

Analysis of variance

(Module 8)



Statistics (MAST20005) &
Elements of Statistics
(MAST90058)

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Aims of this module

- Introduce the **analysis of variance** technique, which builds upon the variance decomposition ideas in previous modules.
- Revisit linear regression and apply the ideas of hypothesis testing and analysis of variance.
- Discuss ways to derive optimal hypothesis tests.

Overview

- **Analysis of variance (ANOVA).**
Comparisons of more than two groups
- **Regression.**
Hypothesis testing for simple linear regression
- **Likelihood ratio tests.**
A method for deriving the best test for a given problem

Outline

Analysis of variance (ANOVA)

- Introduction

- One-way ANOVA

- Two-way ANOVA

- Two-way ANOVA with interaction

Hypothesis testing in regression

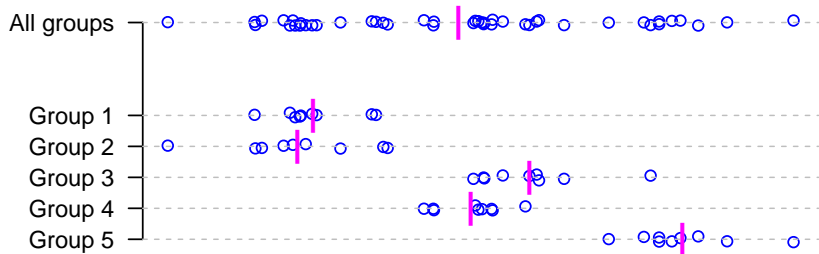
- Analysis of variance approach

Likelihood ratio tests

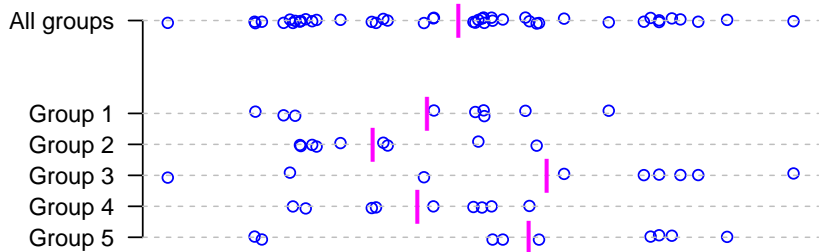
Analysis of variance: introduction

- Initial aim: compare the means of **more than two** populations
- Broader and more advanced aims:
 - Explore components of variation
 - Evaluate the fit a (general) linear model
- Formulated as hypothesis tests
- Referred to as **analysis of variance**, or **ANOVA** for short
- Involves comparing different summaries of variation
- Related to the 'analysis of variance' and 'variance decomposition' formulae we derived previously

Example: large variation between groups



Example: smaller variation between groups



ANOVA: setting it up

- We have random samples from k populations, each having a normal distribution
- We sample n_i iid observations from the i th population, which has mean μ_i
- All populations assumed have the **same** variance, σ^2
- Question of interest: do the populations all have the same mean?
- Hypotheses:

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_k = \mu \quad \text{versus} \quad H_1: \bar{H}_0$$

(\bar{H}_0 means 'not H_0 '))

- This model is known as a **one-way ANOVA**,
or **single-factor ANOVA**

Notation

Population	Sample	Statistics	
$N(\mu_1, \sigma^2)$	$X_{11}, X_{12}, \dots, X_{1n_1}$	$\bar{X}_{1\cdot}$	S_1^2
$N(\mu_2, \sigma^2)$	$X_{21}, X_{22}, \dots, X_{2n_2}$	$\bar{X}_{2\cdot}$	S_2^2
\vdots	\vdots	\vdots	\vdots
$N(\mu_k, \sigma^2)$	$X_{k1}, X_{k2}, \dots, X_{kn_k}$	$\bar{X}_{k\cdot}$	S_k^2
Overall		$\bar{X}_{..}$	

$$n = n_1 + \cdots + n_k \quad (\text{total sample size})$$

$$\bar{X}_{i\cdot} = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij} \quad (\text{group means})$$

$$\bar{X}_{\cdot\cdot} = \frac{1}{n} \sum_{i=1}^k \sum_{j=1}^{n_i} X_{ij} = \frac{1}{n} \sum_{i=1}^k n_i \bar{X}_{i\cdot} \quad (\text{grand mean})$$

Sum of squares (definitions)

- We now define statistics each called a **sum of squares (SS)**
- The **total SS** is:

$$SS(TO) = \sum_{i=1}^k \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{..})^2$$

- The **treatment SS**, or **between groups SS**, is:

$$SS(T) = \sum_{i=1}^k \sum_{j=1}^{n_i} (\bar{X}_{i.} - \bar{X}_{..})^2 = \sum_{i=1}^k n_i (\bar{X}_{i.} - \bar{X}_{..})^2$$

- The **error SS**, or **within groups SS**, is:

$$SS(E) = \sum_{i=1}^k \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i.})^2 = \sum_{i=1}^k (n_i - 1) S_i^2$$

Analysis of variance decomposition

- It turns out that:

$$SS(TO) = SS(T) + SS(E)$$

- This is similar to the analysis of variance formulae we derived earlier, in simpler scenarios (iid model, regression model)
- We will use this relationship as a basis to derive a hypothesis test
- Let's first prove the relationship. . .

