Fewest Repeaters for a Circular Area: Iterative Extremal Optimization Based on Voronoi Diagrams

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Abstract

We propose a two-tiered network in which lower-power users communicate with one another through repeaters, which amplify signals and retransmit them, have limited capacity, and may interfere with one another if their transmitter frequencies are close and they share the same private-line tone.

Our objective is the fewest repeaters so that either every user is covered by at least one repeater or else every user can communicate with any other user anywhere in the considered area.

Motivated by cellular networks, we give a naïve solution where the number of repeaters and their positions can be obtained analytically. In a circular area with radius 40 miles, 12 repeaters can accommodate 1,000 simultaneous users.

We further propose an iterative refinement algorithm consisting of three fundamental modules that draw the Voronoi diagram, determine the centers of the circumscribed circles of the Voronoi regions, and escape the local optimum by using extremal optimization. The algorithm obtains a solution with 11 repeaters, which we prove to be the absolute minimum. For 10,000 users, it uses 104 repeaters, better than the naïve solution's 108.

We further discuss how to assign frequencies and private-line tones (based on maximum and minimum spanning tree techniques), accommodating simultaneous users, the fluctuation of user density in reality, how the landscape can affect repeaters' locations, and the strengths and weaknesses of the model and the algorithms.

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Introduction

Amplify-and-forward relay networks are very helpful for long-distance communication. Through relay nodes, low-power users can communicate with one another in situations where direct user-to-user communication would not be possible. For example, amateur radio "hams" communicate through relay nodes (repeaters), and in wireless sensor networks relay nodes help information dissemination [Akyildiz et al. 2002].

In many such networks, the nodes are homogeneous, and a node can simultaneously play the roles of source node, sink node, and relay node; a typical example is a cellphone. In other scenarios, the relay nodes usually do not initiate communications but only help communications between other nodes. For example, in amateur radio, repeaters can be considered relay nodes, and hams usually carry their own radios (transceivers); the functionalities of repeaters and radios are different, as are their power specifications.

Pan et al. [2003] proposed a two-tiered relay network model, where base stations are considered low-power users and some interapplication nodes play the role of relay nodes. However, they did not address the issue of covering all users. Gupta and Younis considered fault-tolerant [2003a] and traffic load balance [2003b] problems in a two-tiered relay network model but did not address the placement of relay nodes. Tang et al. [2006] proposed two algorithms for placing the fewest relay nodes.

Those works differ from ours because in a more general scenario the capacity of a repeater and interference among nearby repeaters should be taken into account. We propose a model in which each repeater can simultaneously manage at most C users and two nearby repeaters interfere with each other if their transmitter frequencies are close and they share the same private-line tone. Our objective is the fewest repeaters that can satisfy the users' communication requirement. We consider two such requirements:

- **Weak requirement:** Every user is covered by at least one repeater; this is equivalent to a circle-covering problem.
- **Strong requirement:** Every user can communicate with any other user.

The circle-covering problem is NP-hard [Fowler et al. 1981] and is usually very time-consuming even for a small number of circles. For example, Nurmela and Östergård [2000] proposed a simulated annealing algorithm to obtain near-optimal solutions to cover the unit square with up to 30 equal circles; their algorithm has to run more than 2 weeks for 27 circles.

Motivated by cellular networks, we give a naïve solution in which the number of repeaters and their positions can be obtained analytically. In numerical simulation, this naïve solution performs unexpectedly well. For a circular area with radius 40 miles, 12 repeaters accommodate 1,000 simultaneous users; for 10,000 users, 108 suffice. We propose an iterative refinement algorithm that draws the Voronoi diagram, determine the centers of the circumscribed circles of the Voronoi regions, and escapes the local optimum by using extremal optimization. For 1,000 users, this method has 11 repeaters, which we prove is optimal. For 10,000 users, it has 104 repeaters.

Problem Description

Given a circular flat area Γ of radius Φ, we are to determine the fewest radio repeaters to accommodate N users. A repeater is a combination receiver/transmitter that picks up weak signals, amplifies them, and retransmits them on a different frequency. The three parameters characterizing a repeater are receiver frequency f_r, transmitter frequency f_t, and private-line (PL) tone n_{PL}. A repeater responds only to signals on its receiver frequency that contain its PL tone and retransmits with the same PL tone. Both f_r and f_t are in the range [145 MHz, 148 MHz], we have $|f_r - f_t| = 0.6$ MHz, and there are $N_{\text{PL}} = 54$ PL tones available.

The maximal communication distance r from a user to a repeater is the same for every user, and it is considerably smaller than the communication radius R of a repeater. Every user should be covered by at least one repeater.

The primary problem is to determine the minimum number of repeaters to satisfy the communication requirement.

