MAST20005/MAST90058: Week 3 Solutions

For all of the solutions below, we use the notation $\ell(\theta) = \ln L(\theta)$ for log-likelihood functions and $s(\theta) = \frac{\partial \ell}{\partial \theta}$ for their first derivatives.

1. (a)

$$L(\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}$$

$$\ell(\mu) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$s(\mu) = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)$$

Setting $s(\mu) = 0$ and solving gives $\hat{\mu} = \bar{X}$.

(b) i.

$$L(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i} \frac{1}{\prod_{i=1}^{n} x_i!}$$

$$\ell(\lambda) = -n\lambda + \left(\sum_{i=1}^{n} x_i\right)\ln\lambda - \ln\prod_{i=1}^{n} x_i!$$

$$s(\lambda) = -n + \frac{1}{\lambda}\sum_{i=1}^{n} x_i$$

Setting $s(\lambda) = 0$ and solving gives $\hat{\lambda} = \bar{X}$.

ii. $\bar{x} = (5\cdot 0 + 7\cdot 1 + 12\cdot 2 + 9\cdot 3 + 5\cdot 4 + 1\cdot 5 + 1\cdot 6)/40 = 2.225$

(c) i.

$$L(\theta) = \left(\frac{1}{\theta^2}\right)^n \prod_{i=1}^{n} x_i \exp(-x_i/\theta)$$

$$\ell(\theta) = -2n\ln(\theta) + \sum_{i=1}^{n}\ln(x_i) - \frac{1}{\theta}\sum_{i=1}^{n} x_i$$

$$s(\theta) = -\frac{2n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i$$

Setting $s(\theta) = 0$ and solving gives the estimator $\hat{\theta} = \sum_{i=1}^{n} X_i/(2n) = \bar{X}/2$.

ii.

$$L(\theta) = \left(\frac{1}{2\theta^3}\right)^n \prod_{i=1}^{n} x_i^2 \exp(-x_i/\theta)$$

$$\ell(\theta) = -n\ln 2 - 3n\ln(\theta) + \sum_{i=1}^{n} 2\ln(x_i) - \frac{1}{\theta}\sum_{i=1}^{n} x_i$$

$$s(\theta) = -\frac{3n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i$$

Setting $s(\theta) = 0$ and solving gives the estimator $\hat{\theta} = \sum_{i=1}^{n} X_i/(3n) = \bar{X}/3$.

iii.

$$L(\theta) = \left(\frac{1}{2}\right)^n \prod_{i=1}^{n} \exp(-|x_i - \theta|)$$

$$\ell(\theta) = -n \ln 2 - \sum_{i=1}^{n} |x_i - \theta|$$

$$s(\theta) = \sum_{i=1}^{n} \operatorname{sgn}(x_i - \theta)$$

where $\operatorname{sgn}(\cdot)$ is the sign function: $\operatorname{sgn}(x) = 1$ if $x > 0$, $\operatorname{sgn}(x) = -1$ if $x < 0$, and $\operatorname{sgn}(0) = 0$. Note that $\ell(\theta)$ is piecewise linear and not differentiable when θ equals any x_i, so $s(\theta)$ is not defined at those points. If n is even, $s(\theta)$ is zero when there are an equal number of positive and negative signs, so $\hat{\theta}$ is between the middle two ordered values (i.e. any value between these will maximise the likelihood), and we would typically pick their average. If n is odd, for $s(\theta)$ to be zero $\hat{\theta}$ must equal the middle value. So, in general, $\hat{\theta}$ is the sample median.

2. The population mean and variance are $\mathbb{E}(X) = 5\theta/4$ and $\operatorname{var}(X) = (7\theta/4) - (5\theta/4)^2$.

