Bayesian methods

(Module 10)

Statistics (MAST20005) & Elements of Statistics (MAST90058)

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Contents

Aims of this module

- Explain two different ways to use probability for modelling
- Introduce the Bayesian approach to statistical inference
- Review the probability tools required to carry this out
- Show examples of Bayesian inference for simple models
- Discuss how to chose an appropriate prior
- Compare and contrast Bayesian & classical inference

1 Review of probability

From our last lecture...

- Disease testing example
- Tree diagrams

Review some probability definitions

- Let A and B be two events
- Often these are in terms of random variables, e.g. $A = `X = 3'$
- *Joint probability*

$$\Pr(A, B) = \Pr(A \cap B) = \Pr(A \text{ and } B \text{ both occur})$$

- *Marginal probability*

$$\Pr(A) = \Pr(A \text{ occurs irrespective of } B) = \Pr(A, B) + \Pr(A, \bar{B})$$

- *Conditional probability*

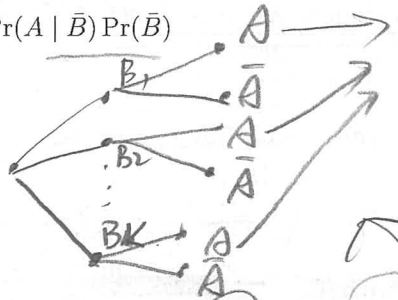
$$\Pr(A \mid B) = \Pr(A \text{ occurs given that } B \text{ occurs}) = \frac{\Pr(A, B)}{\Pr(B)}$$

Bayes' theorem

$$\Pr(B \mid A) = \frac{\Pr(A \mid B)\Pr(B)}{\Pr(A)}$$

The denominator can be written out using:

$$\begin{aligned}\Pr(A) &= \Pr(A, B) + \Pr(A, \bar{B}) \\ &= \Pr(A \mid B)\Pr(B) + \Pr(A \mid \bar{B})\Pr(\bar{B})\end{aligned}$$



Partitions

- Let $B_1, B_2, \ldots, B_k$ be a *partition* of the sample space
- This 'splits up' the sample space into distinct events
- More precisely, the events **cover** the whole sample space ($B_1 \cup B_2 \cup \cdots \cup B_k = \Omega$) and are **mutually exclusive** ($B_i \cap B_j = \emptyset$ when $i \neq j$).
- Example: roll a die and let the outcome be X, the events B_1 = 'X is even' and B_2 = 'X is odd' form a partition.
- The *law of total probability* relates marginal and conditional probabilities,

$$\Pr(A) = \sum_{i=1}^{k} \Pr(A, B_i) = \sum_{i=1}^{k} \Pr(A \mid B_i)\Pr(B_i)$$

Bayes' theorem again

$$\Pr(B_i \mid A) = \frac{\Pr(A \mid B_i)\Pr(B_i)}{\sum_{i=1}^{k} \Pr(A \mid B_i)\Pr(B_i)}$$

Sometimes write this more compactly as:

$$\Pr(B_i \mid A) \propto \Pr(A \mid B_i)\Pr(B_i)$$

Continuous random variables

Analogous definitions in terms of density functions (for rvs X and Y):

- *Joint pdf*

$$f(x, y)$$

- *Marginal pdf (law of total probability)*

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_{-\infty}^{\infty} f(x \mid y) f(y)\, dy$$

- *Conditional pdf*

$$f(x \mid y) = f(x \mid Y = y) = \frac{f(x, y)}{f(y)}$$

- *Bayes' theorem*

$$f(x \mid y) = \frac{f(y \mid x) f(x)}{f(y)}$$

2 Interpretations of probability

How do we use probability?

- Modelling *variation* (frequentist probability)
- Representing *uncertainty* (Bayesian probability)

Classical inference only uses frequentist probability

Bayesian inference uses both

Frequentist probability

- The relative **frequency of occurrence in the long run**, under hypothetical repetitions an experiment
- This is what we usually have in mind when devising a statistical model for the data
- Example: $X \sim \mathrm{N}(\mu, \sigma^2)$, specifies a model for variation across multiple observations of X
- Known as *frequentist* probability
- Also known as *aleatory*, *physical* or *frequency* probability
- Needs a well-defined random experiment / repetition mechanism
- The interpretation for one-off events, and those that have already occurred, is problematic (recall the 'card trick')

Bayesian probability

- The **degree of plausibility**, or **strength of belief**, of a given statement based on existing knowledge and evidence, expressed as a probability
- Known as *Bayesian* probability
- Also known as *epistemic* or *evidential* probability
- Can be assigned to any statement, even when no random process is involved, and irrespective of whether the event has yet occurred or not
- Example: what is the probability the dinosaurs were wiped out by an asteroid?
- Popularly expressed in terms of betting: if you were forced to make a bet on the outcome, what odds would you accept?

Remarks

- Probability also has a mathematical definition, in terms of **axioms**. This is separate to its interpretation as a model of reality.
- When using mathematical probability, it is not self-evident that the 'long-run relative frequency' actually exists and is equal to the underlying probability you start with as part of the axioms; this is something that needs to be proved. It turns out to be true and this fact is known as the Law of Large Numbers.
- Most people only learn about the frequentist notion of probability. However, in practice they often **naturally use the Bayesian notion**, as the card trick demonstrated. They do so without necessarily knowing about the different notions of probability, which can sometimes lead to confusion.

Why use Bayesian probability?

- **We do it naturally.** Card trick, gambling odds,...
- **Asking the right question.** Allows us to directly answer the question of interest
- **Going beyond true/false.** Can be viewed as an extension of formal logic that allows reasoning under uncertainty

3 Bayesian inference: an introduction

3.1 The Bayesian 'recipe'

The elements of Bayesian inference

- Take our existing statistical models and add:
  - Parameters & hypotheses are **modelled as random variables**
- In other words:
  - Parameters will have probability distributions
  - Hypotheses will have probabilities
- These are Bayesian probabilities
- They quantify and express our uncertainty, both before ('prior') and after ('posterior') seeing any data
- Requires the use of Bayes' theorem

Motivating example

- A coin is either **fair** or **unfair**

$$\theta = \Pr(\text{heads}) = \begin{cases} 0.5 & \text{if coin is fair} \\ 0.7 & \text{if coin is unfair} \end{cases}$$

- Flip the coin 20 times
- The number of heads is $X \sim \text{Bi}(20, \theta)$
- In light of the data, what can we say about whether the coin is fair?
- What does X tell us about θ?

