LINEARLY REDUCTIVE GROUPS

1. Motivation – linear representations of compact topological groups

In this section we fix a compact topological group $\mathbf{G}$. Assume that $\rho : \mathbf{G} \to \mathrm{GL}_n(\mathbb{C})$ is a continuous homomorphism i.e. a complex, n-dimensional linear representation of $\mathbf{G}$. For every $g \in \mathbf{G}$ we get a matrix

$$\rho(g) = \big[c_{ij}(g)\big]_{1 \leq i,j \leq n}$$

For i, j function $c_{ij} : \mathbf{G} \to \mathbb{C}$ is a continuous complex valued function. Alternatively suppose that $\{e_1, e_2, ..., e_n\}$ is the standard basis of $\mathbb{C}^n$ on which $\mathrm{GL}_n(\mathbb{C})$ act. Then c_{ij} is equal to a function

$$\mathbf{G} \ni g \mapsto \langle g \cdot e_i, e_j \rangle \in \mathbb{C}$$

Fix now $g_1, g_2 \in \mathbf{G}$ and note that

$$\big[c_{ij}(g_2 \cdot g_1)\big]_{1 \leq i,j \leq n} = \rho(g_2 \cdot g_1) = \rho(g_2) \cdot \rho(g_1) = \left[\sum_{k=1}^{n} c_{ik}(g_2) \cdot c_{kj}(g_1)\right]_{1 \leq i,j \leq n}$$

Hence

$$c_{ij}(g_2 \cdot g_1) = \sum_{k=1}^{n} c_{ik}(g_2) \cdot c_{kj}(g_1)$$

for every $1 \leq i, j \leq n$. This implies that $\sum_{1 \leq i,j \leq n} \mathbb{C} \cdot c_{ij} \subseteq \mathcal{L}^2(\mathbf{G}, \mathbb{C})$ is a linear $\mathbf{G} \times \mathbf{G}^{\mathrm{op}}$-subrepresentation of the regular representation $\mathcal{L}^2(\mathbf{G}, \mathbb{C})$. We call it *the matrix coefficients of* ρ.

2. Matrix coefficients of a representation

Proposition 2.1. *Let $\mathfrak{X}$ be a monoid k-functor and let V be a finitely generated, projective k-module. Fix a morphism of monoids $\rho : \mathfrak{X} \to \mathcal{L}_V$. Fix k-algebra A and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. For every A-algebra B and $x \in \mathfrak{X}_A(B)$ we consider the formula*

$$c_{v,w}(x) = \langle \rho_A(x) \cdot v_B, w_B \rangle$$

Then $c_{v,w}$ defines a regular function on $\mathfrak{X}_A$ for every k-algebra A.

Proof. Suppose that $f : B \to C$ is a morphism of A-algebras and pick $x \in \mathfrak{X}_A(B)$. Since ρ_A is natural and $w : A \otimes_k V \to A$ is a morphism of A-modules, we derive that the diagram

$$\begin{tikzcd}
V_B \arrow[r, "\rho_A(x)"] \arrow[d, "1_{V_A} \otimes_A f"'] & V_B \arrow[r, "w_B"] \arrow[d, "1_{V_A} \otimes_A f"] & B \arrow[d, "f"] \\
V_C \arrow[r, "\rho_A(\mathfrak{X}_A(f)(x))"'] & V_C \arrow[r, "w_C"'] & C
\end{tikzcd}$$

is commutative. Hence

$$c_{v,w}\big(\mathfrak{X}_A(f)(x)\big) = \langle \rho_A\big(\mathfrak{X}_A(f)(x)\big) \cdot v_C, w_C \rangle = f\big(\langle \rho_A(x) \cdot v_B, w_B \rangle\big) = f\big(c_{v,w}(x)\big)$$

and this implies that $c_{v,w} : \mathfrak{X}_A \to \mathbb{A}^1_A$ is natural. $\square$

Definition 2.2. Let $\mathfrak{X}$ be a monoid k-functor and let (V,ρ) be its representation with finitely generated, projective underlying k-module V. Fix k-algebra A and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. Then the regular function $c_{v,w}$ on $\mathfrak{X}_A$ is called *the matrix coefficient of v and w*.

Proposition 2.3. *Let $\mathfrak{X}$ be a monoid k-functor and let (V,ρ) be its representation with finitely generated projective underlying k-module V. Then the following assertions holds.*

(1) *For every k-algebra A map*

$$(A \otimes_k V) \times \left(A \otimes_k V^{\vee}\right) \ni (v,w) \mapsto c_{v,w} \in \mathrm{Mor}_A\left(\mathfrak{X}_A, \mathbb{A}^1_A\right)$$

is A-bilinear.

