ECE 10C
Fall 2020
Slide Set 11
Instructor: Galan Moody
TA: Kamyar Parto

1st correct answer in chat = +1 quiz point:
What is this, and what happened to it today? What's it designed to do?

Previously
- Bode Plot
- 1st-order Filters

This week
- 2nd-order RLC (resonant) filters

Important Items:

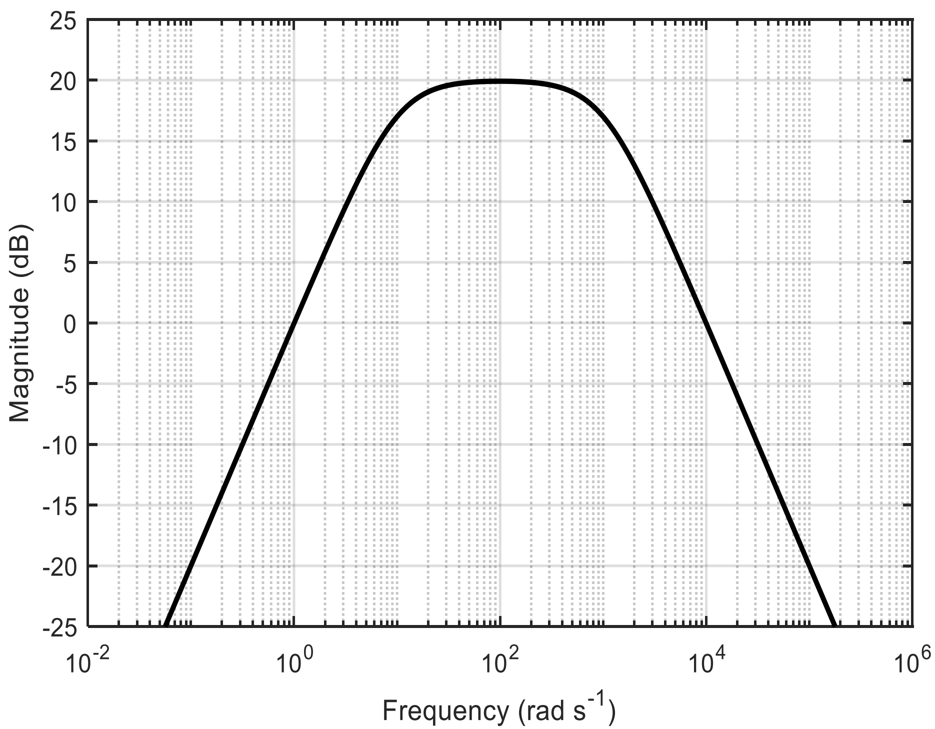
- HW #5 due Thurs, 12/3
- Lab #5 due Friday, 12/4
- ESCI questionnaire
- HW # 6 posted, due 12/10

Quiz Time

Q1: True or False: An RC circuit with the output voltage measured across the capacitor is an example of a low-pass filter.

Q2: True or False: An RL circuit with the output voltage measured across the inductor is an example of a low-pass filter.

Q3: Pick the correct transfer function for the gain showed in the plot below:



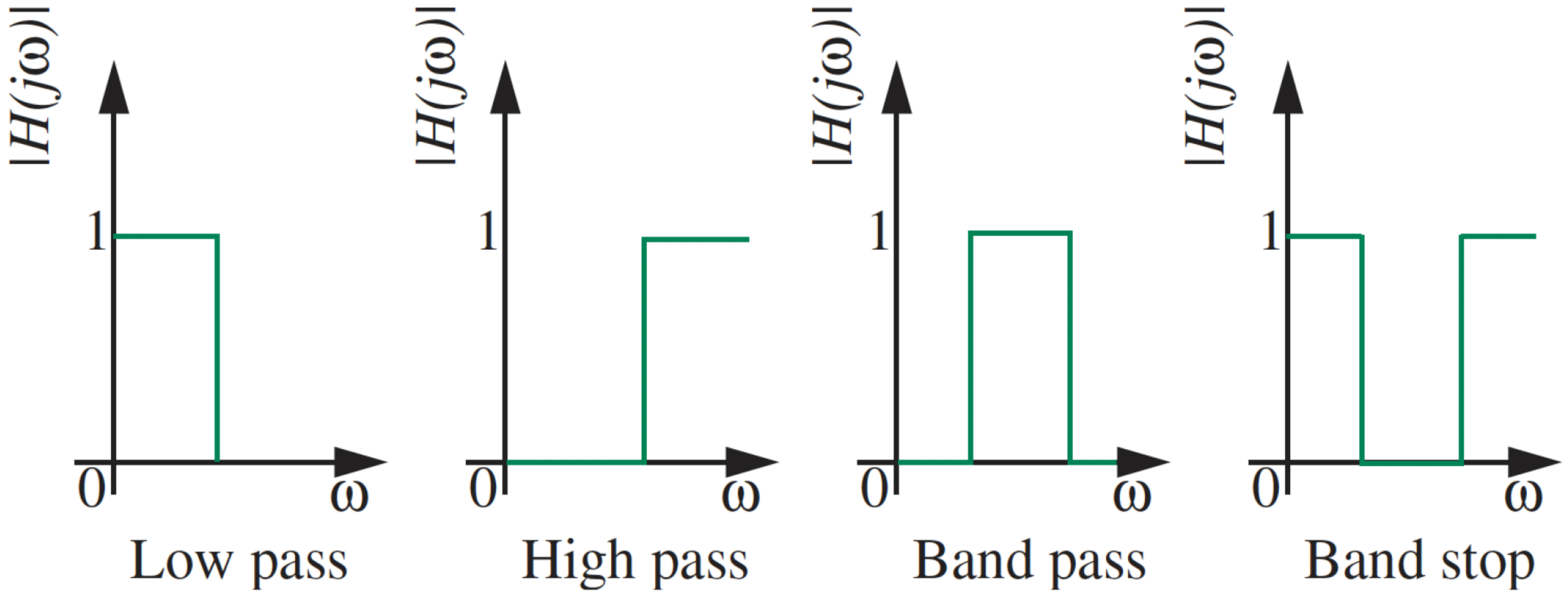
a) $\dfrac{(1+j\omega)}{\left(1+\frac{j\omega}{10}\right)\left(1+\frac{j\omega}{100}\right)}$

b) $\left(1+\dfrac{j\omega}{10}\right)\left(1+\dfrac{j\omega}{100}\right)$

c) $\dfrac{j\omega}{\left(1+\frac{j\omega}{10}\right)\left(1+\frac{j\omega}{1000}\right)}$

d) $\dfrac{j\omega}{\left(1+\frac{j\omega}{10}\right)\left(1+\frac{j\omega}{100}\right)}$