Posterior distribution

- Goal: calculate $\Pr(\text{coin is fair} \mid X) = \Pr(\theta \mid X)$
- More broadly, Pr(parameter or hypothesis | data)
- This is known the *posterior distribution* (or just the *posterior*)
- Quantifies our knowledge in light of the data we observe
- **Posterior** means 'coming after' in Latin
- In Bayesian inference, the posterior distribution summarises all of the information about the parameters of interest

Calculating the posterior

- Use Bayes' theorem,

$$\Pr(\theta = 0.5 \mid X = x) = \frac{\Pr(X = x \mid \theta = 0.5)\Pr(\theta = 0.5)}{\Pr(X = x)}$$

- The denominator is (law of total probability),

$$\Pr(X = x) = \Pr(X = x \mid \theta = 0.5)\Pr(\theta = 0.5) + \Pr(X = x \mid \theta = 0.7)\Pr(\theta = 0.7)$$

- We need to specify:
  - The *likelihood*, $\Pr(X \mid \theta)$.
  - The *prior distribution* (or just the *prior*), $\Pr(\theta)$.
- In our example, the likelihood is a binomial distribution

Specifying the prior

- Also need a prior to get the whole thing off the ground
- **Prior** means 'before' in Latin
- Specifying an appropriate prior requires some thought (more details later)
- For now, let's assume either outcome is equally plausible,

$$\Pr(\text{fair coin}) = \Pr(\text{unfair coin}) = 0.5$$

$$\Pr(\theta = 0.5) = \Pr(\theta = 0.7) = 0.5$$

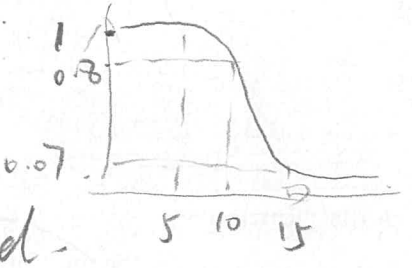
Putting it together

- This gives,

$$\Pr(\theta = 0.5 \mid X = x) = \frac{\Pr(X = x \mid \theta = 0.5)}{\Pr(X = x \mid \theta = 0.5) + \Pr(X = x \mid \theta = 0.7)}$$

- For example,

$$\Pr(\theta = 0.5 \mid X = 15) = 0.076$$
$$\Pr(\theta = 0.5 \mid X = 10) = 0.851$$
$$\Pr(\theta = 0.5 \mid X = 5) = 0.997$$



Example (card experiment)

- Select 5 cards at random (don't look at them!)
- Sample from these n times with replacement
- Let X be the number of times you see a red card
- Likelihood: $X \sim \text{Bi}(n, \theta)$
- $\theta \in \{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$
- Use a uniform prior again,

$$\Pr(\theta = a) = \frac{1}{6} \quad \text{(for all } a\text{)}$$

- Calculate posterior,

$$\Pr(\theta = a \mid X = x) = \frac{\Pr(X = x \mid \theta = a)\Pr(\theta = a)}{\Pr(X = x)}$$

- The denominator is always just a sum/integral of the numerator,

$$\Pr(X = x) = \sum_b \Pr(X = x \mid \theta = b)\Pr(\theta = b)$$

- For convenience, we often omit it,

$$\Pr(\theta = a \mid X = x) \propto \Pr(X = x \mid \theta = a)\Pr(\theta = a)$$

- This gives,

$$\Pr(\theta = a \mid X = x) \propto \binom{n}{x} a^x (1-a)^{n-x} \frac{1}{6}$$

$$\propto a^x (1-a)^{n-x}$$

- Only need the terms that refer to the parameter values, a
- Now try it out...

Example (beta-binomial)

- $X \sim \text{Bi}(n, \theta)$
- $\theta \in [0, 1]$
- Start with a uniform prior again (now a pdf, since continuous),

$$f(\theta) = 1, \quad 0 \leqslant \theta \leqslant 1$$

- Calculate posterior pdf,

$$\begin{aligned} f(\theta \mid X = x) &\propto \Pr(X = x \mid \theta) f(\theta) \\ &\propto \theta^x (1-\theta)^{n-x} \end{aligned}$$

- Calculate the normalising constant by integrating w.r.t. θ,

$$\int_0^1 \theta^x (1-\theta)^{n-x} \, d\theta = \cdots = \frac{x!\,(n-x)!}{(n+1)!}$$

- The posterior therefore has pdf,

$$f(\theta \mid X = x) = \frac{(n+1)!}{x!\,(n-x)!} \theta^x (1-\theta)^{n-x}, \quad 0 \leqslant \theta \leqslant 1$$

- This is a *beta distribution*

Beta distribution

- A distribution over the unit interval, $p \in [0, 1]$
- Two parameters: $\alpha, \beta > 0$
- Notation: $P \sim \text{Beta}(\alpha, \beta)$
- The pdf is:

$$f(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1}, \quad 0 \leqslant p \leqslant 1$$

- Γ is the gamma function, a generalisation of the factorial function. Note that $\Gamma(n) = (n-1)!$
- Properties:

$$\mathbb{E}(P) = \frac{\alpha}{\alpha+\beta}$$

$$\text{mode}(P) = \frac{\alpha-1}{\alpha+\beta-2} \quad (\alpha, \beta > 2)$$

$$\text{var}(P) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

- Draw some pdfs to get an idea.

Inference from the posterior

- For our example, the posterior is:

$$\theta \mid X = x \sim \text{Beta}(x+1, n-x+1)$$

- Could use the *posterior mean* as a point estimate:

$$\mathbb{E}(\theta \mid X = x) = \frac{x+1}{n+2}$$

- More options later...

Different priors

- Let's use a beta distribution as our prior, $\theta \sim \text{Beta}(\alpha, \beta)$
- This gives posterior pdf,

$$\begin{aligned} f(\theta \mid X = x) &\propto \Pr(X = x \mid \theta) f(\theta) \\ &\propto \theta^x (1-\theta)^{n-x} \times \theta^{\alpha-1}(1-\theta)^{\beta-1} \\ &= \theta^{x+\alpha-1}(1-\theta)^{n-x+\beta-1} \end{aligned}$$

- This is again in the form a beta distribution!

$$\theta \mid X = x \sim \text{Beta}(x+\alpha, n-x+\beta)$$

Conjugate distributions

- Beta prior + binomial likelihood $\Rightarrow$ beta posterior
- This a convenient property
- We say that the beta distribution is a *conjugate prior* for the binomial distribution
- Note: we initially used a uniform prior, which is equivalent to $\alpha = \beta = 1$

Pseudodata

- Can think of the prior as being equivalent to unobserved data
- It has the same influence on the posterior as an actual sample with some sample size and particular (pseudo-)observations
- Provides an intuitive interpretation for the prior
- Works particularly well with conjugate priors
- A Beta$(1,1)$ prior is equivalent to a sample of size of 2, with 1 observed success and 1 observed failure.
- A Beta(α, β) prior is equivalent to a sample of size of $\alpha+\beta$. The parameters α and β are often called *pseudocounts*.
- Pseudocounts can be non-integer

Remarks

- The likelihood is sometimes called the 'model'. But we sometimes refer to the whole setup (including the prior) as the 'model'. In any case, we can at least call it the 'model for the data'.
- Classical inference only works with a likelihood, but entails other choices about how to do inference (see later for a more detailed discussion of the differences between approaches)
- Parameters are **modelled** as random variables. This expresses our uncertainty of their value. We don't actually think of them as being truly random quantities, as many textbooks suggest! We still think of them as representing some **fixed underlying true value**, but one we can never know for certain.

