

DIFFERENTIABILITY

1. INTRODUCTION

In these notes we collect basic results on derivatives of functions defined on open subsets of real or complex Banach spaces.

Symbol \mathbb{K} denotes the base field which is either \mathbb{R} or \mathbb{C} .

2. BOUNDED MULTILINEAR FORMS ON NORMED SPACES

In this section we fix a positive integer n and consider normed spaces $\mathfrak{D}_1, \dots, \mathfrak{D}_n, \mathfrak{X}$ over \mathbb{K} . Let $\pi_i : \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \rightarrow \mathfrak{D}_i$ be the projection onto i -th axis.

Definition 2.1. Let $L : \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \rightarrow \mathfrak{X}$ be a \mathbb{K} -multilinear form. Suppose that there exists $c \in \mathbb{R}_+$ such that

$$\|L(\mathbf{x})\| \leq c \cdot \|\pi_1(\mathbf{x})\| \cdot \dots \cdot \|\pi_n(\mathbf{x})\|$$

for every $\mathbf{x} \in \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n$. Then L is *bounded \mathbb{K} -multilinear form*.

The following result characterizes bounded multilinear forms.

Theorem 2.2. Let $L : \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \rightarrow \mathfrak{X}$ be a \mathbb{K} -multilinear form. Then the following assertions are equivalent.

- (i) L is continuous.
- (ii) L is continuous at zero n -tuple in $\mathfrak{D}_1 \times \dots \times \mathfrak{D}_n$.
- (iii) L is bounded.

Proof. Left for the reader as an exercise. □

Definition 2.3. Let $L : \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \rightarrow \mathfrak{X}$ be a bounded \mathbb{K} -multilinear form. Then we define

$$\|L\| = \sup \{ \|L(\mathbf{x})\| \mid \mathbf{x} \in \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \text{ and } \|\pi_i(\mathbf{x})\| = 1 \text{ for all } i \}$$

and call it *the operator norm of L* .

Fact 2.4. Let $L : \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \rightarrow \mathfrak{X}$ be a bounded \mathbb{K} -multilinear form. Then

$$\|L(\mathbf{x})\| \leq \|L\| \cdot \|\pi_1(\mathbf{x})\| \cdot \dots \cdot \|\pi_n(\mathbf{x})\|$$

for every $\mathbf{x} \in \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n$.

Proof. Left for the reader as an exercise. □

Theorem 2.5. Let $L(\mathfrak{D}_1, \dots, \mathfrak{D}_n; \mathfrak{X})$ be a \mathbb{K} -vector space of bounded \mathbb{K} -multilinear forms $\mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \rightarrow \mathfrak{X}$ with respect to operations defined pointwise. Suppose that \mathfrak{X} is a Banach space over \mathbb{K} . Then

$$L(\mathfrak{D}_1, \dots, \mathfrak{D}_n; \mathfrak{X}) \ni L \mapsto \|L\| \in \mathbb{R}_+ \cup \{0\}$$

is a norm which makes $L(\mathfrak{D}_1, \dots, \mathfrak{D}_n; \mathfrak{X})$ into a Banach space over \mathbb{K} .

Proof. We left as an exercise the proof that operator norm is well defined vector space norm on $L(\mathfrak{D}_1, \dots, \mathfrak{D}_n; \mathfrak{X})$. Consider a Cauchy's sequence $\{L_m\}_{m \in \mathbb{N}}$ with respect to operator norm. Fix $\mathbf{x} \in \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n$. Then by Fact 2.4

$$\|(L_m - L_k)(\mathbf{x})\| \leq \|L_m - L_k\| \cdot \|\pi_1(\mathbf{x})\| \cdot \dots \cdot \|\pi_n(\mathbf{x})\|$$

for every $m, k \in \mathbb{N}$. This implies that $\{L_m(\mathbf{x})\}_{m \in \mathbb{N}}$ is a Cauchy's sequence in \mathfrak{X} . Since \mathfrak{X} is a Banach space over \mathbb{K} , we derive that this sequence is convergent. We define

$$L(\mathbf{x}) = \lim_{m \rightarrow +\infty} L_m(\mathbf{x})$$

We also have

$$|\|L_m\| - \|L_k\|| \leq \|L_m - L_k\|$$

for every $m, k \in \mathbb{N}$. Thus $\{\|L_m\|\}_{m \in \mathbb{N}}$ is a Cauchy's sequence in \mathbb{R} and hence it is convergent in \mathbb{R} . Note that we have

$$\|L(\mathbf{x})\| = \lim_{m \rightarrow +\infty} \|L_m(\mathbf{x})\| \leq \left(\lim_{m \rightarrow +\infty} \|L_m\| \right) \cdot \|\pi_1(\mathbf{x})\| \cdot \dots \cdot \|\pi_n(\mathbf{x})\|$$

