

# QUOTIENTS OF ALGEBRAIC GROUPS

## 1. INTRODUCTION

Throughout this notes  $k$  denote a field and  $\mathbf{G}$  denote a group scheme over  $k$ . We denote by  $e$  the identity of  $\mathbf{G}$ . We also fix a  $k$ -scheme  $X$  equipped with an action of  $\mathbf{G}$  determined by morphism  $a : \mathbf{G} \times_k X \rightarrow X$ .

## 2. BASIC PROPERTIES OF SCHEME GROUP QUOTIENTS

The following result gives scheme-theoretic criterion for topological quotient in the case of group scheme actions.

**Proposition 2.1.** *Let  $Y$  be a  $k$ -scheme with the trivial action of  $\mathbf{G}$  and let  $q : X \rightarrow Y$  be a  $\mathbf{G}$ -equivariant morphism. Assume that  $q$  is submersive and the morphism  $\mathbf{G} \times_k X \rightarrow X \times_Y X$  induced by  $a$  and  $\text{pr}_X$  is surjective. Then the diagram*

$$\mathbf{G} \times_k X \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{\text{pr}_X} \end{array} X \xrightarrow{q} Y$$

is a cokernel in the category topological spaces.

*Proof.* Let  $\pi_1$  and  $\pi_2$  be distinct projections  $X \times_Y X \rightarrow X$ . Pick points  $x_1$  and  $x_2$  in  $X$  such that  $q(x_1) = q(x_2)$ . Then there exists a field extension  $K$  over  $k$  such that  $k(x_1) \subseteq K$  and  $k(x_2) \subseteq K$ . These give rise to  $K$ -points  $\bar{x}_1$  and  $\bar{x}_2$  of  $X$  such that their images under  $q$  is the same  $K$ -point of  $Y$ . Since we have an identification

$$(X \times_Y X)(K) = X(K) \times_{Y(K)} X(K)$$

induced by  $\pi_1$  and  $\pi_2$ , we derive that there exists a  $K$ -point  $\bar{z}$  of  $X \times_Y X$  such that  $\pi_1(\bar{z}) = \bar{x}_1$  and  $\pi_2(\bar{z}) = \bar{x}_2$ . Let  $z$  be the point of  $X \times_Y X$  corresponding to  $\bar{z}$ . Then  $\pi_1(z) = x_1$  and  $\pi_2(z) = x_2$ . By assumption  $a$  and  $\text{pr}_X$  induce surjection  $\mathbf{G} \times_k X \twoheadrightarrow X \times_Y X$ . Thus there exists a point  $u$  of  $\mathbf{G} \times_k X$  such that  $a(u) = x_1$  and  $\text{pr}_X(u) = x_2$ . Thus  $x_1$  and  $x_2$  are identified by an equivalence relation on the underlying set of  $X$  which is determined by the pair  $(a, \text{pr}_X)$ . Therefore, fibers of  $q$  are equivalence classes with respect to this relation. Since  $q$  is submersive, this implies that the diagram

$$\mathbf{G} \times_k X \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{\text{pr}_X} \end{array} X \xrightarrow{q} Y$$

is a cokernel in the category topological spaces. □

Now we prove a series results concerning fpqc descent. For this we fix a  $k$ -scheme  $Y$  with the trivial action of  $\mathbf{G}$  and a  $\mathbf{G}$ -equivariant morphism  $q : X \rightarrow Y$ . Let  $g : Y' \rightarrow Y$  be a morphism of  $k$ -schemes and consider the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ q' \downarrow & & \downarrow q \\ Y' & \xrightarrow{g} & Y \end{array}$$

of  $k$ -schemes. Note that  $X'$  admits a unique action  $a'$  of  $\mathbf{G}$  such that the square above consists of  $\mathbf{G}$ -equivariant morphism (we consider  $g$  as a  $\mathbf{G}$ -equivariant morphism between trivial  $\mathbf{G}$ -schemes).

**Fact 2.2.** *Suppose that  $g$  is faithfully flat and quasi-compact. Assume that  $q'$  is (universally) submersive. Then  $q$  is (universally) submersive.*

*Proof.* It suffices to prove that submersive morphisms have descent property. This follows from the fact that  $g$  (as faithfully flat and quasi-compact morphism) and  $q'$  are submersive. Details are left for the reader.  $\square$

**Fact 2.3.** *Suppose that  $g$  is faithfully flat and quasi-compact. Then the canonical morphism  $X' \times_{Y'} X' \rightarrow X \times_Y X$  is faithfully flat and quasi-compact and there is the cartesian square*

$$\begin{array}{ccc} \mathbf{G} \times_k X' & \longrightarrow & \mathbf{G} \times_k X \\ \downarrow & & \downarrow \\ X' \times_{Y'} X' & \longrightarrow & X \times_Y X \end{array}$$

in which the left vertical arrow is induced by  $\langle a', \text{pr}_{X'} \rangle : \mathbf{G} \times_k X' \rightarrow X' \times_k X'$ , the right vertical arrow is induced by  $\langle a, \text{pr}_X \rangle : \mathbf{G} \times_k X \rightarrow X \times_k X$  and the bottom horizontal morphism is the canonical morphism.

*Proof.* Note that squares

$$\begin{array}{ccc} X' \times_{Y'} X' & \longrightarrow & X' \times_Y X' \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{g} & Y \end{array} \quad \begin{array}{ccc} X' \times_Y X' & \longrightarrow & X \times_Y X \\ \downarrow & & \downarrow \\ X' \times_k X' & \xrightarrow{g' \times_k g'} & X \times_k X \end{array}$$

are cartesian. Since both  $g$  and  $g' \times_k g'$  are faithfully flat and quasi-compact, we derive that both morphisms  $X' \times_{Y'} X' \rightarrow X' \times_Y X'$  and  $X' \times_Y X' \rightarrow X \times_Y X$  are faithfully flat and quasi-compact. Then their composition i.e. the canonical morphism  $X' \times_{Y'} X' \rightarrow X \times_Y X$  is faithfully flat and quasi-compact.  $\square$

Finally we need the following notion

**Definition 2.4.** Let  $Y$  be a  $k$ -scheme with the trivial action of  $\mathbf{G}$  and let  $q : X \rightarrow Y$  be a  $\mathbf{G}$ -equivariant morphism. Consider a pair

$$q_* \mathcal{O}_X \xrightleftharpoons[q_* \text{pr}_X^\#]{q_* a^\#} q_* (\text{pr}_X)_* \mathcal{O}_{\mathbf{G} \times_k X} = q_* a_* \mathcal{O}_{\mathbf{G} \times_k X}$$

of morphisms of sheaves of rings on  $Y$ . Suppose that  $q^\# : \mathcal{O}_Y \rightarrow q_* \mathcal{O}_X$  is a kernel of this pair. Then  $\mathcal{O}_Y$  is the sheaf of  $\mathbf{G}$ -invariants for  $q$ .

**Proposition 2.5.** *Suppose that  $g$  is faithfully flat and quasi-compact. Assume that  $q'$  is quasi-compact, semiseparated and  $\mathcal{O}_{Y'}$  is the sheaf of  $\mathbf{G}$ -invariants for  $q'$ . Then  $\mathcal{O}_Y$  is the sheaf of  $\mathbf{G}$ -invariants for  $q$ .*

*Proof.* We denote by  $a'$  the action of  $\mathbf{G}$  on  $X'$ . First note that  $q$  is semiseparated and quasi-compact morphism as these classes of morphisms admit descent along quasi-compact and faithfully flat

morphisms. Since  $q$  is quasi-compact, semiseparated and  $g$  is flat, we derive that for every quasi-coherent sheaf  $\mathcal{F}$  on  $X$  the canonical morphism  $q'_* g'^* \mathcal{F} \rightarrow g^* q_* \mathcal{F}$  is an isomorphism. Thus the diagram

$$\mathcal{O}_{Y'} \xrightarrow{q^\#} q'_* \mathcal{O}_{X'} \xrightarrow[q'_* \text{pr}_{X'}^\#]{q'_* a'^\#} q'_* (\text{pr}_{X'})_* \mathcal{O}_{\mathbf{G} \times_k X'} = q'_* a'_* \mathcal{O}_{\mathbf{G} \times_k X'}$$

is isomorphic to the diagram

$$g^* \mathcal{O}_Y \xrightarrow{g^* q^\#} g^* (q_* \mathcal{O}_X) \xrightarrow[g^* q_* \text{pr}_X^\#]{g^* q_* a^\#} g^* (q_* (\text{pr}_X)_* \mathcal{O}_{\mathbf{G} \times_k X}) = g^* (q_* a_* \mathcal{O}_{\mathbf{G} \times_k X})$$

Since  $\mathcal{O}_{Y'}$  is the sheaf of  $\mathbf{G}$ -invariants for  $q'$ , the first diagram is a kernel diagram. Hence the second is a kernel diagram. According to the fact that  $g$  is faithfully flat we deduce that the diagram

$$\mathcal{O}_Y \xrightarrow{q^\#} q_* \mathcal{O}_X \xrightarrow[q_* \text{pr}_X^\#]{q_* a^\#} q_* (\text{pr}_X)_* \mathcal{O}_{\mathbf{G} \times_k X} = q_* a_* \mathcal{O}_{\mathbf{G} \times_k X}$$

is also a kernel diagram. Thus  $\mathcal{O}_Y$  is the sheaf of  $\mathbf{G}$ -invariants for  $q$ . □

### 3. CATEGORICAL AND GEOMETRIC QUOTIENTS

**Definition 3.1.** Let  $Y$  be a  $k$ -scheme with the trivial action of  $\mathbf{G}$  and let  $q : X \rightarrow Y$  be a  $\mathbf{G}$ -equivariant morphism. Suppose that the following conditions hold.

