Portfolio Optimization

Lagrange Multipliers

Let $U \subset \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}$ be a smooth function i.e infinitely many times differentiable.

Let $g : U \to \mathbb{R}^m$ be another smooth function

Find $x_0 \in U$ such that

$$\max_{\substack{g(x)=0 \\ x \in U}} f(x) = f(x_0)$$

or

$$\min_{\substack{g(x)=0 \\ x \in U}} f(x) = f(x_0) \qquad (10.1)$$

Constrained optimization problem.

Assumption : $m < n$ i.e. number of constraints is smaller than number of degrees of freedom n

Let $\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix} \in R^m$

The Lagrangian is the function

$F : U \times R^m \to R$

$$F(x,\lambda) = f(x) + \lambda^T g(x)$$

λ : Lagrange multiplier vector

$$F(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \qquad (10.2)$$

The constrained extremum point x_0 of (10.1) is found by identifying the critical points of $F(x,\lambda)$.

For this to work

<u>Necessary condition</u> The gradient $\nabla g(x)$ has full rank at any point x where the constraint $g(x)=0$ is satisfied, i.e.

$$\text{rank}\left(\nabla g(x)\right) = m \qquad \forall\, x \in U \text{ such that } g(x)=0$$

Example

Find the maximum and minimum values of

$$f(x_1,x_2,x_3) = 4x_2 - 2x_3$$

subject to $\left.\begin{array}{l} 2x_1 = x_2 + x_3 \\ x_1^2 + x_2^2 = 13 \end{array}\right\} \quad g(x) = \begin{pmatrix} 2x_1 - x_2 - x_3 \\ x_1^2 + x_2^2 - 13 \end{pmatrix} = 0$

Let $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$

$$F(x,\lambda) = 4x_2 - 2x_3 + \lambda_1(2x_1 - x_2 - x_3) + \lambda_2(x_1^2 + x_2^2 - 13)$$

$$\nabla g(x) = \begin{pmatrix} 2 & -1 & -1 \\ 2x_1 & 2x_2 & 0 \end{pmatrix}$$

rank $\nabla g(x) = 2$ for all x_1, x_2 unless $x_1 = x_2 = 0$

But $x_1 = x_2 = 0$ is not possible due to $x_1^2 + x_2^2 = 13$

$\therefore$ nec. condition of $\text{rank}(\nabla g(x)) = 2$ is satisfied.

Gradient of $F(x,\lambda)$ w.r.t x, λ

$$\nabla_{(x,\lambda)} F(x,\lambda) = \left(\nabla_x F(x,\lambda), \nabla_\lambda F(x,\lambda) \right)$$

$$\frac{\partial F(x,\lambda)}{\partial x_j} = \frac{\partial f(x)}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x)}{\partial x_j}, \quad \forall j = 1 \dots n$$

$$\frac{\partial F}{\partial \lambda_i}(x,\lambda) = g_i(x), \quad \forall i = 1 \dots m$$

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \cdots \frac{\partial f}{\partial x_n} \right) \quad 1 \times n$$

$$\nabla g(x) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}, & \frac{\partial g_1}{\partial x_2}, & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix} \quad m \times n$$

$$\nabla_x F(x,\lambda) = \left(\frac{\partial F}{\partial x_1}, \cdots \frac{\partial F}{\partial x_n} \right) = \nabla f(x) + \lambda^T \nabla g(x)$$

$$\nabla_\lambda F(x,\lambda) = \left(\frac{\partial F}{\partial \lambda_1}, \cdots \frac{\partial F}{\partial \lambda_m} \right) = g^T(x)$$

$\therefore$ $$\nabla_{(x,\lambda)} F(x,\lambda) = \left(\nabla f(x) + \lambda^T \nabla g(x),\ g^T(x) \right)$$

<u>Theorem 10.1</u> Assume $g(x)$ satisfies the necessary condition. If $x_0 \in U$ is a constrained extremum point of $f(x)$ w.r.t constraint $g(x)=0$, then there exists a Lagrange multiplier $\lambda_0 \in R^m$ such that the point (x_0, λ_0) is a critical point for the Lagrangian function $F(x,\lambda)$ i.e.,

$$\nabla_{(x,\lambda)} F(x_0,\lambda_0) = 0 .$$

Note that $\nabla_{(x,\lambda)} F : R^{m+n} \to R^{m+n}$

$\nabla_{(x,\lambda)} F(x_0,\lambda_0) = 0$ is a nonlinear equation and can be solved numerically using Newton's method.