- Start with the 'add and subtract' trick:

$$\begin{aligned}SS(TO) &= \sum_{i=1}^k \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{..})^2 \\&= \sum_{i=1}^k \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i.} + \bar{X}_{i.} - \bar{X}_{..})^2 \\&= \sum_{i=1}^k \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i.})^2 + \sum_{i=1}^k \sum_{j=1}^{n_i} (\bar{X}_{i.} - \bar{X}_{..})^2 \\&\quad + 2 \sum_{i=1}^k \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i.})(\bar{X}_{i.} - \bar{X}_{..}) \\&= SS(E) + SS(T) + CP\end{aligned}$$

- The cross-product term is:

$$\begin{aligned} CP &= 2 \sum_{i=1}^k \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i.})(\bar{X}_{i.} - \bar{X}_{..}) \\ &= 2 \sum_{i=1}^k (\bar{X}_{i.} - \bar{X}_{..}) \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i.}) \\ &= 2 \sum_{i=1}^k (\bar{X}_{i.} - \bar{X}_{..})(n_i \bar{X}_{i.} - n_i \bar{X}_{i.}) \\ &= 0 \end{aligned}$$

- Thus, we have:

$$SS(TO) = SS(T) + SS(E)$$

Sampling distribution of $SS(E)$

- The sample variance from the i th group, S_i^2 , is an unbiased estimator of σ^2 and we know that $(n_i - 1)S_i^2/\sigma^2 \sim \chi_{n_i-1}^2$
- The samples from each group are independent, so we can usefully combine them,

$$\sum_{i=1}^k \frac{(n_i - 1)S_i^2}{\sigma^2} = \frac{SS(E)}{\sigma^2} \sim \chi_{n-k}^2$$

- Note that: $(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) = n - k$
- This also gives us an unbiased pooled estimator of σ^2 ,

$$\hat{\sigma}^2 = \frac{SS(E)}{n - k}$$

- These results are true irrespective of whether H_0 is true or not

Null sampling distribution of $SS(TO)$

- If we assume H_0 , we can derive simple expressions for the sampling distributions of the other sums of squares
- The combined data would be a sample of size n from $N(\mu, \sigma^2)$. Hence $SS(TO)/(n - 1)$ is an unbiased estimator of σ^2 and

$$\frac{SS(TO)}{\sigma^2} \sim \chi_{n-1}^2$$

Null sampling distribution of $SS(T)$

- Under H_0 , we have $\bar{X}_{i.} \sim N(\mu, \frac{\sigma^2}{n_i})$
- $\bar{X}_{1.}, \bar{X}_{2.}, \dots, \bar{X}_{k.}$ are independent
- (Can think of this as a sample of sample means, and then think about what its variance estimator is)
- It is possible to show that (proof not shown):

$$\sum_{i=1}^k \frac{n_i(\bar{X}_{i.} - \bar{X}_{..})^2}{\sigma^2} = \frac{SS(T)}{\sigma^2} \sim \chi_{k-1}^2$$

and that this is independent of $SS(E)$

Null sampling distributions

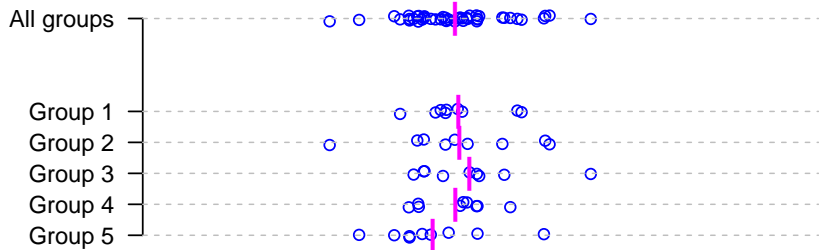
In summary, under H_0 :

$$\frac{SS(TO)}{\sigma^2} = \frac{SS(E)}{\sigma^2} + \frac{SS(T)}{\sigma^2}$$

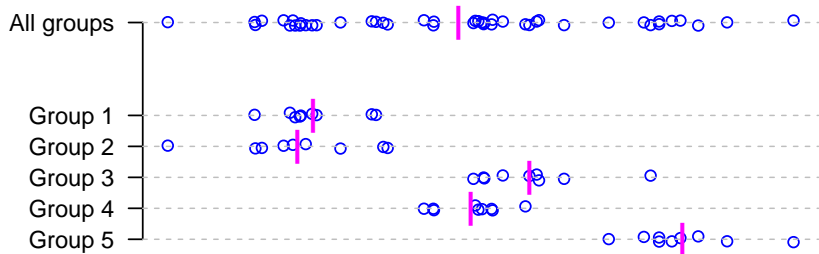
$$\frac{SS(TO)}{\sigma^2} \sim \chi_{n-1}^2, \quad \frac{SS(E)}{\sigma^2} \sim \chi_{n-k}^2, \quad \frac{SS(T)}{\sigma^2} \sim \chi_{k-1}^2,$$

$SS(E)$ and $SS(T)$ are independent

H_0 is true



H_1 is true



$SS(T)$ under H_1

- What happens if H_1 is true?
- The population means differ, which will make $SS(T)$ larger
- Let's make this precise. . .

