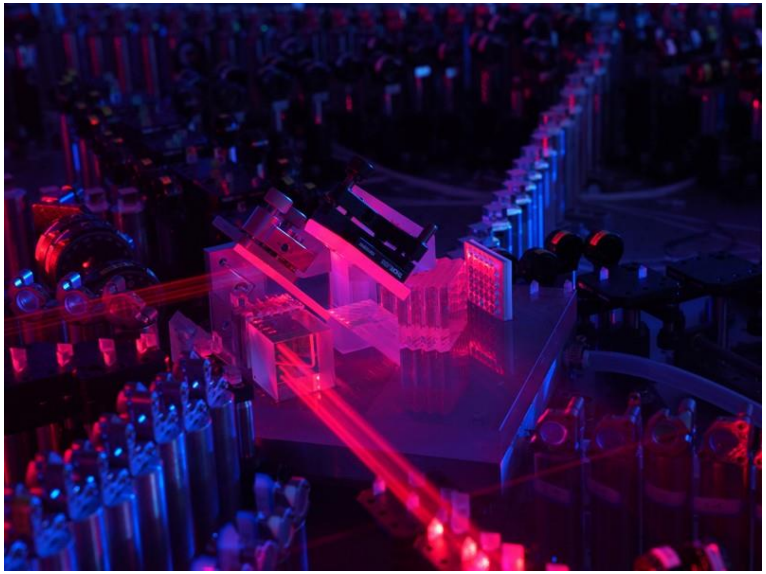
ECE 10C
Fall 2020
Slide Set 13
Instructor: Galan Moody
TA: Kamyar Parto



1st correct answer in chat = +2 quiz points:
Who can tell me what this is? What was it used for recently?

Topics

- RC/RL Circuits
- Power and Energy
- RLC Circuits
- Impedance Method
- Log/Bode Plots
- Filters

Important Items:

- HW #6 due Thursday
- Lab #6 due Friday

Midterm Exam

- 4 quiz-like problems
- 4 multi-part problems

Final Exam

- 4-5 quiz-like problems
- 3-4 multi-part problems
- 1-2 qualitative problems

Other important items:

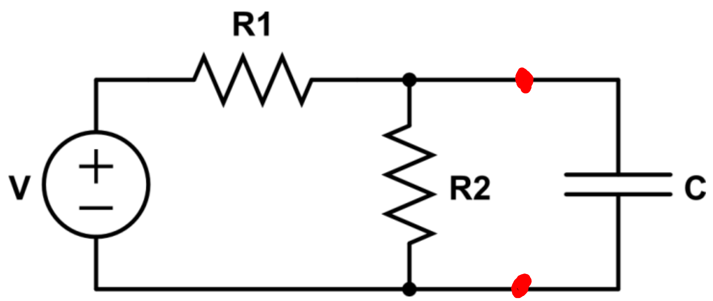
- **Wednesday, Dec. 16th. I'll email everyone the final at 5 pm. Due by 10:30 pm.**
- Quizzes count towards up to 10% final exam credit (4.5% of total course grade)
- If your grade is better without HW scores, then I'll calculate your grade just based on your exam scores
- Grading course on a curve? Maybe, but most likely not much
  - Overall you've done well as a class, so average is high

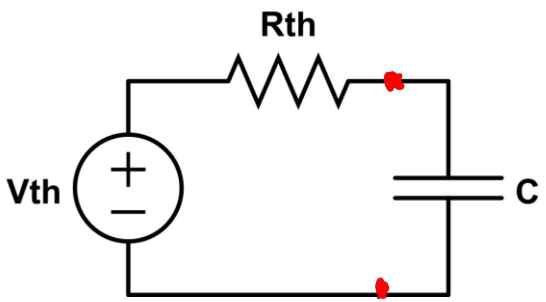
First-Order ODEs/Circuits

Thevenin Equivalent Circuits

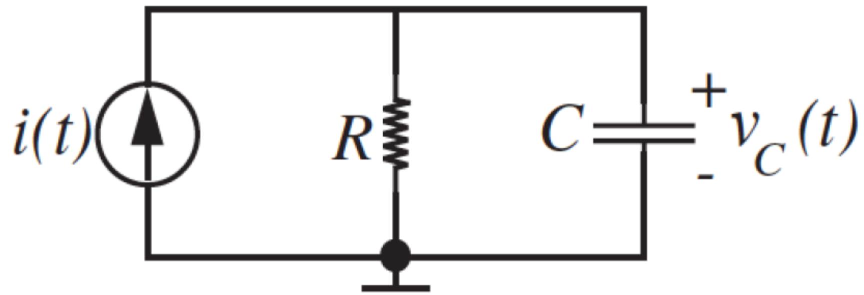
- Write an equivalent circuit connected to terminals of a device/element that you are analyzing
- Set voltage source = short, current source = open, find equiv resistance
- Replace source(s), find voltage between the terminals

The power dissipation of the Thevenin equivalent is not necessarily identical to the power dissipation of the real system. However, the power dissipated by an external element connected between the output terminals is the same





Review: Simple RC Circuit – Step Response (I_0)



- Find $v_c(t)$ when the circuit is driven by $i(t) = I_0 f(t) = \begin{cases} 0, & t < 0 \\ I_0, & t \geq 0 \end{cases}$
- Initial condition: $v_c(t<0) = 0$ (circuit initially uncharged and at rest)

Solution Steps

1. Use KVL/KCL/node methods to find the ODE describing v_c
2. Find the homogeneous solution v_{ch} (set the drive to zero)
3. Find the particular solution v_{cp} (educated guess)
4. The total solution is the sum of the homogeneous and particular solutions. Use the initial conditions to solve for the constants.

Review: RC versus RL Circuit Transient Response

Often the initial conditions aren't given. Instead, we're told that the circuit has reached a steady state before being disturbed.

Principle #1: Initial response of C and L

A capacitor acts like a short circuit and an inductor like an open circuit when they are "at rest"

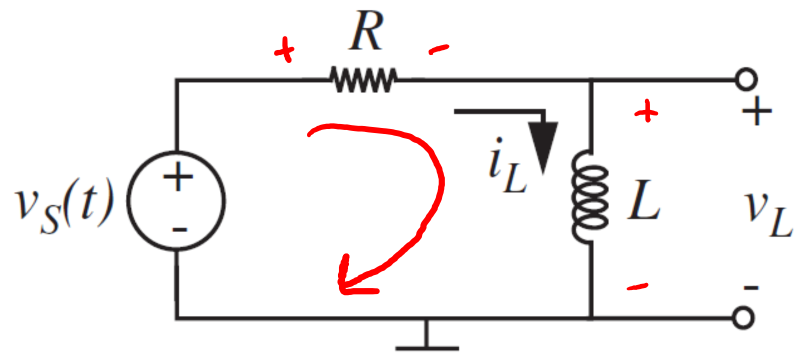
Principle #2: Steady-state of C and L

A capacitor acts like an open circuit and an inductor like a short circuit when they are in their steady state

Principle #3: Continuity

1. Voltage across C is continuous (unless an impulse current)
2. Current through L is continuous (unless impulse voltage)

Review: Simple RL Circuit – Step Response (V_0)

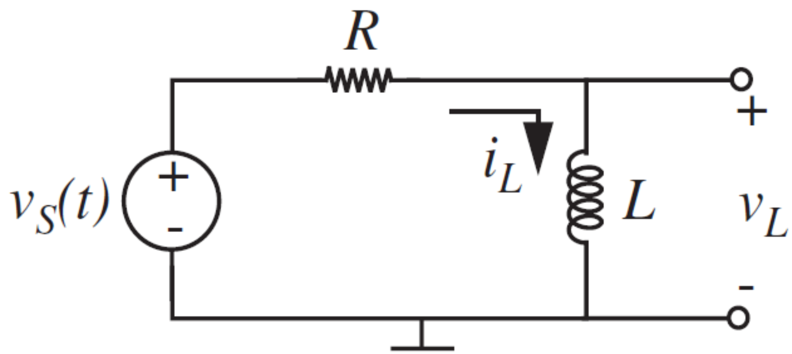


$i = C\frac{dv}{dt}$

$V = L\frac{di}{dt}$

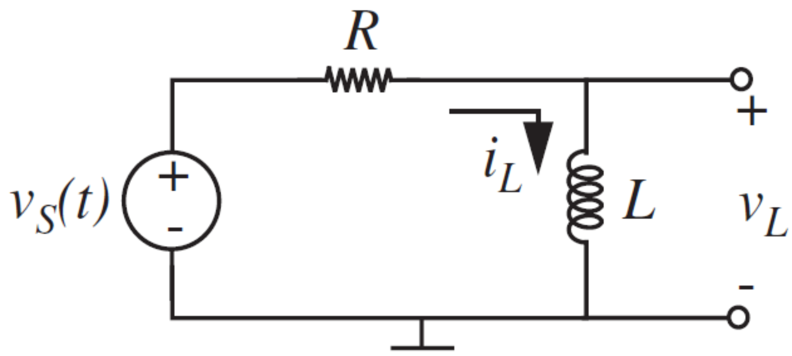
- Find $i_L(t)$ when the circuit is driven by $v(t) = V_0 f(t) = \begin{cases} 0, & t < 0 \\ V_0, & t \geq 0 \end{cases}$
- Initial condition: $i_L(t<0) = 0$ (circuit initially uncharged and at rest)
- From KVL**: $v_s - i_L R - L\frac{di}{dt} = 0$ | $v_s = i_L R + L\frac{di_L}{dt}$
- Homogeneous solution: $i_L(t) = Ae^{st}$ | $0 = Ae^{st}R + LsAe^{st}$
- Characteristic equation: $s = -\frac{R}{L}$
- Homogeneous solution: $i_L(t) = Ae^{-R/L\,t}$ $\quad t_c = \frac{L}{R}$

Review: Simple RL Circuit – Step Response (V_0)



- Find $i_L(t)$ when the circuit is driven by $v(t) = V_0 f(t) = \begin{cases} 0, & t < 0 \\ V_0, & t \geq 0 \end{cases}$
- Initial condition: $i_L(t<0) = 0$ (circuit initially uncharged and at rest)
- Particular equation: $$i_L R + L\frac{di_L}{dt} = v_s$$
- Because source is step response, try: $i_{L,p} = K \;\Big|\; KR = V_o \Rightarrow K = \frac{V_o}{R}$
- Total solution: $i_L(t) = Ae^{-t/t_c} + \frac{V_o}{R}$
- Initial conditions: $i_L(t) = -\frac{V_o}{R} e^{-t/t_c} + \frac{V_o}{R}$

Review: Simple RL Circuit – Step Response (V_0)

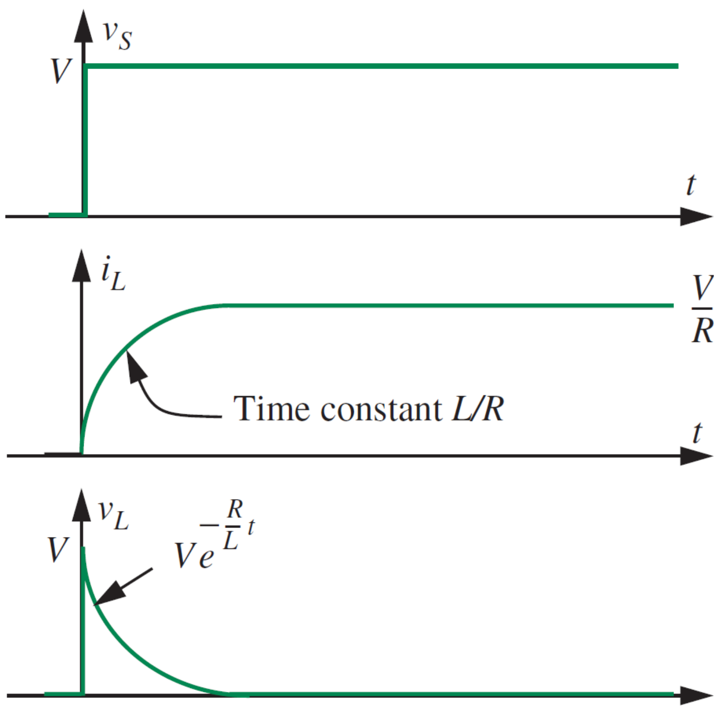


Current response:

$$i_L = -\frac{V}{R}e^{-\frac{Rt}{L}} + \frac{V}{R}$$

Voltage response:

$$v_L = L\frac{di_L}{dt} = Ve^{-\frac{Rt}{L}}$$



Summary: Capacitor and Inductor 1st Order ODEs

$$y(t) = \text{final value} + (\text{initial value} - \text{final value})e^{-t/t_c}$$

$$v_c(t) = v_c(\infty) + (v_c(0) - v_c(\infty))e^{-t/RC}$$

$$i_L(t) = i_L(\infty) + (i_L(0) - i_L(\infty))e^{-Rt/L}$$

****applies to branch variables such as C or L current, resistor V, etc.****

Capacitor

- Behaves as instantaneous **short** circuit if initially at rest for a long time
- Behaves as **open** circuit at long times when driven by DC voltage source

Inductor

- Behaves as instantaneous **open** circuit if initially at rest for a long time
- Behaves as **short** circuit at long times when driven by DC current source

Power and Energy Relation in a Two-Terminal Element

Power:

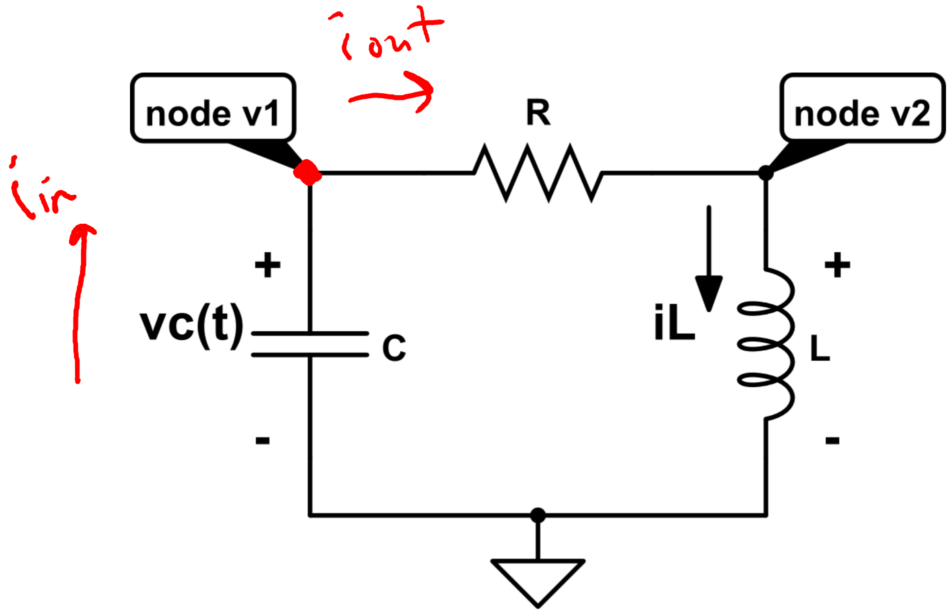
$$P(t) = i(t)v(t) = \frac{v(t)^2}{R}$$

Energy:

$$w(t) = \int_0^T p(t)dt$$

Second-Order ODEs/Circuits

Undriven RLC Circuit: 2nd-Order Transients



$-i_{in} - i_{out} = 0$

RLC circuit behaves similarly to an LC circuit, but energy is now dissipated by R, thus the energy stored in the circuits decays in time

Let's find v1:

- Node 1 KCL: $$-C\frac{dv_1(t)}{dt} - \frac{v_1(t) - v_2(t)}{R} = 0$$

- Node 2 KCL: $$\frac{v_1(t) - v_2(t)}{R} - \frac{1}{L}\int_{-\infty}^{t} v_2(t')dt' = 0$$

$i_{in} - i_{out} = 0$

Undriven RLC Circuit: 2nd-Order Transients

- Node 1 KCL: $$-C\frac{dv_1(t)}{dt} - \frac{v_1(t) - v_2(t)}{R} = 0$$

- Node 2 KCL: $$\frac{v_1(t) - v_2(t)}{R} - \frac{1}{L}\int_{-\infty}^{t} v_2(t')dt' = 0$$

- We can eliminate v_2 using the first equation:

$$\frac{d^2v_1(t)}{dt^2} + \frac{R}{L}\frac{dv_1(t)}{dt} + \frac{1}{LC}v_1(t) = 0$$

- By substituting Ae^{st}, we get the characteristic equation:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Undriven RLC Circuit: 2nd-Order Transients

- We can also write the characteristic equation as:

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

- Where we have defined:

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- The roots of the characteristic equation are:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

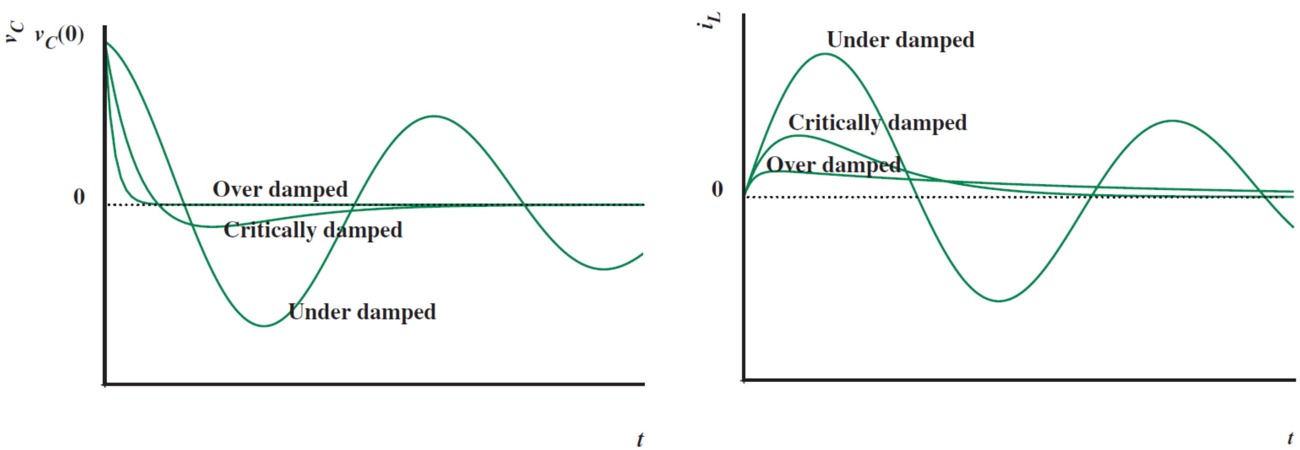
Dynamic Behavior

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

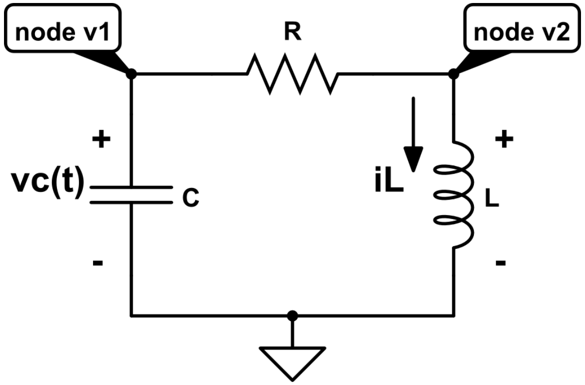
$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \alpha = \frac{R}{2L}$$

