FLATNESS

1. Filtered colimits in the category of modules

Definition 1.1. Let I be a category. Suppose that the following conditions are satisfied.

(1) For any objects $i, j \in I$ there exists an object $k \in I$ and a diagram

$$\begin{tikzcd} & k & \\ i \arrow[ur] & & j \arrow[ul] \end{tikzcd}$$

(2) For any pair of parallel morphisms in I

$$\begin{tikzcd} i \arrow[r, shift left] \arrow[r, shift right] & j \end{tikzcd}$$

there exist an object $k \in I$ and a morphism $j \to k$ such that, the following diagram is commutative

$$\begin{tikzcd} i \arrow[r, shift left] \arrow[r, shift right] & j \arrow[r] & k \end{tikzcd}$$

Then we say that I is *a filtered category*.

Let R be a ring.

Proposition 1.2. *Let I be a small filtered category. Then the functor sending I-indexed diagram of left R-modules to its colimit is exact.*

Proof. Suppose that

$$\left\{ 0 \longrightarrow K_i \xrightarrow{r_i} M_i \xrightarrow{p_i} N_i \longrightarrow 0 \right\}_{i \in I}$$

is an I-indexed family of short exact sequences. Consider a complex

$$0 \longrightarrow K \xrightarrow{r} M \xrightarrow{p} N \longrightarrow 0$$

where $K = \mathrm{colim}_{i\in I} K_i$, $M = \mathrm{colim}_{i\in I} M_i$, $N = \mathrm{colim}_{i\in I} N_i$, $r = \mathrm{colim}_{i\in I} r_i$ and $p = \mathrm{colim}_{i\in I} p_i$. Clearly the complex is right exact. It suffices to prove that r is a monomorphism. For $i \in I$ denote by $v_i : K_i \to K$, $u_i : M_i \to M$ structural morphisms. Pick $k \in K$ such that $r(k) = 0$. Since I is filtered, we have

$$K = \sum_{i \in I} v_i(K_i),\ M = \sum_{i\in I} u_i(M_i)$$

Thus there exists $i_0 \in I$ and $k_{i_0} \in K_{i_0}$ such that $v_{i_0}(k_{i_0}) = k$. We have $u_{i_0}\left(r_{i_0}(k_{i_0})\right) = r(k) = 0$. Again using the fact that I is filtered, we deduce that there exist $i_1 \in I$ and a morphism $\alpha : i_0 \to i_1$ such that $u_\alpha\left(r_{i_0}(k_{i_0})\right) = 0$, where $u_\alpha : M_{i_0} \to M_{i_1}$ is a morphism in the I-indexed diagram $\{M_i\}_{i\in I}$. Now let $k_{i_1} = v_\alpha(k_{i_0})$, where $v_\alpha : K_{i_0} \to K_{i_1}$ is a morphism in the I-indexed diagram $\{K_i\}_{i\in I}$. Then

$v_{i_1}(k_{i_1}) = k$ and $r_{i_1}(k_{i_1}) = 0$. Since r_{i_1} is a monomorphism, we derive that $k_{i_1} = 0$ and hence $k = v_{i_1}(k_{i_1}) = 0$. Thus r is a monomorphism. □

Corollary 1.3. *Let M be a right R-module. Then for every $i \in \mathbb{N}$ functor $\mathrm{Tor}_i^R(M, -)$ defined on the category of left R-modules and with values in the category of abelian groups preserves filtered colimits.*

Proof. Let I be a small filtered category and $\{N_i\}_{i\in I}$ be an I-indexed diagram of left R-modules. Fix a projective resolution $P_\bullet \to M$ of M. Since tensor product commutes with colimits, we have

$$\mathrm{colim}_{i\in I}\left(P_\bullet \otimes_R N_i\right) = P_\bullet \otimes_R \mathrm{colim}_{i\in I} N_i$$

in the category of complexes of abelian groups. Since exact functors preserve kernels, cokernels and images, we derive by Proposition 1.2 that for every $n \in \mathbb{N}$ there is an identification

$$\mathrm{Tor}_n^R\left(M, \mathrm{colim}_{i\in I} N_i\right) = H_n\left(P_\bullet \otimes_R \mathrm{colim}_{i\in I} N_i\right) = H_n\left(\mathrm{colim}_{i\in I}\left(P_\bullet \otimes_R N_i\right)\right) =$$

$$= \mathrm{colim}_{i\in I} H_n\left(P_\bullet \otimes_R N_i\right) = \mathrm{colim}_{i\in I} \mathrm{Tor}_n^R\left(M, N_i\right)$$

of cocones. □

2. Homological characterizations of flatness

Let R be a ring with unit.