- Let $\bar{\mu} = n^{-1} \sum_{i=1}^k n_i \mu_i$, and then,

$$\begin{aligned}
 \mathbb{E}[SS(T)] &= \mathbb{E} \left[\sum_{i=1}^k n_i (\bar{X}_{i.} - \bar{X}_{..})^2 \right] = \mathbb{E} \left[\sum_{i=1}^k n_i \bar{X}_{i.}^2 - n \bar{X}_{..}^2 \right] \\
 &= \sum_{i=1}^k n_i \mathbb{E}(\bar{X}_{i.}^2) - n \mathbb{E}(\bar{X}_{..}^2) \\
 &= \sum_{i=1}^k n_i [\text{var}(\bar{X}_{i.}) + \mathbb{E}(\bar{X}_{i.})^2] - n [\text{var}(\bar{X}_{..}) + \mathbb{E}(\bar{X}_{..})^2] \\
 &= \sum_{i=1}^k n_i \left[\frac{\sigma^2}{n_i} + \mu_i^2 \right] - n \left[\frac{\sigma^2}{n} + \bar{\mu}^2 \right] \\
 &= (k-1)\sigma^2 + \sum_{i=1}^k n_i (\mu_i - \bar{\mu})^2
 \end{aligned}$$

$$\mathbb{E}[SS(T)] = (k - 1)\sigma^2 + \sum_{i=1}^k n_i(\mu_i - \bar{\mu})^2$$

- Under H_0 the second term is zero and we have,

$$\frac{\mathbb{E}(SS(T))}{k - 1} = \sigma^2$$

- Otherwise (under H_1), the second term is positive and gives,

$$\frac{\mathbb{E}(SS(T))}{k - 1} > \sigma^2$$

- In contrast, we always have,

$$\frac{E(SS(E))}{n - k} = \sigma^2$$

F-test statistic

- This motivates using the following as our test statistic:

$$F = \frac{SS(T)/(k-1)}{SS(E)/(n-k)}$$

- Under H_0 , we have $F \sim F_{k-1, n-k}$, since it is the ratio of independent χ^2 random variables
- Under H_1 , the numerator will tend to be larger
- Therefore, reject H_0 if $F > c$
- This is known as an **F-test**

ANOVA table

The test quantities are often summarised using an **ANOVA table**:

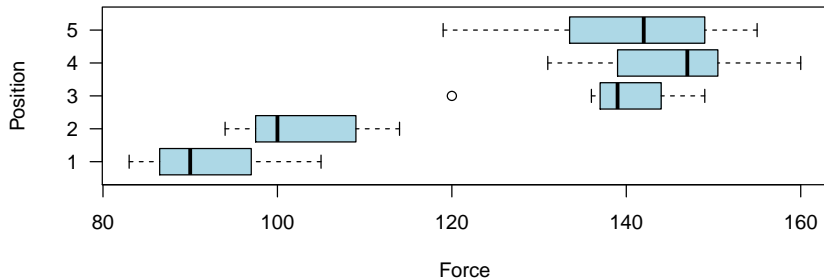
Source	df	SS	MS	F
Treatment	$k - 1$	$SS(T)$	$MS(T) = \frac{SS(T)}{k-1}$	$\frac{MS(T)}{MS(E)}$
Error	$n - k$	$SS(E)$	$MS(E) = \frac{SS(E)}{n-k}$	
Total	$n - 1$	$SS(TO)$		

Notes:

- MS = 'Mean square'
- $\hat{\sigma}^2 = MS(E)$ is an unbiased estimator

Example (one-way ANOVA)

Force required to pull out window studs in 5 positions on a car window.



```
> head(data1)
  Position Force
1         1    92
2         1    90
3         1    87
4         1   105
5         1    86
6         1    83
```

```
> table(data1$Position)
```

```
1 2 3 4 5
7 7 7 7 7
```

```
> model1 <- lm(Force ~ factor(Position), data = data1)
```

```
> anova(model1)
```

Analysis of Variance Table

Response: Force

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
factor(Position)	4	16672.1	4168.0	44.202	3.664e-12 ***
Residuals	30	2828.9	94.3		

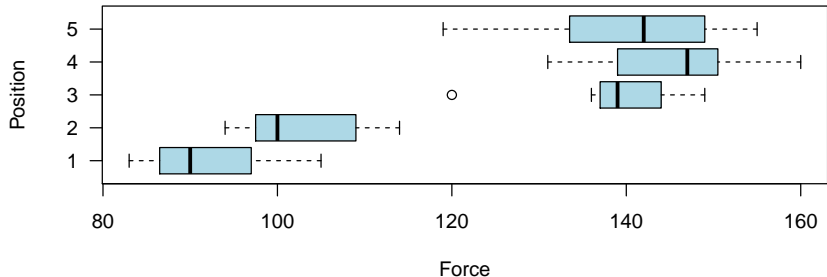
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Notes:

- Need to use `factor()` to denote categorical variables
- R doesn't provide a 'Total' row, but we don't need it
- Residuals is the 'Error' row
- `Pr(>F)` is the p-value for the F-test

We conclude that the mean force required to pull out the window studs varies between the 5 positions on the car window (e.g. $p\text{-value} < 0.01$)

This was obvious from the boxplots: positions 1 & 2 are quite different from 3, 4 & 5



Two factors

- In one-way ANOVA, the observations were partitioned into k groups
- In other words, they were defined by a single categorical variable ('factor')
- What if we had two such variables?
- We can extend the procedure to give **two-way ANOVA**, or **two-factor ANOVA**
- For example, the fuel consumption of a car may depend on type of petrol and the brand of tyres

Two-way ANOVA: setting it up

- Factor 1 has a levels, Factor 2 has b levels
- Suppose we have exactly one observation per factor combination
- Observe X_{ij} with factor 1 at level i and factor 2 at level j
- Gives a total of $n = ab$ observations
- Assume $X_{ij} \sim N(\mu_{ij}, \sigma^2)$, $i = 1, \dots, a$, $j = 1, \dots, b$, and that these are independent
- Consider the model:

$$\mu_{ij} = \mu + \alpha_i + \beta_j$$

$$\text{with } \sum_{i=1}^a \alpha_i = 0, \sum_{j=1}^b \beta_j = 0$$

- μ is an overall effect, α_i is the effect of the i th row and β_j the effect of the j th column.
- For example, $a = 4$ and $b = 4$,

