ISyE 6644 — Summer 2019 — Test #1 Solutions

(revised 6/18/21)

This test is 120 minutes. You're allowed one cheat sheet (both sides).

This test requires a proctor. All questions are 3 points, except 33, which is 4 points. ☺

Good luck! I want you to make this test wish that it had never been born!!!

1. TRUE or FALSE? Simulation can be used to analyze supply chain models that are too complicated to solve analytically.

   Solution: TRUE (of course!) $\Box$

2. Use bisection (or any other method) to find x such that $e^x = x^2$.

   (a) $x \doteq -0.703$

   (b) $x \doteq 0.567$

   (c) $x = e/2$

   (d) $x = e^2$

   (e) None of the above.

   Solution: Let's use bisection to find the zero of $g(x) = e^x - x^2$.

x	$g(x)$	comments
-1	-0.6321	
0	1	look in $[-1, 0]$
-0.5	0.3565	look in $[-1, -0.5]$
-0.75	-0.09013	look in $[-0.75,\ -0.5]$
-0.625	0.1446	look in $[-0.75,\ -0.625]$
-0.6875	0.0302	look in $[-0.75,\ -0.6875]$
-0.71875	-0.0292	look in $[-0.71875,\ -0.6875]$
-0.703125	$0.00065 \doteq 0$	OK, stop here.

Since $g(-0.703125) \doteq 0$, we can stop and declare that $x \doteq -0.703$ does the job. This is answer (a). $\square$

We can also do the problem via Newton's method:

$$x_{n+1} \leftarrow x_n - \frac{g(x_n)}{g'(x_n)} = x_n - \frac{e^{x_n} - x_n^2}{e^{x_n} - 2x_n}$$

Suppose $x_0 = 1$ (which is a terrible choice, actually. ☺) Then

$$x_1 \leftarrow x_0 - \frac{e^{x_0} - x_0^2}{e^{x_0} - 2x_0} = 1 - \frac{e-1}{e-2} = -1.39221,$$

$$x_2 \leftarrow x_1 - \frac{e^{x_1} - x_1^2}{e^{x_1} - 2x_1} = -0.83509,$$

and, similarly, $x_3 = -0.70983$, $x_4 = -0.70348$, and $x_5 = -0.70347$, and we seem to have converged very quickly!

In any case, $x \doteq -0.70347$; so the answer is (a). $\square$

3. Suppose that X is a continuous random variable with p.d.f. $f(x) = x/2$ for $0 < x < 2$. Find $\Pr(X < 1 \,|\, X > 1/2)$.

   (a) 0
   (b) 0.2
   (c) 0.5
   (d) 0.8
   (e) 1/16

Solution: We have

$$\begin{aligned}
\Pr(X < 1 \,|\, X > 1/2) &= \frac{\Pr(X < 1 \cap X > 1/2)}{\Pr(X > 1/2)} \\
&= \frac{\Pr(1/2 < X < 1)}{\Pr(X > 1/2)} \\
&= \frac{\int_{1/2}^{1} (x/2)\, dx}{\int_{1/2}^{2} (x/2)\, dx} \\
&= 0.2,
\end{aligned}$$

after the smoke clears. So the answer is (b). $\square$

4. Suppose I conduct a series of independent experiments, each of which has a 60% chance of success. What's the probability that I'll see my first success on the 3rd run of the experiment?

   (a) 0.096
   (b) 0.192
   (c) 0.819
   (d) 0.973
   (e) 1

   Solution: The number of trials until the first success is $X \sim \text{Geom}(0.6)$. Thus,

   $$\mathsf{Pr}(X = 3) \;=\; q^{x-1}p \;=\; (0.4)^2(0.6) \;=\; 0.096.$$

   So the answer is (a). $\square$

5. If $X \sim \text{Bern}(0.5)$, find $\mathsf{E}[e^X]$.

   (a) 1
   (b) $e/2$
   (c) $(1 + e)/2$
   (d) 0.347
   (e) 1.38
   (f) None of the above.

   Solution: By the Unconscious Statistician and the fact that $X \sim \text{Bern}(0.5)$, we have

   $$\mathsf{E}[e^X] \;=\; \sum_{x=0}^{1} e^x \Pr(X = x) \;=\; \frac{e^0}{2} + \frac{e^1}{2} \;=\; \frac{1+e}{2},$$

   so the answer is (c). $\square$

6. If X has a mean of -2 and a variance of 3, find $\mathsf{E}[-3X-7]$.

   (a) -13
   (b) -7
   (c) -1
   (d) 6
   (e) 27

   Solution: $\mathsf{E}[-3X-7] = -3\mathsf{E}[X] - 7 = -1$, so the answer is (c). $\square$

7. Toss a 10-sided Dungeons and Dragons die repeatedly. What is the *expected value* of the number of tosses until you observe a 5?