The dynamic behavior of an RLC circuit is different for three cases:

$\boldsymbol{\alpha < \omega_0} \rightarrow$ **under-damped dynamics**
$\boldsymbol{\alpha = \omega_0} \rightarrow$ **critically damped dynamics**
$\boldsymbol{\alpha > \omega_0} \rightarrow$ **over-damped dynamics**



Review: 2nd Order ODEs, Series RLC



$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \alpha = \frac{R}{2L}$$

**if critically damped, $A_1 e^{s_1 t} + A_2 \boldsymbol{t} e^{s_2 t}$

For 2nd-order (RLC) circuits, we need to find:

$$i_L(0^-) = i_L(0^+) \qquad \frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C}$$

$$v_c(0^-) = v_c(0^+) \qquad \frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L}$$

1. Once we find $v_C(0)$ and $i_L(0)$, then we can use capacitor element law to determine $dv_C(0)/dt$ using $i_C(0)$
2. We can use the inductor element law to determine $di_L(0)/dt$ using $v_L(0)$

Review: 2nd Order ODEs, Parallel RLC

- We can also write the characteristic equation as:

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

- Where we have defined:

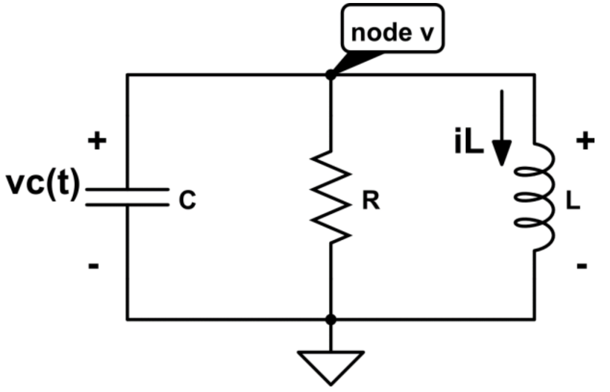
$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- The roots of the characteristic equation are:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

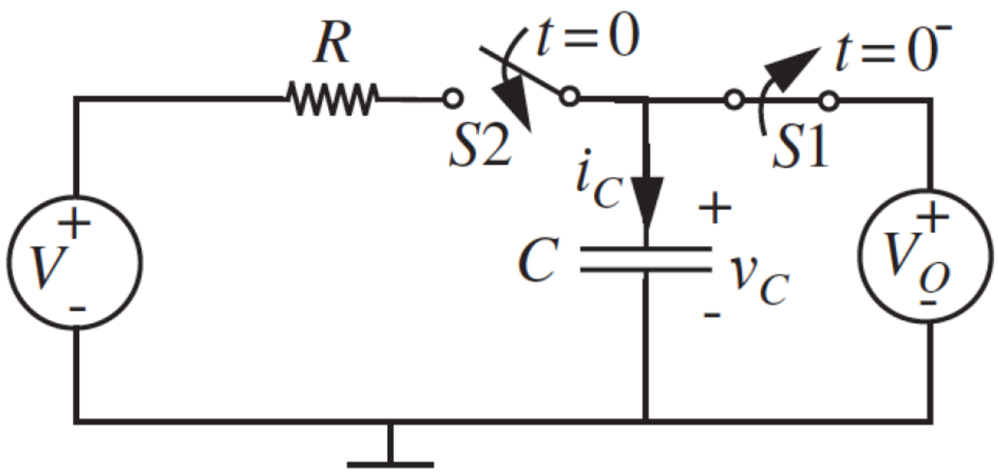
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$



**if critically damped, $A_1 e^{s_1 t} + A_2 t e^{s_2 t}$

Intuitive Analysis – RC Step Response

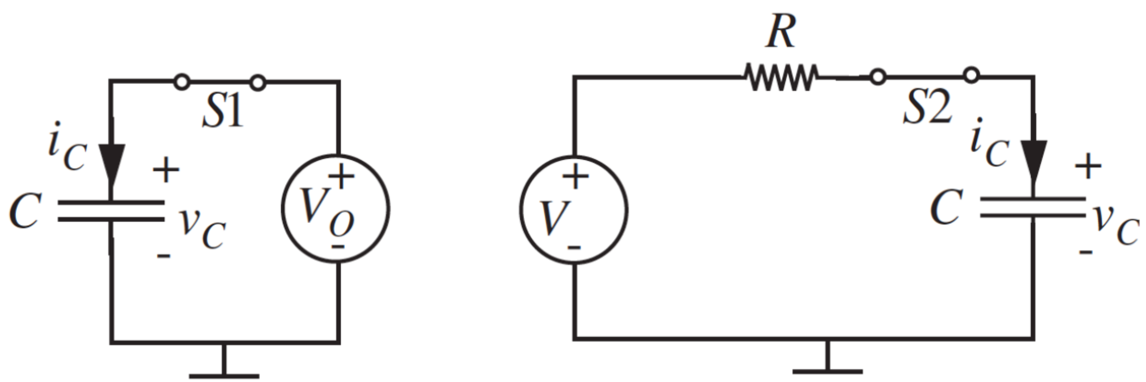


Steps for Intuitive Analysis

1. Initial interval ($t \le 0+$)
2. Final interval ($t \gg \infty$)
3. Transition interval ($t > 0+$) → need to have an understanding of how the circuit will behave

Intuitive Analysis – RC Step Response

$t \leq 0-$ $\qquad\qquad$ $t \gg \infty$



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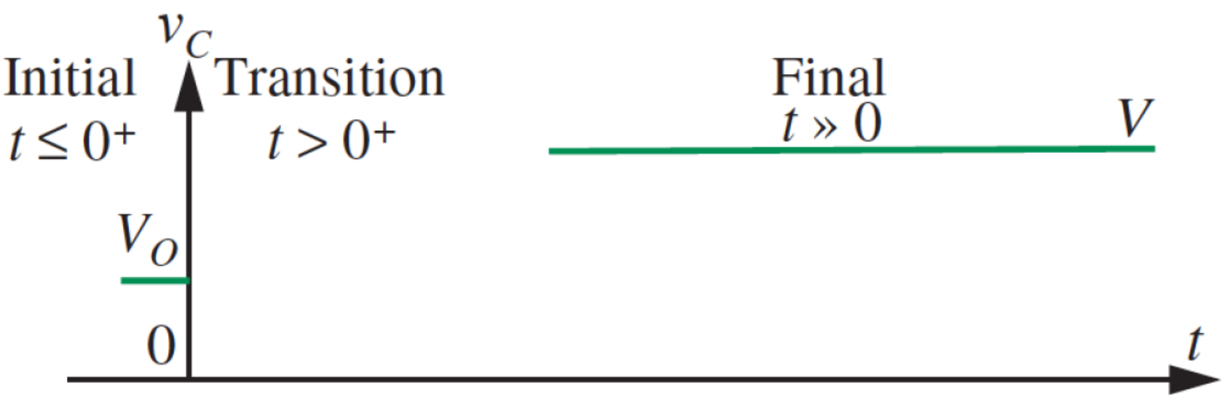
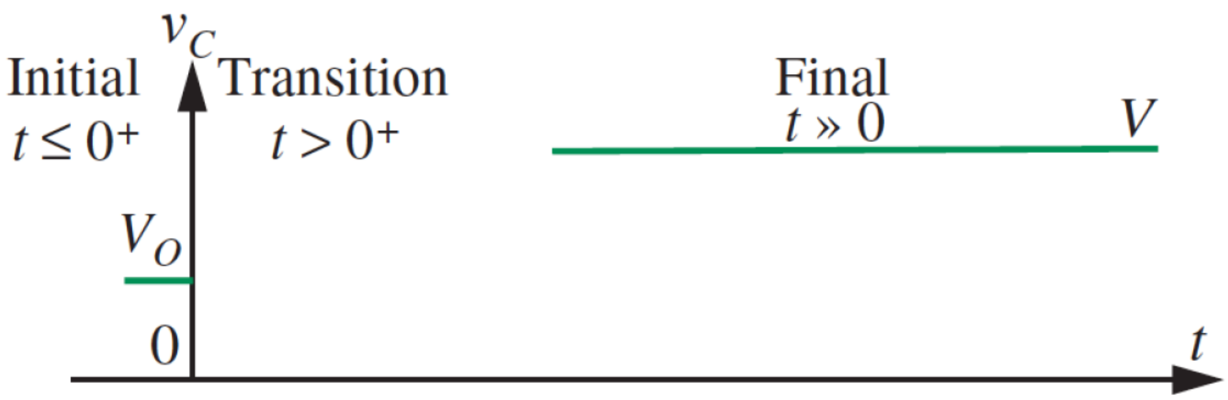
Intuitive Analysis – RC Step Response



We know that for an RC circuit the transient follows an exponential form of either rising ($1 - e^{-t/RC}$) or falling ($e^{-t/RC}$) with time constant *RC*

Intuitive Analysis – RC Step Response



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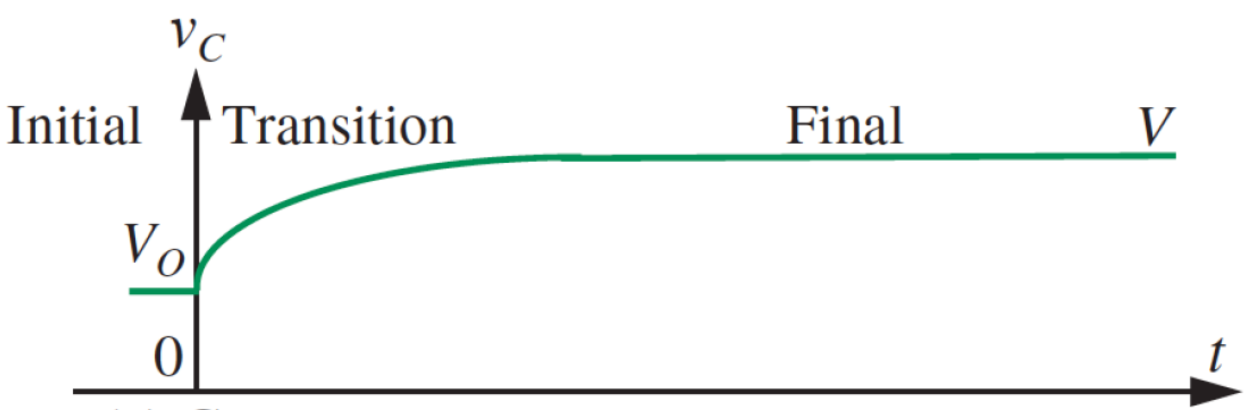
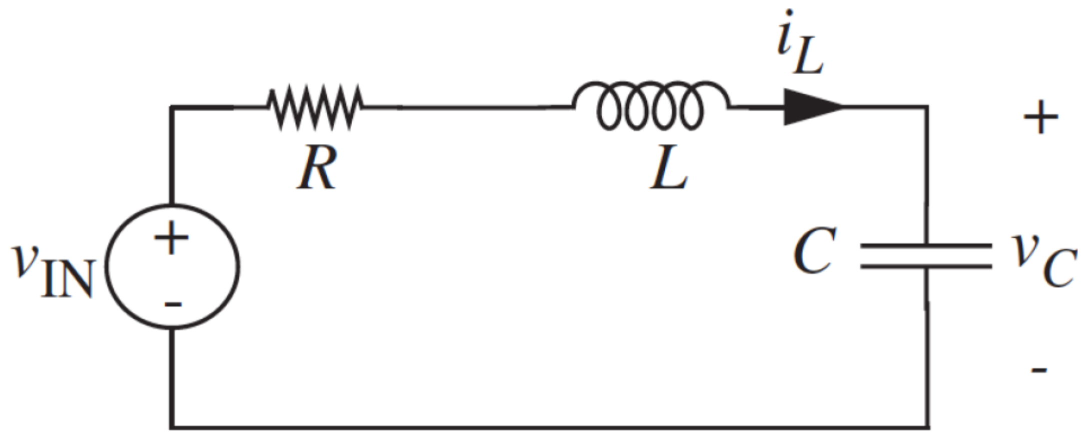


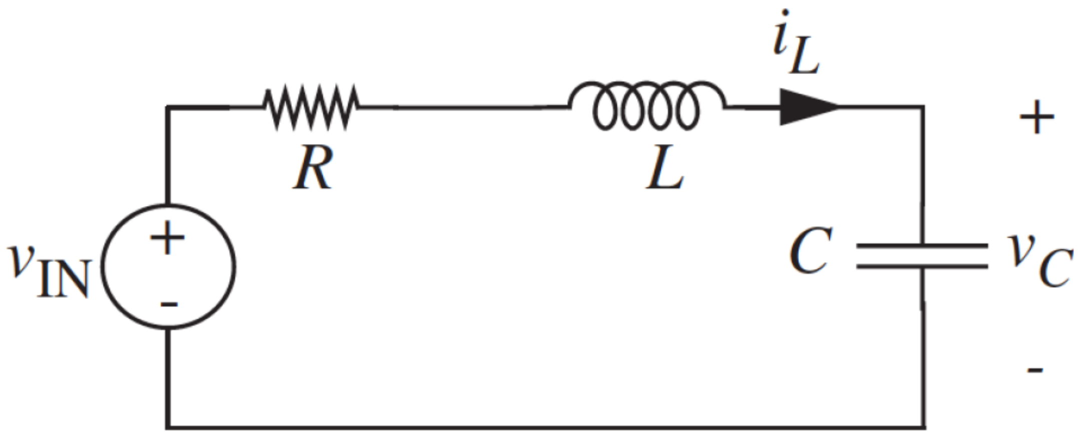
Intuitive Analysis – Driven Series RLC



Steps for Intuitive Analysis

1. Initial interval ($t \le 0+$)
2. Final interval ($t \gg \infty$)
3. Find the initial trajectory of the transient
4. Find the frequency of oscillation (if any)
5. Find the approximate length of time over which the oscillations last (if any)

Intuitive Analysis – Driven Series RLC



- Suppose that:

$$L = 100\ \mu H$$
$$C = 100\ \mu F$$
$$R = 0.2\ \Omega$$

- With initial conditions:

$$v_C(0) = 0.5V$$
$$i_L(0) = -0.5A$$

- And an input step response:

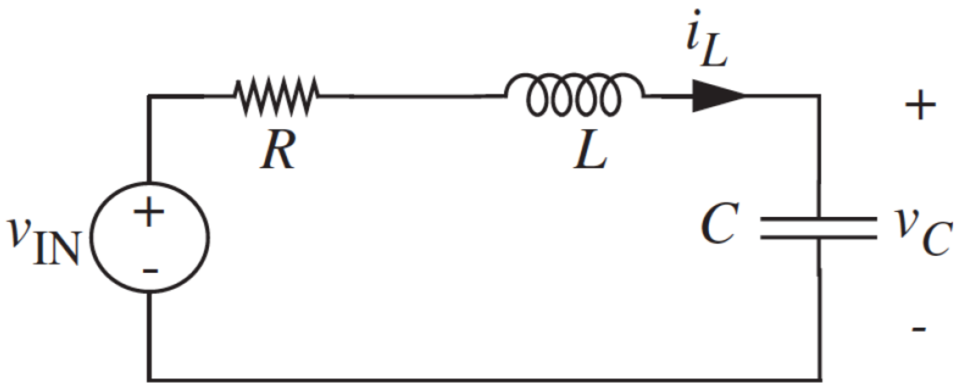
$$v_{IN} = 1V$$

Intuitive Analysis – Driven Series RLC

Initial Interval

No instantaneous jump in v_C and i_L

$$v_c(0+) = 0.5 \text{ V}$$
$$i_L(0+) = -0.5 \text{ A}$$



$$v_C(0) = 0.5V$$
$$i_L(0) = -0.5A$$

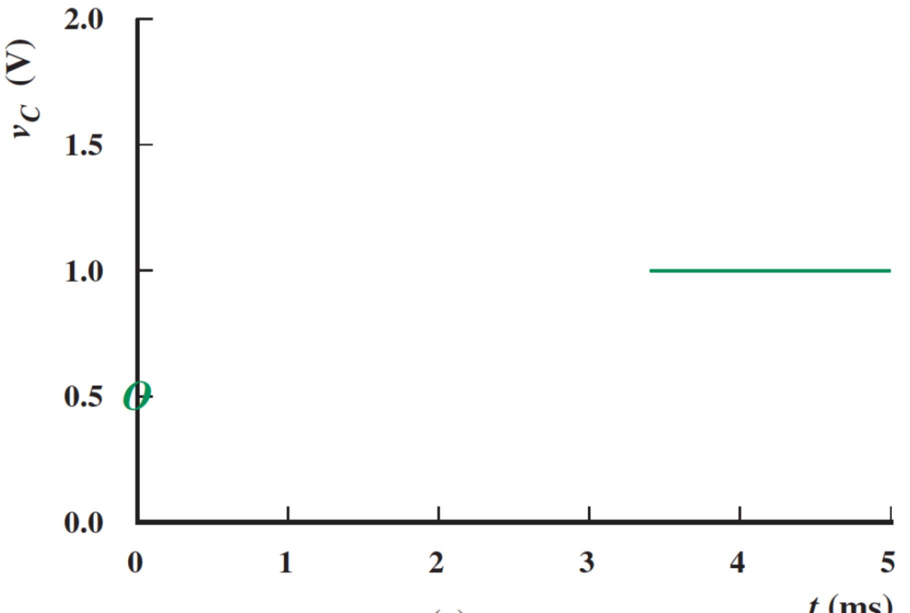
$$v_{IN} = 1V$$

Final Interval

C = open circuit in steady state

$$v_c(\infty) = 1 \text{ V}$$
$$i_L(\infty) = 0 \text{ A}$$

Intuitive Analysis – Driven Series RLC



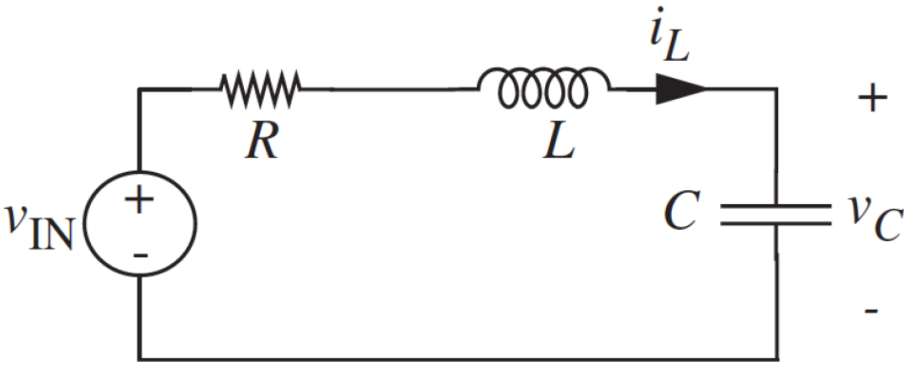
(a)