3.2 Using the posterior

Summarising the posterior

- We've worked out the posterior... now what?
- Visualise it
- Summarise it
- Think about what you wanted to learn
- What was your original question?

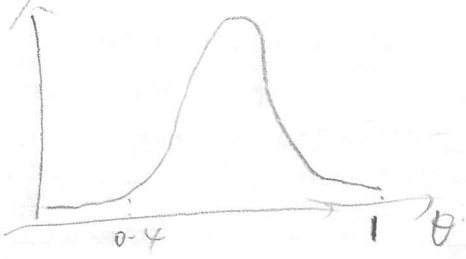
Point estimates

- Can calculate single-number (point) summaries
- Popular choices:
  - *Posterior mean* $\mathbb{E}(\theta \mid X = x)$
  - *Posterior median* $\text{median}(\theta \mid X = x)$
  - *Posterior mode*, $\text{mode}(\theta \mid X = x)$
- Uniform prior $\Rightarrow$ posterior mode = MLE
- The *posterior standard deviation* $\text{sd}(\theta \mid X = x)$, gives a measure of uncertainty (analogous to the standard error)
- For example, with $n = 20$, $x = 15$ and a uniform prior,

$$\theta \mid X = 15 \sim \text{Beta}(16, 6)$$

$$\mathbb{E}(\theta \mid X = 15) = \frac{16}{22} = 0.73$$

$$\text{sd}(\theta \mid X = 15) = \sqrt{\frac{16 \cdot 6}{22^2 \cdot 23}} = 0.093$$



Interval estimates (credible intervals)

- Can calculate intervals to represent the uncertainty
- Simply take probability intervals from the posterior, referred to as *credible intervals*
- A 95% credible interval (a, b) is given by:

$$0.95 = \Pr(a < \theta < b \mid X)$$

- For example, with $n = 20$, $x = 15$ and a uniform prior, the central 95% credible interval is given by:

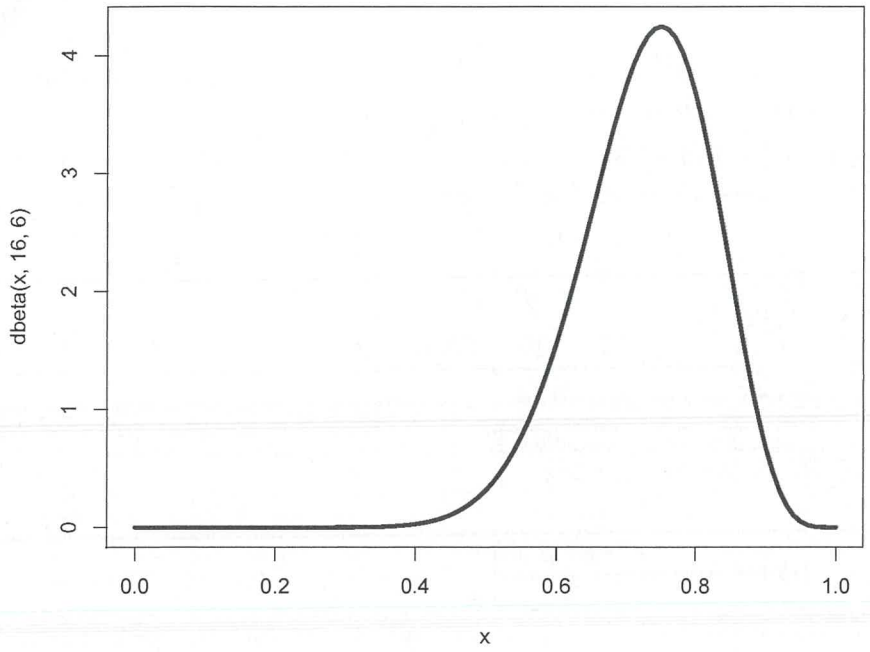
```
> qbeta(c(0.025, 0.975), 16, 6)
[1] 0.5283402 0.8871906
```

- Analogous to confidence intervals, but easier to interpret/explain
- Can calculate one-sided or two-sided intervals, as required

Visual summaries

- Not always necessary to summarise into one or two numbers
- Plot the posterior (bar plot, density curve, box plot,...)
- This is often more informative
- Helps to avoid placing too much emphasis on the tails
- For example:

```
> curve(dbeta(x, 16, 6), from = 0, to = 1)
```



Specific posterior probabilities

- Posterior probabilities of events relevant to the problem, for example:

$$\Pr(\mu > 0 \mid \text{data})$$

- More generally, can calculate posterior distributions for arbitrary functions of the parameters, for example:

$$f\left(\frac{\theta}{1-\theta} \,\middle|\, \text{data}\right)$$

Computation

- Often cannot derive or write down an expression for the posterior
- True for nearly all modern applications of Bayesian analysis!
- Use computational techniques instead (like resampling methods)
- Typically work with simulations ('samples') from the posterior (see the lab)
- Most common class of methods: Markov chain Monte Carlo (MCMC)
- The ability to do this, due to advances in computation, has led to a surge in popularity of Bayesian methods
- Topic is too advanced for this subject, but watch out for it in later years

4 Bayesian inference: further examples

Overview

- Only consider single-parameter models
- Only consider conjugate priors
- Examples are intentionally similar to those from earlier modules

4.1 Normal

Normal, single mean, known σ

- Random sample: $X_1, \ldots, X_n \sim \mathrm{N}(\theta, \sigma^2)$, with σ^2 known
- For simplicity, summarise the data by: $Y = \bar{X} \sim \mathrm{N}(\theta, \sigma^2/n)$
- Prior: $\theta \sim \mathrm{N}(\mu_0, \sigma_0^2)$
- Deriving the posterior:

$$\begin{aligned} f(\theta \mid y) &\propto f(y \mid \theta) f(\theta) \\ &= \frac{1}{\frac{\sigma}{\sqrt{n}}\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2/n}(y-\theta)^2} \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2\sigma_0^2}(\theta-\mu_0)^2} \\ &\propto \exp\left[-\frac{(y-\theta)^2}{2\sigma^2/n} - \frac{(\theta-\mu_0)^2}{2\sigma_0^2}\right] \end{aligned}$$