Therefore, $L : \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \rightarrow \mathfrak{X}$ is a bounded \mathbb{K} -multilinear form. We claim that L is the limit of $\{L_m\}_{m \in \mathbb{N}}$ with respect to operator norm. For the proof fix $\mathbf{x} \in \mathfrak{D}_1 \times \dots \times \mathfrak{D}_n$ such that $\|\pi_1(\mathbf{x})\| = \dots = \|\pi_n(\mathbf{x})\| = 1$. Then

$$\|(L - L_m)(\mathbf{x})\| \leq \|L(\mathbf{x}) - L_k(\mathbf{x})\| + \|L_k - L_m\|$$

Thus we have

$$\|(L - L_m)(\mathbf{x})\| \leq \limsup_{k \rightarrow +\infty} \|L_k - L_m\|$$

Since the left hand side does not depend on \mathbf{x} , we deduce that

$$\|L - L_m\| \leq \limsup_{k \rightarrow +\infty} \|L_k - L_m\|$$

Invoking once again the assumption that $\{\|L_m\|\}_{m \in \mathbb{N}}$ is Cauchy's sequence we infer

$$\lim_{m \rightarrow +\infty} \|L - L_m\| \leq \lim_{m \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \|L_k - L_m\| = 0$$

This completes the proof. \square

Proposition 2.6. *The canonical map $L(\mathfrak{D}_1, \dots, \mathfrak{D}_{n-1}; L(\mathfrak{D}_n, \mathfrak{X})) \rightarrow L(\mathfrak{D}_1, \dots, \mathfrak{D}_n; \mathfrak{X})$ which sends L in $L(\mathfrak{D}_1, \dots, \mathfrak{D}_{n-1}; L(\mathfrak{D}_n, \mathfrak{X}))$ to a \mathbb{K} -multilinear form given by formula*

$$\mathfrak{D}_1 \times \dots \times \mathfrak{D}_n \ni \mathbf{x} \mapsto L(\pi_1(\mathbf{x}), \dots, \pi_{n-1}(\mathbf{x}))(L(\pi_n(\mathbf{x})) \in \mathfrak{X}$$

is an isometry of normed spaces.

Proof. Left for the reader as an exercise. \square

3. FRÉCHET DERIVATIVE

In this section we introduce derivatives and prove their basic properties. We fix Banach spaces $\mathfrak{D}, \mathfrak{X}$ over \mathbb{K} . Let U be an open subset of \mathfrak{D} and let V be an open subset of \mathfrak{X} .

Fact 3.1. *Let x be a point in U and let $f : U \rightarrow V$ be a function. Suppose that $L_i : \mathfrak{D} \rightarrow \mathfrak{X}$ for $i = 1, 2$ are \mathbb{K} -linear maps. If both functions*

$$\{h \in \mathfrak{D} \mid h \neq 0 \text{ and } x + h \in U\} \ni h \mapsto \frac{f(x+h) - f(x) - L_i(h)}{\|h\|} \in \mathfrak{X}$$

tend to zero as $h \rightarrow 0$, then $L_1 = L_2$.

Proof. By assumption the function

$$\{h \in \mathfrak{D} \mid h \neq 0 \text{ and } x+h \in U\} \ni h \mapsto (L_1 - L_2) \left(\frac{h}{\|h\|} \right) \in \mathfrak{X}$$

tends to zero as $h \rightarrow 0$. This implies that $L_1 - L_2$ sends each vector of the unit sphere in \mathfrak{D} to zero. Thus $L_1 - L_2 = 0$ and hence $L_1 = L_2$. \square

Definition 3.2. Let x be a point in U . A function $f : U \rightarrow V$ is *differentiable at point x* if there exists a continuous \mathbb{K} -linear map $L : \mathfrak{D} \rightarrow \mathfrak{X}$ such that the function

$$\{h \in \mathfrak{D} \mid h \neq 0 \text{ and } x+h \in U\} \ni h \mapsto \frac{f(x+h) - f(x) - L(h)}{\|h\|} \in \mathfrak{X}$$

tends to zero as $h \rightarrow 0$. Moreover, the unique continuous \mathbb{K} -linear map L is *the derivative of f at x* .