- (1)  $q$  is submersive.
- (2) The morphism  $\mathbf{G} \times_k X \rightarrow X \times_Y X$  induced by  $\langle a, \text{pr}_X \rangle : \mathbf{G} \times_k X \rightarrow X \times_k X$  is surjective.
- (3)  $\mathcal{O}_Y$  is the sheaf of  $\mathbf{G}$ -invariant for  $q$ .

Then  $q$  is a *geometric quotient* of  $X$ .

**Corollary 3.2.** Let  $q$  be a geometric quotient of  $X$ . Then the diagram

$$\mathbf{G} \times_k X \xrightarrow[\text{pr}_X]{a} X \xrightarrow{q} Y$$

is a cokernel in the category of ringed spaces.

*Proof.* Due to the fact that  $\mathcal{O}_Y$  is the sheaf of  $\mathbf{G}$ -invariants for  $q$  it suffices to prove that

$$\mathbf{G} \times_k X \xrightarrow[\text{pr}_X]{a} X \xrightarrow{q} Y$$

is the cokernel in the category of topological spaces. This follows from Proposition 2.1. □

**Definition 3.3.** Let  $q : X \rightarrow Y$  be a morphism of  $k$ -schemes such that the diagram

$$\mathbf{G} \times_k X \xrightarrow[\text{pr}_X]{a} X \xrightarrow{q} Y$$

is a cokernel in the category of  $k$ -schemes. Then  $q : X \rightarrow Y$  is a *categorical quotient* of  $X$ .

**Fact 3.4.** *Every geometric quotient is categorical.*

*Proof.* Categorical quotient is a cokernel in the category of  $k$ -schemes. On the other hand geometric quotient is a cokernel in the category of locally ringed spaces and hence it also satisfies cokernel property in its full subcategory of  $k$ -schemes. Thus every geometric quotient is categorical.  $\square$

Let  $q : X \rightarrow Y$  be a morphism of  $k$ -schemes such that  $q \cdot \text{pr}_X = q \cdot a$ . For a morphism  $g : Y' \rightarrow Y$  of  $k$ -schemes consider the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ q' \downarrow & & \downarrow q \\ Y' & \xrightarrow{g} & Y \end{array}$$

Then there exists a unique action  $a' : \mathbf{G} \times_k X' \rightarrow X'$  of  $\mathbf{G}$  on  $X'$  such that the square above consists of  $\mathbf{G}$ -equivariant morphism (we consider  $Y, Y'$  as  $\mathbf{G}$ -schemes equipped with trivial  $\mathbf{G}$ -actions). Keeping this in mind we have the following.

**Definition 3.5.** A morphism  $q : X \rightarrow Y$  is a *uniform categorical (geometric) quotient* of  $X$  if for every flat morphism  $g : Y' \rightarrow Y$  its base change  $q' : X' \rightarrow Y'$  is a categorical (geometric) quotient of  $X'$ .

**Definition 3.6.** A morphism  $q : X \rightarrow Y$  is a *universal categorical (geometric) quotient* of  $X$  if for every morphism  $g : Y' \rightarrow Y$  its base change  $q' : X' \rightarrow Y'$  is a categorical (geometric) quotient of  $X'$ .

**Corollary 3.7.** *Let  $g : Y' \rightarrow Y$  be a faithfully flat and quasi-compact morphism. Suppose that  $q'$  is a geometric quotient, then  $q$  is a geometric quotient.*

*Proof.* This follows from Facts 2.2, 2.3 and Proposition 2.5.  $\square$

In the next result we give a simple example of a universal geometric quotient.

**Proposition 3.8.** *Suppose that  $\mathbf{G}$  is a quasi-compact group scheme over  $k$ . Let  $Y$  be a  $k$ -scheme and consider  $\mathbf{G} \times_k Y$  with the action of  $\mathbf{G}$  induced by the regular action on the left factor. Then  $\text{pr}_Y : \mathbf{G} \times_k Y \rightarrow Y$  is a universal geometric quotient.*

*Proof.* Clearly  $\text{pr}_Y$  is universally submersive (it is even universally open). Let  $\mu : \mathbf{G} \times_k \mathbf{G} \rightarrow \mathbf{G}$  be the multiplication morphism and let  $\pi_{23} : \mathbf{G} \times_k \mathbf{G} \times Y \rightarrow \mathbf{G} \times_k Y$  be the projection on the last two factors. Then the morphism

$$\mathbf{G} \times_k \mathbf{G} \times_k Y \rightarrow (\mathbf{G} \times_k Y) \times_Y (\mathbf{G} \times_k Y) = \mathbf{G} \times_k \mathbf{G} \times_k Y$$

induced by  $\langle \mu \times_k 1_Y, \pi_{23} \rangle : \mathbf{G} \times_k \mathbf{G} \times_k Y \rightarrow (\mathbf{G} \times_k Y) \times_k (\mathbf{G} \times_k Y)$  is an isomorphism. We show that  $\mathcal{O}_Y$  is the sheaf of  $\mathbf{G}$ -invariants for  $\text{pr}_Y$ . For this pick an affine open subset  $V$  of  $Y$ . It suffices to check that the diagram

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{\text{pr}_Y^\#} \Gamma(\mathbf{G} \times_k V, \mathcal{O}_{\mathbf{G} \times_k Y}) \xrightleftharpoons[\pi_{23}^\#]{(\mu \times_k 1_Y)^\#} \Gamma(\mathbf{G} \times_k \mathbf{G} \times_k V, \mathcal{O}_{\mathbf{G} \times_k \mathbf{G} \times_k Y})$$

is a kernel. Since  $\mathbf{G}$  is quasi-compact and separated (every group  $k$ -scheme is separated), we derive that the diagram above is isomorphic with

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{f \mapsto 1 \otimes f} \Gamma(\mathbf{G}, \mathcal{O}_{\mathbf{G}}) \otimes_k \Gamma(V, \mathcal{O}_Y) \xrightleftharpoons[\chi \otimes f \mapsto 1 \otimes \chi \otimes f]{\chi \otimes f \mapsto \mu^\#(\chi) \otimes f} \Gamma(\mathbf{G}, \mathcal{O}_{\mathbf{G}}) \otimes_k \Gamma(\mathbf{G}, \mathcal{O}_{\mathbf{G}}) \otimes_k \Gamma(V, \mathcal{O}_Y)$$

Thus the first diagram is the kernel diagram if  $f \mapsto 1 \otimes f$  induces an isomorphism of  $\Gamma(V, \mathcal{O}_Y)$  with subspace of  $\Gamma(\mathbf{G}, \mathcal{O}_{\mathbf{G}}) \otimes_k \Gamma(V, \mathcal{O}_Y)$  given by formula

$$\{\chi \in \Gamma(\mathbf{G}, \mathcal{O}_{\mathbf{G}}) \mid \mu^\#(\chi) = 1 \otimes \chi\} \otimes_k \Gamma(V, \mathcal{O}_Y)$$

Hence it suffices to prove that

$$\{\chi \in \Gamma(\mathbf{G}, \mathcal{O}_{\mathbf{G}}) \mid \mu^\#(\chi) = 1 \otimes \chi\} = \text{constant functions on } \mathbf{G}$$

For this pick a  $k$ -algebra  $A$  and let  $g : \text{Spec } A \rightarrow \mathbf{G}$  be an  $A$ -point. Next let  $e : \text{Spec } A \rightarrow \mathbf{G}$  be an  $A$ -point of  $\mathbf{G}$  which corresponds to the identity element of  $\mathbf{G}$ . Suppose that a regular function  $\chi$  in  $\mathbf{G}$  satisfies  $\mu^\#(\chi) = 1 \otimes \chi$ . Then

$$g^\#(\chi) = \langle g, e \rangle^\# \mu^\#(\chi) = \langle g, e \rangle^\#(1 \otimes \chi) = e^\#(\chi)$$

Recall that  $e$  is given by the composition of the structural morphism  $\text{Spec } A \rightarrow \text{Spec } k$  and the  $k$ -point  $\text{Spec } k \rightarrow \mathbf{G}$  determined by the identity of  $\mathbf{G}$ . Thus  $g^\#(\chi)$  is an element of  $k$ . Since this follows for every  $g : \text{Spec } A \rightarrow \mathbf{G}$ , we derive that  $\chi$  is a constant function. This completes the proof of our claim that