Example

$$f(x) = 4x_2 - 2x_3$$

$$g(x) = \begin{pmatrix} 2x_1 - x_2 - x_3 \\ x_1^2 + x_2^2 - 13 \end{pmatrix}$$

$$F(x,\lambda) = 4x_2 - 2x_3 + \lambda_1(2x_1 - x_2 - x_3) + \lambda_2(x_1^2 + x_2^2 - 13)$$

$$\nabla_{(x,\lambda)} F(x,\lambda) = \begin{pmatrix} 2\lambda_1 + 2\lambda_2 x_1 \\ 4 - \lambda_1 + 2\lambda_2 x_2 \\ -2 - \lambda_1 \\ 2x_1 - x_2 - x_3 \\ x_1^2 + x_2^2 - 13 \end{pmatrix}^T$$

$\Rightarrow \quad \nabla F(x_0,\lambda_0) = 0$

$$2\lambda_{0,1} + 2\lambda_{0,2}x_{0,1} = 0$$

$$4 - \lambda_{0,1} + 2\lambda_{0,2}x_{0,2} = 0$$

$$-2 - \lambda_{0,1} = 0 \quad \Longrightarrow \quad \lambda_{0,1} = -2$$

$$2x_{0,1} - x_{0,2} - x_{0,3} = 0$$

$$x_{0,1}^2 + x_{0,2}^2 - 13 = 0$$

$$\lambda_{0,2} x_{0,1} = 2$$
$$\lambda_{0,2} x_{0,2} = -3$$
$$x_{0,3} = 2x_{0,1} - x_{0,2}$$
$$x_{0,1}^2 + x_{0,2}^2 = 13$$

Since $\lambda_{0,2} \neq 0$

$$x_{0,1} = \frac{2}{\lambda_{0,2}}, \quad x_{0,2} = \frac{-3}{\lambda_{0,2}}, \quad x_{0,3} = \frac{7}{\lambda_{0,2}}, \quad x_{0,1}^2 + x_{0,2}^2 = \frac{13}{\lambda_{0,2}^2} = 13$$

$\lambda_{0,2}^2 = 1$ works giving

$\lambda_{0,2} = +1 \implies x_{0,1} = 2,\ x_{0,2} = -3,\ x_{0,3} = 7,\ \lambda_{0,1} = -2.$
$\lambda_{0,2} = 1$

$\lambda_{0,2} = -1 \implies x_{0,1} = -2,\ x_{0,2} = 3,\ x_{0,3} = -7,\ \lambda_{0,1} = -2$
$\lambda_{0,2} = -1$

Critical points of $F(x, \lambda)$

$$x_0 = \begin{pmatrix} 2 \\ -3 \\ 7 \end{pmatrix}; \quad \lambda_0 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

and

$$x_0 = \begin{pmatrix} -2 \\ 3 \\ -7 \end{pmatrix}; \quad \lambda_0 = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

Finding sufficient conditions for a critical point (x_0, λ_0) of $F(x, \lambda)$ to correspond to a constrained extremum point x_0 of $f(x)$ is more complicated (rarely checked in practice).

Consider $F_0 : U \to R$

$$F_0(x) = F(x, \lambda_0) = f(x) + \lambda_0^T g(x)$$

Let $D^2 F_0(x_0)$ be the Hessian of $F_0(x)$ at x_0

i.e

$$D^2 F_0(x_0) = \begin{pmatrix} \frac{\partial^2 F_0(x_0)}{\partial x_1^2} & \frac{\partial^2 F_0(x_0)}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F_0(x_0)}{\partial x_n \partial x_1} \\ \vdots & & & \vdots \\ \frac{\partial^2 F_0(x_0)}{\partial x_1 \partial x_n} & - & - & \frac{\partial^2 F_0(x_0)}{\partial x_n^2} \end{pmatrix} \in R^{n \times n}$$

Let

$$q(v) = v^T D^2 F_0(x_0) v = \sum_{1 \le i, j \le n} \frac{\partial^2 F_0(x_0)}{\partial x_i \partial x_j} v_i v_j$$

$$v = [v_1, v_2 \,-\,-\, v_n]^T$$

Restrict v to those vectors that satisfy

$\nabla g(x_0) v = 0$ ie to the vector space

$$V_0 = \{ v \in \mathbb{R}^n \mid \nabla g(x_0) v = 0 \}$$

$\nabla g(x_0) \in \mathbb{R}^{m \times n}$ with $m < n$

and

rank $\nabla g(x_0) = m$ ie it has m linearly independent columns.

Let us assume w.l.g that the first m columns are linearly independent.

