ECE 10C
Fall 2020
Slide Set 1 (cont'd)
Instructor: Galan Moody
TA: Kamyar Parto

Last Week
- Introduction
- Review

This Week
- Finish review
- RLC circuits

Important Items:

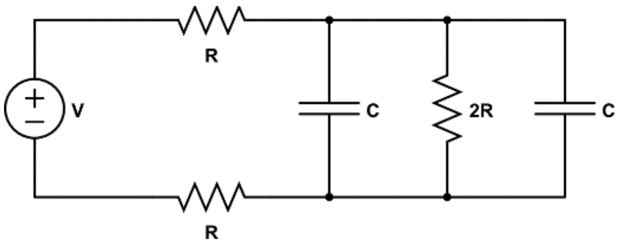
Homework #1 Due Thurs
Homework #2 Due 10/22

Lab #1 due Friday
Lab #2 due 10/23

Last Quiz

Q1 [1 point]. True or False? An inductor stores energy in its magnetic field and reacts to the change in current.

Q2 [2 point]. Pick the correct Thevenin voltage and resistance as would be seen across the two capacitors.

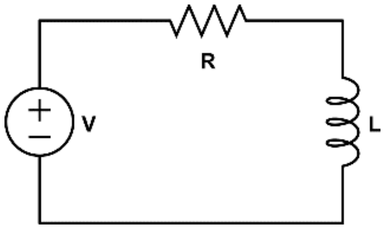


(a) $V_{th} = V/2$, $R_{th} = 2R$

(b) $V_{th} = V$, $R_{th} = R$

(c) $V_{th} = V/2$, $R_{th} = R$

Q3 [2 points]. For a step voltage response from $V = 0$ to $V = V_0$, choose the correct statements for the RL circuit below. Assume the system had been at "rest" for a long time.



At the instance $V = V_0$:

(a) The current through L is zero.

(b) The current through L is not zero.

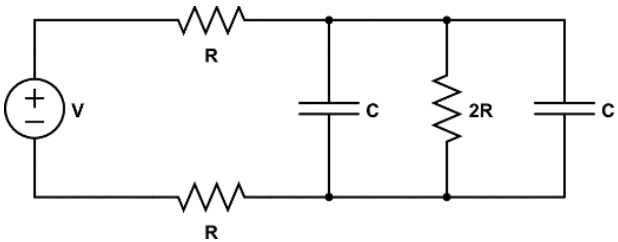
At long times (steady-state):

(a) The voltage across L is not zero.

(b) The voltage across L is zero.

Q1 [1 point]. True or False? An inductor stores energy in its magnetic field and reacts to the change in current.

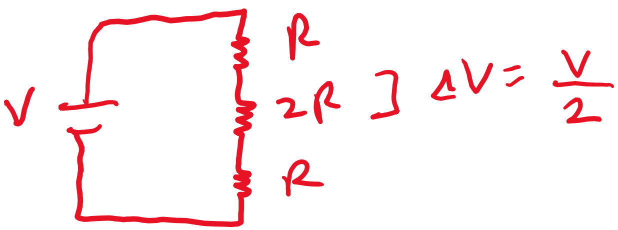
Q2 [2 point]. Pick the correct Thevenin voltage and resistance as would be seen across the two capacitors.



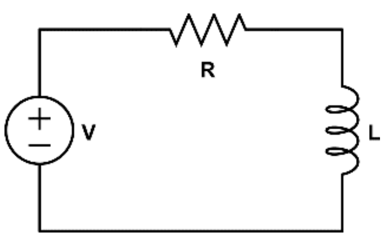
(a) $V_{th} = V/2$, $R_{th} = 2R$

(b) $V_{th} = V$, $R_{th} = R$

(c) $V_{th} = V/2$, $R_{th} = R$



Q3 [2 points]. For a step voltage response from $V = 0$ to $V = V_0$, choose the correct statements for the RL circuit below. Assume the system had been at "rest" for a long time.



At the instance $V = V_0$:

(a) The current through L is zero.

(b) The current through L is not zero.

At long times (steady-state):

(a) The voltage across L is not zero.

(b) The voltage across L is zero.

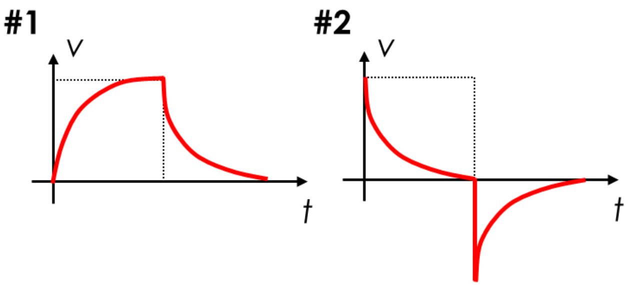
$i_L(0^-) = i_L(0^+)$

at steady state, L = short.
Thus no ΔV across L.

Quiz Time!

Q1 [1 point]. True or False? The energy stored in an inductor is equal to $1/2 \times L \times I^2$.

Q2 [1 point]. The figure shows the voltage response of two circuits to a square pulse source. Choose the correct answer:



(a) #1 = RC, #2 = RL

(b) #1 = RL, #2 = RC

Q3 [1 point]. True or False? If initially “at rest”, an inductor behaves as an instantaneous **open** circuit when the input makes an abrupt change.

Q4 [1 point]. True or False? An inductor behaves as a **short** circuit at long times when driven by a DC voltage source.

Q5 [1 point]: What is the correct expression for the RC and RL circuit time constants?

(a) $\tau_{RC} = RC, \tau_{RL} = RL$ **(b)** $\tau_{RC} = RC, \tau_{RL} = L/R$ **(c)** $\tau_{RC} = RC, \tau_{RL} = R/L$ **(d)** $\tau_{RC} = R/C, \tau_{RL} = RL$

Note About Time Before/After Switches Open/Close

Time right ***before*** a switch changes position usually denoted as t = 0^-

Time right ***after*** a switch changes position usually denoted as t = 0+

Capacitor

Element law: $i(t) = C\dfrac{dv(t)}{dt}$

Stored energy: $w_E(t) = \dfrac{Cv(t)^2}{2}$

Inductor

Element law: $v(t) = L\dfrac{di(t)}{dt}$

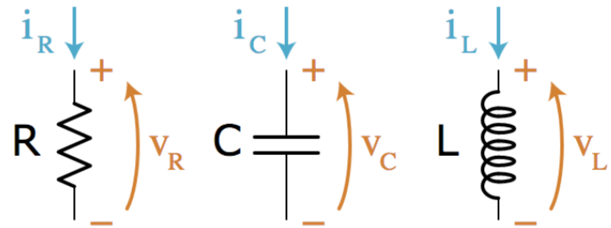
Stored energy: $w_M(t) = \dfrac{Li(t)^2}{2}$

For a capacitor, voltage $v_c(0^-) = v_c(0^+)$ [*V = Q/C --> charge stored in capacitor is same right before and after a switch opens/closes*]

For an inductor, current $i_L(0^-) = i_L(0^+)$ [*I = Flux/L --> magnetic flux in inductor is same right before and after a switch opens/closes*]

Power and Energy

Power = rate of change of energy per unit time



Power and Energy Relation in a Two-Terminal Element

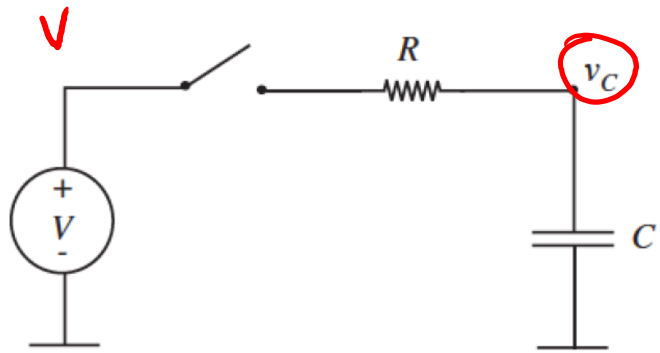
Power: $P(t) = i(t)v(t) = \frac{v(t)^2}{R} = i(t)^2 R$ [J/s or W]

where *i(t)* is defined to be positive if it enters at the positive terminal.

Energy: $w(t) = \int_0^T p(t)dt$

circuit elements can consume or release power/energy if *p(t) > 0* or *p(t) < 0*

Power and Energy in Simple RC Circuit



$\frac{V - V_c}{R}$

- Assume $v_c(0^-) = 0$ (zero state)

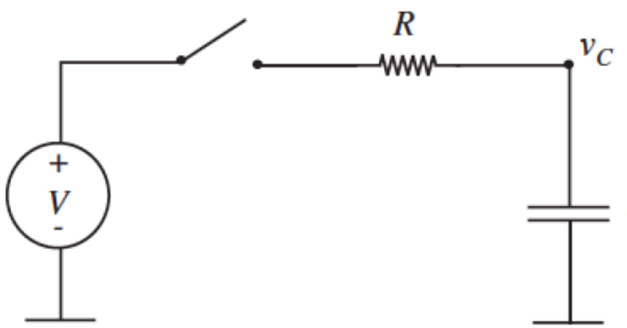
- Let's find the instantaneous power drawn from the voltage source when we close the switch at $t = 0$

- We know: $$p(t) = i(t)V$$

  where $$i(t) = [V - v_c(t)]/R$$

- We need to find $v_c(t)$ when $v_c(0^-) = 0$

Power and Energy in Simple RC Circuit

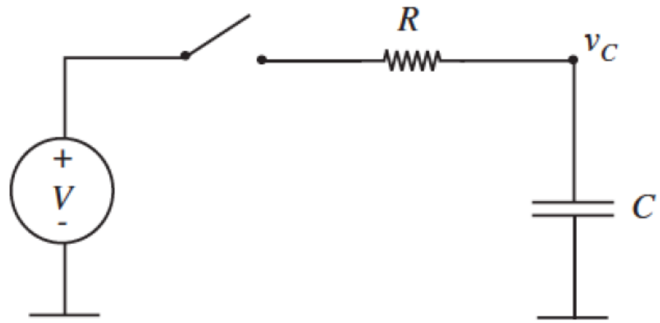


We can find the zero-state response of $v_C(t)$ to the step input $V_0 f(t)$ following the same steps as the parallel RC circuit using the generic expression:

$$v_c(t) = V_0 - V_0 e^{-\frac{t}{RC}}$$

$$\left[v(t) = \underbrace{v(\infty)}_{V_0} + [\underbrace{v(0)}_{0} - \underbrace{v(\infty)}_{V_0}]\, e^{-t/\underbrace{t_c}_{RC}} \right]$$

Power and Energy in Simple RC Circuit



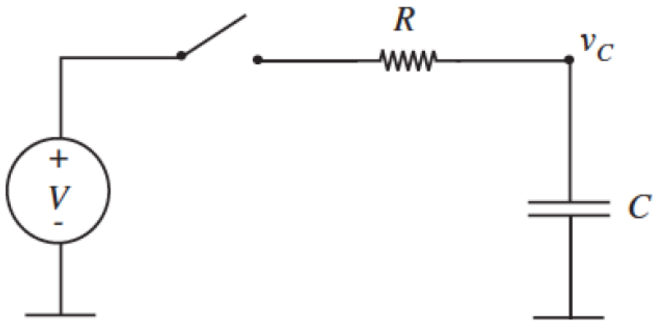
Thus, the instantaneous power ***drawn*** from the source:

$$p(t) = i(t)V = \frac{V_0 - v_c(t)}{R} \times V_0 = \frac{V_0 e^{-\frac{t}{RC}}}{R} \times V_0 = \frac{{V_0}^2 e^{-\frac{t}{RC}}}{R}$$

Energy consumption by an element

Energy: $w(t) = \int_0^T p(t)dt$

Power and Energy in Simple RC Circuit



$$p(t) = \frac{V_0{}^2 e^{-\frac{t}{RC}}}{R}$$

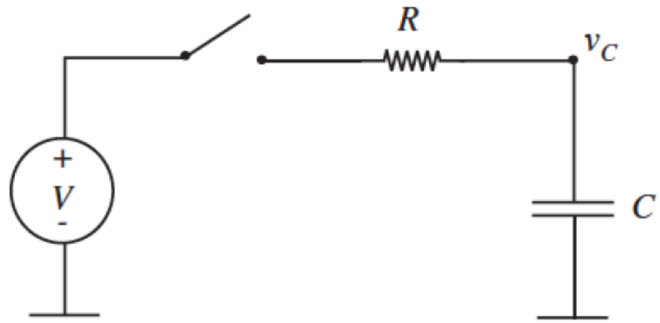
- Total energy **supplied by the voltage source**:

$$w = \int_0^\infty \frac{V_0{}^2 e^{-\frac{t}{RC}}}{R} dt = -\frac{V_0{}^2 RC e^{-\frac{t}{RC}}}{R} = CV_0{}^2$$

- Energy **stored in the capacitor**:

$$w = CV_0{}^2/2$$

Power and Energy in Simple RC Circuit



$$p(t) = \frac{V_0{}^2 e^{-\frac{t}{RC}}}{R}$$

- Total energy **supplied by the voltage source**:

$$w = \int_0^\infty \frac{V_0{}^2 e^{-\frac{t}{RC}}}{R} dt = -\frac{V_0{}^2 RC e^{-\frac{t}{RC}}}{R} = CV_0{}^2$$

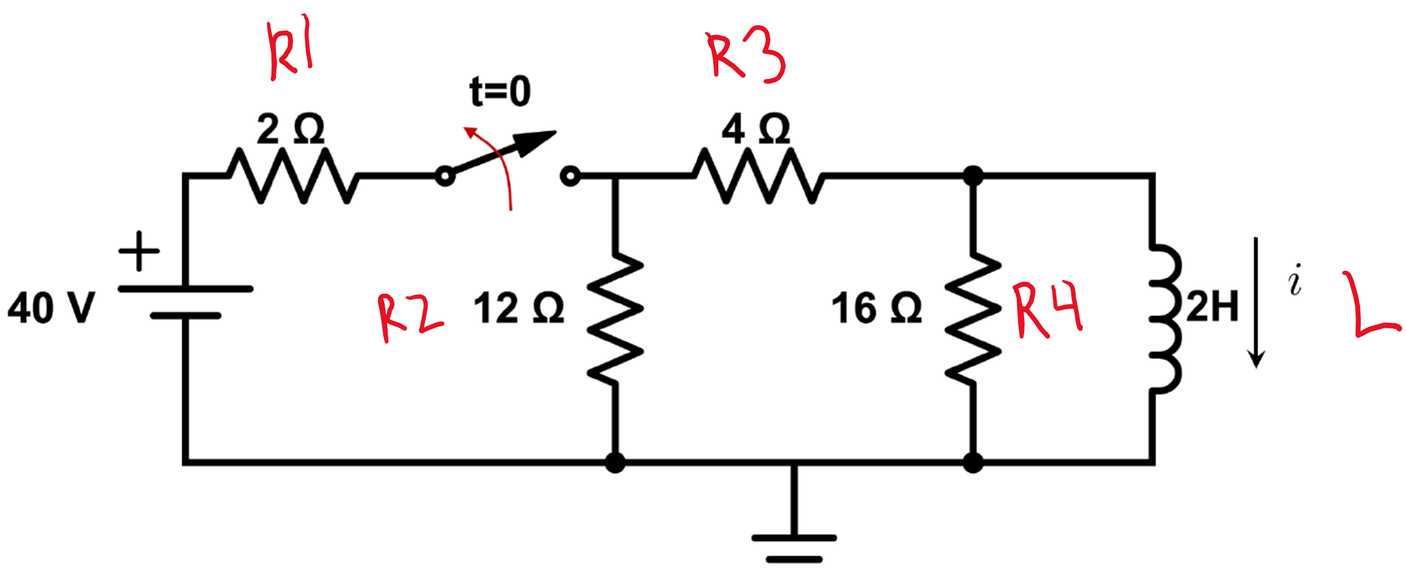
- Energy **stored in the capacitor**:

$w = CV_0{}^2/2$ **Doesn't equal energy supplied by source.** **Why???**

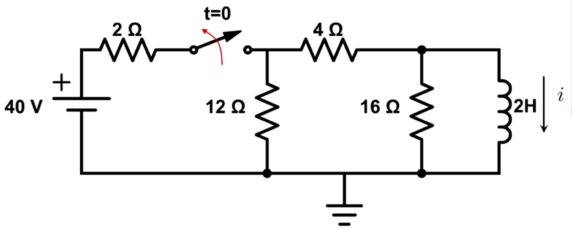
In Class Exercise

The switch in the circuit is opened at $t = 0$ after being closed for a long time.

