# Channel Assignment Strategies for Cellular Phone Systems

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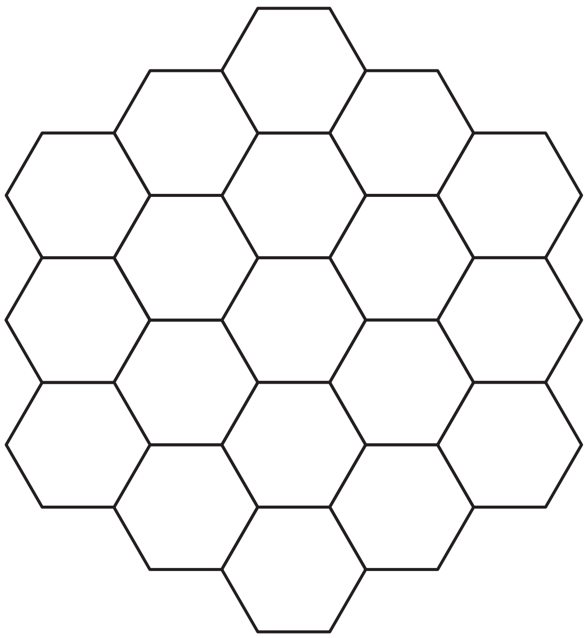

# Abstract

Nowadays people benefit a lot from mobile communication systems. One of recent major breakthroughs in solving the problem of spectral congestion is the cellular concept, which means replacing a large transmitter with many little transmitters, each providing coverage to a small piece of the service area. Neighboring transmitters are assigned different groups of channels so that the interference between transmitters is minimized. Each cellular transmitter is assigned a radio channel to be used within a small geographic area called a cell. By limiting the coverage area to within the boundaries of a cell, the same channel may be used to cover different cells that are separated from one another by distances large enough to keep interference levels within tolerable limits. The design process of selecting and allocating channel groups for all of the cellular transmitters in a system is called frequency reuse or frequency planning. The hexagonal cell shape shown in the picture of the cover is conceptual and is a simplistic model of radio coverage for each transmitter, but it has been universally adopted since the hexagon permits easy and manageable analysis of a cellular system.

Our team has made a comprehensive study on a practical problem of designing mobile communication systems. We know that there are several constraints on frequency assignment. First, any two transmitters within a distance four times as much as the length of the hexagon side cannot be assigned the same channel. Second, due to spectral spreading, transmitters within distance twice as much as the length of the hexagon side must be assigned channels which differ by at least $k$. We have developed quite an efficient strategy dependent on $k$ to assign channels as few as possible to a given service area according to these rules.

If we put our strategy into practice under simple situations, we have 7 channels to be the fewest when $k$ equals 1, 9 channels when $k$ equals 2, and 12 channels when $k$ equals 3. When the situation becomes complicated as $k > 3$, we find the smallest number of channels needed is a linear function $2k + 7$, which is a pretty simple solution. Thus we have the entire results as follows:

$$span = \begin{cases} 2k+5, & k = 1, 2 \\ 2k+6, & k = 3 \\ 2k+7, & k \geq 4. \end{cases}$$

Patterns of our assignments are as follows:

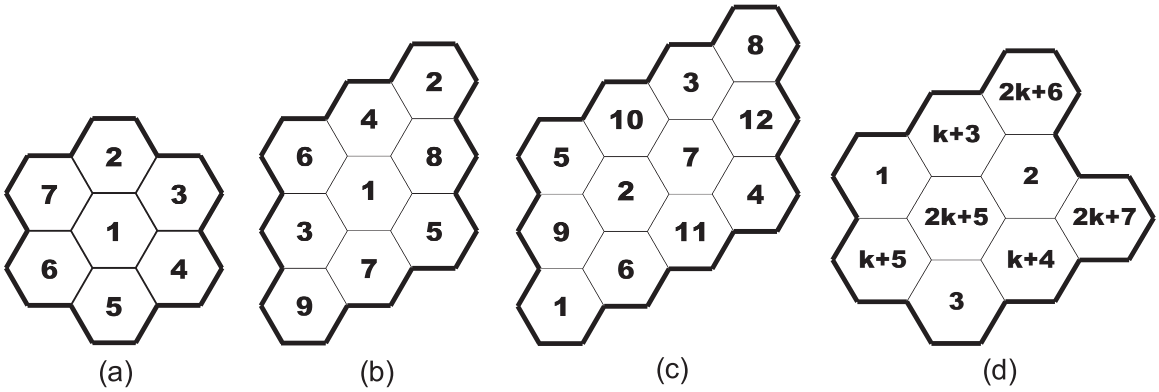


Figure 1: Channel assignment patterns. (a) $k = 1$; (b) $k = 2$; (c) $k = 3$; (d) $k \geq 4$.

We also study several general problems with more constraints. An algorithm aimed at getting a lower bound is devised through a branch-and-bound strategy. This algorithm is very useful as it suffices to provide a good estimate of the lower bound, which may be a great help when we apply integer programming to solve this problem.

We also developed a method to deal with the situation that the number of the channels is fixed. A new strategy is devised to reduce interference as greatly as possible. We have first converted this problem into a nonlinear program. A lot of algorithms are available, such as SA (Simulated Annealing) and LG (Least Gradient). We choose GA (Genetic Algorithm) to solve this problem. A simple procedure of the algorithm is given.

# 1 Introduction

The design objective of early mobile radio systems was to achieve a large coverage area by using a single, high-powered transmitter with an antenna mounted on a tall tower. While this approach achieved very good coverage, it also meant that it was impossible to reuse those same frequencies throughout the system since any attempts to frequency reuse would result in interference. For example, the Bell mobile system in New York City in the 1970s could only support a maximum of 12 simultaneous calls over a thousand square miles [2]. Faced with the fact that government regulatory agencies could not make spectrum allocations in proportion to the increasing demand for mobile services, it became imperative to restructure the radiotelephone system to achieve high capacity with limited radio spectrum, and at the same time cover very large areas.

The cellular concept has been one of the revolutionary breakthroughs in developments of mobile communication systems. Allowing a group of lower power transmitters (small cells) to replace a single, high power transmitter (large cell), the cellular system may provide good coverage to only a small portion of a given area with a portion of the total number of channels being allocated to each base station. Here the design process of selecting and allocating channel groups for all of the cellular base stations in a system is very critical, which is called frequency reuse or frequency planning [6].

Our task is to model the assignment of channels to a symmetric network of transmitter locations over a large planar area, which has been partitioned into hexagons. An interval of the frequency spectrum is to be allotted for transmitter frequency. The interval will be divided into regularly spaced channels, represented by integer 1, 2, 3... Each transmitter will be assigned one positive integer channel. The same channel can be reused at many locations, provided that interference is avoided.

Our goal is to minimize the width of the interval in the frequency spectrum which is needed to assign channels subject to some constraints. This is achieved by the concept of a span. The span is the minimum, over all assignments satisfying the constraints, of the largest channel used at any location. It is not required that every channel smaller than the span be used in an assignment that attains the span.

# 2 Assumptions and Definitions

Assumptions:

- The honeycomb-style is hexagon.
- In Solving Requirement B, C, D, we assume that the grid always extends to the infinity in all directions. The reason is explained below.
- The channels are regularly spaced.

- All of the transmitter use omni-directional antennas.
- All of the cells are identical.
- The transmitter is located over a large planar area.