	1	2	3	4
1	$\mu + \alpha_1 + \beta_1$	$\mu + \alpha_1 + \beta_2$	$\mu + \alpha_1 + \beta_3$	$\mu + \alpha_1 + \beta_4$
2	$\mu + \alpha_2 + \beta_1$	$\mu + \alpha_2 + \beta_2$	$\mu + \alpha_2 + \beta_3$	$\mu + \alpha_2 + \beta_4$
3	$\mu + \alpha_3 + \beta_1$	$\mu + \alpha_3 + \beta_2$	$\mu + \alpha_3 + \beta_3$	$\mu + \alpha_3 + \beta_4$
4	$\mu + \alpha_4 + \beta_1$	$\mu + \alpha_4 + \beta_2$	$\mu + \alpha_4 + \beta_3$	$\mu + \alpha_4 + \beta_4$

- We are usually interested in $H_{0A}: \alpha_1 = \alpha_2 = \dots = \alpha_a = 0$ or $H_{0B}: \beta_1 = \beta_2 = \dots = \beta_b = 0$
- Let

$$\bar{X}_{..} = \frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b X_{ij}, \quad \bar{X}_{i.} = \frac{1}{b} \sum_{j=1}^b X_{ij}, \quad \bar{X}_{.j} = \frac{1}{a} \sum_{i=1}^a X_{ij}$$

- Arguing as before,

$$\begin{aligned}
 SS(TO) &= \sum_{i=1}^a \sum_{j=1}^b (X_{ij} - \bar{X}_{..})^2 \\
 &= \sum_{i=1}^a \sum_{j=1}^b [(\bar{X}_{i.} - \bar{X}_{..}) + (\bar{X}_{.j} - \bar{X}_{..}) \\
 &\quad + (X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})]^2 \\
 &= b \sum_{i=1}^a (\bar{X}_{i.} - \bar{X}_{..})^2 + a \sum_{j=1}^b (\bar{X}_{.j} - \bar{X}_{..})^2 \\
 &\quad + \sum_{i=1}^a \sum_{j=1}^b (X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})^2 \\
 &= SS(A) + SS(B) + SS(E)
 \end{aligned}$$

- If both $\alpha_1 = \cdots = \alpha_a = 0$ and $\beta_1 = \cdots = \beta_b = 0$, then we have $SS(A)/\sigma^2 \sim \chi_{a-1}^2$, $SS(B)/\sigma^2 \sim \chi_{b-1}^2$ and $SS(E)/\sigma^2 \sim \chi_{(a-1)(b-1)}^2$ and these variables are independent (proof not shown)
- Reject H_{0A} : $\alpha_1 = \cdots = \alpha_a = 0$ at significance level α if:

$$F_A = \frac{SS(A)/(a-1)}{SS(E)/((a-1)(b-1))} > c$$

where c is the $1 - \alpha$ quantile of $F_{a-1, (a-1)(b-1)}$

- Reject H_{0B} : $\beta_1 = \cdots = \beta_b = 0$ at significance level α if:

$$F_B = \frac{SS(B)/(b-1)}{SS(E)/((a-1)(b-1))} > c$$

where c is the $1 - \alpha$ quantile of $F_{b-1, (a-1)(b-1)}$

ANOVA table

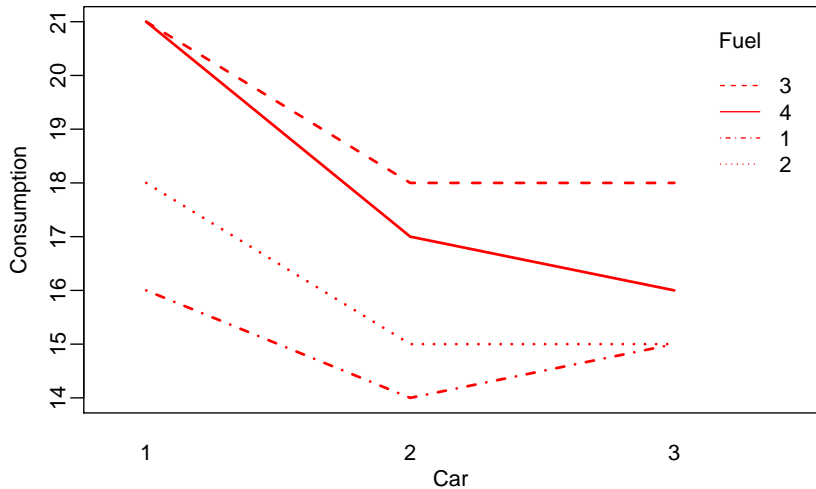
Source	df	SS	MS	F
Factor A	$a - 1$	$SS(A)$	$MS(A) = \frac{SS(A)}{a-1}$	$\frac{MS(A)}{MS(E)}$
Factor B	$b - 1$	$SS(B)$	$MS(B) = \frac{SS(B)}{b-1}$	$\frac{MS(B)}{MS(E)}$
Error	$(a - 1)(b - 1)$	$SS(E)$	$MS(E) = \frac{SS(E)}{(a-1)(b-1)}$	
Total	$ab - 1$	$SS(TO)$		

Example (two-way ANOVA)

Data on fuel consumption for three types of car (A) and four types of fuel (B).

```
> head(data2)
```

	Car	Fuel	Consumption
1	1	1	16
2	1	2	18
3	1	3	21
4	1	4	21
5	2	1	14
6	2	2	15



```
> model2 <- lm(Consumption ~ factor(Car) + factor(Fuel),  
+ data = data2)
```

```
> anova(model2)
```

Analysis of Variance Table

Response: Consumption

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
factor(Car)	2	24	12.0000	18	0.002915 **
factor(Fuel)	3	30	10.0000	15	0.003401 **
Residuals	6	4	0.6667		

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

From this we conclude there is a clear difference in fuel consumption between cars (we reject $H_{0A}: \alpha_1 = \alpha_2 = \alpha_3$) and also between fuels (we reject $H_{0B}: \beta_1 = \beta_2 = \beta_3 = \beta_4$).