Remark 3.3. Notion of differentiability defined above is named by some authors *Fréchet differentiability* after french mathematician Maurice Fréchet.

Remark 3.4. Let x be a point in U and let $f : U \rightarrow V$ be a function differentiable at point x . Then the derivative of f at x is usually denoted by $f'(x)$.

Fact 3.5. Let x be a point in U and let $f : U \rightarrow V$ be a function differentiable at x . Then f is continuous at x .

Proof. The set $\{h \in \mathfrak{D} \mid x+h \in U\}$ contains a neighborhood of zero in \mathfrak{D} and consider the function $\phi_f(h)$ defined on this set and given by formula

$$f(x+h) - f(x) = f'(x)(h) + \phi_f(h) \cdot \|h\|$$

Since f is differentiable at x , we have

$$\lim_{h \rightarrow 0} \phi_f(h) = 0$$

and thus

$$\lim_{h \rightarrow 0} (f(x+h) - f(x)) = \lim_{h \rightarrow 0} (f'(x)(h) + \phi_f(h) \cdot \|h\|) = 0$$

This implies that f is continuous at x . \square

Definition 3.6. A function $f : U \rightarrow V$ is *differentiable* if it is differentiable at each point of U .

4. CHAIN RULE

Chain rule is a basic tools for calculating Fréchet derivatives of a superposition of two functions.

Theorem 4.1. Let $U \subseteq \mathfrak{D}$, $V \subseteq \mathfrak{X}$, $W \subseteq \mathfrak{Z}$ be open subsets of Banach spaces over \mathbb{K} and let $f : U \rightarrow V$, $g : V \rightarrow W$ be functions. Suppose that f is differentiable at some point x in U and g is differentiable at $f(x)$. Then $g \cdot f$ is differentiable at x and the chain rule

$$(g \cdot f)'(x) = g'(f(x)) \cdot f'(x)$$

holds.

Proof. Let L be derivative of f at x and let K be a derivative of g at $f(x)$. For h in \mathfrak{D} such that $x+h \in U$ define $\phi_f(h)$ by formula

$$f(x+h) - f(x) - L(h) = \phi_f(h) \cdot \|h\|$$

Similarly for s in \mathfrak{X} such that $f(x) + s \in V$ define $\phi_g(s)$ by formula

$$g(f(x) + s) - g(f(x)) - K(s) = \phi_g(s) \cdot \|s\|$$

Now pick nonzero h in \mathfrak{D} such that $x + h \in U$ and $f(x + h) \in V$. Then

$$\begin{aligned}
 & g(f(x + h)) - g(f(x)) - K(L(h)) = \\
 &= g(f(x + h) - f(x) + f(x)) - g(f(x)) - K(L(h)) = \\
 &= K(f(x + h) - f(x)) + \phi_g(f(x + h) - f(x)) \cdot \|f(x + h) - f(x)\| - K(L(h)) = \\
 &= K(\phi_f(h)) \cdot \|h\| + \phi_g(f(x + h) - f(x)) \cdot \|f(x + h) - f(x)\| = \\
 &= K(\phi_f(h)) \cdot \|h\| + \phi_g(f(x + h) - f(x)) \cdot \|L(h) + \phi_f(h) \cdot \|h\|\|
 \end{aligned}$$

Thus

$$\begin{aligned}
 & \left\| \frac{g(f(x + h)) - g(f(x)) - K(L(h))}{\|h\|} \right\| \leq \\
 & \leq \left\| K(\phi_f(h)) \right\| + \|\phi_g(f(x + h) - f(x))\| \cdot (\|L\| + \|\phi_f(h)\|)
 \end{aligned}$$

Note that for $h \rightarrow 0$ we have

$$K(\phi_f(h)) \rightarrow 0$$

according to differentiability of f at x and

$$\phi_g(f(x + h) - f(x)) \rightarrow 0$$

which follows from continuity of f at x and differentiability of g at $f(x)$. This implies that

$$\lim_{h \rightarrow 0} \frac{g(f(x + h)) - g(f(x)) - K(L(h))}{\|h\|} = 0$$

and hence the proof is completed. □

5. LAGRANGE'S MEAN VALUE THEOREM

The main topic of this section is very useful result, which connects derivatives with local change of a function.