AI Planning for Autonomy

Problem Set X: Game Theory

1. Auctions are a classic example of game theory at work. Although the analysis of auctions is more advanced than what we have looked at in these note — mainly because the set of strategies (the bids) is so large —, we can paint a picture of how they work.

   Probably most of you are familiar with the concept of an *English auction*, in which the bidding starts at a price and bids are incremented until nobody wants to bid any higher. The winner is the person who bids the highest.

   But how many of you have heard of *Dutch auctions*? This is where the bidding starts at a (presumably high value) and the auctioneer *decrements* the bids by a certain amount. The first bidder to signal wins the auction at that price.

   Another common form of auction is a *Vickrey auction*, also known as a *sealed-bid, second-price auction.* In Vickrey auctions, each bidder submits a single bid privately to the auctioneer, and the person with the highest bid wins. However, they only pay what the price of the second-highest bid.

   Why do they only pay the second-highest bid? To eliminate the dreaded "*winner's curse*", which essentially is that if you pay more than anyone else in a market, then you must have overpaid, because you cannot possibly turn around and sell that item back to the same market for what you paid for it (otherwise someone else would have bid for it). However, if you only pay the second-price, then you could (in theory) sell the item to the second-highest bidder for what you paid for it.

   For the first task in the workshop, your tutor will lead you through some auction exercises.

2. Consider a Vickrey auction between two bidders, Bidder A and Bidder B, who each value the item at \$50 and \$51 respectively. Assume that the buyer has no 'reserve' price.

   a) Are there (weakly or strictly) dominant strategies for these two bidders? If so, what are they? To simplify your analysis, assume that bids can only be whole numbers between 49 and 52.

   Bidder B (val = 51)

		49	50	51	52
	49				
Bidder A	50				
(val = 50)	51				
	52				

   hints: Your first task should be to determine the payoffs in matrix. For example, what is the payoff for Bidder A if they bid 51 and win the auction? Assume that if BOTH players win the auction (because they bid the same amount), the item is divided between them evenly (they both receive half of the payoff) and they both pay half of the bid. Note that if they both bid the same amount, the second price is equal to the first price.

b) What are the equilibrium states in this simplified game?

c) **[Challenge question]** (not covered in the workshop): Use the concepts introduced in lectures, can you argue what the (weakly) dominant strategy is for each bidder in a Vickrey auction for N number of players and any value? For simplicity, assume that no two bidders have the same private value (in practice this is not much more difficult to reason about).

3. Consider the following two games, and assume that you are playing as the Row player.

Game 1:

		Column player	
		L	R
Row	T	320, 40	40, 80
player	B	40, 80	80, 40

Game 2:

		Column player	
		L	R
Row	T	44, 40	40, 80
player	B	40, 80	80, 40

Note that the only difference between the two games is the payoff 320 vs. 44 in the top-left cell.

a) What strategy would you choose in each game?

b) Calculate the mixed strategy that both players should play in both of these games. How close was your intuition to the mixed strategy that you calculated?

payoff = value − payment.

only pay the second bid.

Bidder B (value=51)

(2).

Bidder A (value=50)

	49	50	51	52
49	$\frac{1}{2}, 1$	0, 2	0, 2	0, 2
50	1, 0	$0, \frac{1}{2}$	0, 1	0, 1
51	1, 0	0, 0	$-\frac{1}{2}, 0$	0, 0
52	1, 0	0, 0	-1, 0	$-1, \frac{1}{2}$

49:

① $A = \frac{50}{2} - \frac{49}{2} = \frac{1}{2}$

$B = \frac{51}{2} - \frac{49}{2} = 1$

Game 1:

Column player

		L (y, $\frac{1}{8}$)	R ($1-y$, $\frac{7}{8}$)
Row (x)	T	320, 40	40, 80
player ($1-x$)	B	40, 80	80, 40

$E_c(L) = E_c(R)$

$320y + 40(1-y)$

$= 40y + 80(1-y)$

$320y = 40$

$y = \frac{1}{8}$ $\quad \frac{7}{8}$

Game 2:

Column player

		L	R
Row	T	44, 40	40, 80
player	B	40, 80	80, 40

$40x + 80(1-x)$

$= 80x + 40(1-x)$

$-80x = -40$

$x = \frac{1}{2}$