1.

(a) $y=x^4+2e^x$, $(1,1+2e)$

$y'=4x^3+2e^x$

when $x=1$, $y'=4+2e$ so $k_1=4+2e$

Because the point $(1,1+2e)$ be on the tangent line, so:

Tangent line: $y=(4+2e)(x-1)+1+2e=(4+2e)x-3$

The normal line is perpendicular to the tangent line

so $k_1 \times k_2=-1 \Rightarrow k_2=-\frac{1}{4+2e}$

Normal line: $y=-\frac{1}{4+2e}(x-1)+1+2e$

(b) $y=\sqrt{x}e^x$, $(1,e)$

$y'=e^x(\sqrt{x}+\frac{1}{2\sqrt{x}})$

when $x=1$ $y'=\frac{3}{2}e$ so $k_1=\frac{3}{2}e$

Because the point $(1,e)$ be on the tangent line, so:

Tangent line: $y=\frac{3}{2}e(x-1)+e=\frac{3}{2}ex-\frac{1}{2}e$

The normal line is perpendicular to the tangent line

so $k_1 \times k_2=-1 \Rightarrow k_2=-\frac{2}{3e}$

Normal line: $y=-\frac{2}{3e}(x-1)+e$

(c). $y=sin(x)+cos(x), (0,1)$

We know $y'=cos(x)-sin(x)$. And $y'(0)=1$.

Therefore, for the tangent line, that is: $y=x+1$.

For the normal line, that is: $y=-x+1$

(d). $y = 2^x, (0,1)$

We know $y' = 2^x ln(2)$. And $y'(0) = ln(2)$.

Therefore, for the tangent line, that is: $y = ln(2) + 1$

For the normal line, that is: $y = \frac{-1}{ln(2)} x + 1$

2.(a) $y = x^3 f(x)$

$y' = 3x^2 f(x) + x^3 f'(x)$

(b) $y = \frac{f(x)}{x^3}$

$y' = \frac{x^3 f'(x) - 3x^2 f(x)}{x^6} = \frac{x f'(x) - 3f(x)}{x^4}$

(c) $y = \frac{x^3}{f(x)}$

$y' = \frac{3x^2 f(x) - x^3 f'(x)}{[f(x)]^2}$

(d) $y = \frac{1 + x[f(x)]^2}{\sqrt{x}}$

$y = x^{-\frac{1}{2}} + x^{\frac{1}{2}}[f(x)]^2$

$y' = -\frac{1}{2}x^{-\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}}[f(x)]^2 + x^{\frac{1}{2}} \cdot 2[f(x)] \cdot f'(x)$

$= \frac{4x^2 f(x) f'(x) + x[f(x)]^2 - 1}{2x^{\frac{3}{2}}}$

3. $Q(x) = \dfrac{1+x+x^2+xe^x}{1-x+x^2-xe^x}$

let $f(x) = 1+x+x^2+xe^x$ be the numerator of $Q(x)$

$g(x) = 1-x+x^2-xe^x$ be the denominator of $Q(x)$

then $Q'(x) = \left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$

$f'(x) = 1+2x+e^x(1+x)$ $\quad g'(x) = -1+2x+e^x(1-x)$

$f'(0) = 2$ $\quad g'(0) = 0$ $\quad f(0) = 1$ $\quad g(0) = 1$

then $Q'(0) = \dfrac{f'(0)g(0) - g'(0)f(0)}{g(0)} = 2$

3.

$$Q(x) = \frac{1+x+x^2+xe^x}{1-x+x^2-xe^x} \tag{1}$$

Assume $f(x) = 1+x+x^2+xe^x$, $g(x) = 1-x+x^2-xe^x$

$$Q'(x) = \frac{(1+2x+e^x(1+x))g(x) - (-1+2x-e^x(1+x))f(x)}{g^2(x)} \tag{2}$$

$$Q'(0) = 4 \tag{3}$$

4. $F(t) = e^{t\sin 2t}$

$F'(t) = e^{t\sin 2t}(\sin 2t + 2t\cos 2t)$

$F''(t) = e^{t\sin 2t}(\sin 2t + 2t\cos 2t + 2\cos 2t + 2\cos 2t - 4t\sin 2t)$

$= e^{t\sin 2t}(2t\cos 2t - 4t\sin 2t + \sin 2t + 4\cos 2t)$

$$\lim_{x \to 0} \frac{sin5x}{3x}$$

Using the equation: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

$$\lim_{x \to 0} \frac{sin5x}{3x} = \lim_{x \to 0} \frac{sin(5x)}{5x}\frac{5}{3} = \frac{5}{3}$$

5.

$$\lim_{\theta \to 0} \frac{cos(\theta)-1}{sin(\theta)}$$

Using the equations $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$

$$\lim_{\theta \to 0} \frac{cos(\theta) - 1}{sin(\theta)} = \lim_{\theta \to 0} \frac{cos(\theta) - 1}{\theta} \lim_{\theta \to 0} \frac{\theta}{sin(\theta)} = 0$$

5. (c) $\lim_{x \to 0} \frac{\sin(3+x)^2 - \sin 9}{x}$

$$= \left[\lim_{x \to 0} \frac{\sin(y+x)^2 - \sin y^2}{x - 0}\right]_{y=3}$$

$$= \frac{d}{dy}\left[\sin(y^2)\right]\Big|_{y=3}$$

$$= 2y\cos(y^2)\Big|_{y=3} = 6\cos 9$$

6.