   (a) 1/10.
   (b) 1/6.
   (c) 6.
   (d) 10.
   (e) 90.

   Solution: Let $X \sim \text{Geom}(p = 1/10)$ denote the number of tosses. Thus,

   $$\mathsf{E}[X] \;=\; 1/p \;=\; 10,$$

   so that the answer is (d). $\square$

8. Suppose that X and Y are identically distributed with a mean of -2, a variance of 3, and $\mathsf{Cov}(X,Y) = 0$. Find $\mathsf{Corr}(X,Y)$.

   (a) 0
   (b) 1/9
   (c) $-1/3$
   (d) 1/3
   (e) -1

(f) 3

Solution: $\mathsf{Corr} = \mathsf{Cov}/\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)} = 0$, so (a) is the answer. $\square$

9. If X and Y are both Normal(4,10) with $\mathsf{Cov}(X,Y) = 6$, find $\mathsf{Var}(X - 2Y)$.
   (a) -10
   (b) 10
   (c) 20
   (d) 26
   (e) 46

Solution:

$$\begin{aligned} \mathsf{Var}(X - 2Y) &= \mathsf{Var}(X) + \mathsf{Var}(-2Y) + 2\mathsf{Cov}(X, -2Y) \\ &= \mathsf{Var}(X) + 4\mathsf{Var}(Y) - 4\mathsf{Cov}(X, Y) \\ &= 26. \end{aligned}$$

So the answer is (d). $\square$

10. Consider a Poisson process with rate $\lambda = 1/2$. What is the probability that the time between the 1st and 2nd arrivals is less than 1?
    (a) $1/e$
    (b) $1 - (1/e)$
    (c) $1 - (1/\sqrt{e})$
    (d) $1/e^2$
    (e) $1 - (1/e^2)$

Solution: All interarrivals are i.i.d. Exp(λ). In particular, let $X \sim \text{Exp}(\lambda = 1/2)$ denote the time between the 1st and 2nd arrivals. Then

$$\Pr(X < 1) = 1 - e^{-\lambda x} = 1 - e^{-(1/2)(1)} = 0.393.$$

Thus, the answer is (c). $\square$

11. If X is Nor(2,4), find $\mathsf{Pr}(X > 0)$.

    (a) 0.05
    (b) 0.159
    (c) 0.5
    (d) 0.841
    (e) 0.95

Solution: $\Pr(X > 0) = \Pr(Z > \frac{0-2}{\sqrt{4}}) = \Pr(Z > -1) = 0.8413$. So the answer is (d). □

12. Suppose X and Y are i.i.d. Exp($\lambda = 1$). Find $\mathsf{Pr}(X + Y < 2)$.

    (a) $1/e^2$
    (b) $1 - (1/e^2)$
    (c) 0.5
    (d) 0.406
    (e) 0.594

Solution: Note that $S = X + Y \sim \text{Erlang}_{k=2}(\lambda = 1)$. Thus,

$$\Pr(S < 2) \;=\; 1 - \sum_{i=0}^{k-1} \frac{e^{-\lambda s}(\lambda s)^i}{i!} \;=\; 1 - \sum_{i=0}^{1} \frac{e^{-2}2^i}{i!} \;=\; 1 - 3e^{-2} \;=\; 0.594.$$

So the answer is (e). □

13. Suppose $X_1, \ldots, X_n$ are i.i.d. from a Bern(0.9) distribution. What is the *approximate* distribution of the sample mean $\bar{X}$ for $n = 100$?

    (a) Bern(0.009)
    (b) $\text{Bin}(100, 0.9)$
    (c) $\text{Nor}(0.9, 0.9)$
    (d) $\text{Nor}(0.9, 0.09)$

(e) Nor(0.9, 0.0009)

Solution: If X_i has mean μ and variance σ^2, then the Central Limit Theorem implies that

$$\bar{X} \approx \text{Nor}\Big(\mu, \frac{\sigma^2}{n}\Big) \sim \text{Nor}\Big(0.9, \frac{(0.9)(0.1)}{100}\Big).$$

So the answer is (e). $\Box$

14. Suppose U and V are i.i.d. Unif(0,1) random variables. What does $\lceil 10U \rceil + \lceil 10V \rceil$ do? (Recall that $\lceil x \rceil$ is the "ceiling" function.)