Circuit Parameters to Determine Transient Behavior

$$\alpha = \frac{R}{2L} = 10^3 \text{ rad/s} \qquad \omega_0 = \sqrt{\frac{1}{LC}} = 10^4 \text{ rad/s} \qquad \boldsymbol{\alpha < \omega_0}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \approx 9950 \text{ rad/s} \qquad Q = \frac{\omega_0}{2\alpha} = 5$$

Transition Analysis

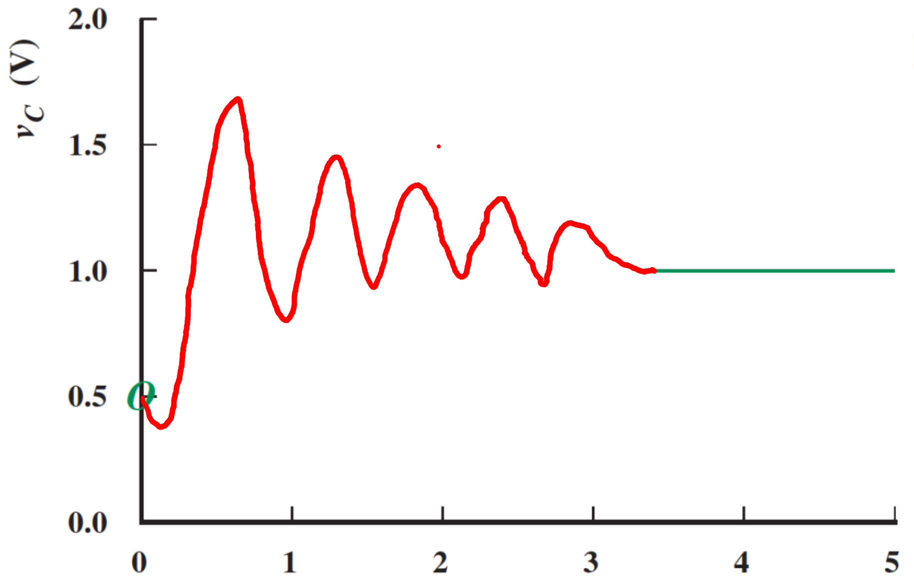


Remember: $i_L(0)$ = -0.5 A, thus capacitor will discharge initially, so $v_C(t)$ will *decrease*

Steps for Intuitive Analysis

1. Initial interval ($t \leq 0+$)
2. Final interval ($t \gg \infty$)
3. **Find the initial trajectory of the transient**
4. Find the frequency of oscillation (if any)
5. Find the approximate length of time over which the oscillations last (if any)

Transition Analysis



(a)

What We Know

Oscillation period = $\frac{2\pi}{9950} \approx 0.6$ ms and oscillations last ~ Q = 5 cycles

We won't work on finding the amplitude (too complex)

Steps for Analyzing these Circuits So Far:

If you're asked to find equation for, e.g. $v_c(t)$:

1. Find the ODE using the node method
2. Characteristic equation $s^2 + 2\alpha s + \omega_o^2 = 0 \rightarrow$ Roots + Response
3. General form*: $v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ (or $A_1 e^{s_1 t} + A_2 t e^{s_2 t}$) **[homogeneous]**
4. Find $v(0+)$ and $dv(0+)/dt$ based on inspecting circuit. Relate back to v(t) to find constants A_1 and A_2
5. Determine instantaneous power, energy dissipated

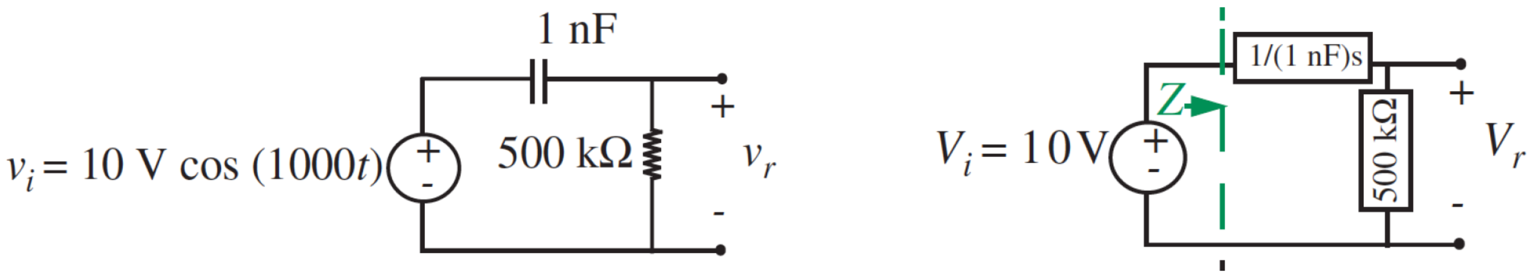
For driven circuit, try intuitive analysis:

1. Initial interval $(t \leq 0+)$ [**FIND**: $v_C(0+)$ & $dv_C(0+)/dt$; same for i_L; KVL + KCL]
2. Final interval $(t \gg \infty)$ [**FIND**: $v_C(\infty)$; $i_L(\infty)$]
3. Inspect circuit to determine quantities of interest:
    1. α, ω_0, ω_d
    2. Damping condition
    3. Q, period = $2\pi/\omega_d$ (if applicable)
    4. Total solution = homogeneous + particular [$v_c(\infty)$ for step input]

Impedance Method

The Concept of Impedance

Impedance: A generalization of resistance for sinusoidal steady state inputs. Associated with each element is a characteristic impedance that affects the flow of current/voltage drop across the element, analogous to a resistor.



$$v_0(t) = |V_0|\cos(\omega t + \angle V_0) = Re\{|V_0|e^{j\angle V_0}e^{j\omega t}\}$$

Generalization of Ohm's Law for Sinusoidal Steady State

Resistor: $V = IR \rightarrow Z_R = R$

Capacitor: $V = I\frac{1}{sC} \rightarrow Z_C = \frac{1}{sC} = \frac{1}{j\omega C}$

Inductor: $V = sLI \rightarrow Z_L = sL = j\omega L$

Impedance Method

$$v_0(t) = |V_0|\cos(\omega t + \angle V_0) = Re\{V_0(j\omega)e^{j\omega t}\}$$

1. First, **replace** the (sinusoidal) sources by the complex amplitudes

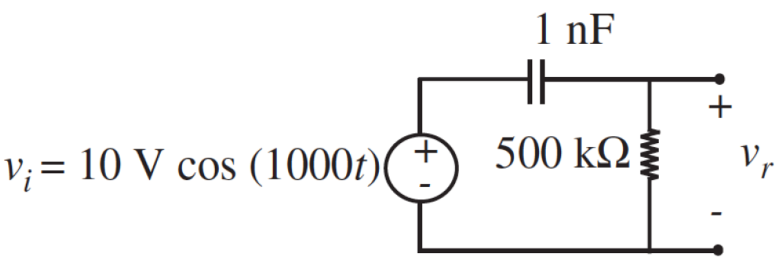
2. Then, replace the circuit elements by their impedances. The resulting diagram is called **the impedance model of the network.**

3. Now, **determine the complex amplitudes** of the voltages and currents in the circuits (e.g. $V_c(j\omega)$). You can use any standard linear circuit analysis technique you like—Node method, KVL, KCL, Thevenin, intuitive method based on series/parallel simplifications, etc.

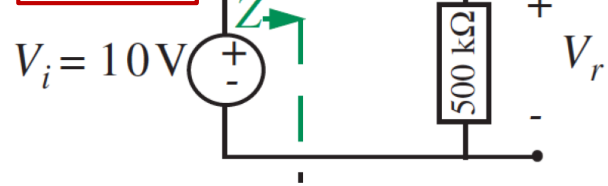
4. Finally, we can obtain the time variables from the complex amplitudes and **plug into the general expression for the dynamics**. This isn't usually necessary though. As an example, the time domain response corresponding to the node variable V_c is given by:

$$v_c(t) = Re\{V_c(j\omega)e^{j\omega t}\} = |V_c|\cos(\omega t + \angle V_c)$$

Example: RC Circuit



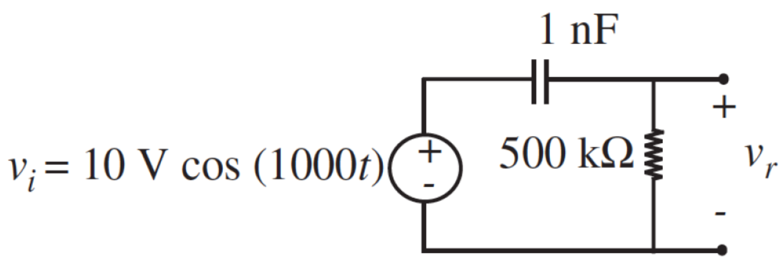
STEP 1



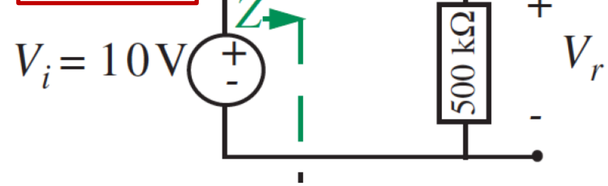
STEP 2 The impedances are:

$$Z_R = 500 * 10^3 \Omega, \quad Z_C = \frac{1}{Cj\omega} = \frac{1}{1 \times 10^{-9} j1000} \Omega$$

Example: RC Circuit



STEP 1



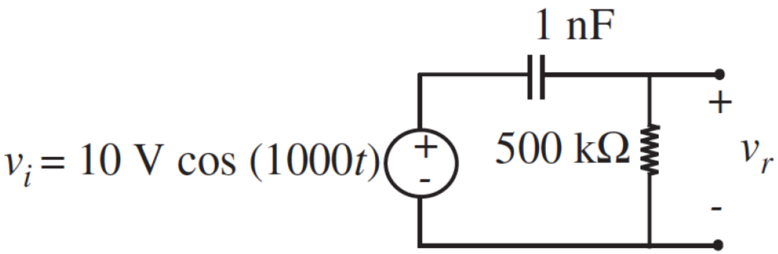
STEP 2 The impedances are:

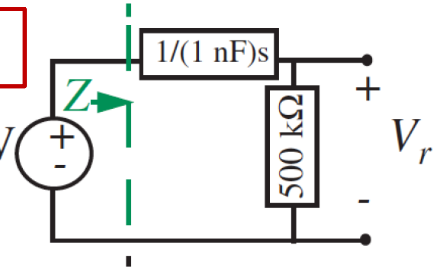
$$Z_R = 500 * 10^3 \Omega, \quad Z_C = \frac{1}{Cj\omega} = \frac{1}{1 \times 10^{-9} j1000} \Omega$$

STEP 3 Using the voltage divider relation for V_r

$$V_r = \frac{Z_R}{Z_R + Z_C} V_i = \frac{500 * 10^3}{500 * 10^3 + \frac{1}{1 \times 10^{-9} j1000}} 10 = \frac{5}{0.5 - j} = 4.47 e^{j1.1}$$

Example: RC Circuit





$\phi = \tan^{-1}\left(\frac{Y}{X}\right)$

$z = X + jY$

$|z| = \sqrt{X^2 + Y^2}$

STEP 2 The impedances are:

$$Z_R = 500 * 10^3 \Omega, \quad Z_C = \frac{1}{Cj\omega} = \frac{1}{1 \times 10^{-9} j1000} \Omega$$

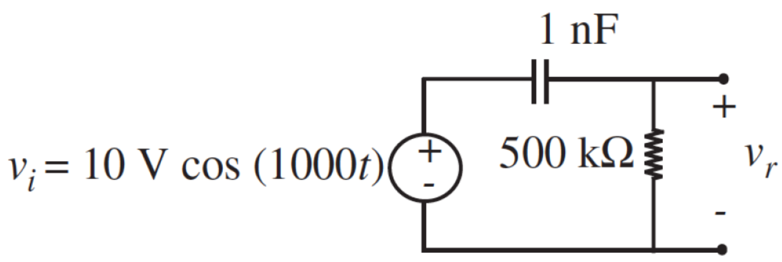
STEP 3 Using the voltage divider relation for V_r

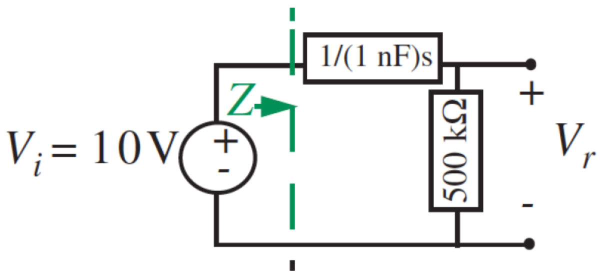
$$V_r = \frac{Z_R}{Z_R + Z_C} V_i = \frac{500 * 10^3}{500 * 10^3 + \frac{1}{1 \times 10^{-9} j1000}} 10 = \frac{5}{0.5 - j} = 4.47 e^{j1.1}$$

Complex amplitude of the voltage $V_r = 4.47 e^{j1.1}$

Magnitude $|V_r| = 4.47$ and Phase $\angle V_r = 1.1$ rad

Example: RC Circuit



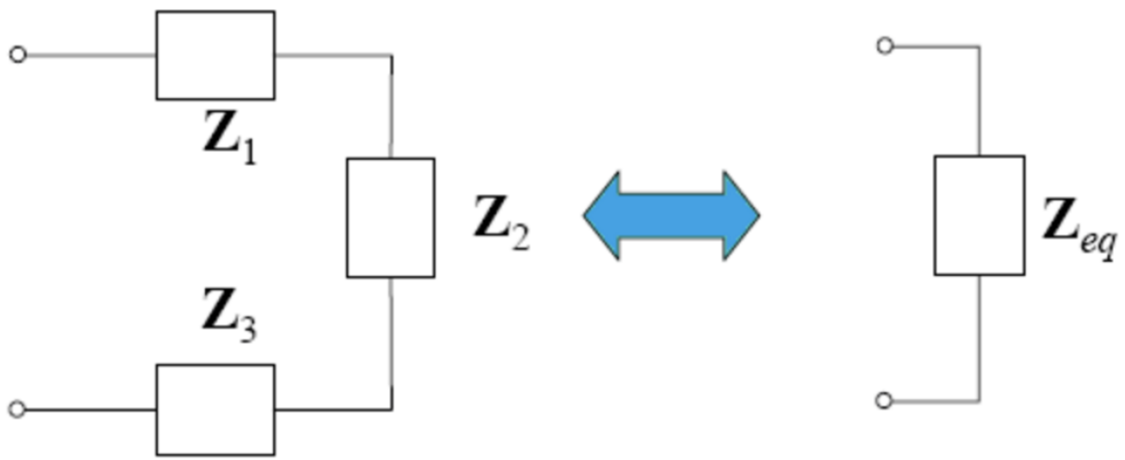


$$V_r = 4.47e^{j1.1}$$

STEP 4 So in the time domain, $v_r(t)$ is given by:

$$v_r = \Re\left[V_r e^{j\omega t}\right] = \Re\left[4.47e^{j1.1}e^{j1000t}\right] = 4.47\cos(1000t + 1.1)V$$

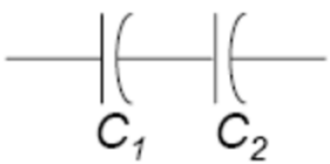
Impedance Combinations: Series



$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3$$

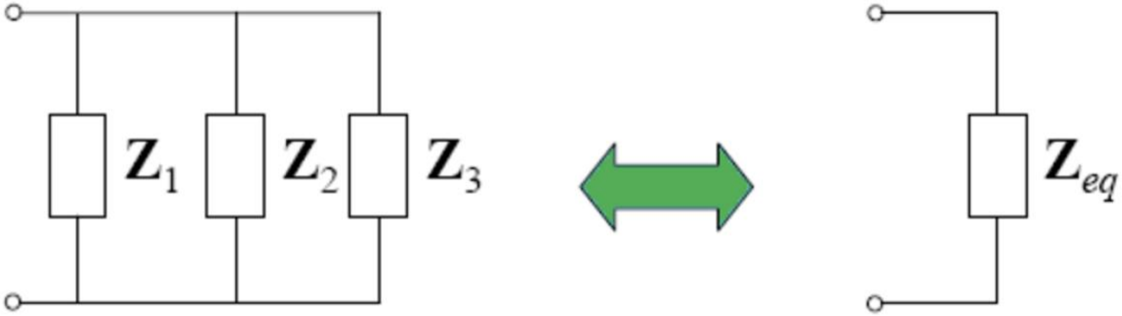
For example:

$$Z_{eq} = j\omega(L_1 + L_2)$$



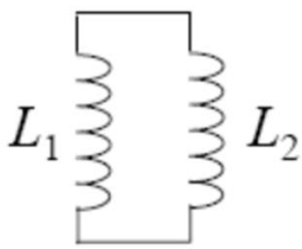
$$Z_{eq} = \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2}$$