Filters



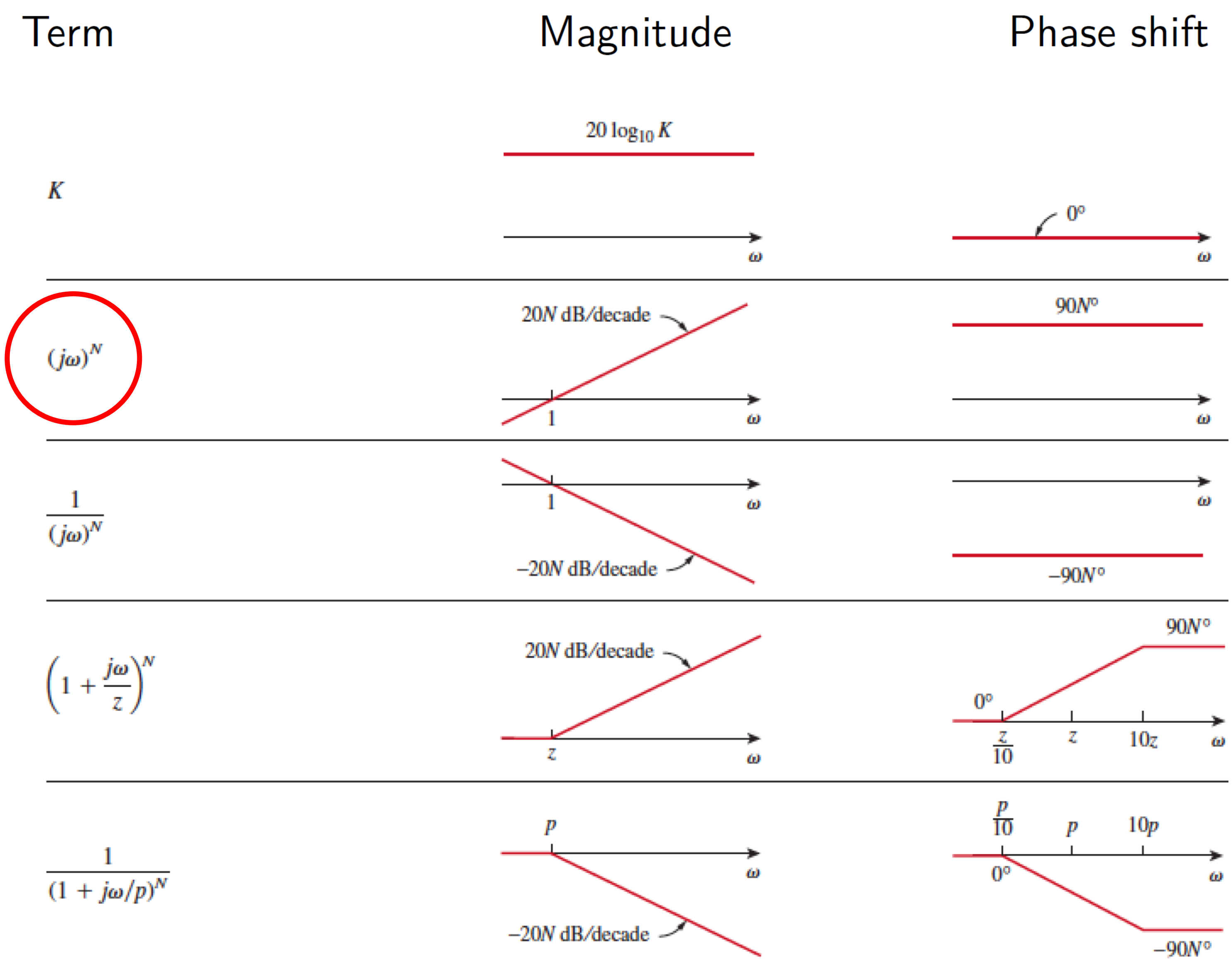
RL and RC circuits

- ✓ First-order ODEs
- ✓ Exponential behavior
- X Not oscillatory

RLC circuits

- ✓ Secord-order ODEs
- ✓ Oscillatory behavior
- ✓ Damped behavior

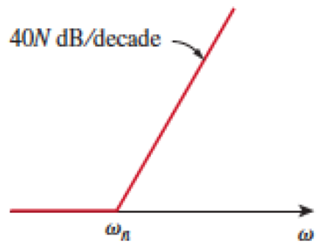
Frequency Response of General Functions

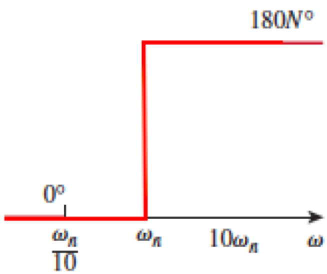


Second-order terms (ω^2)

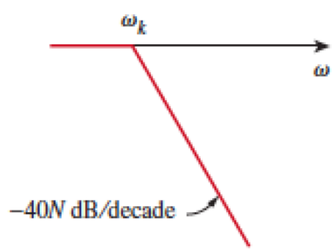
Term | Magnitude | Phase shift

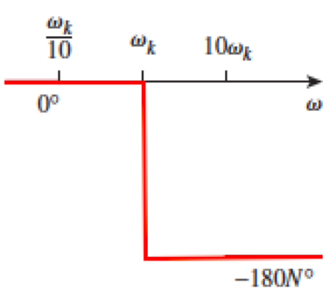
$$\left[1+\frac{2j\omega\zeta}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right]^N$$





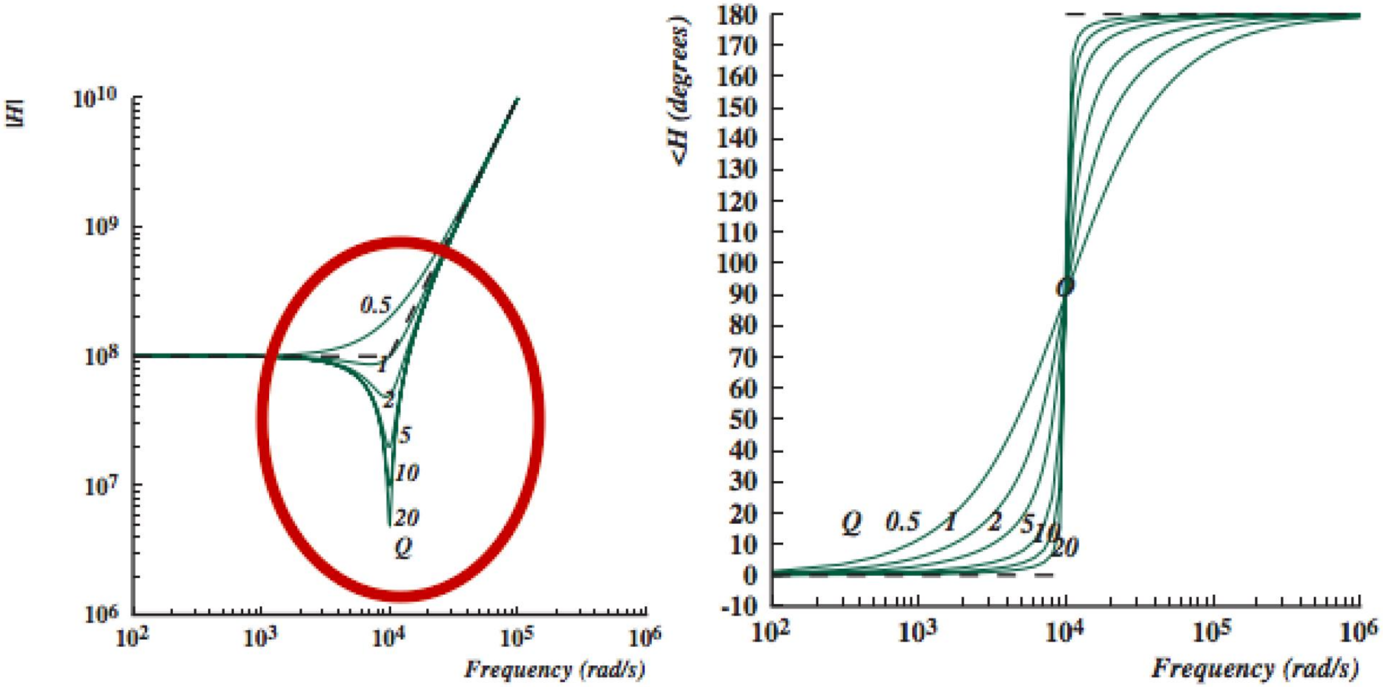
$$\frac{1}{[1+2j\omega\zeta/\omega_k+(j\omega/\omega_k)^2]^N}$$





Frequency Response with 2nd-Order Terms

The actual frequency response of resonant circuits “peaks” at the resonance frequency

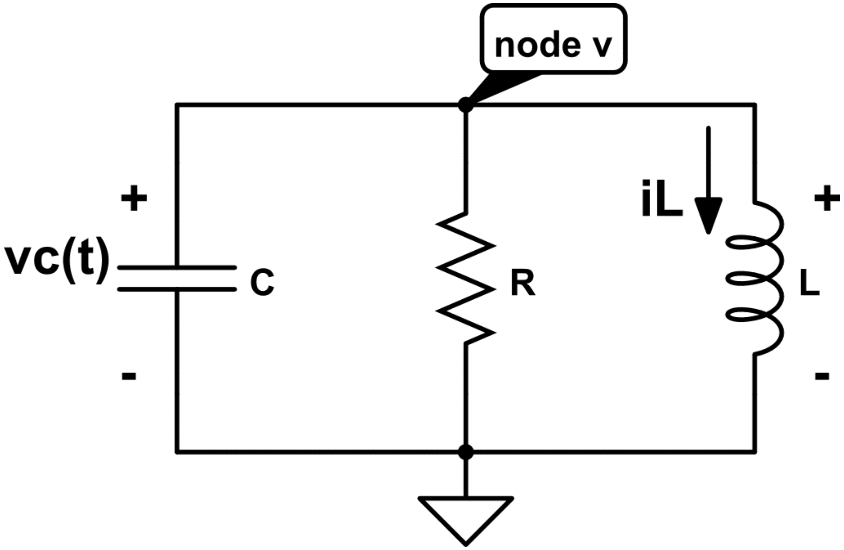


(a) (b)

