ISyE 6644 — Summer 2019 — Test #2 Solutions

(revised 10/26/20)

This test is 120 minutes. The first 22 questions are each worth 3 points. The following 17 questions (all Arena) are each worth 2 points.

Let's get some lawyer stuff out of the way…

You're allowed the following items:

- Pencil / pen and scratch paper.
- A reasonable calculator.
- *Two* cheat sheets (4 sides total).
- Normal, t, and χ^2 tables. (I will supply these.)

But note that

- You are not allowed to use Arena, even though I'm asking questions about it.
- This test requires some sort of proctor.
- If you encounter a ProctorTrack issue, contact us immediately (but don't get an ulcer over it).

Good luck! I want you to make this test sorry that it ever tried to mess around with you!!!

1. Consider the following joint p.m.f.

$f(x,y)$	$X=1$	$X=2$	$f_Y(y)$
$Y=0$	0.4	0.2	0.6
$Y=1$	0.1	0.3	0.4
$f_X(x)$	0.5	0.5	1.0

Let X represent how many times you watch Netflix in a week and let Y represent the number of times you miss your bus to work in a week (note that you always watch Netflix at least once).

YES or NO? Are watching Netflix and missing the bus independent?

Solution: Note that $f(1,0) = 0.4 \neq (0.5)(0.6) = f_X(1)f_Y(0)$. Thus, the answer is NO. $\Box$

2. Again consider the following joint p.m.f.

$f(x,y)$	$X=1$	$X=2$	$f_Y(y)$
$Y=0$	0.4	0.2	0.6
$Y=1$	0.1	0.3	0.4
$f_X(x)$	0.5	0.5	1.0

Find $\mathsf{Cov}(X,Y)$.

(a) -1

(b) -0.1

(c) 0

(d) 0.1

(e) 2

Solution: The table immediately gives us

$$\begin{aligned}
\mathsf{E}[X] &= \sum_x x f_X(x) \;=\; 1(0.5)+2(0.5) \;=\; 1.5 \\
\mathsf{E}[Y] &= \sum_y y f_Y(y) \;=\; 0(0.6)+1(0.4) \;=\; 0.4 \\
\mathsf{E}[XY] &= \sum_x \sum_y xyf(x,y) \\
&= 1(0)(0.4)+2(0)(0.2)+1(1)(0.1)+2(1)(0.3) \;=\; 0.7.
\end{aligned}$$

Thus,

$$\mathsf{Cov}(X,Y) \;=\; \mathsf{E}[XY]-\mathsf{E}[X]\mathsf{E}[Y] \;=\; 0.7-(1.5)(0.4) \;=\; 0.1.$$

So the answer is (d). □

3. TRUE or FALSE? Most discrete-event simulations proceed by moving the simulation clock to the most-computationally-intensive event on the future events list, i.e., the event that takes the "most work" to deal with; executing that event (including any adds, deletes, or swaps to the FEL); and then repeating this cycle.

Solution: FALSE. The simulation clock moves to the most-*imminent* event. □

4. Consider the PRN generator $X_{i+1} = (aX_i + 5)\text{mod}(b)$. Using $X_0 = 0$, which values of a and b yield a cycle length of 4?

   (a) $a = 1, b = 2$

   (b) $a = 3, b = 4$

   (c) $a = 3, b = 6$

   (d) $a = 3, b = 8$

   Solution: By inspection (plugging the values in), we find that (d) is the only answer that works. □

5. Consider our desert island generator $X_{i+1} = 16807\,X_i\,\text{mod}(2^{31} - 1)$. If $X_0 = 1234321$, find the value of X_1.