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{\text{pr}_Y^\#} \Gamma(\mathbf{G} \times_k V, \mathcal{O}_{\mathbf{G} \times_k Y}) \xrightleftharpoons[\pi_{23}^\#]{(\mu \times_k 1_Y)^\#} \Gamma(\mathbf{G} \times_k \mathbf{G} \times_k V, \mathcal{O}_{\mathbf{G} \times_k \mathbf{G} \times_k Y})$$

is the kernel diagram and hence  $\mathcal{O}_Y$  is the sheaf of  $\mathbf{G}$ -invariants for  $\text{pr}_Y$ . Therefore, we proved that  $\text{pr}_Y$  is a geometric quotient of  $\mathbf{G} \times_k Y$ . Consider any morphism  $Y' \rightarrow Y$ . Then base change of  $\text{pr}_Y$  along this morphism is  $\text{pr}_{Y'}$ . We conclude that  $\text{pr}_Y$  is a universal geometric quotient for  $\mathbf{G} \times_k Y$ .  $\square$

#### 4. GEOMETRIC QUOTIENTS OF SEPARATED ACTIONS

**Definition 4.1.** The action of  $\mathbf{G}$  on  $X$  is *separated* if the morphism  $\langle a, \text{pr}_X \rangle : \mathbf{G} \times_k X \rightarrow X \times_k X$  has closed set-theoretic image.

**Theorem 4.2.** Let  $q : X \rightarrow Y$  be a geometric quotient of  $X$ . Assume that  $q$  is universally submersive. Then the following assertions are equivalent.

- (i) The action of  $\mathbf{G}$  on  $X$  is separated.
- (ii)  $Y$  is separated.

*Proof.* We have a cartesian square

$$\begin{array}{ccc} X \times_Y X & \hookrightarrow & X \times_k X \\ \downarrow & & \downarrow q \times_k q \\ Y & \xrightarrow{\Delta_Y} & Y \times_k Y \end{array}$$

It follows that  $X \times_Y X \hookrightarrow X \times_k X$  is a locally closed immersion. Since  $q$  is a geometric quotient, we derive that  $\langle a, \text{pr}_X \rangle$  factors as a surjective morphism  $\mathbf{G} \times_k X \twoheadrightarrow X \times_Y X$  followed by the immersion  $X \times_Y X \hookrightarrow X \times_k X$ . Thus the action of  $\mathbf{G}$  on  $X$  is separated if and only if  $X \times_Y X$  is a closed subscheme of  $X \times_k X$ . Since  $q$  is universally submersive, we derive that  $q \times_k q$  is submersive. As

the square above is cartesian we derive that  $\Delta_Y(Y) \subseteq Y \times_k Y$  is closed if and only if  $X \times_Y X \subseteq X \times_k X$  is closed. Therefore,  $Y$  is separated if and only if the action of  $\mathbf{G}$  on  $X$  is separated.  $\square$

## 5. GEOMETRIC QUOTIENTS OF FREE ACTIONS AND PRINCIPAL BUNDLES

**Definition 5.1.** The action of  $\mathbf{G}$  on  $X$  is *free* if the morphism  $\langle a, \text{pr}_X \rangle : \mathbf{G} \times_k X \rightarrow X \times_k X$  is a closed immersion.

**Definition 5.2.** Let  $x$  be a  $k$ -point of  $X$ . Suppose that the orbit morphism  $\mathbf{G} \rightarrow X$  of  $x$  given by the composition

$$\mathbf{G} = \mathbf{G} \times_k \text{Spec } k \xrightarrow{\text{induced by } x} \mathbf{G} \times_k X \longrightarrow X$$

is a closed immersion. Then the action of  $\mathbf{G}$  on  $X$  has a *closed free orbit* at  $x$ .

**Fact 5.3.** If the action of  $\mathbf{G}$  on  $X$  is free, then every  $k$ -point of  $X$  has a closed free orbit.

The following result states that over special type of local complete noetherian  $k$ -algebras geometric quotients of free actions correspond to trivial  $\mathbf{G}$ -bundles.

**Theorem 5.4.** Suppose that  $k$  is an algebraically closed field and  $\mathbf{G}$  is a smooth algebraic group over  $k$ . Let  $q : X \rightarrow Y$  be a geometric quotient locally of finite type and let  $Y$  be the spectrum of a complete local noetherian  $k$ -algebra such that the residue field of the closed point of  $Y$  is  $k$ . Then the following assertions hold.

- (1) If  $x$  is a  $k$ -point of  $X$  which has a closed free orbit, then there exists a  $\mathbf{G}$ -equivariant, étale and surjective morphism  $f : \mathbf{G} \times_k Y \rightarrow X$  such that the triangle

$$\begin{array}{ccc} \mathbf{G} \times_k Y & \xrightarrow{f} & X \\ \text{pr}_Y \searrow & & \swarrow q \\ & Y & \end{array}$$

is commutative and the morphism

$$Y = \text{Spec } k \times_k Y \xrightarrow{e \times_k 1_Y} \mathbf{G} \times_k Y \xrightarrow{f} X$$

is a section of  $q$ .

- (2) If the action of  $\mathbf{G}$  on  $X$  is free, then  $f$  is an isomorphism.

The proof relies on two algebraic lemmas. The first describe free actions in context of complete rings (it is [Mumford et al., 1994, lemma on page 18]) and the second is a version of Hensel's lemma.

**Lemma 5.4.1.** Let  $(R, \mathfrak{m}, k)$  be a complete local noetherian  $k$ -algebra and let  $\sigma : R \rightarrow R[[x_1, \dots, x_n]]$  be a local morphism into a ring of formal power series over  $R$ . Assume that the composition

$$R \xrightarrow{\sigma} R[[x_1, \dots, x_n]] \xrightarrow{f \mapsto f \bmod (x_1, \dots, x_n)} R$$

is the identity and the composition

$$R \xrightarrow{\sigma} R[[x_1, \dots, x_n]] \xrightarrow{f \mapsto f \bmod \mathfrak{m}} (R/\mathfrak{m})[[x_1, \dots, x_n]] = k[[x_1, \dots, x_n]]$$

is surjective. Consider elements  $y_1, \dots, y_n$  of  $R$  such that  $\sigma(y_i) \bmod \mathfrak{m} = x_i$  for  $i = 1, \dots, n$ . Then the composition

$$R \xrightarrow{\sigma} R[[x_1, \dots, x_n]] \xrightarrow{f \mapsto f \bmod (y_1, \dots, y_n)} (R/(y_1, \dots, y_n))[[x_1, \dots, x_n]]$$

is an isomorphism.

*Proof of the lemma.* For convenience let  $\phi$  denote the morphism given by the rule  $r \mapsto \sigma(r) \bmod (y_1, \dots, y_n)$ . Also denote  $R/(y_1, \dots, y_n)$  by  $S$ . According to assumptions we have

$$\sigma(y_i) = x_i + y_i + \sum_{j=1}^n x_j \cdot \mathfrak{m}[[x_1, \dots, x_n]]$$

for each  $i$ . Thus  $\phi(y_i) = \sum_{j=1}^n f_{ij} \cdot x_j$  where  $f_{ij} \in S$  are elements such that the matrix  $[f_{ij}]_{1 \leq i, j \leq n}$  is invertible in  $S$ . Hence

$$S[[x_1, \dots, x_n]] = S[[\phi(y_1), \dots, \phi(y_n)]]$$

and  $\phi$  composed with  $S[[\phi(y_1), \dots, \phi(y_n)]] \rightarrow S$  is the quotient morphism  $R \rightarrow S$ . From this observations we derive that  $\phi$  is surjective. It remains to prove that it is injective. Consider  $z$  in  $R$  such that  $\phi(z) = 0$ . Suppose that  $z \in (y_1, \dots, y_n)^m$  for some  $m \in \mathbb{N}$ . Write

$$z = \sum_{\alpha \in \Lambda} c_\alpha \cdot y_1^{\alpha_1} \dots y_n^{\alpha_n}$$

for some  $c_\alpha \in R$  where  $\Lambda = \{\alpha \in \mathbb{N}^n \mid \alpha_1 + \dots + \alpha_n = m\}$ . Hence

$$0 = \phi(z) = \sum_{\alpha \in \Lambda} \phi(c_\alpha) \cdot \phi(y_1)^{\alpha_1} \dots \phi(y_n)^{\alpha_n}$$