Definitions:

- $sp$: the value of span.
- $s$: the length of a side of one of the hexagons.
- $N$: the number of cells in a single tile.
- $tile$: pre-assigned network pieces, which are described later.
- $f(x)$: the channel number assigned to the cell $x$.

# 3 Preliminary Analysis

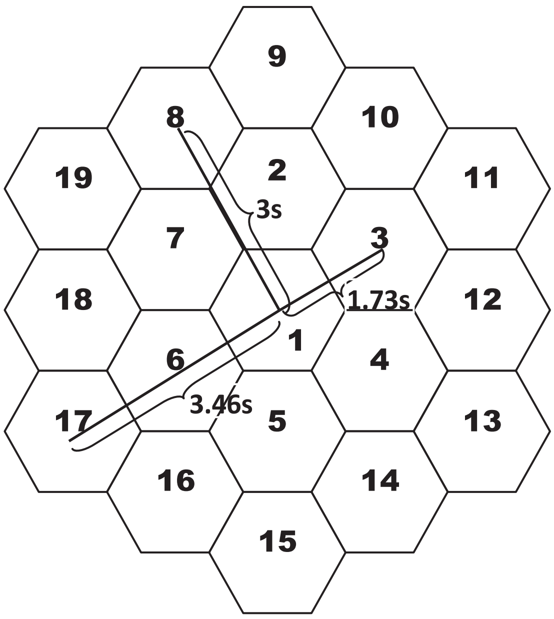


Figure 2: The first and second loop.

## 3.1 Distance Between Cells

First, we aim to calculate the distance of two certain cells. Look at Fig. 2. It is easy to work out the distance between cell 1 and its adjacent cells 2, 3, 4, 5, 6, 7, which is $\sqrt{3}s$. Furthermore, the distance between cell 1 and cells 8, 10, 12, 14, 16, 18 is $3s$ while the distance between cell 1 and cells 9, 11, 13, 15, 17, 19 are $2\sqrt{3}s$. We call cells 2, 3, 4, 5, 6, 7 surrounding cell 1 *the first loop* and the cells 8-19 surrounding cell 1 *the second loop*. We may easily find that only cells in the first loop have a closer distance than $2s$ with cell 1. With the same reason only cells in the second loop have a closer distance than $4s$ with cell 1. So when we consider the $2s(4s)$-constraint, we may only concentrate on those cells in the first (second) loop. This makes our efforts to solve the problem much easier.

## 3.2 Developments

### 3.2.1 Tiling Technique

It is obvious that the same frequency cannot be used in the adjacent cells because of the $4s$-constraint. In order to keep the cells assigned with the same frequency far enough from each other, several surrounding cells cannot be assigned that frequency too. We call these cells a "*tile*". Only cells in different tiles can be **frequency-reused**.

Formation of the tile should satisfy the following two conditions: 1. Tiles and tiles can connect without gaps between adjacent cells; 2. The distance between any two cells with the same frequency should be larger than $4s$. After we have covered the area with tiles, there are only certain tile sizes and cell layouts which are possible [6], they can only have values which satisfy equation: $N = i^2 + ij + j^2$, where $i$ and $j$ are non-negative integers. All the tiles in our models satisfy the above equation.

It is obvious that this tiling technique is always effective no matter whether the region can be covered by a single basic block or not. We now assume that the grid always extends to the infinity in all directions.

#### 3.2.2 $sp = 7$ When $k = 1$

If we have $k = 1$ then we can use integer channels 1, 2, 3, 4, 5, 6, 7 to fill in a infinite grid with the following tile. Refer to Appendix B-1 for a more detailed picture.

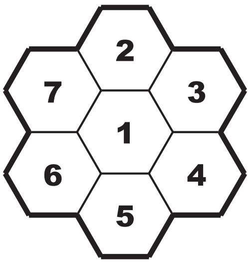


Figure 3: $sp = 7$ when $k = 1$, where $N = 7$, $i = 1$, $j = 2$.

It is clear that 6 integers are not enough to fill this tile. If not so, there must be repetitive integers in the tile, which will surely violate the $4s$-constraint.

#### 3.2.3 $sp = 9$ When $k = 2$

It is very easy to prove that we cannot satisfy both the constraints if we try to assign channels with a span of 8 to the transmitters in a infinite grid. If this is not true, let us suppose that we used a certain way of assignment with span 8 to fill all the cells. Then we are sure to find an integer 1 in one of the cells which we called cell A. If we cannot find the integer 1 in any of these cells, simply subtract 1 from every integer assigned to all the cells. Then we get a better assignment with a new span of 7 that is smaller than the previous one. We know that 2 cannot be placed in any of cell A's adjacent cells (in order to satisfy the $2s$-constraint). We name the A's adjacent cells B, C, D, E, F, G, shown in Fig. 4. So 3, 4, 5, 6, 7, 8 must be used to be filled in these 6 adjacent cells. We may notice that no repetitive integers can be used to fill these 6 cells because any two of these cells are within the distance of $4s$, so the $4s$-constraint will not be satisfied if repetitive integers are used. Thus we know that integer 3 must be used in one of the cells adjacent to cell A accompanying integer 1. But we may still notice that only 1, 5, 6, 7, 8 are available to be filled in the adjacent cells of the cell filled with 3. Since we are unable to assign these 5 integers into 6 cells without repetition we learn that the span 8 cannot be achieved.

So at least 9 channels are needed. There are several ways to get the appropriate assignment of

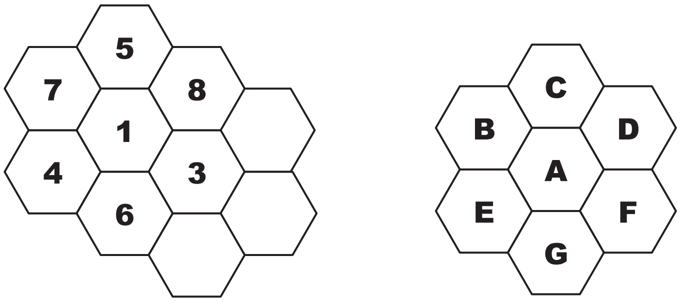


Figure 4: Adjacent cells.

these 9 channels. After a few trials we have successfully found a right assignment as shown in Fig. 5, a more detailed one also included in Appendix B-2.

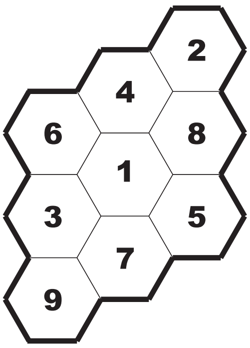


Figure 5: $sp = 9$ when $k = 2$, where $N = 9$, $i = 0$, $j = 3$.

#### 3.2.4 $sp = 12$ When $k = 3$ and Lower Bound of Span for $k \geq 3$

We have got $sp = 7$ when $k = 1$ and $sp = 9$ when $k = 2$. Assuming that $sp$ depends on $k$ linearly we may guess that $sp = 2k + 5$, But when it applies to the situation $k \geq 3$ we find that $2k + 5$ is not a tight lower bound for the span and $2k + 6$ is a tighter one. An effective proof is included in Appendix A-1. We also have designed a certain pattern of assignment to show $sp = 12 = 2k + 6$ when $k = 3$ (See Fig. 6, a more detailed one is in Appendix B-3). Since we have got a realistic example we are sure to say $sp = 2k + 6$ when $k = 3$ and $sp \geq 2k + 6$ when $k \geq 4$.