This peaking behavior is ideal for filter design, especially for bandpass and bandstop filters → **resonant** filters

Recall: Parallel RLC Circuit

Write the ODE, the general expression for $v_c(t)$, and the solutions to the characteristic equation for the undriven parallel RLC circuit below:



- Node KCL:

$$-C\frac{dv(t)}{dt}-\frac{v(t)}{R}-\frac{1}{L}\int_{-\infty}^{t}v(t')dt'=0$$

- Take the derivative:

$$\frac{d^2v(t)}{dt^2}+\frac{1}{RC}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=0$$

- By substituting Ae^{st}, we get the characteristic equation:

$$s^2+\frac{1}{RC}s+\frac{1}{LC}=0$$

Recall: Parallel RLC Circuit

- We can also write the characteristic equation as:

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

- Where we have defined:

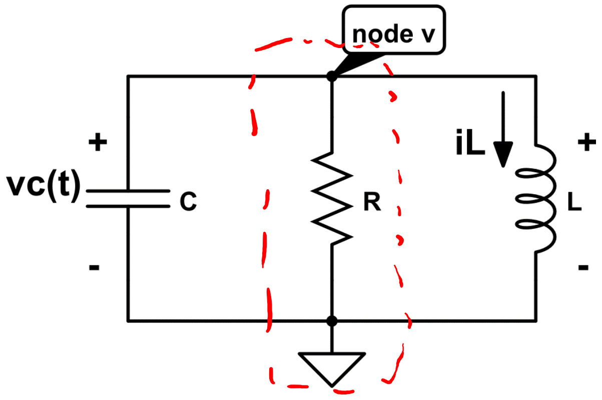
$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- The roots of the characteristic equation are:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$



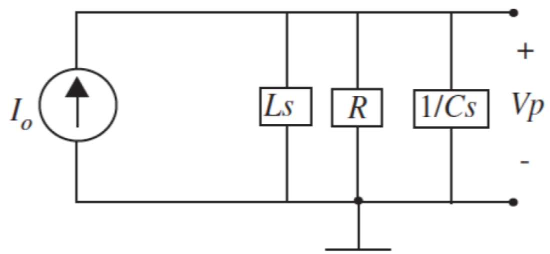
Checking 2nd-Order Terms

Are all 2nd-order circuits resonant?

For a circuit to be resonant, we need to have terms in *H(s)* such as:

$$s^2 + 2\alpha s + \omega_n^2$$

If $Q = \frac{\omega_n}{2\alpha} > 0.5$, the roots of the equation above are complex, and we will have 2nd-order terms in *H(s)*, thus resonant behavior



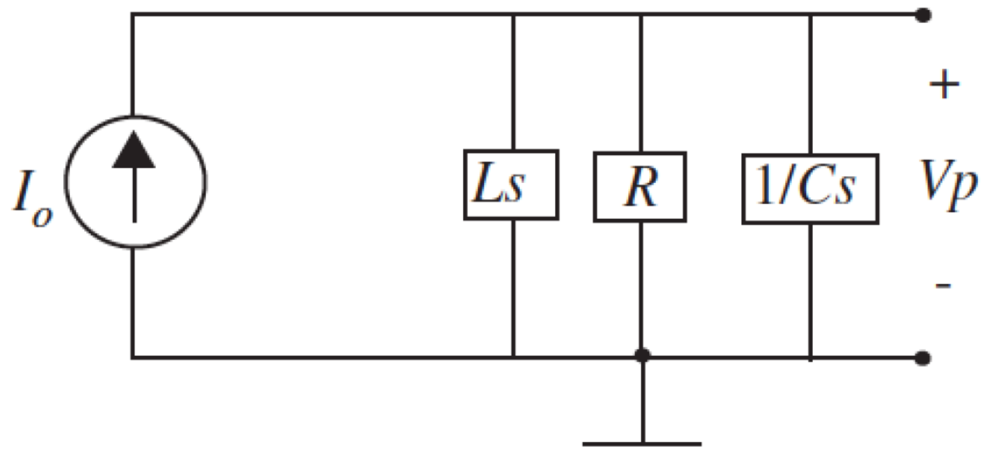
Recall for Parallel RLC

$$\alpha = \frac{1}{2RC}$$

A parallel RLC is not resonant unless the resistance (R) is large enough to meet the following condition:

$$R > 0.5\sqrt{\frac{L}{C}}$$

Resonant Parallel RLC Circuit



We always start with finding the transfer function:

$$H(j\omega) = \frac{V_p}{I} = \frac{1}{\frac{1}{j\omega L} + \frac{1}{R} + j\omega C} = \frac{1}{\frac{1}{R} + j(\omega C - \frac{1}{\omega L})}$$

Sketch Frequency Response: Asymptotic Method

$$H(j\omega) = \frac{V_p}{I} = \frac{1}{\frac{1}{j\omega L} + \frac{1}{R} + j\omega C} = \frac{1}{\frac{1}{R} + j(\omega C - \frac{1}{\omega L})}$$

High frequency asymptote?

$$H(j\omega) = \frac{1}{j\omega C} \rightarrow |H(j\omega)| = \frac{1}{\omega C}, \quad \angle H(j\omega) = -90^\circ$$

Low frequency asymptote?

$$H(j\omega) = j\omega L \rightarrow |H(j\omega)| = \omega L, \quad \angle H(j\omega) = 90^\circ$$

The high and low frequency asymptotes intersect at $\omega_\circ = \frac{1}{\sqrt{LC}}$

Sketch Frequency Response: Asymptotic Method

High frequency asymptote?

$$H(j\omega) = \frac{1}{j\omega C} \to |H(j\omega)| = \frac{1}{\omega C}, \quad \angle H(j\omega) = -90^\circ$$

Low frequency asymptote?

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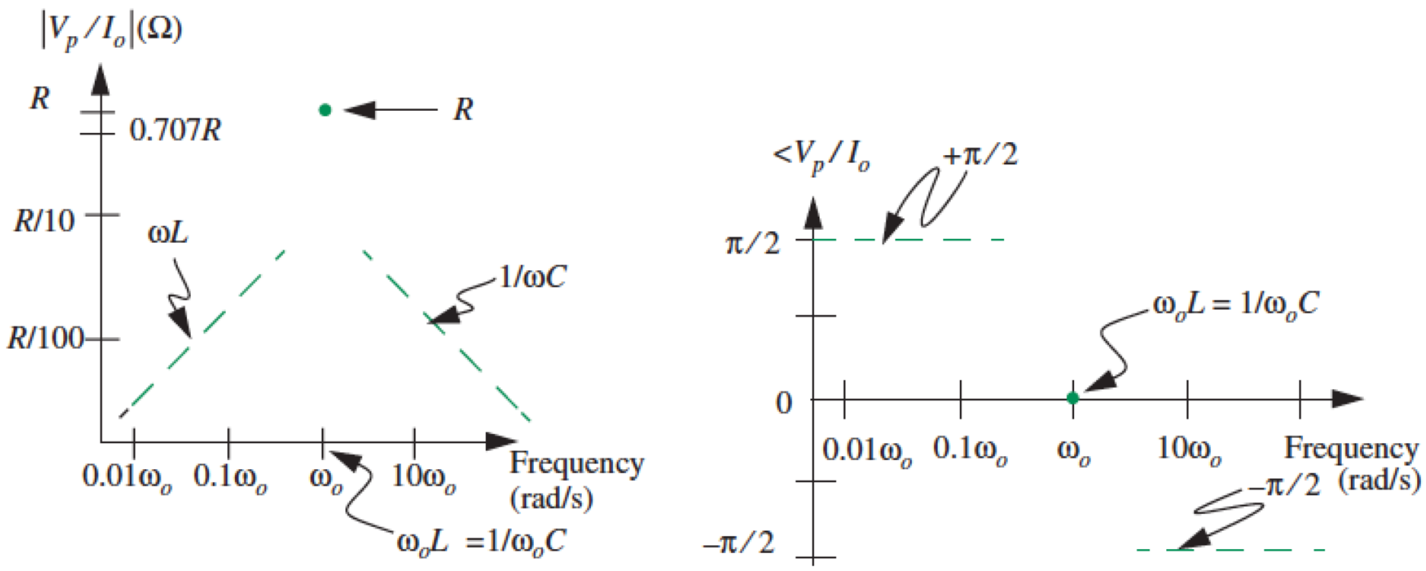