Definition 2.1. Let M be a right R-module. We say that M is *flat* if the functor $M \otimes_R (-)$ defined on the category of left R-modules and with values in the category of abelian groups is exact.

Proposition 2.2. *Let I be a filtered category and $\{M_i\}_{i\in I}$ be an I-indexed diagram of flat right R-modules. Then $\mathrm{colim}_{i\in I} M_i$ is a flat right R-module.*

Proof. Proposition 1.2 implies that filtered colimits of short exact sequences of abelian groups are short exact seqeunces. Thus filtered colimits of flat right R-modules are flat. □

Proposition 2.3. *Let M be a right R-module. Then the following are equivalent.*

- **(i)** *For every finitely generated left ideal $I \subseteq R$ morphism $M \otimes_R I \to M$ induced by the inclusion of I in R is a monomorphism.*
- **(ii)** $\mathrm{Tor}_1^R(M, R/I) = 0$ *for every finitely generated left ideal $I \subseteq R$.*
- **(iii)** *M is flat.*
- **(iv)** $\mathrm{Tor}_i^R(M, N) = 0$ *for every left R-module N and $i > 0$.*

Proof. The implication **(i)** ⇒ **(ii)** is straightforward.
Suppose that **(ii)** holds. Then for every left ideal $I \subseteq R$ we can write $I = \mathrm{colim}_{\lambda\in\Lambda} I_\lambda$, where $\{I_\lambda\}_{\lambda\in\Lambda}$ is a filtered set of all finitely generated left ideals of R contained in I. This induces a presentation of R/I as a filtered colimit of the system $\{R/I_\lambda\}_{\lambda\in\Lambda}$ and thus by Corollary 1.3 we have

$$\mathrm{Tor}_1^R(M, R/I) = \mathrm{colim}_{\lambda\in\Lambda} \mathrm{Tor}_1^R(M, R/I_\lambda) = 0$$

Now suppose that N is a finitely generated left module over R. Then we can decompose N such that it fits in an exact sequence

$$0 \longrightarrow K \longrightarrow N \xrightarrow{q} R/I \longrightarrow 0$$

Now we have $\mathrm{Tor}_1^R(M,K) = 0$ implies that $\mathrm{Tor}_1^R(M,N) = 0$. Therefore, using induction on the minimal number of generators of finitely generated left R-module we may prove that $\mathrm{Tor}_1^R(M,N) = 0$ for every finitely generated left R-module. Since every left R-module is a filtered colimit of its finitely generated left R-submodules, we derive by Corollary 1.3 that $\mathrm{Tor}_1^R(M,N) = 0$ for every left R-module N. Using first terms of the long exact sequence for Tor associated with $M \otimes_R (-)$ we deduce **(iii)**.
Now if M is flat, then tensoring with M is exact. This means that tensor product of a free resolution of a left R-module N with M has trivial higher homologies. Thus $\mathrm{Tor}_i^R(M,N) = 0$ for $i > 0$. This proves **(iii)** $\Rightarrow$ **(iv)**.
Finally **(iv)** $\Rightarrow$ **(i)** is obvious. □

3. Flatness in terms of equations

Let R be a ring with unit.

Proposition 3.1. *Let M be a right R-module and N be a left R-module. Suppose that $\{y_i\}_{i\in I}$ is a set of generators for N and $\{x_i\}_{i\in I}$ is a set of elements of M. Suppose that all x_i for $i \in I$ except of finitely many are zero. Assume that*

$$\sum_{i\in I} x_i \otimes y_i = 0$$

in tensor product $M \otimes_R N$. Then there exist $n \in \mathbb{N}$, $\{a_{ik}\}_{i\in I, 1\leq k\leq n}$ in R and $\{z_k\}_{1\leq k\leq n}$ in M such that

$$x_i = \sum_{k=1}^{n} z_k a_{ki}$$

for every $i \in I$ and

$$\sum_{i\in I} a_{ki} y_i = 0$$

for every $1 \leq k \leq n$.

