MAST20005/MAST90058: Week 10 Problems

1. Let $X_{(1)} < \cdots < X_{(5)}$ be the order statistics of 5 independent observations from an exponential distribution that has a mean of $\theta = 3$.

   (a) Find the pdf of the sample median $X_{(3)}$.

   (b) Compute the probability that $X_{(4)} < 5$.

   (c) Determine $\Pr(1 < X_{(1)})$.

2. Let $X_1, \dots, X_{10}$ be a random sample from a shifted exponential distribution with pdf $f(x \mid \theta) = e^{-(x-\theta)}$, $\theta \leqslant x < \infty$.

   (a) Show that $Y = \min(X_i) = X_{(1)}$ is the maximum likelihood estimator of θ.

   (b) Find the pdf of Y.

   (c) Show that $\mathbb{E}(Y) = \theta + \frac{1}{10}$ and that $Y - \frac{1}{10}$ is an unbiased estimator of θ.

   (d) Compute $\Pr(\theta < Y < \theta + c)$ and use it to construct a 95% confidence interval for θ.

   (e) Where have you seen this example before?

3. Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics of n independent observations from the uniform distribution $\mathrm{Unif}(0, 1)$.

   (a) Find the pdf of $X_{(1)}$.

   (b) Verify that $\mathbb{E}(X_{(1)}) = \frac{1}{n+1}$.

4. Let X have a Laplace distribution with pdf $f(x \mid \theta) = \frac{1}{2}e^{-|x-\theta|}$. (This is also known as a *double exponential distribution*, can you see why?) Suppose we have a random sample of n observations on X.

   (a) Show that $\mathbb{E}(X) = \theta$ and $\mathrm{var}(X) = 2$. (Hint: $\int_0^\infty z^2 e^{-z} dz = 2$)

   (b) Consider the estimator, $\hat{\theta}_1 = \bar{X}$. Find its mean and variance.

   (c) Consider the estimator, $\hat{\theta}_2 = \hat{M}$. Find its approximate mean and variance.

   (d) Which estimator is better?

   (e) What is the maximum likelihood estimator of θ?

5. The following times (in minutes) between tram arrivals were observed at a particular tram stop:

   ```
   0.67, 2.46, 1.00, 8.89, 8.85, 28.45, 2.95,
   2.36, 0.37, 5.66, 6.26, 1.80,  1.88, 4.66
   ```

   Find an approximate 95% confidence interval for the median and state its exact confidence level. You may use the following information:

   ```
   > pbinom(0:6, size = 14, prob = 0.5)
   [1] 0.0001 0.0009 0.0065 0.0287 0.0898 0.2120 0.3953
   > pbinom(13:7, size = 14, prob = 0.5)
   [1] 0.9999 0.9991 0.9935 0.9713 0.9102 0.7880 0.6047
   ```