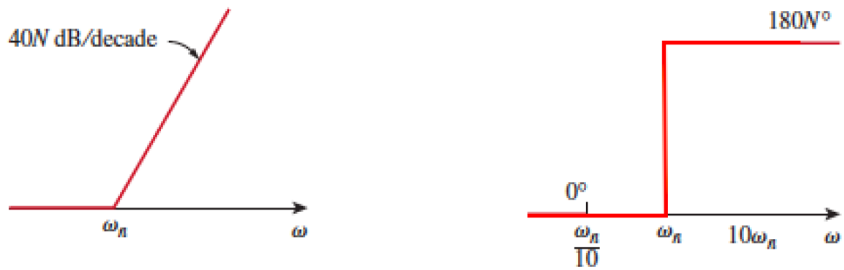
If we connect these asymptotes, we get an approximate sketch of the frequency response

Let's Double-Check Our Method

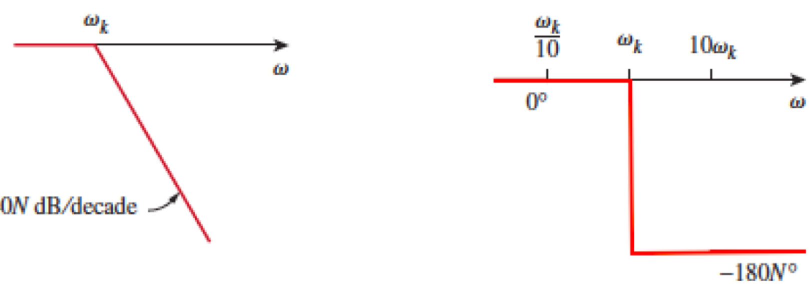
$$H(j\omega) = \frac{V_p}{I} = \frac{1}{\frac{1}{R} + j\omega C + \frac{1}{j\omega L}}$$

$$H(j\omega) = L\frac{j\omega}{1 + \frac{L}{R}j\omega + (j\omega)^2 LC}$$

$$\left[1 + \frac{2j\omega\zeta}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2\right]^N$$

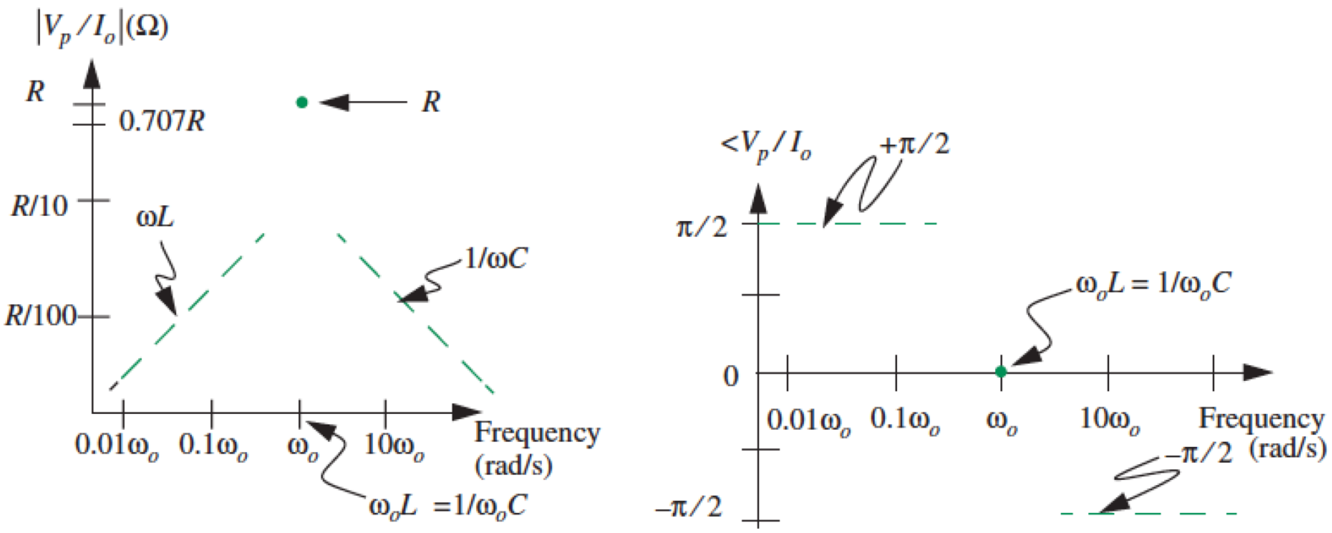


$$\frac{1}{[1 + 2j\omega\zeta/\omega_k + (j\omega/\omega_k)^2]^N}$$



Parallel RLC: Adding the Resonance Peak

$$H(j\omega) = \frac{V_c}{I} = \frac{1}{\frac{1}{R} + j(\omega C - \frac{1}{\omega L})}$$

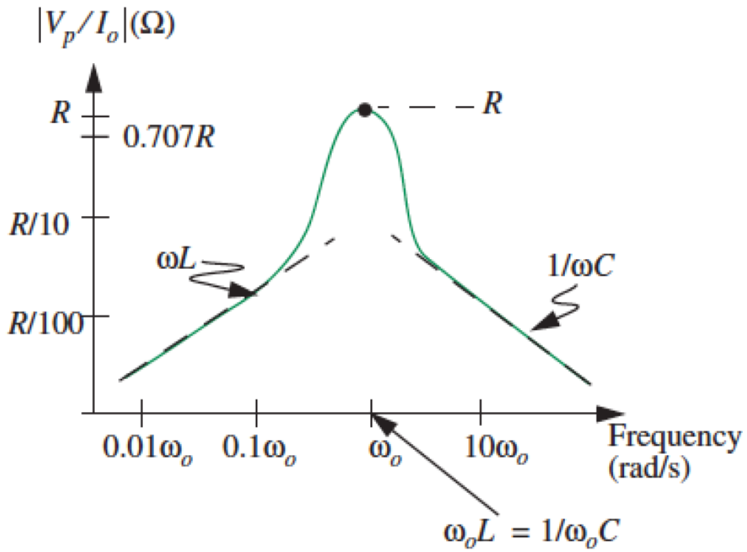


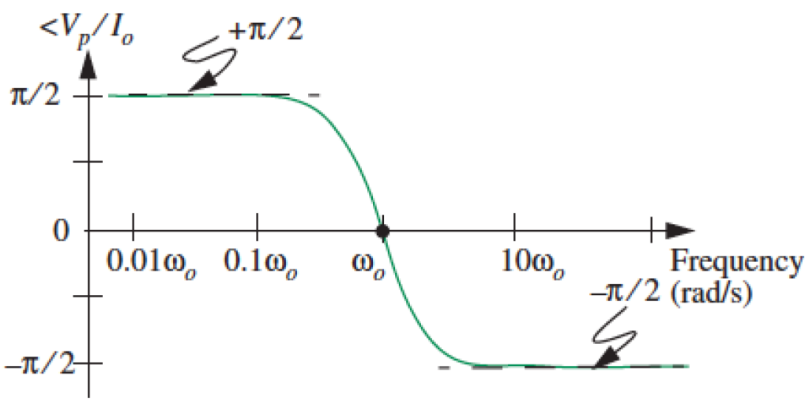
Value of $H(j\omega)$ at $\omega_\circ = \frac{1}{\sqrt{LC}}$?

$$\omega_\circ C - \frac{1}{\omega_\circ L} = 0 \rightarrow \quad H(j\omega_\circ) = R$$