Thus  $\phi(c_\alpha) \in (\phi(y_1), \dots, \phi(y_n))$  for every  $\alpha \in \Lambda$ . Since  $\phi$  composed with  $S[[\phi(y_1), \dots, \phi(y_n)]] \rightarrow S$  is the quotient morphism  $R \rightarrow S$ , we derive that

$$c_\alpha \bmod (y_1, \dots, y_n) = \phi(c_\alpha) \bmod (\phi(y_1), \dots, \phi(y_n)) = 0$$

for every  $\alpha \in \Lambda$ . Thus  $c_\alpha \in (y_1, \dots, y_n)$  for every  $\alpha \in \Lambda$ , which implies that  $z \in (y_1, \dots, y_n)^{m+1}$ . Thus we proved that

$$\phi(z) = 0 \text{ and } z \in (y_1, \dots, y_n)^m \Rightarrow z \in (y_1, \dots, y_n)^{m+1}$$

By  $\mathfrak{m}$ -adic completeness of  $R$  this implies that  $\phi(z) = 0$  if and only if  $z = 0$ . Hence  $\phi$  is also injective.  $\square$

**Lemma 5.4.2.** *Let  $(R, \mathfrak{m})$  be a complete local noetherian ring and let  $R \rightarrow S$  be a local morphism into a local noetherian ring. Suppose that there exists a finitely generated  $R$ -submodule  $N$  of  $S$  such that*

$$S = N + \mathfrak{m}S$$

*Then  $S = N$ .*

*Proof of the lemma.* Pick  $s$  in  $S$ . Since  $S = N + \mathfrak{m}S$ , there exists a sequence  $\{x_n\}_{n \in \mathbb{N}}$  such that  $x_n \in \mathfrak{m}^n N$  and

$$s - \sum_{i \leq n} x_i \in \mathfrak{m}^{n+1} S$$

According to the assumption that  $(R, \mathfrak{m})$  is complete with respect to  $\mathfrak{m}$ -adic topology and  $N$  is finitely generated over  $R$ , we deduce that  $N$  is complete with respect to  $\mathfrak{m}$ -adic topology. Hence there exists a unique element  $x$  in  $N$  such that

$$x = \sum_{n \in \mathbb{N}} x_n$$

where above series is convergent with respect to  $\mathfrak{m}$ -adic topology. Note also that

$$x - \sum_{i \leq n} x_i \in \mathfrak{m}^{n+1} N$$

for every  $n \in \mathbb{N}$ . Thus we have

$$s - x = \left( s - \sum_{i \leq n} x_i \right) - \left( x - \sum_{i \leq n} x_i \right) \in \mathfrak{m}^{n+1}S + \mathfrak{m}^{n+1}N = \mathfrak{m}^{n+1}S$$

for every  $n \in \mathbb{N}$ . Hence

$$x - s \in \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n S$$

Since  $R \rightarrow S$  is local morphism and  $S$  is a local ring, we deduce that  $\mathfrak{m}S$  is contained in the maximal ideal of  $S$ . By assumptions  $S$  is noetherian. Therefore,  $S$  is separated with respect to  $\mathfrak{m}$ -adic topology. This implies that

$$0 = \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n S$$

Thus  $s - x = 0$  and we infer that  $s$  is an element of  $N$ . This completes the proof that  $S = N$ .  $\square$

In what follows we shall denote by  $\mathbf{G}x$  the closed subscheme determined by the orbit morphism  $\mathbf{G} \rightarrow X$  of a  $k$ -point  $x$  of  $X$  which has a closed free orbit. For readers convenience we include the following lemmas, which have topological content.

**Lemma 5.4.3.** *Let  $q : X \rightarrow Y$  be a geometric quotient and assume that  $Y$  is the spectrum of a local  $k$ -algebra such that the residue field of the closed point  $o$  of  $Y$  is  $k$ . Let  $x$  be a  $k$ -point of  $X$  with free closed orbit, then  $q^{-1}(o) = \mathbf{G}x$  as closed subschemes of  $X$ .*

*Proof of the lemma.* Morphism  $q$  induces the morphism of residue fields  $k(q(x)) \hookrightarrow k(x) = k$  over  $k$ . This implies that  $k(q(x)) = k$  and hence  $q(x)$  is a  $k$ -point of  $Y$ . Note that  $o$  is the unique  $k$ -point of  $Y$ . Thus  $q(x) = o$ . Clearly  $q^{-1}(o)$  is a closed  $\mathbf{G}$ -stable subscheme of  $X$  (it is the preimage of  $o$  under  $\mathbf{G}$ -equivariant  $q$ ), that contains  $x$ . Since  $\mathbf{G}x$  is the smallest closed  $\mathbf{G}$ -stable subscheme of  $X$  containing  $x$ , we deduce that  $\mathbf{G}x \subseteq q^{-1}(o)$  scheme-theoretically. Consider the pair of arrows

$$\mathbf{G} \times_k X \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{\text{pr}_X} \end{array} X$$

Passing to functors of points we obtain that  $a^{-1}(\mathbf{G}x) = \text{pr}_X^{-1}(\mathbf{G}x)$ . Since  $q$  is the cokernel of the pair  $(a, \text{pr}_X)$  in the category of topological spaces, we deduce that there exists a closed subset  $Z$  of  $Y$  such that  $q^{-1}(Z) = \mathbf{G}x$ . Clearly  $o \in Z$  and hence  $q^{-1}(o) \subseteq \mathbf{G}x$  set-theoretically. On the other hand above we proved that  $\mathbf{G}x \subseteq q^{-1}(o)$  scheme-theoretically. This can only happen if  $q^{-1}(o) = \mathbf{G}x$  as closed subschemes of  $X$ .  $\square$

**Lemma 5.4.4.** *Let  $q : X \rightarrow Y$  be a geometric quotient and assume that  $Y$  is the spectrum of a local  $k$ -algebra such that the residue field of the closed point  $o$  of  $Y$  is  $k$ . Let  $U$  be an open  $\mathbf{G}$ -stable subset of  $X$  which contain a  $k$ -point. Then  $U = X$ .*

*Proof of the lemma.* Consider the pair of arrows

$$\mathbf{G} \times_k X \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{\text{pr}_X} \end{array} X$$

Since  $U$  is  $\mathbf{G}$ -stable open subset of  $X$ , we derive that  $\text{pr}_X^{-1}(U) = a^{-1}(U)$ . Next by definition  $q$  is the cokernel of the above pair in the category of topological spaces. Hence there exists an open subset  $V$  of  $Y$  such that  $U = q^{-1}(V)$ . Since  $U$  contains a  $k$ -point of  $X$ , we deduce as in Lemma 5.4.3 that  $o \in V$ . Thus  $V = Y$  and finally  $U = q^{-1}(V) = X$ .  $\square$



*Proof of the theorem.* We first prove (1). Denote by  $o$  the closed point of  $Y$ . Assume that  $x$  is a  $k$ -point of  $X$  which has a closed free orbit. Consider the surjective morphism  $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{G,e}$  induced by the orbit morphism  $\mathbf{G} \hookrightarrow X$  of  $x$ . Since  $\mathbf{G}$  is smooth over  $k$ , the ring  $\mathcal{O}_{G,e}$  is regular. Pick a system of parameters  $x_1, \dots, x_n$  of  $\mathcal{O}_{G,e}$  and let  $y_1, \dots, y_n$  be elements of  $\mathcal{O}_{X,x}$  such that  $y_i$  is sent to  $x_i$  by the morphism  $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{G,e}$  for  $1 \leq i \leq n$ . Define  $S$  to be the quotient ring  $\mathcal{O}_{X,x}/(y_1, \dots, y_n)$ . The morphism  $q$  induces the morphism  $q^\# : \mathcal{O}_{Y,o} \rightarrow \mathcal{O}_{X,x}$  and hence the morphism  $\mathcal{O}_{Y,o} \rightarrow S$ . By Lemma 5.4.3 we have

$$S/\mathfrak{m}_o S = k$$

where  $\mathfrak{m}_o$  is the maximal ideal of  $\mathcal{O}_{Y,o}$ . According to Lemma 5.4.2 we derive that  $\mathcal{O}_{Y,o} \rightarrow S$  is surjective. Let  $f : \mathbf{G} \times_k \text{Spec } S \rightarrow X$  be the unique  $\mathbf{G}$ -equivariant morphism induced by the surjection  $\mathcal{O}_{X,x} \twoheadrightarrow S$ . We have a commutative square

$$\begin{array}{ccc} \mathbf{G} \times_k \text{Spec } S & \xrightarrow{f} & X \\ \text{pr}_{\text{Spec } S} \downarrow & & \downarrow q \\ \text{Spec } S & \xrightarrow{j} & Y \end{array}$$

where  $j$  is a closed immersion induced by  $\mathcal{O}_{Y,o} \twoheadrightarrow S$ . According to assumptions  $q$  is locally of finite type. Moreover,  $\mathbf{G}$  is an algebraic group over  $k$  and hence  $\text{pr}_{\text{Spec } S}$  is locally of finite type. These two assertions together with the fact that  $\text{Spec } S \hookrightarrow Y$  is a closed immersion of noetherian schemes (and thus is of finite type) imply that  $f$  is locally of finite type. Then by Lemma 5.4.1 we deduce that  $f$  induces an isomorphism

$$\widehat{\mathcal{O}_{X,x}} \longrightarrow \hat{S}[[x_1, \dots, x_n]] = \hat{S} \hat{\otimes}_k \widehat{\mathcal{O}_{G,e}}$$

of complete local rings. Since  $f$  is locally of finite type, it follows that  $f$  is étale at a  $k$ -point of  $\mathbf{G} \times_k \text{Spec } S$  determined by the unique  $k$ -point of  $\text{Spec } S$  and  $e \in \mathbf{G}$ . Let  $U$  be an étale locus of  $f$ . It contains a  $k$ -point and hence it is nonempty. Moreover,  $U$  is open (it is étale locus) subset of  $X$ . Since  $f$  is  $\mathbf{G}$ -equivariant, we derive that  $U$  is  $\mathbf{G}$ -stable. Similarly  $f(U)$  is open  $\mathbf{G}$ -stable subset of  $X$  and  $x \in f(U)$ . Thus by Lemma 5.4.4 we deduce that

$$U = \mathbf{G} \times_k \text{Spec } S, f(U) = X$$

Therefore,  $f$  is étale and surjective. Now we pullback  $q : X \rightarrow Y$  along the closed immersion  $\text{Spec } S \hookrightarrow Y$ . We obtain a cartesian square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{j}} & X \\ \tilde{q} \downarrow & & \downarrow q \\ \text{Spec } S & \xrightarrow{j} & Y \end{array}$$

