

MATH 221 Review

Scope

- 2.1 ~ 3.10(at most)
- actually I don't know
- $\rightarrow 2.1 \sim x$
- ask your friends
- But this doc is going to focus on 2.1~3.10
- which is sufficient in any cases

Qualification of Writing this Document(The reason why you might want to take my words seriously)

- Experience of 100 on this exam
- And many exams afterwards in this course and following mathematic course

What's the core of achieving high score

Characteristic of exams in MATH 221

- Too easy problems
 - pupils in middle school with prefix knowledge can solve all problems in this exam (midterm and final)
 - difficulty seems to be consistent in all years
- Some investigation on definition
 - some **details** are required to know

Strategies on this exam

- Book
 - you will look at it in days before exam
- Ordinary everyday problem solving
 - no way to save it right now
 - if you don't take it seriously, you've already missed the chance of thinking of certain types of problems
 - **but I believe you did a good job in the past few weeks even if you didn't realize it, so relax**

Why do you want to review

To do one thing, you have to think the reason to do it.

So you can see your target clearly.

1. Re-solve and reflect on some problems that might be crucial.
2. Re-gain and review some knowledge about literals, that is definition and formulae etc., in textbook.

Book oriented review

2.1 Tangent and Velocity

- Use the definition of tangent line to solve problem
 - why are you so crazy about taking derivative and L's Hospital Law
 - **use the definition!!!**

2.2 Limit of Function

In general, we use the following notation.

1 Intuitive Definition of a Limit Suppose $f(x)$ is defined when x is near the number a . (This means that f is defined on some open interval that contains a , except possibly at a itself.) Then we write

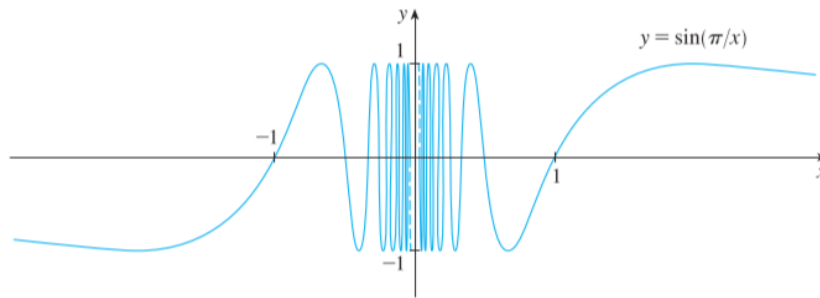
$$\lim_{x \rightarrow a} f(x) = L$$

and say “the limit of $f(x)$, as x approaches a , equals L ”

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a .

- see the last line **but not equal to a**
- oscillating curve

❌ but this time **our guess is wrong**. Note that although $f(1/n) = \sin n\pi = 0$ for any integer n , it is also true that $f(x) = 1$ for infinitely many values of x (such as $2/5$ or $2/101$) that approach 0. You can see this from the graph of f shown in Figure 7.



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SECTION 2.2 The Limit of a Function

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The dashed lines near the y -axis indicate that the values of $\sin(\pi/x)$ oscillate between 1 and -1 infinitely often as x approaches 0. (See Exercise 51.)

Since the values of $f(x)$ do not approach a fixed number as x approaches 0,

$$\lim_{x \rightarrow 0} \sin \frac{\pi}{x} \text{ does not exist}$$

■

- one side limit

2 Definition of One-Sided Limits We write

$$\lim_{x \rightarrow a^-} f(x) = L$$

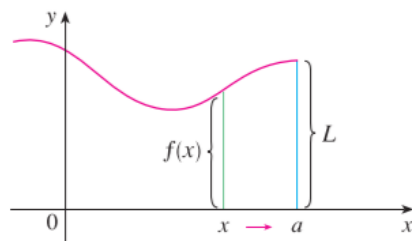
and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x less than a .

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than a . Similarly, if we require that x be greater than a , we get “the **right-hand limit of $f(x)$ as x approaches a** is equal to L ” and we write

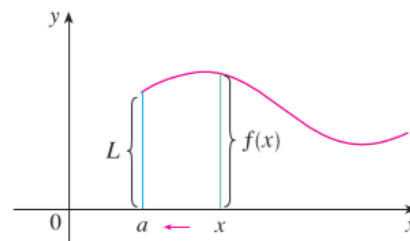
$$\lim_{x \rightarrow a^+} f(x) = L$$

Thus the notation $x \rightarrow a^+$ means that we consider only x greater than a . These definitions are illustrated in Figure 9.

o



(a) $\lim_{x \rightarrow a^-} f(x) = L$



(b) $\lim_{x \rightarrow a^+} f(x) = L$

By comparing Definition 1 with the definitions of one-sided limits, we see that the following is true.

