\to B$ of $k$-algebras and every morphism $\phi:A\otimes_k V\to A\otimes_k V$ of $A$-modules we define $\mathcal{L}_V(f)(\phi)$ as a unique morphism of $B$-modules such that the diagram

$$\begin{tikzcd}
A\otimes_k V \arrow[r, "\phi"] \arrow[d, "f\otimes_k 1_V"'] & A\otimes_k V \arrow[d, "f\otimes_k 1_V"] \\
B\otimes_k V \arrow[r, "\mathcal{L}_V(\phi)"'] & B\otimes_k V
\end{tikzcd}$$

is commutative. Note also that $\mathcal{L}_V(A)$ is an $A$-algebra for every $k$-algebra $A$. Hence $\mathcal{L}_V$ is a monoid $\mathfrak{O}_k$-algebra. Note that we have natural identification

$$\mathcal{L}_V(A) = \mathrm{Hom}_k(V, A \otimes_k V)$$

for every $k$-algebra. One can describe $\mathfrak{O}_k$-algebra structure on $\mathcal{L}_V$ in terms of this identification as follows. Since $\mathrm{Hom}_k(V, A \otimes_k V)$ carries canonical structure of $A$-module it suffices to describe the multiplication. For this suppose that $d_1, d_2 \in \mathrm{Hom}_k(V, A \otimes_k V)$. Then their product is given by

$$(\mu_A \otimes_k 1_V) \cdot (1_A \otimes d_2) \cdot d_1$$

where $\mu_A : A \otimes_k A \to A$ is the multiplication on $A$.

**Remark 9.4.** Let $V$ be a $k$-module. Proposition 9.2 implies that there are bijective maps that make the square

$$\begin{tikzcd} \mathcal{L}_V(A) \arrow[r, "\cong"] \arrow[d, "\mathcal{L}_V(f)"'] & \mathcal{E}nd_{\mathfrak{O}_A}\left((V_{\mathrm{a}})_A, (V_{\mathrm{a}})_A\right) \arrow[d, "\sigma \mapsto \sigma_B"] \\ \mathcal{L}_V(B) \arrow[r, "\cong"'] & \mathcal{E}nd_{\mathfrak{O}_B}\left((V_{\mathrm{a}})_B, (V_{\mathrm{a}})_B\right) \end{tikzcd}$$

commutative for every morphism $f : A \to B$ of $k$-algebras. This induces an idenitification $\mathcal{L}_V = \mathcal{E}nd_{\mathfrak{O}_k}(V_{\mathrm{a}})$ of $\mathfrak{O}_k$-algebras.

**Definition 9.5.** Let $\mathfrak{G}$ be a monoid $k$-functor. A pair $(V, \rho)$ consisting of a $k$-module $V$ and a morphism $\rho : \mathfrak{G} \to \mathcal{L}_V$ of $k$-monoids is called *a linear representation of* $\mathfrak{G}$.

Next result characterizes linear representations of monoid $k$-functors.

**Corollary 9.6.** *Let $\mathfrak{G}$ be a monoid $k$-functor and let $V$ be a $k$-module. Then the following classes are in canonical bijections.*

**(1)** *Linear actions of $\mathfrak{G}$ on $V_{\mathrm{a}}$.*

**(2)** *$\mathfrak{O}_k$-linear actions $\mathfrak{O}_k[\mathfrak{G}] \times V_{\mathrm{a}} \to V_{\mathrm{a}}$. These are precisely $\mathfrak{O}_k[\mathfrak{G}]$-modules.*

**(3)** *Morphisms $\mathfrak{O}_k[\mathfrak{G}] \to \mathcal{L}_V$ of $\mathfrak{O}_k$-algebras.*

**(4)** *Morphisms $\mathfrak{G} \to \mathcal{L}_V$ of monoids.*

*Proof.* This follows from Proposition 7.4. □

**Definition 9.7.** Let $\mathfrak{G}$ be a monoid $k$-functor and let $(V, \rho)$, $(W, \delta)$ be its linear representations. A morphism $\phi : V \to W$ of $k$-modules such that

$$\phi_{\mathrm{a}}^A \cdot \rho(g) = \delta(g) \cdot \phi_{\mathrm{a}}^A$$

for every $k$-algebra $A$ and $g \in \mathfrak{G}(A)$ is called *a morphism of linear representations of* $\mathfrak{G}$.