Then  $f$  factors as a morphism  $\mathbf{G} \times_k \text{Spec } S \rightarrow \tilde{X}$  followed by a closed immersion  $\tilde{j}$ . Since  $f$  is étale and surjective, we deduce that  $\tilde{j}$  is étale and surjective. This implies that  $\tilde{j}$  is an isomorphism of schemes. By definition of geometric quotient we derive that the diagram

$$\mathcal{O}_Y \xrightarrow{q^\#} q_* \mathcal{O}_X \xrightarrow[q_* \text{pr}_X^\#]{q_* a^\#} q_* (\text{pr}_X)_* \mathcal{O}_{\mathbf{G} \times_k X} = q_* a_* \mathcal{O}_{\mathbf{G} \times_k X}$$

is the kernel in the category of sheaves on  $Y$ . Hence  $q^\# : \mathcal{O}_Y \rightarrow q_* \mathcal{O}_X$  is a monomorphism of sheaves. On the other hand we have

$$q^\# = j_* q_* (\tilde{j}^{-1})^\# \cdot j_* \tilde{q}^\# \cdot j^\#$$

and thus  $j^\#$  is a monomorphism. Since  $j$  is a closed immersion, we infer that  $j$  is an isomorphism. Therefore, we can identify  $\text{Spec } S$  with  $Y$ . Then  $f$  is a morphism which makes the triangle

$$\begin{array}{ccc} \mathbf{G} \times_k Y & \xrightarrow{f} & X \\ \text{pr}_Y \searrow & & \swarrow q \\ & Y & \end{array}$$

commutative. This completes the proof of (1).

For the proof of (2) consider the section  $s : Y \hookrightarrow X$  described in (1). Then  $f$  fits into a cartesian square

$$\begin{array}{ccc} \mathbf{G} \times_k Y & \xrightarrow{f} & X \times_Y Y = X \\ 1_{\mathbf{G}} \times_Y s \downarrow & & \downarrow 1_X \times_Y s \\ \mathbf{G} \times_k X & \xrightarrow{\phi} & X \times_Y X \end{array}$$

where  $\phi$  is a closed immersion induced by the closed immersion  $\langle a, \text{pr}_X \rangle : \mathbf{G} \times_k X \hookrightarrow X \times_k X$  (the action of  $\mathbf{G}$  on  $X$  is free). Thus  $f$  is a closed immersion. By (1) it is étale and surjective. Therefore,  $f$  is an isomorphism.  $\square$

**Remark 5.5.** We expect that Theorem 5.4 holds for prime spectra of strictly henselian rings.

Now we introduce sufficient condition for smoothness of geometric quotient in case of locally algebraic  $k$ -schemes.

**Corollary 5.6.** *Suppose that  $\mathbf{G}$  is a smooth algebraic group over  $k$ . Let  $q : X \rightarrow Y$  be a morphism of finite type between  $k$ -schemes locally of finite type. Assume that  $q$  is a uniform geometric quotient of  $X$  and  $x$  is a  $k$ -point of  $X$  with closed free orbit. Then  $q$  is smooth at  $x$ .*

*Proof.* Since smoothness descent along faithfully flat morphisms, we may assume that  $k$  is algebraically closed. Let  $y = q(x)$ . Then  $y$  is a  $k$ -point of  $Y$ . Now  $1_{\text{Spec } \widehat{\mathcal{O}_{Y,y}}} \times_k q$  is a geometric quotient and  $\widehat{\mathcal{O}_{Y,y}}$  is a complete local noetherian  $k$ -algebra with  $k$  as a residue field. Moreover,  $x$  is a  $k$ -point of  $\text{Spec } \widehat{\mathcal{O}_{Y,y}} \times_k X$  with a closed free orbit. By Theorem 5.4 we deduce that  $1_{\text{Spec } \widehat{\mathcal{O}_{Y,y}}} \times_k q$  is smooth. From descent for smoothness we infer that  $q$  is smooth at  $x$ .  $\square$

**Definition 5.7.** Let  $q : X \rightarrow Y$  be a  $\mathbf{G}$ -equivariant morphism into a  $k$ -scheme  $Y$  equipped with the trivial  $\mathbf{G}$ -action. Suppose that  $q$  is faithfully flat, quasi-compact morphism and the square

$$\begin{array}{ccc} \mathbf{G} \times_k X & \xrightarrow{a} & X \\ \text{pr}_X \downarrow & & \downarrow q \\ X & \xrightarrow{q} & Y \end{array}$$

is cartesian. Then  $q$  is a principal  $\mathbf{G}$ -bundle.

Now we use Theorem 5.4 to describe principal  $\mathbf{G}$ -bundles in the category of locally algebraic  $k$ -schemes.

**Theorem 5.8.** *Suppose that  $\mathbf{G}$  is a smooth algebraic group over  $k$ . Let  $q : X \rightarrow Y$  be a morphism of finite type between  $k$ -schemes locally of finite type. Then the following assertions are equivalent.*

- (i)  $q$  is a universal geometric quotient and the action of  $\mathbf{G}$  on  $X$  is free.
- (ii)  $q$  is a uniform geometric quotient and the action of  $\mathbf{G}$  on  $X$  is free.
- (iii)  $q$  is a principal  $\mathbf{G}$ -bundle.

*Proof.* Clearly (i)  $\Rightarrow$  (ii). Suppose that (ii) holds. Let  $\bar{k}$  be an algebraic closure of  $k$ . Then  $1_{\text{Spec } \bar{k}} \times_k q$  is a uniform quotient and the action of  $\text{Spec } \bar{k} \times_k \mathbf{G}$  on  $\text{Spec } \bar{k} \times_k X$  induced by the action of  $\mathbf{G}$  on  $X$  is free. Moreover, if  $1_{\text{Spec } \bar{k}} \times_k q$  is a principal  $\text{Spec } \bar{k} \times_k \mathbf{G}$ -bundle, then  $q$  is a  $\mathbf{G}$ -bundle. This follows from the observation that property of being a principal bundle descends along faithfully flat and quasi-compact base extensions. Thus we may assume that  $k$  is algebraically closed. Next we pick a  $k$ -point  $y$  of  $Y$  and consider base change  $1_{\text{Spec } \widehat{\mathcal{O}_{Y,y}}} \times_Y q$ . This is a geometric quotient (because morphism  $\text{Spec } \widehat{\mathcal{O}_{Y,y}} \rightarrow Y$  is flat) and a morphism of finite type. Moreover, the action of  $\mathbf{G}$  on  $\text{Spec } \widehat{\mathcal{O}_{Y,y}} \times_Y X$  is free. Since  $\widehat{\mathcal{O}_{Y,y}}$  is a complete noetherian  $k$ -algebra with residue field  $k$ , we derive by Theorem 5.4 that  $\text{Spec } \widehat{\mathcal{O}_{Y,y}} \times_Y q$  is isomorphic as a  $\mathbf{G}$ -equivariant morphism with  $\text{pr}_{\text{Spec } \widehat{\mathcal{O}_{Y,y}}}$ . By faithfully flat descent for flat morphism we deduce that  $q$  is flat at every point in the fiber  $q^{-1}(\text{Spec } \mathcal{O}_{Y,y})$ . Since  $y$  is an arbitrary  $k$ -point, it follows that  $q$  is flat at every point of  $X$  which specializes to a  $k$ -point. Every point of  $X$  is a generization of a  $k$ -point ( $X$  is locally of finite type and  $k$  is algebraically closed). Thus  $q$  is flat. It is also surjective (as it is a geometric quotient) and quasi-compact (it is of finite type). Therefore, it is faithfully flat and quasi-compact morphism. In order to obtain (iii) it remains to prove that the morphism  $\Phi : \mathbf{G} \times_k X \rightarrow X \times_Y X$  induced by  $a$  and  $\text{pr}_X$  is an isomorphism. Note that it is a closed immersion (the action of  $\mathbf{G}$  on  $X$  is closed). Moreover,  $1_{\text{Spec } \widehat{\mathcal{O}_{Y,y}}} \times_Y \Phi$  is an isomorphism due to the fact that  $1_{\text{Spec } \widehat{\mathcal{O}_{Y,y}}} \times_Y q$  is isomorphic as a  $\mathbf{G}$ -equivariant morphism with  $\text{pr}_{\text{Spec } \widehat{\mathcal{O}_{Y,y}}}$ . By faithfully flat descent we infer that  $1_{\text{Spec } \mathcal{O}_{Y,y}} \times_Y \Phi$  is an isomorphism. This holds for every  $k$ -point  $y$  in  $Y$ . Thus  $\Phi$  induces an isomorphism  $\mathcal{O}_{X \times_Y X, \Phi(z)} \rightarrow \mathcal{O}_{\mathbf{G} \times_k X, z}$  for every  $k$ -point  $z$  of  $X \times_Y X$ . Hence a closed immersion  $\Phi$  is an isomorphism. This completes the proof of (ii)  $\Rightarrow$  (iii).