- We can simplify this as:

$$f(\theta \mid y) \propto \exp\left[-\frac{(\theta-\mu_1)^2}{2\sigma_1^2}\right]$$

by defining,

$$\mu_1 = \frac{\frac{\mu_0}{\sigma_0^2} + \frac{y}{\sigma^2/n}}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma^2/n}} \quad \text{and} \quad \frac{1}{\sigma_1^2} = \frac{1}{\sigma_0^2} + \frac{1}{\sigma^2/n}$$

- Recognise this as a normal pdf (so, we immediately know the normalising constant)
- Posterior: $\theta \mid y \sim \mathrm{N}(\mu_1, \sigma_1^2)$
- '1/ var' is called the *precision*
- Posterior precision is the sum of the prior and data precisions:

$$\frac{1}{\sigma_1^2} = \frac{1}{\sigma_0^2} + \frac{1}{\sigma^2/n}$$

- Posterior mean is a weighted average of the sample mean, $y = \bar{x}$, and the prior mean, μ_0, weighted by their precisions:

$$\mu_1 = \left(\frac{\frac{1}{\sigma_0^2}}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma^2/n}}\right)\mu_0 + \left(\frac{\frac{1}{\sigma^2/n}}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma^2/n}}\right) y$$

- More data $\Rightarrow$ higher data precision $\Rightarrow$ more influence on the posterior
- Credible intervals: probability intervals from the posterior (normal)
- For example, a central 95% credible interval for θ looks like:

$$\mu_1 \pm 1.96\sigma_1$$

Example (normal, single mean, known σ)

- $X \sim \mathrm{N}(\theta, \sigma^2 = 36^2)$ is the lifetime of a light bulb, in hours.
- Suppose we knew from experience that the lifetime is somewhere between 1200 and 1600 hours. We could summarise this with a prior $\theta \sim \mathrm{N}(\mu_0 = 1400, \sigma_0^2 = 100^2)$, which places 95% probability on that range.
- Test $n = 27$ light bulbs and get $y = \bar{x} = 1478$
- Posterior: $\theta \mid y \sim \mathrm{N}(1478, 6.91^2)$
- 95% credible interval: $(1464, 1491)$
- If we use a more informative prior: $\theta \sim \mathrm{N}(\mu_0 = 1400, \sigma_0^2 = 10^2)$
- Posterior: $\theta \mid y \sim \mathrm{N}(1453, 5.69^2)$
- 95% credible interval: $(1442, 1464)$

A less informative prior

- Can make the prior progressively less informative by reducing its precision: $\sigma_0 \to \infty$
- In the limit, we get a 'flat' prior across the whole real line
- (μ_0 disappears from the model)
- Not a valid probability distribution, cannot integrate to 1
- But it works: it gives us a valid posterior,

$$\sigma_1^2 = \sigma^2/n \quad \text{and} \quad \mu_1 = y = \bar{x}$$

and the credible intervals are the same as the confidence intervals.

- This type of prior (cannot integrate to 1) is called an *improper* prior
- If the prior **does** integrate to 1, it is called a *proper* prior
- Can think of improper priors as approximations to very uninformative proper priors

4.2 Binomial

Binomial (again)

- $X \sim \text{Bi}(n, \theta)$
- Prior: $\theta \sim \text{Beta}(\alpha, \beta)$
- Posterior: $\theta \mid x \sim \text{Beta}(\alpha + x, \beta + n - x)$
- Posterior mean:

$$\begin{aligned}\mathbb{E}(\theta \mid x) &= \frac{\alpha + x}{\alpha + \beta + n} \\ &= \left(\frac{\alpha + \beta}{\alpha + \beta + n}\right)\left(\frac{\alpha}{\alpha + \beta}\right) + \left(\frac{n}{\alpha + \beta + n}\right)\left(\frac{x}{n}\right)\end{aligned}$$

which is a weighted average of the prior mean and the MLE (x/n).

Example (binomial)

- In a survey of $n = 351$ voters, $x = 185$ favour a particular candidate
- Use uniform prior
- Posterior: $\theta \mid x \sim \text{Beta}(1 + 185, 1 + 351 - 185) = \text{Beta}(186, 167)$
- 95% credible interval: $(0.475, 0.579)$
- Posterior probability of a majority:

$$\Pr(\theta > 0.5 \mid x) = 0.84$$

- R code:

```
> 1 - pbeta(0.5, 186, 167)
[1] 0.8444003
```

- Suppose an initial survey suggested support was only 45%
- Include this knowledge as a prior
- Deem it to be worth equivalent to a (pseudo) sample size of 20
- Therefore, our prior should satisfy:

$$\begin{cases} \frac{\alpha}{\alpha+\beta} = 0.45 \\ \alpha + \beta = 20 \end{cases}$$

$$\Rightarrow \alpha = 9, \quad \beta = 11$$

- Posterior: $\theta \mid x \sim \text{Beta}(9+185, 11+351-185) = \text{Beta}(194, 177)$
- 95% credible interval: $(0.472, 0.574)$
- Posterior probability of a majority: $\Pr(\theta > 0.5 \mid x) = 0.81$

4.3 Other

Challenge problem (exponential distribution)

Random sample: $X_1, \ldots, X_n \sim \text{Exp}(\lambda)$

Find a conjugate prior distribution for λ

What is the resulting posterior mean?

Challenge problem (boundary problem)

Random sample of size n from the shifted exponential distribution, with pdf:

$$f(x) = e^{-(x-\theta)} \quad (x > \theta)$$

Equivalently:

$$X_i \sim \theta + \text{Exp}(1)$$

Use a 'flat' improper prior for θ and derive the posterior.

Derive a one-sided 95% credible interval.

5 Prior distributions

Aspects of prior distributions

- We already defined:
  - Conjugate priors
  - Improper priors
  - Proper priors
- We now cover:
  - Choosing appropriate priors
  - Seeking 'noninformative' priors
  - Sensitivity analysis

How do we choose an appropriate prior?

- Considerations:
  - Existing knowledge (try to encapsulate/quantify)
  - Plausability of various values
  - Ability of data to 'overwhelm' the prior
- The prior should be diffuse enough to allow the data, **if sufficient enough**, to overwhelm it
- Usually the prior will be much less precise than the data (otherwise, why are we bothering to collect data?)
- If the prior and the data are (vastly) in conflict, something has likely gone wrong: go back and check your assumptions
- Since we expect the data to dominate, we don't need to be overly worried with the exact shape of the prior

- (All of this becomes more delicate in higher dimensions...)