(2) *The collection of maps*

$$\left\{ (A \otimes_k V) \times \left(A \otimes_k V^{\vee}\right) \ni (v,w) \mapsto c_{v,w} \in \mathrm{Mor}_A\left(\mathfrak{X}_A, \mathbb{A}^1_A\right) \right\}_{A \in \mathbf{Alg}_k}$$

gives rise to a morphism of k-functors

$$V_{\mathrm{a}} \times V^{\vee}_{\mathrm{a}} \longrightarrow \mathcal{M}\mathrm{or}_k\left(\mathfrak{X}, \mathbb{A}^1_k\right)$$

Proof. We left the proof of **(1)** to the reader.
We prove **(2)**. Consider k-algebra A and an A-algebra B with structural morphism $f : A \to B$. Fix $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. We prove that restriction of $c_{v,w} : \mathfrak{X}_A \to \mathbb{A}^1_A$ to the category $\mathbf{Alg}_B$ is c_{v_B,w_B}. For this pick a B-algebra C and an element $x \in \mathfrak{X}_A(C) = \mathfrak{X}_B(C)$. Note that

$$c_{v,w}(x) = \langle \rho_A(x) \cdot v_C, w_C \rangle = \langle \rho_B(x) \cdot v_C, w_C \rangle = \langle \rho_B(x) \cdot (v_B)_C, (w_B)_C \rangle = c_{v_B,w_B}(x)$$

and hence $c_{v,w|\mathbf{Alg}_B} = c_{v_B,w_B}$. Consider the square

$$\begin{array}{ccc} V_{\mathrm{a}}(A) \times V^{\vee}_{\mathrm{a}}(A) & \longrightarrow & \mathcal{M}\mathrm{or}_k\left(\mathfrak{X}, \mathbb{A}^1\right)(A) \\ {\scriptstyle V_a(f) \times V^{\vee}_a(f)} \big\downarrow & & \big\downarrow {\scriptstyle \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathbb{A}^1)(f)} \\ V_{\mathrm{a}}(B) \times V^{\vee}_{\mathrm{a}}(B) & \longrightarrow & \mathcal{M}\mathrm{or}_k\left(\mathfrak{X}, \mathbb{A}^1\right)(B) \end{array}$$

in which both horizontal arrows are given by formula $(v,w) \mapsto c_{v,w}$. We proved that the square commutes. Since f is an arbitrary morphism of k-algebras, we conclude the assertion. □

Corollary 2.4. *Let $\mathfrak{X}$ be a monoid k-functor and let (V,ρ) be its representation with finitely generated projective underlying k-module V. Then there exists a morphism of k-functors*

$$\left(V \otimes_k V^{\vee}\right)_{\mathrm{a}} \xrightarrow{c} \mathcal{M}\mathrm{or}_k\left(\mathfrak{X}, \mathbb{A}^1_k\right)$$

given by formula

$$(A \otimes_k V) \otimes_A \left(A \otimes_k V^{\vee}\right) \ni (v,w) \mapsto c_{v,w} \in \mathrm{Mor}_A\left(\mathfrak{X}_A, \mathbb{A}^1_A\right)$$

Moreover, c is a morphism of k-functors equipped with $\mathfrak{X} \times \mathfrak{X}^{\mathrm{op}}$-actions.

Proof. The first part is an immediate consequence of Proposition 2.3. We prove that c is a morphism of k-functors equipped with $\mathfrak{X} \times \mathfrak{X}^{\mathrm{op}}$-actions. For this we fix a k-algebra k and elements $v \in A \otimes_k V$, $w \in A \otimes_k V^{\vee}$. Pick a morphism of k-algebras $f : A \to B$, $(y,z) \in \mathfrak{X}(A) \times \mathfrak{X}(A)^{\mathrm{op}}$ and $x \in \mathfrak{X}_A(B)$. Then we have

$$c_{\rho(y)\cdot v, w\cdot\rho(z)}(x) = \left\langle \rho_A(x) \cdot \left(\rho(y) \cdot v\right)_B, \left(w \cdot \rho(z)\right)_B \right\rangle =$$

