Explanation on δ-ϵ definition

Hearing that most of you didn't understand what δ-ϵ definition is and how to solve problems related to it, I decided to write some explanation on the definition as well as solution. Although this kind of problems might hardly appear on exam paper, ~~it's enjoyable to solve them, perhaps,~~ I see some of you really want~~ed~~ to know the meaning of the solution to this type of problem.

What does this definition really mean?

> $$lim_{x \to a} f(x) = L$$
>
> if for every number ϵ there is a number $\delta > 0$ such that
>
> if $\quad 0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon \qquad (*)$

This is what the book tells us.

But, what does it mean?

Well, we want to prove the limit of $f(x)$ is L, then we try to find an vertical interval $I_v = (m, n)$ that include the limit, meaning that $L \in (m, n)$ .That vertical interval should be a subset of the range of the function $f(x)$. We choose the **continuous** part of the function (marked blue in fig.1) **near point (a,L)** if it exists, whose head and tail has the corresponding relationship with I_v as shown in fig.1.(I don't want to rigorously explain things here because that will make it more abstract). This blue part of the function corresponds to interval I_h as shown in fig 1. This is what (*) tries to say.

So what do we want to prove? we want to prove that, for every I_v we can find horizontal interval I, satisfying that $I \subseteq I_h$ and $a \in I$.