Parallel RLC: Adding the Resonance Peak

$$H(j\omega) = \frac{V_c}{I} = \frac{1}{\frac{1}{R} + j(\omega C - \frac{1}{\omega L})}$$





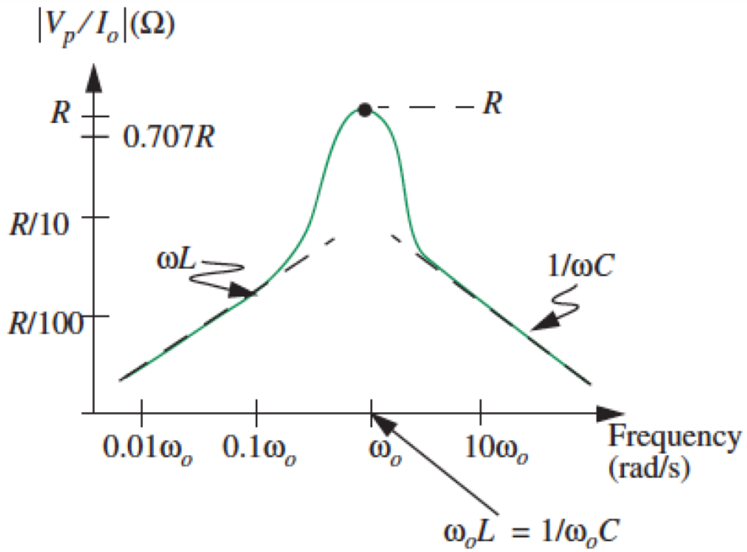
Value of $H(j\omega)$ at $\omega_\circ = \frac{1}{\sqrt{LC}}$?

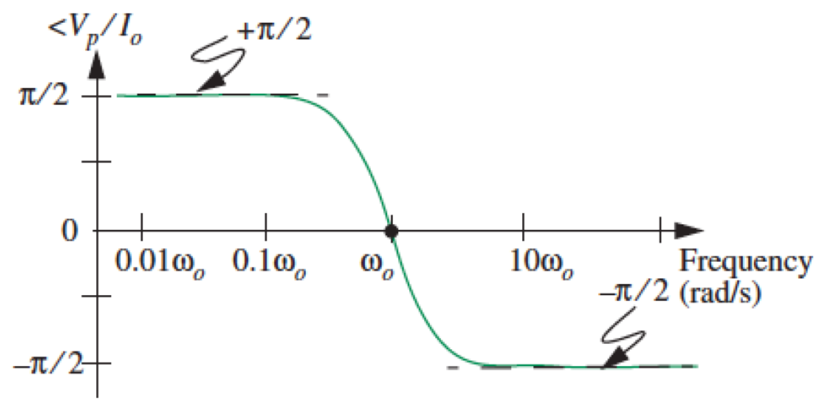
$$\omega_\circ C - \frac{1}{\omega_\circ L} = 0 \rightarrow \quad H(j\omega_\circ) = R$$

Resonant Filters

Note

You don't *need* a resonant term to design a bandpass or bandstop filter. You can do this through combinations of RL or RC first-order circuits. But these require a lot more components in more complex circuits to get similar effects.



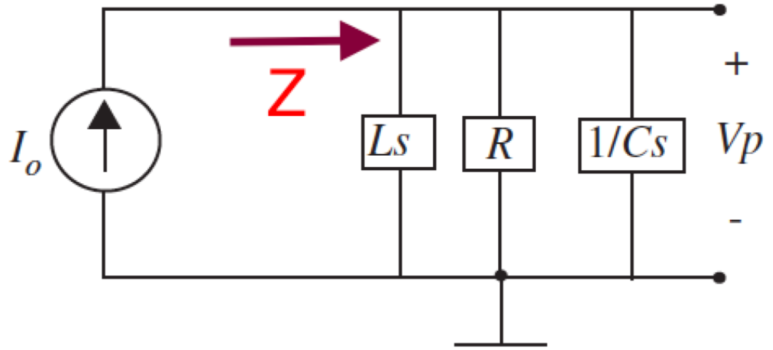


Takeaway Lesson

Resonant Filters can help us design "highly selective" bandpass and bandstop filters. With RLC circuits, by expressing *H(s)* in its standard form, we can identify the important parameters for the filters through inspection, and then plot using methods we've learned (log-log or Bode plots)

Resonance Frequency

$$H(j\omega) = \frac{V_c}{I} = Z = \frac{1}{\frac{1}{R} + j\left(\omega_\circ C - \frac{1}{\omega_\circ L}\right)}$$



This is why the natural frequency $\omega_\circ$ is also called the **resonance frequency** in parallel RLC:

$$\omega_\circ C - \frac{1}{\omega_\circ L} = 0 \rightarrow \text{Input impedance} = R$$

System gain is at its peak

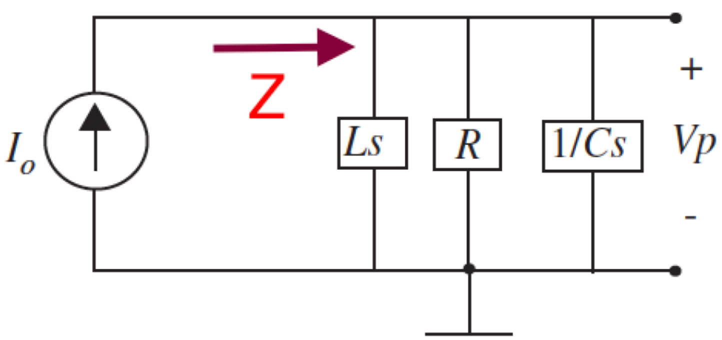
Resonance Frequency

At what frequency does resonance occur?

Multiple definitions of resonance frequency exist (usually very similar)

Definition: The resonance frequency is the frequency at which $|H(s)|$ is at a relative maximum or minimum. In a series or a parallel RLC circuit, the impedance seen from the input is purely real (i.e. resistive)

$$Im(Z) = 0 \text{ at resonance}$$



At resonant frequencies, small periodic driving forces have the potential to produce large amplitude oscillations

Peak Gain

How large is the gain peak at the resonance frequency?

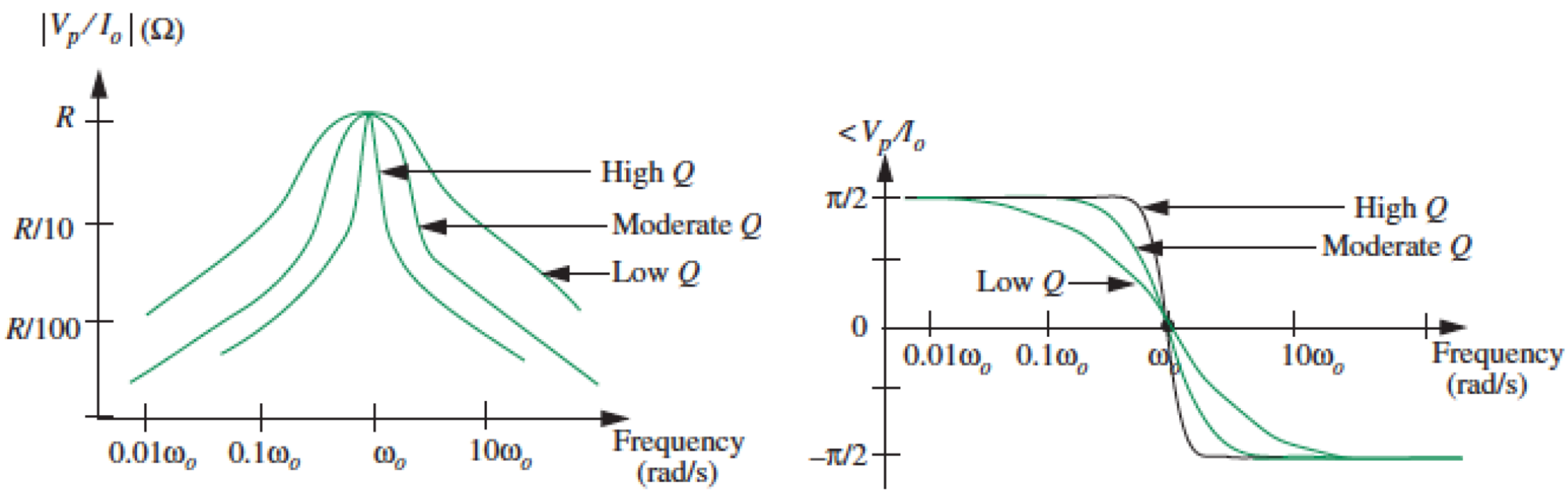
Role of the Q-factor in 2nd-Order Resonant Systems