Interaction terms

- In the previous example we assumed an additive model:

$$\mu_{ij} = \mu + \alpha_i + \beta_j$$

- This assumes, for example, that the relative effect of petrol 1 is the same for all cars.
- If it is not true, then there is a **statistical interaction** (or simply an **interaction**) between the factors

- A more general model, which includes interactions, is:

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$$

where γ_{ij} is the **interaction term** associated with combination (i, j) .

- In addition to our previous assumptions, we also impose:

$$\sum_{i=1}^a \gamma_{ij} = 0, \quad \text{and} \quad \sum_{j=1}^b \gamma_{ij} = 0$$

- The terms α_i and β_j are called **main effects**
- When written out as a table they are also often referred to as the **row effects** and **column effects** respectively

- Writing this out as a table:

	1	2	...
1	$\mu + \alpha_1 + \beta_1 + \gamma_{11}$	$\mu + \alpha_1 + \beta_2 + \gamma_{12}$...
2	$\mu + \alpha_2 + \beta_1 + \gamma_{21}$	$\mu + \alpha_2 + \beta_2 + \gamma_{22}$...
3	$\mu + \alpha_3 + \beta_1 + \gamma_{31}$	$\mu + \alpha_3 + \beta_2 + \gamma_{32}$...
4	$\mu + \alpha_4 + \beta_1 + \gamma_{41}$	$\mu + \alpha_4 + \beta_2 + \gamma_{42}$...

- We are now interested in testing whether:
 - the row effects are zero
 - the column effects are zero
 - the interactions are zero (do this first!)
- To make inferences about the interactions we need more than one observation per cell
- Let X_{ijk} , $i = 1, \dots, a$, $j = 1, \dots, b$, $k = 1, \dots, c$ be the k th observation for combination (i, j)

- Let

$$\bar{X}_{ij\cdot} = \frac{1}{c} \sum_{k=1}^c X_{ijk}$$

$$\bar{X}_{i..} = \frac{1}{bc} \sum_{j=1}^b \sum_{k=1}^c X_{ijk}$$

$$\bar{X}_{\cdot j\cdot} = \frac{1}{ac} \sum_{i=1}^a \sum_{k=1}^c X_{ijk}$$

$$\bar{X}_{...} = \frac{1}{abc} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c X_{ijk}$$

- and as before

$$\begin{aligned}
 SS(TO) &= \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c (X_{ijk} - \bar{X}...)^2 \\
 &= bc \sum_{i=1}^a (\bar{X}_{i..} - \bar{X}...)^2 + ac \sum_{j=1}^b (\bar{X}_{.j.} - \bar{X}...)^2 \\
 &\quad + c \sum_{i=1}^a \sum_{j=1}^b (\bar{X}_{ij.} - \bar{X}_{i..} - \bar{X}_{.j.} + \bar{X}...)^2 \\
 &\quad + \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c (X_{ijk} - \bar{X}_{ij.})^2 \\
 &= SS(A) + SS(B) + SS(AB) + SS(E)
 \end{aligned}$$

Test statistics

- Familiar arguments show that to test

$$H_{0AB}: \gamma_{ij} = 0, \quad i = 1, \dots, a, \quad j = 1, \dots, b$$

we may use the statistic

$$F = \frac{SS(AB)/[(a-1)(b-1)]}{SS(E)/[ab(c-1)]}$$

which has a F distribution with $(a-1)(b-1)$ and $ab(c-1)$ degrees of freedom.

- To test

$$H_{0A}: \alpha_i = 0, \quad i = 1, \dots, a$$

we may use the statistic

$$F = \frac{SS(A)/[(a-1)]}{SS(E)/[ab(c-1)]}$$

which has a F distribution with $(a-1)$ and $ab(c-1)$ degrees of freedom.

- To test

$$H_{0B}: \beta_j = 0, \quad j = 1, \dots, b$$

we may use the statistic

$$F = \frac{SS(B)/[(b-1)]}{SS(E)/[ab(c-1)]}$$

which has a F distribution with $(b-1)$ and $ab(c-1)$ degrees of freedom.

ANOVA table

Source	df	SS	MS	F
Factor A	$a - 1$	$SS(A)$	$MS(A) = \frac{SS(A)}{a-1}$	$\frac{MS(A)}{MS(E)}$
Factor B	$b - 1$	$SS(B)$	$MS(B) = \frac{SS(B)}{b-1}$	$\frac{MS(B)}{MS(E)}$
Factor AB	$(a - 1)(b - 1)$	$SS(AB)$	$MS(AB) = \frac{SS(AB)}{(a-1)(b-1)}$	$\frac{MS(AB)}{MS(E)}$
Error	$ab(c - 1)$	$SS(E)$	$MS(E) = \frac{SS(E)}{ab(c-1)}$	
Total	$abc - 1$	$SS(TO)$		

Example (two-way ANOVA with interaction)

- Six groups of 18 people
- Each person takes an arithmetic test: the task is to add three numbers together
- The numbers are presented either in a down array or an across array; this defines 2 levels of factor A
- The numbers have either one, two or three digits; this defines 3 levels of factor B
- The response variable, X , is the average number of problems completed correctly over two 90-second sessions