1. Find $i(0)$
2. Find $i(t)$ for $t > 0$
3. Find the total energy dissipated by the circuit after $t > 0$



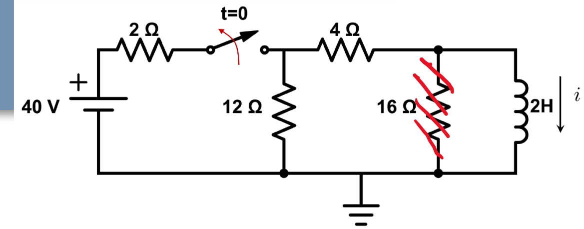
In Class Exercise



Find $i(0)$.

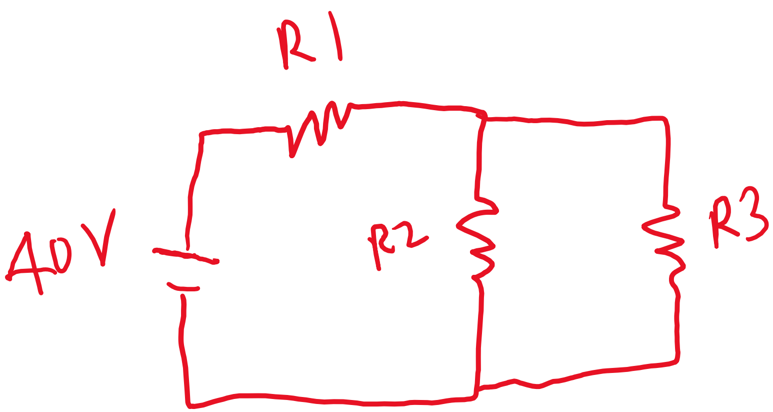
Switch has been closed for a long time.
What's the voltage across L?
What's the current through R4 (16 Ω resistor)?

In Class Exercise



Find $i(0)$.

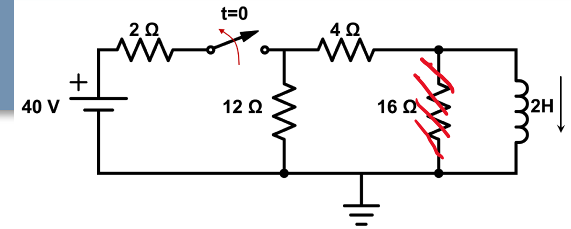
Switch has been closed for a long time.
What's the voltage across L?
What's the current through R4 (16 Ω resistor)?



let's find current drawn from V_s,

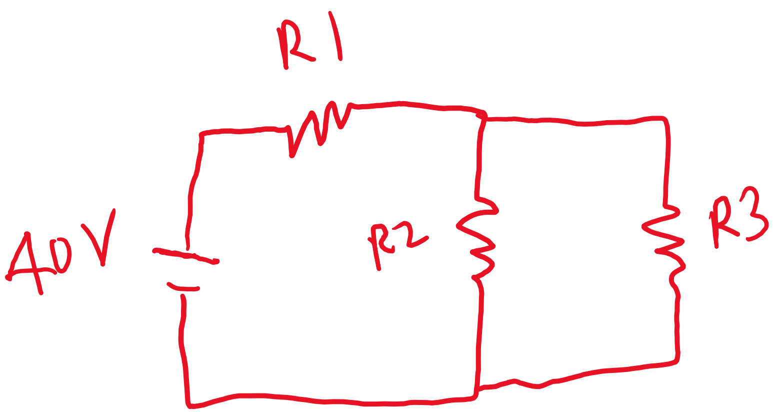
$$R = R_1 + R_2 /\!/ R_3 = 2 + \frac{12 \cdot 4}{16} = 5\,\Omega$$

In Class Exercise



Find *i*(0).

Switch has been closed for a long time.
What's the voltage across L?
What's the current through R4 (16 Ω resistor)?



Let's find current drawn from V_s.

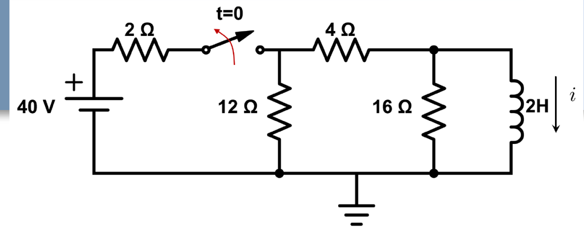
$$R = R_1 + R_2 /\!/ R_3 = 2 + \frac{12 \cdot 4}{16} = 5\,\Omega$$

$i = \frac{40V}{5\Omega} = 8A$. How much through R3?

Same V across R_2 & R_3. $\frac{R_3}{R_2+R_3} = \frac{1}{4}$

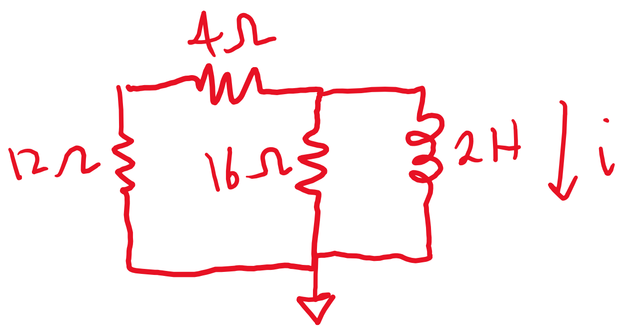
Thus 3/4 i through R_3 $\longrightarrow$ $\boxed{6A = i(0)}$

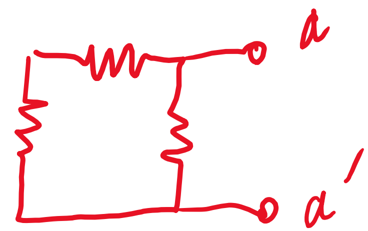
In Class Exercise



Switch opens. Find *i*(*t*).

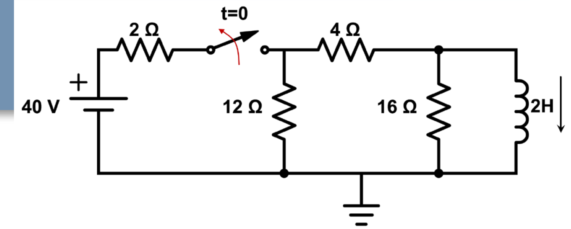
$$i_L(0^-) = i_L(0^+) = 6A.$$





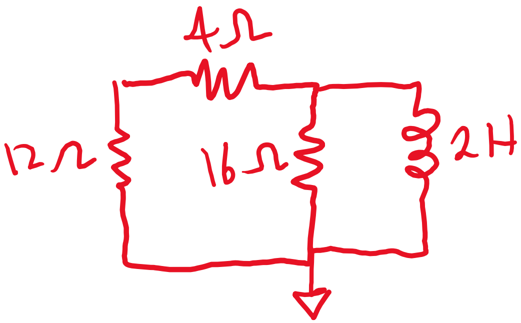
$$\rightarrow R_{th} = 16 // 16 = 8\,\Omega$$

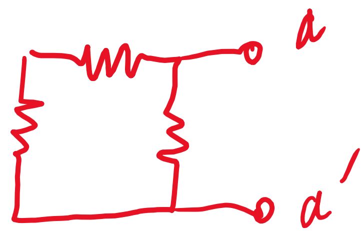
In Class Exercise



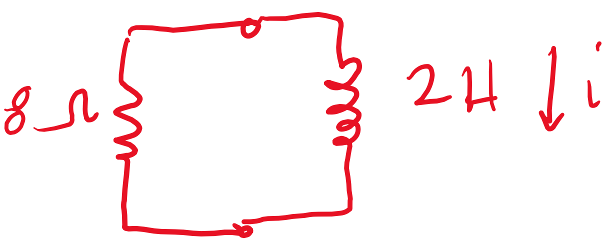
Switch opens. Find *i*(*t*).

$i_L(0^-) = i_L(0^+) = 6A.$





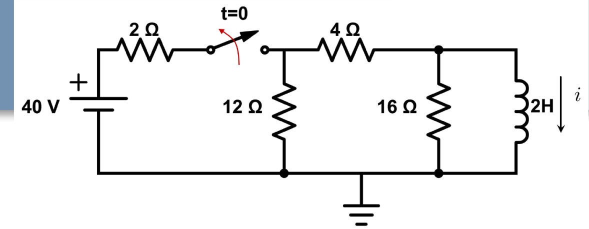
→ $R_{th} = 16 // 16 = 8\,\Omega$



$t_c = \frac{L}{R} = 0.25$ sec.

$i_L(t) = i(\infty) + (i(0) - i(\infty))\, e^{-t/t_c} = 6A\, e^{-t/t_c}$

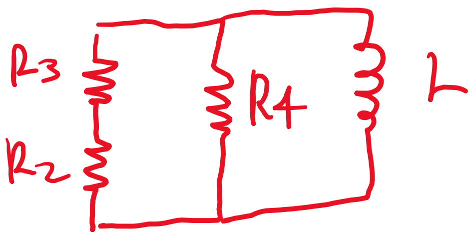
In Class Exercise



Determine power and energy dissipated.

$$p(t) = i(t)\,V(t) = \frac{V(t)^2}{R}$$

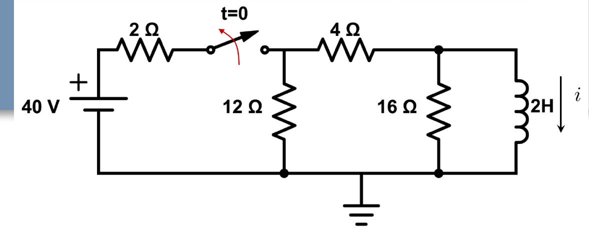
$$i(t) = 6e^{-4t}\ \text{A}$$



V_L = V across $R_2 + R_3$ = V across R_4

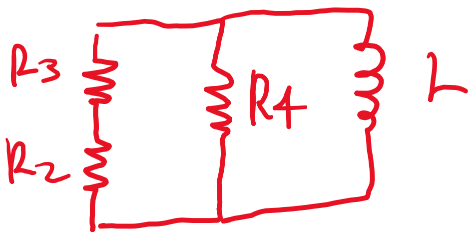
$$V_L = L\frac{di}{dt} = 2 \cdot 6 \cdot -4e^{-4t} = -48e^{-4t}\ \text{A}.$$

In Class Exercise



Determine power and energy dissipated.

$$p(t) = i(t)\,V(t) = \frac{V(t)^2}{R} \qquad i(t) = 6e^{-4t}\ \text{A}$$

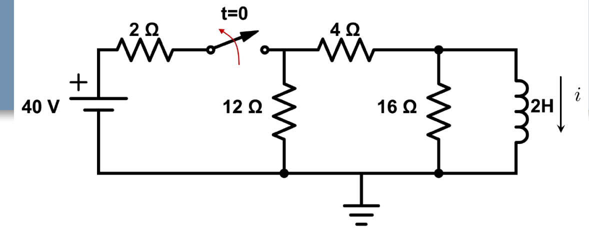


V_L = V across $R_2 + R_3$ = V across R_4

$$V_L = L\frac{di}{dt} = 2 \cdot 6 \cdot -4e^{-4t} = -48e^{-4t}\ \text{A.}$$

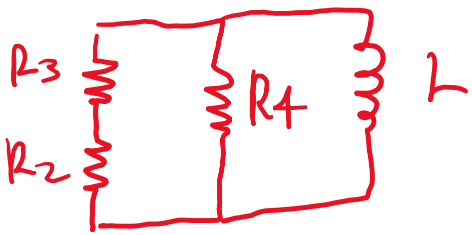
$$p(t) = \frac{(-48e^{-4t})^2}{R} = \frac{(-48e^{-4t})^2}{16\,\Omega} \cdot 2 = 288e^{-8t}$$

In Class Exercise



Determine power and energy dissipated.

$$p(t) = i(t)\,V(t) = \frac{V(t)^2}{R} \qquad i(t) = 6e^{-4t}\ \text{A}$$



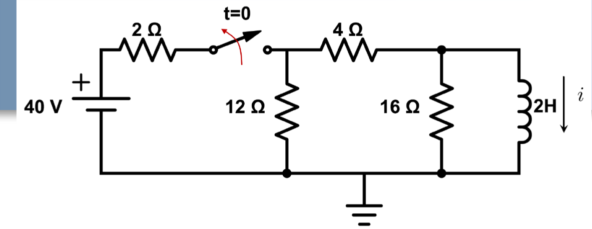
$$V_L = V \text{ across } R_2 + R_3 = V \text{ across } R_4$$

$$V_L = L\frac{di}{dt} = 2 \cdot 6 \cdot -4e^{-4t} = -48e^{-4t}\ \text{A}.$$

$$p(t) = \frac{(-48e^{-4t})^2}{R} = \frac{(-48e^{-4t})^2}{16\,\Omega} \cdot 2 = 288e^{-8t}$$

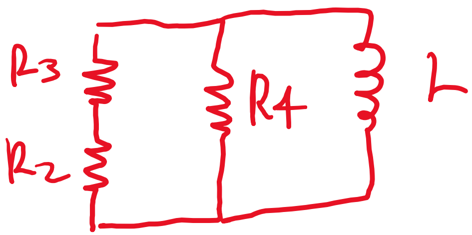
$$W = \int_0^\infty p(t)\,dt = \int_0^\infty 288e^{-8t}\,dt = \frac{288}{-8}e^{-8t}\Big|_0^\infty = 36\ \text{J}.$$

In Class Exercise



Determine power and energy dissipated.

$$p(t) = i(t)\,V(t) = \frac{V(t)^2}{R} \qquad i(t) = 6e^{-4t}\ \text{A}$$



V_L = V across $R_2 + R_3$ = V across R_4

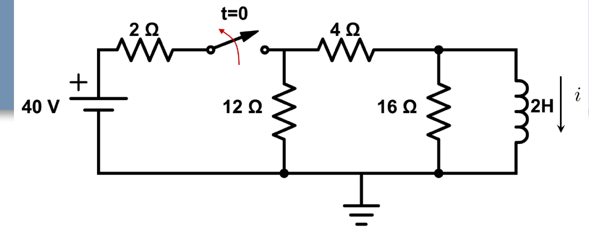
$$V_L = L\frac{di}{dt} = 2 \cdot 6 \cdot -4e^{-4t} = -48e^{-4t}\ \text{A}.$$

$$p(t) = \frac{(-48e^{-4t})^2}{R} = \frac{(-48e^{-4t})^2}{16\,\Omega} \cdot 2 = 288e^{-8t}$$

$$W = \int_0^\infty p(t)\,dt = \int_0^\infty 288e^{-8t}\,dt = \frac{288}{-8}e^{-8t}\Big|_0^\infty = 36\,\text{J}.$$

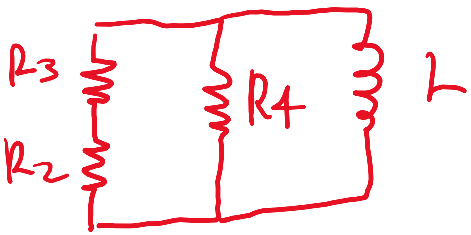
Energy stored in inductor?