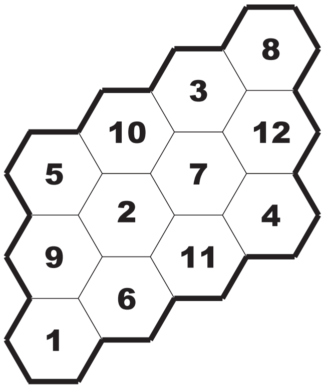


Figure 6: $sp = 12$ when $k = 3$, where $N = 12$, $i = 3$, $j = 1$.

#### 3.2.5 Preliminary Estimate of Upper Bound of The Span

In order to avoid the situation that 2 integers assigned to adjacent cells which may violate the $2s$-constraint we generate an integer sequence $1, k+1, 2k+1, 3k+1, 4k+1, 5k+1, 6k+1$, in which any two integers satisfy the $2s$-constraint because the minimal difference of them is $k$. Assign these integers into the basic tile shown in Fig. 7 (More detailed one is in Appendix B-4), we may notice that the $2s$-constraint is satisfied. Since we also adopt the tiling technique, we may be quite sure that this assignment satisfies the $4s$-constraint too. Thus we get a preliminary estimate of the upper bound of the span ($sp \leq 6k + 1$).

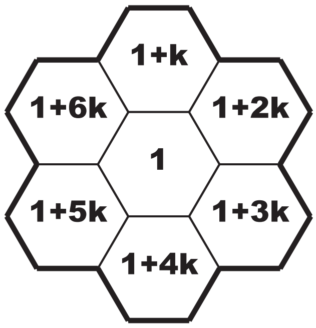


Figure 7: Upper bound of the span.

#### 3.2.6 $sp = 2k + 7$ When $k \geq 4$

We can clearly see that there are a great few of channels wasted when we apply the tile in Fig. 7. So we managed to lower it to some extent. Considering the geometry catachrestic of hexagon and

$sp = 2k + 6$ when $k = 3$, we find that coloring algorithm in the graph theory can be used to prove that $2k + 7$ is the span when $k \geq 4$.

Before we proved it, we first use this algorithm to find the strategy that $span = 2k + 7$ when $k \geq 4$. (See Fig. 8, a more detailed one is in Appendix B-5.)

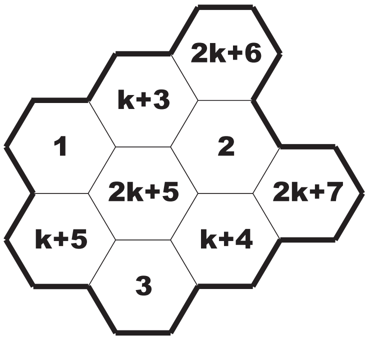


Figure 8: $sp = 2k + 7$ when $k \geq 4$.

This special case tell us that $2k + 7$ is the upper bound of the span when $k \geq 4$. So we only need to prove that $2k + 7$ is also the lower bound of span. The proof is included in Appendix A-2. Now we know that $2k + 7$ is not only the upper bound of span but also the lower bound. Hence, $sp = 2k + 7$ when $k \geq 4$.

#### 3.2.7 Solution of The Span and Assignment Strategies

Up to now we have come up to the following conclusions:

- 1. $sp = 7 = 2k + 5$ when $k = 1$ (in section 3.2.2).
- 2. $sp = 9 = 2k + 5$ when $k = 2$ (in section 3.2.3).
- 3. $sp \geq 2k + 6$ when $k \geq 3$, $sp = 12$ when $k = 3$ (in section 3.2.4).
- 4. $sp = 2k + 7$ when $k \geq 4$ (in section 3.2.6).

With these conclusions, we derive the span as a function of $k$ as follows:

$$span = \begin{cases} 2k + 5, & k = 1, 2 \\ 2k + 6, & k = 3 \\ 2k + 7, & k \geq 4. \end{cases} \tag{1}$$

Using tiling technique, we develop the strategies, which are:

When $k = 1$, we use the tile in Fig. 3 (also included in Appendix B-1).

When $k = 2$, we use the tile in Fig. 5 (also included in Appendix B-2).

When $k = 3$, we use the tile in Fig. 6 (also included in Appendix B-3).

When $k \geq 4$, we use the tile in Fig. 8 (also included in Appendix B-5).

The above are the solutions for Requirement C, while Requirement A and B have been solved in the previous discussion. The span of Requirement A is 9, and the strategy is shown in Fig. 9.

Since we have adopted the tiling technique, The result when $k = 2$ can also be applied to Requirement B assuming that the grid extends arbitrarily far in all directions. The span of Requirement B is also 9.

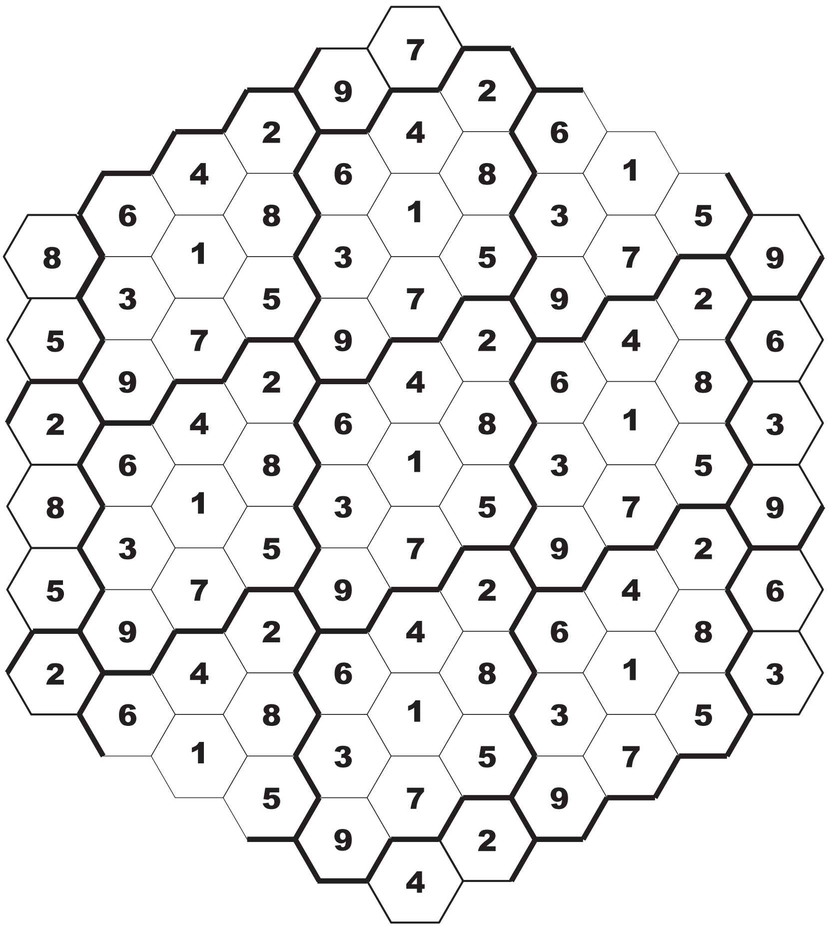


Figure 9: Channel assignment strategy for requirement A.

# 4 Generalized Model Analysis

## 4.1 Generalize The Two-Level-Interference Model

As a most direct extension of the former channel assignment problem, we have a more generalized $4s$-constraint that channels for transmitters within distance $4s$ must differ by at least given integer $e$. So we have:

$4s$-*constraint*: if $d(u, v) \leq 4s$, $f(u) - f(v) \geq e$

$2s$-*constraint*: if $d(u, v) \leq 2s$, $f(u) - f(v) \geq k$

$k, e$ are integers.