Solving $\nabla g(x_0) v = 0$ the entries $v_1, v_2 \ldots v_m$ can be written as linear combinations of $v_{m+1}, v_{m+2}, \ldots v_n$ the other $n-m$ entries of v. Let

$$v_{red} = \begin{bmatrix} v_{m+1} \\ v_{m+2} \\ \vdots \\ v_n \end{bmatrix}$$

For $v \in V_0$ we can write

$$q(v) = q_{red}(v_{red}) = \sum_{m+1 \le i,j \le n} q_{red}(i,j)\, v_i v_j \quad , \forall v \in V_0$$

To see that let us revisit the example

where $x_0 = (2,-3,7)^T$, $\lambda_0 = (-2,1)^T$

$$F_0(x) = 4x_2 - 2x_3 + (-2)(2x_1 - x_2 - x_3) + 1\cdot(x_1^2 + x_2^2 - 13)$$
$$= x_1^2 + x_2^2 - 4x_1 + 6x_2 - 13$$

$$D^2 F_0(x_0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \nabla g(x_0) = \begin{pmatrix} 2 & -1 & -1 \\ 4 & -6 & 0 \end{pmatrix}$$

if $v = (v_1, v_2, v_3)^T$

$$q(v) = v^T D^2 F_0(x_0)\, v = 2v_1^2 + 2v_2^2$$

The condition

$$\nabla g(x_0)\, v = 0 \implies \begin{pmatrix} 2 & -1 & -1 \\ 4 & -6 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

$$\implies \left.\begin{aligned} 2v_1 - v_2 - v_3 &= 0 \\ 4v_1 - 6v_2 &= 0 \end{aligned}\right\} \implies \begin{aligned} v_3 &= 2v_2 \\ v_1 &= \tfrac{3}{2} v_2 \end{aligned}$$

Let $v_{red} = v_2$

$$q_{red}(v_{red}) = 2v_1^2 + 2v_2^2 = \frac{13}{2} v_2^2$$

The second critical point of $F(x,\lambda)$ is
$x_0 = (-2, 3, -7)$ and $\lambda_0 = (-2, -1)$

$$F_0(x) = -x_1^2 - x_2^2 - 4x_1 + 6x_2 + 13$$

$$D^2F_0(x) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \nabla g(x_0) = \begin{pmatrix} 2 & -1 & -1 \\ -4 & 6 & 0 \end{pmatrix}$$

$$q(U) = U^T D^2F_0(x_0) U = -2U_1^2 - 2U_2^2$$

$$\nabla g(x_0) = 0 \implies \begin{pmatrix} 2 & -1 & -1 \\ -4 & 6 & 0 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = 0$$

$$\implies \left.\begin{array}{l} 2U_1 - U_2 - U_3 = 0 \\ -4U_1 + 6U_2 = 0 \end{array}\right\} \implies \begin{array}{l} U_3 = 2U_2 \\ U_1 = \frac{3}{2}U_2 \end{array}$$

Let $U_{red} = U_2$ then

$$q_{red}(U_{red}) = -2U_1^2 - 2U_2^2 = -\frac{13}{2}U_2^2$$

Whether x_0 is a constrained extremum for $f(x)$ will depend on whether q_{red} is positive semidefinite i.e

$$q_{red} \geq 0 \quad \forall\, v_{red} \in R^{n-m}$$

or

negative semidefinite i.e

$$q_{red} \leq 0 \quad \forall\, v_{red} \in R^{n-m}$$

<u>Theorem 10.2</u> Assume $g(x)$ satisfies the necessary condition. Let (x_0, λ_0) be a critical point of $F(x, \lambda) = f(x) + \lambda^T g(x)$

- If $q_{red}(v_{red}) \geq 0$ for (x_0, λ_0) then x_0 is a constrained minimum for $f(x)$ w.r.t $g(x) = 0$

- If $q_{red}(v_{red}) \leq 0$ for (x_0, λ_0) then x_0 is a constrained maximum for $f(x)$ w.r.t $g(x) = 0$

- If $q_{red}(v_{red})$ is not positive semidefinite or negative semidefinite then x_0 is not a constrained extremum point for $f(x)$ w.r.t $g(x) = 0$.

Example continued

In the example the point $x_0 = (2, -3, 7)$ and $\lambda_0 = (-2, 1)$ led to $q_{red}(v_{red}) = \frac{13}{2} v_2^2 \geq 0$

$\therefore$ $x_0 = (2, -3, 7)$ is a minimum for $f(x)$ and $f(x_0) = -26$.