   (a) 0

   (b) 0.61359

   (c) 25,152,971

   (d) 116,194,254

   (e) 1,417,880,224

   (f) $2^{31} - 582$

   Solution: By hook or by crook (e.g., by the algorithm given in the notes), we find that $X_1 =$ 1,417,880,224, so that the answer is (e). □

6. What is (0 `XOR` 1) `XOR` (1 `XOR` 1)?

   (a) 0

   (b) 1

   (c) 2

   (d) 4

   Solution: (0 `XOR` 1) `XOR` (1 `XOR` 1) = 1 `XOR` 0 = 1. So the answer is (b). □

7. Consider a Tausworthe generator with $r = 1$, $q = 3$, $B_1 = 1$, $B_2 = 1$, and $B_3 = 1$. Find B_{39}.

   (a) 0

   (b) 1

   (c) 2

   (d) 4

   Solution: $B_i = B_{i-r} \,\texttt{XOR}\, B_{i-q} = B_{i-1} \,\texttt{XOR}\, B_{i-3}$. Thus,

$$\begin{aligned}
B_4 &= B_3 \,\texttt{XOR}\, B_1 = 1 \,\texttt{XOR}\, 1 = 0 \\
B_5 &= B_4 \,\texttt{XOR}\, B_2 = 0 \,\texttt{XOR}\, 1 = 1 \\
B_6 &= B_5 \,\texttt{XOR}\, B_3 = 1 \,\texttt{XOR}\, 1 = 0 \\
B_7 &= B_6 \,\texttt{XOR}\, B_4 = 0 \,\texttt{XOR}\, 0 = 0 \\
B_8 &= B_7 \,\texttt{XOR}\, B_5 = 0 \,\texttt{XOR}\, 1 = 1 \\
B_9 &= B_8 \,\texttt{XOR}\, B_6 = 1 \,\texttt{XOR}\, 0 = 1 \\
B_{10} &= B_9 \,\texttt{XOR}\, B_7 = 1 \,\texttt{XOR}\, 0 = 1
\end{aligned}$$

   So we see that the thing starts a new cycle at B_8 (which makes sense in light of a theorem in the notes that says that the period is of length $2^q - 1 = 7$). Thus, $B_{39} = B_{32} = \cdots = B_4 = 0$; and so the answer is (a). □

8. TRUE or FALSE? There are various PRN generators out there having periods greater than 2^{19900}, but since those periods are so big we can't really use such generators at all.

   Solution: FALSE. The period only describes how long it takes for the generator to cycle, but doesn't have an effect on generating the PRNs or the quality of the PRNs. □

9. Suppose that we have a sequence of $n = 100$ PRN's, and we observe 80 runs up and down. Using $\alpha = 0.05$, do we ACCEPT (i.e., fail to reject) or REJECT the null hypothesis of independence?

Solution: By class notes, under the null hypothesis of independence, $A \approx \text{Nor}(\frac{2n-1}{3}, \frac{16n-29}{90}) \sim \text{Nor}(66.33, 17.46)$. Thus, the test statistic is

$$Z_0 = \frac{A - \mathsf{E}[A]}{\sqrt{\mathsf{Var}(A)}} = \frac{80 - 66.33}{\sqrt{17.46}} = 3.27.$$

Since $|Z_0| > z_{\alpha/2} = 1.96$, we REJECT H_0. In other words, we conclude that they're *not independent.* □

10. Suppose we sample 1000 PRN's and we wish to conduct a χ^2 goodness-of-fit test at level $\alpha = 0.10$ of the hypothesis that the numbers are Unif(0,1). Here are the results, divided into 4 intervals (convenient chose to be equal-probability intervals under H_0):

interval	O_i
$[0.00, 0.25]$	275
$(0.25, 0.50]$	250
$(0.50, 0.75]$	250
$(0.75, 1.00]$	??

Use the tabled results to find the value of the g-o-f statistic, χ^2_0.

