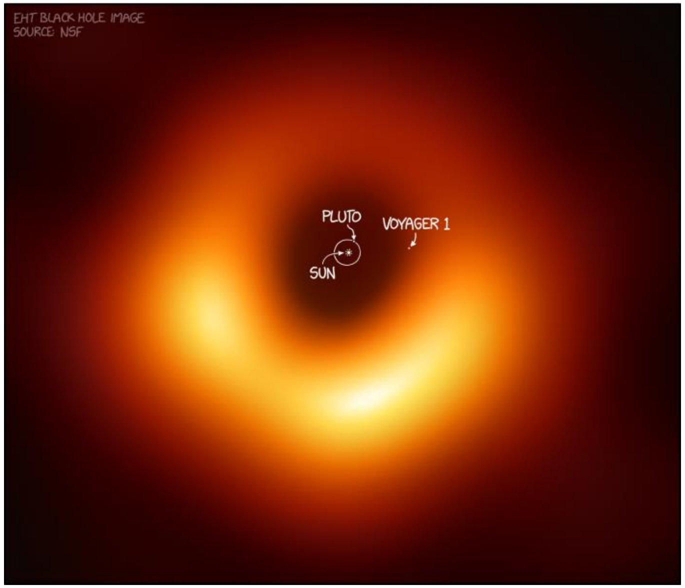
ECE 10C
Fall 2020
Slide Set 4
Instructor: Galan Moody
TA: Kamyar Parto

SIZE COMPARISON:
THE M87 BLACK HOLE
AND
OUR SOLAR SYSTEM



Last week
- Driven RLC circuits

This week
- Sinusoidal input to RLC
- Impedance method

Important Items:

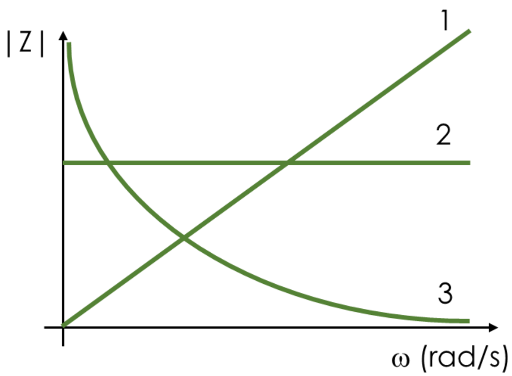
- **HW #3 due Friday, 11/6**
- **Lab #3 due 11/6**
- **Midterm next week**
  - **Format TBD**
- **HW 2 grading by Thurs**

Quiz Time!

Q1 [2 points]. What is the magnitude and phase of the complex number $Z = 3e^{j0} + 4e^{j\pi/2}$?

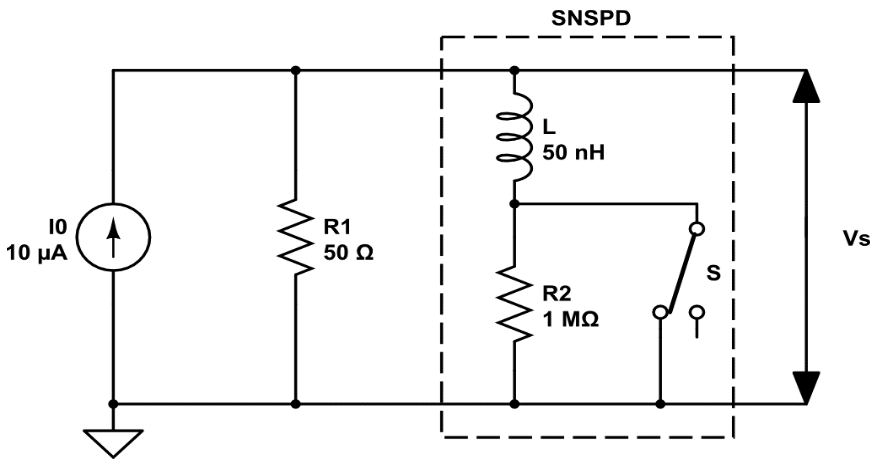
(a) $|Z| = 4,\ \varphi = \tan^{-1}(4/3)$
(b) $|Z| = 5,\ \varphi = \tan^{-1}(3/4)$
(c) $|Z| = 5,\ \varphi = \tan^{-1}(4/3)$

Q2 [3 points]. In the figure below, the three curves represent the frequency dependence of the impedance for a capacitor, an inductor, and a resistor. Label which curve corresponds to which element. [**HINT #1:** the impedance Z of an element can be thought of as its "resistance", which allows us to apply a generalization of Ohm's law V = I×Z to each element.] [**HINT #2:** how do the capacitor, inductor, and resistor behave at DC, *i.e.* as $\omega \to 0$ rad/s?]



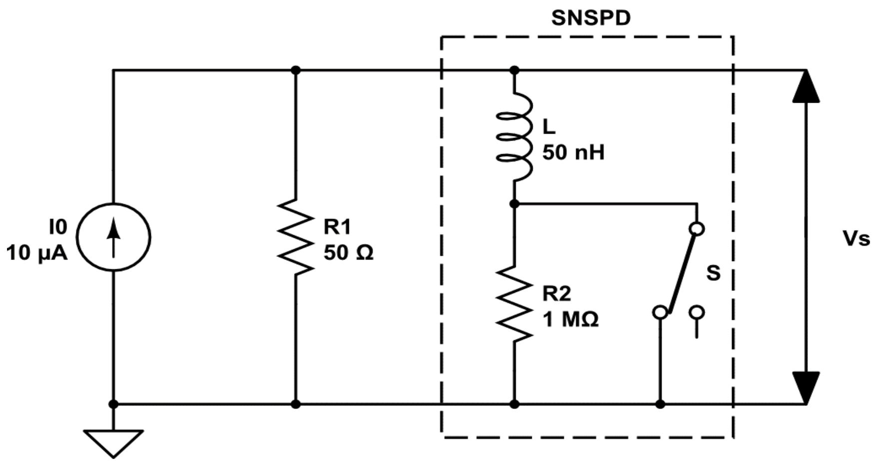
(a) 1 = R, 2 = L, 3 = C
(b) RCL
(c) LRC
(d) LCR
(e) CRL
(f) CLR

HW 2 Problem 4



1. Find i(0-) and v(0-).
2. Find i(0+) and v(0+). Find v(t) and t_C.
3. Find v(inf), t_r. Find v(t). Find max rate.
4. Find optimal L for shortest t_r. Find rate.