If $d(u, v) \leq 2s$ is satisfied then $d(u, v) \leq 4s$ will be surely satisfied so $k \geq e$.

Making a few adjustments to the former model we have the following results as the upper bound for the this problem:

When $k > 3e$, $sp = 2k + 6e + 1$.

When $2e < k \leq 3e$, $sp = 2k + 5e + 1$.

When $e < k \leq 2e$, $sp = 2k + 4e + 1$.

The corresponding tiles are shown in Fig. 10.

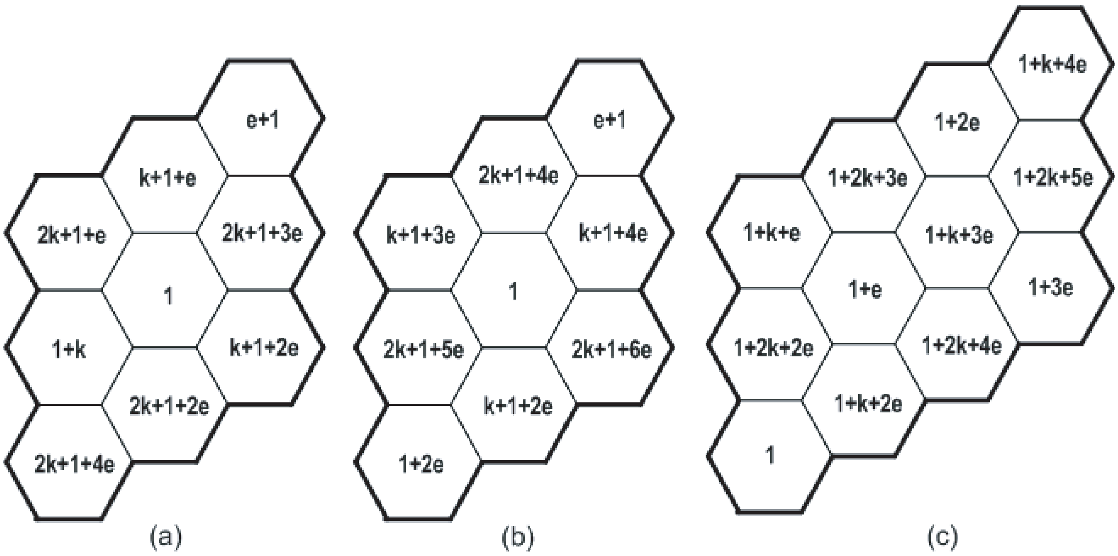


Figure 10: Two level interference model. (a) $e < k \leq 2e$; (b) $k > 3e$; (c) $2e < k \leq 3e$.

## 4.2 Irregular Locations and Several Levels of Interference

### 4.2.1 One Level of Interference and Irregular Located Transmitters

We use graph theory to model this channel assignment problem with irregular locations of the transmitters. Let $G(V)$ be a graph with a vertex set $V$ of points in the plane denoting the actual transmitters and with distinct vertices $u$ and $v$ adjacent whenever there may be interference if the same channel is assigned to $u$ and $v$. The most simplified and basic version of the channel assignment problem involves coloring such "proximity" graphs. We need to assign radio channels (colors) to transmitters (points in $V$) to avoid interference.

### 4.2.2 Clique Configuration Method

We use graph theory to model a certain type of the channel assignment problems with irregular locations of the transmitters. Given a set $V$ of points in the plane and given $d > 0$, we use a function $f$ to represent the assignment of channels $f$: $V \to \{1, 2, 3, \cdots, t\}$. If $d(u, v) < d$, we have $|f(u) - f(v)| \geq k$ and $sp = \min_f(t)$.

Let $G(V, d)$ be a graph with vertex set $V$ of points in the plane denoting the actual transmitters and with distinct vertices $u$ and $v$ adjacent if the Euclidean distance $d(u, v)$ between them is less than $d$. The weight of the edge $(u, v)$ is the least allowed frequency difference of $u$ and $v$.

We develop an algorithm based on the CLIQUE question as follows:

**I.** Find the largest clique of the given graph.

**II.** Work out the length of the shortest Hamiltonian path in this clique.

Assume the number of the vertices of the largest clique found is $\alpha$. Find one of its Hamilton Path. Suppose that the optimal strategy has been found and the vertex sequence $A_\alpha$ on this Hamilton Path are assigned with a weight $f(A_i)(1 \leq i \leq \alpha)$. Sort $f(A_i)$ in an ascending way to get a new vertex sequence $A_{i,1}, A_{i,2}, \cdots, A_{i,\alpha}$. Since the clique is a complete sub graph, $A_{i,j}$ and $A_{i,j+1}$ must be adjacent. So

$$sp \geq f(A_{j,\alpha}) \geq f(A_{j,\alpha-1})+k \geq f(A_{j,\alpha-2})+2k \geq \cdots \geq f(A_{j,1})+(\alpha-1)*k \geq 1+(\alpha-1)*k. \quad (2)$$

Thus we get a lower bound. This result has become known as the Traveling Salesman Bound. This is often tight for cellular problems, but may need improvement for other types of problem.

It is important that this bound is applied to a suitable clique or "near clique" and not to the whole graph, or the bound may be very weak. Once the clique or "near clique" that give the best bound has been identified, a frequency assignment algorithm may be more effective if it is applied to this sub graph first. The assignment of the sub graph can then be fixed and the assignment can be extended to the full graph.

If there are $h$ levels of interference, we have a vector $\mathbf{d} = (d_1, \cdots, d_h)^T$ and a vector $\mathbf{k} = (k_1, \cdots, k_h)^T$. Apply the above algorithm to every $(d_i, k_i)$ to get a distance graph $G_i(V, d_i)$, then we have $sp \geq 1 + (\alpha_i - 1) * k_i$. Therefore we get:

$$sp \geq \max_{1 \leq i \leq h} \{1 + (\alpha_i - 1) * k_i\}. \tag{3}$$

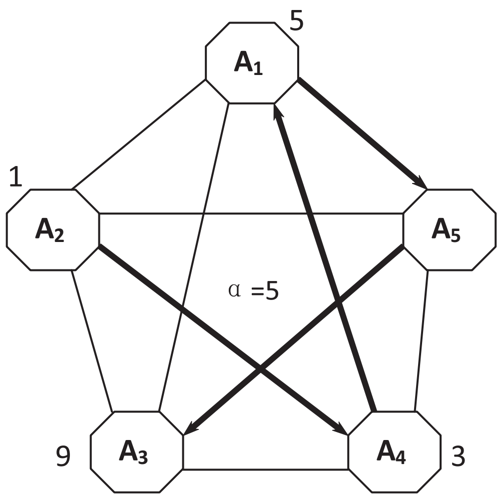


Figure 11: An example of a clique ($\alpha = 5$). The number adjacent to the vertex is $f(A_i)$. This figure shows the Hamilton Path after sorting the vertices by $f(A_i)$.

#### 4.2.3 Lower Bounding with Integer Programming

Lower bounding techniques can be used both to give a measure of the quality of any given solution and to give a meaningful comparison of the quality of different algorithms. The channel assignment problem can be formulated as an integer program and analyzed using polyhedral methods to obtain lower bounds. Lower bounds are important for the evaluation of heuristic methods, to identify bottlenecks, and to provide cuts for linear relaxations of integer programs.