Model

We use a two-tiered directed network $D\{V_u, V_r, E_{ur}, E_{rr}\}$, where $V_u = \{u_1, u_2, ..., u_N\}$ and $V_r = \{r_1, r_2, ..., r_M\}$ denote the sets of users and repeaters, and E_{ur} and E_{rr} are the sets of directed links from users to repeaters and between repeaters. A user u_i is identified by a location $\big(x(u_i), y(u_i)\big)$ in the plane, and a repeater r_j is identified by its location, receiver frequency, transmitter frequency and PL tone as $\big(x(r_j), y(r_j), f_r(r_j), f_t(r_j), n_{PL}(r_j)\big)$. The frequencies of a repeater r_j satisfy $f_r(r_j), f_t(r_j) \in [145\,\text{MHz}, 148\,\text{MHz}]$ and $|f_r(r_j - f_r)| = 0.6$ Mhz. Since the considered area is circular with radius Φ, the location of a user or a repeater satisfies $x^2 + y^2 \leq \Phi^2$, where $\Phi = 40$ mi. A directed link from u_i to r_j exists if $|u_i - r_j| \leq r$. A directed link from r_j to r_k exists (i.e., $(r_j, r_k \in E_{rr})$) if $|r_j - r_k \leq R$, the transmitter frequency of r_j equals the receiver frequency of r_k (i.e., $f_t(r_j) = f_r(r_k)$), and they share the same PL tone (i.e., $n_{PL}(r_j) = n_{PL}(r_k)$). Clearly, we need to know the locations of users and repeaters.

A network D is a solution if the following three conditions $(\Omega_1, \Omega_2, \Omega_3)$ are all satisfied.

- Ω_1 **- Capacity**. For simplicity, we assume that users are uniformly distributed and each communicates with the nearest repeater. Each repeater can manage at most C users at the same time. For repeater r_j, there is a connected area $S_V(r_j)$, the *Voronoi region* of r_j, such that for every point inside r_j is the nearest repeater and for every point outside, r_j is definitely not the nearest repeater. The number of users inside the Voronoi region of a repeater must be no more than its capacity C.

- Ω_2 **- Interference avoidance**. If two repeaters share the same PL tone and are less than $2R$ apart, the difference between their transmitter frequencies must be no less than the threshold $f_c = 0.6$ MHz.
- Ω_3 **- Connectivity**. Every user is covered by at least one repeater. That is, for every u_i, $\exists r_j$ such that $(u_r, r_j) \in E_{ur}$.

Our goal is a solution with the minimum number of repeaters M. Although every user is covered by at least one repeater, the user's signals may not reach the desired position in Γ, since the coverage of a repeater is also limited and the solution does not guarantee a multi-hop path through several repeaters to reach the desired position. Ignoring the small area that can be reached directly by a user without the help of a repeater, the reachable area $S_r(u_i)$ of user u_i is the area in Γ that can be covered by at least one reachable repeater of u_i (each repeater covers a circle with radius R).

The set of reachable repeaters $R_r(u_i)$ for u_i consists of:

- repeaters directly reachable by u_i (i.e., the repeaters located within the circle with radius r and centered at u_i), and
- repeaters reachable through links in E_{rr} from the directly-reachable repeaters.

Figure 1 illustrates a simple example where r_2 can be directly reached by u_1, and r_4 and r_5 can be further reached starting from r_2. Two nearby repeaters, r_2 and r_3, may not have a link between them, since they may not match in frequency or PL tone. The reachable repeaters of u_1 are r_2, r_4, and r_5, and the reachable area of u_1 is the union of their coverage areas. The following condition must be satisfied to guarantee every user can in principle reach any position of the considered area through multi-hop repeaters.

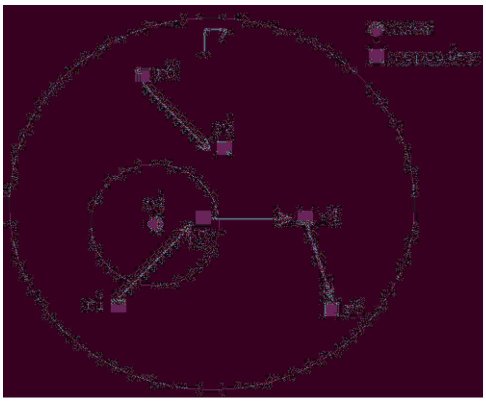

Figure 1. An illustration of repeaters reachable from u_1. The circle centered at u_1 has radius r.

- Ω_4 **- Global reachability**. The reachable area of every user is all of Γ.

A network D is a *strong solution* if $(\Omega_1, \Omega_2, \Omega_4)$ are all satisfied. When $R \geq 2\Phi$, any solution is a strong solution.

To find a strong solution is much more difficult than to find a solution, and the two tasks are equivalent only if $R \geq 2\Phi$.

Analysis

We calculate the communication ranges for repeaters and users, as well as the repeater's capacity, using Shannon's information theory. Taking into consideration mobility of users, we show that continuous approximation of the distribution of users' locations is necessary to address the problem. We present a naïve solution with repeaters arranged in a cellular network.

Communication Radius

We assume that there is no interference from fog, rivers, hills, buildings, sunspots, etc.