- Example of adding **one-digit** numbers in an **across** array:

$$2 + 5 + 1 = ?$$

- Example of adding **two-digit** numbers in an **down** array:

$$\begin{array}{r} 13 \\ 87 \\ + 51 \\ \hline ? \end{array}$$

```
> head(data3)
```

	A	B	X
1	down	1	19.5
2	down	1	18.5
3	down	1	32.0
4	down	1	21.5
5	down	1	28.5
6	down	1	33.0

```
> table(data3[, 1:2])
```

	B		
A	1	2	3
down	18	18	18
across	18	18	18

```
> model3 <- lm(X ~ factor(A) * factor(B), data = data3)
```

```
> anova(model3)
```

Analysis of Variance Table

Response: X

	Df	Sum Sq	Mean Sq	F value	Pr(>F)	
factor(A)	1	48.7	48.7	2.8849	0.09246	.
factor(B)	2	8022.7	4011.4	237.7776	< 2e-16	***
factor(A):factor(B)	2	185.9	93.0	5.5103	0.00534	**
Residuals	102	1720.8	16.9			

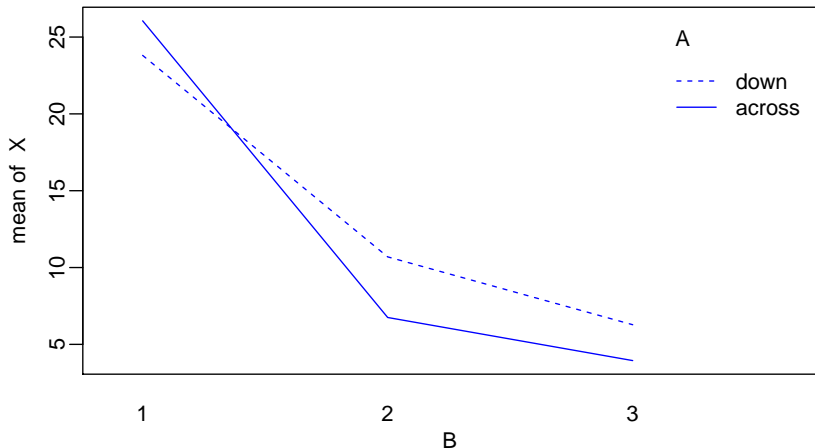
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '1'

Note the use of '*' in the model formula.

The interaction is significant at a 5% level (or even at 1%).

Interaction plot

```
with(data3, interaction.plot(B, A, X, col = "blue"))
```



Beyond the F-test

- We have rejected the null. . . now what?
- This is often only the beginning of a statistical analysis of this type of data
- Will be interested in more detailed inferences, e.g. CIs/tests about individual parameters
- You know enough to be able to work some of this out. . .
- . . . and later subjects will go into this in more detail (e.g. MAST30025)

Outline

Analysis of variance (ANOVA)

- Introduction

- One-way ANOVA

- Two-way ANOVA

- Two-way ANOVA with interaction

Hypothesis testing in regression

- Analysis of variance approach

Likelihood ratio tests

Recap of simple linear regression

- Y a response variable, e.g. student's grade in first-year calculus
- x a predictor variable, e.g. student's high school mathematics mark
- Data: pairs $(x_1, y_1), \dots, (x_n, y_n)$
- Linear regression model:

$$Y_i = \alpha + \beta(x_i - \bar{x}) + \epsilon_i$$

where $\epsilon_i \sim N(0, \sigma^2)$ is a random error

- **Note:** α here plays the same role as α_0 from Module 5. We have dropped the '0' subscript for convenience, and also to avoid confusion with its use to denote null hypotheses.

- The MLE (and OLS) estimators are:

$$\hat{\alpha} = \bar{Y}, \quad \hat{\beta} = \frac{\sum_{i=1}^n Y_i(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

- and

$$\hat{\sigma}^2 = \frac{1}{n-2} \sum_{i=1}^n [Y_i - \hat{\alpha} - \hat{\beta}(x_i - \bar{x})]^2$$

- We also derived:

$$\hat{\alpha} \sim N\left(\alpha, \frac{\sigma^2}{n}\right)$$
$$\hat{\beta} \sim N\left(\beta, \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{x})^2}\right)$$

- and

$$\frac{(n-2)\hat{\sigma}^2}{\sigma^2} = \frac{\sum_{i=1}^n \left[Y_i - \hat{\alpha} - \hat{\beta}(x_i - \bar{x}) \right]^2}{\sigma^2} \sim \chi_{n-2}^2$$

- From these we obtain,

$$T_{\alpha} = \frac{\hat{\alpha} - \alpha}{\hat{\sigma}/\sqrt{n}} \sim t_{n-2}$$

$$T_{\beta} = \frac{\hat{\beta} - \beta}{\hat{\sigma}/\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}} \sim t_{n-2}$$

- We used these previously to construct confidence intervals
- We can also use them to construct hypothesis tests
- For example, to test $H_0: \beta = \beta_0$ versus $H_1: \beta \neq \beta_0$ (or $\beta > \beta_0$ or $\beta < \beta_0$), we use T_{β} as the test statistic

Example: testing the slope parameter (β)

- Data: 10 pairs of scores on a preliminary test and a final exam
- Estimates: $\hat{\alpha} = 81.3$, $\hat{\beta} = 0.742$, $\hat{\sigma}^2 = 27.21$
- Test $H_0: \beta = 0$ versus $H_1: \beta \neq 0$ with a 1% significance level
- Reject H_0 if:

$$|T_\beta| \geq 3.36 \quad (0.995 \text{ quantile of } t_8)$$

- For the observed data,

$$t_\beta = \frac{0.742 - 0}{\sqrt{27.21/756.1}} = 3.91$$

so we reject H_0 , concluding there is sufficient evidence that the slope differs from zero.