We now establish a model for channel assignment problems using integer programming. There exists a matrix of separation constraint $C = (c_{i,j})_{n\times n}$ ($n$ is the number of transmitters). We still use a function $f$ to represent the assignment of channels $f : V \to \{0, \cdots, t\} \Rightarrow |f(v_i) - f(v_j)| \geq c_{i,j}$.

Obviously $C$ is a symmetric matrix and all the elements on the diagonal are zeros.

Suppose $x_i = f(v_i)(1 \leq i \leq n)$, we formulate the following integer program

Min $z = t + 1$

Subject to

$$\begin{cases} |x_i - x_j| \geq c_{i,j},\ 1 \leq i < j \leq n \\ x_i \leq t,\ 1 \leq i \leq n \\ x_i, t \geq 0,\ 1 \leq i \leq n \\ x_i, t\ are\ integers \end{cases} \tag{4}$$

There are $n(n-1)/2$ separation constraints $|x_i - x_j| \geq c_{i,j}$ in this integer programming. Each $|x_i - x_j| \geq c_{i,j}$ can be replaced by two inequalities: $x_i - x_j \geq c_{i,j}$ and $x_j - x_i \geq c_{i,j}$. Considering the binary tree in Fig. 12, we notice that each leaf is corresponded with one lower bound for the problem, and each level of nodes in the tree are associated with a constraint respectively. To every node of the tree, we use the above two inequalities to cut the current available-domain into two available-domains for its two son nodes respectively. We apply linear search algorithm to each node of the tree to get a non-integer lower bound respectively. This tree is generated by the sequence shown in the matrix below.

$$\begin{bmatrix} 0 & c_{1,2} & c_{1,3} & \cdots & c_{1,n-1} & c_{1,n} \\ c_{1,2} & 0 & c_{2,3} & \cdots & c_{2,n-1} & c_{2,n} \\ c_{1,3} & c_{2,3} & 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & c_{n-2,n-1} & c_{n-2,n} \\ c_{1,n-1} & c_{2,n-1} & \cdots & c_{n-2.n-1} & 0 & c_{n-1,n} \\ c_{1,n} & c_{2,n} & \cdots & c_{n-2,n} & c_{n-1,n} & 0 \end{bmatrix}$$

We search the entire tree adopting the Branch-and-Bound strategy to find the largest lower bound $\Phi$ of the entire solution space, which is non-integer. At last we get an integral lower bound $\Psi = \lceil \Phi \rceil$ for the problem.

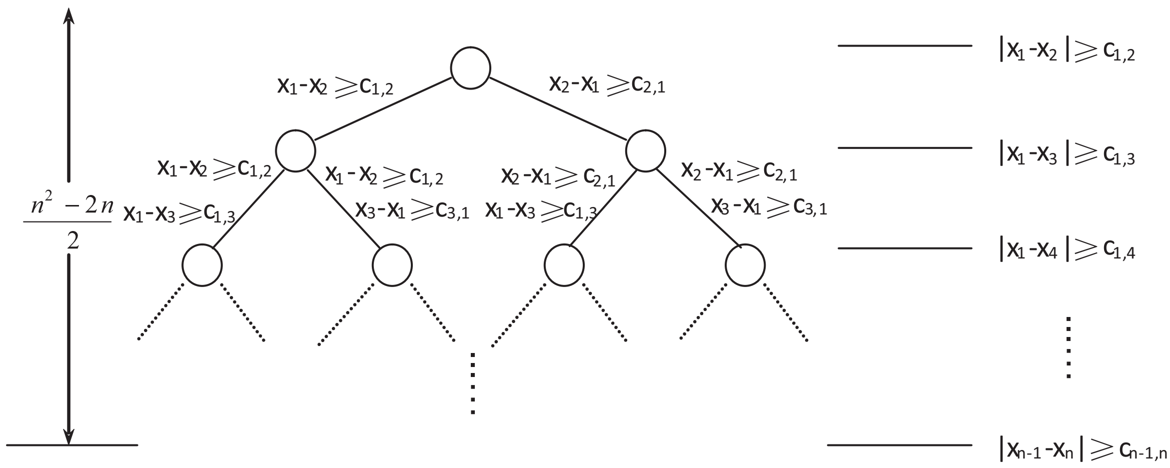


Figure 12: Binary search tree.

### 4.2.4 Genetic Algorithm: What If Channels Are Fixed

In some situations there may not be enough channels to be assigned to the mobile phone network. A bandwidth of a certain size is allocated for the mobile phone use. What's more, a real channel will take up some certain space in the spectrum so the maximal number of the channels available must be a fixed one. For instance if a total bandwidth of 1MHZ is allocated for mobile phone usage and a single channel occupies 5KHZ, then we may work out that at most 200 channels are available. But applying our model we may get a span of 201, so under such circumstances, the above models seem a little weak.

Considering this, we must sacrifice the quality of the system bearing a little interference to fulfill the fixed-channels constraint. Under such circumstances, we want to find an assignment with which the interference can be reduced as greatly as possible. Suppose there are $N$ transmitters dotted throughout a district. We have only $M(\ll N)$ frequency channels available to utilize and we need to assign each transmitter with one channel. Our requirement is that their mutual interference be minimized. The further away we reuse a certain frequency channel the better, but this is not always the case. The characteristic of the whole transmitter network can be represented by a "Mutual Interference Matrix" $A = (a_{i,j})(i, j \leq N)$ where $a_{i,j}$ is the mutual interference between transmitters $i$ and $j$ if they are assigned with the same channel. Usually this number is getting larger as the transmitters are placed closer. We can use some figures of merit to evaluate a certain frequency assignment scheme (for instance, sum up the interference values in the matrix for all co-frequency pairs), and try to minimize that figure by changing the frequency assignments. An iterative algorithm is needed to give us the best channel assignment (global minimum of that figure of merit).

Let $x_{i,j}$ equals 1 if transmitter $i$ is assigned to frequency $f$ and if not $x_{i,j}$ equals 0. The objectivity function $F$ is defined below:

$$F = \min \sum_{i,j=1,f=1}^{i,j=N,f=M} a_{i,j}\, x_{i,f}\, x_{j,f} \tag{5}$$

Subject to:

for $i = 1, \cdots, N, \quad \sum_{f=1}^{M} x_{i,f} = 1$

for $f = 1, \cdots, M, \quad \sum_{i=1}^{N} x_{i,f} \leq K_f$

for $i = 1, \cdots, N, \quad f = 1, \cdots, M, x_{i,j} \in \{0, 1\}$.

$K_f$ is the maximum number of transmitters you can assign to frequency $f$, it may be vacuous. In our case this constraint may not work.

Having set up the above model, we now adopt the Genetic Algorithm taking the following steps:

**Step 1.** Coding: We use a binary number to denote a certain way of assignment. Call it an individual.

**Step 2.** Initiate a group containing $n$ individuals. Create every individual $D_i$ randomly. Notice that $D_i$ is a matrix.

**Step 3.** Calculate $F(D_i)$ for every $i$. Let $f_{old}$ be the smallest value in this group.

**Step 4.** Generate new individuals by Reproduction and Mutation to get $2n$ offsprings. (Reproduction and Mutation are certain binary operations such as Or, And, ExclusiveOr, etc.)

**Step 5.** Select the best $n$ of the $2n$ offsprings which have the lowest values of $F$. Let $f_{new}$ be the smallest value in this new group.

