Numerical Methods for Nonlinear Equations

- compute yield of a bond (y)
- compute implied volatility (σ_{imp})
- compute zero rate curves from bond prices

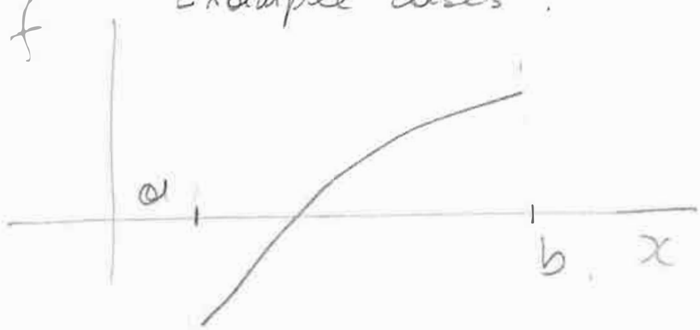
Involve solving a nonlinear equation of the form

$$f(x) = 0 \quad \text{for } x.$$

Bisection Method

Let $f : [a,b] \longrightarrow \mathbb{R}$ such that $f(a) = -\mathrm{sgn}(f(b))$

Example cases.



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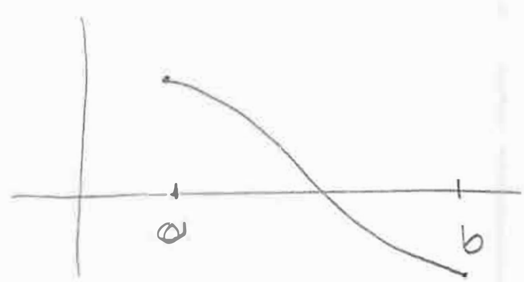
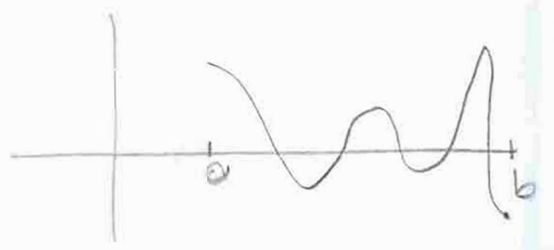




From the intermediate value theorem there exists at least one point $x \in (a, b)$ for which $f(x) = 0$

There are might be more points but the bisection method finds only one ie it considers the case of only one point over $[a, b]$.

set $c = \frac{a+b}{2}$ compute $f(c)$

- If $f(c) = 0$ then solution is $x = c$
- If $f(c)$ has different sign than $f(a)$ then new $c_{new} = \frac{c+a}{2}$ and repeat process
- If $f(c)$ has same sign as $f(a)$ then new $c_{new} = \frac{c+b}{2}$ and repeat process for interval (c_{new}, b)

Process stops when $|f(c_{new})| < ftoler$.

and interval is less than Int_tolerance.

Theorem 8.1 If $f : [a, b] \to R$ is a continuous function and $f(a)$ and $f(b)$ have opposite signs, then the bisection method converges to a solution $f(x) = 0$ for some $x \in (a, b)$.

Outline of proof

At each step the interval is halved. After n-steps the active interval is

$$x_R - x_L = \frac{b-a}{2^n}, \qquad x_R > x_L$$

One stopping condition $x_R - x_L \le$ tol_Interval

$$\therefore \quad \frac{b-a}{2^n} \le \text{tol\_interval}$$

$\Rightarrow$ $\exists$ an n such that stopping criterion is satisfied

Assuming $f(x)$ is differentiable ie $|f'(x)| \le c$ $\forall x \in [a, b]$ for some constant $c > 0$.

From MVT $\exists$ a point $\xi \in (x_L, x_R)$

$$\frac{f(x_R) - f(x_L)}{x_R - x_L} = f'(\xi)$$

$$|f(x_R) - f(x_L)| \leq |f'(\xi)|\,|x_R - x_L| \leq C\,|x_R - x_L|$$

ie

$$|f(x_R) - f(x_L)| \leq C\,\frac{(b-a)}{2^n}$$

$$\max(|f(x_R)|, |f(x_L)|) \leq |f(x_R) - f(x_L)| \leq C\,\frac{(b-a)}{2^n}$$

due to $f(x_R)$ having opposite sign with $f(x_L)$

$\therefore$ for large n the second stopping criterion

of $\max(|f(x_R)|, |f(x_L)|) \leq C\,\frac{(b-a)}{2^n} \leq f\text{-tolerance}$

is satisfied.