The point $x_0 = (2, -3, -7)$ and $\lambda_0 = (-2, -1)$ led to $q_{red}(v_{red}) = -\frac{13}{2} v_2^2 \leq 0$

$\Rightarrow$ $x_0 = (2, -3, -7)$ is a maximum for $f(x)$ and $f(x_0) = 26$.

Theorem 10.2 is simpler if $D^2 F_0(x_0)$ is either positive definite or negative definite.

Corollary 10.1 Assume $g(x)$ satisfies the nec. condition

If $D^2 F_0(x_0)$ is positive definite then x_0 is a minimum of $f(x)$ with constraint $g(x) = 0$

If $D^2 F_0(x_0)$ is negative definite then x_0 is a maximum of $f(x)$ with constraint $g(x) = 0$

Steps

1. Check that rank $\nabla g(x) = m$ $\forall x$ that satisfy $g(x) = 0$

2. Find (x_0, λ_0) such that $\nabla_{(x,\lambda)} F(x_0, \lambda_0) = 0$ where $F(x, \lambda) = f(x) + \lambda^T g(x)$

3. Compute $D^2 F_0(x_0)$ and check if pos. or neg. definite then use Corollary 10.1 for the answer otherwise go to the next step

4. Compute $q_{red}(v_{red})$ from $\nabla g(x_0) v = 0$

5. Check whether $q_{red}(v_{red})$ is pos. semidefinite or neg. semidefinite or neither and use Theorem 10.2 for the answer.

Example

minimize $\quad f(x) = x_1x_2 + x_2x_3 + x_3x_1$

subject to

$$g(x) = x_1x_2x_3 - 1 = 0$$

$g(x) \in R^1 \qquad g(x) : R^3 \to R^1$

Step 1 $\quad \nabla g(x) = (x_2x_3, x_1x_3, x_1x_2) \in R^{1\times 3}$

$m = 1, n = 3$

rank $\nabla g(x) = 1 \quad \forall x$ that satisfy $x_1x_2x_3 = 1$

nec. condition is satisfied.

Step 2 $\quad F(x,\lambda) = x_1x_2 + x_2x_3 + x_3x_1 + \lambda(x_1x_2x_3 - 1)$

$$\nabla F(x,\lambda) = \begin{pmatrix} x_2 + x_3 + \lambda x_2x_3 \\ x_1 + x_3 + \lambda x_1x_3 \\ x_1 + x_2 + \lambda x_1x_2 \\ x_1x_2x_3 - 1 \end{pmatrix}$$

For $\nabla F(x_0, \lambda_0) = 0$

$$\left.\begin{aligned} x_{02} + x_{03} + \lambda_0 x_{02}x_{03} &= 0 \\ x_{01} + x_{03} + \lambda_0 x_{01}x_{03} &= 0 \\ x_{01} + x_{02} + \lambda_0 x_{01}x_{02} &= 0 \\ x_{01}x_{02}x_{03} &= 1 \end{aligned}\right\}$$

$x_{01} = x_{02} = x_{03}$

From $x_{01}x_{02}x_{03} = 1$

$\Rightarrow x_{01} = x_{02} = x_{03} = 1$

$\lambda_0 = -(x_{01}x_{02} + x_{01}x_{03})$

$= -2$

<u>Step 3.</u> Compute $q(v) = v^T D^2 F_0(x_0)\, v$

For $\lambda_0 = -2$

$$F_0(x) = x_1x_2 + x_2x_3 + x_3x_1 - 2x_1x_2x_3 + 2$$

$$D^2F_0(x) = \begin{pmatrix} 0 & 1-2x_3 & 1-2x_2 \\ 1-2x_3 & 0 & 1-2x_1 \\ 1-2x_2 & 1-2x_1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad D^2F_0(1,1,1) = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$q(v) = v^T D^2F_0(1,1,1)\, v = -2v_1v_2 - 2v_2v_3 - 2v_1v_3$$

<u>Step 4</u> Compute $q_{red}(v_{red})$

$$\nabla g(1,1,1)\, v = 0$$

$$\nabla g(x) = (x_2x_3,\ x_1x_3,\ x_2x_3)$$

$$\nabla g(1,1,1) = (1,1,1)$$

$$\nabla g(1,1,1)\, v = v_1 + v_2 + v_3 = 0$$

$$\Rightarrow \quad v_1 = -v_2 - v_3$$

Let $v_{red} = \begin{pmatrix} v_2 \\ v_3 \end{pmatrix}$

$$q_{red}(v_{red}) = 2v_2^2 + 2v_2v_3 + 2v_3^2 = v_2^2 + v_3^2 + (v_2+v_3)^2$$

$\Rightarrow \quad q_{red}(v_{red}) > 0 \quad \forall\ (v_2, v_3) \neq (0,0)$

$\Rightarrow$ positive semidefinite

$\therefore \quad x_0 = (1,1,1)$ is a minimum

and $\quad f(x_0) = 3$

Optimal investment Portfolios

Consider a portfolio with investments in n assets.

w_i : proportion of the portfolio invested in asset i.

$$\sum_{i=1}^{n} w_i = 1$$

Possible to take large long or short positions

$\therefore$ w_i can be negative too.