Let $P_{r,\text{out}}$ be the power of the signal transmitted by a repeater. Its average power P in a unit area at a distance D from the repeater is

$$P = \frac{P_{r,\text{out}}}{4\pi d^2}.$$

According to antenna theory [Balanis 2005], the effective receiving area of an antenna is $\lambda^2/4\pi$, where λ is the wavelength of the signal. So the receiving power of the signal is

$$P' = \frac{P_{r,\text{out}}}{4\pi d^2} \times \frac{\lambda^2}{4\pi}.$$

Replacing λ by c/f, where c is the velocity of light and f is the frequency of the signal, we obtain

$$P' = P_{r,\text{out}} \left(\frac{c}{4\pi d f} \right)^2.$$

In terms of Shannon's information theory, the loss L_s is

$$L_s = 10 \log_{10} \left(\frac{P_{r,\text{out}}}{P'} \right) = 92.4 + 20 \log_{10} d + 20 \log_{10} f,$$

where L_s is in dB, d is in km, and f is in GHz. The actual power of the received signal $P_{r,\text{in}}$ is

$$P_{r,\text{in}} = P_{r,\text{out}} + (G_{\text{out}} + G_{\text{in}}) - (L_{f,\text{out}} + L_{f,\text{in}}) - (L_{b,\text{out}} + L_{b,\text{in}}) - L_s.$$

From one repeater (transmitter) to another repeater (receiver), the equations that hold, and the conventional values, are

$$\begin{aligned} L_{f,\text{out}} &= L_{f,\text{in}} = 20 \text{ db} && \text{(the loss of the feed system)},\\ L_{b,\text{out}} &= L_{b,\text{in}} = \ \ 1 \text{ db} && \text{(other loss of the system)},\\ G_{\text{out}} &= G_{\text{in}} \ = 39 \text{ db} && \text{(the gain of the antenna)}. \end{aligned}$$

For this problem, the frequency of signals is about 146.5 MHz (the midpoint of the available spectrum 145–148 MHz), and thus

$$d = 10^{\frac{10\log_{10}\left(\frac{P_{r,\text{out}}}{P_{r,\text{in}}}\right) - 37.2328}{20}}. \tag{1}$$

The effective radiated power of most repeaters is $P_{r,\text{out}} = 100$ W [Utah VHF Society 2011], and normally a repeater can receive a signal with power no less than $1\mu W$ (i.e., $P_{r,in} \geq 1\mu W$). According to **(1)**, the communication radius of a repeater is $R \approx 85.5$ mi. Analogously, the average working power for a user (according to several wireless devices) is $P_{u,\text{out}} = 3.2W$ and $P_{u,\text{in}} \geq 1\mu W$, resulting in a communication radius $r \approx 15.28$ miles.

Repeater's Capacity

We calculate the capacity C of a repeater. Ignoring background noise and the interference, we assume that signals from one repeater do not affect others. A mainstream method to estimate the capacity of information over a noisy channel, according to Shannon's theory, is

$$\phi = B\log_2(1 + \text{SNR}) \tag{2}$$

where ϕ is the information bit rate (dB), SNR is the signal-to-noise ratio (dimensionless), and B is the total bandwidth (Hz).

The transmitter frequency in a repeater is an exact value rather than in a broad band. We use the equation

$$\frac{E_b}{N_0} = \frac{G_{\text{out}}}{V(C-1)(1 + I_{\text{other}}/I_{\text{self}})},$$

where E_b/N_0 is the level that ensures operation of bit-performance at the level required for digital voice transmission, G is the gain of the antenna, V is the gain of voice, I_{other} is the interference from other repeaters, and I_{self} is the interference of a repeater with itself. The SNR can be regarded as the ratio of effective information in the total received signal:

$$\text{SNR} = \frac{P_{ur}}{(C-1)P_{ur}} = \frac{1}{C-1},$$

where P_{ur} is the power of the signal from a single user as received by the repeater (measured in watts). Normally, $G = 39$ dB $= 7966.40$ W and $V = 0.4$ dB $= 1.07$ W (in the calculation, the units must be watts). We ignore interference from other repeaters: We set $I_{\text{other}}/I_{\text{self}} = 0$. Setting $E_b/E_0 = 18$ dB $= 63.01$ W (the bit energy-to-noise density ratio always ranges from 5 to 30 dB; we set it at the midpoint 18 dB), we get the capacity of a repeater as

$$C = 1 + \frac{G_{\text{out}}}{V(1 + I_{\text{other}}/I_{\text{self}} E_b/N_0)} \approx 119.$$

Continuous Approximation

Because users are mobile, the (fixed) repeaters must cover all of the considered area Γ. A solution for one distribution of users may not be a solution for another distribution. Therefore, a practical solution should not depend on a specific distribution.

Consequently, we use a continuous uniform distribution instead of a discrete uniform distribution of users. The user density is $\rho = N/\pi\Phi^2$. Omitting the bow-irrelevant Ω_2, the other three constraints changed to

- Ω_1^* **- Capacity**. Every repeater r_j, which has $S_V(r_j)$ as the area of its Voronoi region, must satisfy $\rho S_V(r_j) \leq C$.
- Ω_2^* **- Connectivity**. Every point is covered by a repeater.
- Ω_3^* **- Global reachability**. The reachable area of every point (considering that at every point, there can be user) is equal to the considered area Γ.