Next result characterizes morphisms of linear representations of monoid $k$-functor.

**Corollary 9.8.** *Let $\mathfrak{G}$ be a monoid $k$-functor and let $V$, $W$ be $k$-modules. Suppose that $\alpha_1 : \mathfrak{G} \times V_{\mathrm{a}} \to V_{\mathrm{a}}$, $\alpha_2 : \mathfrak{G} \times W_{\mathrm{a}} \to W_{\mathrm{a}}$ are linear actions of $\mathfrak{G}$. Suppose that $\phi : V \to W$ is a morphism of $k$-modules. Then the following assertions are equivalent.*

**(i)** *The square*

$$\begin{tikzcd}
\mathfrak{G} \times V_{\mathrm{a}} \arrow[r, "1_{\mathfrak{G}} \times \phi_{\mathrm{a}}"] \arrow[d, "\alpha_1"'] & \mathfrak{G} \times W_{\mathrm{a}} \arrow[d, "\alpha_2"] \\
V_{\mathrm{a}} \arrow[r, "\phi_{\mathrm{a}}"'] & W_{\mathrm{a}}
\end{tikzcd}$$

*is commutative.*

**(ii)** *The square*

$$\begin{tikzcd}
\mathfrak{O}_k[\mathfrak{G}] \times V_{\mathrm{a}} \arrow[r, "1_{\mathfrak{O}_k[\mathfrak{G}]} \times \phi_{\mathrm{a}}"] \arrow[d, "\tilde{\alpha_1}"'] & \mathfrak{O}_k[\mathfrak{G}] \times W_{\mathrm{a}} \arrow[d, "\tilde{\alpha_2}"] \\
V_{\mathrm{a}} \arrow[r, "\phi_{\mathrm{a}}"'] & W_{\mathrm{a}}
\end{tikzcd}$$

*is commutative, where $\tilde{\alpha_1}$ and $\tilde{\alpha_2}$ are $\mathfrak{O}_k$-linear actions of $\mathfrak{O}_k[\mathfrak{G}]$ corresponding to $\alpha_1$ and $\alpha_2$, respectively.*

**(iii)** *For every $k$-algebra $A$ and $g \in \mathfrak{G}(A)$ we have*

$$\phi_{\mathrm{a}}^A \cdot \tilde{\rho}_1(g) = \tilde{\rho}_2(g) \cdot \phi_{\mathrm{a}}^A$$

*where $\tilde{\rho}_1 : \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{L}_V$ and $\tilde{\rho}_2 : \mathfrak{O}_k[\mathfrak{G}] \to \mathcal{L}_W$ are morphism of $\mathfrak{O}_k$-algebras corresponding to $\tilde{\alpha_1}$ and $\tilde{\alpha_2}$, respectively.*

**(iv)** *For every $k$-algebra $A$ and $g \in \mathfrak{G}(A)$ we have*

$$\phi_{\mathrm{a}}^A \cdot \rho_1(g) = \rho_2(g) \cdot \phi_{\mathrm{a}}^A$$

*where $\rho_1 : \mathfrak{G} \to \mathcal{L}_V$ and $\rho_2 : \mathfrak{G} \to \mathcal{L}_W$ are restrictions of $\tilde{\rho}_1$ and $\tilde{\rho}_2$, respectively. This states that $\phi$ is a morphism of linear representations of $\mathfrak{G}$.*

*Proof.* This follows from Proposition 7.5. □

Let $\mathfrak{G}$ be a monoid $k$-functor. We denote by $\mathbf{Rep}(\mathfrak{G})$ its category of linear representations. Note that $\mathbf{Rep}(\mathfrak{G})$ is a full subcategory of $\mathbf{Mod}(\mathfrak{O}_k[\mathfrak{G}])$.