In Class Exercise



Determine power and energy dissipated.

$$p(t) = i(t)\,V(t) = \frac{V(t)^2}{R} \qquad i(t) = 6e^{-4t}\ \text{A}$$



$$V_L = V_{\text{across } R_2+R_3} = V_{\text{across } R_4}$$

$$V_L = L\frac{di}{dt} = 2\cdot 6\cdot -4e^{-4t} = -48e^{-4t}\ \text{A.}$$

$$p(t) = \frac{(-48e^{-4t})^2}{R} = \frac{(-48e^{-4t})^2}{16\,\Omega}\cdot 2 = 288e^{-8t}$$

$$w = \int_0^\infty p(t)\,dt = \int_0^\infty 288e^{-8t}\,dt = \frac{288}{-8}e^{-8t}\Big|_0^\infty = 36\,J.$$

Energy stored in inductor? $w = \frac{1}{2}L I_0^2 = \frac{1}{2}\cdot 2\cdot 6^2 = 36J$

Good!

Summary: Capacitor and Inductor 1st Order ODEs

$$y(t) = \text{final value} + (\text{initial value} - \text{final value})e^{-t/t_c}$$

$$v_c(t) = v_c(\infty) + (v_c(0) - v_c(\infty))e^{-t/RC}$$

$$i_L(t) = i_L(\infty) + (i_L(0) - i_L(\infty))e^{-Rt/L}$$

****applies to branch variables such as C or L current, resistor V, etc.****

Capacitor

- Behaves as instantaneous **short** circuit when inputs make abrupt changes
- Behaves as **open** circuit at long times when driven by DC voltage source

Inductor

- Behaves as instantaneous **open** circuit when inputs make abrupt changes
- Behaves as **short** circuit at long times when driven by DC current source

Power and Energy Relation in a Two-Terminal Element

Power:

$$P(t) = i(t)v(t) = \frac{v(t)^2}{R}$$

Energy:

$$w(t) = \int_0^T p(t)dt$$

ECE 10C
Fall 2020
Slide Set 2
Instructor: Galan Moody
TA: Kamyar Parto

Last Week

- Review of RL/RC
- Energy + power

This Week

- Undriven RLC circuits

Important Items:

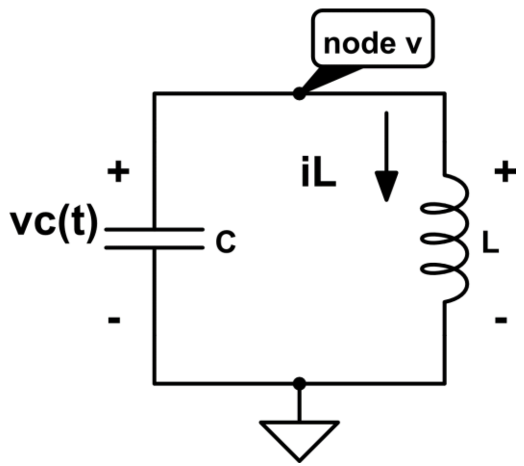
Homework #1 Due Thurs
Homework #2 Due 10/22

Lab #1 due Friday
Lab #2 due 10/23

This Week

- Undriven LC circuit (*i.e.* no source)
- Undriven series RLC circuit
- Finding initial conditions in 2nd-order circuits
- General expressions for 2nd-order circuits

Undriven LC Circuit: 2nd-Order Transients

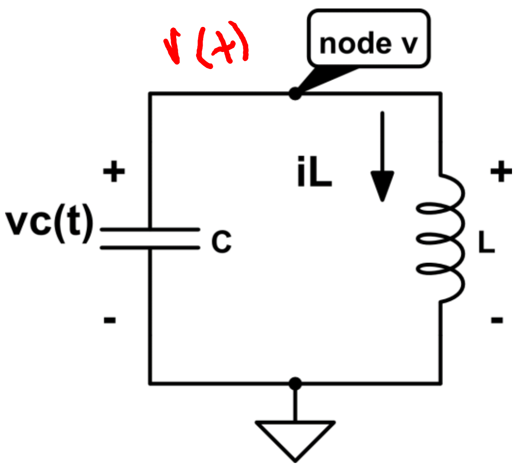


Second-order systems contain two energy storage elements with independent states

Solution Steps to find v

1. Use node method to find the differential equation describing v
2. Find the homogeneous solution v_h (set the drive to zero)
3. Find the particular solution v_p (educated guess)
4. The total solution is the sum of the homogeneous and particular solutions. Use the initial conditions to solve for the constants.

Undriven LC Circuit: 2nd-Order Transients



Element Laws

Capacitor: $i_c(t) = C\frac{dv_c(t)}{dt}$

Inductor: $v_L(t) = L\frac{di_L(t)}{dt}$

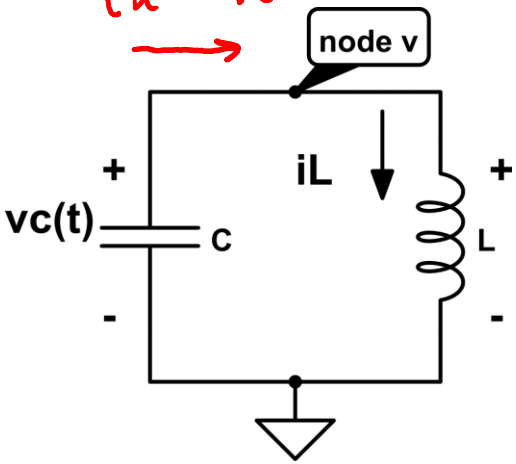
- Using KCL/node analysis, we want to calculate *v(t)* at the node v. If we do this, then we can determine $v_c(t)$ and $i_L(t)$, which are the quantities of interest that we care about:

$$v(t) = v_c(t)$$

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v(t')dt'$$

Undriven LC Circuit: 2nd-Order Transients

$i_{in} - i_{out} = 0$



Element Laws

Capacitor: $$i_c(t) = C\frac{dv_c(t)}{dt}$$

Inductor: $$v_L(t) = L\frac{di_L(t)}{dt}$$

- Let's use KCL in terms of v at the top node:

$$\frac{d}{dt}\left[C\frac{dv(t)}{dt} + \frac{1}{L}\int_{-\infty}^{t} v(t')dt' = 0\right]$$

- By differentiating with respect to time, we get the 2nd-order differential equation:

$$\frac{d^2v(t)}{dt^2} + \frac{1}{LC}v(t) = 0$$

Undriven LC Circuit: 2nd-Order Transients

- There's no drive/source, so this is the homogeneous equation:

$$\frac{d^2v(t)}{dt^2} + \frac{1}{LC}v(t) = 0$$

Remember!

The homogeneous solution to any constant coefficient ODE is always a superposition of functions of the form Ae^{st}

- Insert Ae^{st} to get the characteristic equation:

$$s^2 \cdot Ae^{st} + \frac{1}{LC}Ae^{st} = 0 \quad \Big| \quad s^2 + \frac{1}{LC} = 0$$

Undriven LC Circuit: 2nd-Order Transients

- Characteristic equation: $$s^2 + \frac{1}{LC} = 0$$ $\omega_0^2 \equiv \frac{1}{LC}$
- There are two roots (natural frequencies):

$$s^2 + \omega_0^2 = 0 \rightarrow s = \pm\sqrt{-\omega_0^2}$$

$$s_1 = +j\omega_0 \qquad s_2 = -j\omega_0$$

- Thus, the solution for *v(t)* is a linear combination of two functions:

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{+j\omega_0 t} + A_2 e^{-j\omega_0 t}$$

We can work with complex exponentials, but it can be more intuitive to work with sinusoidal functions

Undriven LC Circuit: 2nd-Order Transients

- Using Euler's relation: $e^{j\omega_0 t} = cos(\omega_0 t) + jsin(\omega_0 t)$

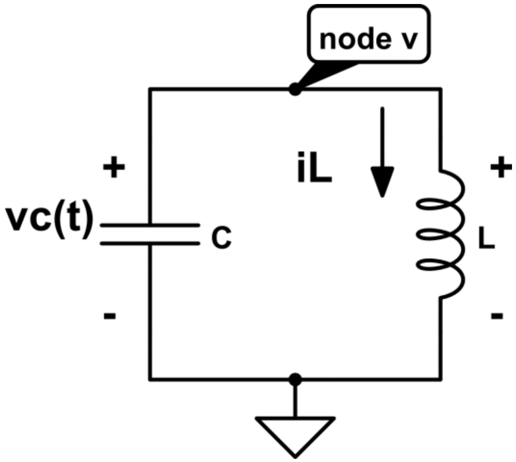
$$v(t) = A_1 e^{+j\omega_0 t} + A_2 e^{-j\omega_0 t} = K_1 cos(\omega_0 t) + K_2 sin(\omega_0 t)$$

- We need to determine K_1 and K_2 using initial conditions
- Mathematically, this means we need v and dv/dt at a particular time (usually this is the initial time *t* = 0)
- For example, let's assume that we know $v(0)$ and $dv(0)/dt$:

$K_1 = v(0)$

$K_2 = \frac{1}{\omega_0} \cdot \frac{dv(0)}{dt}$

Undriven LC Circuit: 2nd-Order Transients



$$K_1 = v(0) \qquad K_2 = \frac{1}{\omega_0}\frac{dv(0)}{dt}$$

$$v(t) = v(0)cos(\omega_0 t) + \frac{1}{\omega_0}\frac{dv(0)}{dt}sin(\omega_0 t)$$

- What we really want though is not $v(t)$, but $v_c(t)$ and $i_L(t)$
- Let's evaluate these at time $t = 0$

$$v_c(0) = v(0), \qquad i_c(0) = C\frac{dv(0)}{dt} = -i_L(0) \rightarrow \frac{dv(0)}{dt} = \frac{-i_L(0)}{C}$$

$$v(t) = v_c(t) = v_c(0)\,cos(\omega_0 t) - \sqrt{\frac{L}{C}}\,i_L(0)sin(\omega_0 t)$$

Undriven LC Circuit: 2nd-Order Transients

$$v(t) = v_c(t) = v_c(0)cos(\omega_0 t) - \sqrt{\frac{L}{C}}i_L(0)sin(\omega_0 t)$$

- Let's express $v_c(t)$ as a single sinusoidal function:

Two useful trig identities:

$$a\cos(x) - b\sin(x) = \sqrt{a^2 + b^2}\cos(x + tan^{-1}(b/a))$$
$$a\sin(x) + b\cos(x) = \sqrt{a^2 + b^2}\sin(x + tan^{-1}(b/a))$$

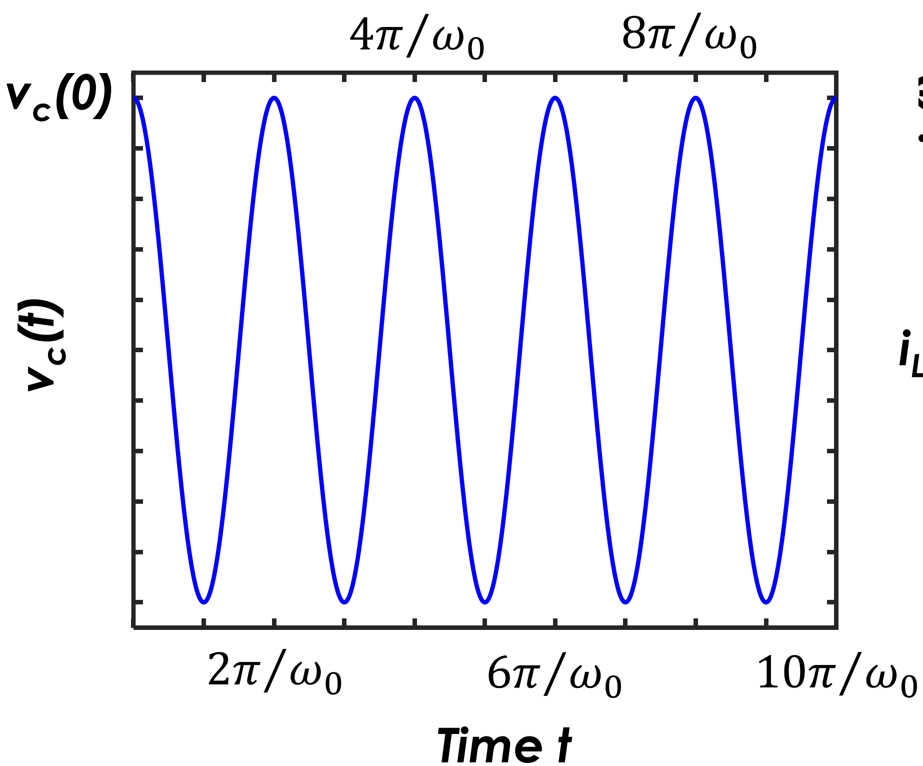
- So we get:

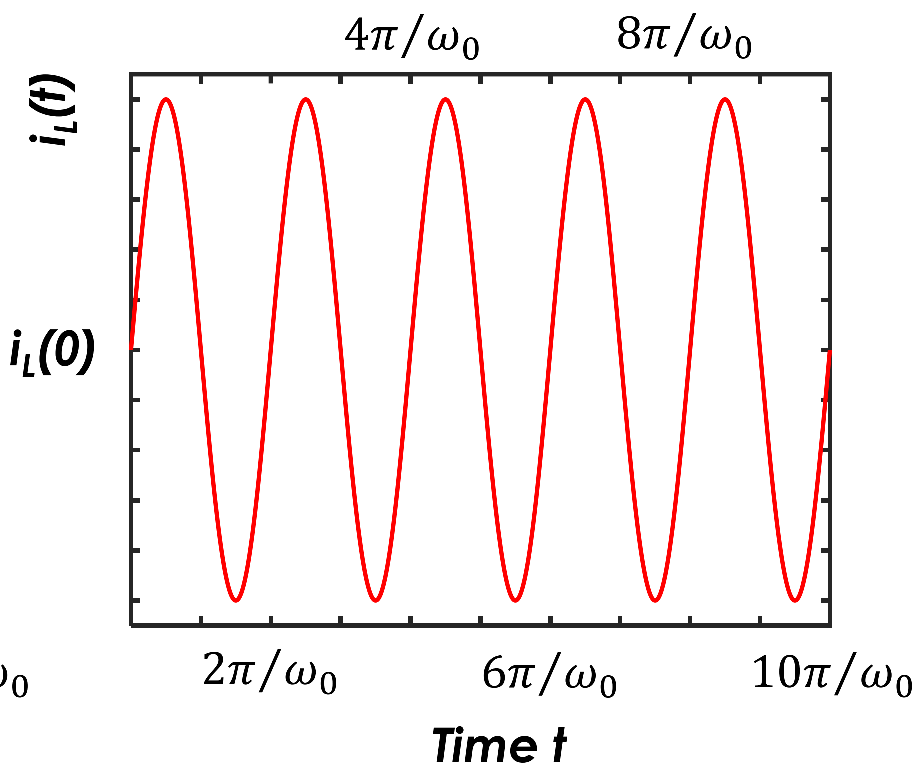
$$v_c(t) = \sqrt{v_c^2(0) + \frac{L}{C}i_L^2(0)}\cos\left(\omega_0 t + tan^{-1}\left(\sqrt{\frac{L}{C}}\frac{i_L(0)}{v_c(0)}\right)\right)$$

$$i_L(t) = \sqrt{\frac{C}{L}}\sqrt{v_c^2(0) + \frac{L}{C}i_L^2(0)}\sin\left(\omega_0 t + tan^{-1}\left(\sqrt{\frac{L}{C}}\frac{i_L(0)}{v_c(0)}\right)\right)$$