3 $\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^-} f(x) = L$ and $\lim_{x \rightarrow a^+} f(x) = L$

o this connect left and right hand limit to the general definition of limit

•

4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a , except possibly at a itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a , but not equal to a .

o if you can evaluate the limit to ∞ or $-\infty$ then do **NOT** write *DNE*

- this is very very very **important**. I see 50% of you still make this kind of mistake in worksheet.

One important remark

Note that each definition about limit, there is one sentence that's commonly appeared.

except possibly at a itself

This means limit isn't about $f(a)$ but the **points in vicinity to $(a, f(a))$*

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$, provided the limits exist.

- vertical asymptote

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

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$$\begin{array}{lll} \lim_{x \rightarrow a} f(x) = \infty & \lim_{x \rightarrow a^-} f(x) = \infty & \lim_{x \rightarrow a^+} f(x) = \infty \\ \lim_{x \rightarrow a} f(x) = -\infty & \lim_{x \rightarrow a^-} f(x) = -\infty & \lim_{x \rightarrow a^+} f(x) = -\infty \end{array}$$

2.3 Limit laws

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \rightarrow a} f(x) \quad \text{and} \quad \lim_{x \rightarrow a} g(x)$$

exist. Then

$$1. \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$2. \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

$$3. \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$$

$$4. \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$5. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0$$

- the prerequisite is really important
 - $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ must exist
 - most of you didn't notice that

$$6. \lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n \quad \text{where } n \text{ is a positive integer}$$

In applying these six limit laws, we need to use two special limits:

$$7. \lim_{x \rightarrow a} c = c$$

$$8. \lim_{x \rightarrow a} x = a$$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$), but proofs based on the precise definition are requested in the exercises for Section 2.4.

If we now put $f(x) = x$ in Law 6 and use Law 8, we get another useful special limit.

$$9. \lim_{x \rightarrow a} x^n = a^n \quad \text{where } n \text{ is a positive integer}$$

A similar limit holds for roots as follows. (For square roots the proof is outlined in Exercise 2.4.37.)

$$10. \lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a} \quad \text{where } n \text{ is a positive integer}$$

(If n is even, we assume that $a > 0$.)

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SECTION 2.3 Calculating Limits Using the Limit Laws

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More generally, we have the following law, which is proved in Section 2.5 as a consequence of Law 10.

Root Law

$$11. \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \quad \text{where } n \text{ is a positive integer}$$

[If n is even, we assume that $\lim_{x \rightarrow a} f(x) > 0$.]

- **You should know Power Law and Root Law is.**
- One trick is almost all the structure can be evaluated using limit law.
 - But this trick can have some damage on your problem-solving procedure.
 - Why?
 - Because you might implicit use the law without specifying it.
 - Specifying it doesn't **only** mean you should write the name of limit law
 - You must state the prerequisite clearly!

Direct Substitution Property

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

- See prerequisite
 - polynomial and rational function

How to compute the limit?

$$\boxed{1} \text{ Theorem } \lim_{x \rightarrow a} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x)$$

Usually you just need to compute a limit directly

But if the function is like

$$\bullet \quad f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- you can use this theorem

Squeeze Theorem

2 Theorem If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a , then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$$

3 The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L$$

2.4 Precise definition of limit

- only you only have to know how to get δ
- More info please refer to $\epsilon - \delta$ definition document I gave it to you.

2.5 Continuity

1 Definition A function f is **continuous at a number a** if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Notice that Definition 1 implicitly requires three things if f is continuous at a :

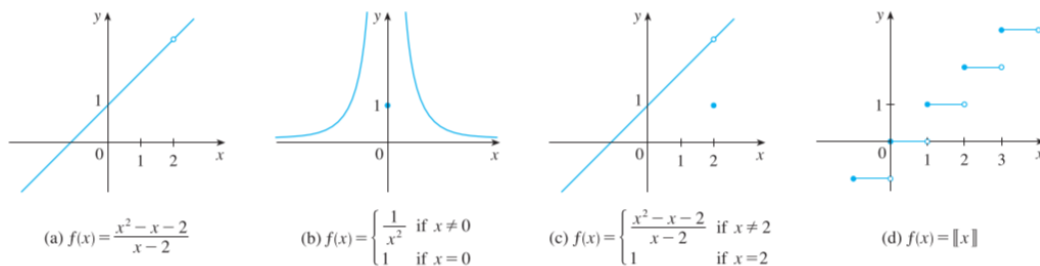
1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \rightarrow a} f(x)$ exists
3. $\lim_{x \rightarrow a} f(x) = f(a)$

This is **point continuity**(made up name)

Examples on discontinuity

Exercise 2.3.53.)