Assume now that (iii) holds. Then the square

$$\begin{array}{ccc} \mathbf{G} \times_k X & \xrightarrow{a} & X \\ \text{pr}_X \downarrow & & \downarrow q \\ X & \xrightarrow{q} & Y \end{array}$$

is cartesian and  $q$  is faithfully flat and quasi-compact. By Proposition 3.8 morphism  $\text{pr}_X$  is a universal geometric quotient. According to Corollary 3.7 we derive that  $q$  is universal geometric quotient. Moreover, the cartesian square above shows that the morphism  $\mathbf{G} \times_k X \rightarrow X \times_Y X$  induced by  $a$  and  $\text{pr}_X$  is an isomorphism. Thus the action of  $\mathbf{G}$  on  $X$  is free. This is (i). Hence (iii)  $\Rightarrow$  (i) holds.  $\square$

## 6. GEOMETRICALLY REDUCTIVE GROUPS AND NAGATA'S THEOREM

We start by proving the following result which give yet another characterization of linearly reductive groups.

**Theorem 6.1.** *Let  $\mathbf{G}$  be a smooth affine algebraic group over  $k$ . Then the following assertions are equivalent.*

- (i)  $\mathbf{G}$  is linearly reductive.
- (ii) For every finitely dimensional linear representation  $V$  of  $\mathbf{G}$  and for every nonzero  $\mathbf{G}$ -invariant element  $v$  in  $V$  there exists a  $\mathbf{G}$ -invariant linear function  $f : V \rightarrow k$  such that  $f(v) \neq 0$ .

We need the following easy result.

**Lemma 6.1.1.** *Let  $\mathbf{G}$  be an algebraic group over  $k$  which satisfies (ii). Suppose that  $V$  is a finitely dimensional representation of  $\mathbf{G}$ . Then the map*

$$\mathrm{Hom}_k(V, k)^{\mathbf{G}} \ni f \mapsto f|_{V^{\mathbf{G}}} \in \mathrm{Hom}_k(V^{\mathbf{G}}, k)$$

*is an isomorphism of vector spaces over  $k$ .*

*Proof of the lemma.* The image of the map in the statement is a  $k$ -vector subspace  $W \subseteq \mathrm{Hom}_k(V^{\mathbf{G}}, k)$  such that for every nonzero element  $v$  in  $V^{\mathbf{G}}$  there exists  $f$  in  $W$  such that  $f(v) \neq 0$  (this is a consequence of (ii)). It follows that  $W$  cannot be proper subspace of  $\mathrm{Hom}_k(V^{\mathbf{G}}, k)$ . Hence the map in the statement is an epimorphism. Now fix a nonzero  $\mathbf{G}$ -invariant linear function  $f : V \rightarrow k$ . By (ii) there exists a  $\mathbf{G}$ -invariant linear function  $w : \mathrm{Hom}_k(V, k) \rightarrow k$  such that  $w(f) = 0$ . Note that the canonical isomorphism

$$V \cong \mathrm{Hom}_k(\mathrm{Hom}_k(V, k), k)$$

of  $k$ -vector spaces is a morphism of representations of  $\mathbf{G}$ . Thus  $w$  is defined in terms of evaluation in some  $\mathbf{G}$ -invariant vector  $v$  in  $V$ . Therefore,  $f(v) \neq 0$  and hence  $f|_{V^{\mathbf{G}}} \neq 0$ . Thus the map described in the statement is also a monomorphism.  $\square$

*Proof of the theorem.* Suppose that (i) holds. Consider a  $\mathbf{G}$ -invariant nonzero vector  $v$  in a finitely dimensional representation  $V$  of  $\mathbf{G}$ . Then  $k \cdot v \subseteq V$  is a  $\mathbf{G}$ -subrepresentation. Since  $\mathbf{G}$  is linearly reductive, there exists a morphism of  $\mathbf{G}$ -representations which is a left inverse of  $k \cdot v \hookrightarrow V$ . This morphism can be identified with a  $\mathbf{G}$ -invariant linear function  $f : V \rightarrow k$  such that  $f(v) \neq 0$ . Hence (i)  $\Rightarrow$  (ii).

Now suppose that (ii) holds. Pick an epimorphism  $\theta : V \twoheadrightarrow W$  of finitely dimensional representations  $V$  of  $\mathbf{G}$ . Assume that there exists a nonzero  $\mathbf{G}$ -invariant vector  $w$  in  $W$  such that  $w \notin \theta(V^{\mathbf{G}})$ . By Lemma 6.1.1 there exists  $f$  in  $\mathrm{Hom}_k(W, k)^{\mathbf{G}}$  such that  $f|_{\theta(V^{\mathbf{G}})} = 0$  and  $f(w) \neq 0$ . Then  $f \cdot \theta$  is a nonzero element of  $\mathrm{Hom}(V, k)^{\mathbf{G}}$  such that  $(f \cdot \theta)|_{V^{\mathbf{G}}} = 0$ . This is impossible according to Lemma 6.1.1. Hence  $\theta^{\mathbf{G}} : V^{\mathbf{G}} \rightarrow W^{\mathbf{G}}$  is an epimorphism. Now assume that  $\theta : V \twoheadrightarrow W$  is an epimorphism of arbitrary linear representations of  $\mathbf{G}$ . Since  $\mathbf{G}$  is affine, every linear representation of  $\mathbf{G}$  is rational (i.e. it is a sum of its finitely dimensional subrepresentations). This together with the finitely dimensional case considered above imply that  $\theta^{\mathbf{G}} : V^{\mathbf{G}} \rightarrow W^{\mathbf{G}}$  is an epimorphism. Thus the functor  $(-)^{\mathbf{G}} : \mathbf{Rep}(\mathbf{G}) \rightarrow \mathbf{Vect}_k$  is exact.  $\square$

The result above motivates the following notion.

**Definition 6.2.** Let  $\mathbf{G}$  be a smooth affine algebraic group. Suppose that for every finitely dimensional representation  $V$  of  $\mathbf{G}$  and for every nonzero  $\mathbf{G}$ -invariant vector  $v$  of  $V$  there exists a homogenous  $\mathbf{G}$ -invariant polynomial  $f : V \rightarrow k$  such that  $f(v) \neq 0$ . Then  $\mathbf{G}$  is *geometrically reductive*.

We state here the following celebrated result.

**Theorem 6.3.** *If  $\mathbf{G}$  is reductive, then it is geometrically reductive.*

The result above is due to Haboush and its proof can be found in [Haboush, 1975].

The following theorem shows that geometric reductivity admits up to an integral extension the same property as linear reductivity (see also Remark 6.5 below).