'Noninformative' priors

- Can we use a prior that has **no influence** on the posterior?
- Sometimes: e.g. improper prior for θ in $N(\theta, \sigma^2)$
- But usually not
- 'Noninformative' depends on the parameterisation
- What is noninformative on one scale can be informative on a different scale for the same parameter!
- Example, for binomial sampling, $Bi(n, \theta)$:
  - $\theta \sim \text{Beta}(1, 1)$ is uniform for θ
  - $\theta \sim \text{Beta}(0, 0)$ is uniform for $\log(\theta/(1-\theta))$
  - $\theta \sim \text{Beta}(\frac{1}{2}, \frac{1}{2})$ is invariant under reparameterisation ("Jeffreys' prior")
- So, generally talk about **'diffuse'** priors rather than 'noninformative'

Sensitivity analysis

- Not sure about your prior?
- Worried that it might be too influential?
- Try a range of different priors
- This is *a sensitivity analysis*
- Useful to cover a reasonable set of 'extreme' views, plus typical diffuse priors

Sensitivity to the prior

- The (potential) sensitivity to the prior is a key feature of Bayesian inference.
- If the prior is influential, and you don't really believe it, then you have insufficient data.
- Either need more data, or a more reliable prior.
- This is not a 'bug', it is a feature!
- It alerts you to the relative amount of information in your data (or the lack of it)

6 Comparing Bayesian & classical inference

Goals & philosophies

- Shared goals:
  - Learning about the population
  - Estimating parameters
  - Making decisions
  - Prediction
- Different underlying philosophies:
  - Use of probability
  - Manner of inference
  - Interpretation of results

Assumptions

- Bayesian inference needs a prior.
- Sometimes seen as a weakness, but this aspect is usually overplayed or misrepresented as being overly subjective.
- Classical inference (a.k.a. frequentist inference) requires further choices (e.g. which estimator to use), which can be just as arbitrary or ad hoc as a choice of prior.
- Choice of likelihood is also crucial, and involves similar considerations (and problems) to choosing a prior, but people often overlook this.
- Complex models often start blurring the boundary between the two anyway.

Making assumptions explicit

> Bayesian approaches need to be explicit about the assumptions they make, whereas many of the assumptions underlying frequentist approaches are often implicit (I.J. Good, 1976)
>
> "Bayesian analyses should not be penalized for openness, particularly when the corresponding frequentist analysis would evade criticism by keeping issues hidden" (Stephens & Balding, 2009)

Advantages of Bayesian inference

- Forces you to be upfront about your assumptions
- If you have useful prior knowledge, provides a principled way of including it
- Once you have a prior, how to do inference is (in theory) automatic: calculate the posterior
- Interpretation generally easier, since we answer the question directly

Disadvantages of Bayesian inference

- Need to write a full probability model, can be difficult to specify for complex problems
- Typically much more computation required
- Usually harder to set up and implement, need more experience
- Strong focus on parametric models, although 'nonparametric Bayes' techniques exist (but require some expertise)

Reconciliation?

- Posterior summaries $\leftrightarrow$ estimators
- E.g. take a posterior summary and evaluate its operating characteristics (bias, variance, etc., under repeated sampling)
- Or, take an estimator and ask what prior/likelihood it is equivalent to.
- We saw very clear correspondances in the examples here. This is not always possible, esp. in higher-dimensional models.

Which one should I use?

- Be agnostic: **learn both**
- Use whatever works for the problem at hand
- Many statisticians say they 'use the right tool for the right job'
- In practice, classical techniques get used more often because of convenience, familiarity or convention... not necessarily because they are the 'right tool'!
- In simple settings (includes everything we have covered in this subject), both approaches lead to similar procedures, so the question is moot.
- It becomes a more relevant question as the problems start getting more complex.

When not to use Bayes?

- If a specific method is expected and is a strong convention (e.g. clinical drug trials).
- If you are more familiar and proficient with non-Bayesian approaches (different approaches can often solve the problem adequately, use the ones you are most proficient at).
- Bayesian methods can be computationally demanding, which can put them out of reach for very large/complex problems (although there are approximations that help speed things up) or for non-experienced users.

Damjan's approach

- Bayesian inference is preferred
- If impractical/infeasible, fall back to non-Bayesian methods
- Think about what implicit assumptions exist
- Think about what Bayesian model it might be similar/equivalent to

Asymptotics & optimality

(Module 11)

Statistics (MAST20005) & Elements of Statistics (MAST90058)

Semester 2, 2019

Contents

Aims of this module

- Explain some of the theory that we skipped in previous modules
- Show why the MLE is usually a good (or best) estimator
- Explain some related important theoretical concepts

1 Likelihood theory

Previous claims (from modules 2 & 4)

The MLE is asymptotically:

- unbiased
- efficient (has the optimal variance)
- normally distributed

Can use the 2nd derivative of the log-likelihood (the 'observed information function') to get a standard error for the MLE.

Motivating example (non-zero binomial)

- Consider a factory producing items in batches. Let θ denote the proportion of defective items. From each batch 3 items are sampled at random and the number of defectives is determined. However, records are only kept if there is at least one defective.
- Let Y be the number of defectives in a batch.
- Then $Y \sim \text{Bi}(3, \theta)$,

$$\Pr(Y = y) = \binom{3}{y} \theta^y (1-\theta)^{3-y}, \quad y = 0, 1, 2, 3$$

- But we only take an observation if $Y > 0$, so the pmf is

$$\Pr(Y = y \mid Y > 0) = \frac{\binom{3}{y} \theta^y (1-\theta)^{3-y}}{1 - (1-\theta)^3}, \quad y = 1, 2, 3$$

- Let X_i be the number of times we observe i defectives and let $n = X_1 + X_2 + X_3$ be the total number of observations.
- The likelihood is,

$$L(\theta) = \frac{n!}{x_1!x_2!x_3!}\left(\frac{3\theta(1-\theta)^2}{1-(1-\theta)^3}\right)^{x_1}\left(\frac{3\theta^2(1-\theta)}{1-(1-\theta)^3}\right)^{x_2}\left(\frac{\theta^3}{1-(1-\theta)^3}\right)^{x_3}$$

- This simplifies to,

$$L(\theta) \propto \frac{\theta^{x_1+2x_2+3x_3}(1-\theta)^{2x_1+x_2}}{(1-(1-\theta)^3)^n}$$

- After taking logarithms and derivatives, the MLE is found to be the smaller root of

$$t\theta^2 - 3t\theta + 3(t-n) = 0$$

where $t = x_1 + 2x_2 + 3x_3$.