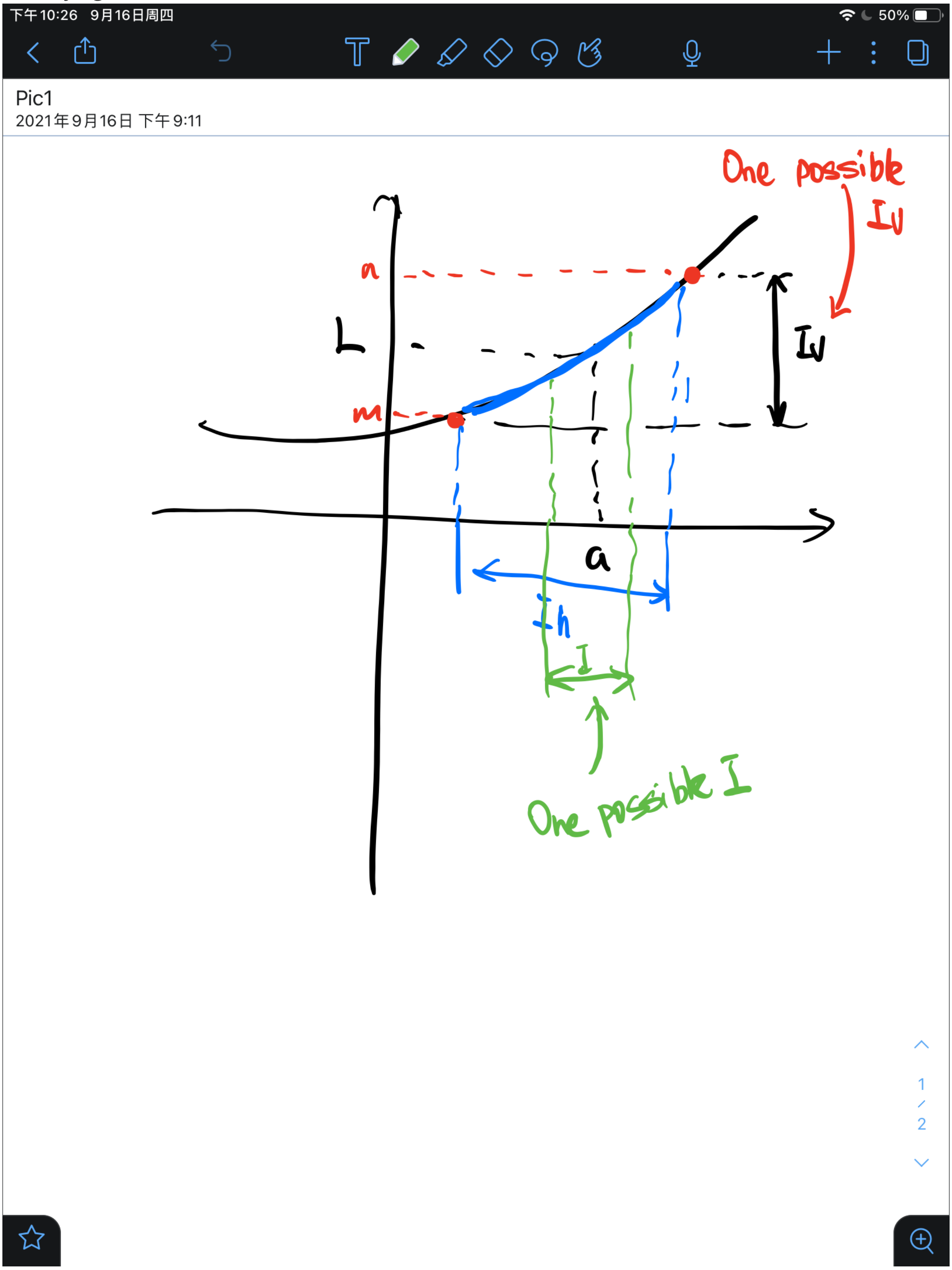

fig.1

To this point, you should know what the definition means.

How to use this definition to solve problems

Problems are real, and they can be complex.

Sure, you can use the previous definition to prove simple problems. But it will take you some time to think, or even make you stuck at some steps and stop you eventually. You might not want this to happen. Here are my understanding that may help you simplify the problem.

Do we need to investigate all possible I_v to see if there can be I?

No, when the interval gets big enough, if its sub-interval can make $(*)$ condition true, the big interval must also make $(*)$ condition true. Why? Bigger I_v means bigger I_h. If bigger I_v' contains I_v then I_h' must contain I_h. If there is an interval that satisfies the $(*)$ condition for I_h' then it must satisfies the same condition for I_h. That's why.

This means, what makes it a problem, is very small I_v, if we can prove that for very small I_v we can find I, then bigger I_v shouldn't be a problem. Smaller I_v means it's harder to find I_h because we have less candidates, but it also gives us an advantage, which is the core to calculus. **The curve you see will be monotonic!** That's really awesome because you don't have deal with something like fig2.