**Theorem 6.4.** *Suppose that  $\mathbf{G}$  is geometrically reductive. Let  $A$  be a  $k$ -algebra such that  $\text{Spec } A$  admits an action of  $\mathbf{G}$  and let  $\mathfrak{a}$  be a  $\mathbf{G}$ -stable ideal of  $A$ . We consider  $A^{\mathbf{G}}/A^{\mathbf{G}} \cap \mathfrak{a}$  as a  $k$ -subalgebra of  $(A/\mathfrak{a})^{\mathbf{G}}$  by means of the canonical inclusion  $A^{\mathbf{G}}/A^{\mathbf{G}} \cap \mathfrak{a} \hookrightarrow A/\mathfrak{a}$ . For every element  $x \in (A/\mathfrak{a})^{\mathbf{G}}$  there exists positive integer  $r$  such that  $x^r \in A^{\mathbf{G}}/A^{\mathbf{G}} \cap \mathfrak{a}$ .*

*Proof.* Let  $d : A \rightarrow k[\mathbf{G}] \otimes_k A$  be the coaction of  $\mathbf{G}$  on  $A$ . Pick an element  $x_0 \in A$  which maps to  $x$  modulo  $\mathfrak{a}$ . Consider finitely dimensional vector subspace  $V \subseteq A$  over  $k$  such that  $V$  is a  $\mathbf{G}$ -subrepresentation of  $A$  and  $x_0 \in V$ . Since  $x$  is  $x_0$  modulo  $\mathfrak{a}$ , we derive that  $c(x_0) - 1 \otimes x_0$  is in ideal of  $k[\mathbf{G}] \otimes_k A$  generated by  $k[\mathbf{G}] \otimes_k \mathfrak{a}$ . Thus  $W = k \cdot x_0 + V \cap \mathfrak{a} \subseteq A$  is finitely dimensional  $\mathbf{G}$ -subrepresentation of  $A$ . Let  $\lambda : W \rightarrow k$  be a  $k$ -linear form such that  $\lambda(x_0) = 1$  and  $\lambda|_{V \cap \mathfrak{a}} = 0$ . Since  $\mathbf{G}$  is geometrically reductive there exists  $f \in \text{Sym}_r(W)^{\mathbf{G}}$  such that  $f(\lambda) = 1$ . Since the canonical morphism  $\text{Sym}_r(W) \rightarrow A$  is a morphism of representations of  $\mathbf{G}$ , we deduce that  $f$  is mapped under this morphism to some  $\mathbf{G}$ -invariant element  $y$  in  $A$ . Note that  $f$  is sum of an  $r$ -th symmetric power of  $x_0$  and some element of  $\text{Sym}_r(V \cap \mathfrak{a})$ . Thus  $y \bmod \mathfrak{a} = x^r$ . Hence  $x^r \in A^{\mathbf{G}}/A^{\mathbf{G}} \cap \mathfrak{a}$ .  $\square$

**Remark 6.5.** Let  $\mathbf{G}$  be an algebraic group  $\mathbf{G}$  which acts on  $\text{Spec } A$  for some  $k$ -algebra  $A$  and let  $\mathfrak{a}$  be a  $\mathbf{G}$ -stable ideal of  $A$ . Then the sequence

$$0 \longrightarrow \mathfrak{a}^{\mathbf{G}} \longrightarrow A^{\mathbf{G}} \longrightarrow (A/\mathfrak{a})^{\mathbf{G}}$$

is left exact and it induces a monomorphism  $A^{\mathbf{G}}/\mathfrak{a} \cap A^{\mathbf{G}} = A^{\mathbf{G}}/\mathfrak{a}^{\mathbf{G}} \hookrightarrow (A/\mathfrak{a})^{\mathbf{G}}$ . If  $\mathbf{G}$  is linearly reductive, then the sequence is exact and this monomorphism is an isomorphism. Theorem 6.4 states that if  $\mathbf{G}$  is geometrically reductive, then the monomorphism  $A^{\mathbf{G}}/\mathfrak{a} \cap A^{\mathbf{G}} \hookrightarrow (A/\mathfrak{a})^{\mathbf{G}}$  is integral.

Now we are going to formulate the main result of this section.

**Theorem 6.6.** *Suppose that  $\mathbf{G}$  is geometrically reductive. Let  $A$  be a finitely generated  $k$ -algebra such that  $\text{Spec } A$  admits an action of  $\mathbf{G}$ . Then  $A^{\mathbf{G}}$  is finitely generated  $k$ -algebra.*

The theorem above was first proved by Nagata and here we follow Nagata's original proof. In the argument we denote the coaction of  $k[\mathbf{G}]$  on  $A$  by  $d : A \rightarrow k[\mathbf{G}] \otimes_k A$ . The proof relies on a series of partial results.

**Lemma 6.6.1.** *Let  $A \hookrightarrow B$  be an integral morphism of  $k$ -algebras and suppose that  $B$  is finitely generated over  $k$ . Then  $A$  is finitely generated.*

*Proof of the lemma.* Suppose that  $b_1, \dots, b_r$  are generators of  $B$  as a  $k$ -algebra. For every  $1 \leq i \leq r$  we have a polynomial relation

$$b_i^{n_i} + a_{i,n_i-1}b_i^{n_i-1} + \dots + a_{i,1}b_i + a_{i,0} = 0$$

where  $n_i > 0$  and  $a_{i,j} \in A$  for  $0 \leq j \leq n_i - 1$ . Suppose that  $\tilde{A}$  is a  $k$ -subalgebra of  $A$  generated by  $a_{i,j}$  for  $1 \leq i \leq r$  and  $0 \leq j \leq n_i - 1$ . Then  $B$  is finite over  $\tilde{A}$ . Since  $\tilde{A} \subseteq A \subseteq B$  and  $\tilde{A}$  is noetherian, we derive that  $A$  is finite over  $\tilde{A}$ . Hence  $A$  is finitely generated over  $k$ .  $\square$

**Lemma 6.6.2.** *Suppose that  $\mathbf{G}$  is geometrically reductive. Let  $A$  be a  $k$ -algebra such that  $\text{Spec } A$  admits an action of  $\mathbf{G}$ . Assume that  $A$  contains  $\mathbf{G}$ -invariant zero divisor and that for every proper  $\mathbf{G}$ -stable ideal  $\mathfrak{a}$  of  $A$  the  $k$ -algebra  $(A/\mathfrak{a})^{\mathbf{G}}$  is finitely generated over  $k$ . Then  $A^{\mathbf{G}}$  is finitely generated over  $k$ .*

*Proof of the lemma.* Let  $f$  be a  $\mathbf{G}$ -invariant zero divisor of  $A$ . By assumption both  $k$ -algebras  $(A/fA)^{\mathbf{G}}$  and  $(A/\text{ann}(f))^{\mathbf{G}}$  are finitely generated over  $k$ . Now by combination of Lemma 6.6.1 and Theorem 6.4 we obtain that  $A^{\mathbf{G}}/A^{\mathbf{G}} \cap fA$  and  $A^{\mathbf{G}}/A^{\mathbf{G}} \cap \text{ann}(f)$  are finitely generated over  $k$ . Let  $B$  be a finitely generated  $k$ -subalgebra of  $A^{\mathbf{G}}$  which maps surjectively onto  $A^{\mathbf{G}}/A^{\mathbf{G}} \cap fA$  and  $A^{\mathbf{G}}/A^{\mathbf{G}} \cap \text{ann}(f)$ . Let  $u_1, \dots, u_n$  be elements in  $A$  such that the image of  $B \cdot u_1 + \dots + B \cdot u_n \subseteq A$  modulo  $\text{ann}(f)$  contains a finite  $B$ -module  $(A/\text{ann}(f))^{\mathbf{G}}$ . Fix  $a \in A^{\mathbf{G}}$ . Since  $B$  maps surjectively onto  $A^{\mathbf{G}}/A^{\mathbf{G}} \cap fA$ , there exist  $b \in B$  and  $c \in A$  such that  $a - b = fc$ . Note that  $fc \in A^{\mathbf{G}}$  and thus

$$(1 \otimes f)(d(c) - 1 \otimes c) = 0$$

This implies that  $c$  is sent to  $(A/\text{ann}(f))^{\mathbf{G}}$  modulo  $\text{ann}(f)$ . Then  $c \in B \cdot u_1 + \dots + B \cdot u_n$ . Hence  $a - b \in B \cdot fu_1 + \dots + B \cdot fu_n$ . Therefore,  $a \in B[f u_1, \dots, f u_n]$ . This completes the proof that  $A^{\mathbf{G}}$  is finitely generated over  $k$ .  $\square$

**Lemma 6.6.3.** *Suppose that  $\mathbf{G}$  is geometrically reductive. Let  $A = \bigoplus_{n \in \mathbb{N}} A_n$  be a  $\mathbb{N}$ -graded  $k$ -algebra such that  $A$  admits an action of  $\mathbf{G}$ . Assume that  $A_n$  is a  $\mathbf{G}$ -subrepresentation of  $A$  for every  $n \in \mathbb{N}$  and that for every proper  $\mathbf{G}$ -stable homogenous ideal  $\mathfrak{a}$  of  $A$  the  $k$ -algebra  $(A/\mathfrak{a})^{\mathbf{G}}$  is finitely generated over  $k$ . If  $A$  contains  $\mathbf{G}$ -invariant zero divisor, then  $A^{\mathbf{G}}$  is finitely generated over  $k$ .*

*Proof of the lemma.* Let  $f$  be a  $\mathbf{G}$ -invariant zero divisor of  $A$ . We may pick  $f$  such that it is homogenous. Then both ideals  $fA$  and  $\text{ann}(f)$  are homogenous,  $\mathbf{G}$ -stable and proper in  $A$ . Now we proceed as in the proof of Lemma 6.6.2.  $\square$

*Proof of the theorem.* We first prove the theorem in case of  $\mathbb{N}$ -graded  $k$ -algebras and then reduce the general case to this graded case.