Undriven LC Circuit: 2nd-Order Transients

$$v_c(t) \propto \cos(\omega_0 t + \varphi) \qquad i_L(t) \propto \sqrt{\frac{C}{L}} \sin(\omega_0 t + \varphi)$$

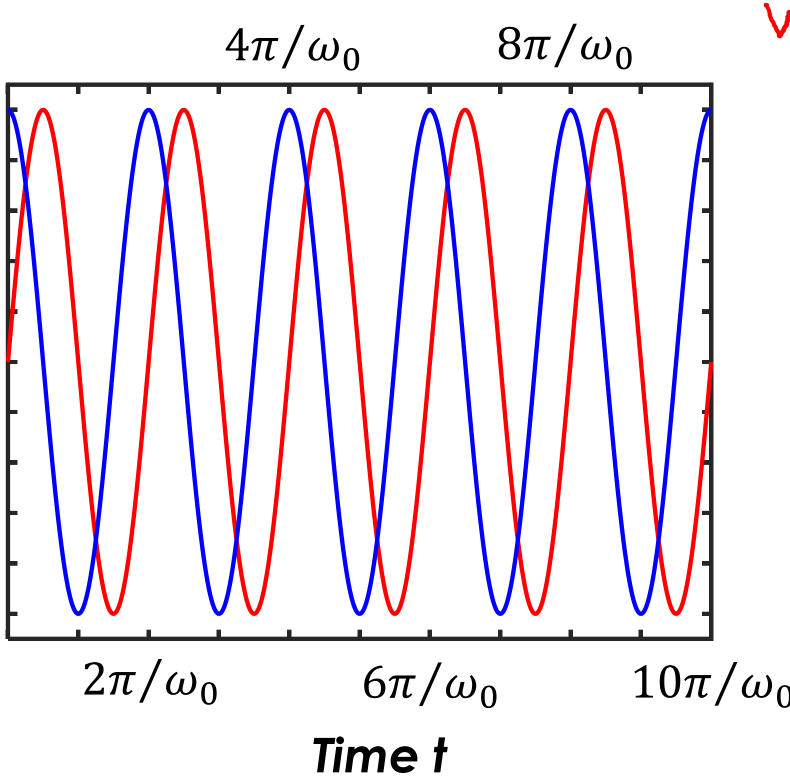


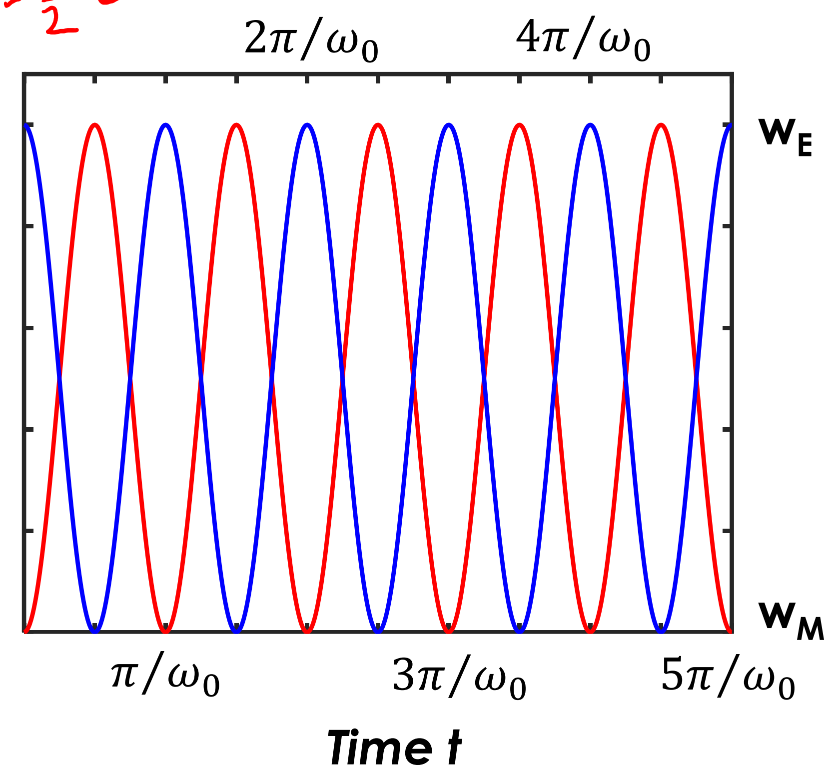


Undriven LC Circuit: 2nd-Order Transients

- $v_c(t)$ and $i_L(t)$ are a quarter cycle out of phase with each other
- Repetitive exchange of energy = lossless oscillation

$W_E = \frac{1}{2} C V^2(t)$ $W_M = \frac{1}{2} L I^2(t)$

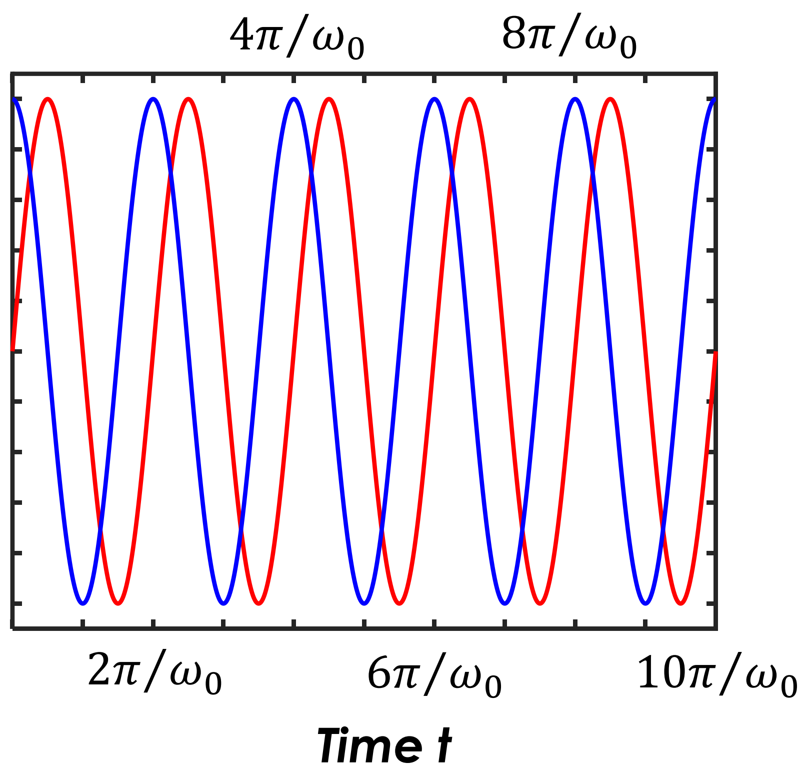


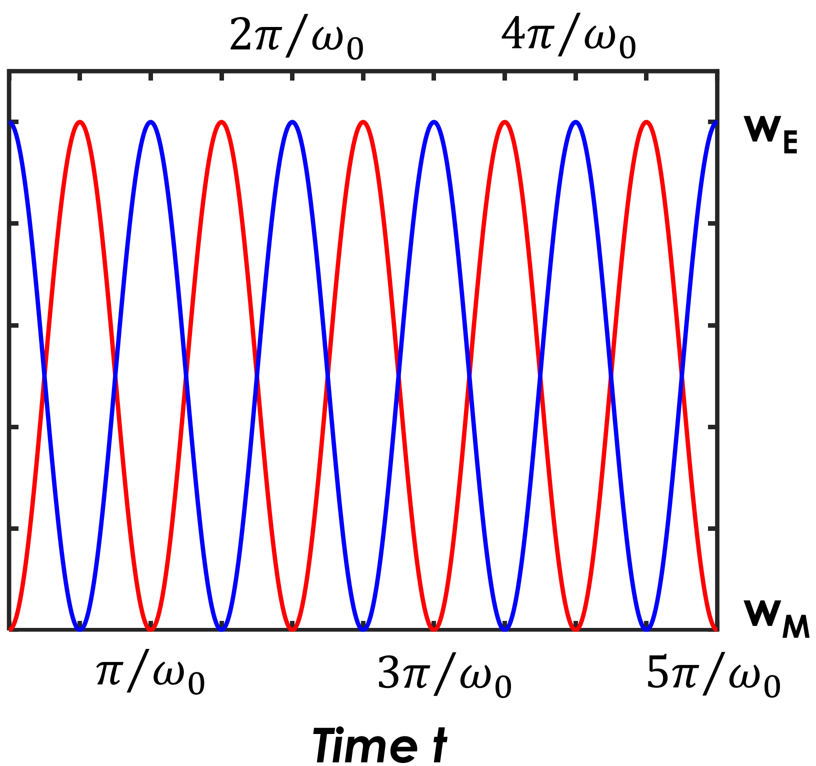


Undriven LC Circuit: 2nd-Order Transients

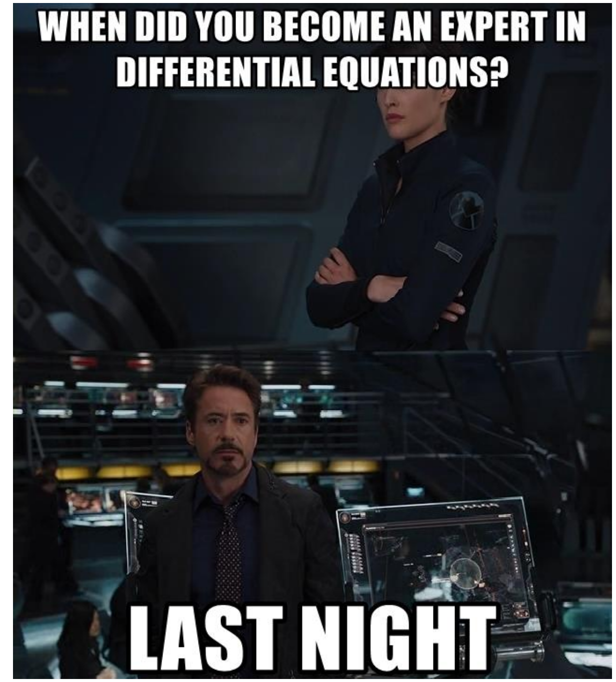
- **Total energy stored in circuit remains a constant:**

$$w_T = w_E + w_M = {}^{1}\!/_{2}\, C v_c^2(0) + {}^{1}\!/_{2}\, L i_L^2(0)$$





ECE 10C
Fall 2020
Slide Set 2
Instructor: Galan Moody
TA: Kamyar Parto



Tuesday

- Finished energy/power
- Undriven LC circuit

Today

- RLC circuit

Important Items:

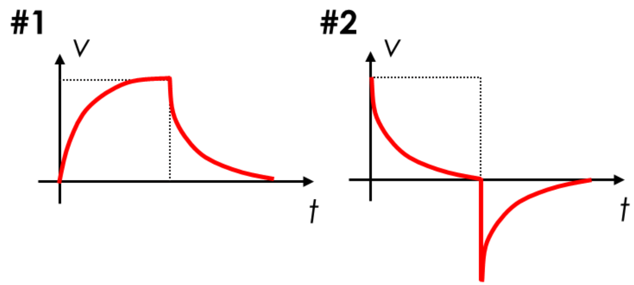
Homework #1 Due!
Homework #2 Due 10/22

Lab #1 due Tomorrow
Lab #2 due 10/23

Last Quiz

Q1 [1 point]. True or False? The energy stored in an inductor is equal to $1/2 \times L \times I^2$.

Q2 [1 point]. The figure shows the voltage response of two circuits to a square pulse source. Choose the correct answer:



(a) #1 = RC, #2 = RL

(b) #1 = RL, #2 = RC

Q3 [1 point]. True or False? If initially "at rest", an inductor behaves as an instantaneous **open** circuit when the input makes an abrupt change.

Q4 [1 point]. True or False? An inductor behaves as a **short** circuit at long times when driven by a DC voltage source.

Q5 [1 point]: What is the correct expression for the RC and RL circuit time constants?

(a) $\tau_{RC} = RC, \tau_{RL} = RL$ **(b)** $\tau_{RC} = RC, \tau_{RL} = L/R$ **(c)** $\tau_{RC} = RC, \tau_{RL} = R/L$ **(d)** $\tau_{RC} = R/C, \tau_{RL} = RL$

Quiz Time!

Q1 [1 point]. True or False? The resonant frequency of an LC circuit is $\sqrt{LC}$.

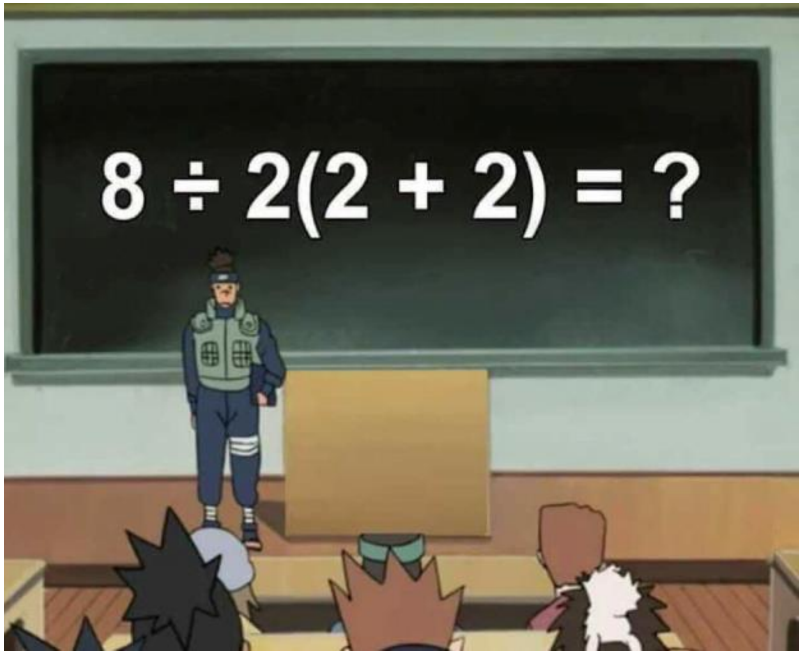
Q2 [1 point]. True or False? In an LC circuit, the capacitor voltage and inductor current are π out of phase.

Q3 [1 point]. True or False? An LC circuit is analogous to a mass on a spring, where the energy stored in the spring corresponds to the energy stored in the capacitor, and the kinetic energy of the mass corresponds to the magnetic energy in the inductor.