$$= \left\langle \rho_A(x) \cdot \rho_A((\mathfrak{X}_A(f)(y))) \cdot v_B, w_B \cdot \rho_A\left(\mathfrak{X}_A(f)(z)\right) \right\rangle = w_B\left(\rho_A\left(\mathfrak{X}_A(f)(z)\right) \cdot \rho_A(x) \cdot \rho_A\left(\mathfrak{X}_A(f)(y)\right) \cdot v_B\right) =$$

$$= w_B\left(\rho_A\left(\mathfrak{X}_A(f)(z) \cdot x \cdot \mathfrak{X}_A(f)(y)\right) \cdot v_B\right) = \left\langle \rho_A\left(\mathfrak{X}_A(f)(z) \cdot x \cdot \mathfrak{X}_A(f)(y)\right) \cdot v_B, w_B \right\rangle =$$

$$= c_{v,w}\big(\mathfrak{X}_A(f)(z)\cdot x\cdot \mathfrak{X}_A(f)(y)\big)$$

and hence c is a morphism of k-functors equipped with actions of $\mathfrak{X}\times\mathfrak{X}^{\mathrm{op}}$. □

3. Algebra of regular functions of a k-functor

Example 3.1. For every k-algebra A we denote by $|A|$ its underlying set. We denote by $\mathbb{A}^1_k$ a k-functor given by assignment $\mathbb{A}^1_k(A) = |A|$ for every A. We call $\mathbb{A}^1_k$ *the affine line over* k. Let $k[x]$ be a polynomial k-algebra with variable x. For every k-algebra A map of sets

$$\mathrm{Mor}_k\,(k[x],A)\ni f\mapsto f(x)\in |A|$$

is a bijection. The family of such maps gives rise to an isomorphism of k-functors

$$\mathrm{Mor}_k\,(\mathrm{Spec}\,(-),\mathrm{Spec}\,k[x])\cong \mathrm{Mor}_k\,(k[x],-)\cong \mathbb{A}^1_k$$

and hence $\mathbb{A}^1_k$ is representable by an affine k-scheme $\mathrm{Spec}\,k[x]$.

Definition 3.2. Let $\mathfrak{X}$ be a k-functor. Consider $\alpha\in k$ and $f,\,g\in \mathrm{Mor}_k\big(\mathfrak{X},\mathbb{A}^1_k\big)$. Then for every k-algebra A and $x\in\mathfrak{X}(A)$ formulas

$$(f+g)\,(x) = f(x)+g(x),\ (f\cdot g)\,(x) = f(x)\cdot g(x),\ (\alpha\cdot f)\,(x) = \alpha\cdot f(x)$$

define k-algebra operations on the class $\mathrm{Mor}_k\big(\mathfrak{X},\mathbb{A}^1_k\big)$. We call them *pointwise k-algebra operations*. In particular, if $\mathrm{Mor}_k\big(\mathfrak{X},\mathbb{A}^1_k\big)$ is a set, then pointwise k-algebras operations on this set give rise to *the k-algebra of regular functions on* $\mathfrak{X}$.

4. k-functors of monoids and their linear representations

In the sequel we assume that the reader is familiar with notions of a monoid, group etc. in arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 2-5].

Definition 4.1. *A monoid (group) k-functor* is a monoid (group) object in the category of k-functors.

Next we introduce an important notion of a linear representation of a monoid k-functor. For this we define k-functors associated with modules over k and discuss their properties.

Example 4.2. Let V be a k-module. We define a k-functor V_{a}. We set

$$V_{\mathrm{a}}(A) = A\otimes_k V,\ V_{\mathrm{a}}(f) = f\otimes_k 1_V$$

for every k-algebra A and every morphism $f : A\to B$ of k-algebras. Moreover, V_{a} admits a structure of a commutative group k-functor. Indeed, $V_{\mathrm{a}}(A)$ is a commutative group with respect to addition induced by its structure of A-module and $V_{\mathrm{a}}(f) : V_{\mathrm{a}}(A)\to V_{\mathrm{a}}(B)$ preserves the addition.

Suppose now that V, W are k-modules and $\sigma : (V_{\mathrm{a}})_A \to (W_{\mathrm{a}})_A$ is a morphism of A-functors. Then for every A-algebra B we denote by $\sigma^B : B\otimes_k V\to B\otimes_k W$ the component of σ for B.

Definition 4.3. Let V,W be k-modules and let A be a k-algebra. A morphism $\sigma : (V_{\mathrm{a}})_A\to (W_{\mathrm{a}})_A$ of A-functors is *linear* if for every A-algebra B the component $\sigma^B : B\otimes_k V\to B\otimes_k W$ is a morphism of B-modules.