**Step 6.** If $|f_{new} - f_{old}| > \varepsilon$ (which is a pre-set little non-negative number ), then goto Step 4.

**Step 7.** End.

## 5 Strengths and Weaknesses

Our model works excellently under the conditions of this problem. The strategies of our basic model depend heavily on the special geometric character of hexagons and the $2s$, $4s$ constraints. Our basic model gives an accurate description of the relationship between $k$ and the span as a function of $k$, and it also gives us a very simple and effective strategy to assign channels, which is clearly demonstrated in Appendix B. Furthermore we have also considered some more general forms of the channel assignment problem. As most of them are NP-Complete, we developed a way to find a lower bound, which uses linear programming instead of the traditional way of integer programming to find the lower bound fast, and these lower bounds are critical to the branch-and-cut algorithm.

When the situations change, such as that the interference level exceeds two or that the transmitters are placed irregularly, it is clear that the complexities of these problems are not polynomial, because they are NP-Complete. So all of our algorithms dealing with these problems will only give sub-optimal answers in most cases. But we can use these algorithms to find lower bounds for certain cases.

With the limitation of time we cannot carry on a deeper study on this problem. For instance Simulated Annealing and Heuristic Search are also good methods to solve the fixed channel problem, but it is a pity that we don't have enough time to implement them.

## 6 Summary

We develop a model for solving the channel assignment problem. With a tiling technique a simple yet efficient strategy is provided to assign channels to the given regions. Our model is based on number theory and coloring theory. We try to constrain the span in an interval as small as possible, by increasing the lower bound and decreasing the upper bound for the span. We find that the upper bound and the lower bound meet at some certain value which is a function of $k$. Our model can be successfully applied to the situation in which there are only two levels of interference and the transmitters are located regularly. We also examine some general situations, in which the channel assignment problem becomes a NP-Complete problem. Be aware of this, we design several models to find a good lower bound. Integer programming and clique configuration are proposed. A situation is also concerned that there might not be enough channels to be assigned to satisfy all the separation constraints in real practice. Under such circumstances, we want to find an assignment with which the interference can be reduced as greatly as possible. We think that the Genetic Algorithm is an effective alternative, which is included in our models. Finally, strengths and weaknesses are discussed.

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# Appendix A: Rationales and Proofs

## A-1. $2k + 6$ Is A Lower Bound for The Span When $k \geq 3$

**Proof.**

Definition of symbol:

$\Omega(i)$: Define the available set of integer $i$ as a set that holds all the available integers that can be assigned adjacent to the cell of $i$. In other words any $n$ is in the available set of $i$ if it satisfies the constraint $|n - i| \leq k$. Denote this set as $\Omega(i)$. Obviously, if integer $i$ is used, $\Omega(i)$ must have 6 elements.

$d_{min}(i) = \min |f(p_j) - f(p_k)|$ where $p_j, p_k$ are two adjacent cells around cell $i$. If integer $i$ is used, there are 6 different numbers adjacent to cell $i$. The $2s$-constraint requires $d_{min}(i) \geq k \geq 3$.

First let us suppose that integer $k + 6$ is used in this assignment. We remove any integer $n$ in the sequence $1, 2, \cdots, 2k + 5$ if it satisfies $|n - (k + 6)| < k$, leaving those integers as the available set of $k + 6$. $\Omega(k + 6) = \{1, 2, 3, 4, 5, 6\}$. The $4s$-constraint requires that the integers used to be placed exactly around $k + 6$ to be different from each other. Obviously integer 3 must be placed adjacent to $k + 6$. But only 6 can be placed adjacent to 3 since 1, 2, 4, 5 all have too close a distance with 3. Since two different integers are needed to fill the two cells adjacent to integer 3, a single integer 6 is not enough. **So we may assert that integer $k + 6$ is not used in this assignment**.

Second we suppose that integer 6 is used in this assignment. We also remove any integer $n$ in the sequence $1, 2, \cdots, 2k + 5$ if it satisfies $|n - 6| < k$, leaving those integers as the available set of 6. If $k \geq 6$ then $\Omega(6) = \{k + 7, k + 8, k + 9, \cdots, 2k + 5\}$($k + 6$ is not used as we have just proved in the above paragraph). This means $d_{min}(6) \leq (2k + 5) - (k + 7) = k - 2$, it contradicts with $d_{min}(i) \geq k$. So 6 is not used. If $3 \leq k \leq 6$ then $\Omega(6) = \{1, \cdots, 6 - k, k + 7, k + 8, \cdots, 2k + 5\}$. Notice that there are actually 5 elements in $\Omega(6)$ which is not enough to be placed around integer 6 considering the $4s$-constraint. **So we may assert that 6 is not used in this assignment**.

Third we suppose that integer $k + 7$ is used in this assignment. $\Omega(k + 7) = \{1, 2, 3, 4, 5, 7\}$ (6 is not used as we have just proved). Still the $4s$-constraint requires that the integers which are used to be placed exactly around $k + 7$ to be different from each other. Obviously 3 is used. But only 7 can be placed adjacent to 3 since 1, 2, 4, 5 all have too close a distance with 3. There are two adjacent cells of integer 3 which means that one available integer 7 is not enough. **So we may assert that integer** $k+7$ **is not used in this assignment**.

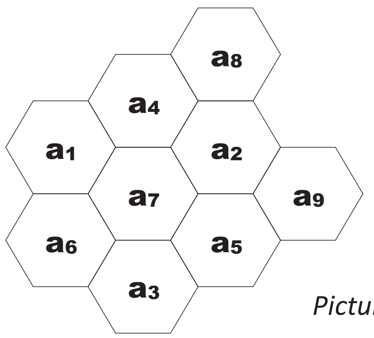
Fourth we suppose that the integer 7 is used in this assignment. If $k \geq 7$ then $\Omega(7) = \{k+8, k+9, \cdots, 2k+5\}$($k+7$ is not used as we have just proved before). This means $d_{min}(7) \leq (2k+5)-(k+8) = k-3$, it contradicts with $d_{min}(i) \geq k$. So 7 is not used. If $3 \leq k \leq 7$ then $\Omega(7) = \{1, \cdots, 7-k, k+8, \cdots, 2k+5\}$. Notice that there are actually 5 elements in $\Omega(7)$ which is not enough to be placed around integer 7 considering the $4s$-constraint. **So we may assert that 7 is not used in this assignment**.

Using the same technique we may assert that none of $k+8, 8, k+9, 9, k+10, 10, \cdots, 2k+5, k+5$ exist in the assignment, so only integers 1, 2, 3, 4 or 5 can be used in this assignment which is, in fact, impossible. The only reason for this is that our presumption is wrong. We cannot fill the grid with only $1, \cdots, 2k+4, 2k+5$. In other words at least $2k+6$ integers are needed to fill the grid. **So at least a span of** $2k+6$ **is necessary when** $k \geq 3$.