(a) 0

(b) 5

(c) 25

(d) 125

(e) 250

Solution: We begin with a trivial completion of the table (to take into account the missing "??").

interval	O_i
$[0.00, 0.25]$	275
$(0.25, 0.50]$	250
$(0.50, 0.75]$	250
$(0.75, 1.00]$	225

Since we have $k = 4$ equiprobable intervals, we obtain $E_i = n/k = 250$ for all i. Thus,

$$\chi_0^2 \;=\; \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \;=\; \sum_{i=1}^{4} \frac{(O_i - 250)^2}{250} \;=\; 5.0,$$

and so the answer is (b). $\Box$

11. Similar to Problem 10, let's do a χ^2 test for uniformity with $n = 1000$ PRN's, $k = 4$ equiprobability intervals, and level $\alpha = 0.10$. But now suppose it turns out that $\chi_0^2 = 10$ (instead of whatever answer you got before). Do we ACCEPT (i.e., fail to reject) or REJECT the null hypothesis of uniformity?

    Solution: First of all, the appropriate test quantile is $\chi^2_{\alpha,k-1} = \chi^2_{0.10,3} = 6.25$. Since $\chi_0^2 > \chi^2_{\alpha,k-1}$, we REJECT uniformity. $\Box$

12. Suppose the random variable X has p.d.f. $f(x) = 2(x-1)$ for $1 \le x \le 2$. Find the inverse of its c.d.f., i.e., $F^{-1}(U)$, where U is a PRN.

    (a) $U^{1/2} + 1$
    (b) $2(X-2)^2$
    (c) $2(U-2)^2$
    (d) $(X-1)^3$
    (e) $(U-1)^3$

    Solution: After a little algebra, we can calculate the c.d.f.,

    $$F(x) \;=\; \int_1^x 2(t-1)\,dt \;=\; (x-1)^2, \quad \text{for } 1 \le x \le 2.$$

    Set $F(X) = (X-1)^2 = U$. Solving, we have $X = F^{-1}(U) = U^{1/2} + 1$. So the answer is (a). $\Box$

13. You are conducting a study on the distribution of sales prices for donut-making machines, and you start things off by using the *inverse transform method* with the PRN $U = 0.27$ to generate a realization of $Z = \Phi^{-1}(U) \sim \text{Nor}(0,1)$. (You'll need the normal tables for this.) Suppose your co-worker then applies a linear

transformation $X = \mu + \sigma Z$ to your observation which results in a final donut machine price of $X = 3.37$.

You need to continue your co-worker's research when he is out of the office, but unfortunately he spilled coffee on his worksheet! All you can see through the smudged mess is that the mean that he used in his transformation was $\mu = 4.6$. What was the approximate standard deviation σ that he used for the transformation?

(a) -2
(b) 2.01
(c) 3.37
(d) 4
(e) 4.6

Solution: We take

$$X \;=\; \mu + \sigma Z \;=\; 4.6 + \sigma\Phi^{-1}(0.27) \;=\; 4.6 - 0.6128\sigma \;=\; 3.37.$$

Solving, we obtain $\sigma \approx 2.01$. Thus, the answer is (b). $\square$

14. Suppose that X has the Beta(α, β) distribution with (messy) c.d.f.

$$F(x) \;=\; \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt, \quad \text{for } 0 \le x \le 1.$$

What is the mean of the random variable $3F(X) - 1$?

(a) $\frac{\alpha}{\alpha+\beta}$
(b) $\frac{3\alpha}{\alpha+\beta} - 1$
(c) 0
(d) $\frac{1}{2}$
(e) $\frac{3}{2}$

Solution: By the Inverse Transform Theorem, $F(X) \sim \text{Unif}(0,1)$. Thus,

$$3F(X) - 1 \;\sim\; \text{Unif}(0,3) - 1 \;\sim\; \text{Unif}(-1,2).$$

Recall that the mean of a Unif(a,b) distribution is simply $\frac{a+b}{2}$. Here, $a = -1$ and $b = 2$; thus the mean is $1/2$. So the answer is (d). $\square$

15. Let's toss a pair of dice and look at the sum. The number of trials X it takes until I see a sum of 2 is easily shown to be a Geom(1/36) random variable. Use the PRN $U = 0.75$ to generate X via inverse transform. [There are actually two related inverse transform answers (depending on whether you use U or $1 - U$), but only one of them appears in the list of choices below.]