HW 2 Problem 4



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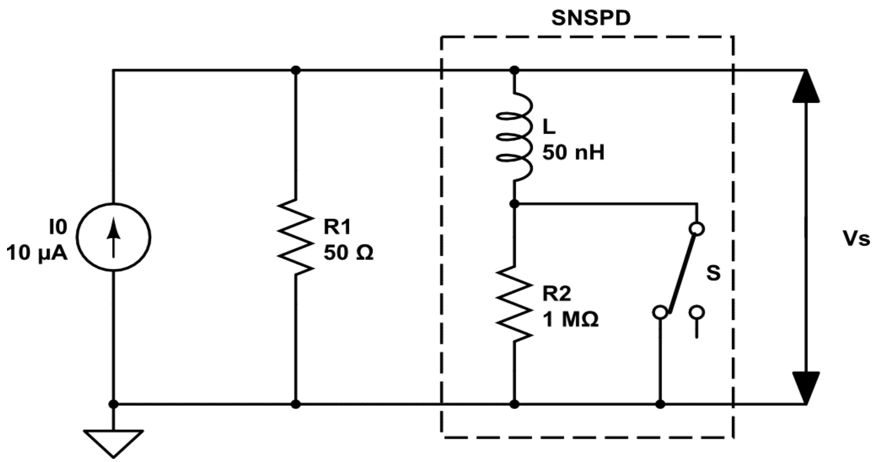
4. Find optimal L for shortest t_r. Find rate.

1. $t<0$, switch is closed. L = short

Thus $i_L(0^-) = I_0 = 10\mu A$.

$V_s(0^-) = 0V$ since $i_{R_1}(0^-) = 0A$; L = short.

HW 2 Problem 4



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3. Find v(inf), t_r. Find v(t). Find max rate.
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2. $t=0$, switch opens.

$i_L(0^+) = i_L(0^-) = 10\,\mu A$.

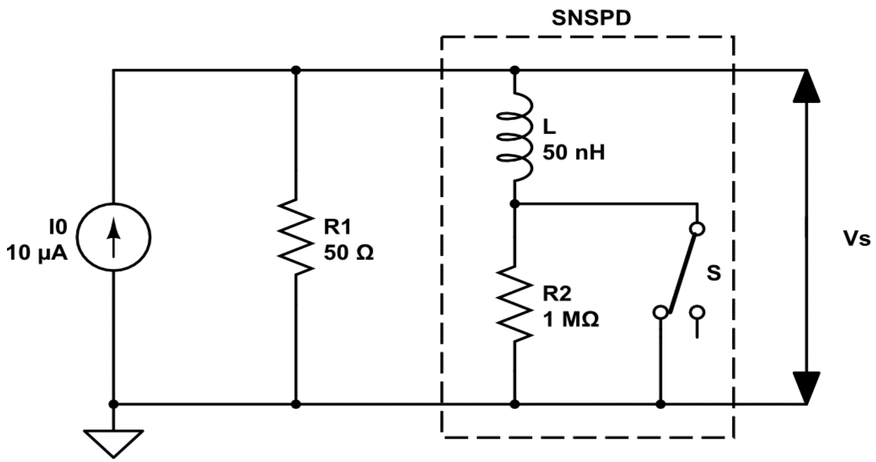
$i_L(\infty) = 0A$ since $R_2 >> R_1$

$V_S = V_{R1}$. Thus we need to find $V_{R1}(t)$.

$V_{R1}(0^+) = 0V$ (no current initially flowing).

$V_{R1}(\infty) = 10\,\mu A \cdot R_1 = 500\,\mu V$.

HW 2 Problem 4



1. Find i(0-) and v(0-).
2. Find i(0+) and v(0+). **Find v(t) and t_c.**
3. Find v(inf), t_r. Find v(t). Find max rate.
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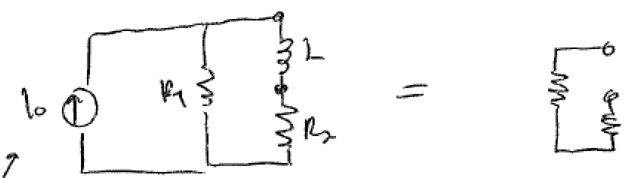
Current will divert from L branch to R_1 branch. What is the time constant? Write thevenin equivalent resistance:



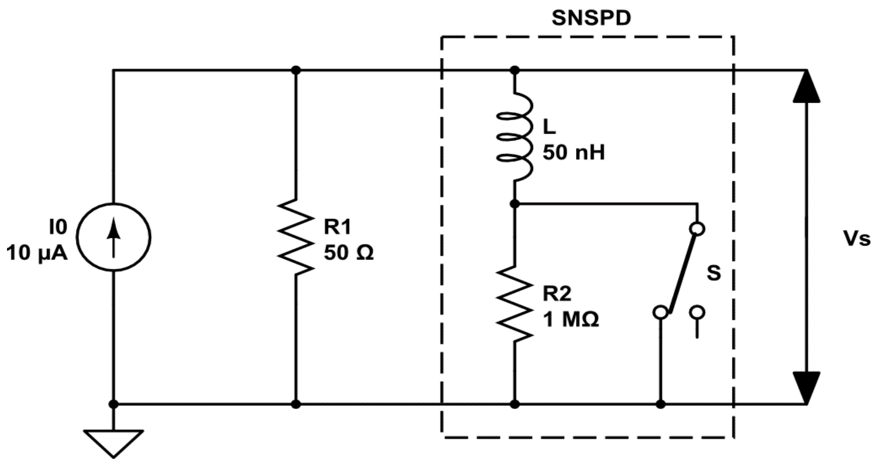
$\rightarrow R_{th} = R_1 + R_2 \sim 1\,M\Omega$.

set to open circuit

Thus: $t_c = \frac{L}{R_{th}} = \frac{50\,nH}{1\,M\Omega} = 50\,fs$.

t_c is time it takes to divert current to R_1 branch. We expect the exponential decay of $i_L(t)$ is reflected in exponential rise of V_{R_1} & thus V_s.