Assume that  $A = \bigoplus_{n \in \mathbb{N}} A_n$  is  $\mathbb{N}$ -graded in such a way that  $A_0 = k$  and  $A_n$  is a  $\mathbf{G}$ -subrepresentation of  $A$  for every  $n \in \mathbb{N}$ . Since  $A$  is finitely generated over  $k$  and by virtue of noetherian induction, we assume that  $(A/\mathfrak{a})^{\mathbf{G}}$  is finitely generated over  $k$  for every homogenous  $\mathbf{G}$ -stable proper ideal  $\mathfrak{a}$  of  $A$ . If there are  $\mathbf{G}$ -invariant zero divisors of  $A$ , then by Lemma 6.6.3 we deduce that  $A^{\mathbf{G}}$  is finitely generated over  $k$ . So we may assume that  $A^{\mathbf{G}}$  contains no zero divisors of  $A$ . Pick a nonzero homogenous element  $f \in A^{\mathbf{G}}$  of positive degree. If there are no such elements, then  $A^{\mathbf{G}} = A_0 = k$  and the result holds. So we may assume that such an element exists. Note that it is noninvertible. Consider  $x \in A$  such that  $fx \in A^{\mathbf{G}}$ . Then

$$0 = d(fx) - 1 \otimes fx = d(f) \cdot d(x) - (1 \otimes f) \cdot (1 \otimes x) = (1 \otimes f)(d(x) - 1 \otimes x)$$

Since  $f$  is not a zero divisor in  $A$ , we derive that  $1 \otimes f$  is not a zero divisor in  $k[\mathbf{G}] \otimes_k A$ . Thus  $d(x) = 1 \otimes x$  and  $x \in A^{\mathbf{G}}$ . This shows that  $fA \cap A^{\mathbf{G}} = fA^{\mathbf{G}}$ . By Theorem 6.4  $(A/fA)^{\mathbf{G}}$  is integral over  $A^{\mathbf{G}}/fA \cap A^{\mathbf{G}} = A^{\mathbf{G}}/fA^{\mathbf{G}}$ . Note that  $(A/fA)^{\mathbf{G}}$  is finitely generated over  $k$  by inductive assumption. According to Lemma 6.6.1 we obtain that  $A^{\mathbf{G}}/fA^{\mathbf{G}}$  is finitely generated over  $k$ . Clearly

$$A^{\mathbf{G}} = \bigoplus_{n \in \mathbb{N}} A_n^{\mathbf{G}}$$

and hence  $A^{\mathbf{G}}/fA^{\mathbf{G}}$  inherits  $\mathbb{N}$ -grading from  $A$ . The ideal generated by elements of positive degree  $(A^{\mathbf{G}}/fA^{\mathbf{G}})_+$  is finitely generated (as is every ideal in noetherian ring). Hence also

$$(A^{\mathbf{G}})_+ = \bigoplus_{n \in \mathbb{N}_+} A_n^{\mathbf{G}}$$

is finitely generated (generating set consists of lifts of generators of  $(A^{\mathbf{G}}/fA^{\mathbf{G}})_+$  and  $f$ ). This implies that  $A^{\mathbf{G}}$  is finitely generated over  $A_0^{\mathbf{G}} = k$ .

Now assume that  $A$  is an arbitrary finitely generated  $k$ -algebra. By noetherian induction we may assume that  $(A/\mathfrak{a})^G$  is finitely generated over  $k$  for every proper  $G$ -stable ideal  $\mathfrak{a}$  of  $A$ . Pick a finitely dimensional  $G$ -subrepresentation  $V$  of  $A$  which contains some finite set of generators of  $A$  as a  $k$ -algebra. Define  $S = \text{Sym}(V)$  and  $S_n = \text{Sym}_n(V)$  for every  $n \in \mathbb{N}$ . Then  $S$  is  $\mathbb{N}$ -graded,  $S_0 = k$  and  $G$  acts on  $\text{Spec } S$  in such a way that  $S_n$  is a  $G$ -subrepresentation of  $S$  for every  $S$ . By the case considered above  $S^G$  is finitely generated over  $k$ . The canonical (induced by  $V \hookrightarrow A$ ) surjective morphism  $S \twoheadrightarrow A$  of  $k$ -algebras is also a morphism of representations of  $G$ . Let  $I$  be its kernel. Then  $I$  is a  $G$ -stable ideal of  $S$ . By Theorem 6.4 we derive that  $A^G = (S/I)^G$  is integral over its finitely generated  $k$ -subalgebra  $S^G/I \cap S^G$ . Moreover, by Lemma 6.6.2 we may assume that  $A^G$  does not contain zero divisors of  $A$ . In particular, it is an integral domain. Hence  $S^G/I \cap S^G$  is a domain. Let  $\overline{A^G}$  be the integral closure of  $S^G/I \cap S^G$  in the field  $Q(A^G)$  of fractions of  $A^G$ . Since  $\overline{A^G}$  is integral over  $A^G$ , Lemma 6.6.1 shows that it suffices to prove that  $\overline{A^G}$  is finitely generated over  $k$ . For this we show that  $\overline{A^G}$  is a finite  $S^G/I \cap S^G$ -module. Since fields are Nagata rings (see [Eisenbud, 1995, Corollary 13.13]), we may reduce this question to proving that  $Q(A^G)$  is a finite extension of the field  $Q(S^G/I \cap S^G)$  of fractions of  $S^G/I \cap S^G$ . Since  $Q(S^G/I \cap S^G) \subseteq Q(A^G)$  is algebraic (due to the fact that  $S^G/I \cap S^G \hookrightarrow A^G$  is integral), it suffices to show that  $Q(A^G)$  is finitely generated field over  $Q(S^G/I \cap S^G)$ . For this pick a set  $T$  of nonzero divisors of  $A$ . Note that  $T$  is a multiplicative subset of  $A$ . Fix a maximal ideal  $\mathfrak{m} \subseteq T^{-1}A$ . Since nonzero elements of  $\mathfrak{m} \cap A^G$  are zero divisors of  $A$ , we derive that  $\mathfrak{m} \cap A^G = 0$ . Thus  $Q(A^G)$  is a subfield of  $T^{-1}A/\mathfrak{m}$ . The inclusion  $A \hookrightarrow T^{-1}A$  induces an isomorphism between the field of fractions  $Q(A/\mathfrak{m} \cap A)$  and the field  $T^{-1}A/\mathfrak{m}$ . By our assumption  $A$  is finitely generated over  $k$ . Thus  $Q(A/\mathfrak{m} \cap A)$  is a field finitely generated over  $k$ . It follows that  $Q(A^G)$  is a field finitely generated over  $k$ . This implies that  $Q(A^G)$  is a field finitely generated over  $Q(S^G/I \cap S^G)$ . This completes the proof that  $A^G$  is a finitely generated  $k$ -algebra.  $\square$

## 7. GOOD CATEGORICAL QUOTIENTS

We first introduce the following important notion.

**Definition 7.1.** Let  $Z$  be a closed subset of  $X$ . Suppose that  $\text{pr}_X^{-1}(Z) = a^{-1}(Z)$ . Then  $Z$  is a  $G$ -stable closed subset of  $X$ .

**Theorem 7.2.** Let  $q : X \rightarrow Y$  be a morphism into a  $k$ -scheme  $Y$  equipped with the trivial  $G$ -action. Assume that the following assertions hold.

- (1)  $q$  is  $G$ -equivariant.
- (2)  $\mathcal{O}_Y$  is the sheaf of  $G$ -invariants for  $q$ .
- (3) If  $Z$  is a  $G$ -stable closed subset of  $X$ , then  $q(Z)$  is a closed subset of  $Y$ .
- (4) If  $\{Z_i\}_{i \in I}$  is a family of closed  $G$ -stable subsets with the empty intersection, then the intersection  $\{q(Z_i)\}_{i \in I}$  is empty.

Then  $q$  is submersive and it is a categorical quotient of  $X$ .

*Proof.* Clearly  $q(X)$  is closed in  $Y$ . Hence  $V = Y \setminus q(X)$  is open. Moreover,  $q^\# : \mathcal{O}_Y \rightarrow q_*\mathcal{O}_X$  is a monomorphism of sheaves of  $k$ -algebras. Thus we have a monomorphism  $\mathcal{O}_V \hookrightarrow (q_*\mathcal{O}_X)_{q^{-1}(V)}$ . We have  $(q_*\mathcal{O}_X)_{q^{-1}(V)} = 0$  and hence  $\mathcal{O}_V = 0$ . This implies that  $V = \emptyset$ . Thus  $q$  is surjective. Suppose that  $Z$  is a subset of  $Y$  such that  $q^{-1}(Z)$  is a closed subset of  $X$ . Then  $q^{-1}(Z)$  is a  $G$ -stable closed subset and hence  $q(q^{-1}(Z))$  is closed. Note that  $q(q^{-1}(Z)) = Z$  because  $q$  is surjective. Thus  $Z$  is closed. This completes the proof that  $q$  is submersive.

Now we show that  $q$  is a categorical quotient of  $X$ .  $\square$



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