    (a) 1
    (b) 27
    (c) 36
    (d) 50
    (e) 72

    Solution: We have

    $$X = \left\lceil \frac{\ell\text{n}(1-U)}{\ell\text{n}(1-p)} \right\rceil = \left\lceil \frac{\ell\text{n}(0.25)}{\ell\text{n}(35/36)} \right\rceil = \lceil 49.21 \rceil = 50.$$

    This is choice (d). $\Box$

    The other obvious possible answer is

    $$\lceil \ell\text{n}(U)/\ell\text{n}(1-p) \rceil = \lceil \ell\text{n}(0.75)/\ell\text{n}(35/36) \rceil = 11,$$

    but this answer did not appear in the list of choices that I offered. $\Box$

16. Recall that the Box–Muller method takes two i.i.d. Unif(0,1) PRNs, U_1 and U_2, and magically transforms them into two i.i.d. Nor(0,1)'s, Z_1 and Z_2. Specifically,

    $$Z_1 = \sqrt{-2\ell\text{n}(U_1)}\cos(2\pi U_2) \quad \text{and} \quad Z_2 = \sqrt{-2\ell\text{n}(U_1)}\sin(2\pi U_2).$$

    It also turns out (I think I mentioned this in class) that $T = Z_2/Z_1$ has the *Cauchy distribution*. Using the B-M expressions for Z_1 and Z_2 above, find a nice simple expression for the Cauchy random variable T.

    (a) $\sqrt{-2\ell\text{n}(U_1)}\cos(2\pi U_2)$
    (b) $\sqrt{-2\ell\text{n}(U_1)}\sin(2\pi U_2)$
    (c) $\sqrt{-2\ell\text{n}(U_1)}$
    (d) $\sin(2\pi U_2)$

(e) $\tan(2\pi U_2)$

Solution: Using elementary trigonometry, we see that

$$T \;=\; \frac{Z_2}{Z_1} \;=\; \frac{\sqrt{-2\ell\mathrm{n}(U_1)}\,\sin(2\pi U_2)}{\sqrt{-2\ell\mathrm{n}(U_1)}\,\cos(2\pi U_2)} \;=\; \frac{\sin(2\pi U_2)}{\cos(2\pi U_2)} \;=\; \tan(2\pi U_2),$$

which is (e). $\square$

17. If U_1 and U_2 are i.i.d. Unif(0,1), find the distribution of $U_1 - U_2$.

(a) Unif(0,1)
(b) Unif(0,2)
(c) Unif$(-1,1)$
(d) Tria(0,1,2)
(e) Tria$(-1,0,1)$
(f) Tria$(-2,-1,0)$

Solution: Since U and $1-U$ are both Unif(0,1), we have

$$U_1 - U_2 \;\sim\; U_1 - (1-U_2) \;=\; U_1 + U_2 - 1 \;\sim\; \text{Tria}(0,1,2) - 1 \;\sim\; \text{Tria}(-1,0,1).$$

This is choice (e). $\square$

18. Consider 48 PRNs, $U_1, U_2, \ldots, U_{48}$, and suppose that $\sum_{i=1}^{48} U_i = 24.8$. Use these PRNs to generate a single approximately Nor(0,1) random variate via our "desert island" technique.