Example

$$f(x) = x^4 - 5x^2 + 4 - \frac{1}{1+e^{x^3}}, \qquad [-2,3]$$

$f(-2) = -0.9997$, $f(3) = 40$

tol_Interval $= 10^{-6}$, tol_function $= 10^{-9}$

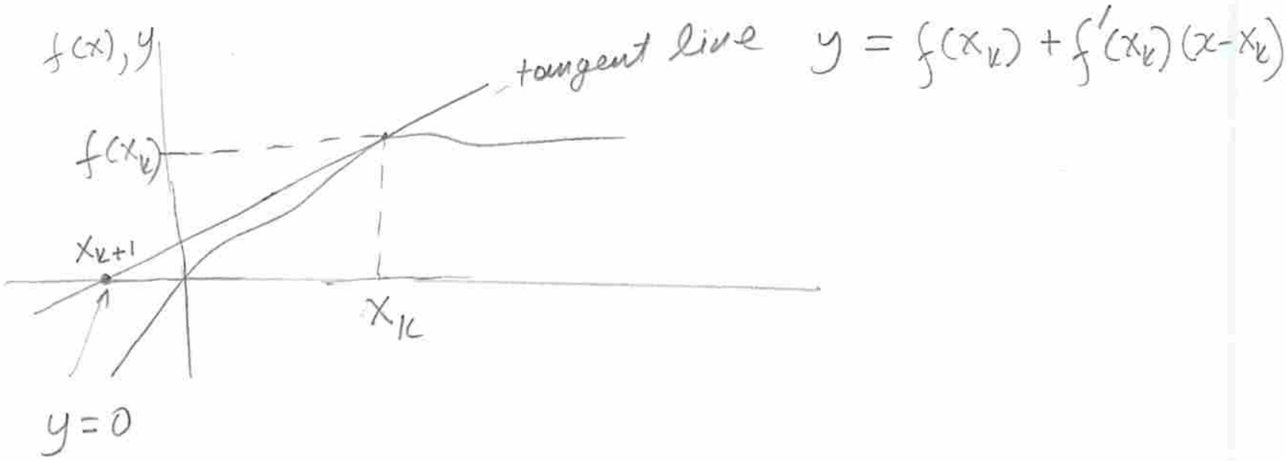
After 33 iterations the solution is $\underline{-0.889642}$

Note that $2.000028 \in [-2,3]$ is another zero for $f(x)$.

Newton's Method

Commonly used. It can extend to high dimensional systems.

Consider $f(x) = 0$



Let x_k be an approximation of the solution $f(x) = 0$. The next value of x denoted by x_{k+1} is taken as the value of x which corresponds to the x-axis intercept of the tangent line $y = f(x_k) + f'(x_k)(x - x_k)$ at the point x_k. ie for

$$y = 0 = f(x_k) + f'(x_k)(x_{k+1} - x_k)$$

$$\Rightarrow \quad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Another way of deriving the above equation is to use the Taylor expansion of $f(x)$ around the point $x = x_k$

$$f(x) \approx f(x_k) + f'(x_k)(x - x_k)$$

set $\quad x = x_{k+1} \quad$ then

$$f(x_{k+1}) \approx f(x_k) + f'(x_k)(x_{k+1} - x_k)$$

setting $f(x_{k+1}) = 0 \quad \Rightarrow \quad x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$

Start with an initial guess x_0

compute $f(x_0)$, $f'(x_0)$ then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

compute $f(x_1)$, $f'(x_1)$ then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

etc.

Given any initial guess x_0 there is no guarantee that the method will converge to an x that satisfies $f(x) = 0$.

Speed and convergence depend on the accuracy of initial guess x_0.

In financial applications usually there is a good initial guess i.e yield is positive within certain possible values etc.

Stopping criteria

$$|x_{N+1} - x_N| \leq \text{tol-consec} . \approx 10^{-6}$$

$$|f(x_{N+1})| \leq \text{tol-approx} . \approx 10^{-9}$$

<u>Theorem 8.2</u> Let x^* be a solution of $f(x) = 0$. Assume that $f(x)$ is twice differentiable with $f''(x)$ continuous. If $f'(x^*) \neq 0$ and if x_0 is close enough to x^*, then Newton's method converges quadratically i.e $\exists$ $M > 0$ and a positive integer n_M such that

$$\left|\frac{x_{k+1} - x^*}{(x_k - x^*)^2}\right| < M \quad , \quad \forall\, k \geq n_M$$