## A-2. $2k+7$ Is A Lower Bound for The Span When $k \geq 4$

**Proof.**

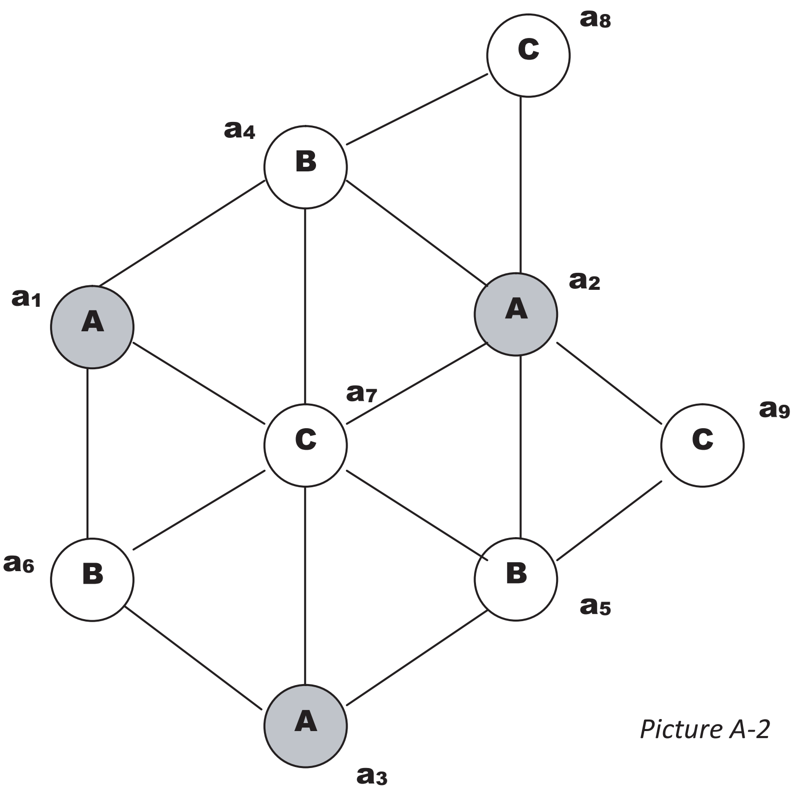
Consider an arbitrary cell $a_7$ and another 8 cells $a_1, a_2, \cdots, a_9$ nearby on an infinite grid, as shown in *picture A-1*.




*Picture A-1*

We construct a adjacent graph $G$ with these cells. $V(G) = \{a_1, a_2, \cdots, a_9\}$. If $a_i$ and $a_j$ are adjacent, there is an edge in $G$ between the vertex $a_i$ and $a_j$. $G$ is shown in *picture A-2*.

According to Coloring Algorithm we need three colors to color the graph $G$. Here we color $a_1, a_2, a_3$ with color $A$, color $a_4, a_5, a_6$ with color $B$ and color $a_7, a_8, a_9$ with color $C$. Cells of the



Picture A-2

same color cannot be assigned with repetitive channels because of the $4s$-constraint. We can use 3 consecutive integers to assign the cells of the same color in order not to waste any integer channel among the cells of the same color. There are three colors and each color is assigned to three cells, **so at least 9 channels are needed to fill these cells**.

Obviously, $a_7$ is adjacent to $a_1, \cdots, a_6$, so we know that $a_7$ has a different color from all its adjacent cells. Since $a_7$ is an arbitrary cell on the infinite grid, every cell on the infinite grid should have the same character with $a_7$ according to the symmetry. We can say that $a_1, \cdots, a_9$, also have this character. That means every cell has a different color from its adjacent cells. **If two cells** $a_i, a_j$ **are painted different colors then it is true** $|f(a_j) - f(a_i)| \geq k$.

Divide $a_1, \cdots, a_9$ according to their colors: set $A = \{a_1, a_3, a_5\}$, set $B = \{a_2, a_4, a_6\}$, and set $C = \{a_7, a_8, a_9\}$. We suppose: $f(a_1) < f(a_3) < f(a_5)$, $f(a_2) < f(a_4) < f(a_6)$, $f(a_7) < f(a_8) < f(a_9)$, and $f(a_1) < f(a_2) < f(a_7)$.

There exist three situations:

(i) *If* $f(a_1) < f(a_2) < f(a_3)$, *then* $f(a_1) < f(a_2) < f(a_4) < f(a_6) < f(a_3) < f(a_5)$.

Otherwise we may get an inequality: $f(a_1) < f(a_2) < f(a_3) < f(a_i)$, $i = 4$ or 6. Insert $a_7$ into this inequality and we now have a new one in an ascending way. Since any two consecutive integer channels belong to two different color sets, so

$$sp \geq f(a_1) + 4k \geq 1 + 4k \geq 1 + 2k + 2k > 1 + 2k + 2 * 3 = 2k + 7,\ k > 3.$$

It is known that $sp \leq 2k + 7$, it leads to a contradiction. Insert $f(a_7)$ to (i), there are 5 places for $f(a_7)$, such as

$$f(a_1) < f(a_2) < f(a_4) < f(a_6) < f(a_3) < f(a_5) < f(a_7).$$

Thus,

$$\begin{aligned} sp \geq f(a_7) \geq k + f(a_5) \geq k + 1 + f(a_3) \geq 1 + f(a_6) + 2k > 1 + 2k + f(a_2) + 2 \\ \geq 2k + 3 + f(a_1) + k \geq 3k + 3 + 1 = 2k + 4 + k > 2k + 4 + 3,\ k > 3. \end{aligned}$$

The result is identical if we assert $a_7$ into the other 5 places. Therefore the assumption $a_1 < a_2 < a_3$ leads to $sp > 2k + 7$. It contradicts with $sp \leq 2k + 7$, so the assumption is actually wrong.

(ii) *If* $a_3 < a_2 < a_5$*, the same proof as above can be given again to show that the assumption is wrong*.

(iii) $a_5 < a_2$. *This inequality holds forever*.

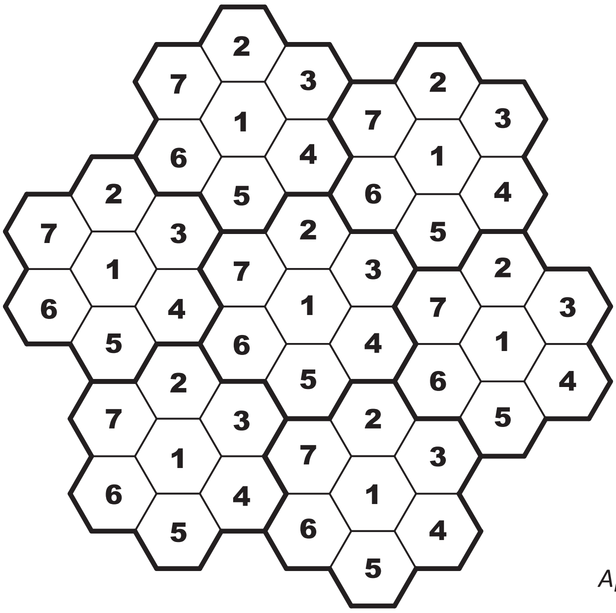
The proof as above shows $f(a_6) < f(a_7)$. Then we have

$$\begin{aligned} f(a_1) < f(a_3) < f(a_5) < f(a_2) < f(a_4) < f(a_6) < f(a_7) < f(a_8) < f(a_9), \\ sp \geq f(a_9) \geq f(a_7) + 2 \geq f(a_6) + k + 2 \geq f(a_2) + k + 4 \\ \geq f(a_5) + 2k + 4 \geq f(a_1) + 2k + 6 \geq 2k + 7. \end{aligned}$$

So $sp = 2k + 7$. A feasible assignment strategy is:

$$\begin{aligned} f(a_1) = 1,\ f(a_3) = 2,\ f(a_5) = 3, \\ f(a_2) = k + 3,\ f(a_4) = k + 4,\ f(a_6) = k + 5, \\ f(a_7) = 2k + 5,\ f(a_8) = 2k + 6,\ f(a_9) = 2k + 7. \end{aligned}$$