Impedance Combinations: Parallel

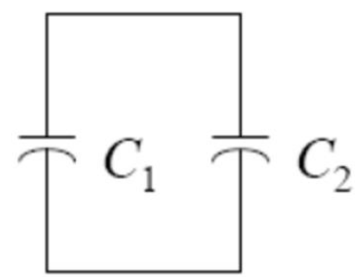


$$1/\mathbf{Z}_{eq} = 1/\mathbf{Z}_1 + 1/\mathbf{Z}_2 + 1/\mathbf{Z}_3$$

For example:



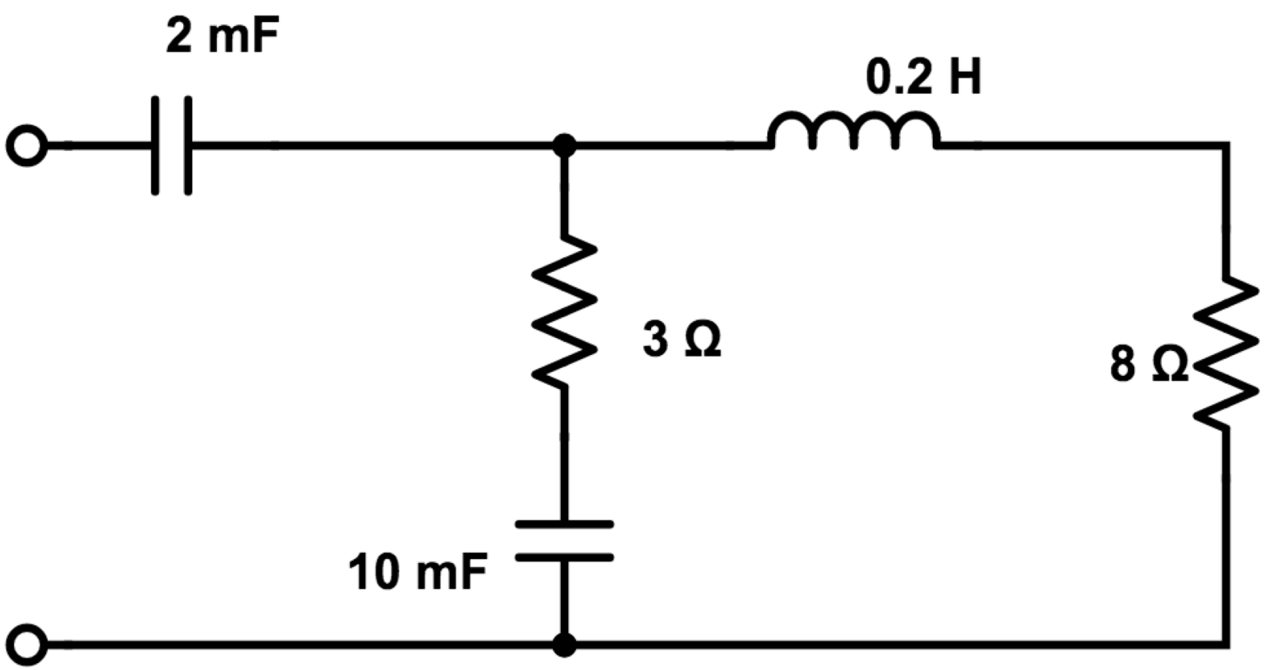
$$\mathbf{Z}_{eq} = j\omega \frac{L_1 L_2}{(L_1 + L_2)}$$



$$\mathbf{Z}_{eq} = \frac{1}{j\omega(C_1 + C_2)}$$

In-Class Exercise #1

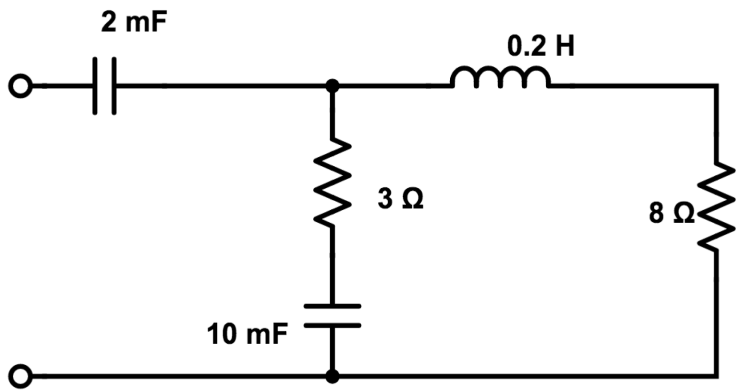
Find the input impedance of the circuit (at frequency $\omega = 50$ rad/s):

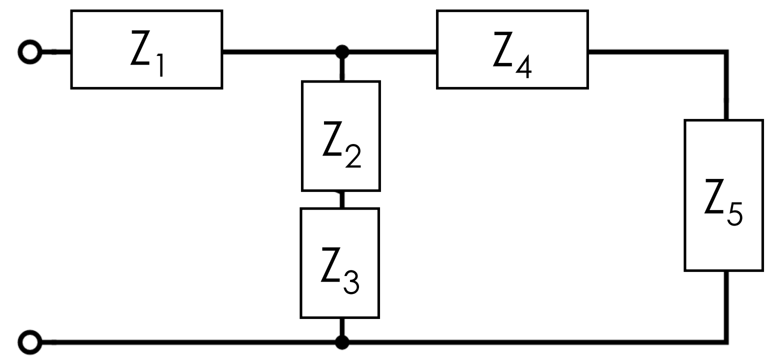


$Z_R = R \qquad Z_C = \frac{1}{j\omega C} \qquad Z_L = j\omega L$

In-Class Exercise #1

Find the input impedance of the circuit (at frequency ω = 50 rad/s):

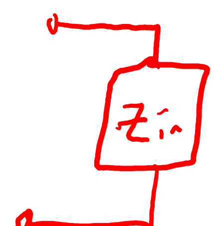




$Z_1 + (Z_2+Z_3) // (Z_4+Z_5)$

$Z_C = \frac{1}{j\omega C}$ $\quad Z_L = j\omega L$

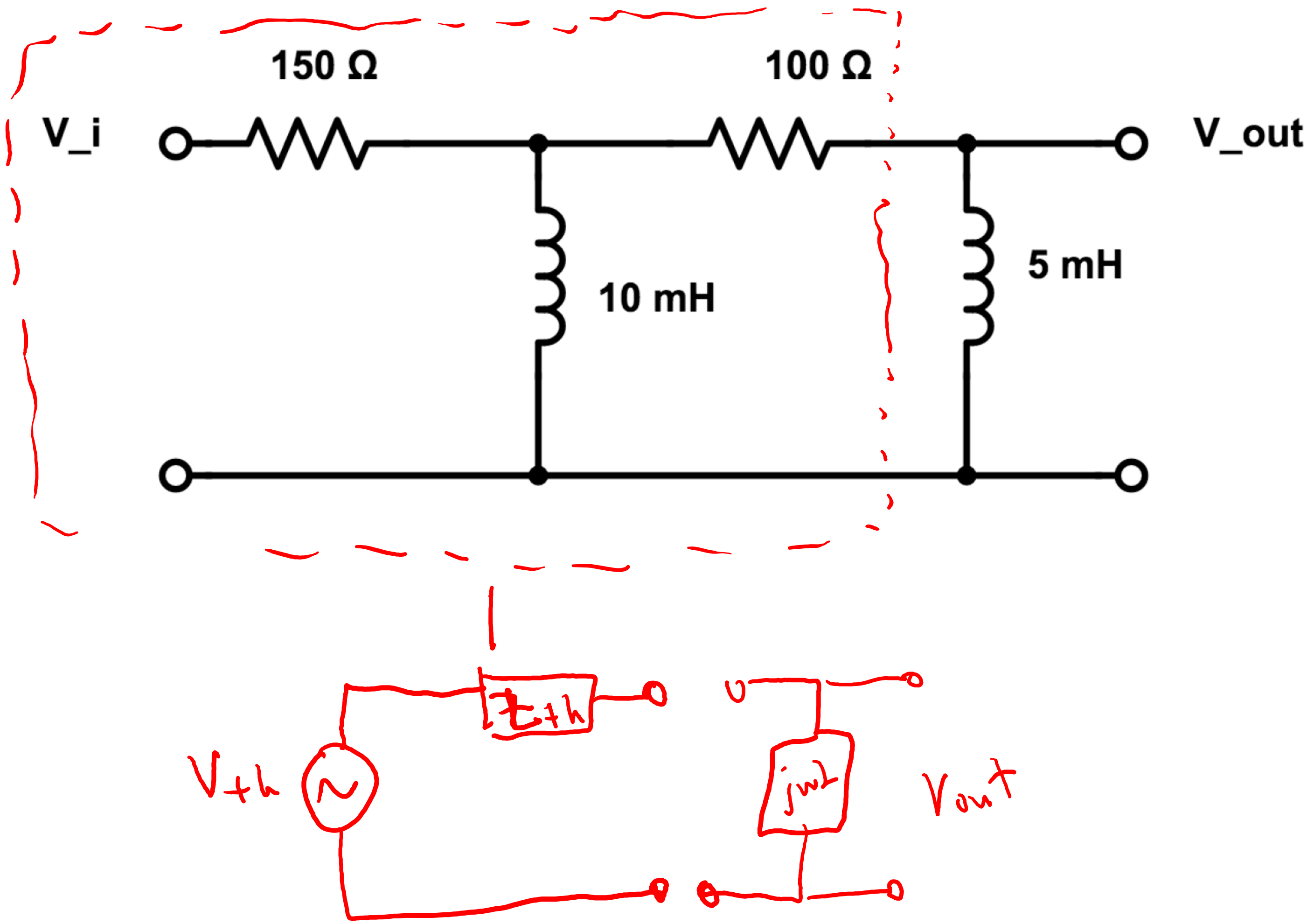
$\cancel{Z_{in} = 2.68 - 9.95j}$



$\mathbf{Z_{in} = 3.22 - 11.07j}$

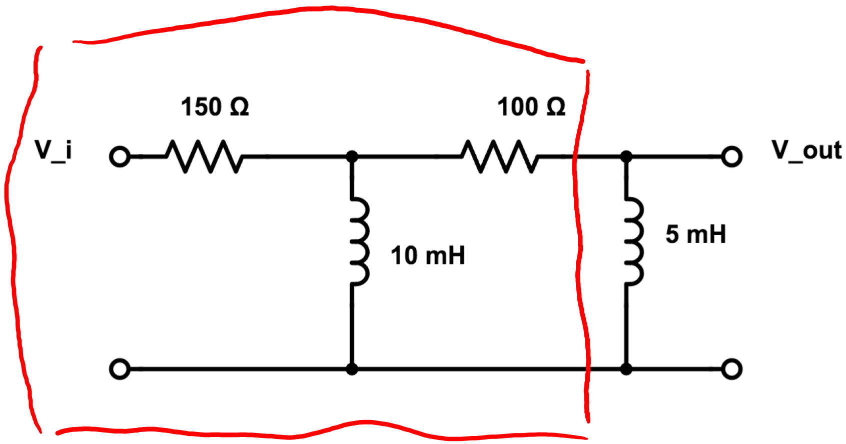
In-Class Exercise #2

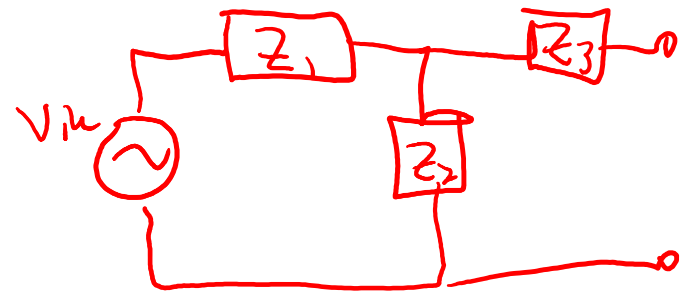
Find the phase shift produced by the following circuit at ω = 2 kHz:



In-Class Exercise #2

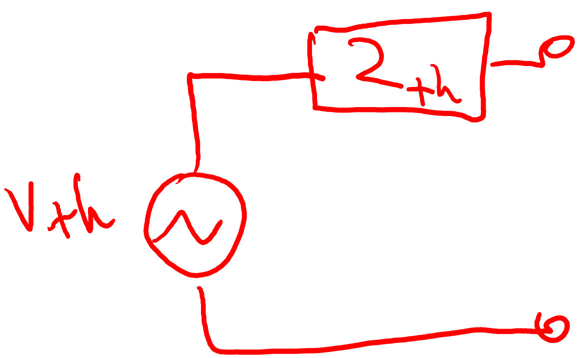
Find the phase shift produced by the following circuit at ω = 2 kHz:

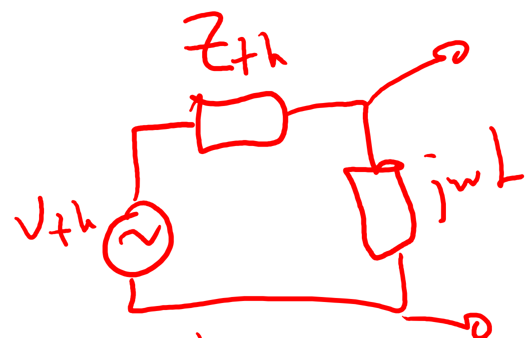




$$Z_{th} = Z_1 \parallel Z_2 + Z_3$$

$$V_{th} = \frac{Z_2}{Z_1 + Z_2} V_{in}$$





$$V_{out} = \left(\frac{j\omega L}{Z_{th} + j\omega L}\right) V_{th}$$

$$X + jY \rightarrow \phi = \tan^{-1}\left(\frac{Y}{X}\right)$$

In-Class Exercise #3

What is we had these two sources instead?

$$i_1(t) = 4\cos(10t + 30^\circ)A, \quad i_2(t) = 5\sin(1000t - 20^\circ)$$

Can you combine then like you just did using the impedance method?

No! But, superposition theorem does apply to sinusoidal steady state circuits the same way it applies to DC circuits. So you can calculate the response to one source at one frequency, and then repeat for each source at a different frequency. Then sum up the final responses.

Transfer Function and Bode Plot

Plotting the Frequency Response

typically V_0 or I_0

transfer function = $H(j\omega) = \dfrac{complex\ amplitude\ of\ the\ output}{complex\ amplitude\ of\ the\ input}$

$$v_i(t) = V_i \cos(\omega t) \longrightarrow \boxed{H(j\omega)} \longrightarrow v_o(t) = |H(j\omega)| V_i \cos\left(\omega t + \angle H(j\omega)\right)$$

When asked to plot the frequency response of a system, you need to draw the approximate sketches of the magnitude and phase as follows:

Plotting the Frequency Response

- Plot $|H(j\omega)|$: the log magnitude plotted against log frequency
- Plot $\angle H(j\omega)$: the angle in linear scale plotted against log frequency

Plotting the Frequency Response

typically V_0 or I_0

$$\text{transfer function} = H(j\omega) = \frac{\textit{complex amplitude of the output}}{\textit{complex amplitude of the input}}$$

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Plotting the Frequency Response

- Plot $|H(j\omega)|$: the log magnitude plotted against log frequency
- Plot $\angle H(j\omega)$: the angle in linear scale plotted against log frequency

Summary of Log Plots

- Imagine if you wanted to plot:

$$|H(j\omega)| = |\frac{j\omega + 1}{2j\omega + 1}|$$

- Summing terms is easy to do graphically; products are harder

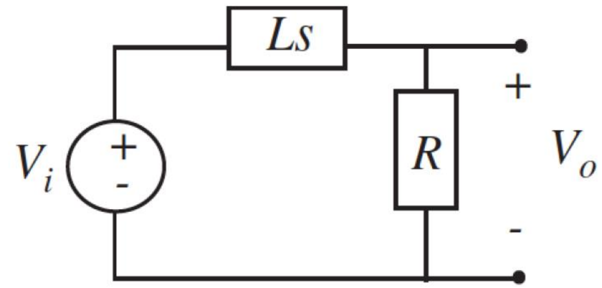
- On a log scale, products (and divisions) turn into a sum:

$$\log|\frac{j\omega + 1}{2j\omega + 1}| = \log|j\omega + 1| - \log|2j\omega + 1|$$

Intuitive Sketching for RL and RC Circuits

- Let's first consider the series RL circuit:

$$H(j\omega) = \frac{V_o}{V_i} = \frac{R}{R + j\omega L}$$
$$= \frac{\frac{R}{L}}{\frac{R}{L} + j\omega}$$



(b) Impedance model

- So we get:

$$|H(j\omega)| = \left| \frac{\frac{R}{L}}{\frac{R}{L} + j\omega} \right|$$

$$\angle H(j\omega) = \tan^{-1} \frac{-\omega L}{R}$$

Intuitive Sketching, Magnitude $|H(j,\omega)|$

- First, find the asymptotes of the magnitude plot:

$$|H(j\omega)| = \left| \frac{\frac{R}{L}}{\frac{R}{L} + j\omega} \right|$$

- At low frequencies ($\omega \to 0$):

$$|H(j\omega)| \approx 1$$

  - Hence the magnitude appears as a horizontal line at low frequencies

- At high frequencies $\omega \to \infty$, ω dominates in the denominator

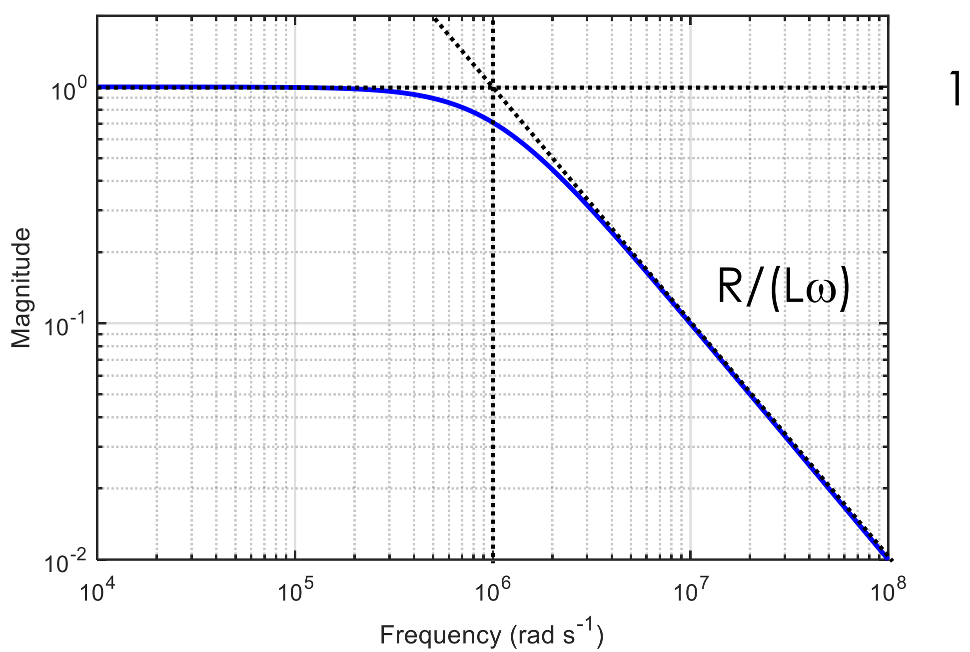
$$|H(j\omega)| \approx \frac{R/L}{\omega} \to \log|H(j\omega)| \approx \log R/L - \log \omega$$