Remember: The Q factor or quality factor is a dimensionless parameter that describes how under-damped an oscillator is:

$$Q = \frac{\omega_0}{2\alpha} = \frac{\tau\omega_0}{2} \approx \text{\# oscillations before energy dissipates}$$

As Q factor ↑, resonance with **larger relative gain** + **smaller bandwidth**

Effect of Q: Frequency Selectivity

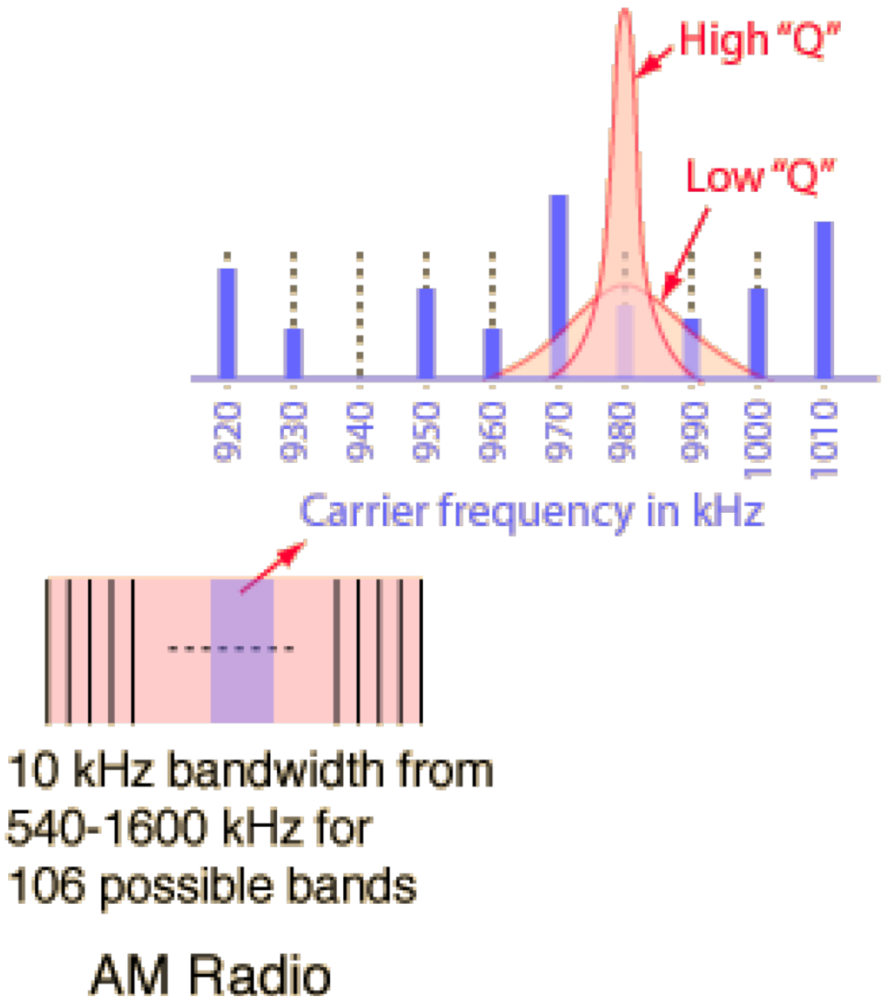


Selectivity is captured through the **bandwidth** of a filter

The bandwidth measures the width of a filter's "passband" (range of freqs that can pass through a filter) $\rightarrow$ but how do we define passing?

Example: Importance of Right Choice of Q

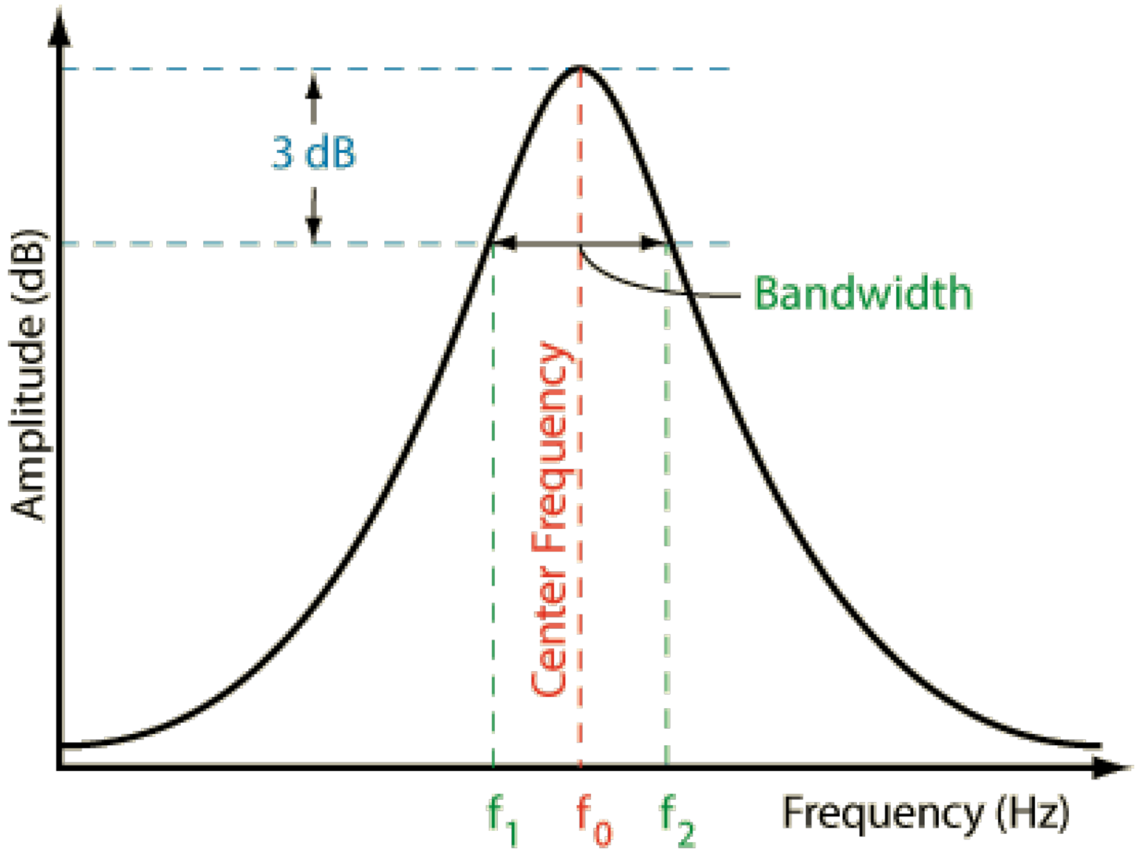
Selection of AM radio stations by the radio receiver $\rightarrow$ The selectivity of the tuning must be high enough