Proof. Consider a free left R-module F on a set I and a morphism $\phi : F \to N$ given by $\phi(e_i) = y_i$, where e_i is a free generator corresponding to $i \in I$. Let $K = \ker(\phi)$. Applying $M \otimes_R (-)$ we derive that $M \otimes_R K$ maps onto the kernel of $1_M \otimes_R \phi$. Next by assumptions $(1_M \otimes_R \phi)(\sum_{i\in I} x_i \otimes e_i) = \sum_{i\in I} x_i \otimes y_i = 0$. Thus $\sum_{i\in I} x_i \otimes e_i$ is equal to $\sum_{k=1}^{n} z_k \otimes f_k$ for $z_k \in M$, $f_k \in K$ and $n \in \mathbb{N}$. We can write $f_k = \sum_{i\in I} a_{ki} e_i$. Then we have

$$\sum_{k=1}^{n} z_k \otimes f_k = \sum_{k=1}^{n} z_k \otimes \left(\sum_{i\in I} a_{ki} e_i\right) = \sum_{k=1}^{n}\sum_{i\in I} (z_k \otimes a_{ki} e_i) = \sum_{i\in I}\sum_{k=1}^{n} (z_k a_{ki} \otimes e_i) = \sum_{i\in I}\left(\sum_{k=1}^{n} z_k a_{ki}\right) \otimes e_i$$

We deduce that $x_i = \sum_{k=1}^{n} z_k a_{ki}$ and $0 = \sum_{i\in I} a_{ki} y_i$ for every $i \in I$ and $1 \leq k \leq n$. □

Theorem 3.2 (Equational criteria for flatness)**.** *Let M be a right R-module. Then the following are equivalent.*

(i) *M is flat.*

(ii) *For every set of elements $\{x_i\}_{i=1,\ldots,n}$ in M and a relation*

$$\sum_{i=1}^{n} x_i a_i = 0$$

where $a_i \in R$ there exist elements $z_k \in M$ and $r_{ki} \in R$ for $1 \leq k \leq l$ such that

$$x_i = \textstyle\sum_{k=1}^{l} z_k r_{ki},\ \sum_{i=1}^{n} r_{ki} a_i = 0$$

for every $1 \leq i \leq n$ and $1 \leq k \leq l$.

(iii) *For every finitely presented right R-module N, every morphism $\phi : N \to M$ and every finitely generated R-submodule $K \subseteq \ker(\phi)$ there exists a factorization*

$$\begin{tikzcd} & G \arrow[dr, "\theta"] & \\ N \arrow[ur, "\psi"] \arrow[rr, "\phi"'] & & M \end{tikzcd}$$

where G is a finitely generated free right R-module and $K \subseteq \ker(\psi)$.

(iv) *For every set of elements $\{x_i\}_{i=1,\dots,n}$ in M and a finite set of relations*

$$\sum_{i=1}^{n} x_i a_{ij} = 0$$

where $1 \le j \le m$ and $a_{ij} \in R$ there exist elements $z_k \in M$ and $r_{ki} \in R$ for $1 \le k \le l$ such that

$$x_i = \sum_{k=1}^{l} z_k r_{ki},\ 0 = \sum_{i=1}^{n} r_{ki} a_{ij}$$

for every $1 \le i \le n$, $1 \le j \le m$ and $1 \le k \le l$.

Proof. Suppose that M is flat. We show then that **(ii)** holds. We have relation

$$\sum_{i=1}^{n} x_i a_i = 0$$

Let $I = \sum_{1 \le i \le n} R a_i \subseteq R$ be a left ideal. Since M is flat, the canonical morphism $M \otimes_R I \to M$ is a monomorphism. It sends $\sum_{i=1}^{n} x_i \otimes a_i$ to $\sum_{i=1}^{n} x_i a_i = 0$. It follows that

$$\sum_{i=1}^{n} x_i \otimes a_i = 0$$

in $M \otimes_R I$. Thus by Proposition 3.1 there exist $\{r_{ki}\}_{1 \le i \le n, 1 \le k \le l}$ in R and $\{z_k\}_{1 \le k \le l}$ in M such that

$$x_i = \sum_{k=1}^{l} z_k r_{ki},\ 0 = \sum_{i=1}^{n} r_{ki} a_i$$

for every $1 \le i \le n$ and $1 \le k \le l$.
Now we prove that **(ii)** $\Rightarrow$ **(iii)**. Suppose first that N is a finitely generated and free right R-module, $\phi : N \to M$ is a morphism and $K \subseteq \ker(\phi)$ is finitely generated. Note that our result easily follows from **(ii)**, if $K \subseteq \ker(\phi)$ is generated by a single element. Now easy induction on the number of generators for $K \subseteq \ker(\phi)$ yields the assertion **(iii)** in the case of finitely generated free right R-module N.
Suppose now that N is a finitely presented right R-module, $\phi : N \to M$ is a morphism and $K \subseteq \ker(\phi)$ is a finitely generated submodule. Take an epimorphism $f : F \to N$ where F is a finitely generated free left R-module. Let $\phi' = \phi f$ and pick a factorization

$$\begin{tikzcd} & G \arrow[dr, "\theta"] & \\ F \arrow[ur, "g"] \arrow[rr, "\phi'"'] & & M \end{tikzcd}$$

where G is a finitely generated free right R-module and $f^{-1}(K) \subseteq \ker(g)$. Such a factorization exists according to the fact that $f^{-1}(K)$ is a finitely generated submodule of $\ker(\phi')$. Since