  - On a log scale, a line with slope = -1 passing through (R/L, 1)

Intuitive Sketching, Magnitude $|H(j, \omega)|$

- At low frequencies ($\omega \to 0$): $|H(j\omega)| = 1$
- At high frequencies $\omega \to \infty$, ω dominates in the denominator

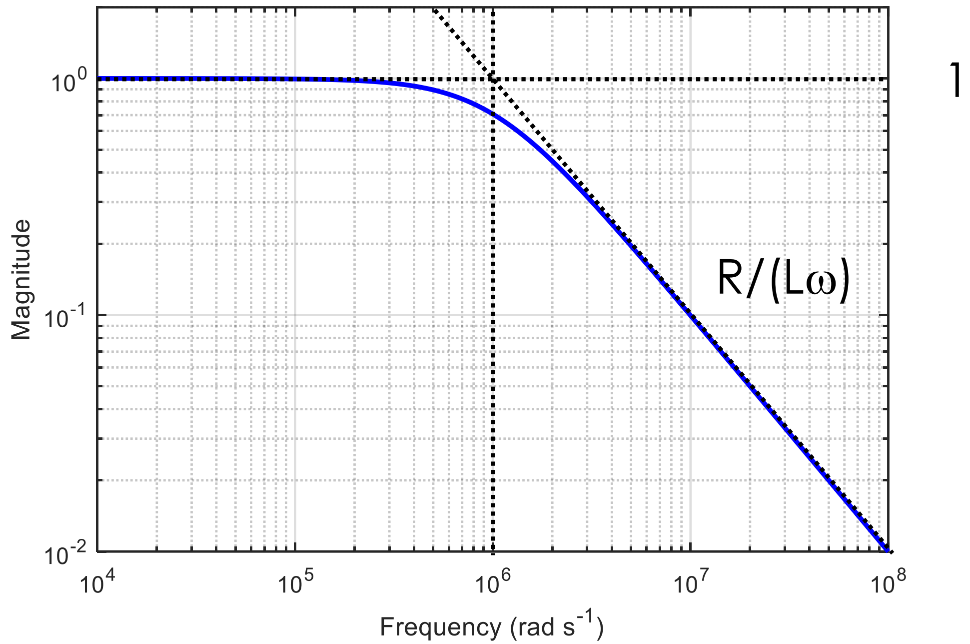
$$|H(j\omega)| \approx \frac{R/L}{\omega} \to \log|H(j\omega)| \approx \log R/L - \log \omega$$



Intuitive Sketching, Magnitude $|H(j,\omega)|$

- The two asymptotes intersect at: $$\omega = \frac{R}{L}$$
- This frequency is called the "corner" or "cutoff" frequency

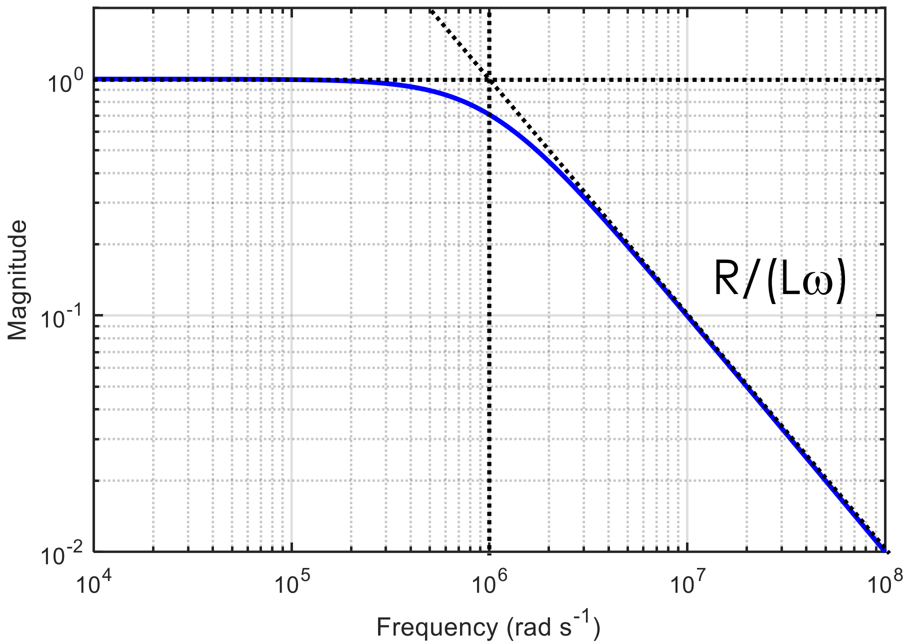
At the cutoff frequency, the real and imaginary components of $H(j,\omega)$ are equal, and $|H(j,\omega)| = \frac{1}{\sqrt{2}} = 0.707$



$$|H(j\omega)| \approx \frac{R/L}{\omega} \rightarrow \log|H(j\omega)| \approx \log R/L - \log\omega$$

Intuitive Sketching, Magnitude $|H(j,\omega)|$

At the cutoff frequency, the real and imaginary components of $H(j,\omega)$ are equal, and $|H(j,\omega)| = \frac{1}{\sqrt{2}} = 0.707$



1

Decibels:

$$P_2 = 10^{L/10\ dB} P_1$$

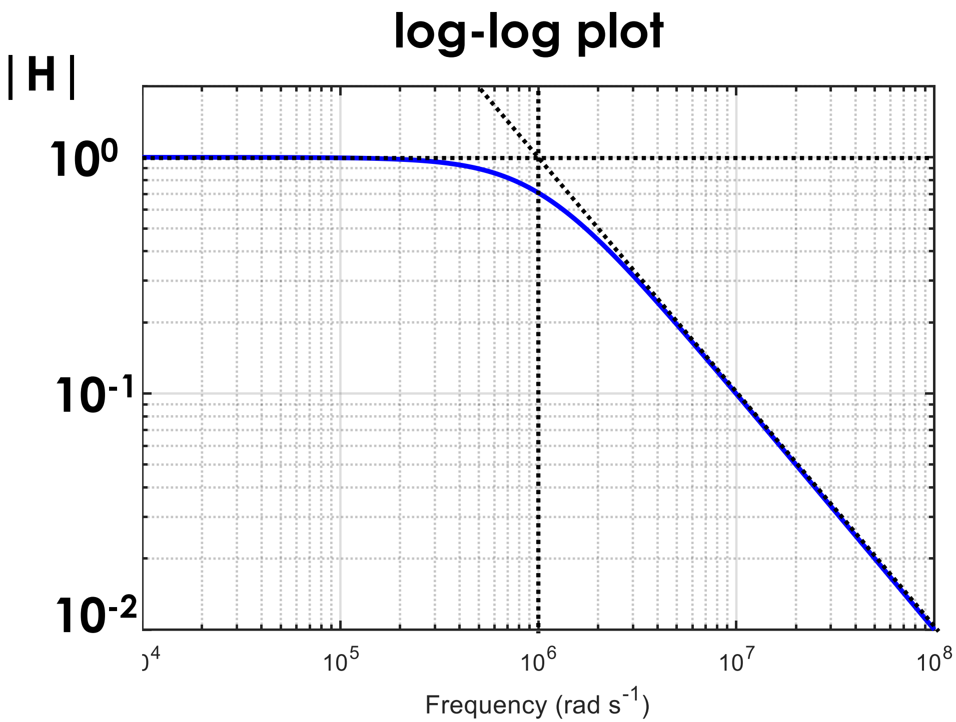
$$L = 10\log(P_2/P_1)\ \text{dB}$$

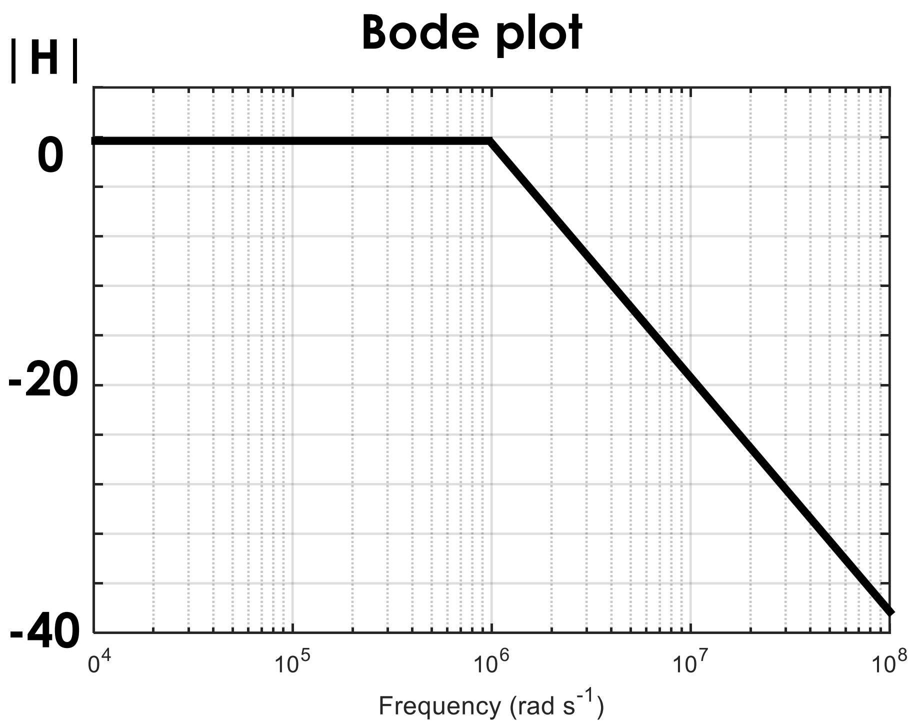
$$10\log\left(\frac{1}{2}\right) \sim 3\ \text{dB}$$

Recall that power dissipated through impedance Z is $\frac{V_0^2}{Z}$, and $V_0^2 \propto |H(j,\omega)|^2$. At the cutoff frequency, $|H(j,\omega)|^2 = 1/2$. **Thus the cutoff frequency also tells you the "3 dB" point**, i.e. the frequency at which the output power is halved.

The Bode Plot

Graph of the frequency response of a system on a Decibel scale using asymptotic approximations and straight-line segments





Provides an easier way to break down a complex response into individual components, sum them, and plot an approximate response

The Bode Plot

Using asymptotes, we now can **approximately plot** $H(s) \in \{1/(s+a), (s+a), s/(s+a), (s+a)/s\}$

How about more general frequency response forms?

$$H(s) = \frac{K_\circ s^l (s+a_1)(s+a_2)\ldots(s^2+2\alpha_1 s+\omega_1^2)\ldots}{(s+a_3)(s+a_4)\ldots(s^2+2\alpha_2 s+\omega_2^2)\ldots}$$

$$\begin{aligned}
log|H(s)| = {} & \log K_o + \\
& \log|s| + \log|s| + \cdots \ (l \text{ terms}) + \\
& \log|s+a_1| + \log|s+a_2| + \cdots - \log|s+a_3| - \log|s+a_4| + \cdots \\
& \log|s^2+2\alpha_1 s+\omega_1^2| + \cdots - \log|s^2+2\alpha_2 s+\omega_2^2| + \cdots
\end{aligned}$$

$$\begin{aligned}
\angle H(s) = {} & \angle K_o + \\
& \angle s + \angle s + \cdots \ (l \text{ terms}) \\
& \angle(s+a_1) + \angle(s+a_2) + \qquad\qquad + a_3) - \angle(s+a_4) - \cdots \\
& + \angle(s^2+2\alpha_1 s+\omega_1^2) + \cdots - \angle(s^2+2\alpha_2 s+\omega_2^2) - \cdots
\end{aligned}$$

The Bode Plot: Rules

- Determine the transfer function: $H(s) = \dfrac{K(s+z_1)}{s(s+p_1)}$
- Rewrite it by factoring numerator and denominator into "standard" form.
  - The "z's" are called zeros
  - The "p's" are called poles

$$H(s) = \frac{Kz_1(s/z_1 + 1)}{sp_1(s/p_1 + 1)}$$

- Replace ***s*** with ***jω***. Find the magnitude of the transfer function.
- Take $\log_{10}$ and multiply by 20.

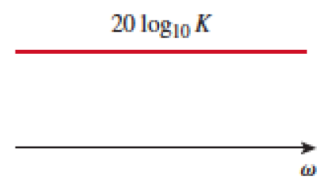
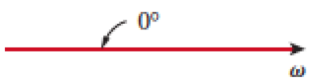
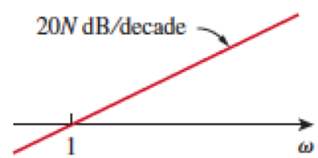
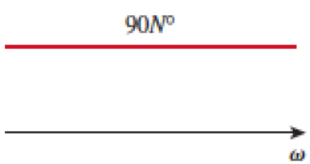
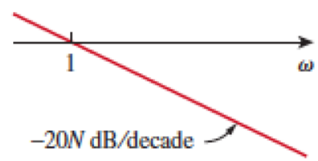
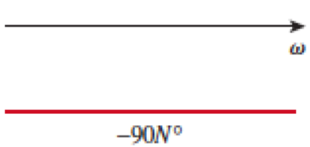
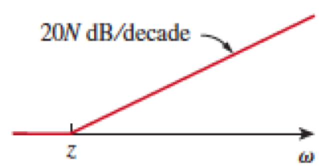
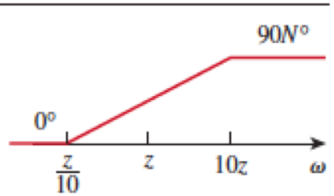
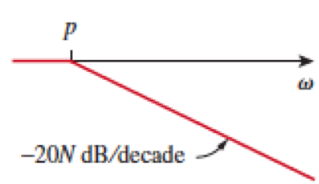
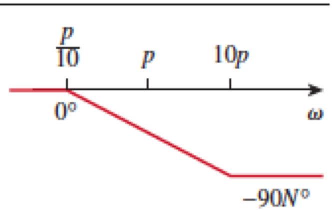
$$20 \log_{10}(H(jw)) = 20\log_{10}\left(\frac{Kz_1(jw/z_1 + 1)}{jwp_1(jw/p_1 + 1)}\right) =$$

$$20\log_{10}|K| + 20\log_{10}|z_1| + 20\log_{10}\left|(jw/z_1 + 1)\right| - 20\log_{10}|p_1| - 20\log_{10}|jw| - 20\log_{10}\left|(jw/z_1 + 1)\right|$$

- Each of these terms is fairly straightforward to show on a log scale. For the Bode plot, graph each one individually, and then connect the straight-line segments. With a little practice, we can do this quickly. Let's look at each term.

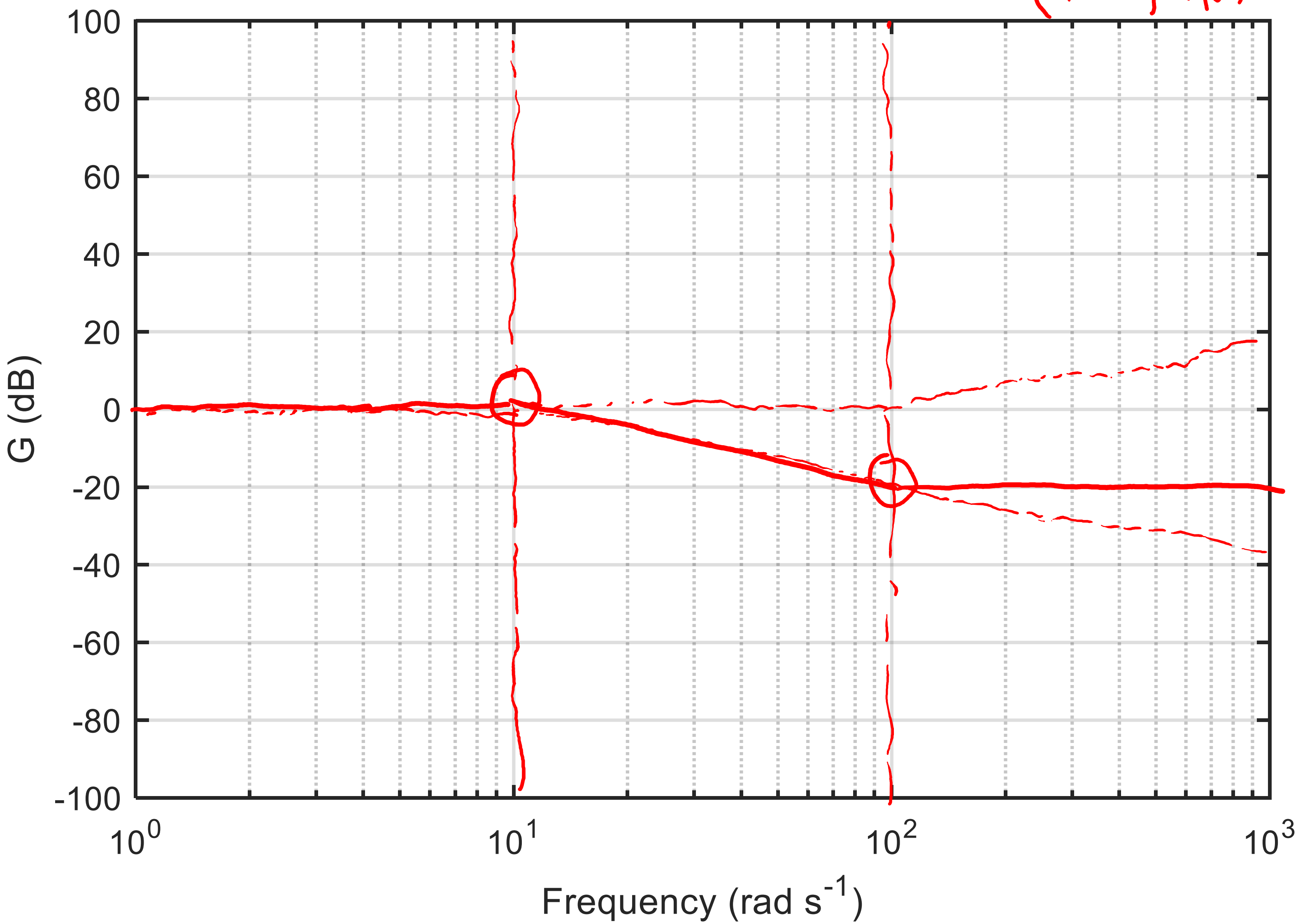
The Bode Plot

$$G_{dB} = 20\ log_{10}\ |H(s)|\ \text{dB}$$

Term	Magnitude	Phase shift
K		
$(j\omega)^N$		
$\dfrac{1}{(j\omega)^N}$		
$\left(1 + \dfrac{j\omega}{z}\right)^N$		
$\dfrac{1}{(1 + j\omega/p)^N}$		