If short selling is not allowed $w_i \geq 0$.

Let

R_i : rate of return of asset i over a fixed period of time.

$\mu_i = E[R_i]$ and $\sigma_i^2 = var(R_i)$, $i = 1, 2 \ldots n$.

Let

ρ_{ij} : correlation between R_i and R_j, $1 \leq i < j \leq n$

$\mu_i, \sigma_i, \rho_{ij}$ can be estimated using hystorical data

Rate of return R is given by

$$R = \sum_{i=1}^{n} w_i R_i$$

It follows that

$$E[R] = \sum_{i=1}^{n} w_i \mu_i$$

$$\mathrm{var}(R) = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{1 \le i < j \le n} w_i w_j \sigma_i \sigma_j \rho_{i,j}$$

In matrix form

$$E[R] = \mu^T w \qquad \mu = [\mu_1, \mu_2 \cdots \mu_n]^T$$

$$w = [w_1, w_2 \cdots w_n]^T$$

$$\mathrm{var}(R) = w^T M w$$

$$M = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{1,2} & \sigma_1\sigma_3\rho_{1,3} & \cdots & \sigma_1\sigma_n\rho_{1,n} \\ \sigma_1\sigma_2\rho_{1,2} & \sigma_2^2 & & & \sigma_2\sigma_n\rho_{2,n} \\ \sigma_1\sigma_3\rho_{1,3} & \sigma_2\sigma_3\rho_{2,3} & \sigma_3^2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \\ \sigma_1\sigma_n\rho_{1,n} & \sigma_2\sigma_n\rho_{2,3} & \sigma_3\sigma_n\rho_{3,n} & \cdots & \sigma_n^2 \end{pmatrix}$$

is the $n \times n$ covariance matrix of the rates of return of the n assets given by

$$M(i,j) = \sigma_i \sigma_j \rho_{i,j} \quad \forall\ 1 \le i \ne j \le n; \quad M(i,i) = \sigma_i^2$$

$\forall\ i = 1:n$.

Optimization problems

1) Given μ_p, find w_i, $i=1:n$ with $E[R]=\mu_p$ such that $var(R)$ is minimal

i.e

$$\min_{w} Var R = w^T M w$$

subject to: $\mu^T w = \mu_p$ or $\mu^T w - \mu_p = 0$

$\sum_{i=1}^{n} w_i = 1$ or $1^T w - 1 = 0$

2) Given σ_p find w_i, $i=1:n$ with $var(R)=\sigma_p^2$ such that $E[R]$ is maximal.

i.e

$$\max_{w} E[R] = \mu^T w$$

subject to: $w^T M w = \sigma_p^2$ or $w^T M w - \sigma_p^2 = 0$

$\sum_{i=1}^{n} w_i = 1$ or $1^T w - 1 = 0$

Assumptions

(i) the covariance matrix of returns is nonsingular

(ii) the assets do not have the same expected rate of return.

Let

$$\min_{w} f(w) = w^T M w$$

$$\text{subject to : } g(w) = \begin{pmatrix} 1^T w - 1 \\ \mu^T w - \mu_p \end{pmatrix} = \begin{pmatrix} g_1(w) \\ g_2(w) \end{pmatrix} = 0$$

Check necessary condition that $\nabla g(w) = 0$ $\forall w$ that satisfy $g(w) = 0$.

$$\nabla g(w) = \begin{bmatrix} 1^T \\ \mu^T \end{bmatrix}$$

rank $\nabla g(w) = 2$ due to assumption (ii)

Let $\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$

$$F(w,\lambda) = w^T M w + \lambda_1 (1^T w - 1) + \lambda_2 (\mu^T w - \mu_p)$$

$$\nabla_{(w,\lambda)} F(w,\lambda) = \left[\nabla f(w) + \lambda^T \nabla g(w) \; , \; g^T(w) \right]$$

$$\nabla f(w) = 2 (M w)^T, \quad \lambda^T \nabla g(w) = \lambda_1 1^T + \lambda_2 \mu^T$$

$$\nabla_{(w,\lambda)} F(w,\lambda) = \left[2(Mw)^T + \lambda_1 \mathbb{1}^T + \lambda_2 \mu^T , (g(w))^T \right]$$

$$= \begin{pmatrix} 2Mw + \lambda_1 \mathbb{1} + \lambda_2 \mu \\ \mathbb{1}^T w - 1 \\ \mu^T w - \mu_p \end{pmatrix}^T$$