Note regarding the intercept parameter (α)

- Software packages (such as R) will typically fit the model:

$$Y_i = \alpha + \beta x_i + \epsilon_i$$

- This is equivalent to

$$Y_i = \alpha^* + \beta(x_i - \bar{x}) + \epsilon_i$$

where $\alpha = \alpha^* - \beta\bar{x}$

- The formulation $Y_i = \alpha^* + \beta(x - \bar{x}) + \epsilon$ is easier to examine theoretically.
- We saw that

$$\hat{\alpha}^* = \bar{Y}, \quad \text{and} \quad \hat{\alpha} = \bar{Y} - \hat{\beta}\bar{x}$$

- $\hat{\alpha}$ or $\hat{\alpha}^*$ are rarely of direct interest

Using R

Use R to fit the regression model for the slope example:

```
> m1 <- lm(final_exam ~ prelim_test)
> summary(m1)
```

Call:

```
lm(formula = final_exam ~ prelim_test)
```

Residuals:

Min	1Q	Median	3Q	Max
-6.883	-3.264	-0.530	3.438	8.470

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	30.6147	13.0622	2.344	0.04714 *
prelim_test	0.7421	0.1897	3.912	0.00447 **

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 5.217 on 8 degrees of freedom

Multiple R-Squared: 0.6567, Adjusted R-squared: 0.6137

F-statistic: 15.3 on 1 and 8 DF, p-value: 0.004471

The t-value and the p-value are for testing $H_0: \alpha = 0$ and $H_0: \beta = 0$ respectively.

Interpreting the R output

- Usually most interested in testing $H_0: \beta = 0$ versus $H_1: \beta \neq 0$
- If we reject H_0 then we conclude there is sufficient evidence of (at least) a linear relationship between the mean response and x
- In the example,

$$t = \frac{0.7421}{0.1897} = 3.912$$

- This test statistic has a t -distribution with $10 - 2 = 8$ degrees of freedom, and the associated p-value is $0.00447 < 0.05$ so at the 5% level of significance we reject H_0
- It is also possible to represent this test using an ANOVA table

Deriving the variance decomposition formula

- Independent pairs $(x_1, Y_1), \dots, (x_n, Y_n)$
- Parameter estimates,

$$\hat{\alpha} = \bar{Y}, \quad \hat{\beta} = \frac{\sum_{i=1}^n Y_i(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

- Fitted value (estimated mean),

$$\hat{Y}_i = \bar{Y} + \hat{\beta}(x_i - \bar{x})$$

- Do the 'add and subtract' trick again:

$$\begin{aligned}\sum_{i=1}^n (Y_i - \bar{Y})^2 &= \sum_{i=1}^n (Y_i - \hat{Y}_i + \hat{Y}_i - \bar{Y})^2 \\ &= \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 + \sum_{i=1}^n (\hat{Y}_i - \bar{Y})^2 \\ &\quad + 2 \sum_{i=1}^n (Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y})\end{aligned}$$

- Deal with the cross-product term,

$$\begin{aligned}\sum_{i=1}^n (Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) &= \sum_{i=1}^n \left[Y_i - \bar{Y} - \hat{\beta}(x_i - \bar{x}) \right] \hat{\beta}(x_i - \bar{x}) \\ &= \hat{\beta} \sum_{i=1}^n \left[Y_i - \bar{Y} - \hat{\beta}(x_i - \bar{x}) \right] (x_i - \bar{x}) \\ &= \hat{\beta} \left[\sum_{i=1}^n (Y_i - \bar{Y})(x_i - \bar{x}) - \hat{\beta} \sum_{i=1}^n (x_i - \bar{x})^2 \right] \\ &= \hat{\beta} \left[\sum_{i=1}^n Y_i(x_i - \bar{x}) - \hat{\beta} \sum_{i=1}^n (x_i - \bar{x})^2 \right] \\ &= 0\end{aligned}$$

- That gives us,

$$\sum_{i=1}^n (Y_i - \bar{Y})^2 = \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 + \sum_{i=1}^n (\hat{Y}_i - \bar{Y})^2$$

- We can write this as follows,

$$SS(TO) = SS(E) + SS(R)$$

where $SS(R)$ is the **regression SS** or **model SS**

- The regression SS quantifies the variation **due to** the straight line
- The error SS quantifies the variation **around** the straight line

- To complete the specification,

$$MS(E) = \frac{SS(E)}{n-2} = \frac{1}{n-2} \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 = \hat{\sigma}^2$$

$$MS(R) = \frac{SS(R)}{1} = \sum_{i=1}^n (\hat{Y}_i - \bar{Y})^2$$

- Then we have the test statistic,

$$F = \frac{MS(R)}{MS(E)} \sim F_{1,n-2}$$

ANOVA table

Source	df	SS	MS	F
Model	1	$SS(R)$	$MS(R) = \frac{SS(R)}{1}$	$\frac{MS(R)}{MS(E)}$
Error	$n - 2$	$SS(E)$	$MS(E) = \frac{SS(E)}{n-2}$	
Total	$n - 1$	$SS(TO)$		

Using R

```
> anova(m1)
```

Analysis of Variance Table

Response: final_exam

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
prelim_test	1	416.39	416.39	15.301	0.004471 **
Residuals	8	217.71	27.21		

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Notes:

- The F-statistic tests the 'significance of the regression'
- That is, $H_0: \beta = 0$ versus $H_1: \beta \neq 0$

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Likelihood ratio tests

Is there a 'best' test?