(a) -0.8
(b) 0.4
(c) 0.8
(d) 18.8
(e) 24.8

Solution: From our class notes, we have

$$Z = \frac{\sum_{i=1}^n U_i - \frac{n}{2}}{\sqrt{n/12}} = \frac{24.8 - 24}{\sqrt{48/12}} = 0.4.$$

Thus, the answer is (b). $\square$

19. Suppose that U is a PRN and $\Phi(x)$ is the Nor(0,1) c.d.f. Find $\Pr(\Phi^{-1}(U) \leq 0)$.

    (a) 0

    (b) 0.25

    (c) 0.5

    (d) 0.9

    (e) 0.95

Solution: By Inverse Transform, $\Phi^{-1}(U)$ is Nor(0,1). Thus,

$$\Pr(\Phi^{-1}(U) \leq 0) = \Pr(\text{Nor}(0,1) \leq 0) = \Phi(0) = 0.5.$$

This is (c). $\square$

20. Suppose that U_1 and U_2 are i.i.d. Unif(0,1). Find $\Pr(\ell\text{n}(U_1U_2) > -2)$. Do not panic.... You know how to do this!

    (a) $1/e^2$

    (b) $1 - (1/e^2)$

    (c) 0.5

    (d) 0.406

    (e) 0.594

Solution: Note that

$$\begin{aligned}
\Pr(\ell\text{n}(U_1U_2) > -2) &= \Pr(-\ell\text{n}(U_1U_2) < 2) \\
&= \Pr(-\ell\text{n}(U_1) - \ell\text{n}(U_2) < 2) \\
&= \Pr(\text{Exp}(1) + \text{Exp}(1) < 2) \\
&= \Pr(\text{Erlang}_2(1) < 2)
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \sum_{i=0}^{k-1} \frac{e^{-\lambda t}(\lambda t)^i}{i!} \\
&= 1 - \sum_{i=0}^{1} \frac{e^{-(1)(2)}[(1)(2)]^i}{i!} \\
&= 1 - 3e^{-2} \;=\; 0.594.
\end{aligned}
$$

So the answer is (e). $\square$

21. Suppose that $X \sim \text{Pois}(\lambda = 0.2)$. I wouldn't usually use *inverse transform* to generate a Poisson, but if λ is pretty small, it ought to work just fine because X is likely to be small. So that's what we're gonna do here — use inverse transform to generate X. Let me be a really nice guy and write out the p.m.f. / c.d.f. table for you.

x	$f(x) = e^{-\lambda}\lambda^x/x!$	$F(x)$	Unif(0,1)'s
0	0.8187	0.8187	$[0, 0.8187]$
1	0.1637	0.9824	$(0.8187, 0.9824]$
2	0.0164	0.9988	$(0.9824, 0.9988]$
3	0.0011	0.9999	$(0.9988, 0.9999]$
≥ 4	0.0001	1.0000	$(0.9999, 1.0000]$

Notice that I'm not very concerned with the "≥ 4" case. Anyway, suppose that I draw a PRN $U = 0.726$. What value of X do I get?

(a) 0

(b) 0.726

(c) 1

(d) 2

(e) 3

Solution: From the table, we see that $U = 0.726$ clearly corresponds to $X = 0$, choice (a). $\square$

22. Not a bonus, but almost surely free points — no excuses, no partial credit this time! What is the only possible choice?

(a) Dinner with that *Australopithecus* Justin Bieber

(b) A worthless Ph.D. in Animal Husbandry from UGA

(c) A gorgeous outdoor concert in the cool delicate breeze of a sublime Atlanta evening with The Zombies

Solution: (c). □

And now, several Arena questions...

23. TRUE or FALSE? In Arena, a `DECIDE` module can route customers probabilistically or conditionally to multiple locations.

    Solution: TRUE. □

24. TRUE or FALSE? In Arena, a primitive `QUEUE` block can connect with a `PROCESS` module.

    Solution: FALSE. (This was discussed during the Call Center example — it's a bit surprising!) □

25. TRUE or FALSE? In Arena, you can pre-assign a service time as an attribute before you actually get to the server that you'll be using.

    Solution: TRUE. (Sure, as discussed in the Electronic Assembly system example.) □

26. TRUE or FALSE? In Arena, it's perfectly OK to have a `CREATE` module generating one-at-a-time customer arrivals, while also having a *different* `CREATE` module with 4-at-a-time customer arrivals elsewhere in the model.