In the frequency range [145 MHz, 148 MHz] with $f_c = 0.6$ MHz, in a PL tone, if $R \geq 2\Phi$, there are at most 6 different repeater transmitter frequencies without interference (145.0 MHz, 145.6 MHz, 146.2 MHz,. . . , 148.0 MHz).

Let repeater i have receiver frequency $f_{i,1}$ and transmitter frequency $f_{i,2}$. With more than 6 repeaters, there must be at least one pair i and j with "inverse frequencies," i.e., $f_{i,1} - f_{i,2} = f_{j,2} - f_{j,1}$. Repeater i may amplify signals and send them to repeater j, and repeater j may amplify those signals and send them back to repeater i, and so on. To avoid this problem, we put only noninteracting repeaters in a PL tone group; the maximum size of such a group is 5. So when $R \geq \Phi$, with N_{PL}=54 different PL tones, the maximum number of repeaters without any interference is $54 \times 6 = 324$, and without any interactions is $54 \times 5 = 270$. Therefore, if the required number of small circles is no more than 324, we do not need to consider interference avoidance but just set repeaters not to interfere with one another; if the required number is no more than 270, we can make sure there will not be interactions between repeaters.

According to the connectivity constraint, the (integer) number of repeaters M should satisfy

$$M \geq \left\lceil \frac{\pi\Phi^2}{\pi r^2} \right\rceil = \left\lceil \frac{\Phi^2}{r^2} \right\rceil \approx \left\lceil \frac{40^2}{15.28^2} \right\rceil \approx \lceil 6.9 \rceil = 7. \tag{3}$$

According to the capacity constraint, M should satisfy

$$M \geq \left\lceil \frac{N}{C} \right\rceil = \left\lceil \frac{N}{118} \right\rceil = \left\lceil \frac{1000}{118} \right\rceil \approx \lceil 8.5 \rceil = 9. \tag{4}$$

For $N = 10{,}000$, we need $M \geq 85$.

naïve Solution

We use a cellular network of equal-size regular hexagons centered at repeaters. The pattern is the *Voronoi diagram* of the repeater sites.

We first consider an inverse problem: to determine the largest circle that can be covered by a number of such hexagons with edge length 1. **Table 1** gives the results up to 13 regular hexagons, calculated by hand.

Table 1.
Radius of the largest circle coverable by a network of M regular hexagons of side 1.

M	1	2	3	4	5	6	7	8	9	10	11	12	13
Radius	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{7}}{2}$	$\frac{7}{5}$	$\sqrt{3}$	2	2	2	$\frac{\sqrt{19}}{2}$	$\frac{\sqrt{19}}{2}$	$\sqrt{7}$	$\sqrt{7}$

We show how to obtain a solution for $R \geq 2\Phi$ and $N = 1{,}000$ by using **Table 1**. The user density is $\rho = N/\pi\Phi^2 \approx 0.1989$. To make sure that each point is covered by at least one repeater, the edge length r_h of the regular hexagon should not exceed the communication range r. In addition, according to the capacity constraint, r_h should satisfy

$$\frac{3\sqrt{3}}{2} r_h^2 \rho \leq C. \tag{5}$$

These two conditions determine the longest possible edge length; to cover as large circle as possible, we always use the longest r_h. For our case, $r = 15.28$ mi, and according to **(5)**, r_h should be 15.28 mi. The circle to be covered has radius $\Phi = 40$ mi. Since

$$\frac{\sqrt{19}}{2} \approx 2.18 < \frac{\Phi}{r_h} \approx \frac{40}{15.28} \approx 2.62 < \sqrt{7} \approx 2.65,$$

according to **Table 1**, 12 repeaters are sufficient (but 11 won't do) (**Figure 2**). (We consider only $R \geq 2\Phi$, for which frequencies and PL tones are easily arranged; in our example, we use 3 PL tones.) The algorithm for the naïve solution is to find the longest allowable r_h and search **Table 2** (extended

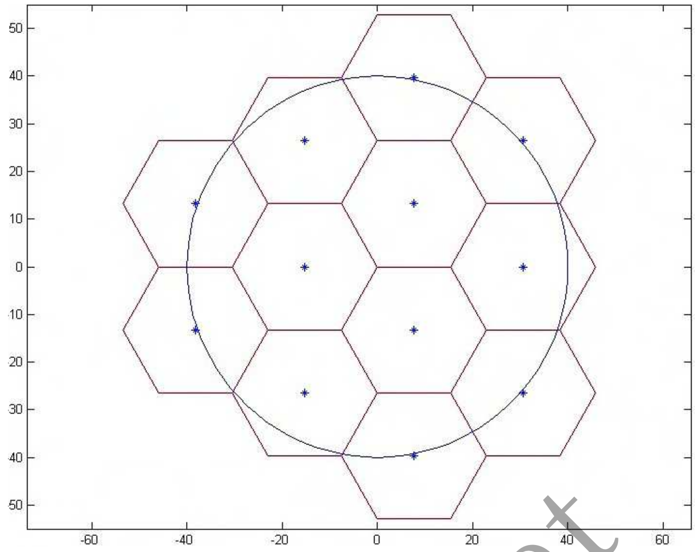

Figure 2. The solution with 12 repeaters arranged in a cellular network.

as necessary). However, extending the table is not easy (at least by hand). We get analytical results, for up to 121 circles, for two particular cases: the covered circle's center is at the center of the central hexagon, or it is at the intersection of three more central hexagons (**Table 3**).