## 10. Monoid $k$-schemes

**Definition 10.1.** *A monoid $k$-scheme* $\mathbf{M}$ is a monoid object in the category of $k$-schemes. If $\mathbf{M}$ is affine, then we say that $\mathbf{M}$ is *an affine monoid $k$-scheme*.

**Definition 10.2.** *A group $k$-scheme* $\mathbf{G}$ is a group object in the category of $k$-schemes. If $\mathbf{G}$ is affine, then we say that $\mathbf{G}$ is *an affine group $k$-scheme*.

**Corollary 10.3.** *The functor*

$$\mathbf{Sch}_k \xrightarrow{\mathfrak{P}} \text{the category of } k\text{-functors}$$

*induces an equivalence of categories*

$$\text{the category of monoid } k\text{-schemes} \cong \text{monoid } k\text{-functors representable by } k\text{-schemes}$$

*Similarly for categories of groups.*

*Proof.* Follows from [Mon19b, Fact 4.1]. □

Recall that by Example 2.3 each monoid $k$-functor $\mathfrak{G}$ has its group $k$-functor $\mathfrak{G}^*$ of units.

**Proposition 10.4.** *Let* $\mathbf{M}$ *be an affine monoid k-scheme. Then the k-functor of units* $\mathfrak{P}^*_{\mathbf{M}}$ *is representable (by an affine k-scheme).*

*Proof.* Note that $\mathfrak{P}^*_{\mathbf{M}}$ fits into a cartesian square

$$\begin{tikzcd}
\mathfrak{P}^*_{\mathbf{M}} \arrow[r] \arrow[d] & \mathbf{1} \arrow[d, "\mathfrak{P}_e"] \\
\mathfrak{P}_{\mathbf{M}} \times \mathfrak{P}_{\mathbf{M}} \arrow[r, "\mathfrak{P}_m"'] & \mathfrak{P}_{\mathbf{M}}
\end{tikzcd}$$

where $m : \mathbf{M} \times \mathbf{M} \to \mathbf{M}$ is the multiplication and $e : \operatorname{Spec} k \to \mathbf{M}$ is the unit. By [Mon19b, Fact 4.1] the functor $\mathfrak{P}$ preserves finite products and hence it preserves fiber-products. This implies that $\mathfrak{P}^*_{\mathbf{M}}$ is represented by a unique (up to an isomorphism) $k$-scheme $\mathbf{M}^*$ that fit into a cartesian square below.

$$\begin{tikzcd}
\mathbf{M}^* \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d, "e"] \\
\mathbf{M} \times \mathbf{M} \arrow[r, "m"'] & \mathbf{M}
\end{tikzcd}$$

Note that if $\mathbf{M}$ is affine, then also $\mathbf{M}^*$ is affine. $\square$

**Remark 10.5.** Under the embedding given in Corollary 10.3 notions defined for monoid $k$-functors can be translated to monoid $k$-schemes.

We give three instances of the use of Remark 10.5 below.

**Definition 10.6.** Let $\mathbf{M}$ be a monoid $k$-scheme. Then the group $k$-scheme $\mathbf{M}^*$ representing $\mathfrak{P}^*_{\mathbf{M}}$ is called *the group of units of* $\mathbf{M}$.

**Definition 10.7.** Let $\mathbf{M}$ be a monoid $k$-scheme. Then *the category of linear representations of* $\mathbf{M}$ is the category of linear representations of the monoid $k$-functor $\mathfrak{P}_{\mathbf{M}}$. We denote this category by $\mathbf{Rep}(\mathbf{M})$.

**Definition 10.8.** Let $\mathbf{M}$ be a monoid $k$-functor and let $\alpha : \mathfrak{P}_{\mathbf{M}} \times \mathfrak{X} \to \mathfrak{X}$ be an action of $\mathfrak{P}_{\mathbf{M}}$ on a $k$-functor $\mathfrak{X}$. Then we say that $\alpha$ is *an action of* $\mathbf{M}$ *on* $\mathfrak{X}$.

## 11. Bialgebras and affine monoid $k$-schemes

We start here with a general notion of $k$-coalgebras.