Thus: $V_s(t) = 500 - 500\,e^{-t/t_c}$ μV.



1. Find i(0-) and v(0-).
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3. **Find v(inf), t_r. Find v(t). Find max rate.**
4. Find optimal L for shortest t_r. Find rate.

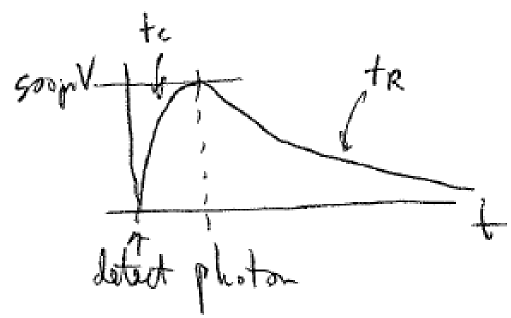
(3) Switch closes. Can ignore R_2 now. Call this time $t=0$.

$V_{R1}(0^+) = V_{R1}(0^-) = 500\,\mu V.$

$V_{R1}(\infty) = 0V.$

$t_R = \frac{L}{R_1} = \frac{50nH}{50\Omega} = 1ns.$

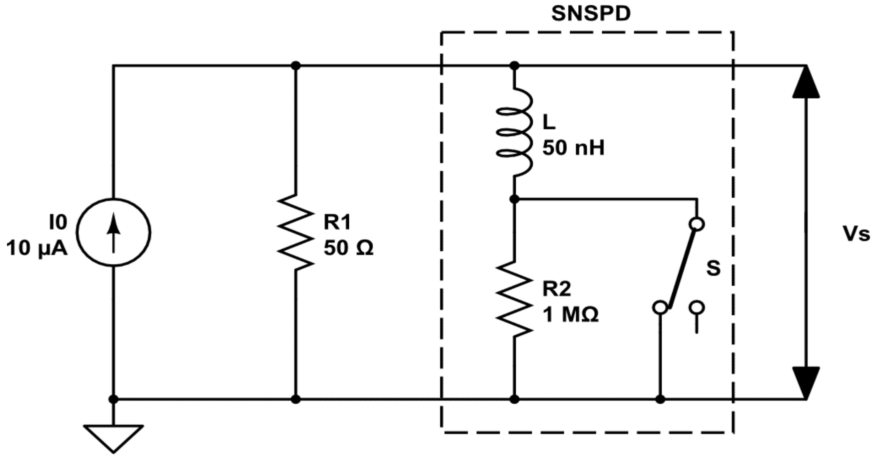
$V_S(t) = V_{R1}(t) = 500\, e^{-t/t_R}\ \mu V.$



$t = 3t_R = 3ns \rightarrow$ max detection

rate $= 1/t = 333.3$ MHz.

HW 2 Problem 4



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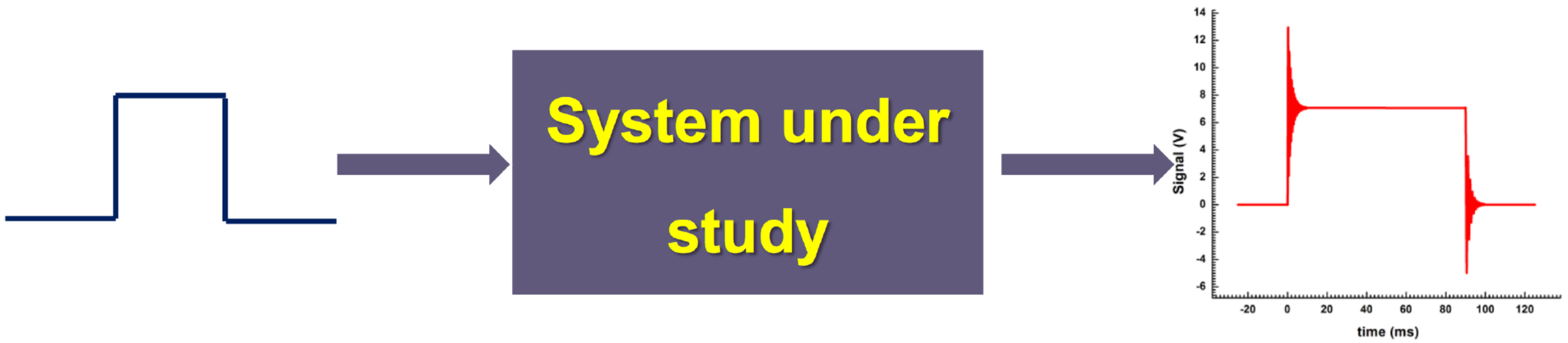
2. Find i(0+) and v(0+). Find v(t) and t_C.
3. Find v(inf), t_r. Find v(t). Find max rate.
4. **Find optimal L for shortest t_r. Find rate.**

(4) If $L = 10nH$, then $t_R = 200\,ps = t_d$. Then count rate max $= \frac{1}{3 \cdot t_R} = 1.67$ GHz.

Transient Analysis

We care about the exact response of a system at all times *t*:



input signal

output signal

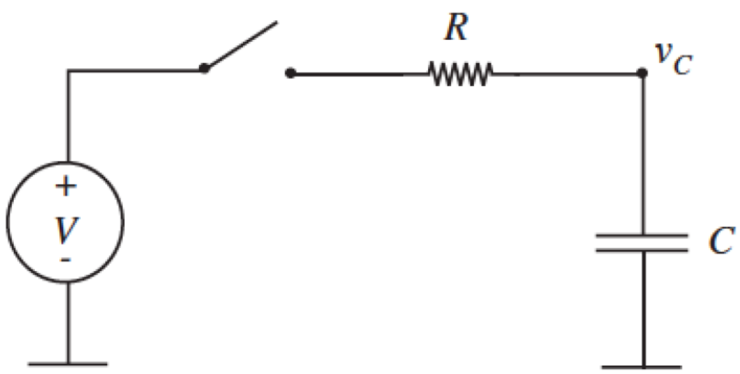
- We can use what we've learned with 1st and 2nd order ODEs to calculate the system's response.
- A different method: impedances. We treat a complex circuit as a bunch of elements that appear similar to resistors.