- This gives:

$$\hat{\theta} = \frac{3t - \sqrt{-3t^2 + 12tn}}{2t}$$

- We now have the MLE...
- ...but finding its sampling distribution is not straightforward!
- In general, finding the exact distribution of a statistic is often difficult.
- We've used the Central Limit Theorem to approximate the distribution of the sample mean.
- Gave us approximate CIs for a population mean μ of the form,

$$\bar{x} \pm \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) \times \frac{s}{\sqrt{n}}$$

- Similar results hold more generally for MLEs (and other estimators)

1.1 Asymptotic distribution of the MLE

Definitions

- Start with the log-likelihood:

$$\ell(\theta) = \ln L(\theta)$$

- Taking the first derivative gives the *score function* (also known simply as the *score*). Let's call it U,

$$U(\theta) = \frac{\partial \ell}{\partial \theta}$$

- Note: we solve $U(\hat{\theta}) = 0$ to get the MLE
- Taking the second derivative, and then it's negative, gives the *observed information function* (also known simply as the *observed information*). Let's call it V,

$$V(\theta) = -\frac{\partial U}{\partial \theta} = -\frac{\partial^2 \ell}{\partial \theta^2}$$

- This represents the **curvature** of the log-likelihood. Greater curvature $\Rightarrow$ narrower likelihood around a certain value $\Rightarrow$ the likelihood is more **informative**.

Fisher information

- All of the above are functions of the data (and parameters). Therefore they are random variables and have sampling distributions.
- For example, we can show that $\mathbb{E}(U(\theta)) = 0$.
- An important quantity is $I(\theta) = \mathbb{E}(V(\theta))$, which is the *Fisher information function* (or just the *Fisher information*). It is also known as the *expected information function* (or simply as the *expected information*).
- Many results are based on the Fisher information.
- For example, we can show that $\text{var}(U(\theta)) = I(\theta)$.
- More importantly, it arises in theory about the distribution of the MLE.

Asymptotic distribution

- The following is a key result:

$$\hat{\theta} \approx \text{N}\left(\theta, \frac{1}{I(\theta)}\right) \quad \text{as } n \to \infty$$

- It requires some conditions for it to hold. The main one being that the parameter should not be defining a boundary of the sample space (e.g. like in the boundary problem examples we've looked at).
- Let's see a proof...

Asymptotic distribution (derivation)

- Assumptions:
  - $X_1, \ldots, X_n$ is a random sample from $f(x, \theta)$
  - Continuous pdf, $f(x, \theta)$
  - θ is not a boundary parameter
- Suppose the MLE satisfies:

$$U(\hat{\theta}) = \frac{\partial \ln L(\hat{\theta})}{\partial \theta} = 0$$

  Note: this requires that θ is not a boundary parameter.
- Taylor series approximation for $U(\hat{\theta})$ about θ:

$$\begin{aligned} 0 = U(\hat{\theta}) = \frac{\partial \ln L(\hat{\theta})}{\partial \theta} &\approx \frac{\partial \ln L(\theta)}{\partial \theta} + (\hat{\theta} - \theta)\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} \\ &= U(\theta) - (\hat{\theta} - \theta)V(\theta) \end{aligned}$$

- We can write this as:

$$V(\theta)\,(\hat{\theta} - \theta) \approx U(\theta)$$

- Remember that we have a random sample (iid rvs), so we have,

$$U(\theta) = \frac{\partial \ln L(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial \ln f(X_i, \theta)}{\partial \theta}$$

- Since the X_i are iid so are:

$$U_i = \frac{\partial \ln f(X_i, \theta)}{\partial \theta}, \quad i = 1, \ldots, n.$$

- And the same for:

$$V_i = -\frac{\partial^2 \ln f(X_i, \theta)}{\partial \theta^2}, \quad i = 1, \ldots, n.$$

- Determine $\mathbb{E}(U_i)$ by integration by substitution and exchanging the order of integration and differentiation,

$$\mathbb{E}(U_i) = \int_{-\infty}^{\infty} \frac{\partial \ln f(x,\theta)}{\partial \theta} f(x,\theta)\, dx = \int_{-\infty}^{\infty} \frac{\partial f(x,\theta)}{\partial \theta} \frac{f(x,\theta)}{f(x,\theta)}\, dx$$
$$= \int_{-\infty}^{\infty} \frac{\partial f(x,\theta)}{\partial \theta}\, dx = \frac{\partial}{\partial \theta} \int_{-\infty}^{\infty} f(x,\theta)\, dx = \frac{\partial}{\partial \theta} 1 = 0$$

- To get the variance of U_i, we start with one of the above results,

$$\int_{-\infty}^{\infty} \frac{\partial \ln f(x,\theta)}{\partial \theta} f(x,\theta) dx = 0$$

- Taking another derivative of both sides gives,

$$\int_{-\infty}^{\infty} \left\{ \frac{\partial^2 \ln f(x,\theta)}{\partial \theta^2} f(x,\theta) + \frac{\partial \ln f(x,\theta)}{\partial \theta} \frac{\partial f(x,\theta)}{\partial \theta} \right\} dx = 0$$

- But,

$$\frac{\partial f(x,\theta)}{\partial \theta} = \frac{\partial \ln f(x,\theta)}{\partial \theta} f(x,\theta)$$

- Combining the previous two equations gives,

$$\int_{-\infty}^{\infty} \left\{ \frac{\partial \ln f(x,\theta)}{\partial \theta} \right\}^2 f(x,\theta)\, dx = -\int_{-\infty}^{\infty} \frac{\partial^2 \ln f(x,\theta)}{\partial \theta^2} f(x,\theta)\, dx$$

- In other words,

$$\mathbb{E}(U_i^2) = \mathbb{E}(V_i)$$

- Since $\mathbb{E}(U_i) = 0$ we also have $\mathbb{E}(U_i^2) = \mathrm{var}(U_i)$, so we can conclude,

$$\mathrm{var}(U_i) = \mathbb{E}(V_i)$$

- Thus $U = \sum_i U_i$ is the sum of iid rvs with mean 0 and this variance.
- Thus,

$$\mathrm{var}(U) = n\, \mathbb{E}(V_i)$$

- Also, since $V = \sum_i V_i$, we can conclude that,

$$\mathbb{E}(V) = n\, \mathbb{E}(V_i)$$

- Note that this is just the Fisher information, i.e.

$$\mathbb{E}(V) = \mathrm{var}(U) = I(\theta)$$

- Looking back at,

$$V(\theta)\,(\hat{\theta} - \theta) \approx U(\theta)$$

We want to know what happens to U and V as the sample size gets large.

- U has mean 0 and variance $I(\theta)$.
- Central Limit Theorem $\Rightarrow U \approx \mathrm{N}(0, I(\theta))$.
- V has mean $I(\theta)$.
- Law of Large Numbers $\Rightarrow V \to I(\theta)$
- Putting these together gives, as $n \to \infty$,

$$I(\theta)\,(\hat{\theta} - \theta) \sim \mathrm{N}(0, I(\theta))$$

- Equivalently,

$$\hat{\theta} \sim \mathrm{N}\left(\theta, \frac{1}{I(\theta)}\right)$$

- This is a very powerful result. For large (or even modest) samples we do not need toa find the exact distribution of the MLE but can use this approximation.
- In other words, as a standard error of the MLE we can use:

$$\text{se}(\hat{\theta}) = \frac{1}{\sqrt{I(\hat{\theta})}}$$

if we know $I(\theta)$, or otherwise replace it with it's realised (observed) version,

$$\text{se}(\hat{\theta}) = \frac{1}{\sqrt{V(\hat{\theta})}}$$

- Furthermore, we use the normal distribution to construct approximate confidence intervals.