Bandwidth Definition

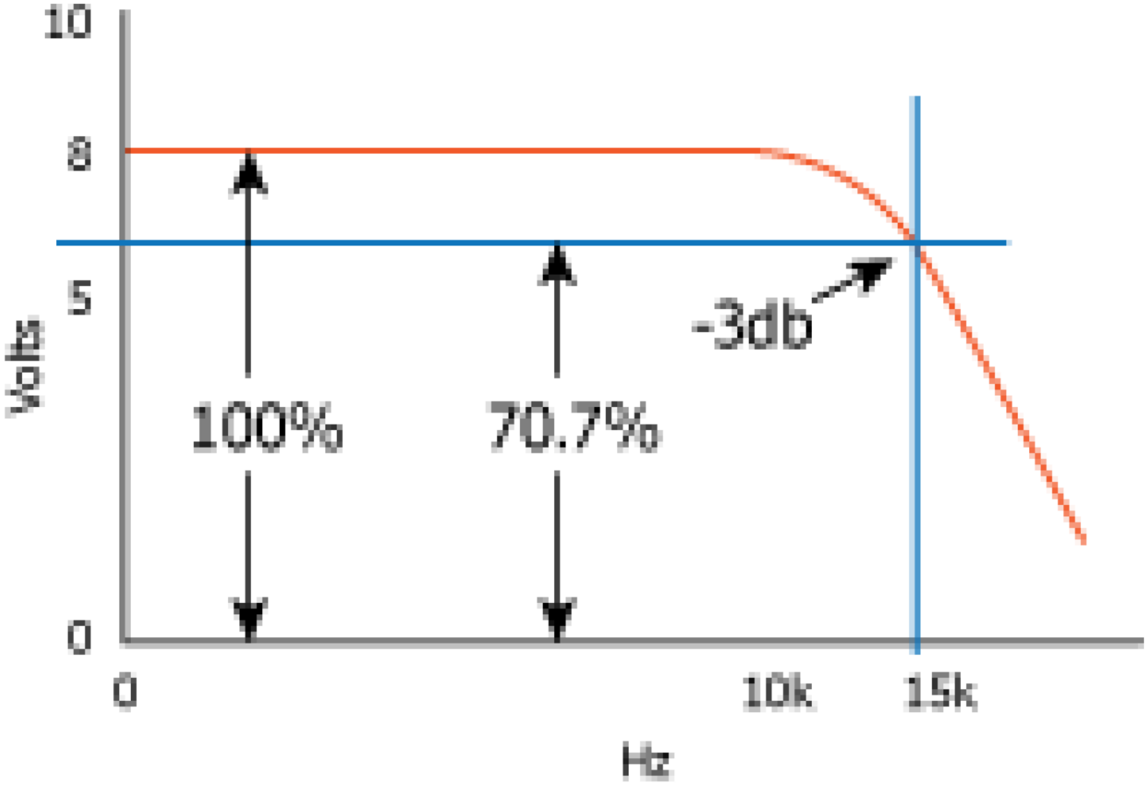
The bandwidth of a bandpass filter is usually defined as the difference between two lower and upper cut-off points where

$$\text{Gain at cut-off} = \frac{1}{\sqrt{2}}(\text{max gain}) = 0.707(\text{max gain})$$



Bandwidth of a Low-Pass Filter

The same idea applies to a lowpass filter (the highpass filter is solely characterized by its cut off frequency and does not have a bandwidth)

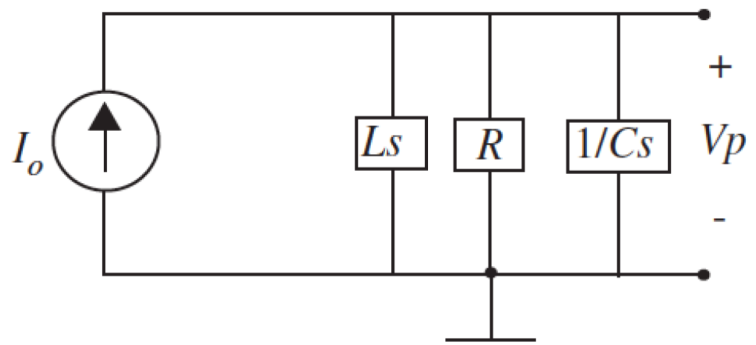


$$\text{Gain at cut off} = \frac{1}{\sqrt{2}}\,(\text{DC gain})$$

Bandwidth of a Parallel RLC Filter

$$H(j\omega) = \frac{V_c}{I} = \frac{1}{\frac{1}{j\omega L} + \frac{1}{R} + j\omega C} = \frac{1}{\frac{1}{R} + j(\omega C - \frac{1}{\omega L})}$$

We know the maximum gain point: $H(j\omega_\circ) = R$



$$|H(j\omega_{cut})| = \frac{R}{\sqrt{2}}$$

$$\left| \frac{1}{\frac{1}{R} + j(\omega_{cut} C - \frac{1}{\omega_{cut} L})} \right| = \frac{1}{\frac{1}{R}\sqrt{2}} \rightarrow \left| (1 + jR(\omega_{cut} C - \frac{1}{\omega_{cut} L}) \right| = \sqrt{2}$$

Bandwidth of a Parallel RLC Filter

$$R(\omega_{cut}C - \frac{1}{\omega_{cut}L}) = \pm 1$$

$$\frac{1}{R} = G \rightarrow \omega_{cut}^2 \pm \frac{G}{C}\omega_{cut} - \frac{1}{LC} = 0$$

Four roots $\rightarrow \omega_{cut} = \pm\frac{G}{2C} \pm \sqrt{(\frac{G}{2C})^2 + \frac{1}{LC}}$ $\rightarrow$ Only two are positive

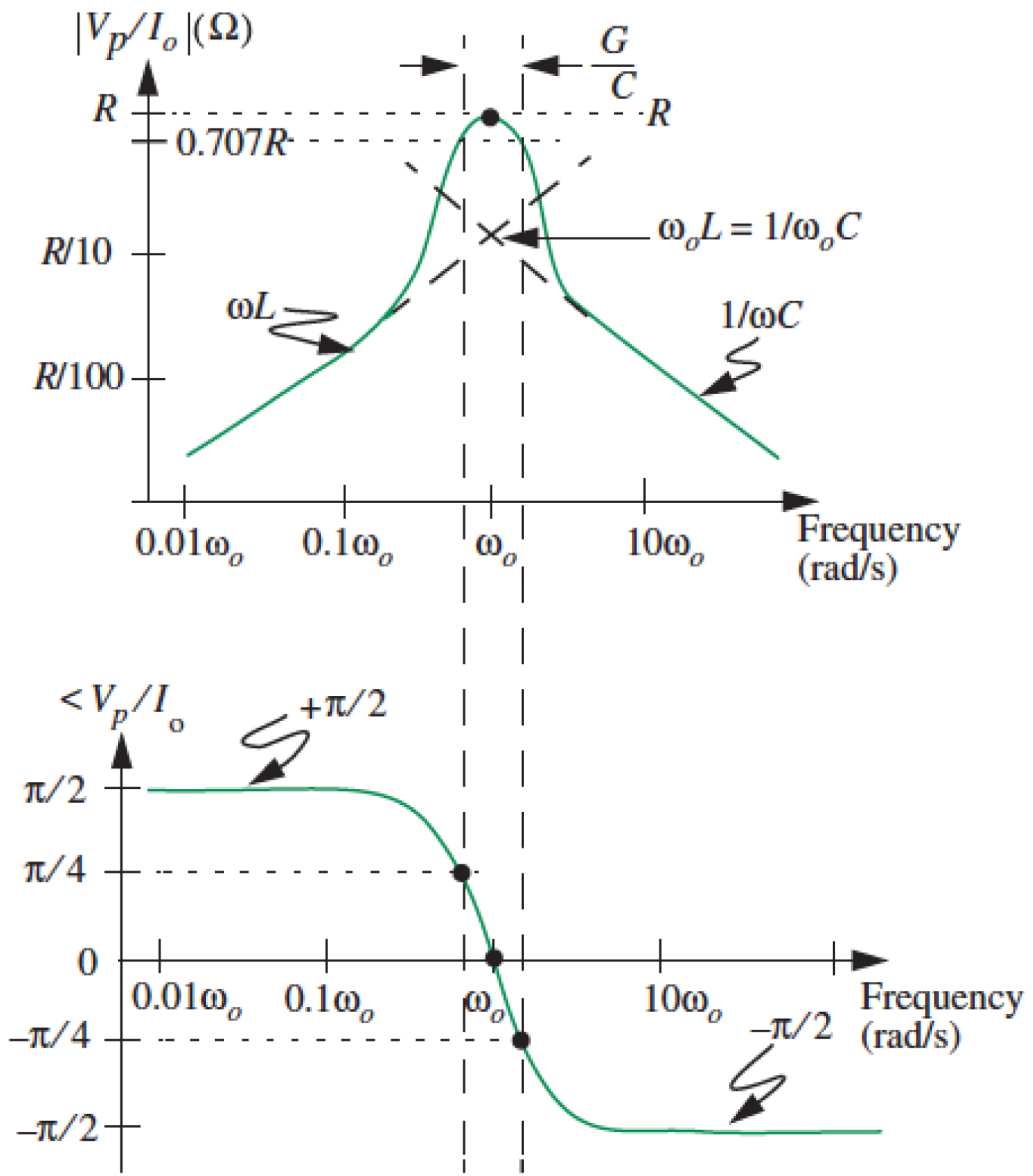
Cut-off frequencies and Bandwidth

$\omega_{cut} = \pm\frac{G}{2C} + \sqrt{(\frac{G}{2C})^2 + \frac{1}{LC}} \rightarrow$ Bandwidth $= \frac{G}{C} = \frac{1}{RC} = 2\alpha = \frac{\omega_\circ}{Q}$