$\ker(f) \subseteq f^{-1}(K)$, we deduce that g factorizes through f. This proves the implication.
Assume that **(iii)** holds. Suppose that $\{x_i\}_{i=1,\dots,n}$ are in M and that we have a finite set of relations

$$\sum_{i=1}^{n} x_i a_{ij} = 0$$

where $1 \le j \le m$ and $a_{ij} \in R$. Let F be a right free R-module of rank n with basis $e_1,\dots,e_n$. Define a morphism $\phi : F \to M$ by $\phi(e_i) = x_i$ for $1 \le i \le n$. Then

$$K = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} e_i a_{ij} \right) R \subseteq \ker(\phi)$$

is finitely generated. Hence by **(iii)** there exist a finitely generated free right R-module G and morphisms $\psi : F \to G$, $\theta : G \to M$ such that $\phi = \theta \cdot \psi$ and $K \subseteq \ker(\psi)$. Next if $f_1,\dots,f_l$ is a basis of G, then we pick $z_k = \theta(f_k)$ for $1 \le k \le l$. There exist $r_{ki} \in R$ for $1 \le k \le l$ and $1 \le i \le n$ such that $\psi(e_i) = \sum_{k=1}^{l} f_k r_{ki}$ for $1 \le i \le n$. Now straightforward verification shows that $z_k \in M$ and $r_{ki} \in R$ for $1 \le k \le l$ and $1 \le i \le n$ satisfy **(iv)**.
Now assume that **(iv)** holds. Let I be a finitely generated left ideal in R. Suppose that a_i for $1 \le i \le n$ are generators of I. We are going to prove that the canonical morphism $M \otimes_R I \to M$ is a monomorphism. This implies **(i)** due to Proposition 2.3. Assume that there exist x_i for $1 \le i \le n$ in M such that $\sum_{i=1}^{n} x_i \otimes a_i \in M \otimes_R I$ is in the kernel of $M \otimes_R I \to M$. This means that $\sum_{i=1}^{n} x_i a_i = 0$ in M. According to **(iv)** there exist $z_k \in M$ and $r_{ki} \in R$ for $1 \le k \le l$ and $1 \le i \le n$ such that

$$x_i = \sum_{k=1}^{l} z_k r_{ki},\ 0 = \sum_{i=1}^{n} r_{ki} a_i$$

Thus

$$\sum_{i=1}^{n} x_i \otimes a_i = \sum_{i=1}^{n} \left(\sum_{k=1}^{l} z_k r_{ki} \right) \otimes a_i = \sum_{i=1}^{n} \sum_{k=1}^{l} (z_k r_{ki} \otimes a_i) = \sum_{k=1}^{l} \sum_{i=1}^{n} (z_k \otimes r_{ki} a_i) = \sum_{k=1}^{l} z_k \otimes \left(\sum_{i=1}^{n} r_{ki} a_i \right) = 0$$

Hence the kernel of the morphism $M \otimes_R I \to M$ is trivial. □

4. Categorical characterizations of flatness

Let R be a ring with unit.

Theorem 4.1 (Lazard's theorem)**.** *A right R-module M is flat if and only if it is a colimit of a filtered diagram of finitely generated free right R-modules.*