Critical points of $F(w,\lambda)$ from $\nabla_{(w,\lambda)} F(w,\lambda) = 0$

$$\left.\begin{aligned} 2Mw + \lambda_1 \mathbb{1} + \lambda_2 \mu &= 0 \\ \mathbb{1}^T w &= 1 \\ \mu^T w &= \mu_p \end{aligned}\right\} \quad \underbrace{\begin{pmatrix} 2M & \mathbb{1} & \mu \\ \mathbb{1}^T & 0 & 0 \\ \mu^T & 0 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} w \\ \lambda_1 \\ \lambda_2 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ \mu_p \end{pmatrix}}_{b}$$

The matrix A is nonsingular iff the rates of return of the assets are not <u>all</u> equal and M is nonsingular.

Unique solution $(w_0, \lambda_{01}, \lambda_{02}) = (w_0, \lambda_0)$

We check whether w_0 is a constrained minimum

$$F_0(w) = F(w, \lambda_0) = f(w) + \lambda_{0,1}\, g_1(w) + \lambda_{0,2}\, g_2(w)$$

$$\nabla f(w) = 2\,(Mw)^T \;;\quad \nabla g_1(w) = \mathbb{1}^T \;;\quad \nabla g_2(w) = \mu^T$$

Then

$$D^2 f(w) = 2M \;,\quad D^2 g_1(w) = D^2 g_2(w) = 0$$

$$\therefore \quad D^2 F_0(w) = 2M$$

$\Rightarrow$ $D^2 F_0(w)$ is positive definite due to M been positive definite

$\therefore$ w_0 is a constrained minimum for $f(w)$ given $g(w)=0$

$$f(w_0) = w_0^T M w_0 = \sigma_p^2$$

<u>Example</u> Find the minimum variance portfolio with 11.5% expected rate of return, if 4 assets can be traded to set up the portfolio, given

$\mu_1 = 0.09$	$\sigma_1 = 0.2$	$e_{1,2} = -0.5$
$\mu_2 = 0.12$	$\sigma_2 = 0.3$	$e_{2,3} = 0.25$
$\mu_3 = 0.15$	$\sigma_3 = 0.35$	$e_{1,3} = 0.35$
$\mu_4 = 0.06$	$\sigma_4 = 0.14$	$e_{i,4} = 0 \;,\; \forall i = 1:3$

$$\mu = \begin{pmatrix} 0.09 \\ 0.12 \\ 0.15 \\ 0.06 \end{pmatrix}, \quad M = \begin{pmatrix} 0.04 & -0.03 & 0.0245 & 0 \\ -0.03 & 0.09 & 0.02625 & 0 \\ 0.0245 & 0.02625 & 0.1225 & 0 \\ 0 & 0 & 0 & 0.0225 \end{pmatrix}$$

Solution for (w_0, λ_0)

$$\begin{pmatrix} 0.08 & -0.06 & 0.049 & 0 & 1 & 0.09 \\ -0.06 & 0.18 & 0.0525 & 0 & 1 & 0.12 \\ 0.049 & 0.0525 & 0.245 & 0 & 1 & 0.15 \\ 0 & 0 & 0 & 0.045 & 1 & 0.06 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0.09 & 0.12 & 0.15 & 0.06 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0.115 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} w_{0,1} \\ w_{0,2} \\ w_{0,3} \\ w_{0,4} \\ \lambda_{0,1} \\ \lambda_{0,2} \end{pmatrix} = \begin{pmatrix} 0.547452 \\ 0.440377 \\ 0.135042 \\ -0.122872 \\ 0.064569 \\ -0.983995 \end{pmatrix}$$

ie 54.75% of portfolio in asset 1
44.03% " " " " 2
13.5% " " " " 3
short 12.29% " " " " 4

\$1,000,000 portfolio

\$547,452
\$440,377
\$135,042
-\$122872

Borrowed asset and sold for cash

Maximum Return Portfolios

$$\max_{w} f(w) = \mu^T w$$

subject to
$$g(w) = \begin{pmatrix} g_1(w) \\ g_2(w) \end{pmatrix} = \begin{pmatrix} \mathbb{1}^T w - 1 \\ w^T M w - \sigma_p^2 \end{pmatrix}$$

i.e. maximize return subject to a variance of return σ_p^2.