General definition of quality factor

$$Q = \frac{\omega_\circ}{\text{Bandwidth}}$$

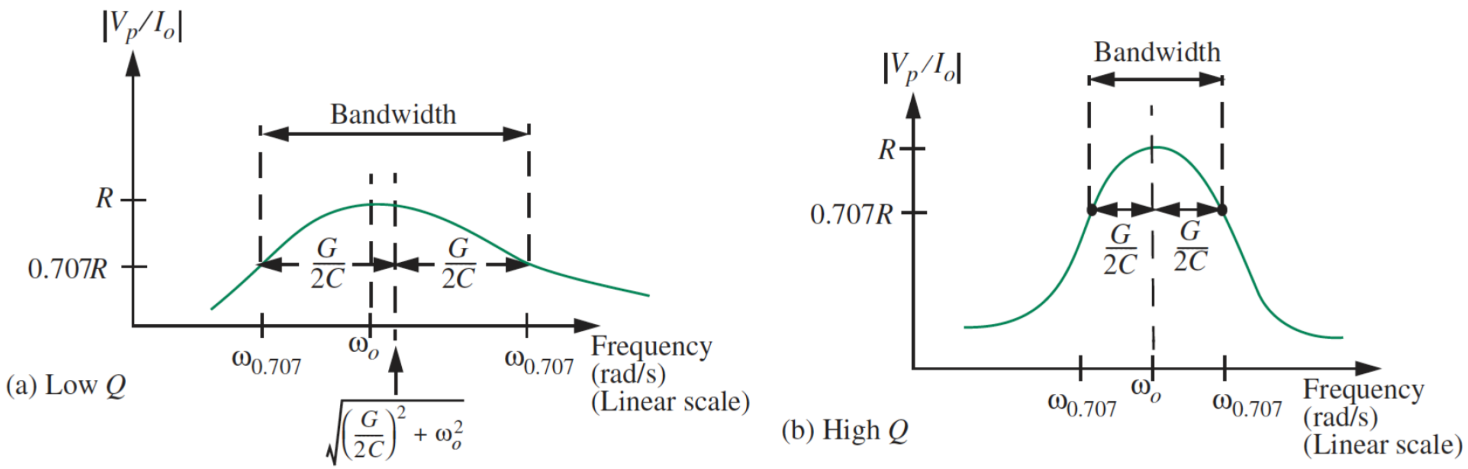
Bandwidth of a Parallel RLC Filter



Cut-Off (Half Power) Frequencies

The cut-off frequencies

$$\omega_{cut} = \pm\frac{\omega_\circ}{2Q} + \omega_\circ\sqrt{1+\left(\frac{1}{2Q}\right)^2}$$



The cut-off frequencies with high Q factor ($Q \geq 10$)

$$\omega_{cut} \approx \omega_\circ \pm \frac{\text{Bandwidth}}{2}$$

Designing a Bandpass Filter

We would like to choose R, L, C to get a certain peak gain, at a certain center frequency $\omega_\circ$, with a given Q factor (or a given bandwidth $\frac{\omega_\circ}{Q}$).

1. Peak gain $= R$
2. Given the Q factor and $\omega_\circ$, choose L and C such that:

$$\text{resonance frequency} \quad \omega_\circ = \frac{1}{\sqrt{LC}} \to LC = \frac{1}{\omega_\circ^2}$$

$$Q = \omega_\circ RC = \frac{1}{\sqrt{LC}} RC \to \sqrt{\frac{C}{L}} = \frac{Q}{R} \to \frac{C}{L} = \left(\frac{Q}{R}\right)^2$$

$$\left(\frac{Q}{R}\right)^2 L^2 = \frac{1}{\omega_\circ^2} \to L = \frac{R}{Q\omega_\circ}$$

Example

Design constraints

Peak gain $= 10$, $\omega_\circ = 10^5$ rad/s, $Q = 1, 2, 0.5$

$$R = 10\ \Omega \quad \text{for all } Q$$

$$\text{For } Q = 1 \rightarrow L = \frac{R}{Q\omega_\circ} = 0.1\text{mH}, C = \frac{1}{\omega_\circ^2 L} = 1\mu\text{F}$$

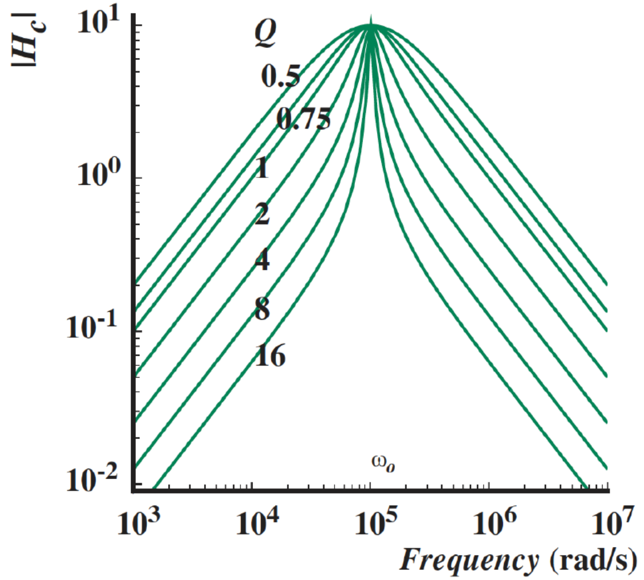
$$\text{For } Q = 2 \rightarrow L = 0.05\text{mH}, C = 2\mu\text{F}$$

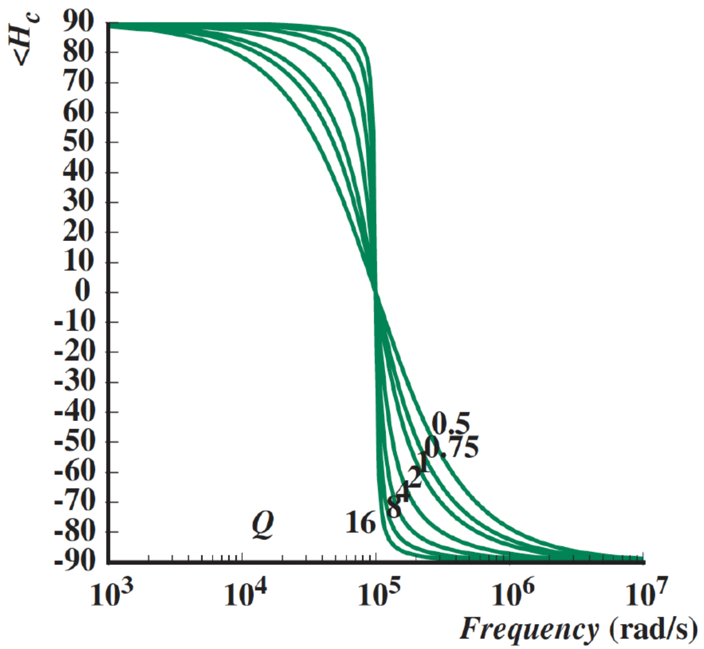
$$\text{For } Q = 0.5 \rightarrow L = 0.2\text{mH}, C = 0.5\mu\text{F}$$

Example

Design constraints

Peak gain $= 10$, $\omega_\circ = 10^5$ rad/s, $Q = 1, 2, 0.5$





Summary

Parallel RLC circuit transfer function: Can act as a resonant band-pass filter **(but not always, make sure to check that circuit's underdamped!)**

Thursday: Series RLC resonant circuit. Measuring output across R, L, or C gives different filter response (bandpass and bandstop).

HW #5 Due Thursday, 12/3
HW#6 Posted, Due Thursday 12/10
(**remember**: I drop your lowest HW score)