Next Fact characterizes linear morphism.

Fact 4.4. *Let V, W be k-modules and let A be a k-algebra. Suppose that $\phi : A\otimes_k V\to A\otimes_k W$ is a morphism of A-modules. Then there exists a unique linear morphism $\sigma : (V_{\mathrm{a}})_A\to (W_{\mathrm{a}})_A$ of A-functors such that $\sigma^A = \phi$.*

Proof. Note that if such σ exists, then by requirement $\sigma^A = \phi$ for every morphism $f : A\to B$ of k-algebras the following diagram

$$\begin{tikzcd} A\otimes_k V \arrow[r, "\phi"] \arrow[d, "f\otimes_k 1_V"'] & A\otimes_k W \arrow[d, "f\otimes_k 1_W"] \\ B\otimes_k V \arrow[r, "\sigma^B"'] & B\otimes_k W \end{tikzcd}$$

must commute. We make this into a definition of a morphism σ^B of B-modules. It is a matter of linear algebra that this diagram uniquely determines σ^B and also that $\sigma^A = \phi$. It remains to verify that $\sigma = \{\sigma^B\}_{B\in\mathbf{Alg}_A}$ defined in such a way is a morphism of A-functors. For this suppose that $f : A \to B$ and $g : B \to C$ are morphisms of k-algebras. Then we have

$$\sigma_C \cdot (g\otimes_k 1_V)\cdot(f\otimes_k 1_V) = \sigma_C\cdot((g\cdot f)\otimes_k 1_V) = ((g\cdot f)\otimes_k 1_W)\cdot\phi =$$

$$= (g\otimes_k 1_W)\cdot(f\otimes_k 1_V)\cdot\phi = (g\otimes_k 1_W)\cdot\sigma_B\cdot(f\otimes_k 1_V)$$

and hence $\sigma_C\cdot(g\otimes_k 1_V) = (g\otimes_k 1_W)\cdot\sigma_B$. Thus σ is a linear morphism of A-functors. □

We restate Fact 4.4 in the form of the following result.

Corollary 4.5. *Let V, W be k-modules and A be a k-algebra. Consider the map*

$$\mathrm{Hom}_A(A\otimes_k V, A\otimes_k W) \longrightarrow \mathrm{Mor}_A\big((V_{\mathrm{a}})_A, (W_{\mathrm{a}})_A\big)$$

that sends morphism ϕ to a unique linear morphism $\sigma : (V_{\mathrm{a}})_A \to (W_{\mathrm{a}})_A$ of A-functors such that $\sigma^A = \phi$. Then this map is injective and its image consists of all linear morphisms of A-functors.

Example 4.6. Let V be a k-module. We define a k-functor $\mathcal{L}_V$. We set

$$\mathcal{L}_V(A) = \mathrm{Hom}_A(A\otimes_k V, A\otimes_k V)$$

for every k-algebra A. Next for every morphism $f : A \to B$ of k-algebras and a morphism $\phi : A\otimes_k V \to A\otimes_k V$ of A-modules we define $\mathcal{L}_V(f)(\phi)$ as a unique morphism of B-modules such that the diagram

$$\begin{tikzcd} A\otimes_k V \arrow[r, "\phi"] \arrow[d, "f\otimes_k 1_V"'] & A\otimes_k W \arrow[d, "f\otimes_k 1_W"] \\ B\otimes_k V \arrow[r, "\mathcal{L}_V(\phi)"'] & B\otimes_k W \end{tikzcd}$$

is commutative. Note also that $\mathcal{L}_v(A)$ is a monoid k-functor with respect to the usual composition of morphism of A-modules and $\mathcal{L}_V(f) : \mathcal{L}_V(A) \to \mathcal{L}_V(B)$ preserves this composition.

Remark 4.7. Corollary 4.5 implies that there are injective maps that make the square

$$\begin{tikzcd} \mathcal{L}_V(A) \arrow[r, hook] \arrow[d, "\mathcal{L}_V(f)"'] & \mathrm{Mor}_A\big((V_{\mathrm{a}})_A, (V_{\mathrm{a}})_A\big) \arrow[d, "\sigma\mapsto\sigma_B"] \\ \mathcal{L}_V(B) \arrow[r, hook] & \mathrm{Mor}_B\big((V_{\mathrm{a}})_B, (V_{\mathrm{a}})_B\big) \end{tikzcd}$$

commutative for every morphism $f : A \to B$ of k-algebras. Also Corollary 4.5 shows that for every k-algebra A this identifies $\mathcal{L}_V(A)$ with a subset of the class $\mathrm{Mor}_A\big((V_{\mathrm{a}})_A, (V_{\mathrm{a}})\big)$ consisting of all linear morphism of A-functor.