Outline of proof

Let x^* be a solution of $f(x^*) = 0$

then the recursion equation satisfies

$$x_{k+1} - x^* = x_k - x^* - \left(\frac{f(x_k) - f(x^*)}{f'(x_k)}\right)$$

$$= \frac{f(x^*) - f(x_k) + (x_k - x^*) f'(x_k)}{f'(x_k)}$$

From Taylor's formula

$$f(x) - P_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

where $c \in (a, x)$ and

$$P_n(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a)$$

For $n = 1$

$$f(x) - P_1(x) = f(x) - f(a) + (x-a) f'(a) = \frac{(x-a)^2}{2} f''(c)$$

Let $x = x^*$ and $a = x_k$

We conclude that $\exists$ constant c_k between x^* and x_k such that

$$f(x^*) - \left(f(x_k) + (x^* - x_k) f'(x_k)\right) = \frac{(x^* - x_k)^2}{2} f''(c_k)$$

$$\therefore \quad x_{k+1} - x^* = (x^* - x_k)^2 \, \frac{f''(c_k)}{2 f'(x_k)}$$

$$\Rightarrow \quad \frac{|x_{k+1} - x^*|}{|x_k - x^*|^2} = \frac{|f''(c_k)|}{2 |f'(x_k)|}$$

Since f'', f' are continuous and $f'(x^*) \neq 0$ it follows that if x_k is close to x^* then $f'(x_k) \neq 0$ by continuity.

$$\Rightarrow \quad \frac{|f''(c_k)|}{2|f'(x_k)|} < \infty$$

Secant Method

For some problems $f'(x)$ cannot be calulated as $f(x)$ may be given in form of data and a closed form for $f(x)$ may not be known.

In such case the Newton's method uses an approximation of $f'(x_k)$ as

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$\therefore \quad x_{k+1} = x_k - \frac{(x_k - x_{k-1})\, f(x_k)}{f(x_k) - f(x_{k-1})} \quad , \forall k \geq 0$$

Two approximate guesses x_{-1} and x_0 with $f(x_{-1}) \neq f(x_0)$ must be used to initialize the secant method.

The stopping criterion are the same as in Newton's case.

Secant method is usually slower than Newton's method.

N- dimensional systems

Let $F : R^N \longrightarrow R^N$, $F = \begin{bmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_N(x) \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$

where $F_i(x) : R^N \rightarrow R$

$F_i(x)$ are continuous

Solve

$$F(x) = 0 \quad \text{for } x.$$

Gradient $DF(x)$ of $F(x)$ is the $N \times N$ matrix

$$DF(x) = \begin{pmatrix} \frac{\partial F_1(x)}{\partial x_1}, & \frac{\partial F_1}{\partial x_2}, & \cdots & \frac{\partial F_1(x)}{\partial x_N} \\ \vdots & & & \vdots \\ \frac{\partial F_N}{\partial x_1} & \cdots & \cdots & \frac{\partial F_N}{\partial x_N} \end{pmatrix}$$

From Taylor expansion around x_k

$$F(x) \approx F(x_k) + DF(x_k)(x - x_k)$$

Setting $F(x_{k+1}) = 0$

$$x_{k+1} = x_k - \left(DF(x_k)\right)^{-1} F(x_k)$$

computing vector $U_k = \left(DF(x_k)\right)^{-1} F(x_k)$

is equivalent to solving the linear equation

$$DF(x_k)\, U_k = F(x_k)$$

More efficient numerical linear algebra methods may be used to solve for U_k for each k and then compute

$$x_{k+1} = x_k - U_k$$

Stopping criteria

$$\| F(x_{new}) \| \leq tol\_appr.$$

$$\| x_{new} - x_{old} \| \leq tol\_consey.$$

$\| \cdot \|$ = Euclidean norm

<u>Theorem 8.3</u> Let x^* be a solution of $F(x) = 0$ where $F(x)$ is a function with continuous 2nd order partial derivatives. If $DF(x^*)$ is invertible and x_0 is close to x^*, then the Newton's method converges quadratically i.e $\exists$ $M > 0$ and n_M pos. integer

$$\| x_{k+1} - x^* \| < M \| x_k - x^* \|^2 \qquad \forall k \geq n_M$$

Approximate Newton's Method

Forward finite difference approx.

$$\frac{\partial F_i(x)}{\partial x_j} \approx \Delta_j F_i(x) = \frac{F_i(x+h e_j) - F_i(x)}{h}$$

$$e_j = (0 \cdots 0 \; 1 \; 0 \cdots 0)^T$$

jth term

$$\Delta F(x) = \begin{pmatrix} \Delta_1 F_1(x) & \Delta_2 F_1(x) & \cdots & \Delta_N F_1(x) \\ \vdots & & & \vdots \\ \Delta_1 F_N(x) & \Delta_2 F_N(x) & \cdots & \Delta_N F_N(x \end{pmatrix}$$

$$x_{k+1} = x_k - \left(\Delta F(x_k)\right)^{-1} F(x_k)$$

Another approximation (central finite differences)

$$\frac{\partial F_i(x)}{\partial x_j} \approx \frac{F_i(x+he_j) - F_i(x-he_j)}{2h}$$

more computations.