# Appendix B: Strategies

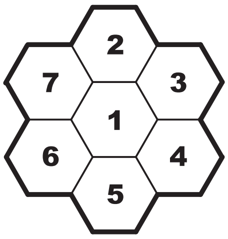


N=7

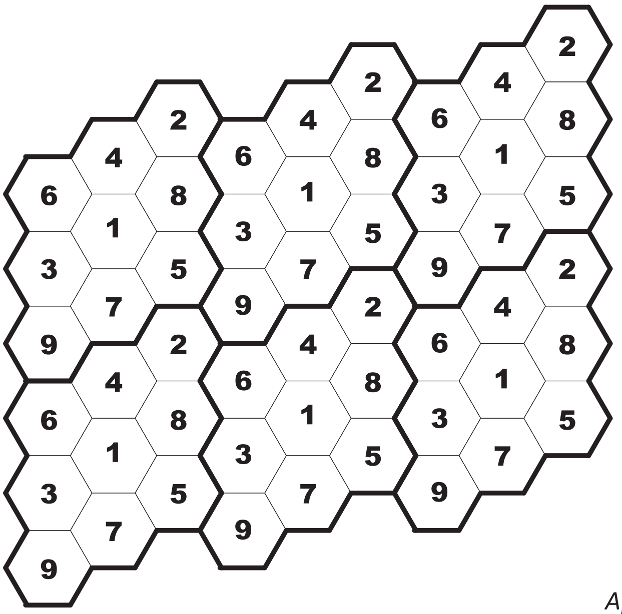
i=1

j=2

k=1



*Appendix B-1*

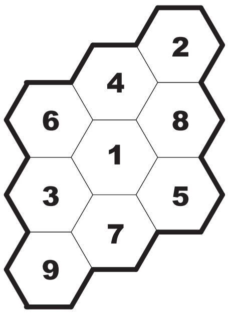


N=9

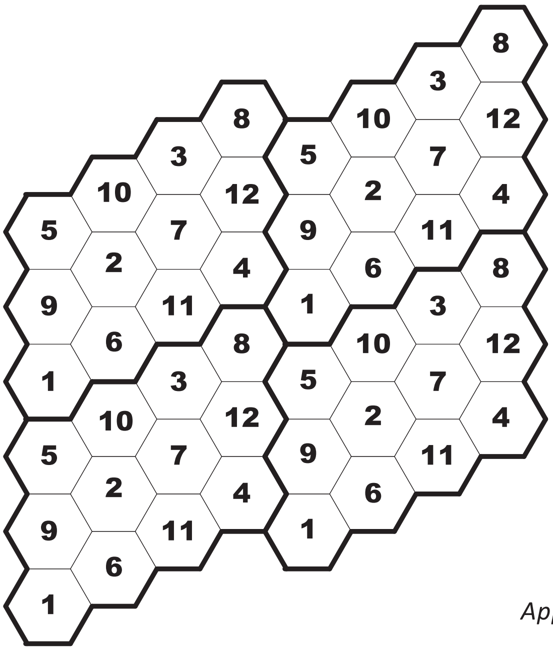
i=0

j=3

k=2



*Appendix B-2*

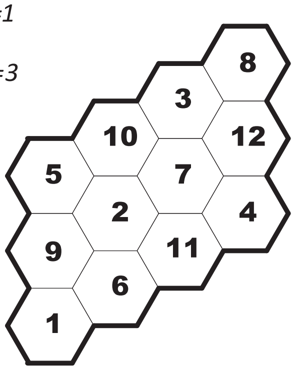


N=12

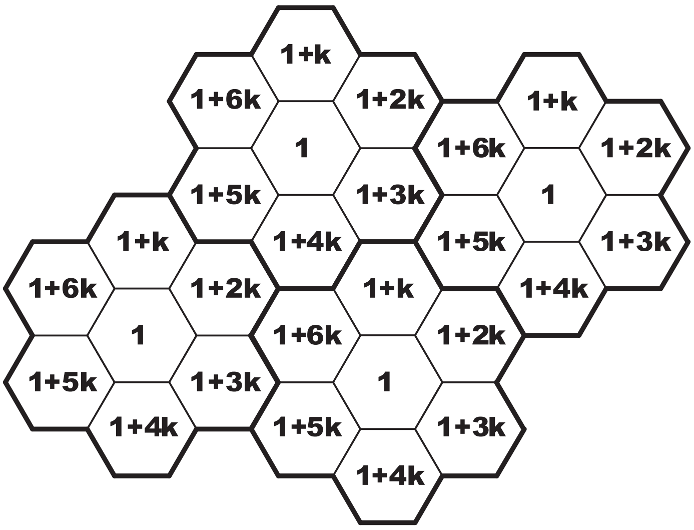
i=3

j=1

k=3



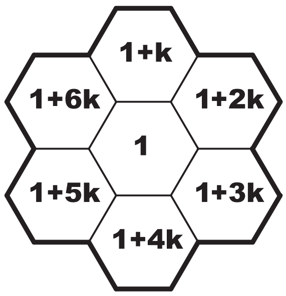
*Appendix B-3*



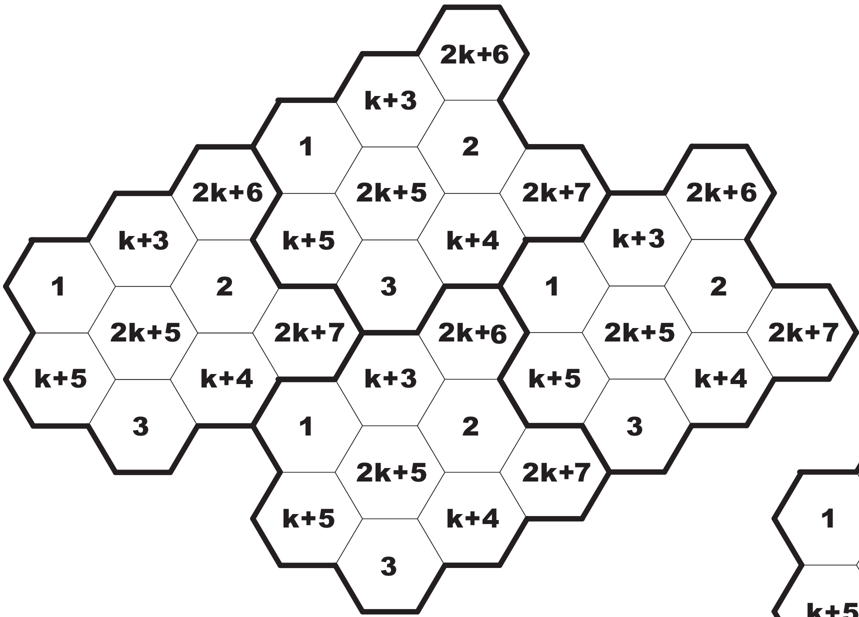
N=7

i=1

j=2



*Appendix B- 4*

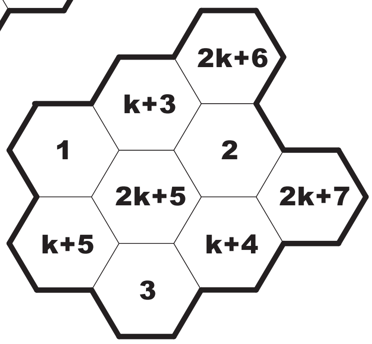



*N=9*

*i=3*

*j=0*

*k>3*




*Appendix B- 5*