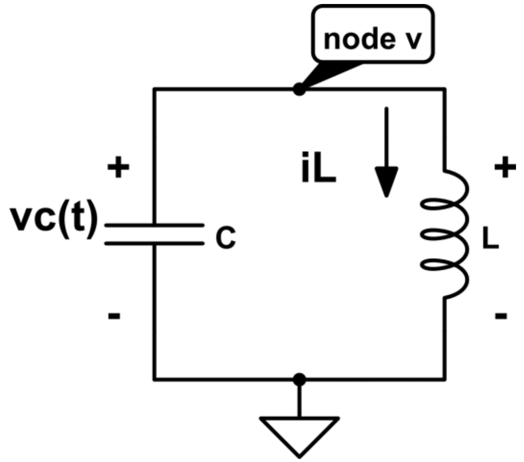
Q4 [1 point]. True or False? The current in an overdamped LC circuit decays to zero more quickly compared to the current in a critically damped system.

Q5 [1 point]. This question broke the internet last year. Did we come to a consensus as a society? I don't think so. Where to you stand?

- **(a)** 16
- **(b)** 1
- **(c)** Something else



Undriven LC Circuit: 2nd-Order Transients

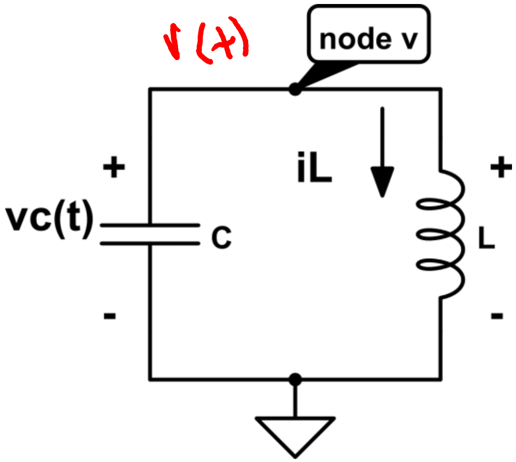


Second-order systems contain two energy storage elements with independent states

Solution Steps to find v

1. Use node method to find the differential equation describing v
2. Find the homogeneous solution v_h (set the drive to zero)
3. Find the particular solution v_p (educated guess)
4. The total solution is the sum of the homogeneous and particular solutions. Use the initial conditions to solve for the constants.

Undriven LC Circuit: 2nd-Order Transients



Element Laws

Capacitor: $i_c(t) = C\dfrac{dv_c(t)}{dt}$

Inductor: $v_L(t) = L\dfrac{di_L(t)}{dt}$

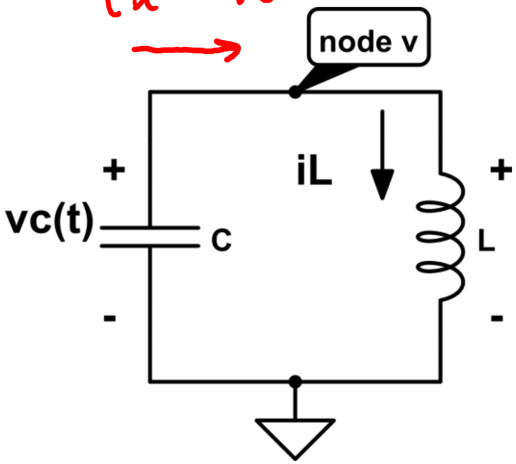
- Using KCL/node analysis, we want to calculate *v(t)* at the node v. If we do this, then we can determine $v_c(t)$ and $i_L(t)$, which are the quantities of interest that we care about:

$$v(t) = v_c(t)$$

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v(t')dt'$$

Undriven LC Circuit: 2nd-Order Transients

$i_{in} - i_{out} = 0$



Element Laws

Capacitor: $i_c(t) = C\frac{dv_c(t)}{dt}$

Inductor: $v_L(t) = L\frac{di_L(t)}{dt}$

- Let's use KCL in terms of v at the top node:

$$\frac{d}{dt}\left[C\frac{dv(t)}{dt} + \frac{1}{L}\int_{-\infty}^{t} v(t')dt' = 0 \right]$$

- By differentiating with respect to time, we get the 2nd-order differential equation:

$$\frac{d^2v(t)}{dt^2} + \frac{1}{LC}v(t) = 0$$

Undriven LC Circuit: 2^{nd}-Order Transients

- There's no drive/source, so this is the homogeneous equation:

$$\frac{d^2v(t)}{dt^2} + \frac{1}{LC}v(t) = 0$$

Remember!

The homogeneous solution to any constant coefficient ODE is always a superposition of functions of the form Ae^{st}

- Insert Ae^{st} to get the characteristic equation:

$$s^2 \cdot Ae^{st} + \frac{1}{LC}Ae^{st} = 0 \quad \Big| \quad s^2 + \frac{1}{LC} = 0$$

Undriven LC Circuit: 2nd-Order Transients

- Characteristic equation: $s^2 + \frac{1}{LC} = 0$ $\quad \omega_0^2 \equiv \frac{1}{LC}$

- There are two roots (natural frequencies):

$$s^2 + \omega_0^2 = 0 \rightarrow s = \pm\sqrt{-\omega_0^2}$$

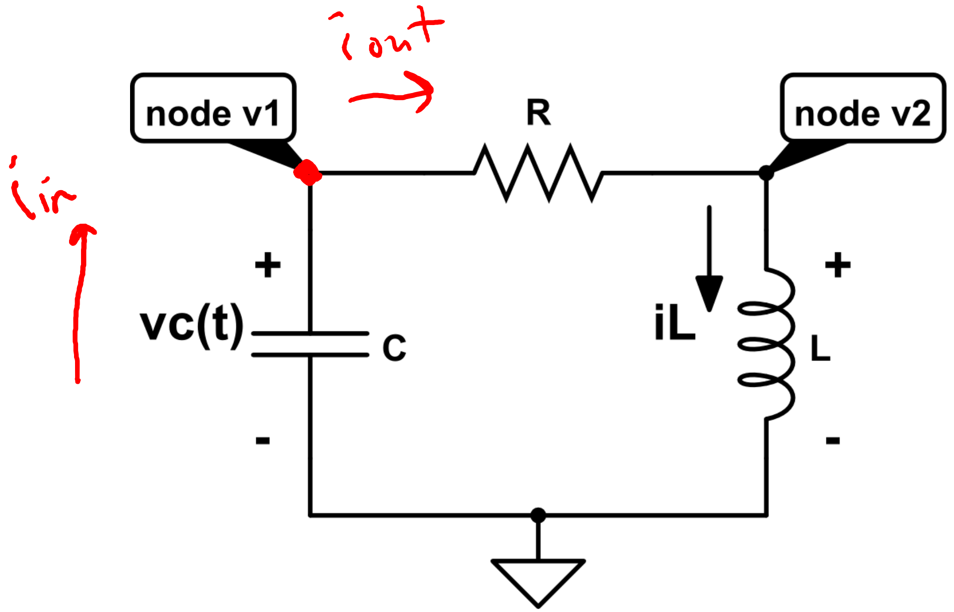
$$s_1 = +j\omega_0 \qquad s_2 = -j\omega_0$$

- Thus, the solution for *v(t)* is a linear combination of two functions:

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{+j\omega_0 t} + A_2 e^{-j\omega_0 t}$$

Undriven RLC Circuit: 2nd-Order Transients

$-i_{in} - i_{out} = 0$



RLC circuit behaves similarly to an LC circuit, but energy is now dissipated by R, thus the energy stored in the circuits decays in time

Let's find v1:

- Node 1 KCL: $$-C\frac{dv_1(t)}{dt} - \frac{v_1(t) - v_2(t)}{R} = 0$$

- Node 2 KCL: $$\frac{v_1(t) - v_2(t)}{R} - \frac{1}{L}\int_{-\infty}^{t} v_2(t')dt' = 0$$

$i_{in} - i_{out} = 0$

Undriven RLC Circuit: 2nd-Order Transients

- Node 1 KCL: $$-C\frac{dv_1(t)}{dt} - \frac{v_1(t) - v_2(t)}{R} = 0$$

- Node 2 KCL: $$\frac{v_1(t) - v_2(t)}{R} - \frac{1}{L}\int_{-\infty}^{t} v_2(t')dt' = 0$$

- We can eliminate v_2 using the first equation:

$$\frac{d^2v_1(t)}{dt^2} + \frac{R}{L}\frac{dv_1(t)}{dt} + \frac{1}{LC}v_1(t) = 0$$

- By substituting Ae^{st}, we get the characteristic equation:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Undriven RLC Circuit: 2nd-Order Transients

- We can also write the characteristic equation as:

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

- Where we have defined:

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- The roots of the characteristic equation are:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

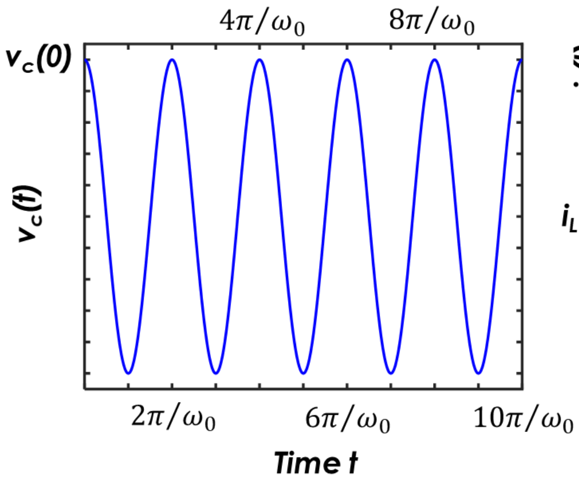
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

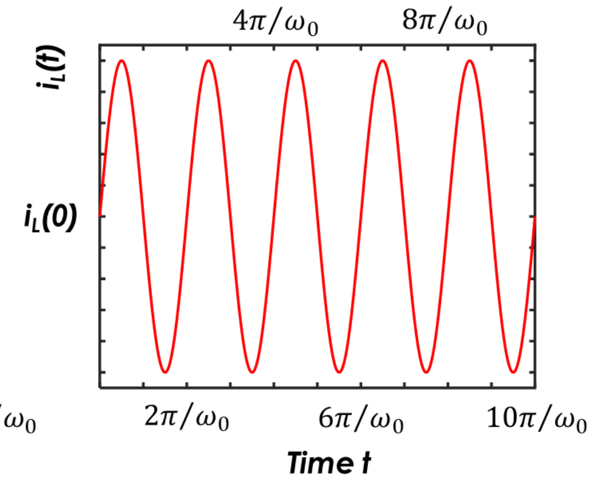
$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

***unless $s_1 = s_2$**

Undriven RLC Circuit: 2^{nd}-Order Transients

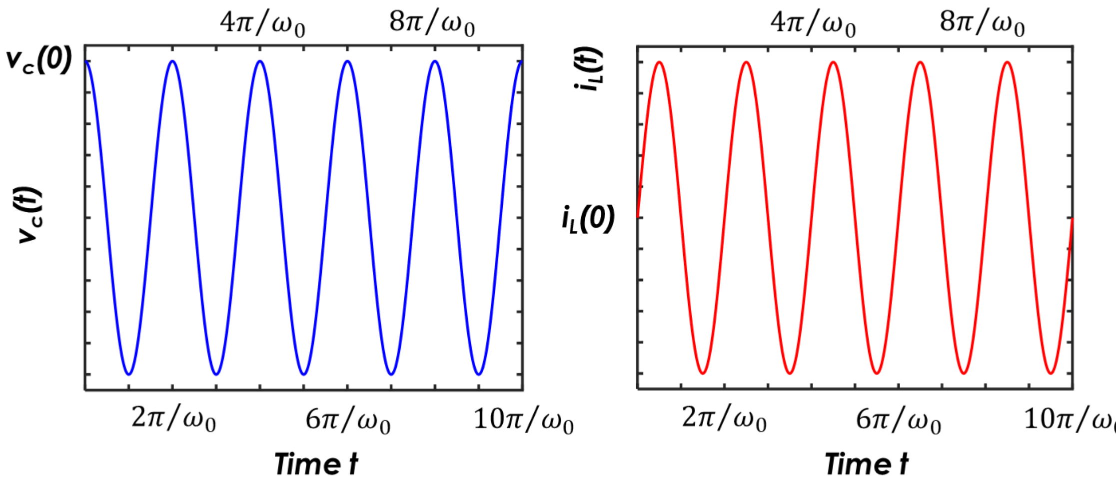
Remember what we had for $v_c(t)$ and $i_L(t)$ for an undriven LC circuit:



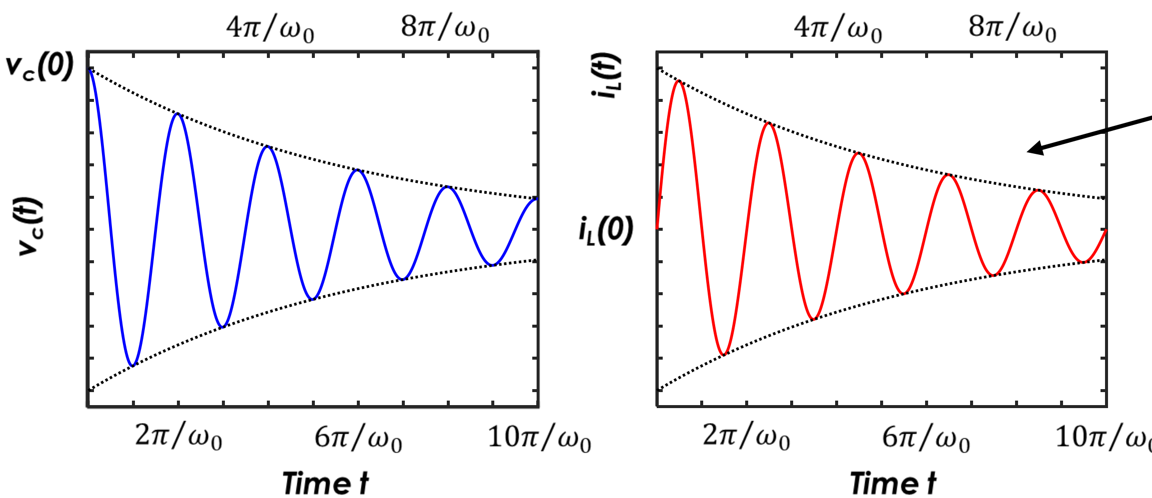


Undriven RLC Circuit: 2nd-Order Transients

Remember what we had for $v_c(t)$ and $i_L(t)$ for an undriven LC circuit:



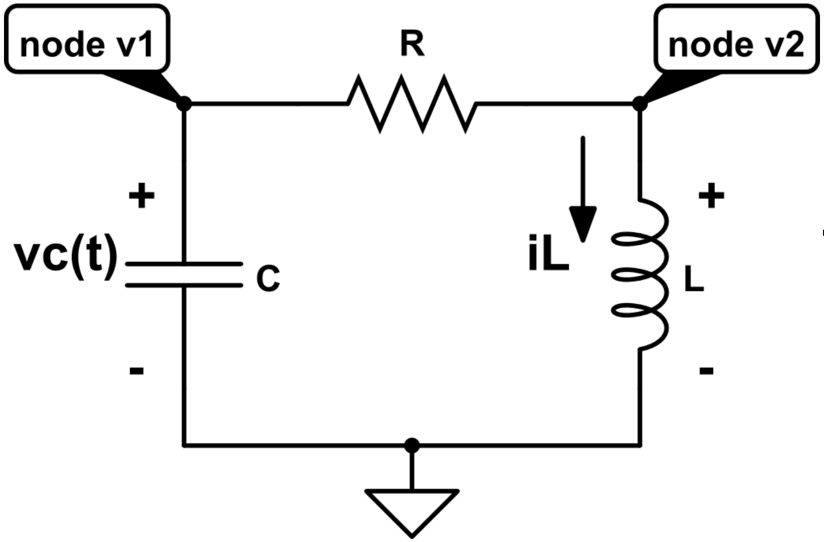
For an undriven RLC circuit, $v_C(t)$ and $i_L(t)$ could look like:



$$e^{-\alpha t}$$

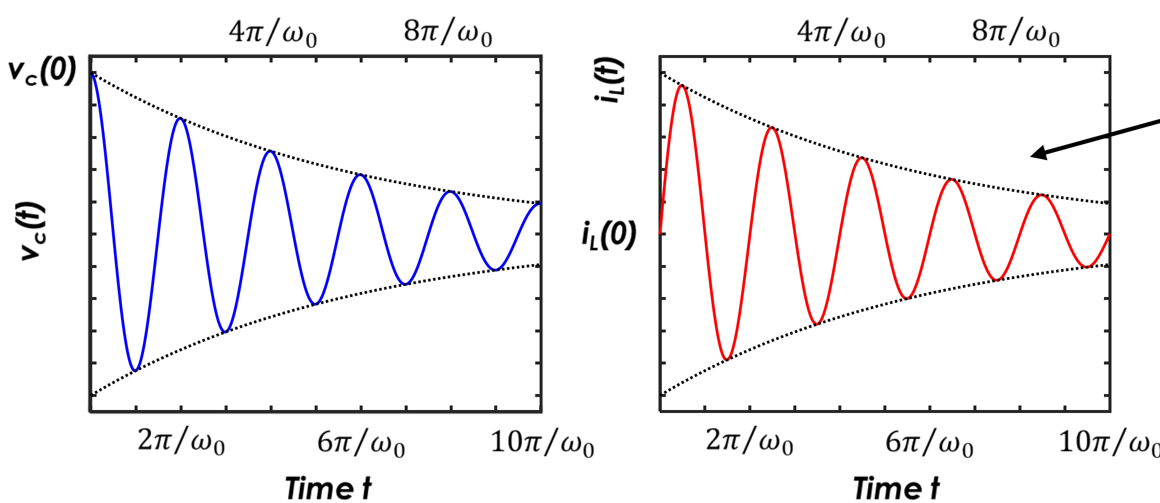
$$\alpha = \frac{R}{2L}$$

Undriven RLC Circuit: 2nd-Order Transients



For a series RLC circuit, why does the damping coefficient only depend on R and L? Why not C?