- We have examined a variety of commonly used tests
- We used test statistics that:
 - Seemed useful
 - We were familiar with
- Did we use the 'best' one?
- Is there a general procedure for finding a good/best test statistic?
- We will introduce a general procedure now, and discuss why it is optimal later in the semester

Likelihood ratio test

- The **likelihood ratio test (LRT)** is a general procedure that can find the best test for a given problem
- Suppose we have H_0 and H_1 and both are composite and of the form:

$$H_0: \theta \in A_0 \quad \text{versus} \quad H_1: \theta \in A_1$$

where A_0 and A_1 are sets of possible parameter values consistent with each of the hypotheses.

- Note: we have mostly dealt with A_0 that has only one element (simple null hypothesis)
- The **likelihood ratio** is:

$$\lambda = \frac{L_0}{L_1} = \frac{\max_{\theta \in A_0} L(\theta)}{\max_{\theta \in A_1} L(\theta)}$$

$$\lambda = \frac{L_0}{L_1} = \frac{\max_{\theta \in A_0} L(\theta)}{\max_{\theta \in A_1} L(\theta)}$$

- L is the likelihood function
- Clearly $\lambda \geq 0$
- Large $\lambda \Rightarrow$ more support for H_0 over H_1
- λ near zero \Rightarrow more support for H_1 over H_0
- Therefore, we want a critical region of the form,

$$\lambda \leq k$$

- Choose k to give the desired significance level

Example 1 (likelihood ratio test)

- $X_i \sim N(\mu, \sigma^2 = 5)$, i.e. σ is known
- $H_0: \mu = 162$ versus $H_1: \mu \neq 162$
- When H_0 is true, $\mu = 162$ so $L_0 = L(162)$
- When H_1 is true, need to maximise the likelihood,
 $L_1 = L(\hat{\theta}) = L(\bar{x})$
- The likelihood ratio is,

$$\begin{aligned}\lambda &= \frac{L_0}{L_1} = \frac{L(162)}{L(\bar{x})} = \frac{(10\pi)^{-n/2} \exp \left[-\frac{1}{10} \sum_{i=1}^n (x_i - 162)^2 \right]}{(10\pi)^{-n/2} \exp \left[-\frac{1}{10} \sum_{i=1}^n (x_i - \bar{x})^2 \right]} \\ &= \exp \left[-\frac{n}{10} (\bar{x} - 162)^2 \right]\end{aligned}$$

$$\lambda = \exp \left[-\frac{n}{10} (\bar{x} - 162)^2 \right]$$

- $\lambda \leq k$ same as

$$\frac{|\bar{x} - 162|}{\sigma/\sqrt{n}} \geq c$$

- A critical region for a size α test is

$$\frac{|\bar{x} - 162|}{\sigma/\sqrt{n}} \geq \Phi^{-1}(1 - \alpha/2)$$

- Note: this required knowledge of the distribution of \bar{X} !

Example 2 (likelihood ratio test)

- $X_i \sim N(\mu, \sigma^2)$, i.e. σ is unknown
- $H_0: \mu = \mu_0$ versus $H_1: \mu \neq \mu_0$
- Under H_0 we have $\mu = \mu_0$, and under H_1 we need to use its MLE
- Under either hypothesis, σ^2 is unspecified, so in both cases we need its MLE (conditional on the specified value of μ).
- So, under H_0 we use:

$$\hat{\mu} = \mu_0, \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \mu_0)^2$$

- And under H_1 we use:

$$\hat{\mu} = \bar{x}, \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

- Some simplification yields

$$\lambda = \left[\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\sum_{i=1}^n (x_i - \mu_0)^2} \right]^{n/2}$$

- and

$$\sum_{i=1}^n (x_i - \mu_0)^2 = \sum_{i=1}^n (x_i - \bar{x} + \bar{x} - \mu_0)^2 = \sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2$$

- Substitute and rearrange to get

$$\lambda = \left[\frac{1}{1 + \frac{n(\bar{x} - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2}} \right]^{n/2}$$

- Therefore, we have $\lambda \leq k$ when,

$$\frac{n(\bar{x} - \mu_0)^2}{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2} \geq c$$

- When H_0 is true, $\sqrt{n}(\bar{X} - \mu_0)/\sigma \sim N(0, 1)$ and $\sum_{i=1}^n (X_i - \bar{X})^2/\sigma^2 \sim \chi_{n-1}^2$, and is independent of \bar{X} .
- Therefore,

$$\begin{aligned} T &= \frac{\sqrt{n}(\bar{X} - \mu_0)/\sigma}{\sqrt{\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2/\sigma^2}} \\ &= \frac{\sqrt{n}(\bar{X} - \mu_0)}{\sqrt{\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2}} = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{n-1} \end{aligned}$$

- So we reject H_0 when $|T|$ is too large, with the following critical region for a test with significance level α ,

$$|T| \geq d, \quad \text{where } d \text{ is the } 1 - \frac{\alpha}{2} \text{ quantile of } t_{n-1}$$

Remarks

- Usually easy to find the **form** of the test
- What is harder is to find the corresponding sampling distribution
- Manipulating λ until we have something whose distribution we know can be tricky!
- Many of the standard tests arise from the likelihood ratio

Asymptotic distribution & optimality

- The likelihood ratio itself is a statistic and therefore has a sampling distribution.
- For large sample sizes, this approaches a known distribution
- Also, the LRT gives the optimal test
- We will cover this theory later in the semester