Transient Analysis

What we'll go over today:

1. Sinusoidal input to RC circuit
2. Expressing source and response as complex numbers
3. Transient + steady state solution
4. Concept of "impedance"
    1. **Allows us to use a generalized version of Ohm's law for R's, L's, and C's. Easier method to analyze circuits, and no more (almost) ODEs!**
    2. **Need to understand how to get to these generalized expressions and impedance method, then we can use it to analyze complex circuits.**

Let's Go Back to Our RC Circuit…



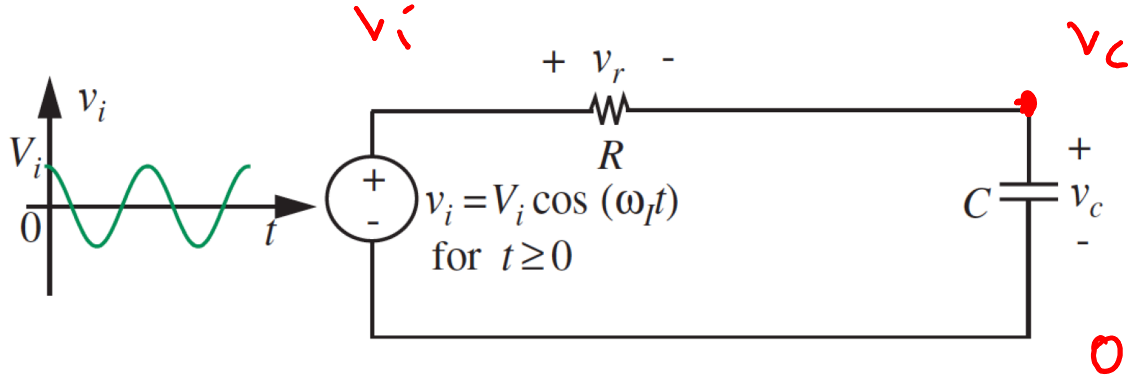
We want to find $v_c(t)$ when $v(t) = V*u(t)$ and $v_c(0-) = V_0$.

Solution Steps

1. Use node method to find the ODE describing v_c
2. Find homogeneous solution v_{ch} (set the source to zero)
3. Find the particular solution v_{cp}
4. The total solution is a sum of the homogeneous solution and the particular solution. Use the initial conditions to solve for the unknown constants.

Sinusoidal Input – A Review (covered in Chpt 10)

We can use this same method to solve for $v_c(t)$ with a sinusoidal input:



Node analysis: $\frac{v_i - v_c}{R} - C\frac{dv_c}{dt} = 0 \rightarrow v_i = v_c + RC\frac{dv_c}{dt}$

$v_c = v_{ch} + v_{cp}$ $\qquad$ $v_{ch} = A_1 e^{-t/RC}$

Particular solution?

$$V_i\cos(\omega t) = v_{cp} + RC\frac{dv_{cp}}{dt}$$

Sinusoidal Input – A Review (covered in Chpt 10)

For particular solution, "guess" sinusoidal response:

Our guess: v_{cp} must be some combination of sines and cosines

$$v_{cp} = K_1 \cos(\omega t) + K_2 \sin(\omega t)$$

$$V_i \cos(\omega t) = v_{cp} + RC\frac{dv_{cp}}{dt}$$

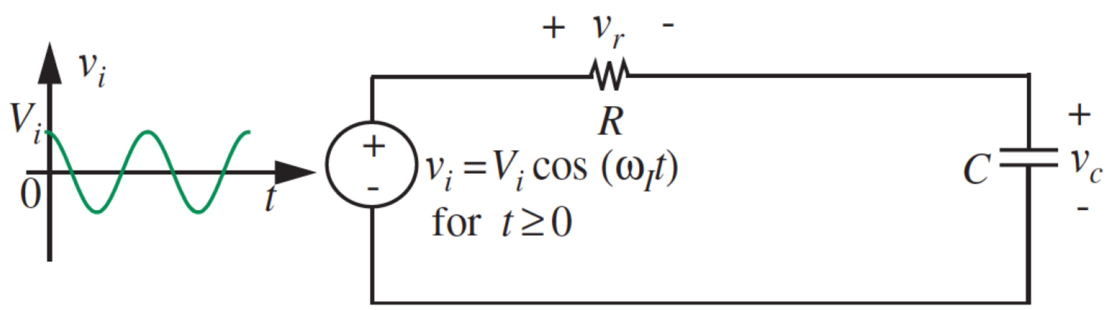
$$V_i \cos(\omega t) = K_1 \cos(\omega t) + K_2 \sin(\omega t) + RC\omega K_2 \cos(\omega t) - RC\omega K_1 \sin(\omega t)$$

Remaining Steps (see page 558)

1. We find K_1 and K_2 by equating sine and cosine terms
2. We use trig equalities to convert solution to a single cosine
3. We find A_1 from homogeneous sol'n using the initial conditions

Sinusoidal Input – A Review (covered in Chpt 10)

We can use this same method to solve for $v_c(t)$ with a sinusoidal input:



$$v_c(t) = A_1 e^{-t/RC} + \frac{V_i}{\sqrt{1 + (\omega RC)^2}} \cos(\omega t + \varphi)$$