Figure 3 shows the graphs of the functions in Example 2. In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.] The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function "jumps" from one value to another.



Partial continuity (made up name)

2 Definition A function f is **continuous from the right at a number a** if

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x \rightarrow a^-} f(x) = f(a)$$

Interval continuity (made up name)

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

Composition of continuity

4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a :

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\frac{f}{g}$ if $g(a) \neq 0$

Function types of continuity

5 Theorem

- (a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.
- (b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials 多项式
- rational functions
- root functions
- trigonometric functions 三角函数
- inverse trigonometric functions 反三角函数
- exponential functions
- logarithmic functions

Continuity chain rule (made up name)

8 Theorem If f is continuous at b and $\lim_{x \rightarrow a} g(x) = b$, then $\lim_{x \rightarrow a} f(g(x)) = f(b)$.
In other words,

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right)$$

Notation:

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a .

The intermediate Value Theorem

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Very Very Important

Although it's not recommended to use heading to stress something since it's against markdown style, I still do it since it's very very important.

- Think about how you can construct functions to use this theorem.

2.6 Limits at infinity asymptotes

- horizontal asymptotes can only extend to infinity
 - think about why

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L$$

Examples

5 Theorem If $r > 0$ is a rational number, then

$$\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x , then

$$\lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0$$

$$\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2} \quad \lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$$

EXAMPLE 10 Find $\lim_{x \rightarrow \infty} (x^2 - x)$.

 **SOLUTION** It would be **wrong** to write

$$\lim_{x \rightarrow \infty} (x^2 - x) = \lim_{x \rightarrow \infty} x^2 - \lim_{x \rightarrow \infty} x = \infty - \infty$$

The Limit Laws can't be applied to infinite limits because ∞ is not a number ($\infty - \infty$ can't be defined). However, we *can* write

$$\lim_{x \rightarrow \infty} (x^2 - x) = \lim_{x \rightarrow \infty} x(x - 1) = \infty$$

because both x and $x - 1$ become arbitrarily large and so their product does too. 

2.7 Derivatives and Rates of Change

1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

4 Definition The **derivative of a function f at a number a** , denoted by $f'(a)$, is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

... f(x+h) - f(x) ...

2.8 The derivatives as a function

2

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

Given any number x for which this limit exists, we assign to x the number $f'(x)$. So we can regard f' as a new function, called the **derivative of f** and defined by Equation 2. We know that the value of f' at x , $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

Differentiable

3 Definition A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval (a, b)** [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

- note that you can only say some function is differentiable in an interval
- such as its domain
 - which might have gap
 - like $[a, x), (x, b]$

Differentiable and Continuous

4 Theorem If f is differentiable at a , then f is continuous at a .

NOTE The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} |x| = 0 = f(0)$$

(See Example 2.3.7.) But in Example 5 we showed that f is not differentiable at 0.

3.1 Derivatives of Polynomials and Exp function

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c \frac{d}{dx}f(x)$$

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

The Constant Multiple Rule, the Sum Rule, and the Difference Rule can be combined with the Power Rule to differentiate any polynomial, as the following examples demonstrate.

Definition of the Number e

$$e \text{ is the number such that } \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

3.2 Product and Quotient Rule

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)]$$

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx}[f(x)] - f(x) \frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf'$$

$$(f + g)' = f' + g'$$

$$(f - g)' = f' - g'$$

$$(fg)' = fg' + gf'$$

$$\left(\frac{f}{g} \right)' = \frac{gf' - fg'}{g^2}$$

3.3 Derivatives of Trigonometric Functions

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0$$

- only use these two equations
- don't make something up
 - transform \tan to \sin and \cos
- There will be hint to inform you not to use L's Hospital Rule
 - don't use it if the explicit warning are given to you