    (a) This gives a continuous Unif(0,20) random variate.
    (b) This gives a continuous triangular random variate.
    (c) This gives a normal random variate.
    (d) This is a simulated Dungeons and Dragons 10-sided die toss.
    (e) This simulates the sum of two Dungeons and Dragons 10-sided dice tosses.

    Solution: (e). $\Box$

15. If U_1, U_2, U_3 are i.i.d. Unif(0,1) random variables, what is the distribution of $-3\ell\text{n}(U_1(1-U_2)U_3)$?

    (a) $\text{Exp}(\lambda = 1/3)$
    (b) $\text{Exp}(\lambda = 3)$
    (c) $\text{Erlang}_3(\lambda = 1/3)$
    (d) $\text{Erlang}_3(\lambda = 1)$
    (e) $\text{Erlang}_3(\lambda = 3)$

    Solution: By symmetry of the Unif(0,1), we note that U_1, $1-U_2$, and U_3 are i.i.d. Unif(0,1). Therefore,

$$\begin{aligned} -3\ell\text{n}(U_1(1-U_2)U_3) &\sim -3\ell\text{n}(U_1) - 3\ell\text{n}(1-U_2) - 3\ell\text{n}(U_3) \\ &\sim -3\ell\text{n}(U_1) - 3\ell\text{n}(U_2) - 3\ell\text{n}(U_3) \\ &\sim \text{Exp}(1/3) + \text{Exp}(1/3) + \text{Exp}(1/3) \\ &\sim \text{Erlang}_3(1/3). \end{aligned}$$

So the answer is (c). $\Box$

16. Suppose X has the Nor(0,1) distribution with c.d.f. $\Phi(x)$. Suppose that U is a Unif(0,1) number. What is the distribution of the inverse $\Phi^{-1}(U)$?

    (a) Unif(0,1)

    (b) Exponential

    (c) Nor(0,1)

    (d) Weibull

    (e) None of the above.

    Solution: By the Inverse Transform Theorem, $\Phi(X) \sim U \sim \text{Unif}(0,1)$. So we have
    $$\Phi^{-1}(U) \ \sim \ \Phi^{-1}(\Phi(X)) \ = \ X \ \sim \ \text{Nor(0,1)}.$$
    (This is why Inverse Transform is so nice!) Thus, the answer is (c). $\Box$

17. If X is a continuous random variable with p.d.f. $f(x)$ and c.d.f. $F(x)$, find $\mathsf{E}[F(X)]$. (Hint: Don't panic on this problem. One approach might be to use Inverse Transform, or another might be to use LOTUS.)

    (a) e

    (b) $e - 1$

    (c) 1/2

    (d) 0.876

    (e) 1.876

    Solution: The answer turns out to be (c). Since it's Summertime Summertime in Atlanta and the livin' is easy, I'll give you *two* methods to prove this!

    Method (i): By Inverse Transform, $F(X) \sim$ Unif(0,1). Thus, if U denotes a Unif(0,1) random variable, we have
    $$\mathsf{E}[F(X)] \ = \ \mathsf{E}[U] \ = \ 1/2. \quad \Box$$

Method (ii): By LOTUS and the Chain Rule for integration,

$$\begin{aligned}
\mathsf{E}[F(X)] &= \int_{-\infty}^{\infty} F(x)f(x)\,dx \\
&= \frac{[F(x)]^2}{2}\Big|_{-\infty}^{\infty} \\
&= \frac{[F(\infty)]^2}{2} - \frac{[F(-\infty)]^2}{2} \\
&= \frac{1^2}{2} - \frac{0^2}{2} \;=\; 1/2. \quad \Box
\end{aligned}$$

18. Consider the linear congruential generator $X_{i+1} = (5X_i + 1)\text{mod}(8)$. Using $X_0 = 0$, calculate the 99th integer X_{99}.

    (a) 0

    (b) 1

    (c) 7

    (d) 8

    (e) 16

Solution: We have

i	0	1	2	3	4	5	6	7	8
X_i	0	1	6	7	4	5	2	3	0

So we see that the generator cycles every 8 PRN's, and then

$$X_{99} \;=\; X_{91} \;=\; \cdots \;=\; X_{11} \;=\; X_3 \;=\; 7.$$

Thus, the answer is (c). $\Box$

19. What does the following algorithm do?

Initialize X_0 (integer) and $i \leftarrow 1$
Repeat
 Set $X_i \leftarrow 16807 X_{i-1}\,\mathtt{mod}(2^{31}-1)$
 Set $U_i \leftarrow X_i/(2^{31}-1)$
 Set $i \leftarrow i+1$

(a) $X_1, X_2, \ldots$ is a sequence of integers that will eventually cycle.