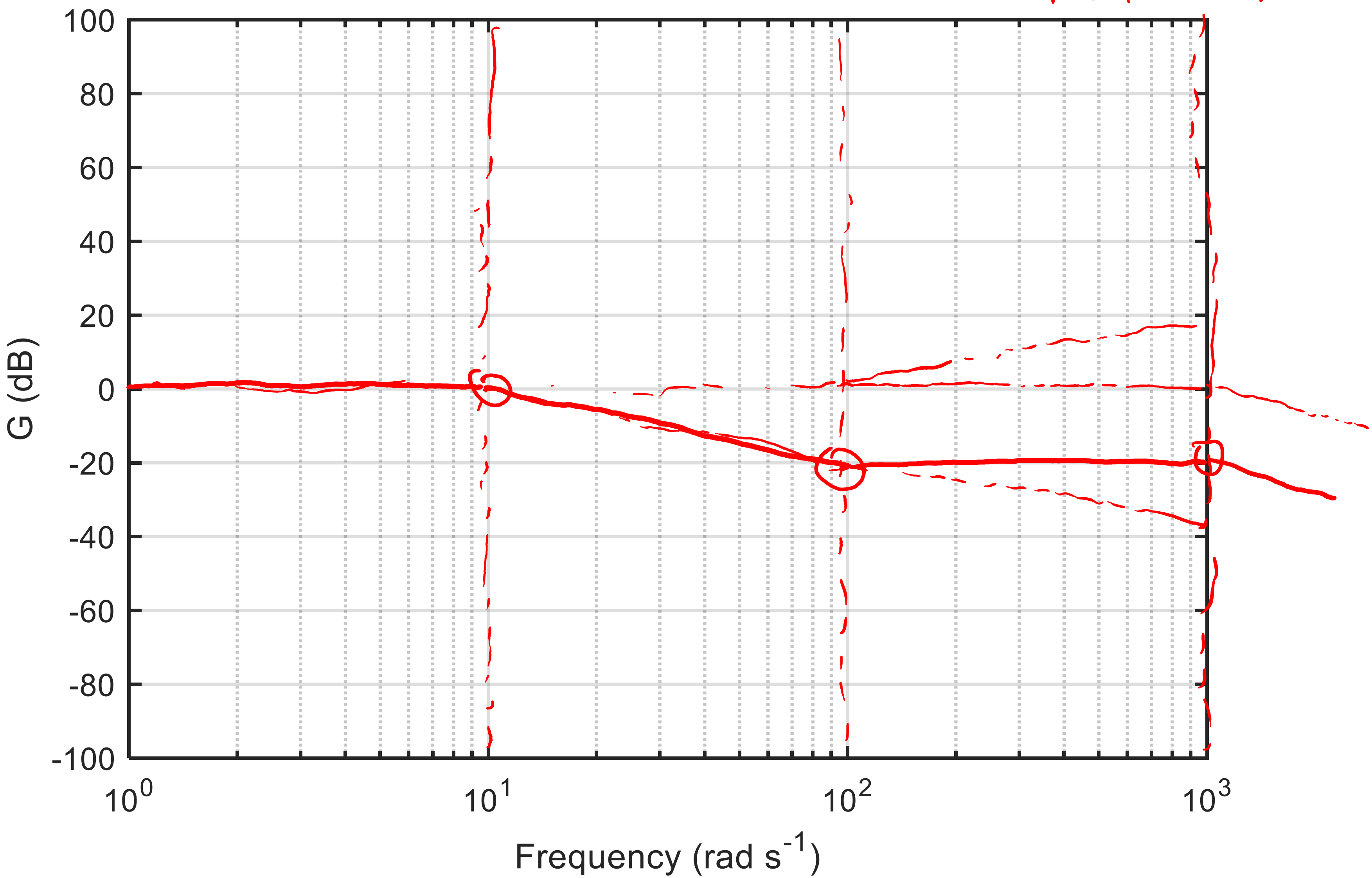
Practice Problem #1

$$H(j,\omega) = \frac{0.1 * (100 + j\omega)}{10 + j\omega} = \frac{(1 + j\omega/100)}{(1 + j\omega/10)}$$



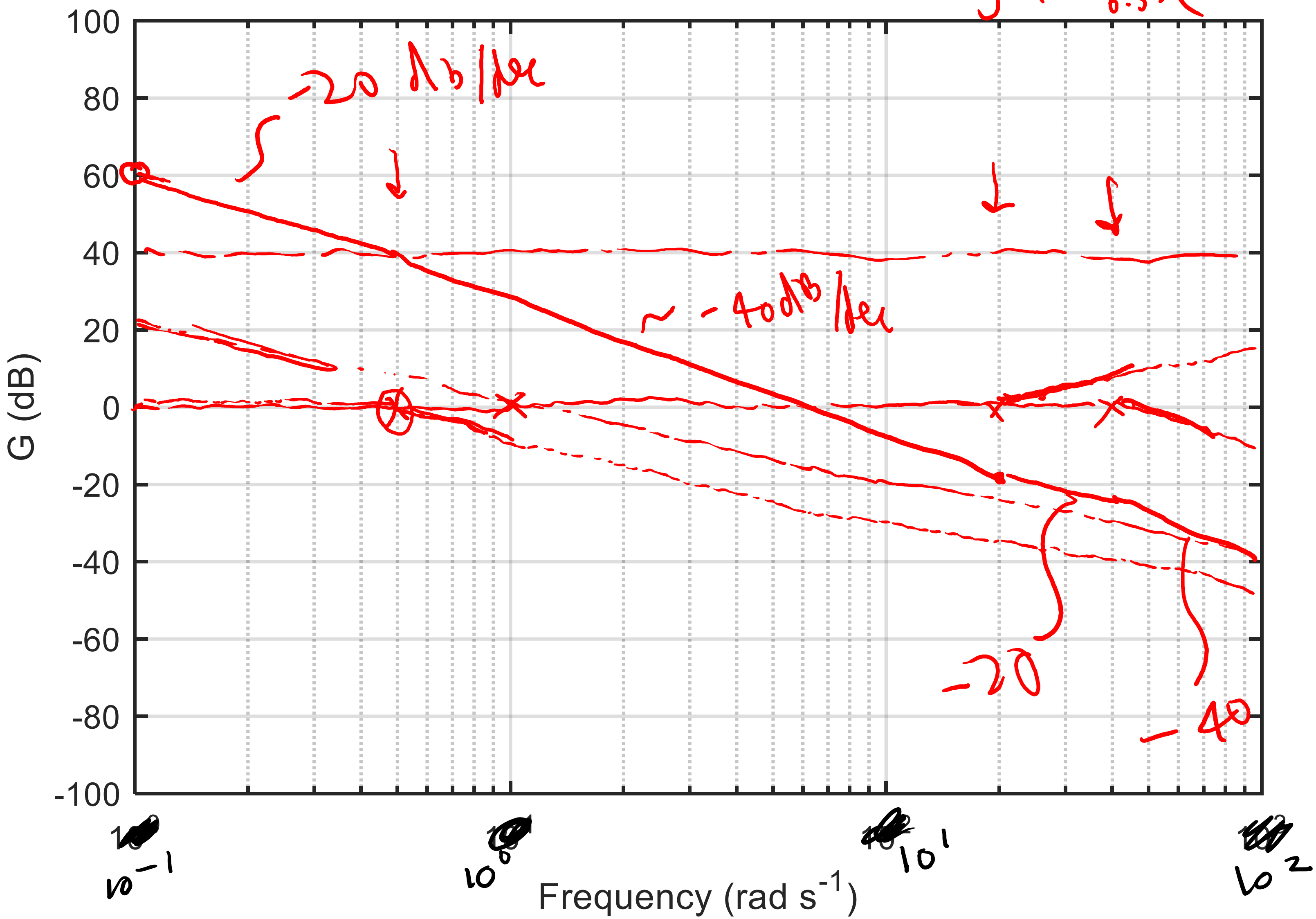
Practice Problem #2

$$H(j,\omega) = \frac{100 * (100 + j\omega)}{(10 + j\omega)(1000 + j\omega)} = \frac{(1 + j\omega/100)}{(1 + j\omega/10)(1 + j\omega/1000)}$$



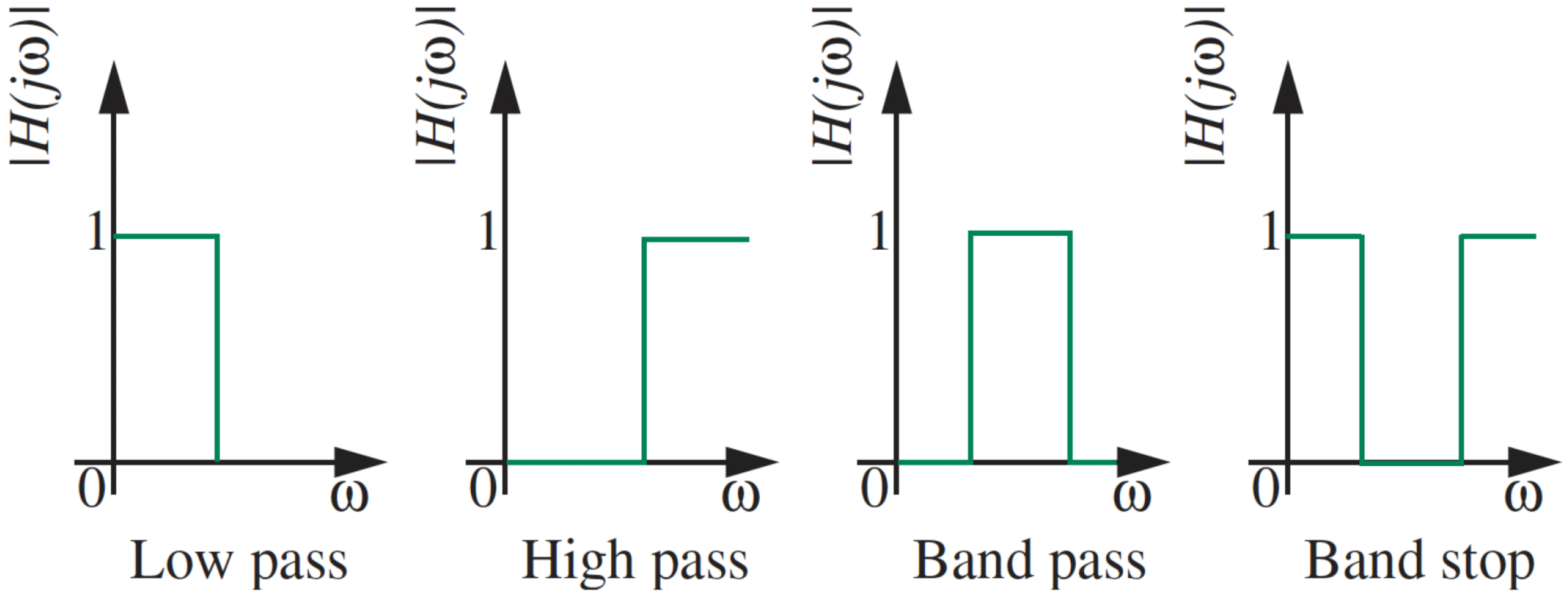
Practice Problem #3

$$H = \frac{200(j\omega + 20)}{j\omega(2j\omega + 1)(j\omega + 40)} = \frac{100(1 + j\omega/20)}{j\omega(1 + \frac{j\omega}{0.5})(1 + j\omega/40)}$$



Filters

Filters



RL and RC circuits

✓ First-order ODEs
✓ Exponential behavior
X Not oscillatory

RLC circuits

✓ Secord-order ODEs
✓ Oscillatory behavior
✓ Damped behavior

RLC Circuits: Electronic Filters

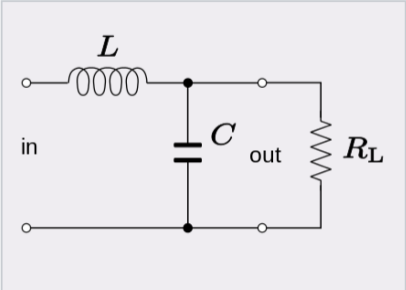


Figure 6. RLC circuit as a low-pass filter

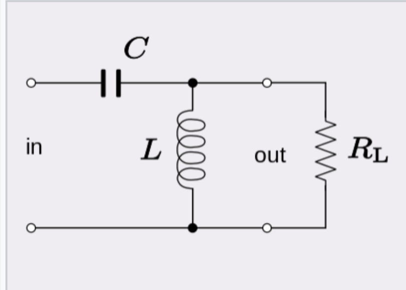


Figure 7. RLC circuit as a high-pass filter

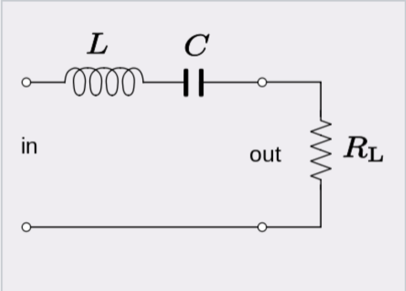


Figure 8. RLC circuit as a series band-pass filter in series with the line

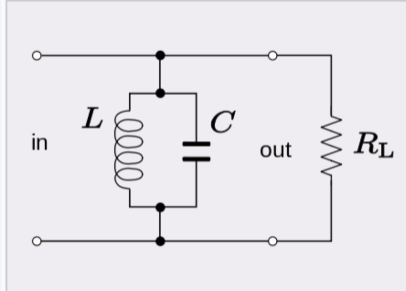


Figure 9. RLC circuit as a parallel band-pass filter in shunt across the line

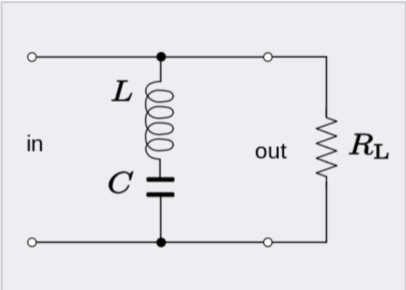


Figure 10. RLC circuit as a series band-stop filter in shunt across the line

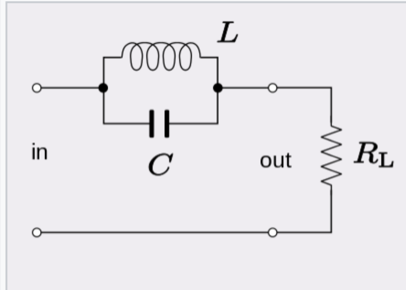


Figure 11. RLC circuit as a parallel band-stop filter in series with the line

- Re-write circuit using impedance model
- Find transfer function as complex amplitude of output divided by complex amplitude of input
- Analyze transfer function asymptotes, zeros, poles, slopes, resonances, etc.
- Plot/sketch on Bode graph

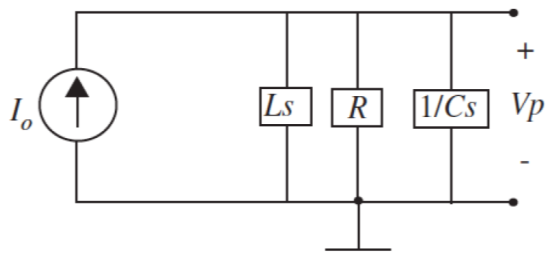
Checking 2nd-Order Terms

Are all 2nd-order circuits resonant?