Table 3.
Analytical results for two particular cases:
Left: The covered circle is centered at the center of the central hexagon.
Right: The covered circle is centered at the intersection of the three central hexagons.

Center of a circle		Intersection of three circles	
Cells	Radius of largest coverable circle	Cells	Radius of largest coverable circle
1	$\sqrt{3}/2$	3	1
7	2	6	$\sqrt{3}$
13	$3\sqrt{3}/2$	12	$\sqrt{7}$
19	$\sqrt{13}$	18	3
31	$5\sqrt{3}/2$	27	4
37	5	36	$\sqrt{21}$
55	$7\sqrt{3}/2$	48	$\sqrt{43}$
61	$\sqrt{43}$	60	6
85	$9\sqrt{3}/2$	75	7
91	8	90	$\sqrt{57} \approx 7.5$
121	$\sqrt{91} \approx 9.5$	108	9

Applying this approach to the case $N = 10{,}000$, we have $\rho = N/\pi\Phi^2 \approx 1.989$ and $r_h = 4.8$ miles. Since $8 < \Phi/r_h < 9$, according to **Table 3**, 108 repeaters are sufficient. Since this number is smaller than 324, it is easy to arrange them when $R \geq 2\Phi$. [EDITOR'S NOTE: We omit the authors' figure.]

Algorithms

We present algorithms

- to adaptively place the repeaters so as to solve the circle-covering problem with the fewest circles; and
- to assign receiver frequencies, transmitter frequencies, and PL tones.

Algorithm for Repeater Locations

1. Randomly place M_0 (the lower bound from **(3)** and **(4)**) repeaters.
2. Determine the Voronoi region of each repeater [Aurenhammer 1991].
3. Determine the circumscribed circle of each Voronoi region [Megiddo 1983].
4. Calculate the coordinates of the center of each circumscribed circle.
5. Calculate the distance from each repeater's current location to the center of its circumscribed circle. Sum up the distances over all repeaters and compare the sum to a threshold ζ. If the sum is less, the current locations are considered to be converged; otherwise, move each repeater to the center of its circumscribed circle and return to Step 2.
6. Check if the number of users in each Voronoi region is less than the repeater's capacity C, and if the radius of the circumscribed circle is less than the communication radius of users. If so, stop the algorithm and output the current solution. Otherwise, go to Step 7.
7. If the number of extremal optimization operations is less than a threshold T_c, pick up the repeater with the smallest Voronoi region, move it to a random position and go to Step 2; and advance the count of extremal optimization operations. Otherwise, add one more repeater, randomize the positions of all repeaters, and go back to Step 2.

Extremal optimization is a method to escape a local optimum by changing the individual with the least fitness (here, the repeater with the smallest Voronoi region). This idea comes from the Bak and Sneppen [1993], who describe the punctuated equilibrium in evolution caused by the annihilation of the least-fit species. In our simulation, $\xi = 0.01$ and $T_c = 100$.

Algorithm for Assignments

This algorithm does not change locations of repeaters as obtained from the first algorithm nor add new repeaters. It tries to maximize the reachable area of users just by rearranging the frequencies and PL tones of repeaters.

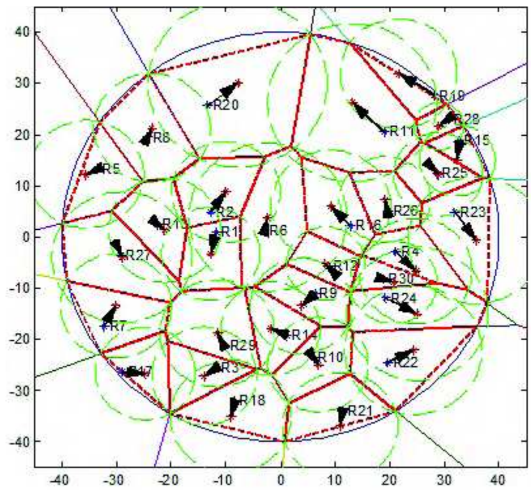

Figure 3. Example of application of the algorithm for repeater placement. The labeled plus signs are the original locations of the repeaters, and the black asterisks are their final locations; arrows show the movements. Voronoi regions are outlined by red (thick) line segments, and circumscribed circles are drawn in green (dashed) arcs (with red asterisks for their centers).