Nec. condition
$$\nabla g(w) = \begin{pmatrix} \mathbb{1}^T \\ 2(Mw)^T \end{pmatrix}$$

$$\left.\begin{array}{l} \text{rank } \nabla g(w) = 1 \\ \text{and} \\ g(w) = 0 \end{array}\right\} \text{ iff } \begin{array}{l} 2Mw = c \cdot \mathbb{1} \quad \text{for some } c \in \mathbb{R} \\ \mathbb{1}^T w = 1 \\ w^T M w = \sigma_p^2 \end{array}$$

Since the returns of the n assets are linearly independent, M is nonsingular.

Therefore from $2Mw = c \cdot \mathbb{1} \Rightarrow w = \frac{c}{2} M^{-1} \cdot \mathbb{1}$

$$\mathbb{1}^T w = \frac{c}{2} \mathbb{1}^T M^{-1} \mathbb{1} = 1 \Rightarrow \mathbb{1}^T M^{-1} \mathbb{1} = \frac{2}{c}$$

$$w^T M w = \sigma_p^2 \Rightarrow \mathbb{1}^T \cdot M^{-1} M M^{-1} \cdot \mathbb{1} \frac{c^2}{4} = \sigma_p^2$$

$$\mathbb{1}^T \cdot M^{-1} \cdot \mathbb{1} = \frac{2}{c} = \frac{4\sigma_p^2}{c^2}$$

$$\Rightarrow \underline{c = 2\sigma_p^2}$$

ie rank $\nabla g(w) = 1$ iff $1^T . M^{-1} . 1 = 1/\sigma_p^2$

otherwise rank $\nabla g(w) = 2$

Assume $1^T . M^{-1} . 1 \neq 1/\sigma_p^2 \Rightarrow$ rank $\nabla g(w) = 2$
for $w \Rightarrow g(w) = 0$.

Let $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$

$$F(w,\lambda) = \mu^T w + \lambda_1 (1^T w - 1) + \lambda_2 (w^T M w - \sigma_p^2)$$

$$\nabla_{(w,\lambda)} F(w,\lambda) = \begin{pmatrix} \mu + \lambda_1 1 + 2\lambda_2 M w \\ 1^T . w - 1 \\ w^T M w - \sigma_p^2 \end{pmatrix}^T$$

Critical points (w_0, λ_0) satisfy

$$G(w_0, \lambda_0) = \begin{pmatrix} \mu + \lambda_{01} 1 + 2\lambda_{0,2} M w_0 \\ 1^T . w_0 - 1 \\ w_0^T M w_0 - \sigma_p^2 \end{pmatrix} = 0$$

Nonlinear problem. Can be solved numerically using Newton's method.

$$\nabla_{(w_0,\lambda_0)} G(w_0, \lambda_{01}, \lambda_{02}) = \begin{pmatrix} 2\lambda_{02} M & 1 & 2 M w_0 \\ 1^T & 0 & 0 \\ 2(M w_0)^T & 0 & 0 \end{pmatrix}$$

It can be shown that $G(w_0, \lambda_0) = 0$ has a unique solution and $\lambda_{0,2} < 0$.

Check whether w_0 with $\lambda_0 = \begin{pmatrix} \lambda_{01} \\ \lambda_{02} \end{pmatrix}$ and $\lambda_{02} < 0$ is a constrained maximum of $\mu^T w$ subject to the constraint $g(w) = 0$

$$F_0(w) = \mu^T w + \lambda_{01}(\mathbb{1}^T w - 1) + \lambda_{0,2}(w^T M w - \sigma_p^2)$$

$$D(w^T M w) = 2(Mw)^T$$

$$\therefore \quad D^2 F_0(w) = 2\lambda_{0,2} M$$

Since $\lambda_{0,2} < 0$ and M is pos. definite

$D^2 F_0(w)$ is negative definite

$\therefore$ w_0 is a maximum.