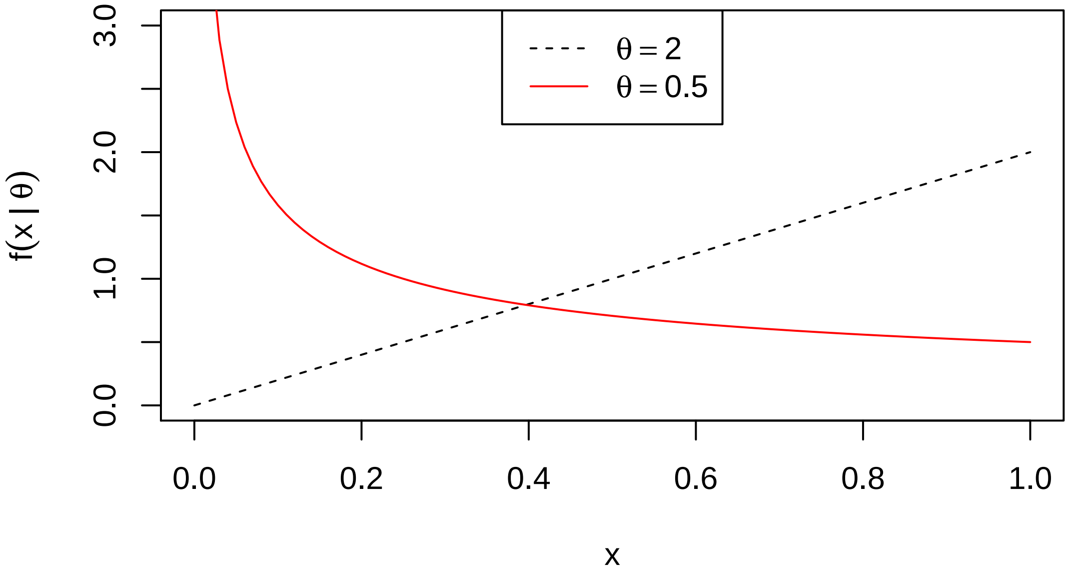
(a) We know that $\mathbb{E}(\bar{X}) = (5/4)\,\theta$ and therefore an unbiased estimator of θ based on $\bar{X}$ is $T_1 = (4/5)\bar{X}$.
Note that $Z \sim \operatorname{Bi}(n, 1-\theta)$, which means that $\mathbb{E}(Z) = n(1-\theta)$ and $\mathbb{E}(Z/n) = 1-\theta$. Therefore, an unbiased estimator of θ based on Z is $T_2 = 1 - Z/n$.

(b) Calculating the variance of the above estimators gives:

$$\operatorname{var}(T_1) = \frac{\theta\left(\frac{28}{25} - \theta\right)}{n}, \quad \operatorname{var}(T_2) = \frac{\theta(1-\theta)}{n}$$

We can therefore see that $\operatorname{var}(T_1) > \operatorname{var}(T_2)$.

3. (a)
```
f1 <- function(x) {2   * x^(2   - 1)}
f2 <- function(x) {0.5 * x^(0.5 - 1)}
curve(f1, 0, 1, col = 1, lty = 2, ylim = c(0, 3),
      ylab = expression(f(x ~ "|" ~ theta)))
curve(f2, 0, 1, col = 2, lty = 1, add = TRUE)
legend("top", c(expression(theta == 2), expression(theta == 0.5)),
       col = c(1, 2), lty = c(2, 1))
```



(b)

$$L(\theta) = \theta^n \prod_{i=1}^{n} x_i^{\theta-1}$$

$$\ell(\theta) = n \ln \theta + (\theta - 1) \sum_{i=1}^{n} \ln(x_i)$$

$$s(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n} \ln(x_i)$$

Setting $s(\theta) = 0$ and solving gives the estimator:

$$\hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \ln X_i}$$

(c) The MLEs are: $\hat{\theta}_X = 0.549$, $\hat{\theta}_Y = 2.210$, $\hat{\theta}_Z = 0.959$.
To find the method of moments estimator, we need to solve $\bar{X} = \theta/(\theta + 1)$ which gives $\tilde{\theta} = \bar{X}/(1 - \bar{X})$. Therefore, the MM estimates are: $\tilde{\theta}_X = 0.598$, $\tilde{\theta}_Y = 2.400$ and $\tilde{\theta}_Z = 0.865$.