**Definition 11.1.** Let $(C, \Delta, \xi)$ be a triple consisting of a module $C$ over $k$ and morphisms

$$\Delta : C \to C \otimes_k C, \xi : C \to k$$

of $k$-modules such that the following diagrams are commutative.

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes_k C \arrow[d, "1_C \otimes_k \Delta"] \\
C \otimes_k C \arrow[r, "\Delta \otimes_k 1_C"'] & C \otimes_k C \otimes_k C
\end{tikzcd}
\quad
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[dr, "\cong"'] & C \otimes_k C \arrow[d, "1 \otimes_k \xi"] \\
& C \otimes_k k
\end{tikzcd}
\quad
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[dr, "\cong"'] & C \otimes_k C \arrow[d, "\xi \otimes_k 1_C"] \\
& k \otimes_k C
\end{tikzcd}$$

Then $(C, \Delta, \xi)$ is called *a $k$-coalgebra*. Morphisms $\Delta$, $\xi$ are called *a comultiplication* and *a counit*, respectively.

**Definition 11.2.** Let $(C_1, \Delta_1, \xi_1)$ and $(C_2, \Delta_2, \xi_2)$ are $k$-coalgebras. Then a morphism $f : C_1 \to C_2$ of $k$-modules is *a morphism of $k$-coalgebras* if the following diagrams are commutative.

$$\begin{tikzcd} C_1 \otimes_k C_1 \arrow[r, "f \otimes_k f"] & C_2 \otimes_k C_2 \\ C_1 \arrow[u, "\Delta_1"] \arrow[r, "f"'] & C_2 \arrow[u, "\Delta_2"'] \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[rr, "f"] \arrow[dr, "\xi_1"'] & & C_2 \arrow[dl, "\xi_2"] \\ & k & \end{tikzcd}$$

By $k$-algebra we mean commutative and unital $k$-algebra.

**Definition 11.3.** Let $B$ be a $k$-module with structures of both $k$-algebra and $k$-coalgebra. Assume that the comultiplication and the counit of $B$ are morphisms of $k$-algebras with respect to $k$-algebra structure of $B$. Then we say that $B$ with these structures is *a $k$-bialgebra*.

**Definition 11.4.** Let $B_1, B_2$ be $k$-bialgebras and let $f : B_1 \to B_2$ be a morphism of $k$-modules. We say that $f$ is *a morphism of $k$-bialgebras* if it is simultaneously morphism of $k$-algebras and $k$-coalgebras.

**Theorem 11.5.** *The functor* Spec : $\mathbf{Alg}_k \to \mathbf{Sch}_k$ *induces an equivalence of categories*

$$k\text{-bialgebras} \cong \text{the category of affine monoid } k\text{-schemes}$$

*Proof.* This is an exercise in translation. For details see [DG70, II, 1.6]. ☐

Let $\mathbf{M}$ be an affine monoid $k$-scheme. Then we denote by $k[\mathbf{M}]$ its coordinate $k$-bialgebra, by $\Delta_{\mathbf{M}}$ its comultiplication and by $\xi_{\mathbf{M}}$ its counit. This is a notation that we consistently use in these notes.

## 12. Comodules over $k$-coalgebras

**Definition 12.1.** Let $C$ be a $k$-coalgebra with the comultiplication $\Delta$ and the counit $\xi$. A pair $(V, d)$ consisting of a $k$-module $V$ and a morphism $d : V \to C \otimes_k V$ of $k$-modules such that the following diagrams are commutative

$$\begin{tikzcd} V \arrow[r, "d"] \arrow[d, "d"'] & C \otimes_k V \arrow[d, "1_C \otimes_k d"] \\ C \otimes_k V \arrow[r, "\Delta \otimes_k 1_V"'] & C \otimes_k C \otimes_k V \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "d"] \arrow[dr, "\cong"'] & C \otimes_k V \arrow[d, "\xi \otimes_k 1_V"] \\ & k \otimes_k V \end{tikzcd}$$

is called *a $C$-comodule*. Morphism $d$ is called *a coaction of $C$ on $V$*.