Derivatives of Trigonometric Functions

$$\frac{d}{dx} (\sin x) = \cos x$$

$$\frac{d}{dx} (\csc x) = -\csc x \cot x$$

$$\frac{d}{dx} (\cos x) = -\sin x$$

$$\frac{d}{dx} (\sec x) = \sec x \tan x$$

$$\frac{d}{dx} (\tan x) = \sec^2 x$$

$$\frac{d}{dx} (\cot x) = -\csc^2 x$$

- note the derivative of \sec and \tan
- \csc and \cot is similar to \sec and \tan respectively
 - the only difference is the minus sign ("−") before the formulae

3.4 The Chain Rule

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

The Chain Rule can be written either in the prime notation

2

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or, if $y = f(u)$ and $u = g(x)$, in Leibniz notation:

3

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

- don't write dx instead of Δx and dy instead of Δy
- don't think this part is trivial since I wrote just few words here
 - this is the most important part in this chapter
 - go through the text book carefully
- One another important thing about Leibniz notation
 - $f''(x)$ is $\frac{d(\frac{dy}{dx})}{dx}$
 - it equals to
 - $\frac{d^2y}{dx^2}$
 - **NOT**
 - $\frac{d^2y}{d^2x}$
 - **NOR**
 - $\frac{dy^2}{dx^2}$
 - OR any other forms
 - I may explain it next session if you don't understand

3.5 Implicit Differentiation

- treat a variable as x and choose another one as $f(x)$
- take derivative on both sides
- watch the domain carefully
 - **don't change the structure of the equation easily**
 - at least Prof don't think that's a good idea
 - and standard solution often don't change the structure
 - meaning you're more likely to get grades if you don't change it
- The most of the time you can't represent y with x (for example), then don't, just leave them there.

- $y = \pm \sqrt{25 - x^2}$.

- when this happened don't substitute y **Check this!**

-

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \quad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

- I don't think you will need to remember these
- I have to check this with prof

3.6 Derivative of Log function

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

EXAMPLE 7 Differentiate $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln(x^2 + 1) - 5 \ln(3x + 2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y} \frac{dy}{dx} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2 + 1} - 5 \cdot \frac{3}{3x + 2}$$

Solving for dy/dx , we get

$$\frac{dy}{dx} = y \left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2} \right)$$

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mic differen- Because we have an explicit expression for y , we can substitute and write
would have

ent Rule
e resulting
zen

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5} \left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2} \right)$$



Another definition of e

■ The Number e as a Limit

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus $f'(1) = 1$. We now use this to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \rightarrow 0} \frac{f(1+x) - f(1)}{x} \\ &= \lim_{x \rightarrow 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \rightarrow 0} \frac{1}{x} \ln(1+x) \\ &= \lim_{x \rightarrow 0} \ln(1+x)^{1/x} \end{aligned}$$

Because $f'(1) = 1$, we have

$$\lim_{x \rightarrow 0} \ln(1+x)^{1/x} = 1$$

Then, by Theorem 2.5.8 and the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \rightarrow 0} \ln(1+x)^{1/x}} = \lim_{x \rightarrow 0} e^{\ln(1+x)^{1/x}} = \lim_{x \rightarrow 0} (1+x)^{1/x}$$

5

$$e = \lim_{x \rightarrow 0} (1+x)^{1/x}$$

Formula 5 is illustrated by the graph of the function $y = (1+x)^{1/x}$ in Figure 4 and a table of values for small values of x . This illustrates the fact that, correct to seven decimal places,

$$e \approx 2.7182818$$

If we put $n = 1/x$ in Formula 5, then $n \rightarrow \infty$ as $x \rightarrow 0^+$ and so an alternative expression for e is

6

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

3.8 ~3.9

- application
- I think that's pretty straightforward
 - **once you master certain models mentioned in lecture**
 - if you don't know which models are those
 - ask someone who know

3.10 Linear approximation and differentials

- dy/dx (1)
- $\Delta y \Delta x$ (2)

1

$$f(x) \approx f(a) + f'(a)(x - a)$$

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is called the **linear approximation** or **tangent line approximation** of f at a . The linear function whose graph is this tangent line, that is,

- why are we doing differentiation

- to unify (1) and (2)
- so we can

is called the **linear approximation** or **tangent line approximation** of f at a . The linear function whose graph is this tangent line, that is,

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2

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a .

- and we can
- use dx dy dt etc. to solve many problem piece by piece
- such as the former applications
- **Match the name to its corresponding concepts**
- **You must know the difference between (1) and (2)**
 - the change of variable and differentials