(b) $U_1, U_2, \ldots$ is a sequence of PRNs.

(c) $U_1, U_2, \ldots$ will appear to be Unif(0,1).

(d) $U_1, U_2, \ldots$ will appear to be independent.

(e) All of the above.

Solution: (e). $\Box$

20. YES or NO? Is it possible for an arrival event to initiate insertions, deletions, *and* swaps of events in a simulation's future events list?

    Solution: YES — for instance, that arrival is deleted from the FEL; a subsequent arrival might be scheduled; some future events might be deleted; and some future events might be re-ordered. $\Box$

21. TRUE or FALSE? Deep, deep down in the heart of almost every discrete-event simulation language, there's a future events list that the simulation maintains in order to do event scheduling.

    Solution: TRUE. $\Box$

22. TRUE or FALSE? Most discrete-event simulations proceed by jumping from event to event in time order; and nothing of significant interest happens between events.

    Solution: TRUE. $\Box$

23. TRUE or FALSE? In an Arena `PROCESS` module, it is possible to do a `DELAY` *without* an accompanying `SEIZE` or `RELEASE`.

    Solution: TRUE. $\Box$

24. TRUE or FALSE? An Arena `DECIDE` module looks like a diamond.

Solution: TRUE. $\square$

25. Consider the differential equation $f'(x) = (x-3)f(x)$ with $f(0) = 2$. Use Euler's method with increment $h = 0.01$ to find the approximate value of $f(0.02)$.

    (a) 1
    (b) 1.88
    (c) 2
    (d) e
    (e) 3.72

Solution: As usual, we start with

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) \\ &= f(x) + h(x-3)f(x) \\ &= f(x)[1 + h(x-3)] \\ &= f(x)(0.97 + 0.01x), \end{aligned}$$

from which we obtain

$$f(0.01) \;=\; f(0)(0.97 + 0.01(0)) \;=\; 2(0.97) \;=\; 1.94$$

and then

$$f(0.02) \;=\; f(0.01)(0.97 + 0.01(0.01)) \;=\; 1.94(0.9701) \;=\; 1.88199.$$

Thus, the answer is (b). $\square$

26. Suppose that you want to estimate the integral

$$I \;=\; \int_0^1 [1 + e^{x^4}]\, dx$$

(which I don't think has a closed form).

Consider the following 4 Unif(0,1)'s:

0.42 0.11 0.73 0.89

Use the Monte Carlo method from class to approximate the integral I via the estimator $\bar{I}_4$.

(a) 1
(b) 1.881
(c) 2.308
(d) e
(e) 3.726

Solution:

$$\begin{aligned}\bar{I}_4 &= \frac{b-a}{n}\sum_{i=1}^{n} g(a+(b-a)U_i) \\ &= \frac{1-0}{4}\sum_{i=1}^{4} g(U_i) \\ &= \frac{1}{4}\sum_{i=1}^{4}[1+e^{U_i^4}] \\ &= 2.308.\end{aligned}$$

Thus, the answer is (c). □

27. Suppose I inscribe a circle of radius 1/2 in a unit square. Now I randomly toss 1000 darts in the square and 796 happen to land in the circle. Use this sample in conjunction with the example we did in class to give me an estimate of π.

(a) π
(b) $\pi/2$
(c) 3.082
(d) 3.184
(e) 3.248

Solution: Let $\hat{p}_n = 0.796$ be the proportion of darts that hit the circle. Then we know that the estimate $\hat{\pi}_n = 4\hat{p}_n = 3.184$. The answer is therefore (d). □

28. If X and Y are i.i.d. Unif(0,1), it turns out that the p.d.f. of the nasty joint random variable $W \equiv X/(X+Y)$ is somewhat interesting looking.

YES or NO? The expected value of this mess is $\mathsf{E}[W] = 1/2$.

Hint: There are various ways to do this problem — either

(i) Analytically (involving a cute trick), or

(ii) "Pretend" Monte Carlo: Select a reasonable grid of X and Y values; then average the resulting W values; and then make a good guess.

Good luck and have fun!

Solution: The answer is YES.

