MAST20005/MAST90058: Week 9 Solutions

1. (a)
$$\mathbb{E}(\bar{X}_{..}) = \mathbb{E}\left(\frac{1}{I}\sum_{i=1}^{I}\bar{X}_{i.}\right) = \frac{1}{I}\sum_{i=1}^{I}\mu_i = \mu$$

   (b)
$$\mathbb{E}(\bar{X}_{i.}^2) = \mathrm{var}(\bar{X}_{i.}) + \mathbb{E}(\bar{X}_{i.})^2 = \frac{\sigma^2}{J} + \mu_i^2$$

   (c)
$$\mathbb{E}(\bar{X}_{..}^2) = \mathrm{var}(\bar{X}_{..}) + \mathbb{E}(\bar{X}_{..})^2 = \frac{\sigma^2}{IJ} + \mu^2$$

   since we have
$$\begin{aligned}\mathrm{var}(\bar{X}_{..}) = \mathbb{E}\left((\bar{X}_{..}-\mu)^2\right) &= \mathbb{E}\left(\left(I^{-1}\sum_{i=1}^{I}\bar{X}_{i\cdot} - I^{-1}\sum_{i=1}^{I}\mu_i\right)^2\right) \\ &= \mathbb{E}\left[\left(I^{-1}\sum_{i=1}^{I}(\bar{X}_{i\cdot}-\mu_i)\right)^2\right] = \frac{I\sigma^2}{I^2J} = \frac{\sigma^2}{IJ}\end{aligned}$$

   (d)
$$\begin{aligned}\mathbb{E}(SS(T)) = \mathbb{E}\left(\sum_{i=1}^{I}J(\bar{X}_{i\cdot}-\bar{X}_{..})^2\right) &= \mathbb{E}\left(\sum_{i=1}^{I}J\bar{X}_{i\cdot}^2 + \sum_{i=1}^{I}J\bar{X}_{..}^2 - 2\sum_{i=1}^{I}J\bar{X}_{i\cdot}\bar{X}_{..}\right) \\ &= \sum_{i=1}^{I}J\,\mathbb{E}(\bar{X}_{i\cdot}^2) + IJ\,\mathbb{E}(\bar{X}_{..}^2) - 2IJ\,\mathbb{E}(\bar{X}_{..}^2) \\ &= \sum_{i=1}^{I}J\,\mathbb{E}(\bar{X}_{i\cdot}^2) - IJ\,\mathbb{E}(X_{..}^2) \\ &= \sum_{i=1}^{I}J\left(\frac{\sigma^2}{J}+\mu_i^2\right) - IJ\left(\frac{\sigma^2}{IJ}+\mu^2\right) \\ &= (I-1)\sigma^2 + J\sum_{i=1}^{I}\left(\mu_i^2-\mu^2\right) \\ &= (I-1)\sigma^2 + J\sum_{i=1}^{I}\left(\mu_i-\mu\right)^2\end{aligned}$$

   as $\sum_{i=1}^{I}\mu_i\mu = I\mu^2$. Hence
$$\mathbb{E}(MS(T)) = \sigma^2 + \frac{J}{I-1}\sum_{i=1}^{I}(\mu_i-\mu)^2$$

   as required. Note that we have used: $\sum_{i=1}^{I}(\mu_i-\mu)^2 = \sum_{i=1}^{I}(\mu_i^2 - 2\mu_i\mu + \mu^2) = \sum_{i=1}^{I}\mu_i^2 - 2I\mu^2 + I\mu^2 = \sum_{i=1}^{I}\mu_i^2 - I\mu^2 = \sum_{i=1}^{I}(\mu_i^2-\mu^2)$.

   (e) When H_0 is true, this is σ^2, otherwise it is larger than σ^2.

2.
$$F = \frac{MS(T)}{MS(E)} = \frac{2573.3}{1394.2} = 1.846$$

Under H_0, $F \sim \mathrm{F}_{4,15}$ so we reject H_0 if $F > 3.056$. Hence we cannot reject H_0.

3. (a) This is the same as a test that the slope parameter is equal to zero (versus not zero). The test statistic for this is equal to $t = 0.75169/0.02846 = 26.41$. This is much larger than the critical value 2.66 (note: this is a two-sided test, so we need the 0.995 quantile of t_{60}). Thus, we reject the null hypothesis and conclude that there is strong evidence of an association between body and brain weights.

   (b) Under $H_0\colon \rho = 0$, we have the following
$$Z = \frac{\frac{1}{2}\ln\frac{1+R}{1-R}}{\sqrt{\frac{1}{n-3}}} \approx \mathrm{N}(0,1).$$
   Therefore, we should reject H_0 if $|Z| > \Phi^{-1}(1-\alpha/2)$.

   (c) From the R output we see that $r^2 = 0.9208$ (under `Multiple R-squared`), so we can calculate that $r = 0.9595$.

   (d) The observed value for the above test statistic is $z = 14.91 > 2.58$. Thus we reject the null hypothesis that there is no correlation between brain and body weight.

   (e) Both tests indicate strong evidence against the null hypothesis of no association between the body and brain weights of mammals.

4. (a) Let $A_0 = \{(\mu,\sigma^2) : -\infty < \mu < \infty,\, \sigma^2 = \sigma_0^2\}$ and $A_1 = \{(\mu,\sigma^2) : -\infty < \mu < \infty,\, \sigma^2 > 0\}$ be the set of parameter values consistent with H_0 and H_1 respectively. To maximise L under H_0, we can show that we need $\hat{\mu} = \bar{x}$ and $\hat{\sigma}^2 = \sigma_0^2$, which gives:
$$L_0 = L(\hat{\mu},\hat{\sigma}^2) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{n/2} e^{-\frac{\sum_i (x_i-\bar{x})^2}{2\sigma_0^2}}.$$
   To maximise L under H_1, we can show that we need $\hat{\mu} = \bar{x}$ and $\hat{\sigma}^2 = n^{-1}\sum_i (x_i - \bar{x})^2$, which gives:
$$L_1 = L(\hat{\mu},\hat{\sigma}^2) = \left[\frac{ne^{-1}}{2\pi\sum_i (x_i-\bar{x})^2}\right]^{n/2}.$$

   (b) The LRT statistic is,
$$\lambda = \frac{L_0}{L_1} = \left(\frac{w}{n}\right)^{n/2} e^{-w/2+n/2}$$
   where $w = \sum_i (x_i - \bar{x})^2/\sigma_0^2$. We reject H_0 if $\lambda \leqslant k$. Solving this inequality in w gives a solution of the form $w \leqslant c_1$ and $w \geqslant c_2$, where c_1 and c_2 are constants depending on k and n. These need to be selected so to achieve the desired significance level. (Note: since $w \sim \chi^2_{n-1}$, for convenience we typically set c_1 and c_2 as equal-tailed quantiles from this distribution, even if these do not exactly correspond to the c_1 and c_2 from the LRT.)