Example

Find a maximum return portfolio with 25% standard deviation of the rate of return for a portfolio of 4 assets with

$$\mu = \begin{bmatrix} 0.09 \\ 0.12 \\ 0.15 \\ 0.06 \end{bmatrix}, \quad M = \begin{pmatrix} 0.04 & -0.03 & 0.0245 & 0 \\ -0.03 & 0.09 & 0.02625 & 0 \\ 0.0245 & 0.02625 & 0.1225 & 0 \\ 0 & 0 & 0 & 0.0225 \end{pmatrix}$$

$$\mathbb{1}^T M^{-1} \mathbb{1} = 122.4056 \neq 16 = \frac{1}{(0.25)^2}, \quad \sigma_P = 0.25$$

$$G(w, \lambda_1, \lambda_2) = \begin{pmatrix} \mu + \lambda_1 \mathbf{1} + 2\lambda_2 M w \\ \mathbf{1}^t w - 1 \\ w^t M w - \sigma_P^2 \end{pmatrix}$$
$$= \begin{pmatrix} 0.09 + \lambda_1 + 0.08\lambda_2 w_1 - 0.06\lambda_2 w_2 + 0.045\lambda_2 w_3 \\ 0.12 + \lambda_1 - 0.06\lambda_2 w_1 + 0.18\lambda_2 w_2 + 0.0525\lambda_2 w_3 \\ 0.15 + \lambda_1 + 0.049\lambda_2 w_1 + 0.0525\lambda_2 w_2 + 0.245\lambda_2 w_3 \\ 0.06 + \lambda_1 + 0.09\lambda_2 w_4 \\ w_1 + w_2 + w_3 + w_4 - 1 \\ 0.04w_1^2 + 0.09w_2^2 + 0.1225w_3^2 + 0.0225w_4^2 \\ \quad -0.06w_1w_2 + 0.049w_1w_3 + 0.0525w_2w_3 - 0.0625 \end{pmatrix}$$

$$\nabla G(w, \lambda) = \begin{pmatrix} 2\lambda_2 M & \mathbb{1} & 2Mw \\ \mathbb{1}^T & 0 & 0 \\ 2(Mw)^T & 0 & 0 \end{pmatrix}$$

Using Newton's method to solve $G(w_0, \lambda_0) = 0$ for (w_0, λ_0)

$$x_{k+1} = x_k - \left(\nabla G(x_k)\right)^{-1} G(x_k)$$

$$x = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \\ 1 \\ 1 \end{pmatrix} \Rightarrow x^* = \begin{pmatrix} 0.6269 \\ 0.5868 \\ 0.3152 \\ -0.5289 \\ -0.0732 \\ -0.5537 \end{pmatrix}$$

$$\Rightarrow \quad w_0 = \begin{pmatrix} 0.6269 \\ 0.5868 \\ 0.3152 \\ -0.5289 \end{pmatrix}, \quad \lambda_{0,1} = -0.0732, \quad \lambda_{0,2} = -0.5537$$

						\$1,000,000
62.69 %	of	portfolio	is	asset	1	\$626900
58.68 %	"	"	"	"	2	\$586800
31.52 %	"	"	"	"	3	\$315200
short 52.89 %	"	"	"	"	4	-\$528900

Borrowed asset to raise cash.

Expected rate of return $\mu_p = \mu^T w_0 = 0.1424$

ie 14.24 %

Another optimization Problem

$$\text{maximize} \quad \mu^T w - \frac{1}{2} w^T M w$$

$$\text{subject to} \quad w^T \mathbb{1} - 1 = 0$$

$$w^T M w - \sigma_p^2 = 0$$

$$F_{(w,\lambda)} = \mu^T w - \frac{1}{2} w^T M w + \lambda_1 (w^T \mathbb{1} - 1) + \lambda_2 (w^T M w - \sigma_p^2)$$

$$\nabla F(w,\lambda) = \begin{pmatrix} \mu - Mw + \lambda_1 \mathbb{1} + 2\lambda_2 M w \\ \mathbb{1}^T w - 1 \\ w^T M w - \sigma_p^2 \end{pmatrix}$$

$$G(w_0, \lambda_0) = \begin{pmatrix} \mu + (2\lambda_{02} - 1) M w_0 + \lambda_{01} \mathbb{1} \\ \mathbb{1}^T w_0 - 1 \\ w_0^T M w_0 - \sigma_p^2 \end{pmatrix} = 0$$

$$\nabla G(w_0, \lambda_0) = \begin{pmatrix} (2\lambda_{0,2} - 1) M & \mathbb{1} & 2 M w_0 \\ \mathbb{1}^T & 0 & 0 \\ 2(M w_0)^T & 0 & 0 \end{pmatrix}$$

Use Newton's method to solve for $w_0, \lambda_{01}, \lambda_{02}$

$$F_0(w) = \mu^T w - \frac{1}{2} w^T M w + \lambda_{01}(\mathbb{1}^T w - 1) + \lambda_{02}(w^T M w - \sigma_p^2)$$

$$D^2 F_0(w) = -M + 2\lambda_{0,2} M = (2\lambda_{02} - 1) M$$

Maximum if $\lambda_{02} < 1/2$