For a circuit to be resonant, we need to have terms in *H(s)* such as:

$$s^2 + 2\alpha s + \omega_n^2$$

If $Q = \frac{\omega_n}{2\alpha} > 0.5$, the roots of the equation above are complex, and we will have 2nd-order terms in *H(s)*, thus resonant behavior



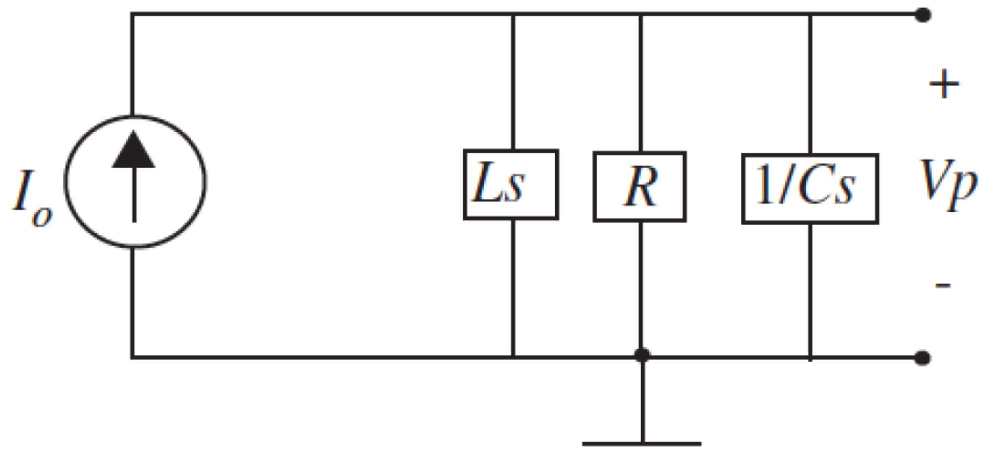
Recall for Parallel RLC

$$\alpha = \frac{1}{2RC}$$

A parallel RLC is not resonant unless the resistance (R) is large enough to meet the following condition:

$$R > 0.5\sqrt{\frac{L}{C}}$$

Resonant Parallel RLC Circuit



We always start with finding the transfer function:

$$H(j\omega) = \frac{V_p}{I} = \frac{1}{\frac{1}{j\omega L} + \frac{1}{R} + j\omega C} = \frac{1}{\frac{1}{R} + j(\omega C - \frac{1}{\omega L})}$$

Sketch Frequency Response: Asymptotic Method

$$H(j\omega) = \frac{V_p}{I} = \frac{1}{\frac{1}{j\omega L} + \frac{1}{R} + j\omega C} = \frac{1}{\frac{1}{R} + j(\omega C - \frac{1}{\omega L})}$$

High frequency asymptote?

$$H(j\omega) = \frac{1}{j\omega C} \rightarrow |H(j\omega)| = \frac{1}{\omega C}, \quad \angle H(j\omega) = -90^\circ$$

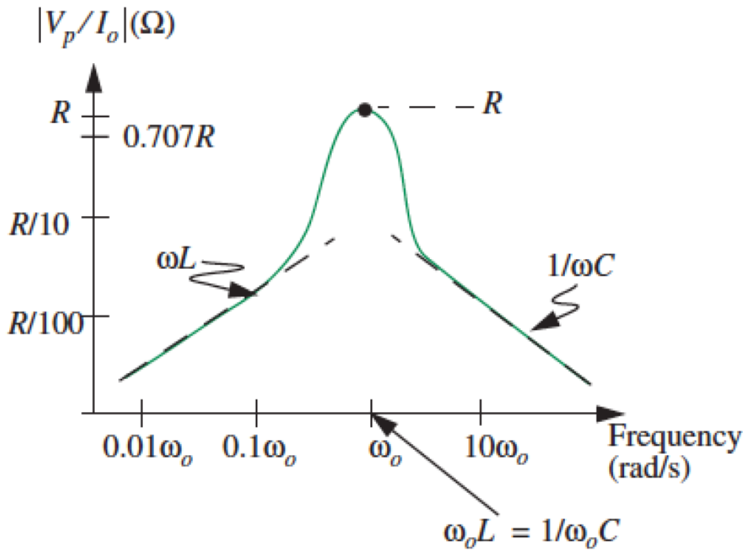
Low frequency asymptote?

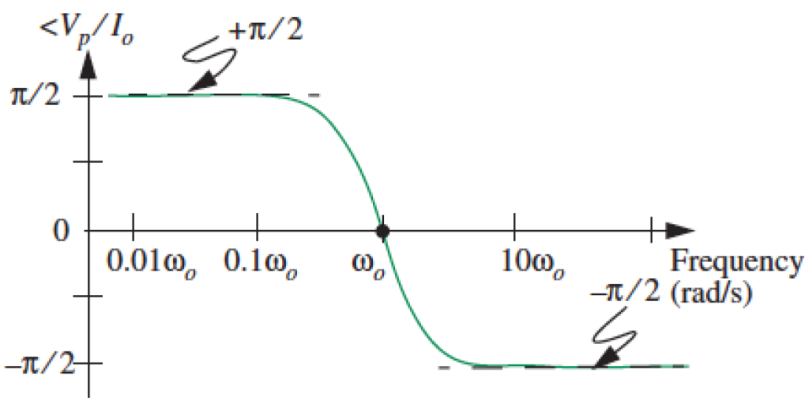
$$H(j\omega) = j\omega L \rightarrow |H(j\omega)| = \omega L, \quad \angle H(j\omega) = 90^\circ$$

The high and low frequency asymptotes intersect at $\omega_\circ = \frac{1}{\sqrt{LC}}$

Parallel RLC: Adding the Resonance Peak

$$H(j\omega) = \frac{V_c}{I} = \frac{1}{\frac{1}{R} + j(\omega C - \frac{1}{\omega L})}$$





Value of $H(j\omega)$ at $\omega_\circ = \frac{1}{\sqrt{LC}}$?

$$\omega_\circ C - \frac{1}{\omega_\circ L} = 0 \rightarrow \quad H(j\omega_\circ) = R$$

Peak Gain

How large is the gain peak at the resonance frequency?

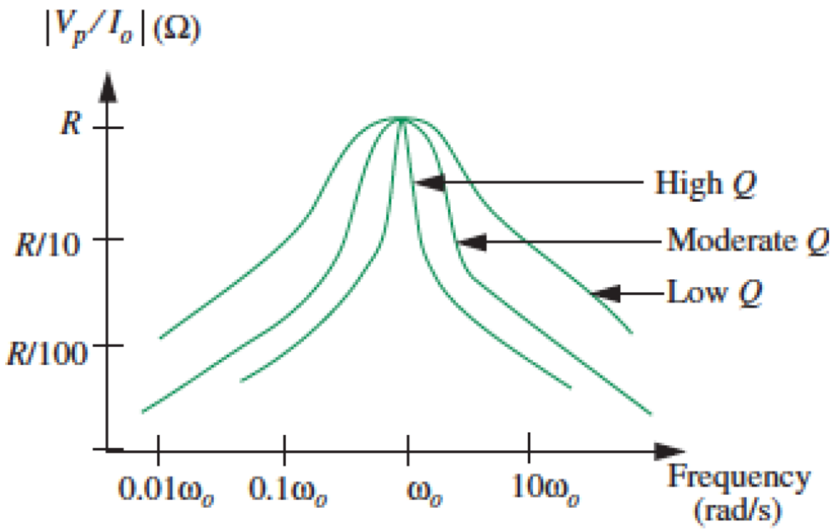
Role of the Q-factor in 2nd-Order Resonant Systems

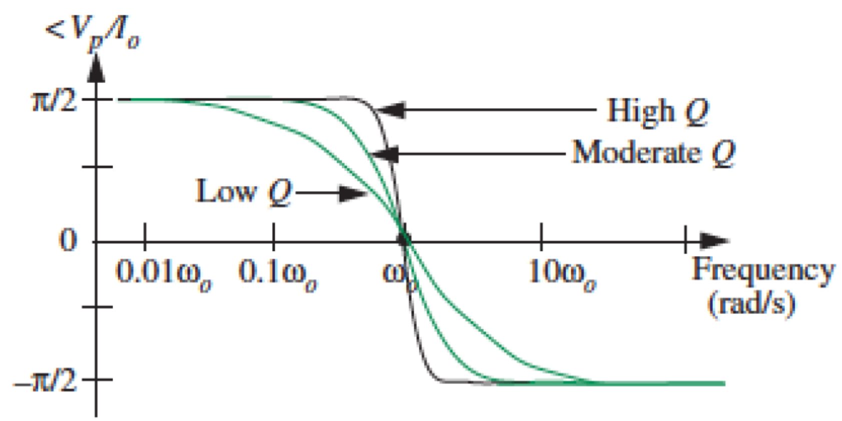
Remember: The Q factor or quality factor is a dimensionless parameter that describes how under-damped an oscillator is:

$$Q = \frac{\omega_0}{2\alpha} = \frac{\tau\omega_0}{2} \approx \text{\# oscillations before energy dissipates}$$

As Q factor ↑, resonance with **larger relative gain** + **smaller bandwidth**

Effect of Q: Frequency Selectivity



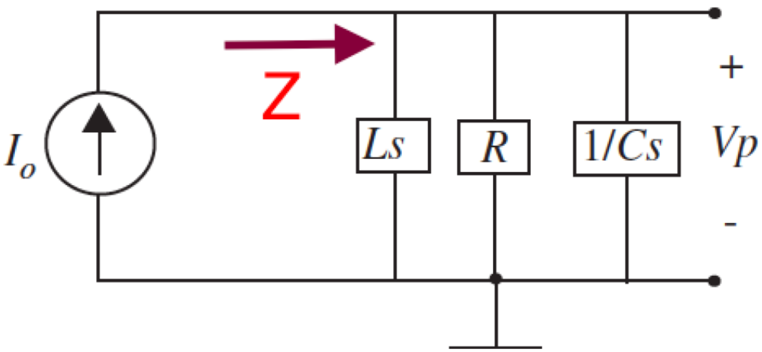


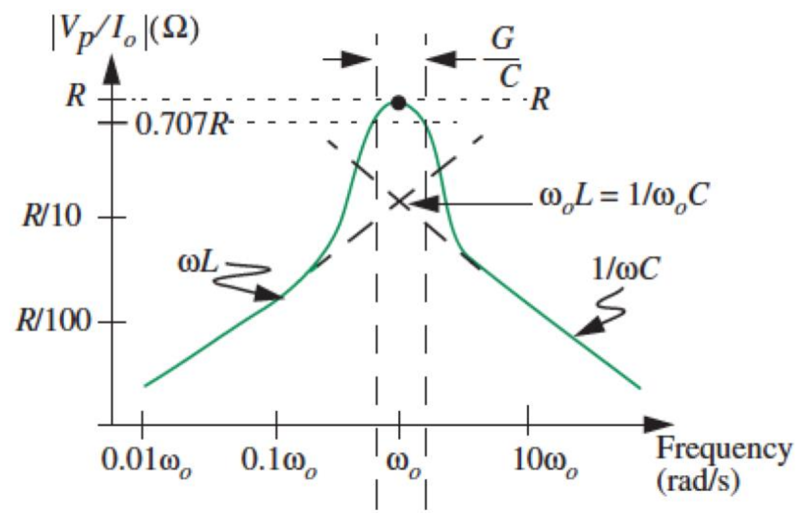
Selectivity is captured through the **bandwidth** of a filter

The bandwidth measures the width of a filter's "passband" (range of freqs that can pass through a filter) → but how do we define passing?

Review: Parallel RLC Filters

$$H(j\omega) = \frac{V_c}{I} = Z = \frac{1}{\frac{1}{R} + j\left(\omega_\circ C - \frac{1}{\omega_\circ L}\right)}$$





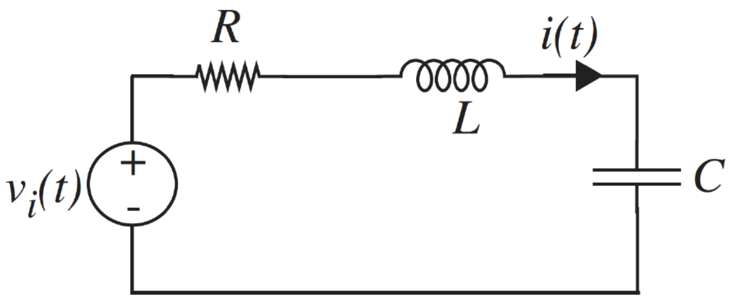
Value of $H(j\omega)$ at $\omega_\circ = \frac{1}{\sqrt{LC}}$?

$$\omega_\circ C - \frac{1}{\omega_\circ L} = 0 \rightarrow \quad H(j\omega_\circ) = R$$

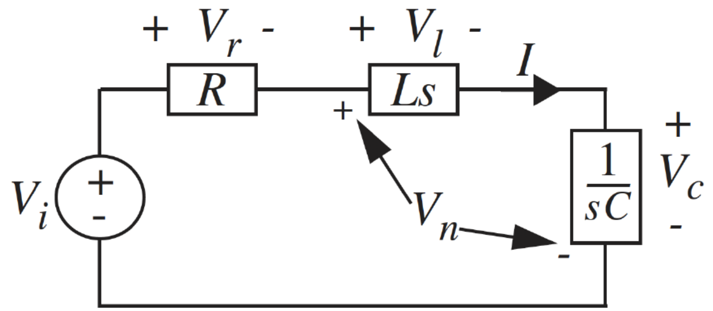
Cut-off frequencies and Bandwidth

$$\omega_{cut} = \pm\frac{G}{2C} + \sqrt{(\frac{G}{2C})^2 + \frac{1}{LC}} \rightarrow \text{Bandwidth} = \frac{G}{C} = \frac{1}{RC} = 2\alpha = \frac{\omega_\circ}{Q}$$

Review: Series RLC Circuit



(a) Circuit

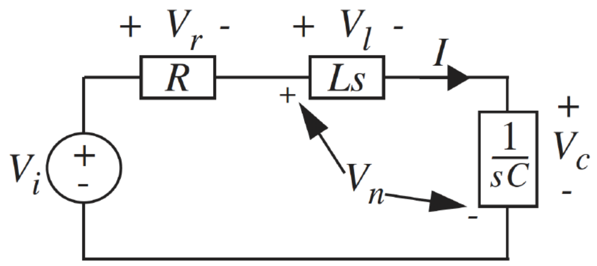


(b) Impedance Model

What we will learn:

- Depending on where the output is taken, the series and parallel RLC resonant circuits can be used as filters of various types.
- The higher the Q factor of the circuit, the higher the selectivity.

Complex Amplitude of the Current

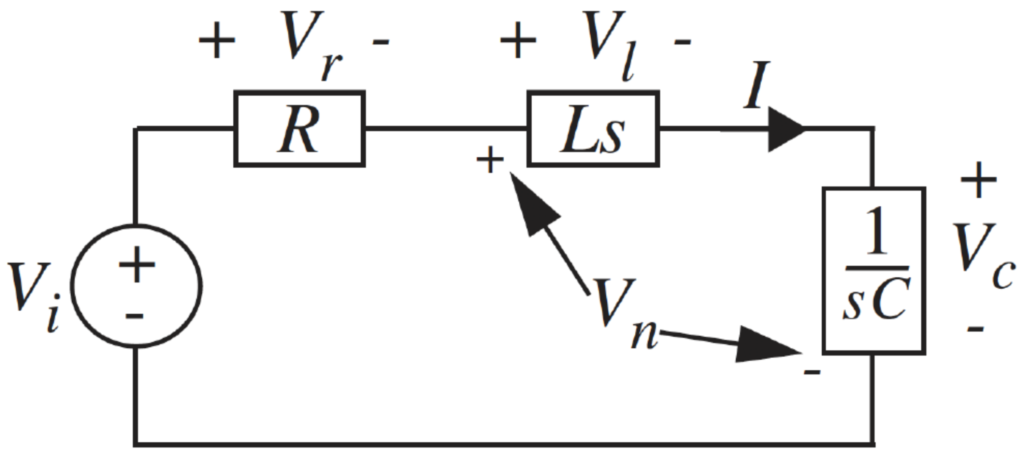


$$\begin{aligned} I &= \frac{V_i}{R + Ls + 1/Cs} \\ &= \frac{(s/L)V_i}{s^2 + sR/L + 1/LC} \\ &= \frac{(s/L)V_i}{s^2 + 2\alpha s + \omega_\circ^2} \end{aligned}$$

$$\omega_\circ = \sqrt{1/LC}, \quad \alpha = \frac{R}{2L}$$

We will assume the second order term has complex roots, e.g.,
$R = 1\Omega, L = 1\mu H, C = 1\mu F \rightarrow \omega_\circ = 10^6 rad/s, \alpha = 5 \times 10^5 s, Q = 1$

Output #1: Resistor Voltage

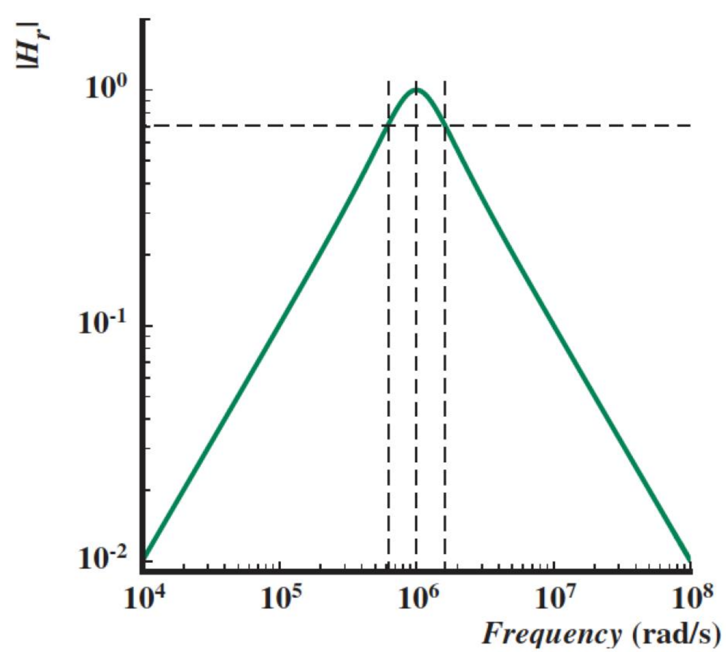


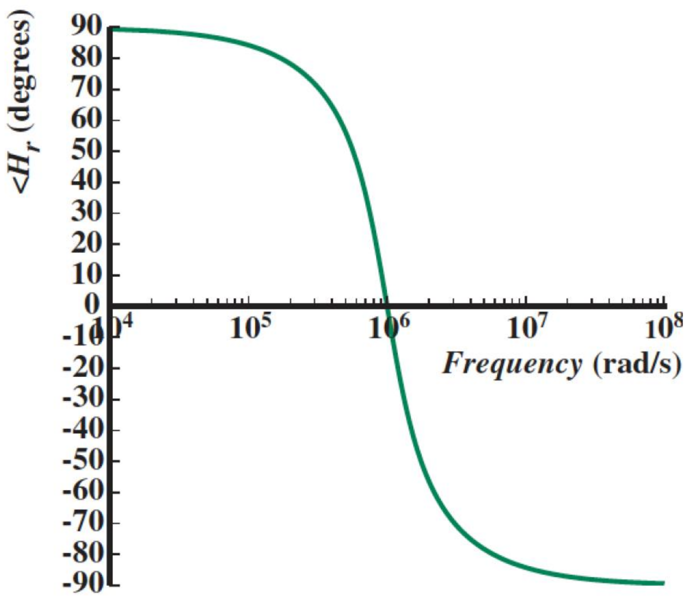
$$V_r = IR = \frac{\frac{sR}{L} V_i}{s^2 + 2\alpha s + \omega_\circ^2}$$

$$H_r(s) = \frac{V_r}{V_i} = \frac{\frac{sR}{L}}{s^2 + 2\alpha s + \omega_\circ^2} = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_\circ^2}$$