**Definition 12.2.** Let $C$ be a $k$-coalgebra and let $(V_1, d_1), (V_2, d_2)$ be two comodules over $C$. A morphism of $k$-modules $f : V_1 \to V_2$ is *a morphism of $C$-comodules* if the diagram

$$\begin{tikzcd} C \otimes_k V_1 \arrow[r, "1_C \otimes_k f"] & C \otimes_k V_2 \\ V_1 \arrow[u, "d_1"] \arrow[r, "f"'] & V_2 \arrow[u, "d_2"'] \end{tikzcd}$$

is commutative.

We denote by $\mathbf{coMod}(C)$ the category of $C$-comodules for a $k$-coalgebra $C$.

**Theorem 12.3.** *Let $C$ be a $k$-coalgebra. Then the forgetful functor $\mathbf{coMod}(C) \to \mathbf{Mod}(k)$ creates colimits.*

*Proof.* Let $\Delta, \xi$ be the comultiplication and the counit of $C$, respectively. Suppose that $I \ni i \mapsto (V_i, d_i) \in \mathbf{coMod}(C)$ is a diagram of $C$-comodules indexed by some category $I$. Let $V$ together with $u_i : V_i \to V$ for $i \in I$ be a colimit of the diagram $I \ni i \mapsto V_i \in \mathbf{Mod}(k)$. By the universal property of colimit we deduce that there exists a unique morphism $d : V \to C \otimes_k V$ such that diagrams

$$\begin{tikzcd} C \otimes_k V_i \arrow[r, "1_C \otimes_k u_i"] & C \otimes_k V \\ V_i \arrow[u, "d_i"] \arrow[r, "u_i"'] & V \arrow[u, "d"'] \end{tikzcd}$$

are commutative for every $i \in I$. In order to verify that diagrams

$$\begin{tikzcd} V \arrow[r, "d"] \arrow[d, "d"'] & C \otimes_k V \arrow[d, "1_C \otimes_k d"] \\ C \otimes_k V \arrow[r, "\Delta \otimes_k 1_V"'] & C \otimes_k C \otimes_k V \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "d"] \arrow[dr, "\cong"'] & C \otimes_k V \arrow[d, "\xi \otimes_k 1_V"] \\ & k \otimes_k V \end{tikzcd}$$

are commutative it suffices to note that for every $i \in I$ we have chains of equalities

$$(1_C \otimes_k d) \cdot d \cdot u_i = (1_C \otimes_k 1_C \otimes_k u_i) \cdot (1_C \otimes_k 1_C \otimes_k d_i) \cdot d_i = (1_C \otimes 1_C \otimes_k u_i) \cdot \left(\Delta \otimes_k 1_{V_i}\right) \cdot d_i = (\Delta \otimes_k 1_V) \cdot d \cdot u_i$$

and

$$(\xi \otimes_k 1_V) \cdot d \cdot u_i = (1_k \otimes_k u_i) \cdot \left(\xi \otimes_k 1_{V_i}\right) \cdot d_i = (1_k \otimes_k u_i) \cdot j_{V_i} = j_V \cdot u_i$$

where $j_W : W \to k \otimes_k W$ is the natural isomorphism for every $k$-module $W$. Hence $(V, d)$ is a $C$-comodule. Suppose now that $(W, e)$ is a $C$-comodule and $w_i : V_i \to W$ for $i \in I$ is a family of $C$-comodule morphisms compatible with the diagram $I \ni i \mapsto (V_i, d_i) \in \mathbf{coMod}(C)$. Since $\{u_i : V_i \to V\}_{i \in I}$ form a colimiting cocone for $I \ni i \mapsto V_i \in \mathbf{Mod}(k)$, there exists a unique morphism of $k$-modules $f : V \to W$ such that $f \cdot u_i = w_i$. Note that

$$e \cdot f \cdot u_i = e \cdot w_i = (1_C \otimes_k w_i) \cdot d_i = (1_C \otimes_k f) \cdot (1_C \otimes_k u_i) \cdot d_i = (1_C \otimes_k f) \cdot d \cdot u_i$$

for every $i \in I$. Hence $e \cdot f = (1_C \otimes_k f) \cdot d$. Thus $f$ is a morphism of $C$-comodules. Thus $(V, d)$ together with family $\left\{u_i : (V_i, d_i) \to (V, d)\right\}_{i \in I}$ is a colimit of the diagram $I \ni i \mapsto (V_i, d_i) \in \mathbf{coMod}(C)$ of $C$-comodules. This implies that the forgetful functor $\mathbf{coMod}(C) \to \mathbf{Mod}(k)$ creates colimits. □