The discussion below is partially an application of the main result in [Mon19, section 6] (Remark 4.7 shows that $\mathcal{L}_V$ is a subcopresheaf of internal endomorphisms of V_{a} and hence the machinery developed in the citation above can be applied), but for the reader's convenience we decide to include all essential details even if this requires repetition.

Let $\mathfrak{X}$ be a monoid k-functor and let be V be a k-module. Suppose that $\alpha : \mathfrak{X} \times V_{\mathrm{a}} \to V_{\mathrm{a}}$ is an action of $\mathfrak{X}$ on V_{a}. Assume that A is a k-algebra and $x \in \mathfrak{X}(A)$. We denote by $i_x : \mathbf{1}_A \to \mathfrak{X}_A$ the morphism of A-functors corresponding to x by means of Fact **??**. Since $\mathbf{1}_A$ is terminal A-functor, a morphism $\alpha_A \cdot \left(i_x \times 1_{(V_{\mathrm{a}})_A}\right)$ is isomorphic to a morphism $\alpha_x : (V_{\mathrm{a}})_A \to (V_{\mathrm{a}})_A$ of A-functors. Suppose now that for any k-algebra A and point $x \in \mathfrak{X}(A)$ morphism α_x is linear. Then we define a morphism $\rho : \mathfrak{X} \to \mathcal{L}_V$ of k-functors by formula $\rho(x) = \alpha_x^A$. We first check that ρ really is a morphism of k-functors. For this fix morphism $f : A \to B$ of k-algebras and $x \in \mathfrak{X}(A)$. Then $\alpha_{\mathfrak{X}(f)(x)}$ is a morphism of B-functors isomorphic with $\alpha_B \cdot \left(i_{\mathfrak{X}(f)(x)} \times 1_{(V_{\mathrm{a}})_B}\right)$ and since

$$\alpha_B \cdot \left(i_{\mathfrak{X}(f)(x)} \times 1_{(V_{\mathrm{a}})_B}\right) = \alpha_B \cdot \left(i_x \times 1_{(V_{\mathrm{a}})_A}\right)_B = \left(\alpha_A \cdot \left(i_x \times 1_{(V_{\mathrm{a}})_A}\right)\right)_B$$

we derive that $\alpha_{\mathfrak{X}(f)(x)} = (\alpha_x)_B$. This implies that

$$\rho\left(\mathfrak{X}(f)(x)\right) = \alpha^B_{\mathfrak{X}(f)(x)} = \left((\alpha_x)_B\right)^B = \alpha_x^B = \mathcal{L}_V(f)(\rho(x))$$

and thus ρ is a morphism of k-functors. Now we show that ρ is a morphism of monoids. For this pick k-algebra A and $x, y \in \mathfrak{X}(A)$. Since α is an action, we deduce that $\alpha_{x\cdot y} = \alpha_x \cdot \alpha_y$ and hence also

$$\rho(x \cdot y) = \alpha^A_{x\cdot y} = \alpha^A_x \cdot \alpha^A_y = \rho(x) \cdot \rho(y)$$

Therefore, ρ is a morphism of monoid k-functors.

Theorem 4.8. *Let $\mathfrak{X}$ be a monoid k-functor and let V be a k-module. Consider the following classes.*

(1) *The class of actions $\alpha : \mathfrak{X} \times V_{\mathrm{a}} \to V_{\mathrm{a}}$ of $\mathfrak{X}$ such that for any k-algebra A and point $x \in \mathfrak{X}(A)$ morphism α_x is linear.*

(2) *The class of morphisms $\rho : \mathfrak{X} \to \mathcal{L}_V$ of monoid k-functors.*

Let α be an element of **(1)** *and $\rho : \mathfrak{X} \to \mathcal{L}_V$ be the element of* **(2)** *such that $\rho(x) = \alpha_x^A$ for any k-algebra A and $x \in \mathfrak{X}(A)$. Then the correspondence $\alpha \mapsto \rho$ is a bijection between these classes.*

Proof. We may refer to [Mon19, Theorem 6.3], but for self-containment of the presentation let us give a direct proof of this important result. □

References

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[Mon19] Monygham. Categories of presheaves. *github repository: "Monygham/Pedo-mellon-a-minno"*, 2019.