    Solution: TRUE. (This was discussed in the Electronic Assembly system lesson.) □

27. TRUE or FALSE? In Arena, "fake" customers can be used to schedule machine breakdowns, keep track of which time period the simulation is currently in, and carry out other duties not associated with actual customers.

Solution: TRUE. $\Box$

28. What is the expected value of the Arena expression `EXPO(DISC(0.5,2,1,3))`?

(a) 0.5
(b) 1
(c) 1.5
(d) 2
(e) 2.5
(f) 3

Solution: Since `DISC(0.5,2,1,3)` evaluates to 2 or 3, each with probability 0.5, we have

$$\mathsf{E}\Big[\mathtt{EXPO}(\mathtt{DISC}(0.5,2,1,3))\Big] = \frac{\mathsf{E}[\mathtt{EXPO}(2)] + \mathsf{E}[\mathtt{EXPO}(3)]}{2} = \frac{2+3}{2} = 2.5.$$

So the answer is (e). $\Box$

29. Suppose there are 7 people in the line called `joe.queue` and 2 people in the line called `tom.queue`. What is the value of the following Arena expression?

$$(\mathtt{NQ}(\mathtt{tom.queue}) >= \mathtt{NQ}(\mathtt{joe.queue})) + (\mathtt{TNOW} > -1)$$

(a) -1
(b) 0
(c) 1
(d) 2
(e) 3
(f) 7

Solution: The logical expression `(NQ(tom.queue) >= NQ(joe.queue))` is false (since $2 < 7$), and the logical expression `(TNOW > -1)` is true (since time is always at least 0). Thus,

$$(\texttt{NQ}(\texttt{tom.queue}) >= \texttt{NQ}(\texttt{joe.queue})) + (\texttt{TNOW} > -1) \ = \ 0 + 1 \ = \ 1.$$

So the answer is (c). □

30. TRUE or FALSE? In Arena, the `SEIZE`, `DELAY`, `RELEASE` functionalities are in multiple templates.

    Solution: TRUE. (This is primarily because the `PROCESS` module does not allow us to use `SEIZE`, `DELAY`, `RELEASE` with sufficient generality.) □

31. TRUE or FALSE? In Arena, you can schedule failures to occur after a random number of customers have used a resource or after a random amount of time has passed.

    Solution: TRUE. □

32. TRUE or FALSE? In Arena, you are only allowed to schedule one type of failure for a particular resource.

    Solution: FALSE. In Arena, you can use multiple rows of the Failures column of the Resource spreadsheet to schedule several types of failures. □

33. TRUE or FALSE? In Arena, it is possible for the resource Joey to be a member of three different resource sets.

    Solution: TRUE. (Joey likes to join a lot of clubs!) □

34. TRUE or FALSE? For easy display and organizational purposes, our Arena Call Center simulation is divided into several submodels.

    Solution: TRUE. □

35. TRUE or FALSE? The customer arrival pattern for our Arena Call Center simulation changes over the day and is therefore modeled as a nonhomogeneous Poisson process.

Solution: TRUE. □

36. TRUE or FALSE? All of the Tech Support servers for our Arena Call Center simulation are equally skilled and have the same concurrent work schedules.

Solution: FALSE. Some of the workers are more talented than others (e.g., can work on multiple products). And all of the servers have different schedules. □

37. In Arena, what kind of module would you use to change an entity's picture?

   (a) `RECORD`
   (b) `SET`
   (c) `ASSIGN`
   (d) `TALLY`
   (e) `PICTURE`

Solution: (c) `ASSIGN`. □

38. In Arena, where would you find the `Expression` spreadsheet?

   (a) `Basic Process` panel
   (b) `Advanced Process` panel
   (c) `Basic Transfer` panel
   (d) `Advanced Transfer` panel

Solution: (b) `Advanced Process` panel. □

39. Consider the Arena Inventory model that we did in class as a demo. Did this model allow for backlogs? YES or NO?

Solution: YES. □