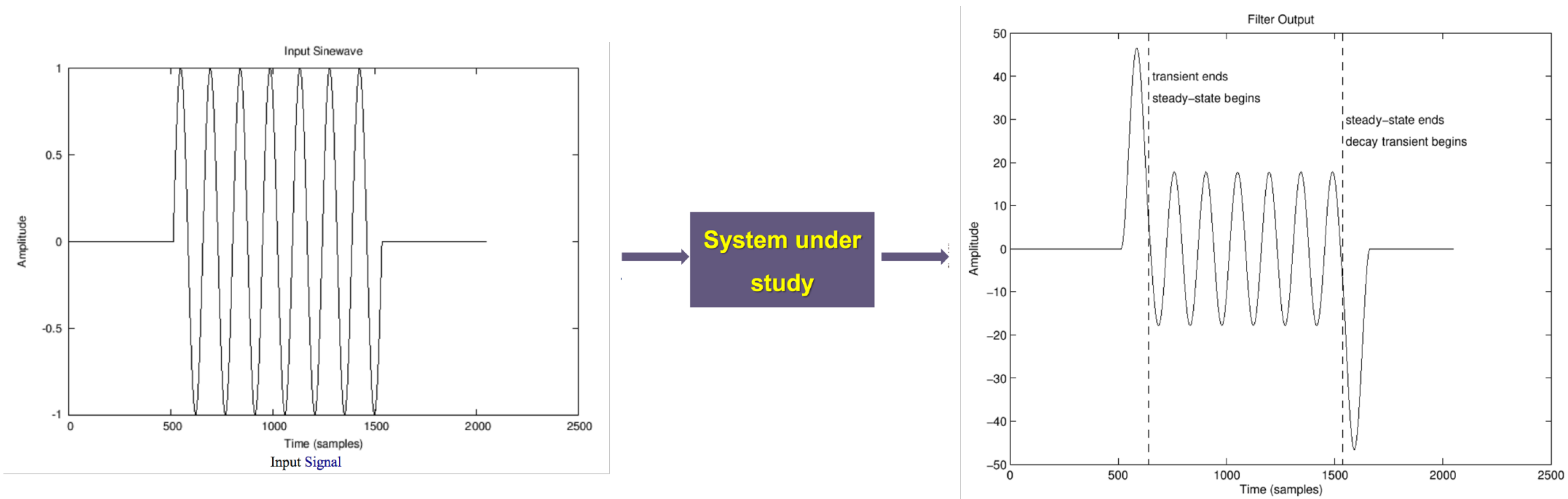
transient solution (v_{ch})

steady state solution (v_{cp})

Transient vs. Steady State Analysis

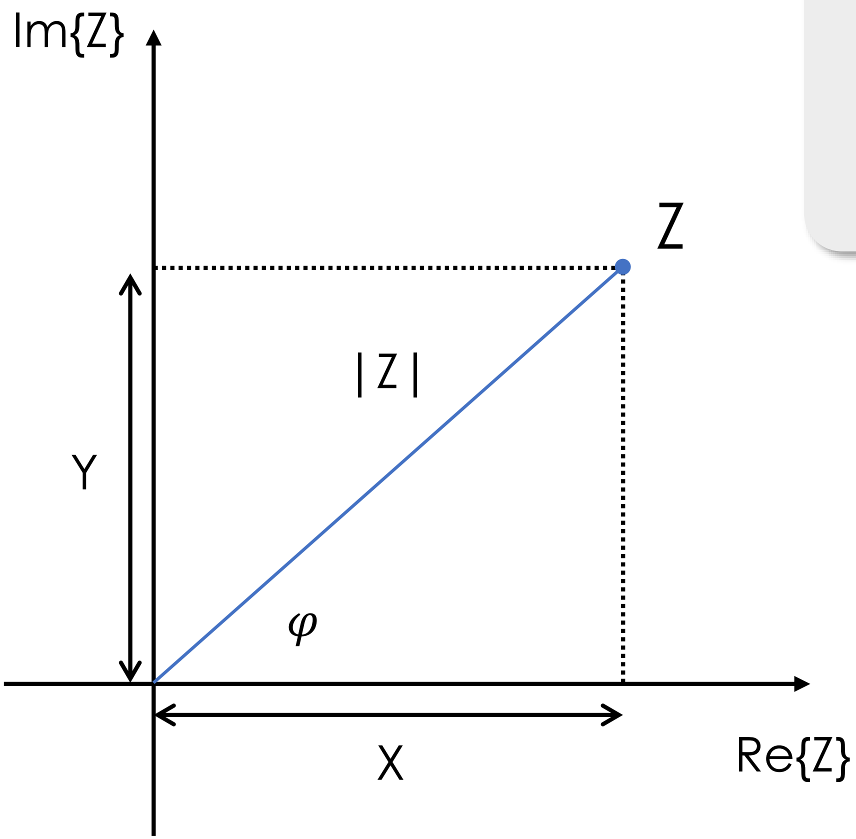
The term transient response and steady state response arise naturally in the context of sinusoidal inputs

When the input sine wave is switched on, the system takes a while to "settle down" to a perfect sine wave response at the same frequency…



$$v_c(t) = A_1 e^{-t/RC} + \frac{V_i}{\sqrt{1+(\omega RC)^2}}\cos(\omega t + \varphi)$$

Let's Make Our Lives Easier with Complex Numbers



- A complex number in Cartesian form:

Z = X + jY = |Z| $e^{j\phi}$ (polar form)
Re(Z) = X
Im(Z) = Y

$$Z^* = X - jY = |Z|e^{-j\varphi}$$

$$|Z| = \sqrt{Z \times Z^*} = \sqrt{X^2 + Y^2}$$

$$\varphi = \tan^{-1}(Y/X)$$

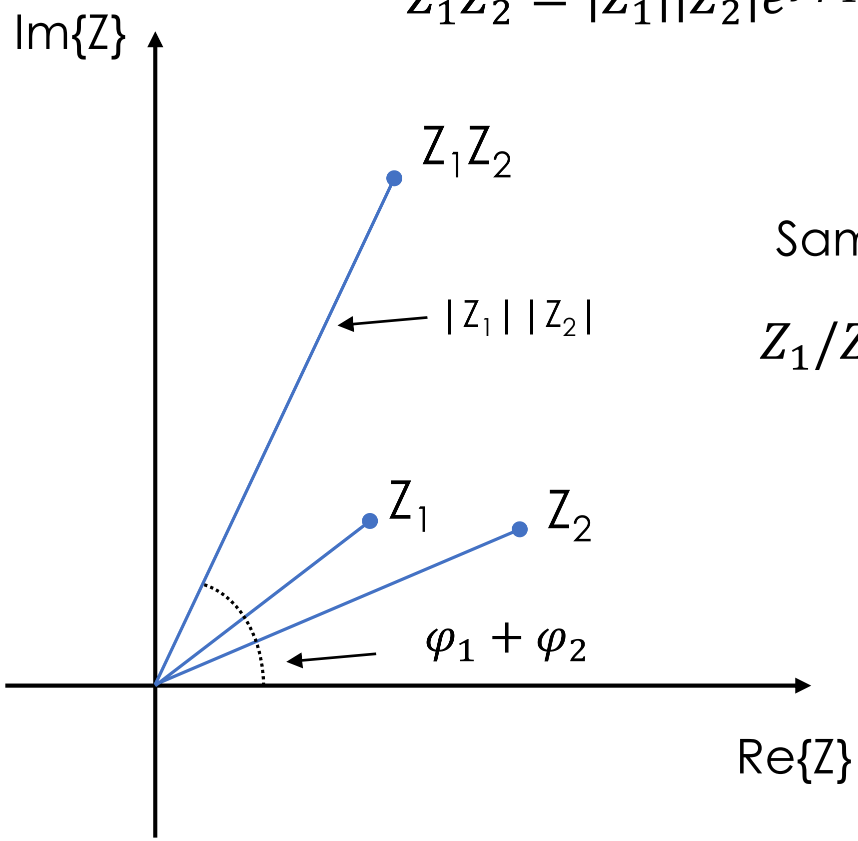
$$X = |Z|\cos(\varphi)$$

$$Y = |Z|\sin(\varphi)$$

Multiplication and Division of Complex Numbers

Easy to multiple/divider in exponential form:

$$Z_1Z_2 = |Z_1||Z_2|e^{j\varphi_1}e^{j\varphi_1} = |Z_1||Z_2|e^{j(\varphi_1+\varphi_2)}$$



Same with division:

$$Z_1/Z_2 = |Z_1|e^{j\varphi_1}/|Z_2|e^{j\varphi_1} = |Z_1|/|Z_2|e^{j(\varphi_1-\varphi_2)}$$

And inversion:

$$1/Z_1 = \frac{1}{|Z_1|}e^{-j\varphi_1}$$

In-Class Exercise

Calculate the magnitude and phase of:

$$Z = \frac{a}{1 + jb\omega}$$

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$$|Z| = \sqrt{Z \times Z^*} = \frac{a}{1+jb\omega}\frac{a}{1-jb\omega} = \frac{a}{\sqrt{1+b^2\omega^2}}$$

In-Class Exercise

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For the phase, Need form like:

$$Z = X + jY$$

In-Class Exercise

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For the phase, Need form like:

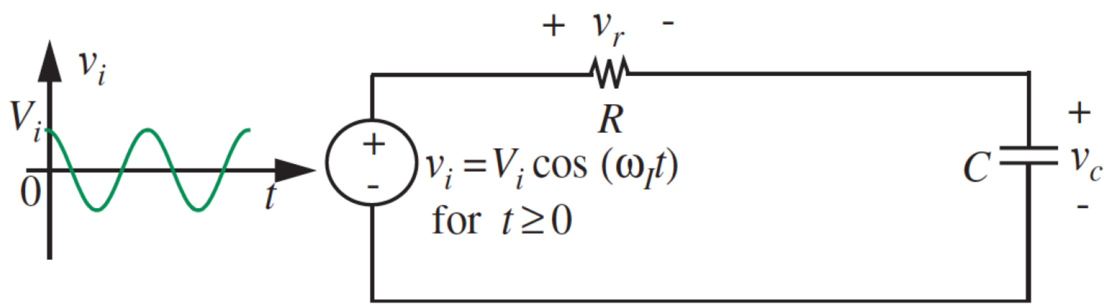
$$Z = X + jY$$

$$\varphi = \tan^{-1}(Y/X) = \frac{a}{1+jb\omega} \frac{1-jb\omega}{1-jb\omega} = \frac{a-jb\omega}{1+b^2\omega^2}$$

$$\varphi = \tan^{-1}(Y/X) = \tan^{-1}(-b\omega/a)$$

Sometimes written as, or asked for, in polar coordinates, i.e. $r\angle\varphi = Z\angle\varphi$

Back to our RC circuit



Let's analyze it now using complex exponentials:

$$V_i e^{j\omega t} = V_i \cos(\omega t) + jV_i \sin(\omega t)$$

Replace the source of $v_i(t) = V_i \cos(\omega t)$ with the form:

$$\tilde{v}_i(t) = V_i e^{j\omega t}$$

We can see that: $v_i(t) = Re\{\tilde{v}_i(t)\}$

Idea! Let's replace the sinusoidal source and the response with complex exponentials, work through the math, and then just take the real part of the complex response at the end

Complex Exponential Drive

The differential equation to find the particular solution is:

$$\tilde{v}_{cp} + RC\frac{d\tilde{v}_{cp}}{dt} = V_i e^{s_1 t} \xrightarrow{\text{was}} v_i(t) = V_i \cos(\omega t)$$

The particular solution is of the form:

$$\tilde{v}_{cp} = V_c e^{st}$$

So we get:

$$V_i e^{s_1 t} = V_c e^{st} + RCsV_c e^{st} \rightarrow s = s_1$$

Since $e^{st} \neq 0$, $\forall$ finite t, we can divide out e^{st}:

$$V_i = V_c + V_c RC s_1 \rightarrow V_c = \frac{V_i}{1 + RCs_1}$$

Since $s_1 = j\omega_1$, the particular solution to the fake input $V_i e^{s_1 t}$ is:

$$\tilde{v}_{cp}(t) = \frac{V_i}{1 + j\omega RC} e^{j\omega t}$$

Complex Exponential Drive – Particular Solution

- No waveform or response that you actually measure will have a “j” associated with it. It’s a mathematical “trick” to make analysis easier.
- Let’s find the particular response:

$$\tilde{v}_{cp}(t) = \frac{V_i}{1 + j\omega RC} e^{j\omega t}$$

$$v_{cp}(t) = Re\{\tilde{v}_{cp}(t)\} = Re\left\{\frac{V_i}{1 + j\omega RC} e^{j\omega t}\right\}$$