For an undriven RLC circuit, $v_C(t)$ and $i_L(t)$ <u>could look like:</u>



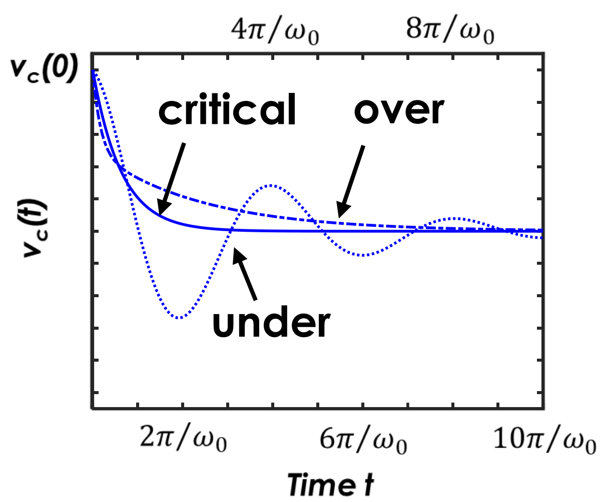
$e^{-\alpha t}$

$\alpha = \dfrac{R}{2L}$

Undriven RLC Circuit: 2^{nd}-Order Transients

The dynamic behavior of an RLC circuit is different for three cases:

$\alpha < \omega_0 \rightarrow$ **under-damped dynamics**
$\alpha = \omega_0 \rightarrow$ **critically damped dynamics**
$\alpha > \omega_0 \rightarrow$ **over-damped dynamics**

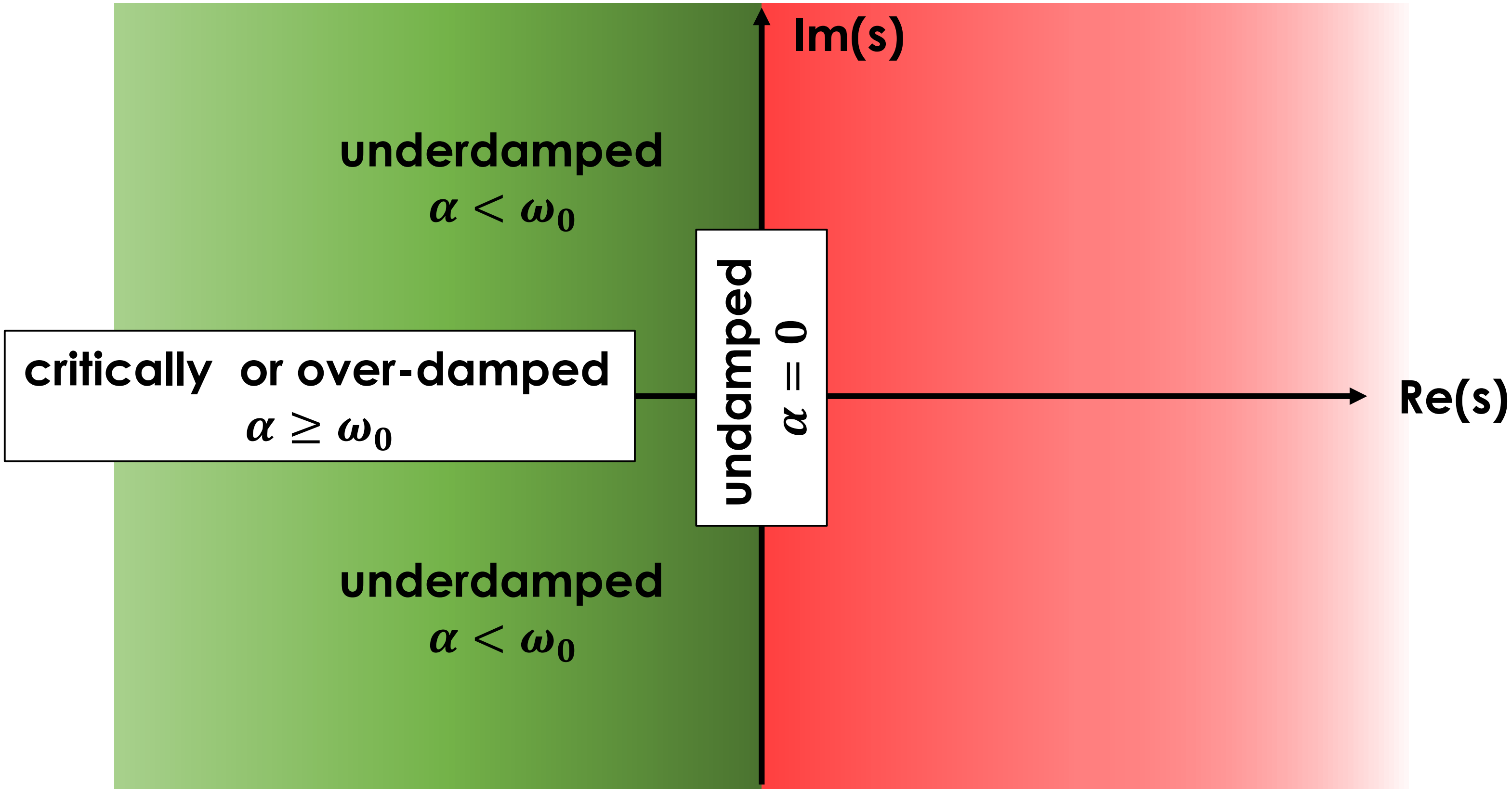


Damping on an oscillatory system has the effect of reducing, restricting, or preventing oscillations. **In physical systems, damping is produced by processes that dissipate energy.**

Undriven RLC Circuit: 2nd-Order Transients

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

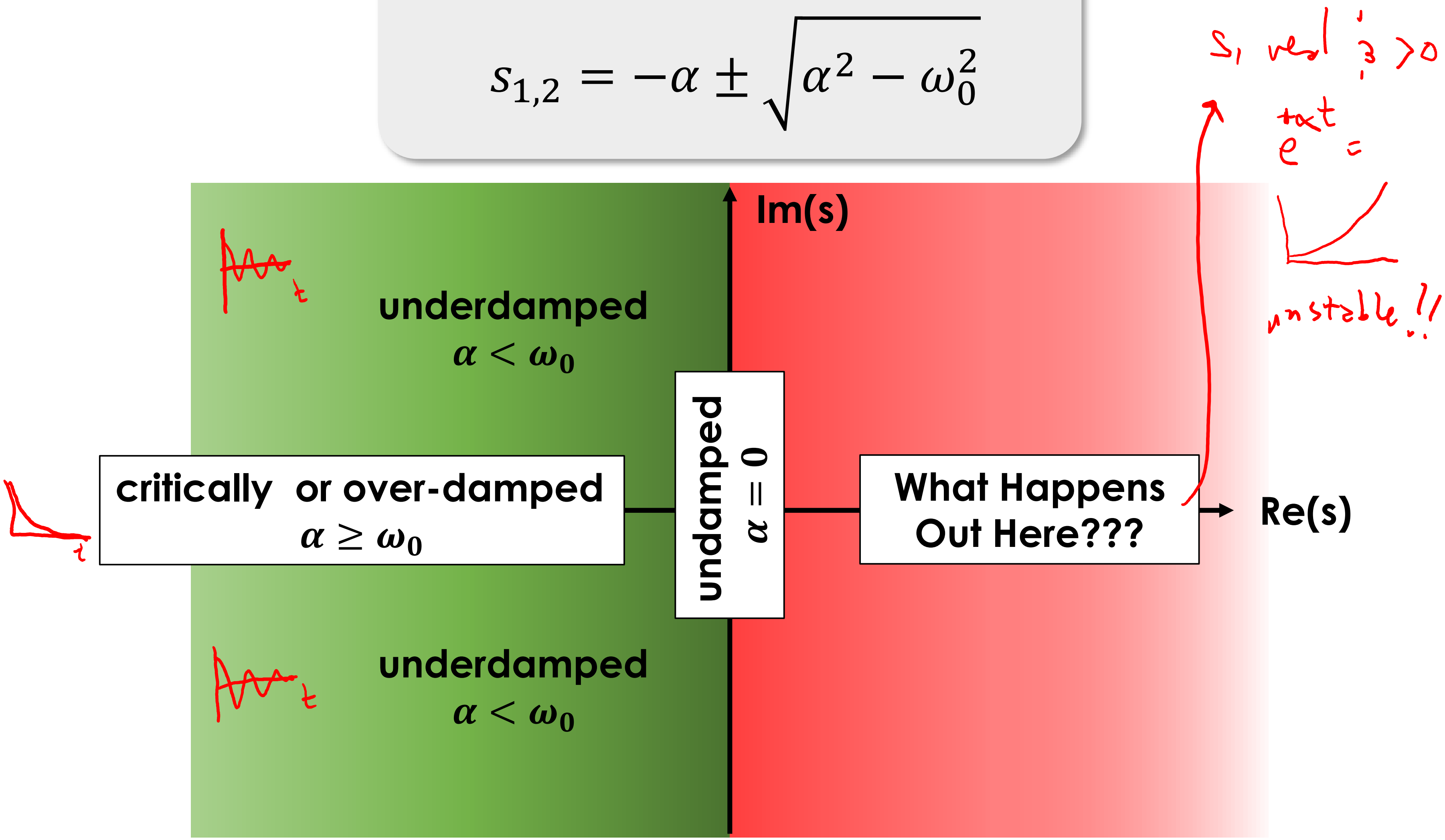
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$



Undriven RLC Circuit: 2nd-Order Transients

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$



Why Care About Damping and Transient Response?

One of the most dramatic failures of engineering: the 1940 collapse of the Narrow Bridge:

This Course: Dynamic Behavior of 2^{nd} Order Systems

Chapter 12: Transient response of undriven systems

Chapter 13: Steady-state response to sinusoidal inputs

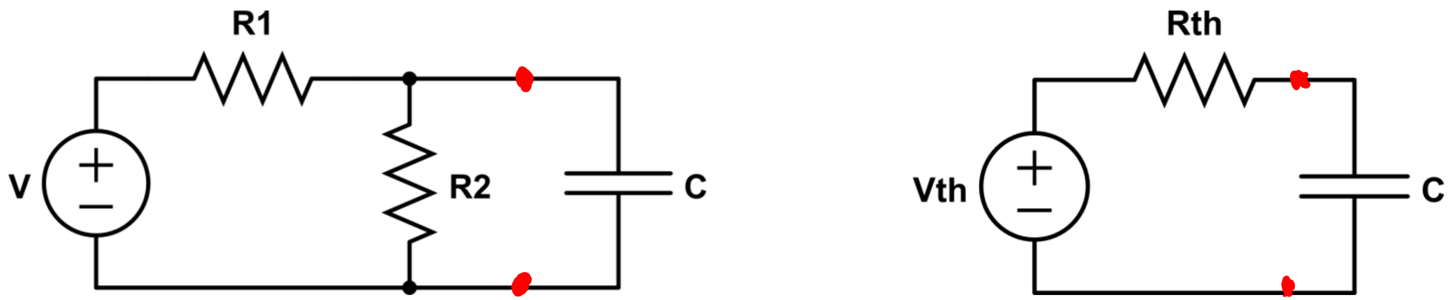
Chapter 14: Specific focus on resonant systems (filters)

What Can We Do With 2^{nd} Order Circuits???