**Theorem 12.4.** *Let $C$ be a $k$-coalgebra such that $C$ is a flat $k$-module. Then the forgetful functor $\mathbf{coMod}(C) \to \mathbf{Mod}(k)$ creates finite limits.*

*Proof.* The proof is similar to the proof of Theorem 12.3. □

**Corollary 12.5.** *Let $C$ be a coalgebra over $k$ and assume that $C$ is flat as a $k$-module. Then $\mathbf{coMod}(C)$ is an abelian category with small colimits.*

*Proof.* This follows from Theorems 12.3 and 12.4. □

The next result is of fundamental importance.

**Theorem 12.6.** *Let $C$ be a $k$-coalgebra that is free as a $k$-module. Suppose that $V$ is a $C$-comodule over $C$. Then for every finitely generated $k$-submodule $U \subseteq V$ there exists a $C$-subcomodule $W$ of $V$ such that $U \subseteq W$ and $W$ is a finitely generated $k$-module.*

The theorem follows from the following simple lemma.

**Lemma 12.6.1.** *Let $C$ be a $k$-coalgebra over $k$ that is free as a $k$-module. Suppose that $V$ is a $C$-comodule over $C$ and fix an element $v \in V$. Then there exists a $C$-subcomodule $W$ of $V$ such that $v \in W$ and $W$ is a finitely generated $k$-module.*

*Proof of the lemma.* Let $\{e_j\}_{j\in J}$ be a free basis of $C$ over $k$ and let $d : V \to C \otimes_k V$ be a left coaction of $C$ on $V$. Denote by $\Delta : C \to C \otimes_k C$ the comultiplication of $C$. Then we have

$$d(v) = \sum_{j\in J} e_j \otimes v_j$$

where $v_j \in V$ are zero for almost all $j \in J$. Next according to

$$(\Delta \otimes_k 1_V) \cdot d = (1_C \otimes_k d) \cdot d$$

we derive that equality

$$\sum_{j\in J} e_j \otimes d(v_j) = (1_C \otimes_k d)\big(d(v)\big) = (\Delta \otimes_k 1_V)\big(d(v)\big) = \sum_{j\in J} \Delta(e_j) \otimes v_j \subseteq \sum_{j\in J} C \otimes_k C \otimes_k k \cdot v_j$$

holds. This implies that $d(v_j) \subseteq C \otimes_k \left(\sum_{j\in J} k \cdot v_j\right)$. Hence $k$-submodule $W$ of $V$ generated by $v$ and $\{v_j\}_{j\in J}$ is $C$-subcomodule of $V$. It is finitely generated as a $k$-module and $v \in W$. □

*Proof of the theorem.* Suppose that $U$ is generated by $\{v_1, ..., v_n\}$ as a $k$-module. For each $i$ pick $C$-subcomodule $W_i$ of $V$ such that $W_i$ is finitely generated as a $k$-module and $v_i \in W_i$. This can be done by Lemma 12.6.1. Next

$$W = W_1 + ... + W_n$$

is a $C$-subcomodule of $V$ that is finitely generated as a $k$-module and contains $U$. □

## 13. Linear representations and comodules

Let $\mathbf{M}$ be an affine monoid $k$-scheme and let $\rho : \mathfrak{P}_{\mathbf{M}} \to \mathcal{L}_V$ be a morphism of functors of sets, where $V$ is a $k$-module. Yoneda Lemma implies that $\rho$ is determined by some element (Example 9.3)

$$d_\rho \in \mathrm{Hom}_k\left(V, k[\mathbf{M}] \otimes_k V\right)$$

**Theorem 13.1.** *Let $\mathbf{M}$ be an affine monoid $k$-scheme. Then the correspondence*

$$\mathbf{Rep}(\mathbf{M}) \ni (V,\rho) \mapsto \left(V, d_\rho\right) \in \mathbf{coMod}\left(k[\mathbf{M}]\right)$$

*is an isomorphism of categories over* $\mathbf{Mod}(k)$.