Proof. If M is a filtered colimit of finitely generated flat right R-modules, then Proposition 2.2 implies that M is flat.
Assume now that M is flat. Consider a set of symbols $E = \{e_m \mid m \in M\}$. For every finite subset $\alpha \subseteq E$ let F_α be a right free R-module generated by symbols in α. Next for every such α let $q_\alpha : F_\alpha \to M$ be a morphism defined by formula $q_\alpha(e_m) = m$ for $e_m \in \alpha$.
Next we define a small diagram category I. Objects of I are finite subsets $\alpha \subseteq E$. Morphisms $f : \alpha \to \beta$ for any two finite subsets $\alpha, \beta \subseteq E$ are morphisms of right R-modules $f : F_\alpha \to F_\beta$ such that $q_\beta \cdot f = q_\alpha$. The composition of morphisms in I is given by the usual composition of morphisms of right R-modules.
We will now show that I is a filtered category. Pick $\alpha_1, \alpha_2 \in I$. Let $\alpha = \alpha_1 \cup \alpha_2$. Then α is well defined object of I. Moreover, canonical inclusions $\alpha_1 \subseteq \alpha$, $\alpha_2 \subseteq \alpha$ give rise to morphisms $f_1 : F_{\alpha_1} \to F_\alpha$ and $f_2 : F_{\alpha_2} \to F_\alpha$ in the category of right R-modules and hence give rise to morphisms $f_1 : \alpha_1 \to \alpha$ and $f_2 : \alpha_2 \to \alpha$ in I. This verifies the first axiom of filtered category for I. Now if $f, g : \alpha \to \beta$ are two morphisms in I, then

$$q_\beta \cdot (f - g) = q_\alpha - q_\alpha = 0$$

in the category of right R-modules. Hence $(f - g)(F_\alpha)$ is a finitely generated right R-submodule of F_β contained in the kernel of q_β. Using Theorem 3.2 we derive that there exists some finite

subset $\gamma \subseteq E$ and a morphism $h : F_\beta \to F_\gamma$ such that $h \cdot (f - g) = 0$ and $q_\gamma \cdot h = q_\beta$. This implies that $h : \beta \to \gamma$ is a morphism in I and $h \cdot f = h \cdot g$. Hence I verifies the second axiom for filtered category.
Now we define a diagram of finitely generated free right R-modules indexed by I. We send each object α of I to right R-module F_α and we send $f : \alpha \to \beta$ in I to $f : F_\alpha \to F_\beta$ in the category of right R-modules. It is clear that it is well defined I-indexed diagram.
Finally it suffices to verify that $q_\alpha : F_\alpha \to M$ for $\alpha \in I$ admit the universal property of colimit for the I-indexed diagram defined above. For this let N be some right R-module and $r_\alpha : F_\alpha \to N$ for $\alpha \in I$ be morphisms such that $r_\beta \cdot f = r_\alpha$ for every $f : \alpha \to \beta$ in I. Now we define a function $s : M \to N$ by formula

$$s(m) = r_\alpha(e_m)$$

for any $m \in M$ and any $\alpha \in I$ such that $e_m \in \alpha$. It is easy to verify that the function s is well defined. Moreover, it is a unique function that satisfies $s \cdot q_\alpha = r_\alpha$.
We will show now that s is a morphism of right R-modules. Pick $x \in R$ and $m \in M$. Consider $\alpha \in I$ such that $e_m, e_{mx} \in \alpha$. Since $q_\alpha(e_m x - e_{mx}) = mx - mx = 0$ and M is flat, by Theorem 3.2 there exist $\beta \in I$ and a morphism $f : \alpha \to \beta$ in I such that $f(e_m x - e_{mx}) = 0$. Hence we deduce that

$$s(m)x - s(mx) = r_\alpha(e_m)x - r_\alpha(e_{mx}) = r_\alpha(e_m x - e_{mx}) = r_\beta\left(f\left(e_m x - e_{mx}\right)\right) = 0$$

Similar argument shows that for $m_1, m_2 \in M$ the relation $s(m_1 + m_2) - (s(m_1) + s(m_2)) = 0$ is satisfied.
Now according to the fact that $s : M \to N$ is a unique morphism of cocones in the category of right R-modules, we deduce that

$$M = \mathrm{colim}_{\alpha \in I} F_\alpha$$

□

Corollary 4.2. *Let M be a right R-module of finite presentation. Then M is flat if and only if it is projective.*

Proof. Using Theorem 4.1 we derive that $M = \mathrm{colim}_{\alpha \in I} F_\alpha$, where I is a filtered category and $\{F_\alpha\}_{\alpha \in I}$ is I-indexed diagram of finitely generated right free R-modules. Next we have that

$$\mathrm{Hom}_R(M, M) = \mathrm{colim}_{\alpha \in I} \mathrm{Hom}_R(M, F_\alpha)$$

by finite presentation of M. Thus there exists an $\alpha \in I$ and a morphism $f : M \to F_\alpha$ such that $q_\alpha \cdot f = 1_M$ for the structural morphism $q_\alpha : F_\alpha \to M$. This means that q_α is a retraction. Hence M is a direct summand of a right free R-module P_β. Thus it is projective. □