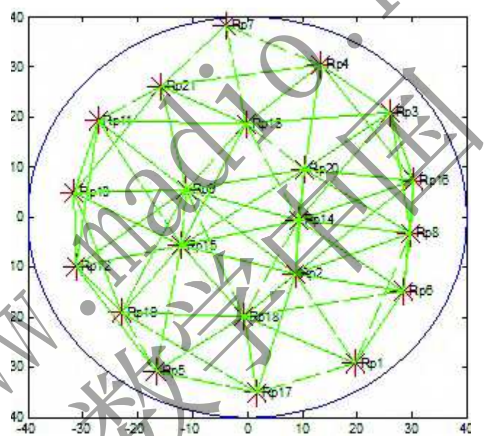

Figure 4. The red asterisks represent repeaters, and a green line connects two repeaters if they can communicate (that is, are less than R apart).

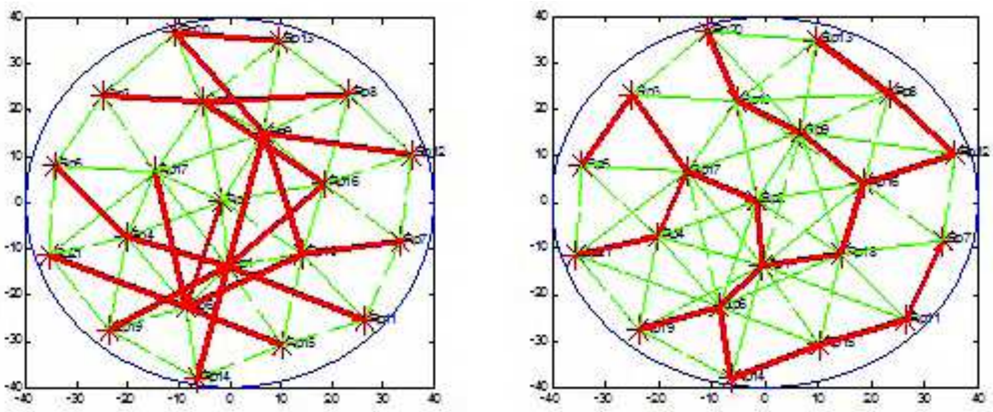

Figure 5. Maximum spanning tree (left) and the minimum spanning tree (right). The red (thick) lines are the edges in the spanning trees and the green (thin) lines are the other edges in the graph.

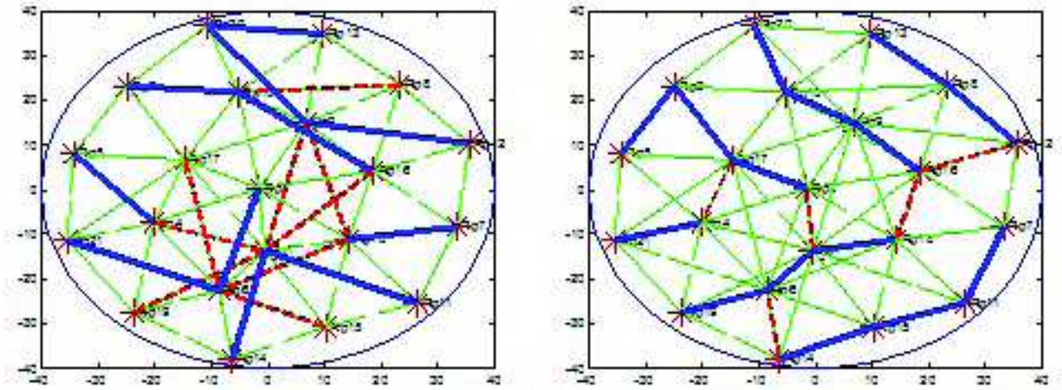

Figure 6. Resulting routings after the removal of edges corresponding to the spanning trees in **Figure 5**. The red (dashed) lines are the removed edges, while the blue (thick solid) lines are the preserved ones. Repeaters connected by a blue line share the same tone.

1. Construct a graph G to represent the relation that two repeaters can transmit to each other: If the distance between them is less than R, add an edge between them.

2. Find the minimum spanning tree (mST) T or the maximum spanning tree (MST) T' of G. From the illustration in **Figure 4**, we see that edges in the minimum spanning tree will not intersect. If we build the transmission paths along the mST, the signals received by repeaters are fewer than for the maximum spanning tree. However, since the distance between two adjacent repeaters along the MST tree is the shortest, the received signals will be much stronger. So the assignment based on the mST is suitable for communication in a local area. In contrast, most edges in the MST intersect with one another, the number of signals in many areas is very large, and signals can cover larger areas. However, this situation increases the chance of interference. Depending on purpose, one can choose either spanning tree to continue the algorithm.

3. Remove edges from the tree. For any node i with degree $k > 3$, delete $k - 2$ edges. Then the node i will be apart from $k - 2$ connected components. Let the size of component j be SC_j, then the method of removing edges can be presented as follows: Find $k - 2$ edges to remove in order to minimize $\sum_{a,b} |\mathrm{SC}_a - \mathrm{SC}_b|$. The results can be found in **Figure 5**.