Example (exponential distribution)

- $X_1, \ldots, X_n$ random sample from

$$f(x \mid \theta) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 < x < \infty, \quad 0 < \theta < \infty$$

- MLE is $\bar{X}$.
- $\ln f(x \mid \theta) = -\ln\theta - x/\theta$, so

$$U_i(\theta) = \frac{\partial}{\partial\theta} \ln f(x \mid \theta) = -\frac{1}{\theta} + \frac{x}{\theta^2}$$

$$V_i(\theta) = -\frac{\partial^2}{\partial\theta^2} \ln f(x \mid \theta) = -\frac{1}{\theta^2} + \frac{2x}{\theta^3}$$

- Since $\mathbb{E}(X) = \theta$,

$$I_i(\theta) = \mathbb{E}(V_i(\theta)) = \mathbb{E}\left(-\frac{1}{\theta^2} + \frac{2X}{\theta^3}\right) = -\frac{1}{\theta^2} + \frac{2\theta}{\theta^3} = \frac{1}{\theta^2}$$

- Then $I(\theta) = n/\theta^2$ and $\hat{\theta} \approx \text{N}(\theta, \theta^2/n)$
- Suppose we observe $n = 20$ and $\bar{x} = 3.7$. An approximate 95% CI is,

$$3.7 \pm 1.96\sqrt{\frac{3.7^2}{20}} = (2.1, 5.3)$$

Example (Poisson distribution)

- Same arguments hold for discrete distributions, e.g. Pn(λ).

$$f(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, \ldots, \quad \lambda > 0$$

We have seen $\hat{\lambda} = \bar{X}$.

- $\ln f(x \mid \lambda) = x\ln\lambda - \lambda - \ln(x!)$, so

$$\frac{\partial \ln f(x \mid \lambda)}{\partial\lambda} = \frac{x}{\lambda} - 1, \quad \text{and} \quad \frac{\partial^2 \ln f(x \mid \lambda)}{\partial\lambda^2} = -\frac{x}{\lambda^2}$$

- Thus

$$I_i(\lambda) = \mathbb{E}\left(\frac{X}{\lambda^2}\right) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

- Then $\hat{\lambda} \approx \text{N}(\lambda, \lambda/n)$
- Suppose we observe $n = 40$ and $\bar{x} = 2.225$. An approximate 90% CI is,

$$2.225 \pm 1.645\sqrt{\frac{2.225}{40}} = (1.837, 2.612)$$

1.2 Cramér–Rao lower bound

Cramér–Rao lower bound

- How good can our estimator get?
- Suppose we know that it is unbiased.
- What is the minimum variance we can achieve?
- Under similar assumptions to before (esp. the parameter must not define a boundary), we can find a lower bound on the variance
- This is known as the *Cramér Rao lower bound*
- It is equal to the asymptotic variance of the MLE.
- In other words, if we take any unbiased estimator T, then

$$\operatorname{var}(T) \geqslant \frac{1}{I(\theta)}$$

Cramér–Rao lower bound (proof)

- Let T be an unbiased estimator of θ
- Consider its covariance with the score function,

$$\begin{aligned}\operatorname{cov}(T, U) &= \mathbb{E}(TU) - \mathbb{E}(T)\,\mathbb{E}(U) = \mathbb{E}(TU) \\ &= \int T \frac{\partial \ln L}{\partial \theta} L \, d\mathbf{x} = \int T \frac{\partial L}{\partial \theta} \, d\mathbf{x} \\ &= \frac{\partial}{\partial \theta} \int TL \, d\mathbf{x} = \frac{\partial}{\partial \theta} \mathbb{E}(T) = \frac{\partial}{\partial \theta} \theta = 1\end{aligned}$$

- Using the fact that $\operatorname{cor}(T, U)^2 \leqslant 1$,

$$\begin{aligned}\operatorname{cov}(T, U)^2 &\leqslant \operatorname{var}(T) \operatorname{var}(U) \\ \operatorname{var}(T) &\geqslant \frac{1}{\operatorname{var}(U)} = \frac{1}{I(\theta)}\end{aligned}$$

Implications of the Cramér–Rao lower bound

- If an unbiased estimator attains this bound, then it is best in the sense that it has minimum variance compared with other unbiased estimators.
- Therefore, MLEs are approximately (or exactly) optimal for large sample size because:
  - They are asymptotically unbiased
  - Their variance meets the Cramér Rao lower bound asymptotically

Efficiency

- We can compare any unbiased estimator against the lower bound
- We define the *efficiency* of the unbiased estimator T as its variance relative to the lower bound,

$$\operatorname{eff}(T) = \frac{1/I(\theta)}{\operatorname{var}(T)} = \frac{1}{I(\theta) \operatorname{var}(T)}$$

- Note that $0 \leqslant \operatorname{eff}(T) \leqslant 1$
- If $\operatorname{eff}(T) \approx 1$ we say that T is an *efficient* estimator

Example (exponential distribution)

- Sampling from an exponential distribution
- We saw that $I(\theta) = n/\theta^2$
- Therefore, the Cramér Rao lower bound is θ^2/n.
- Any unbiased estimator must have variance at least as large as this.
- The MLE in this case is the sample mean, $\hat{\theta} = \bar{X}$
- Therefore, $\text{var}(\hat{\theta}) = \text{var}(X)/n = \theta^2/n$
- So the MLE is efficient (for all sample sizes!)

2 Sufficient statistics

Sufficiency: a starting example

- We toss a coin 10 times
- Want to estimate the probability of heads, θ
- $X_i \sim \text{Be}(\theta)$
- Suppose we use $\hat{\theta} = \frac{1}{2}(X_1 + X_2)$
- Only uses the first 2 coin tosses
- Clearly, we have not used all of the available information!

Motivation

- Point estimation reduces the whole sample to a few statistics.
- Different methods of estimation can yield different statistics.
- Is there a preferred reduction?
- Toss a coin with probability of heads θ 10 times. Observe T H T H T H H T T T.
- Intuitively, knowing we have 4 heads in 10 tosses is all we need.
- But are we missing something? Does the length of the longest run give extra information?