*Proof.* We fix notation in the proof. We denote by $\mu_A : A \otimes_k A \to A$ the multiplication and by $\eta_A : k \to A$ the unit for every $k$-algebra $A$. If $A$ is a $k$-algebra, then we denote by $e_A$ the composition $\eta_A \cdot \xi_{\mathbf{M}} : k[\mathbf{M}] \to A$. Note that $e_A \in \mathfrak{P}_{\mathbf{M}}(A)$ is the neutral element.
We start the proof with some useful remarks. If $V$ is a $k$-module, then

$$\mathcal{L}_V(A) = \mathrm{Hom}_k(V, A \otimes_k V)$$

for every $k$-algebra $A$ with $\mathfrak{O}_k$-algebra structure discussed in Example 9.3. Moreover, if $\rho : \mathfrak{P}_{\mathbf{M}} \to \mathcal{L}_V$ is a morphism of $k$-functors corresponding to $d_\rho : V \to k[\mathbf{M}] \otimes_k V$, then for every $k$-algebra $A$ and a morphism $f : k[\mathbf{M}] \to A$ of $k$-algebras we have

$$\rho(f) = (f \otimes_k 1_V) \cdot d_\rho$$

Our Discussion in Example 9.3 and Yoneda Lemma show that the following assertions hold.

**(1)** For $k$-algebra $A$ and $f_1, f_2 \in \mathrm{Hom}_k(k[\mathbf{M}], A) = \mathfrak{P}_{\mathbf{M}}(A)$ we have

$$\rho(f_1)\cdot\rho(f_2) = (\mu_A \otimes_k 1_V)\cdot(f_2 \otimes_k f_1 \otimes_k 1_V)\cdot\left(1_{k[\mathbf{M}]}\otimes_k d_\rho\right)\cdot d_\rho$$

and

$$\rho(f_1\cdot f_2) = (\mu_A \otimes_k 1_V)\cdot(f_2 \otimes_k f_1 \otimes_k 1_V)\cdot(\Delta_{\mathbf{M}}\otimes_k 1_V)\cdot d_\rho$$

**(2)** For $k$-algebra $A$ we have

$$\rho(e_A) = (\eta_A \otimes_k 1_V)\cdot(\xi_{\mathbf{M}}\otimes_k 1_V)\cdot d_\rho$$

Now **(1)** imply that if $(\Delta_{\mathbf{M}}\otimes_k 1_V)\cdot d_\rho = \left(1_{O_{\mathbf{M}}}\otimes_k d_\rho\right)\cdot d_\rho$ then $\rho(f_1\cdot f_2) = \rho(f_1)\cdot\rho(f_2)$. On the other hand suppose that $\rho(f_1\cdot f_2) = \rho(f_1)\cdot\rho(f_2)$ for any two $f_1, f_2 : k[\mathbf{M}] \to A$ morphism of $k$-algebras and for every $k$-algebra $A$. Pick inclusions $f_1, f_2 : k[\mathbf{M}] \to k[\mathbf{M}]\otimes_k k[\mathbf{M}]$ onto first and second component, respectively. Then

$$\left(\mu_{k[\mathbf{M}]\otimes_k k[\mathbf{M}]}\otimes_k 1_V\right)\cdot(f_2\otimes_k f_1\otimes_k 1_V) = 1_{k[\mathbf{M}]}\otimes_k 1_{k[\mathbf{M}]}\otimes_k 1_V$$