Proof: Hint (i) calls for a cute trick. Since X and Y are identically distributed, it follows that $\frac{X}{X+Y}$ and $\frac{Y}{X+Y}$ are identically distributed. So

$$1 = \mathsf{E}\left[\frac{X+Y}{X+Y}\right] = \mathsf{E}\left[\frac{X}{X+Y}\right] + \mathsf{E}\left[\frac{Y}{X+Y}\right] = 2\mathsf{E}\left[\frac{X}{X+Y}\right] = 2\mathsf{E}[W],$$

so that $\mathsf{E}[W] = 1/2$. $\square$

Another approach is to use Hint (ii)'s suggestion of "pretend" Monte Carlo. For instance, take X and Y equal to all combinations of the reasonable grid 0.2, 0.4, 0.6, 0.8, and then take the average of the resulting 16 values of W. Amazingly, you'll get an average of exactly 0.5! $\square$

29. Suppose that the probability that I make my first foul shot is 0.7. Also suppose that if I make shot i ($i = 1, 2, \ldots$), then I become very confident and will make shot $i+1$ with probability 0.9. However, if I miss shot i, then I become discouraged and make shot $i+1$ with probability of only 0.6.

Let's do Monte Carlo sampling to see how many shots I make. Suppose that I generously give you the following 10 Unif$(0,1)$ random numbers; call them $U_1, U_2, \ldots, U_{10}$:

0.83 0.16 0.95 0.47 0.37 0.65 0.77 0.20 0.14 0.13

Our simulation will declare that I make shot i if $U_i < p_i$, where p_i is the conditional probability that I make shot i (as discussed above). Using the given random numbers, how many shots will I have to take until I make my 4th one?

(a) 3

(b) 4

(c) 5

(d) 6

(e) 7

Solution: $p_1 = 0.7$, so $U_1 = 0.83$ corresponds to a miss.
Then $p_2 = 0.6$, so $U_2 = 0.16$ corresponds to a made shot (my 1st).
Then $p_3 = 0.9$, so $U_3 = 0.98$ corresponds to a miss.
Then $p_4 = 0.6$, so $U_4 = 0.47$ corresponds to a made shot (my 2nd).
Then $p_5 = 0.9$, so $U_5 = 0.37$ corresponds to a made shot (my 3rd).
Then $p_6 = 0.9$, so $U_6 = 0.65$ corresponds to a made shot (my 4th).

Thus, it took 6 shots, so the answer is (d). $\square$

30. Joey works at a chocolate store. Starting at time 0, we have the following 4 customer interarrival times (in minutes):

$$\text{Bill} = 8 \qquad \text{Tom} = 2 \qquad \text{Angie} = 5 \qquad \text{Ursula} = 2$$

Customers are served in *alphabetical order* (though once you start service, you don't get displaced by a higher-priority customer). The 4 customers order the following numbers of chocolate products, respectively:

$$6 \qquad 2 \qquad 3 \qquad 1$$

Suppose it takes Joey 3 minutes to prepare each chocolate product. Further suppose that he charges \$2/chocolate. Unfortunately, the customers are unruly and each customer causes \$0.50 in damage for every minute the customer has to wait in line.

When does the first customer leave?

(a) 8

(b) 18

(c) 26

(d) 41

(e) Not enough information to tell.

Solution: Consider the following table.

cust	arrl time	serv start	serve time	depart	wait	sys time
Bill (1)	8	8	18	26	0	18
Tom (2)	10	35	6	41	25	31
Angie (3)	15	26	9	35	11	20
Ursula (4)	17	41	3	44	24	27

The first customer (Bill) leaves at time 26. So the answer is (c). □

31. Under the same set-up as in Question 30, what is the average number of customers in the system during the first 20 minutes?

(a) 1.5

(b) 17/5

(c) 7

(d) 18

(e) Not enough information to tell.

Solution: Let X_i denote the amount of time Customer i spends in the system during the time interval [0,20]. In particular, $X_1 = 20 - 8 = 12$, $X_2 = 20 - 10 = 10$, $X_3 = 5$, and $X_4 = 3$. The average number of customers in the system during the first 20 minutes is

$$\frac{\text{total customer time}}{20} = \frac{12 + 10 + 5 + 3}{20} = 1.5.$$

Thus, the answer is (a). □

32. Under the same set-up as in Question 30, how much money will Joey make or lose with the above 4 customers?

(a) $-\$6.00$
(b) $\$0.00$
(c) $\$6.00$
(d) $\$36.00$
(e) Not enough information to tell.

Solution: Joey makes $2(6+2+3+1) - 0.5(0+25+11+24) = -\$6.00$. So the answer is (a). $\square$

33. Which incredible musical act was inducted into The Rock and Roll Hall of Fame this year?

(a) Justin Bieber
(b) REO Speedwagon
(c) REO Bieberwagon
(d) The Zombies

Solution: (d). Duh. $\square$