MAST20005/MAST90058: Week 5 Problems

Some useful information for many of the problems is shown at end of this problem sheet.

1. Let X be the length in centimeters of a species of fish when caught in the spring. A random sample of 13 observations yielded the sample variance $s^2 = 37.751$. Find a 95% confidence interval for σ.

2. A test was conducted to determine if a wedge on the end of a plug designed to hold a seal onto that plug was operating correctly. The data were the force required to remove a seal from the plug with the wedge in place (X) and without the wedge (Y). Assume the distributions of X and Y are $\mathrm{N}(\mu_X, \sigma^2)$ and $\mathrm{N}(\mu_Y, \sigma^2)$ respectively. Samples of size 10 on each variable yielded:

	n	$\bar{x}$	s
X	10	2.548	0.323
Y	10	1.564	0.210

   (a) Find a 95% confidence interval for $\mu_X - \mu_Y$.

   (b) Do you think the wedge is operating correctly?

   (c) How could you support the assumption $\mathrm{var}(X) = \mathrm{var}(Y)$ in part (a)?

3. A candy maker produces mints that have a mean weight of 20.4 grams. For quality assurance, $n = 16$ mints were selected at random from the Wednesday morning shift, yielding $\bar{x} = 21.95$ grams and $s_x = 0.197$. On Wednesday afternoon $m = 13$ mints were selected at random, giving $\bar{y} = 21.88$ grams and $s_y = 0.318$. Find a 90% confidence interval for σ_x/σ_y, the ratio for the standard deviations of the mints produced by the morning and afternoon shifts respectively. Is it reasonable to suppose the standard deviations are the same in the two shifts?

4. A machine shop manufactures toggle levers. A lever is flawed if a standard nut cannot be screwed onto the threads. Let p be the proportion of flawed toggle nuts the shop manufactures. There were 24 flawed levers out of a sample of 642 that were selected randomly from the production line.

   (a) Give a point estimate of p.

   (b) Find an approximate 95% confidence interval.

   (c) Find a one-sided 95% confidence interval that gives an upper bound for p.

5. Let X be the length of a male grackle (a type of bird). Suppose $X \sim \mathrm{N}(\mu, 4.84)$. Find the sample size that is needed to produce an estimate of μ that is accurate to within ± 0.4, as measured by a 95% confidence interval.

6. We wish to hold a public opinion poll for a close election. Let p denote the proportion of votes who favour candidate A. How large a sample should be taken if we want the maximum error of the estimate of p to be equal to:

   (a) 0.03, with 95% confidence interval?

   (b) 0.02, with 95% confidence interval?

   (c) 0.03, with 90% confidence interval?

7. We obtain the following random sample on $X \sim \mathrm{N}(\mu, \sigma^2)$.

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33.8  32.2  30.7  35.4  31  30.3  26.8  33.2  27.8  27.2
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   (a) Give a point estimate and 90% confidence interval for μ.
   (b) Give a point estimate and 95% confidence interval for σ.
   (c) Give a 90% prediction interval for a future observation on X.

8. We observe the outcome of n Bernoulli trials with parameter p. In the lectures we obtained an approximate confidence interval for p by using two approximations: (i) the Central Limit Theorem to get a sampling distribution for the sample proportion, $\hat{p}$; and (ii) the 'plug-in' approximation to get a standard error. If we avoid making the second approximation we can derive a more accurate confidence interval. The exercises below take you through the steps in the derivation. We will refer to this as the *quadratic approximation*.

   (a) Write an appropriate probability interval for deriving this confidence interval.
   (b) We need to rearrange this to be in terms of p. Show that this is equivalent to solving the following quadratic inequality:

$$(\hat{p} - p)^2 < c^2 p(1-p)/n,$$

   where $c = \Phi^{-1}(1 - \alpha/2)$.
   (c) Show that the solution to this inequality (in terms of p) is an interval with the following endpoints:

$$\frac{\hat{p} + \frac{c^2}{2n} \pm c\sqrt{\frac{\hat{p}(1-\hat{p})}{n} + \frac{c^2}{4n^2}}}{1 + \frac{c^2}{n}}.$$

Quantiles of various distributions

Standard normal: $\Phi^{-1}(0.975) = 1.960$, $\Phi^{-1}(0.95) = 1.645$

χ^2_{20}: $F^{-1}(0.025) = 9.591$, $F^{-1}(0.975) = 34.17$
χ^2_{19}: $F^{-1}(0.025) = 8.907$, $F^{-1}(0.975) = 32.85$
χ^2_{18}: $F^{-1}(0.025) = 8.231$, $F^{-1}(0.975) = 31.53$
χ^2_{13}: $F^{-1}(0.025) = 5.009$, $F^{-1}(0.975) = 24.74$
χ^2_{12}: $F^{-1}(0.025) = 4.404$, $F^{-1}(0.975) = 23.34$
χ^2_{10}: $F^{-1}(0.025) = 3.247$, $F^{-1}(0.975) = 20.48$
χ^2_{9} : $F^{-1}(0.025) = 2.700$, $F^{-1}(0.975) = 19.02$

t_{20}: $F^{-1}(0.975) = 2.086$, $F^{-1}(0.95) = 1.725$
t_{19}: $F^{-1}(0.975) = 2.093$, $F^{-1}(0.95) = 1.729$
t_{18}: $F^{-1}(0.975) = 2.101$, $F^{-1}(0.95) = 1.734$
t_{13}: $F^{-1}(0.975) = 2.160$, $F^{-1}(0.95) = 1.771$
t_{12}: $F^{-1}(0.975) = 2.179$, $F^{-1}(0.95) = 1.782$
t_{10}: $F^{-1}(0.975) = 2.228$, $F^{-1}(0.95) = 1.812$
t_{9} : $F^{-1}(0.975) = 2.262$, $F^{-1}(0.95) = 1.833$

$\mathrm{F}_{12,15}$: $F^{-1}(0.05) = 0.3821$, $F^{-1}(0.95) = 2.475$
$\mathrm{F}_{13,16}$: $F^{-1}(0.05) = 0.3976$, $F^{-1}(0.95) = 2.397$
$\mathrm{F}_{15,12}$: $F^{-1}(0.05) = 0.4040$, $F^{-1}(0.95) = 2.617$
$\mathrm{F}_{16,13}$: $F^{-1}(0.05) = 0.4171$, $F^{-1}(0.95) = 2.515$