4. Recall that for a random sample $X_1, \ldots, X_n$,

$$E(\bar{X}) = \frac{1}{n} \sum_{i=1}^{n} E(X_i) = E(X_i)$$

and

$$\text{var}(\bar{X}) = \frac{1}{n^2} \sum_{i=1}^{n} \text{var}(X_i) = \frac{\text{var}(X_i)}{n}.$$

(a) The X_i are iid exponential random variables with mean θ. Therefore, $\mathbb{E}(\bar{X}) = \mathbb{E}(X_i) = \theta$ and $\bar{X}$ is unbiased.

(b) We know that $\text{var}(X_i) = \theta^2$. Therefore, $\text{var}(\bar{X}) = \text{var}(X_i)/n = \theta^2/n$.

(c) Based on the above, an estimate of θ is $\hat{\theta} = \bar{x} = 3.48$.

5. From question 4(a)(i) from week 2, we know that:

$$S^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right).$$

Let $\mu = \mathbb{E}(X)$. Since $\sigma^2 = \text{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \mathbb{E}(X^2) - \mu^2$, we see that $\mathbb{E}(X^2) = \sigma^2 + \mu^2$. A similar argument shows that $\mathbb{E}(\bar{X}^2) = \sigma^2/n + \mu^2$. Using the above expression for the sample variance and taking expectations of both sides,

$$E(S^2) = \frac{1}{n-1}\left\{n(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n}+\mu^2\right)\right\} = \frac{(n-1)\sigma^2}{n-1} = \sigma^2$$

which shows that S^2 is unbiased for σ^2.

6. We already know that $\mathbb{E}(S^2) = \theta^2$, meaning that it is unbiased. Note that,

$$\theta^2 = \text{var}(X_i) = \mathbb{E}(X_i^2) - \mathbb{E}(X_i)^2 = \mathbb{E}(X_i^2) - 0 = \mathbb{E}(X_i^2).$$

Therefore,

$$\mathbb{E}(\hat{\theta}^2) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i^2) = \frac{1}{n}\sum_{i=1}^{n}\theta^2 = \theta^2,$$

meaning that it is also unbiased.

To derive the variance of the estimator, first note that,

$$\text{var}(X_i^2) = \mathbb{E}(X_i^4) - \mathbb{E}(X_i^2)^2 = \mathbb{E}(X_i^4) - \theta^4.$$

The 4th moment for a normal distribution (easy to look up) is,

$$\mathbb{E}(X^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4.$$

Here we have $\mu = 0$ and $\sigma^2 = \theta^2$, so we have,

$$\mathbb{E}(X_i^4) = 3\theta^4.$$

Therefore we have,

$$\text{var}(X_i^2) = 2\theta^4.$$

(After covering Module 3, you will learn that another way to do this is to note that $X_i^2/\theta^2 \sim \chi_1^2$ and therefore $\text{var}\left(X_i^2/\theta^2\right) = 2$.) Now we derive the variance of the estimator,

$$\text{var}(\hat{\theta}^2) = \frac{1}{n^2}\sum_i \text{var}(X_i^2) = \frac{2\theta^4}{n}.$$

Also, we know that,

$$\text{var}(S^2) = \frac{2\theta^4}{n-1}.$$

Therefore, $\text{var}(\hat{\theta}^2) < \text{var}(S^2)$ for any $n > 1$.

7. (a) We know this result from the lectures, but to show you some of the derivation: since $\text{var}(X_i) = \sigma^2$ for all $i = 1, \dots, n$ and the observations are independent, we have $\text{var}(\bar{X}) = (\sigma^2 + \cdots + \sigma^2)/n^2 = \sigma^2/n$.

(b) According to the information given, $\operatorname{var}(\hat{\pi}_{0.5}) \approx \pi/2 \times \sigma^2/n$. Since $\sigma^2/n < \pi/2 \times \sigma^2/n$, we see that the sample mean has smaller variance than the sample median.

(c) We already know that the sample mean is unbiased. According to the information given, $\mathbb{E}(\hat{\pi}_{0.5}) \approx \pi_{0.5} = \mu$, so the sample median is at least approximately unbiased.

(d) Both estimators are unbiased (exactly or approximately) but the sample mean has smaller variance, so we would expect it to be more accurate.