4. After Step 3, we may have several signal routes that do not connect. Assign a different PL tone to each route. Then for the repeaters in each route, assign transmitting frequency and receiving frequency. Make sure that the transmitting frequency of a repeater is the receiving frequency of the repeater's neighbors in the same route. (See **Figure 6**.)

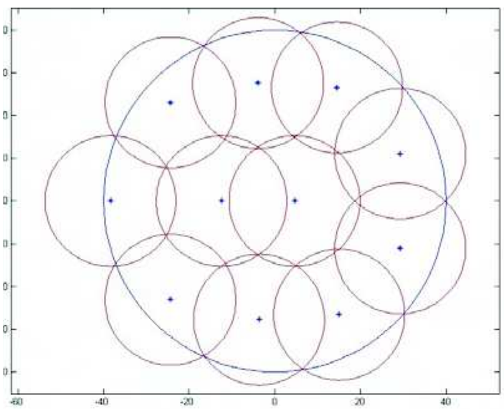

Figure 7. Solution with 11 repeaters obtained by our algorithm.

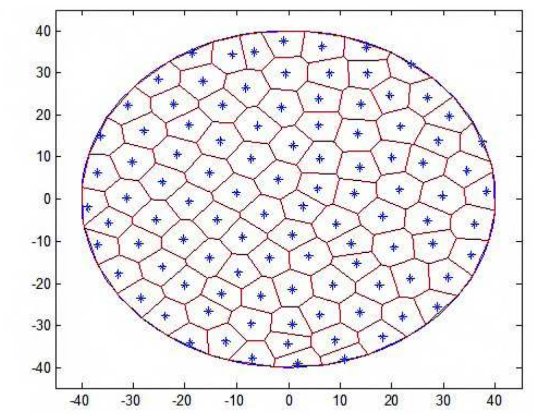

Figure 8. Solution with 104 repeaters obtained by our algorithm.

Simulation

N = 1,000, R = 85.45 mi

Figure 7 shows a solution with 11 repeaters obtained by our algorithm. The maximal Voronoi area is 560.56, the user density is 0.1989, and thus the largest capacity demand is 112, smaller than the repeater's capacity $C = 119$. Compared with the naïve solution, fewer repeaters are required and sizes of the Voronoi regions are more homogeneous.

We prove that 11 repeaters is optimal: 10 repeaters with radius no more than 15.28 miles cannot cover a circle with radius 40 miles.

Lemma [Toth 2005]. *Let $r(n)$ be the maximum radius of a circular disc that can be covered by n closed unit circles, then*

$$r(n) = 1 + 2\cos\left(\frac{2\pi}{n-1}\right)$$

for $n = 8$, $n = 9$, and $n = 10$.

According to the Lemma,

$$r(10) = 1 + 2\cos\left(\frac{2\pi}{9}\right) \approx 2.53 < \frac{40}{15.28} \approx 2.62,$$

so coverage by 10 circles is not possible.

N = 10,000, R = 85.45 mi

Figure 8 shows a solution with 104 repeaters obtained by our algorithm; 21 PL tones are used to guarantee that every pair of repeaters will not interact each other.

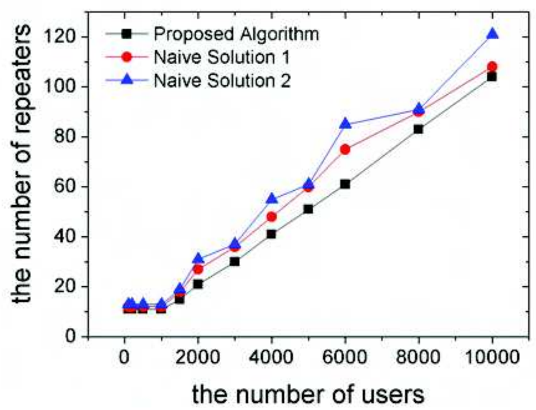

Figure 9. Comparison of fewest required repeaters obtained by our algorithm vs. the naïve solution. The largest coverable circle in the naïve Solution 1 is centered at the center of the central hexagon, while the largest coverable circle in naïve Solution 2 is centered at the intersection of the three more central hexagons.

N =1,000, R = 40 mi

The repeaters' locations are the same as earlier, but the frequencies and PL tones are different.

To test the effectiveness of the secondary algorithm, we randomly pick 100,000 ordered pairs of points (u, v) inside the considered area and see in how many pairs u can send to v. The answer is 90,708. So, given a user, the probability that the system can satisfy this user's requirement to communicate with any other user at random is about 91%.

N = 100,000, R = 40 mi

It seems that when the required number of repeaters increases, the reachable area of a user increases. For example, in this case, for the solution found by our algorithm, the corresponding probability is 97%.