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$$v_{cp}(t) = Re\{\tilde{v}_{cp}(t)\} = Re\left\{\frac{V_i}{1 + j\omega RC} e^{j\omega t}\right\}$$

- First, convert $\tilde{v}_{cp}(t)$ to **polar form**: Z = |Z| x $e^{j\phi}$

$$\tilde{v}_{cp} = \frac{V_i}{1 + j\omega_1 RC} e^{j\omega_1 t} = \left|\frac{V_i}{1 + j\omega_1 RC}\right| e^{j\phi} e^{j\omega_1 t}, \quad \phi = \angle\left(\frac{V_i}{1 + j\omega_1 RC}\right)$$

- Let's find the particular response:

$$\tilde{v}_{cp} = \frac{V_i}{\sqrt{1 + (w_1 RC)^2}} \cos(\omega_1 t + \phi) + j\frac{V_i}{\sqrt{1 + (w_1 RC)^2}} \sin(\omega_1 t + \phi)$$

Complex Exponential Drive – Particular Solution

$$\tilde{v}_{cp} = \frac{V_i}{\sqrt{1+(w_1 RC)^2}}\cos(\omega_1 t+\phi) + j\frac{V_i}{\sqrt{1+(w_1 RC)^2}}\sin(\omega_1 t+\phi)$$

Thus, the particular solution to the $v_i(t) = V_i\cos(\omega t)$ drive is:

$$v_{cp}(t) = Re\{\tilde{v}_{cp}(t)\} = \frac{V_i}{\sqrt{1+(\omega RC)^2}}\cos(\omega t+\varphi) = |V_c|\cos(\omega t + \angle V_c)$$

$$\varphi = \angle V_c = \tan^{-1}\left(\frac{-\omega RC}{1}\right)$$

Finally, our total solution is the sum of homogeneous and particular:

$$v_c(t) = A_1 e^{-t/RC} + \frac{V_i}{\sqrt{1+(\omega RC)^2}}\cos(\omega t+\varphi)$$

A Key Observation to Make Everything Easier…

$$\tilde{v}_{cp}(t) = V_c e^{j\omega t} \qquad V_c = \frac{V_i}{1 + j\omega RC} = \frac{V_i}{1 + RCs} \qquad s = j\omega$$

Let's rewrite V_c:

$$V_c = \frac{1/Cs}{R + 1/Cs} V_i$$

Looks just like a voltage divider expression!

A Key Observation to Make Everything Easier…

$$\tilde{v}_{cp}(t) = V_c e^{j\omega t} \qquad V_c = \frac{V_i}{1 + j\omega RC} = \frac{V_i}{1 + RCs} \qquad s = j\omega$$

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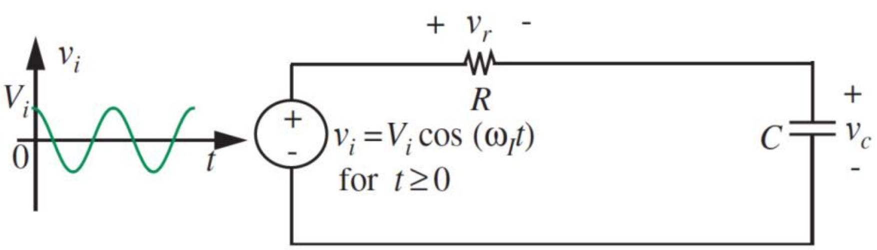

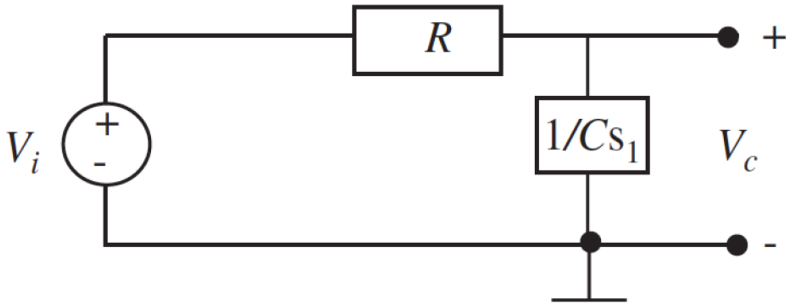

The Concept of Impedance

Let's see what these "boxes" are and what we can do with them!

- For a capacitor, let's assume the voltage and current are of the form:

$$v = Ve^{st}, \quad i = Ie^{st}$$

- We know that: $i = C\frac{dv}{dt} \to Ie^{st} = CsVe^{st}$

- So we can write a simple algebraic expression of: $V = \frac{1}{Cs}I$

Generalization of Ohm's Law for Sinusoidal Steady State

Resistor: $V = IR \to Z_R = R$

Capacitor: $V = I\frac{1}{sC} \to Z_C = \frac{1}{sC} = \frac{1}{j\omega C}$

Inductor: $V = sLI \to Z_L = sL = j\omega L$

The Concept of Impedance

Impedance: A generalization of resistance for sinusoidal steady state inputs. Associated with each element is a characteristic impedance that affects the flow of current/voltage drop across the element, analogous to a resistor.

We can now analyze complex circuits with R's, C's, and L's using our standard tools, like KVL, KCL, Node method, and treat each element in the circuit using Ohm's law with it's characteristic impedance.