and hence $(\Delta_{\mathbf{M}}\otimes_k 1_V)\cdot d_\rho = \left(1_{O_{\mathbf{M}}}\otimes_k d_\rho\right)\cdot d_\rho$ by **(1)**.
Now if $(\xi_{\mathbf{M}}\otimes_k 1_V)\cdot d_\rho$ is the canonical isomorphism $V \cong k\otimes_k V$. Then by **(2)** we derive that $\rho(e_A)$ is the canonical morphism $V \to A\otimes_k V$. On the other hand if $\rho(e_A)$ is $V \to A\otimes_k V$ for every $k$-algebra $A$, then substituting $k$ for $A$ we deduce by **(2)** that $\rho(e_k) = (\xi_{\mathbf{M}}\otimes_k 1_V)\cdot d_\rho$ is the canonical isomorphism $V \cong k\otimes_k V$.
These considerations prove that $\rho$ is a morphism of monoid $k$-functors if and only if $d_\rho$ is a coaction of $k[\mathbf{M}]$ on $V$.
Now suppose that $V_1, V_2$ are $k$-modules and $\rho_1 : \mathfrak{P}_{\mathbf{M}} \to \mathcal{L}_V$, $\rho_2 : \mathfrak{P}_{\mathbf{M}} \to \mathcal{L}_W$ are morphisms of $k$-functors. Suppose that $\phi : V_1 \to V_2$ is a morphism of $k$-modules. Pick a $k$-algebra $A$ and a morphism $f : k[\mathbf{M}] \to A$ of $k$-algebras. Assume that the diagram

$$\begin{tikzcd}
k[\mathbf{M}]\otimes_k V_1 \arrow[r, "1_{k[\mathbf{M}]}\otimes_k \phi"] & k[\mathbf{M}]\otimes_k V_2 \\
V_1 \arrow[u, "d_{\rho_1}"] \arrow[r, "\phi"'] & V_2 \arrow[u, "d_{\rho_2}"']
\end{tikzcd}$$

is commutative. Since the square

$$\begin{tikzcd}
A\otimes_k V_1 \arrow[r, "1_A\otimes_k \phi"] & A\otimes_k V_2 \\
k[\mathbf{M}]\otimes_k V_1 \arrow[u, "f\otimes_k 1_V"] \arrow[r, "1_{k[\mathbf{M}]}\otimes_k \phi"'] & k[\mathbf{M}]\otimes_k V_2 \arrow[u, "f\otimes_k 1_W"']
\end{tikzcd}$$

is commutative, we derive that

$$\begin{tikzcd}
A\otimes_k V_1 \arrow[r, "1_A\otimes_k \phi"] & A\otimes_k V_2 \\
V_1 \arrow[u, "\rho_1(f)"] \arrow[r, "\phi"'] & V_2 \arrow[u, "\rho_2(f)"']
\end{tikzcd}$$

Moreover, if the square above commutes for every $k$-algebra $A$, then it also commutes for $A = k[\mathbf{M}]$ and this recovers the commutativity of the first square. Suppose now that $(V, \rho_1)$ and $(W, \rho_2)$ are linear representations of $\mathbf{M}$, then the discussion above implies that $\phi$ is a morphism of linear representations if and only if $\phi$ is a morphism of $k[\mathbf{M}]$-comodules $(V, d_{\rho_1})$ and $(W, d_{\rho_2})$. □

We obtain immediate consequence.

**Corollary 13.2.** *Let $k$ be a field. Let $(V, \rho)$ be a linear representation of an affine monoid $k$-scheme $\mathbf{M}$. Then for every finitely generated $k$-subspace $U \subseteq V$ there exists a subrepresentation $W$ of $(V, \rho)$ such that $U \subseteq W$ and $W$ is a finitely generated $k$-module.*

*Proof.* This follows from Theorems 13.1 and 12.6. □

## References

[DG70] Michel Demazure and Pierre Gabriel. *Groupes algébriques. Tome I. Géométrie algébrique généralités. Groupes commutatifs*. North-Holland, 1970.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[Mon19a] Monygham. Categories of presheaves. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2019.

[Mon19b] Monygham. Geometry of k-functors. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2019.

[Mon20] Monygham. Monoid k-functors and their representations. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2020.