Sensitivity Analysis

Sensitivity of Parameters

We discuss to what extent the results depend on the parameters. **Figure 9** displays the fewest repeaters required vs. the number of users and shows that our algorithm is better than the naïve solution. There is a transition point at about $N = 1{,}000$, which approximately satisfies the equation

$$\frac{N}{\pi\Phi^2} = \frac{C}{\pi r^2}.$$

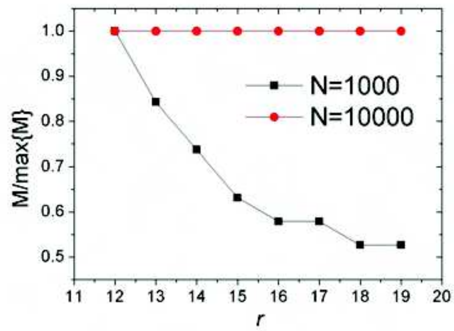

Figure 10. Comparison between the fewest required repeaters obtained by our algorithm vs. the naïve solution. The black points correspond to the case of N = 10,000 and the red points to N = 10,000. The two curves have been normalized.

Up to the transition point, the number of repeaters mainly depends on the communication range r, while beyond it is capacity that becomes the bottleneck determining the fewest required repeaters. Since the capacity constraint plays a major role when there are many users, it is not a surprise that the number of repeaters grows linearly with the number of users.

Figure 10 reports the relation between the fewest required repeaters and the user's communication range r. Each curve has been normalized by dividing by its respective largest value. In the case $N = 10{,}000$, the number of repeaters never changes with r, again indicating that the capacity limitation determines the result, while when $N = 1{,}000$, the number of repeaters decreases with increase of the user's communication range.

User Density Fluctuation

In the analysis of our model, the user density is constant, behaving like a real variable. However, in reality, the number of users can only be an integer. For the discrete case where users are distributed uniformly, each user belongs to Voronoi area S_ν with probability $S_\nu/\pi\Phi^2$, and thus the number X of users in this area obeys a Bernoulli distribution

$$P(X) = \binom{N}{X} \left(\frac{S_\nu}{\pi\Phi^2}\right)^X \left(1 - \frac{S_\nu}{\pi\Phi^2}\right)^{N-X},$$

whose expectation and standard deviation are

$$E(X) = \frac{NS_\nu}{\pi\Phi^2} = \rho S_\nu, \quad \sigma(X) = \sqrt{\frac{NS_\nu}{\pi\Phi^2}\left(1 - \frac{S_\nu}{\pi\Phi^2}\right)} = \sqrt{\rho S_\nu}\sqrt{1 - \frac{S_\nu}{\pi\Phi^2}}.$$

When the total number N of users is small, capacity is not a big problem; and when N is big (corresponding to high user density), the area S_ν should be small due to the capacity limitation. Therefore, the standard deviation is approximately the square root of the expected number of users in the Voronoi region. We are interested in the case when the number of users in S_ν approaches the capacity limitation. For example, in this problem, we have $C = 119$, so the standard deviation is about 11. If we set up a tolerance of two standard deviations, the tolerant capacity C' should satisfy

$$C' + 2\sqrt{C'} = C,$$

leading to $C' = 99$.

Effects of Landscape: Mountainous Areas

[EDITOR'S NOTE: We must omit the team's discussion of this point.]

Conclusion and Discussion

We propose a two-tiered network model, where lower-power users communicate with one another through repeaters, taking into account in our model capacity constraints and interference.

We give a naïve solution in which the number of repeaters and their positions can be obtained analytically. We further develop an algorithm based on Voronoi diagrams, which outperforms the naïve solution. For 1,000 users, the algorithm proposes 11 repeaters, which we prove to be optimal. For the 10,000 users, the algorithm obtains a solution with 104 repeaters.

Moreover, we offer an algorithm, based on maximum and minimum spanning trees, to assign frequencies and private-line tones. This algorithm does not introduce any new repeaters yet can broaden the reachable areas of users.

Compared with the related model for sensor wireless networks and mobile communication networks, our model is more general. Our algorithm is effective and efficient: It runs much faster than the simulated annealing approach [Nurmela and Östergård 2000] and is better able escape the local optimum than another iterative refinement algorithm [Das et al. 2006]. Our algorithm does not require any specific geographical features of the considered area, while many efficient circle-covering algorithms work well only for squares.

There are also two considerable weaknesses in our work:

- We have not developed an algorithm that satisfies the constraint of global reachability without adding too many repeaters.
- Our model does not take into account heterogeneity of users or repeaters, or wave reflection and refraction by the atmosphere.

Appendix

Summary of Assumptions

A1 Users are uniformly distributed in the considered area, and in a more strong assumption, we consider the number of users as a real variable and the user density in the considered area is a constant.

A2 Users prefer to communicate with the nearest repeaters.

A3 Consider two repeaters sharing the same PL tone, with the difference between their transmitter frequencies less than a threshold $f_c = 0.6$ MHz. If the distance between them is less than $2R$, they will interfere with each other.

A4 In the considered circular area, wireless signals can fade freely; there are no other sources of interference such as fogs, rivers, hills, buildings, activities of the Sun, and so forth, so that the fading of signals is due only to the distance involved.

A5 There is no background noise in this system.

A6 Repeaters don't have noisy impact on others.

A7 Functionalities and specifications of users' radios are the same (i.e., homogeneous users' radios). Functionalities and specifications of radio repeaters are the same (i.e., homogeneous repeaters).

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