Ohm's law is easier to use for evaluating circuits. No more ODE's! (mostly)

Generalization of Ohm's Law for Sinusoidal Steady State

Resistor: $V = IR \rightarrow Z_R = R$

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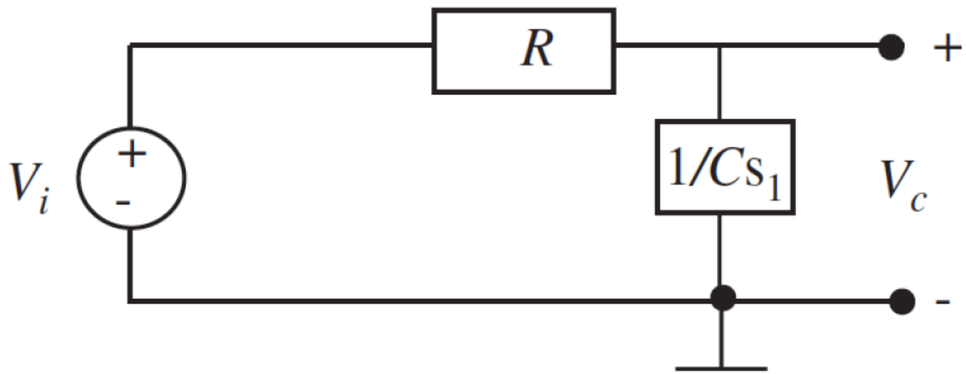
Inductor: $V = sLI \rightarrow Z_L = sL = j\omega L$

A New Circuit with “Boxes”

We can redraw the circuit, replacing resistors with *R* boxes, capacitors with 1/Cs boxes, and cosine sources by their amplitudes



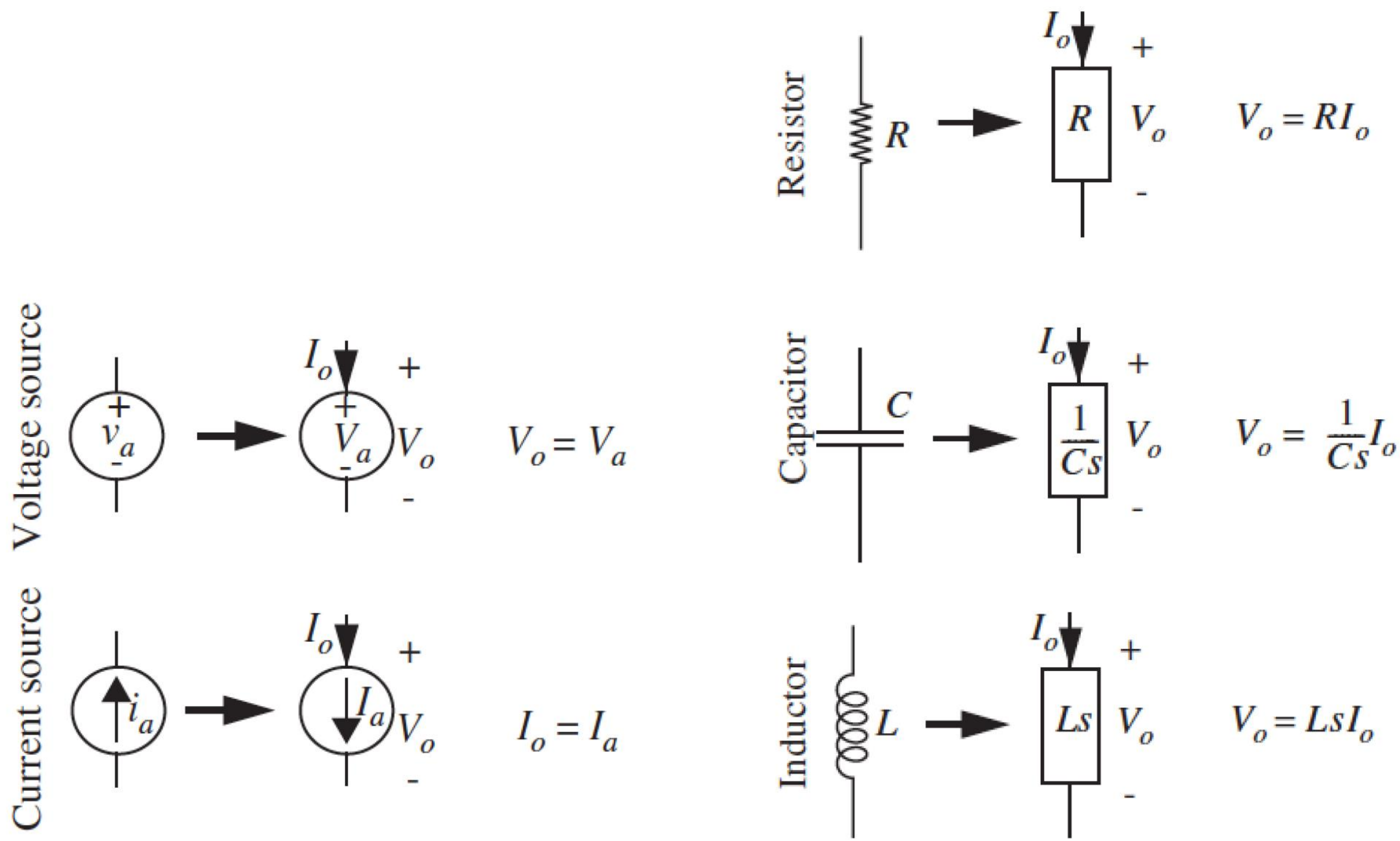
$$V_c = \frac{1/Cs}{R + 1/Cs} V_i$$

Looks just like a voltage divider expression!

(1) Replace components with complex impedance
(2) Replace sources with their amplitudes and phases (remove time dep.)
(3) Solve for complex v, i amplitudes, then add back in time dep. at end

The Concept of Impedance

Impedance: A generalization of resistance for sinusoidal steady state inputs. Associated with each element is a characteristic impedance that affects the flow of current/voltage drop across the element, analogous to a resistor.



THURSDAY

- Summary of our new <u>impedance</u> method and how to use
- How to convert sources, write out sinusoidal inputs as complex amplitudes
- How to determine complex amplitudes of voltage and current throughout the circuit
- Reconstruct the total time-domain signal
- Examples.