Definition

- Intuition: want to find a statistic so that any **other** statistic provides no additional information about the value of the parameter
- Definition: the statistic $T = g(X_1, \ldots, X_n)$ is *sufficient* for an underlying parameter θ if the conditional probability distribution of the data $(X_1, \ldots, X_n)$, given the statistic $g(X_1, \ldots, X_n)$, does not depend on the parameter θ.
- Sometimes need more than one statistic, e.g. T_1 and T_2, in which case we say they are *jointly sufficient* for θ

Example (binomial)

- The pdf is, $f(x \mid p) = p^x(1-p)^{1-x}, \quad x = 0, 1$
- The likelihood is,

$$\prod_{i=1}^{n} f(x_i \mid p) = p^{\sum x_i}(1-p)^{n-\sum x_i}$$

- Let $Y = \sum X_i$, we have that $Y \sim \text{Bi}(n, p)$ and then,

$$\begin{aligned}
&\Pr(X_1 = x_1, \ldots, X_n = x_n \mid Y = y) \\
&= \frac{\Pr(X_1 = x_1, \ldots, X_n = x_n)}{\Pr(Y = y)} \\
&= \frac{p^{x_1}(1-p)^{1-x_1} \ldots p^{x_n}(1-p)^{1-x_n}}{\binom{n}{y} p^y (1-p)^{n-y}} = \frac{1}{\binom{n}{y}}
\end{aligned}$$

- Given $Y = y$, the conditional distribution of $X_1, \cdots X_n$ does not depend on p.
- Therefore, Y is sufficient for p.

2.1 Factorisation theorem

Factorisation theorem

- Let $X_1, \ldots, X_n$ have joint pdf or pmf $f(x_1, \ldots, x_n \mid \theta)$
- $Y = g(x_1, \ldots, x_n)$ is sufficient for θ if and only if

$$f(x_1, \ldots, x_n \mid \theta) = \phi\{g(x_1, \ldots, x_n) \mid \theta\} h(x_1, \ldots, x_n)$$

- ϕ depends on $x_1, \cdots, x_n$ only through $g(x_1, \cdots, x_n)$ and h doesn't depend on θ.

Example (binomial)

- The pdf is, $f(x \mid p) = p^x(1-p)^{1-x}, \quad x = 0, 1$
- The likelihood is,

$$\prod_{i=1}^{n} f(x_i \mid p) = p^{\sum x_i}(1-p)^{n-\sum x_i}$$

- So $y = \sum x_i$ is sufficient for p, since we can factorise the likelihood into:

$$\phi(y, p) = p^y(1-p)^{n-y} \quad \text{and} \quad h(x_1, \cdots, x_n) = 1$$

- So in the coin tossing example, the total number of heads is sufficient for θ.

Example (Poisson)

- $X_1, \ldots, X_n$ random sample from a Poisson distribution with mean λ.
- The likelihood is,

$$\prod_{i=1}^{n} f(x_i \mid \lambda) = \frac{\lambda^{\sum x_i} e^{-n\lambda}}{x_1! \ldots x_n!} = (\lambda^{n\bar{x}} e^{-n\lambda}) \left(\frac{1}{x_1! \ldots x_n!} \right)$$

- We see that $\bar{X}$ is sufficient for λ.

Exponential family of distributions

- We often use distributions which have pdfs of the form:

$$f(x \mid \theta) = \exp\{K(x)p(\theta) + S(x) + q(\theta)\}$$

- This is called the *exponential family.*
- Let $X_1, \cdots, X_n$ be iid from an exponential family. Then $\sum_{i=1}^{n} K(X_i)$ is sufficient for θ.
- To prove this note that the joint pdf is

$$\begin{aligned}
&\exp\left\{p(\theta)\sum K(x_i) + \sum S(x_i) + nq(\theta)\right\} \\
&= \left[\exp\left\{p(\theta)\sum K(x_i) + nq(\theta)\right\}\right] \exp\left\{\sum S(x_i)\right\}
\end{aligned}$$

- The factorisation theorem then shows sufficiency.

Example (exponential)

- The pdf is,

$$f(x \mid \theta) = \frac{1}{\theta} e^{-x/\theta} = \exp\left[x\left(-\frac{1}{\theta}\right) - \ln\theta\right], \quad 0 < x < \infty$$

- This is of the form

$$f(x \mid \theta) = \exp\{K(x)p(\theta) + S(x) + q(\theta)\}$$

- So $K(x) = x$ and $\sum X_i$ is sufficient for θ (and so is $\bar{X} = \sum X_i/n$).

Sufficiency and MLEs

- If there exist sufficient statistics, the MLE will be a function of them.
- Factorise the likelihood:

$$L(\theta) = f(x_1, \ldots, x_n \mid \theta) = \phi\{g(x_1, \ldots, x_n) \mid \theta\}\, h(x_1, \ldots, x_n)$$

- We find the MLE by maximizing $\phi\{g(x_1, \ldots, x_n) \mid \theta\}$ which is a function of the sufficient statistics and θ
- So the MLE must be a function of the sufficient statistics

Importance of sufficiency

- Why are sufficient statistics important?
- Once the sufficient statistics are known there is no additional information on the parameter in the sample
- Samples that have the same values of the sufficient statistic yield the same estimates
- The optimal estimators/tests are based on sufficient statistics (such as the MLE)
- A lot of statistical theory is based on them
- Easy to find the sufficient statistics in some special cases (e.g. exponential family)

Disclaimer

- But... the concept of sufficiency relies on **knowing the population distribution**
- So, it is mostly important for theoretical work.
- In practice, we want to also look at all aspects of our data
- That is, we should go beyond any putative sufficient statistics, as a sanity check of our assumptions (e.g. QQ plots).

3 Optimal tests

Previous claims (from module 8)

- The likelihood ratio test (LRT) gives the optimal test
- The likelihood ratio has a known distribution

Neyman–Pearson lemma

- Comparing **simple** hypotheses:

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1$$

- The *Neyman Pearson lemma* states that the *most powerful test*, for a given significance level, is the LRT
- (Proof of lemma not shown)

Uniformly most powerful tests

- Now consider a **composite alternative** hypothesis,

$$H_1 : \theta \in A_1$$

- If the same test (from the LRT) is most powerful for all $\theta_1 \in A_1$, then we say it is *uniformly most powerful* for $\theta_1 \in A_1$.
- If the form of the LRT **differs** for different values of θ_1, then any given one will only be the best for particular values of θ_1.
- If so, then we do not have a uniformly best test.
- But any given test might still be a reasonably good test for other values of θ_1

Asymptotic distribution of the likelihood ratio*

- Consider the test,

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0$$

- The likelihood ratio is,

$$\lambda = \frac{L_0}{L_1} = \frac{L(\theta_0)}{L(\hat{\theta})}$$

- The function $2\ln(\lambda)$ asymptotically follows a χ^2_1 distribution
- This can be used to set up approximate hypothesis tests
- Is often used to formally compare different models