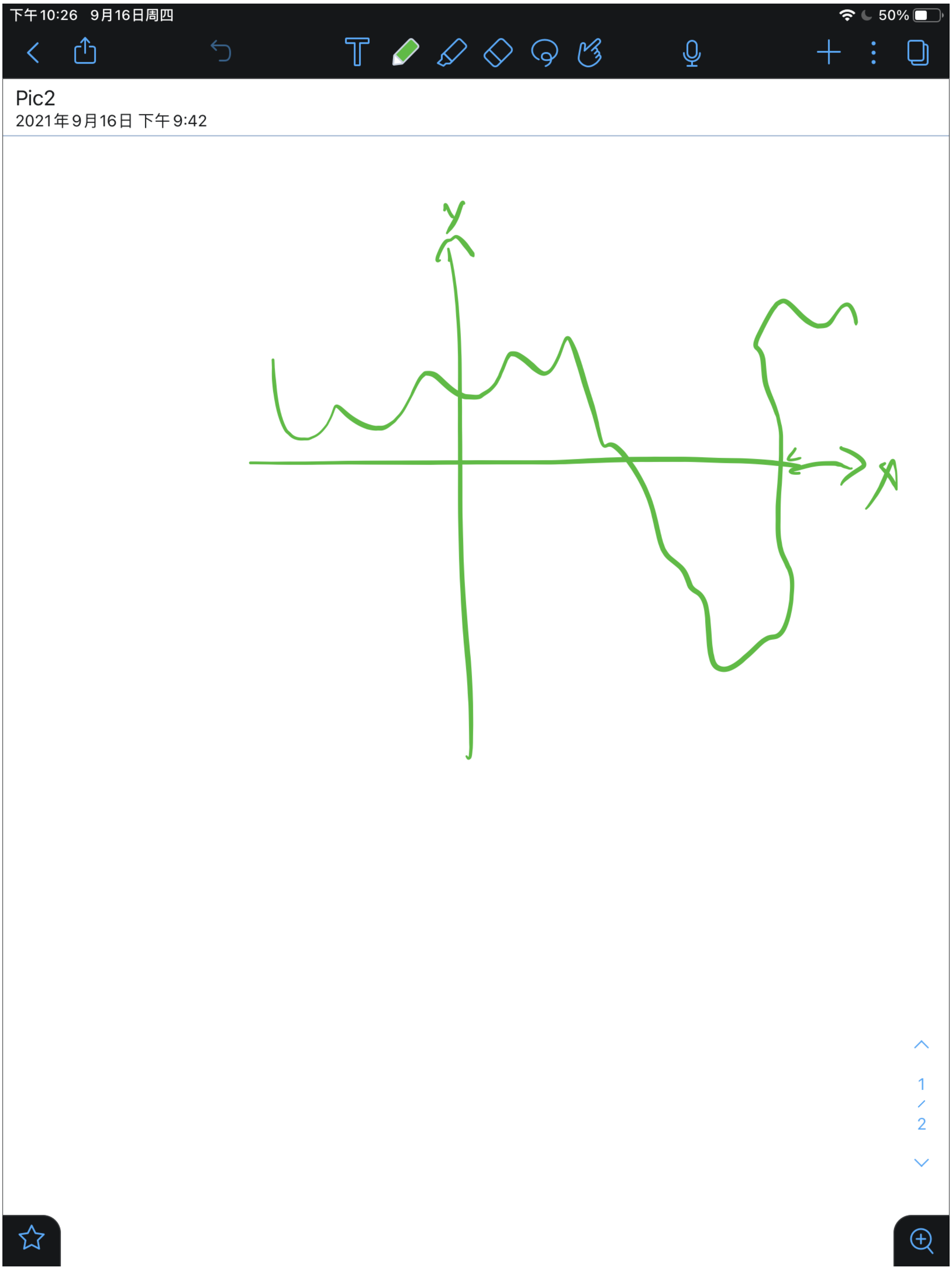

fig.2

One thing that's important is if I_h exists, you don't have to find smaller I satisfying $I \subseteq I_h$. Why? because I_h is easier to find and it's enough to make $(*)$ condition true.

What does this mean? What we need to do is to calculate I_h for I_v

So, here is the problem, given I_v get I_h

Problem 01

$p108$~$p109$

> Prove that $\lim_{x \to 3}(4x-5) = 7$

Step 1

Get I_v

Iv is $|(4x-5)-7|<\epsilon$

Step 2

Get I_h from I_v

$|(4x-5)-7|<\epsilon$, this is I_v , simplify it.

we have$|x-3|<\frac{\epsilon}{4}$

By doing this, we solve I_h , then we have $\delta=\frac{\epsilon}{4}$

Step 3

Show the value of δ can work by the definition

You just have to substitute δ and copy and paste the definition again in this step

I don't want to do this kind of job for you here.

Easy, right.

Problem 02

$p110$ ~ 111

> Prove that $\lim_{x\to 3} x^2=9$

Step 1

Get I_v

I_v is $|x^2-9|<\epsilon$

Step 2

Get I_h from I_v

What? How?

We want to solve I_h, but I_v is nothing like I_h which is $|x-3|<\delta$

Actually you can.

$|(x+3)(x-3)|<\epsilon$

$|x-3|<\frac{\epsilon}{(x+3)}$

We can use any x if $x>3$, because $\frac{\epsilon}{(x+3)}$ suggest the length of I (not I_h)

x=4 is a good choice

so $\delta=\epsilon/7$ is good enough,

but it can also be $\frac{\epsilon}{8}$ $\frac{\epsilon}{9}$ and so forth.

Why, because we only need to know horizontal interval I exists

Step 3

substitute δ and just copy the $\delta - \epsilon$ definition word by word.

Why it can work without your thinking? Because we deduce the value of δ with this definition. So, if your calculation is correct, the step 3 shouldn't be a problem.

It's like when you solve an equation, and get your answer, for example, $x = 4$. Step 1 & 2 is the part when you calculate the answer. Step 3 is substitute the value of x into the original equation so you can check whether your calculation is correct or not.