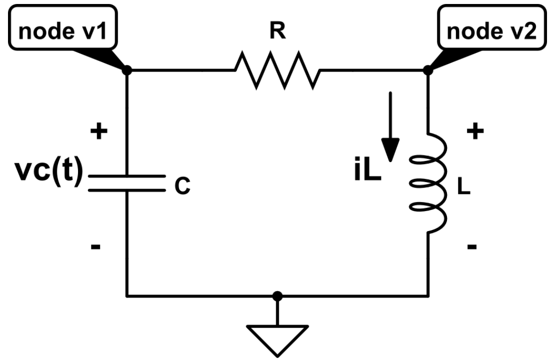
- Tuning circuits (radio, TV)
- Oscillators (create an oscillating signal with a DC source)
- Tank circuits (damp only certain frequencies of a signal)
- Filters (block/pass frequency bands)

Power in Thevenin Equivalent Circuits

The power dissipation of the Thevenin equivalent is not necessarily identical to the power dissipation of the real system. However, the power dissipated by an external element connected between the output terminals is the same



Let's Review the Undriven RLC Circuit

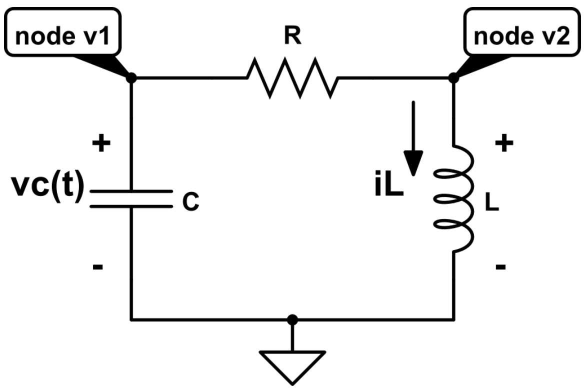


- Node 1 KCL: $$-C\frac{dv_1(t)}{dt} - \frac{v_1(t) - v_2(t)}{R} = 0$$

- Node 2 KCL: $$\frac{v_1(t) - v_2(t)}{R} - \frac{1}{L}\int_{-\infty}^{t} v_2(t')dt' = 0$$

$$\boxed{\frac{d^2 v_1(t)}{dt^2} + \frac{R}{L}\frac{dv_1(t)}{dt} + \frac{1}{LC}v_1(t) = 0}$$

Finding Initial Conditions in 2nd-Order Circuits



$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \alpha = \frac{R}{2L}$$

Recall how we found *v(0)* and *i(0)* for C and L in RC or RL circuits:

Principle #1: Steady-state of C and L

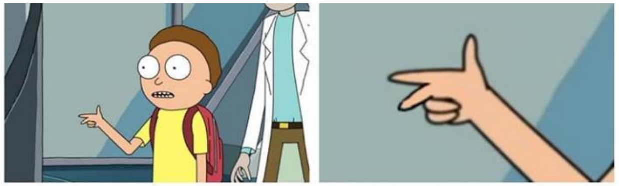
A capacitor acts like an open circuit and an inductor like a short circuit when they are in their steady state

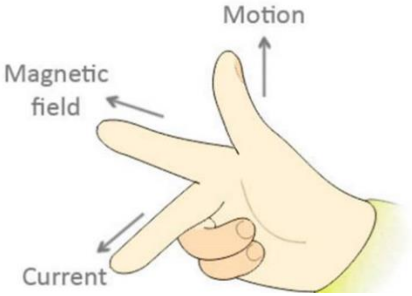
Principle #2: Continuity

1. Voltage across C is continuous (unless an impulse current)
2. Current through L is continuous (unless impulse voltage)

ECE 10C
Fall 2020
Slide Set 2 (cont'd)
Instructor: Galan Moody
TA: Kamyar Parto

Let us all take a moment and thank Morty for teaching us the "Right Hand Rule"





Last Week
- Started undriven RLC circuit

This week
- Finish undriven RLC circuit
- Driven RLC circuit

Important Items:

- **HW #1: 96% average**
- **Homework #2 Due Thurs**
- **Lab #2 due 10/23**
- **Piazza sign-up**
- **Office hours change: Still Tu 10-12 email me when you logon to zoom**

Last Quiz

Q1 [1 point]. True or False? The resonant frequency of an LC circuit is $\sqrt{LC}$.

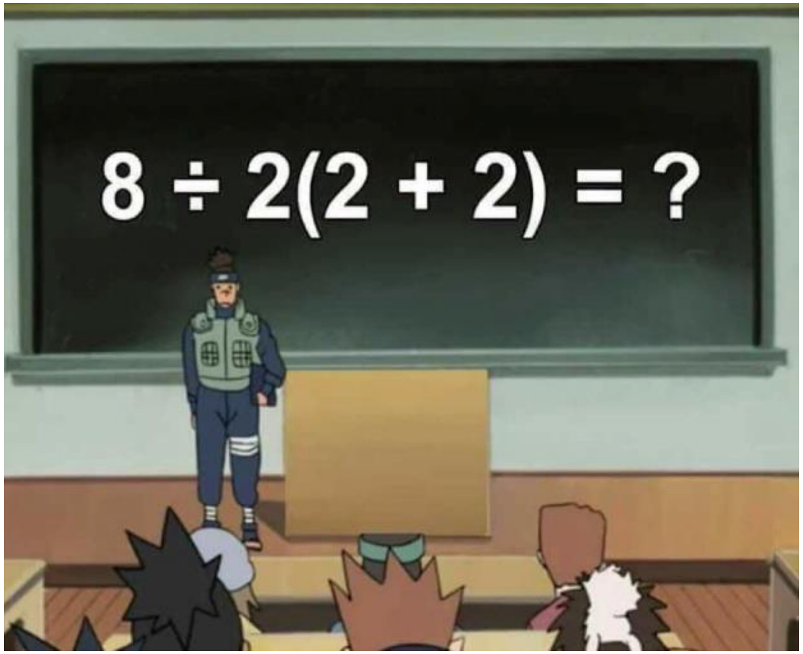
Q2 [1 point]. True or False? In an LC circuit, the capacitor voltage and inductor current are π out of phase.

Q3 [1 point]. True or False? An LC circuit is analogous to a mass on a spring, where the energy stored in the spring corresponds to the energy stored in the capacitor, and the kinetic energy of the mass corresponds to the magnetic energy in the inductor.

Q4 [1 point]. True or False? The current in an overdamped LC circuit decays to zero more quickly compared to the current in a critically damped system.

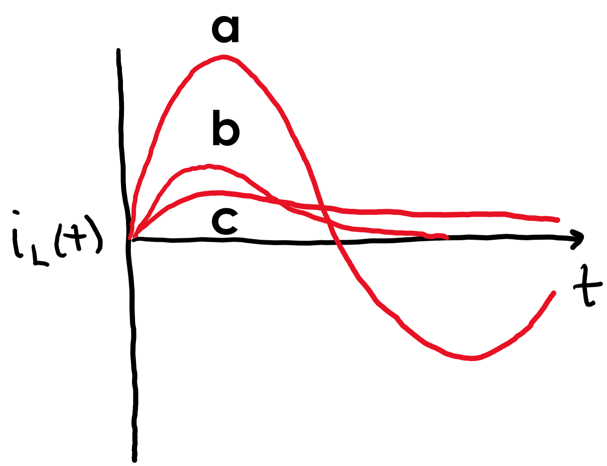
Q5 [1 point]. This question broke the internet last year. Did we come to a consensus as a society? I don't think so. Where to you stand?

- **(a)** 16
- **(b)** 1
- **(c)** Something else



Quiz Time!

[Q1] [3 points] In the diagram below for the current through an inductor in a series RLC circuit, pick the answer that corresponds to under-damped, over-damped, and critically damped (in that order):



[Q2] [2 points] Without damping ($\alpha = 0$), the resonance frequency of an RLC circuit is $\omega_0 = 1/\sqrt{LC}$. For $\alpha \neq 0$ and smaller than ω_0, choose the correct answer:

(a) ω_0 is unaffected
(b) ω_0 is smaller
(c) ω_0 is larger

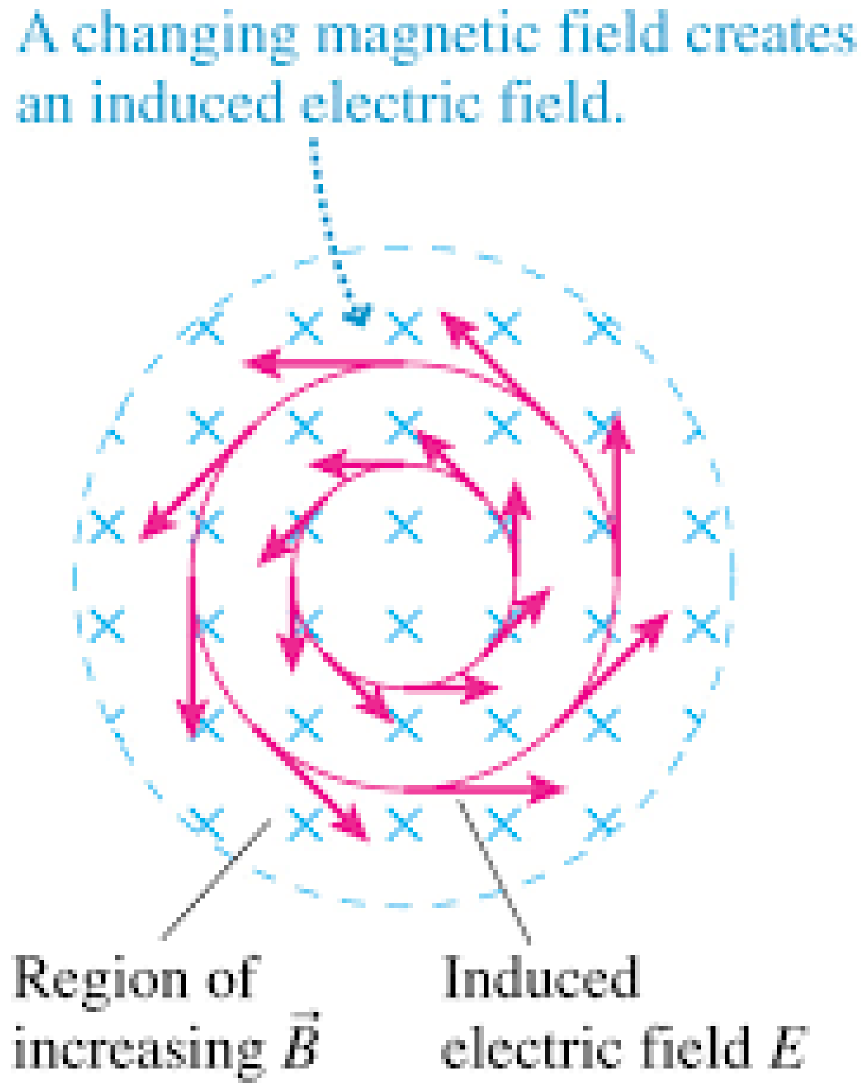
Maxwell's Equations

Differential form	Integral form
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\int_C \mathbf{E}.d\ell = -\int_A (\dfrac{\partial \mathbf{B}}{\partial t}).d\,\mathbf{A}$
$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\int_C \mathbf{H}.d\ell = \int_A (\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}).d$
$\nabla.\mathbf{D} = \rho$	$\int_A \mathbf{D}.d\,\mathbf{A} = \int_V \rho dV$
$\nabla.\mathbf{B} = 0$	$\int_A \mathbf{B}.d\,\mathbf{A} = 0$

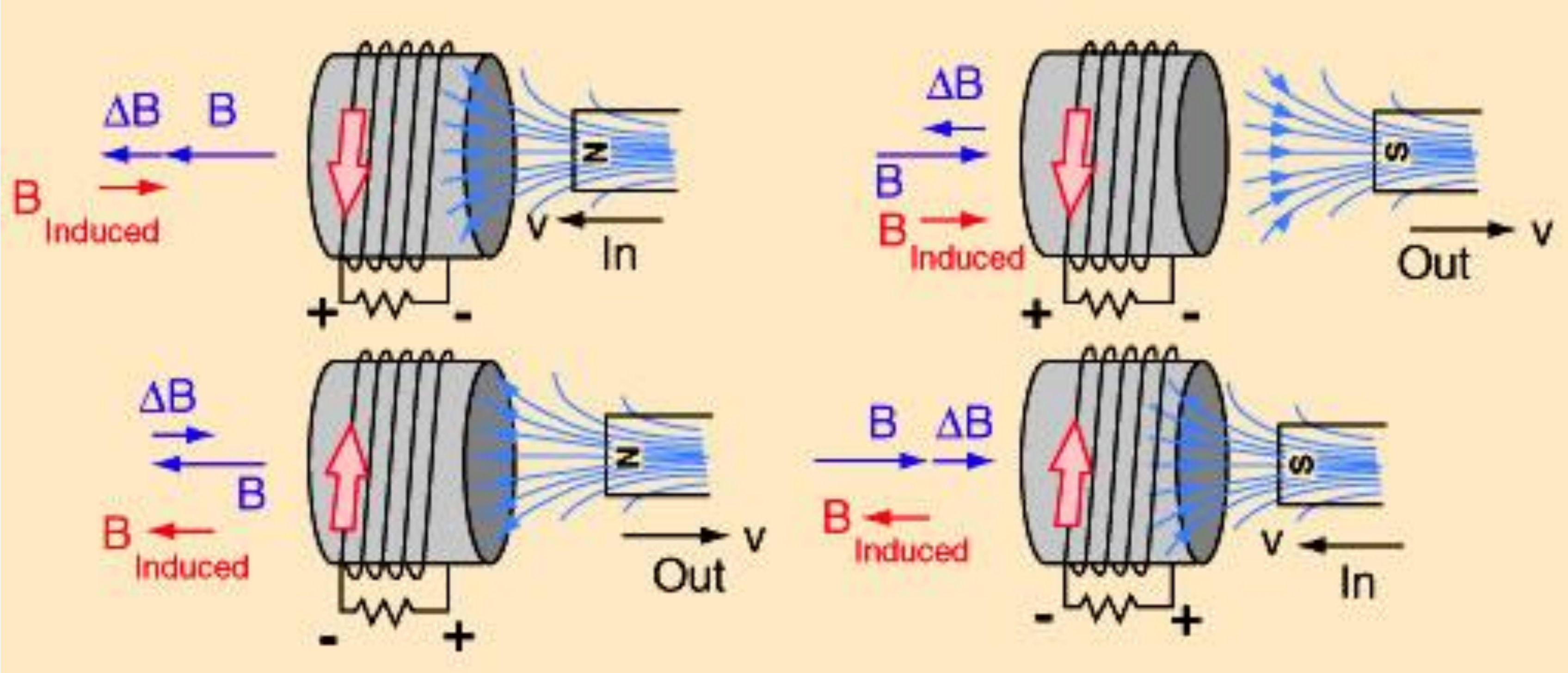
Maxwell's Equations

Differential form	Integral form
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\int_C \mathbf{E} . d\ell = -\int_A (\dfrac{\partial \mathbf{B}}{\partial t}) . d\,\mathbf{A}$
$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\int_C \mathbf{H} . d\ell = \int_A (\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}) . d\,\mathbf{A}$
$\nabla . \mathbf{D} = \rho$	$\int_A \mathbf{D} . d\,\mathbf{A} = \int_V \rho dV$
$\nabla . \mathbf{B} = 0$	$\int_A \mathbf{B} . d\,\mathbf{A} = 0$

Why Does an Inductor Oppose a Change in Current?



Why Does an Inductor Oppose a Change in Current?