(a) $f(x) = (x+1)^2 (x^3+x^2+1)^3$

$$f'(x) = 2(x+1)(x^3+x^2+1)^3 + 3(x^3+x^2+1)^2(3x^2+2x)(x+1)^2$$

$$= (x+1)(x^3+x^2+1)^2\left[2x^3+2x^2+2+3(3x^2+2x)(x+1)\right]$$

$$= (x+1)(x^3+x^2+1)^2(11x^3+17x^2+6x+2)$$

(b)

$$\frac{\sin^2 x}{1+\cot x}+\frac{\cos^2 x}{1+\tan x}$$

Replace $\cot x$ with $\dfrac{\cos x}{\sin x}$ and $\tan x$ with $\dfrac{\sin x}{\cos x}$

$$=\frac{\sin^2 x}{1+\dfrac{\cos x}{\sin x}}+\frac{\cos^2 x}{1+\dfrac{\sin x}{\cos x}}$$

$$=\frac{\sin^2 x}{\dfrac{\sin x}{\sin x}+\dfrac{\cos x}{\sin x}}+\frac{\cos^2 x}{\dfrac{\cos x}{\cos x}+\dfrac{\sin x}{\cos x}}$$

$$=\frac{\sin^2 x}{\dfrac{\sin x+\cos x}{\sin x}}+\frac{\cos^2 x}{\dfrac{\cos x+\sin x}{\cos x}}$$

$$=\sin^2 x\div\frac{\sin x+\cos x}{\sin x}+\cos^2 x\div\frac{\cos x+\sin x}{\cos x}$$

$$=\sin^2 x\cdot\frac{\sin x}{\sin x+\cos x}+\cos^2 x\cdot\frac{\cos x}{\cos x+\sin x}$$

$$=\frac{\sin^3 x}{\sin x+\cos x}+\frac{\cos^3 x}{\cos x+\sin x}$$

Note that the two fractions now have the same denominator

$$=\frac{\sin^3 x+\cos^3 x}{\sin x+\cos x}$$

Recall that: $A^3+B^3=(A+B)\left(A^2-AB+B^2\right)$

$$=\frac{(\sin x+\cos x)\left(\sin^2 x-\sin x\cos x+\cos^2 x\right)}{\sin x+\cos x}$$

Cancel $(\sin x+\cos x)$ from both the numerator and the denominator

$$=\sin^2 x-\sin x\cos x+\cos^2 x$$

$$=\left(\sin^2 x+\cos^2 x\right)-\frac{1}{2}(2\sin x\cos x)$$

Recall that: $\sin^2 x+\cos^2 x=1$ and $2\sin x\cos x=\sin 2x$

$$=1-\frac{1}{2}\sin 2x$$

$$\frac{\sin^2 x}{1+\cot x}+\frac{\cos^2 x}{1+\tan x}=1-\frac{1}{2}\sin 2x$$

Differentiate both sides with respect to x

$$\frac{d}{dx}\left[\frac{\sin^2 x}{1+\cot x}+\frac{\cos^2 x}{1+\tan x}\right]=\frac{d\,(1)}{dx}-\frac{1}{2}\frac{d\,(\sin 2x)}{dx}$$

The **Chain Rule** for differentiation

$$\frac{d\Big[f\,[g(x)]\Big]}{dx}=\frac{d\Big[f\,[g(x)]\Big]}{d\Big[g(x)\Big]}\cdot\frac{d\Big[g(x)\Big]}{dx}$$

$$=0-\frac{1}{2}\frac{d\,(\sin 2x)}{d\,(2x)}\cdot\frac{d\,(2x)}{dx}$$

$$=-\frac{1}{2}\cos 2x\cdot 2$$

$$=-\cos 2x$$

7. (last year 5(a))

5. (a) If n is a positive integer, prove that

$$\frac{d}{dx}(sin^n x\cos nx)=n\,sin^{n-1}x\cos(n+1)x$$

$\frac{d}{dx}(sin^n x\cos nx)=n\,sin^{n-1}x\;cosx\cos nx+sin^n x\,(-n\sin nx)$ [Product Rule]

$=n\,sin^{n-1}x\,(cosx\cos nx-sinx\sin nx)$

$=n\,sin^{n-1}x\cos(nx+x)$ [Addition formula for cosine]

$=n\,sin^{n-1}x\cos(n+1)x$