Output #1: Resistor Voltage

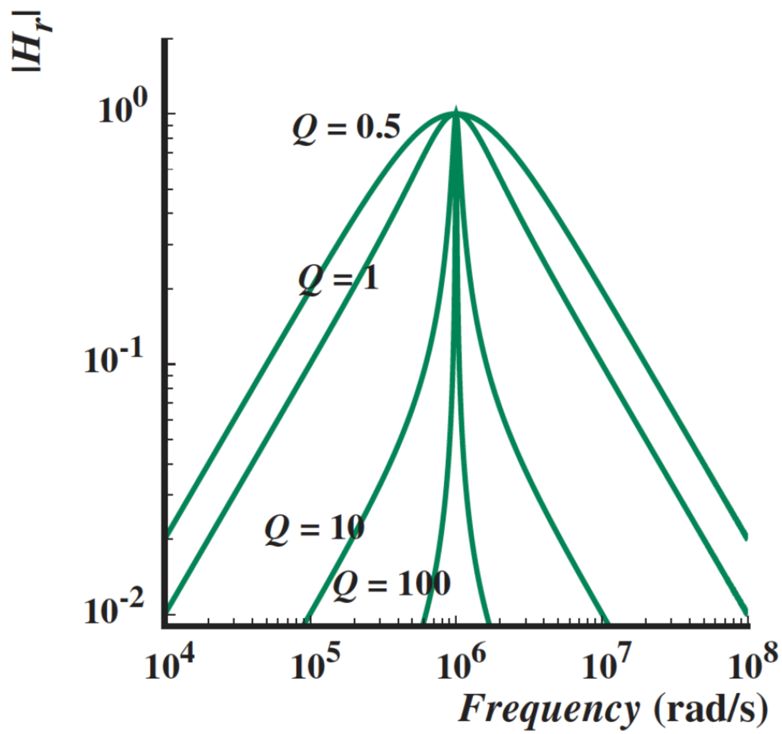
For plots:
$R = 1\Omega, L = 1\mu H, C = 1\mu F \rightarrow \omega_\circ = 10^6 rad/s, \alpha = 5 \times 10^5 s, Q = 1$

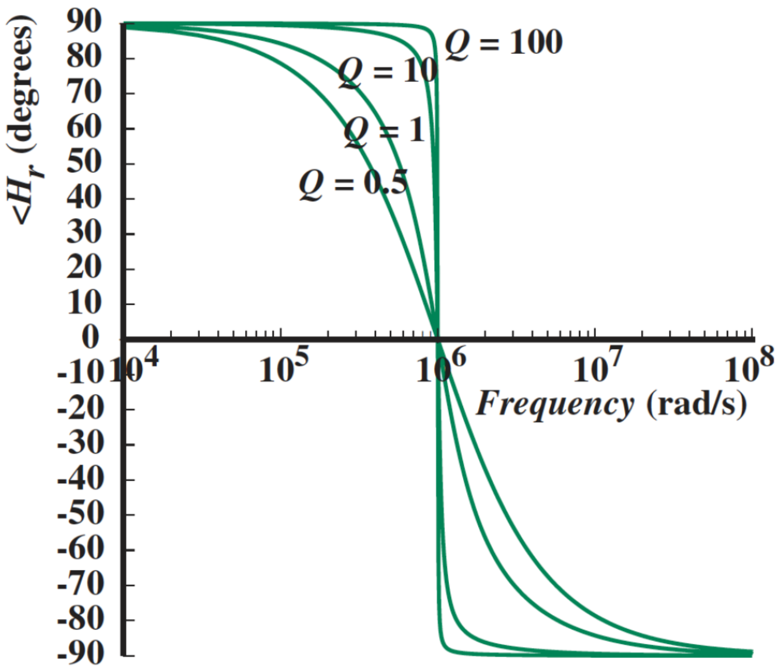




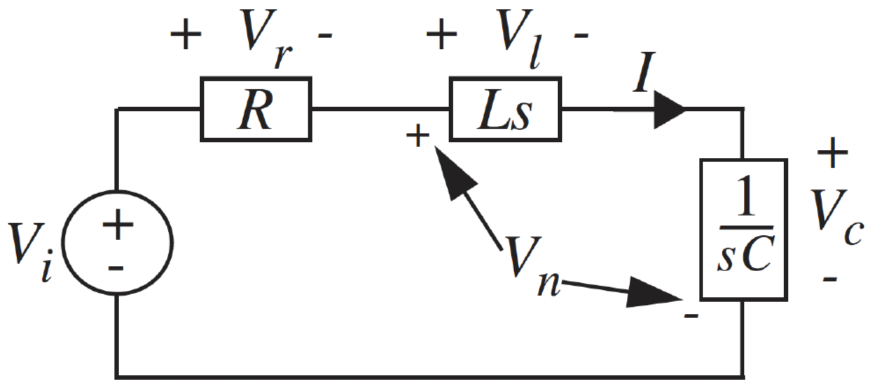
Output #1: Resistor Voltage

Notice that the maximum gain is always equal to 1





Output #2: Capacitor Voltage



$$V_c = \frac{I}{sC} = \frac{\frac{1}{LC} V_i}{s^2 + 2\alpha s + \omega_\circ^2}$$

$$H_c(s) = \frac{V_c}{V_i} = \frac{\frac{1}{LC}}{s^2 + 2\alpha s + \omega_\circ^2} = \frac{\omega_\circ^2}{s^2 + 2\alpha s + \omega_\circ^2}$$

Let's Follow Our General Set of Rules

$$H(s) = \frac{V_r}{V_i} = \frac{\omega_\circ^2}{s^2 + 2\alpha s + \omega_\circ^2}$$

1. Sketch the approximate frequency response, using Bode plots or asymptotes. Be careful to clarify units on the plots;
2. Determine the filter type from approximate sketch (lowpass, highpass, bandpass, bandstop);
3. If bandpass or bandstop $\rightarrow$ Calculate the exact gain at resonance frequency and add in the "peak" in sketch;

$$|H(j\omega_\circ)| = \frac{\omega_\circ^2}{2\alpha\omega_\circ} = Q \quad (20\log Q \ dB)$$

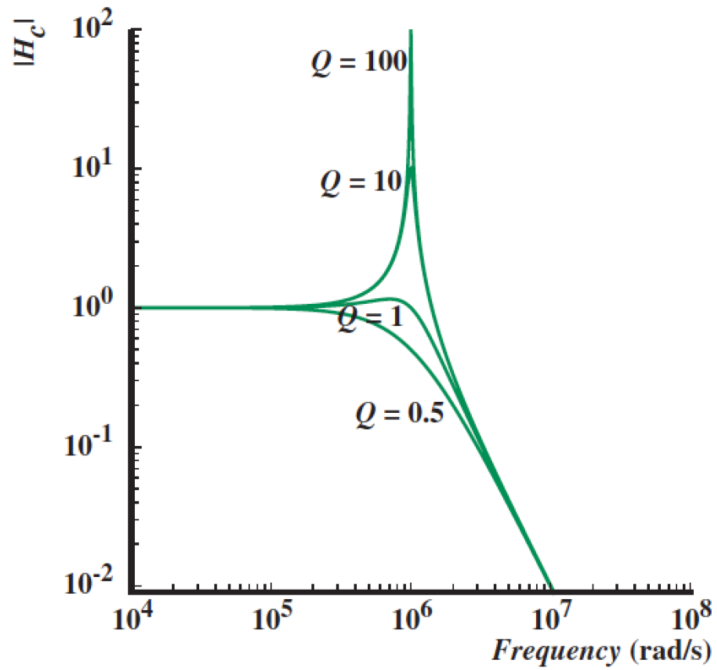
4. Calculate filter bandwidth (by calculating cut off freqs).

$$\text{Bandwidth} \approx \omega_\circ$$

5. Exact BW? Solve $|H(j\omega_c)| = \frac{1}{\sqrt{2}}|H(j0)| = \frac{1}{\sqrt{2}}$

The Gain at Resonance Frequency

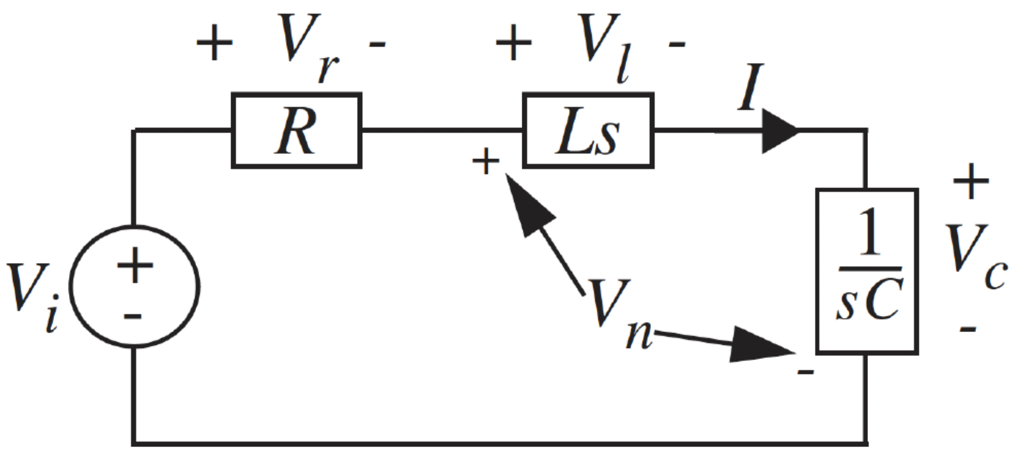
Let's look at the gain:



High Q circuits are not useful low-pass filters (as they produce voltages that are significantly higher than the input when $\omega \approx \omega_\circ$)

Need a good low-pass filter? Choose $Q = 1$

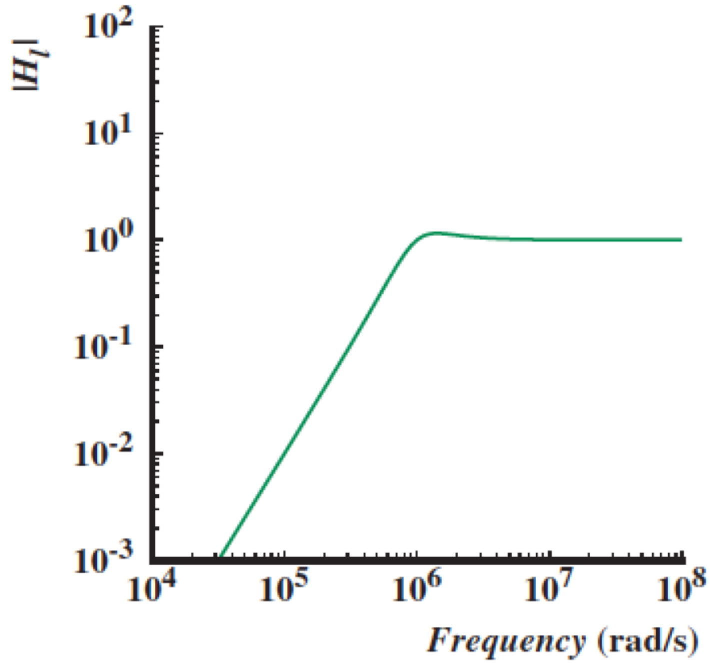
Output #3: Inductor Voltage

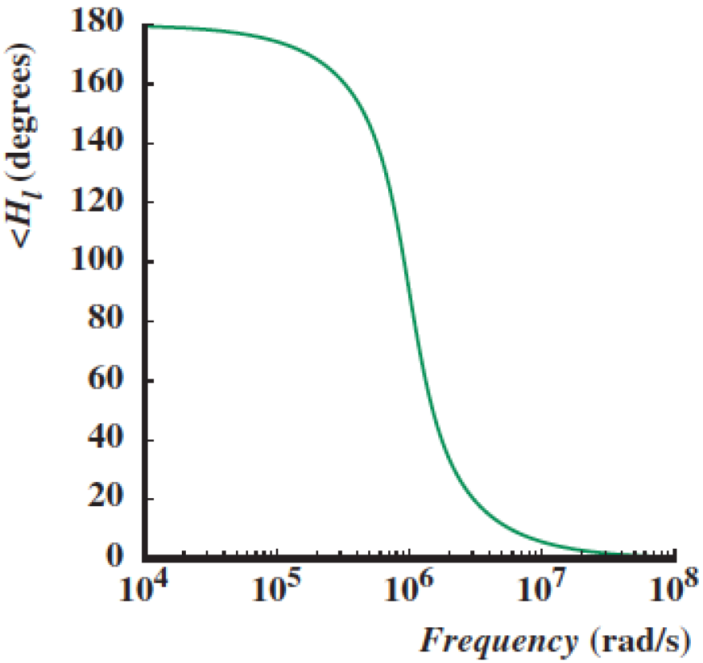


$$V_L = IsL = \frac{s^2 V_i}{s^2 + 2\alpha s + \omega_\circ^2}$$

$$H_L(s) = \frac{V_L}{V_i} == \frac{s^2}{s^2 + 2\alpha s + \omega_\circ^2}$$

Output #3: Inductor Voltage

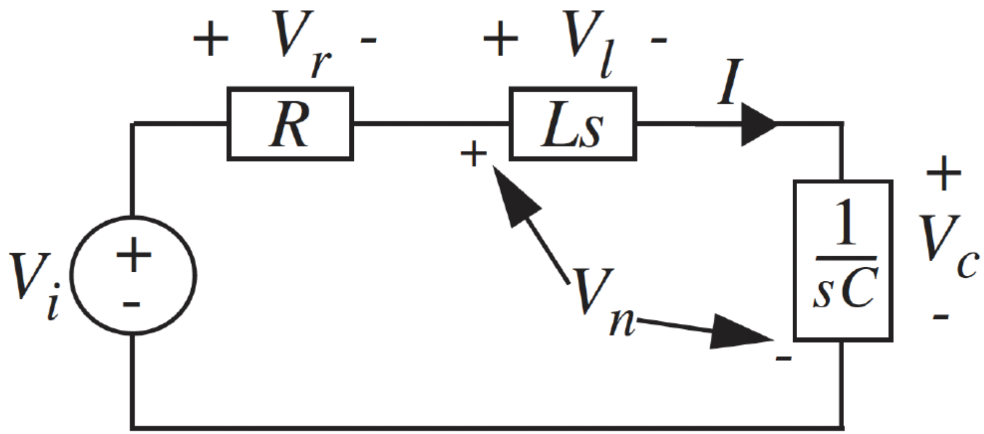




$$H_L(s) = \frac{V_L}{V_i} = \frac{s^2}{s^2 + 2\alpha s + \omega_\circ^2}$$

Need a good high pass filter? Select $Q = 1$

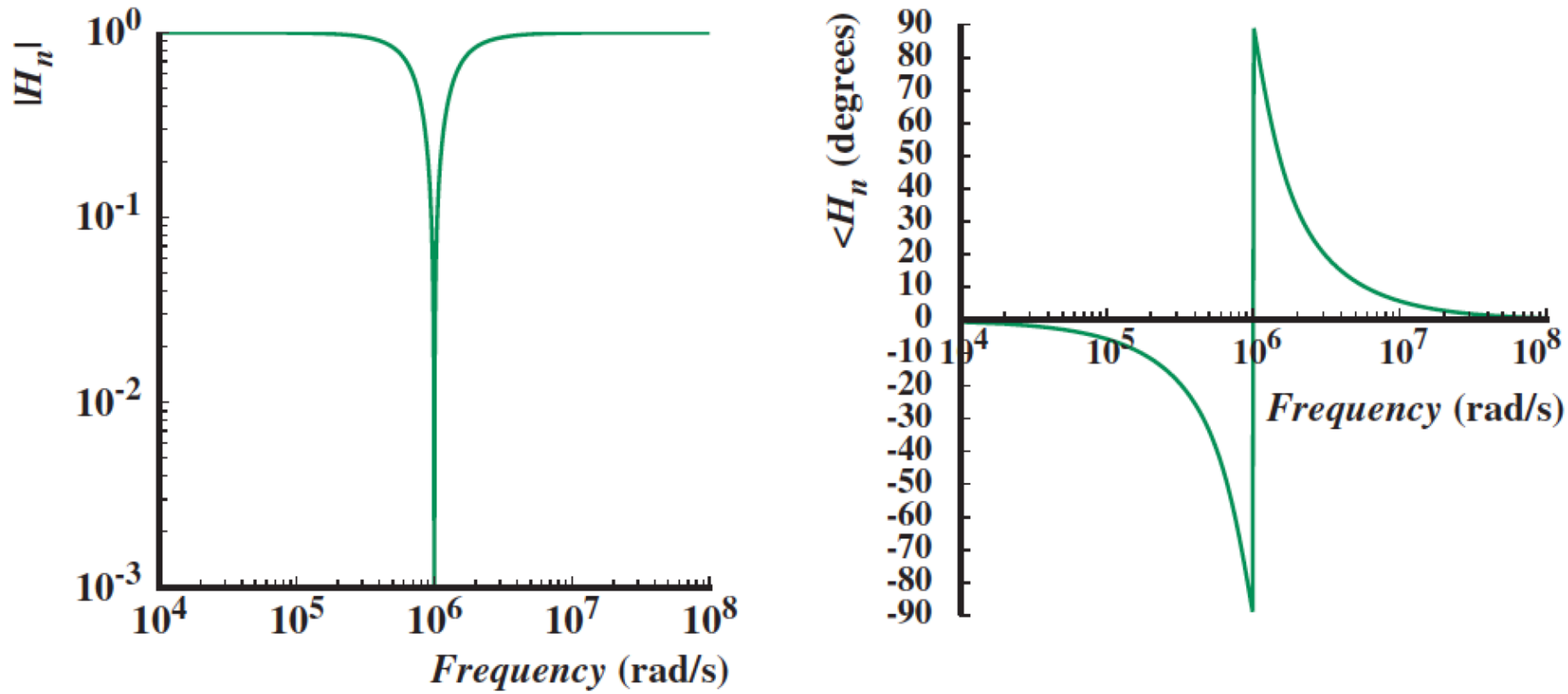
Output #4: Inductor & Capacitor Voltage



$$V_n = I(sL + \frac{1}{sC}) = \frac{(s^2 + \frac{1}{LC})V_i}{s^2 + 2\alpha s + \omega_\circ^2}$$

$$H_n(s) = \frac{V_n}{V_i} = \frac{(s^2 + \frac{1}{LC})}{s^2 + 2\alpha s + \omega_\circ^2}$$

Output #4: Inductor & Capacitor Voltage



$$H_n(s) = \frac{V_n}{V_i} = \frac{(s^2 + \omega_\circ^2)}{s^2 + 2\alpha s + \omega_\circ^2}$$

Recap: All Different Filter Types Using Same Circuit