1. Inductor = coiled wire with *N* loops
2. Current through inductor creates a magnetic field **B**
3. Constant current = steady-state **B** = no induced electric field **E**
4. Current increases = increase in **B**
5. Increase in **B** creates **E** around loop
6. Force from **E** induces current in opposite direction
7. Induced current creates **B** in opposite direction to keep net Δ**B** = 0

Standard Form for the Characteristic Equation

- We can also write the characteristic equation as:

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

- Where we have defined:

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- The roots of the characteristic equation are:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

***unless $s_1 = s_2$**

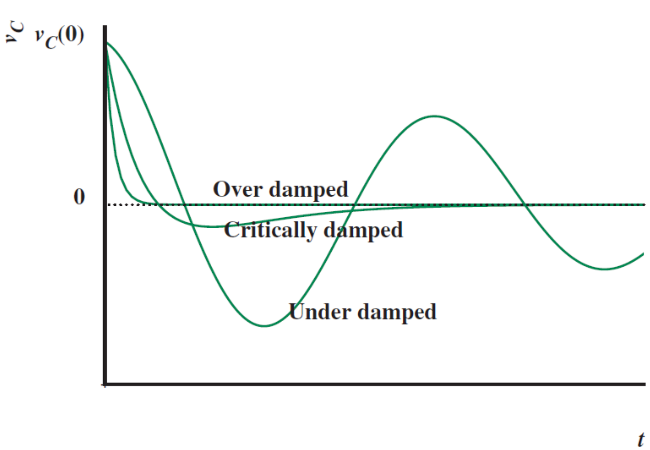
Dynamic Behavior

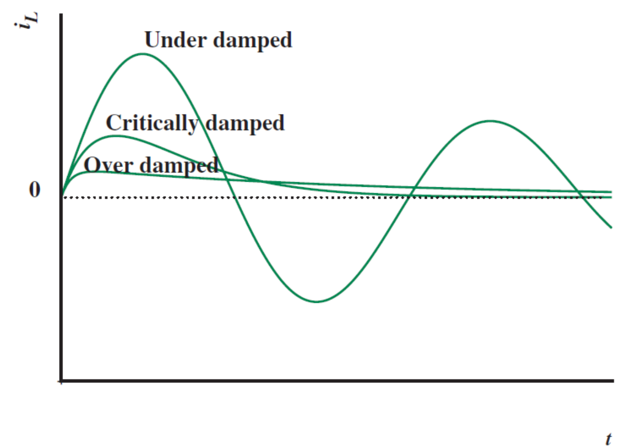
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \alpha = \frac{R}{2L}$$

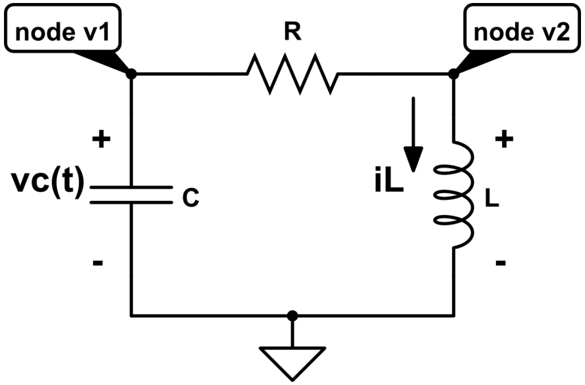
The dynamic behavior of an RLC circuit is different for three cases:

$\boldsymbol{\alpha < \omega_0}$ → **under-damped dynamics**
$\boldsymbol{\alpha = \omega_0}$ → **critically damped dynamics**
$\boldsymbol{\alpha > \omega_0}$ → **over-damped dynamics**





Finding Initial Conditions in 2nd-Order Circuits



$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$\alpha = \frac{R}{2L}$$

For 2nd-order (RLC) circuits, we need to find:

$i_c(t) = C\frac{dv_c(t)}{dt}$

$v_L(t) = L\frac{di_L(t)}{dt}$

$v(0)$ and $\frac{dv(0)}{dt}$

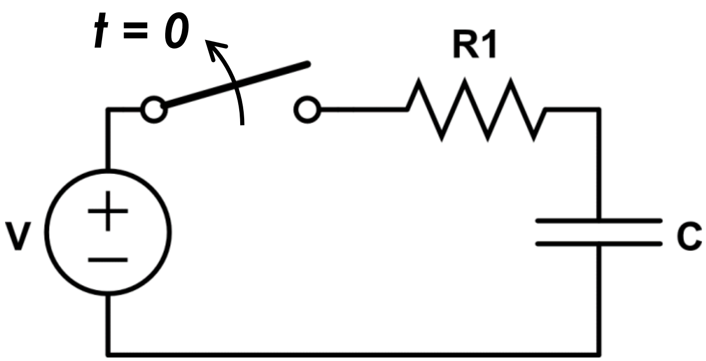
$i(0)$ and $\frac{di(0)}{dt}$

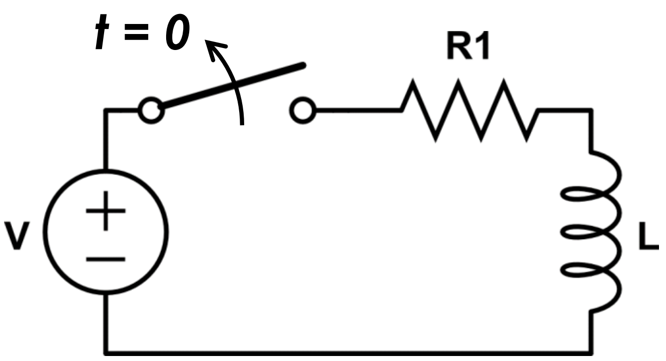
1. Once we find $v_C(0)$ and $i_L(0)$, then we can use capacitor element law law to determine $dv_C(0)/dt$ using $i_C(0)$
2. We can use the inductor element law to determine $di_L(0)/dt$ using $v_L(0)$

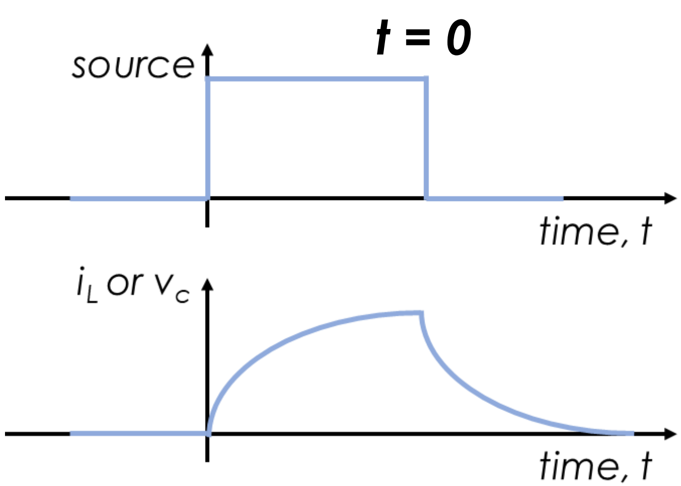
Finding Initial Conditions in 2nd-Order Circuits

Principle #2: Continuity

1. Voltage across C is continuous (unless an impulse current)
2. Current through L is continuous (unless impulse voltage)



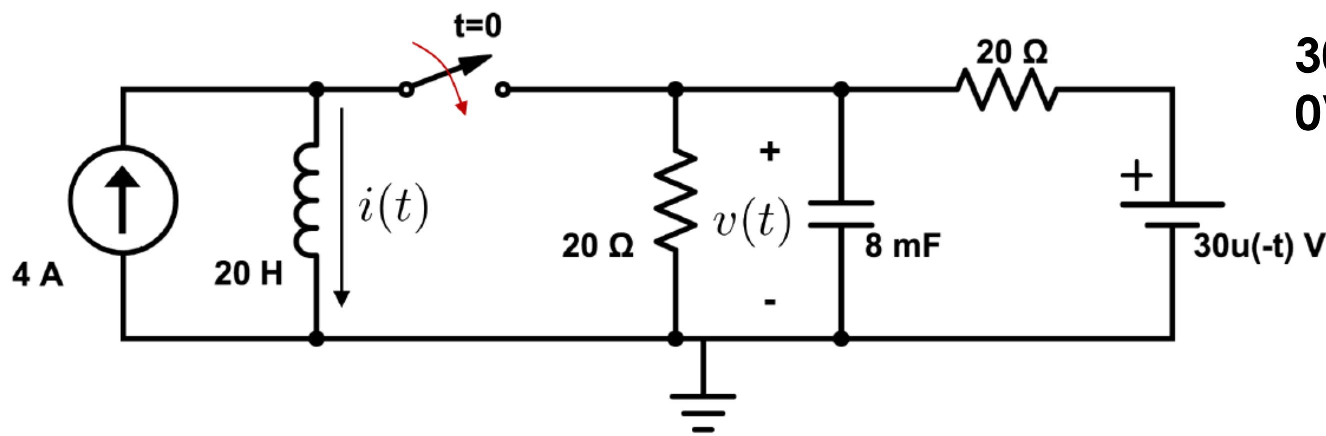




$$v_c(0^-) = v_c(0^+)$$

$$i_L(0^-) = i_L(0^+)$$

Finding Initial Conditions in 2^{nd}-Order Circuits



30V for t < 0
0V for t > 0

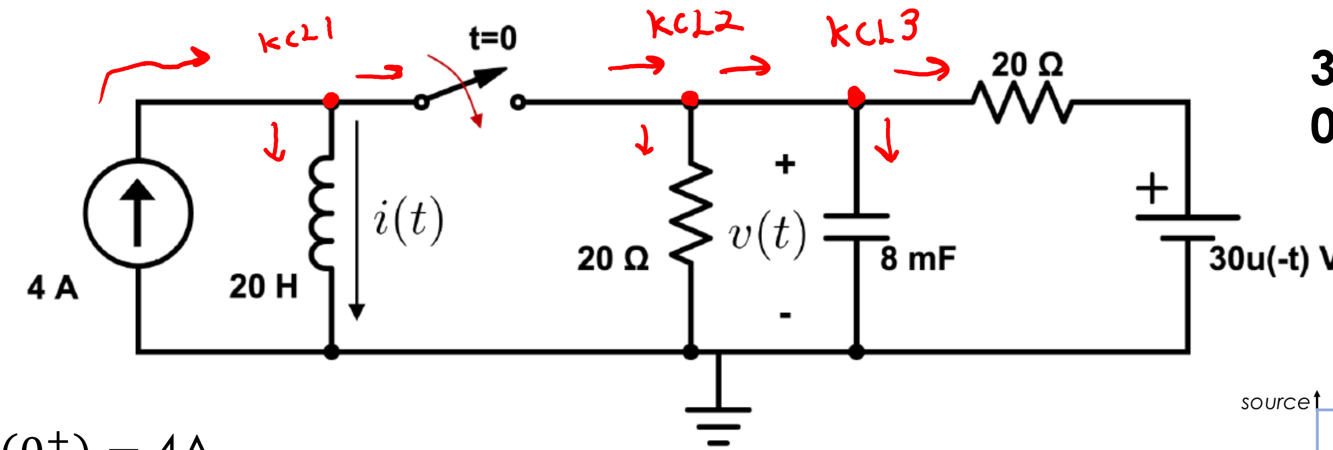
Right before switch is flipped ($t = 0^-$):

$$i(0^-) = 4A = i_L(0^+)$$

$$v(0^-) = 30V \cdot \frac{20\Omega}{40\Omega} = 15V = v(0^+)$$

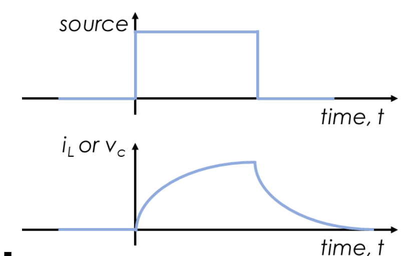
Finding Initial Conditions in 2nd-Order Circuits

$t=0^+$



30V for t < 0
0V for t > 0

$i(0^-) = i(0^+) = 4\text{A}$
$v(0^-) = v(0^+) = 15\text{V}$



What about $\frac{dv(0)}{dt}$ and $\frac{di(0)}{dt}$?

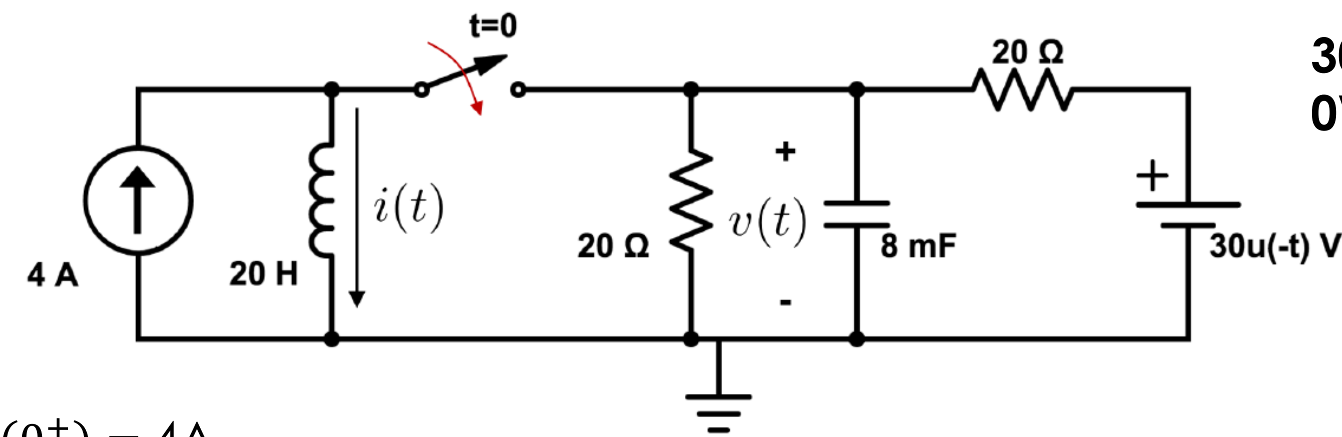
Use t = 0⁺, *i.e.* right after switch is flipped.

This is the slope we care about, since we want to know *v(t)* or *i(t)* for *t* > 0 (including *t* = 0+)

$$i_L(0^-) = i_L(0^+)$$
$$v_c(0^-) = v_c(0^+)$$

$$\frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C}$$
$$\frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L}$$

Finding Initial Conditions in 2^{nd}-Order Circuits



30V for t < 0
0V for t > 0

$i(0^-) = i(0^+) = 4\text{A}$
$v(0^-) = v(0^+) = 15\text{V}$

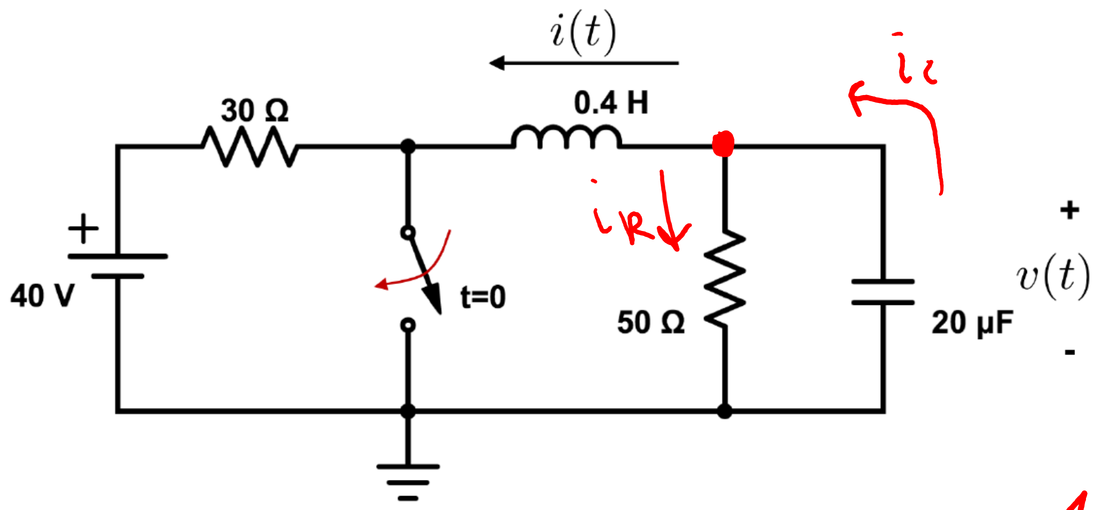
What about $\frac{dv(0)}{dt}$ and $\frac{di(0)}{dt}$?

Use t = 0^+, *i.e.* right after switch is flipped.

This is the slope we care about, since we want to know *v(t)* or *i(t)* for *t* > 0 (including *t* = 0+)

$$4A = i_L(0^+) + \frac{v_C(0^+)}{20\Omega} + i_C(0^+) + \frac{V_C(0^+)}{20\Omega}$$

In-Class Exercise



Find $v(0)$ and $\frac{dv(0)}{dt}$.

$$V_c(0^-) = V_c(0^+) = 40\,V \cdot \frac{50\,\Omega}{80\,\Omega} = 25\,V$$

$$i_L(0^-) = i_L(0^+) = \frac{40\,V}{80\,\Omega} = 0.5\,A$$

$$\frac{d\,V_c(0^+)}{dt} = \frac{i_c(0^+)}{C}$$

$$i_c(0^+) = i_L(0^+) + \frac{V_c(0^+)}{50\,\Omega}$$

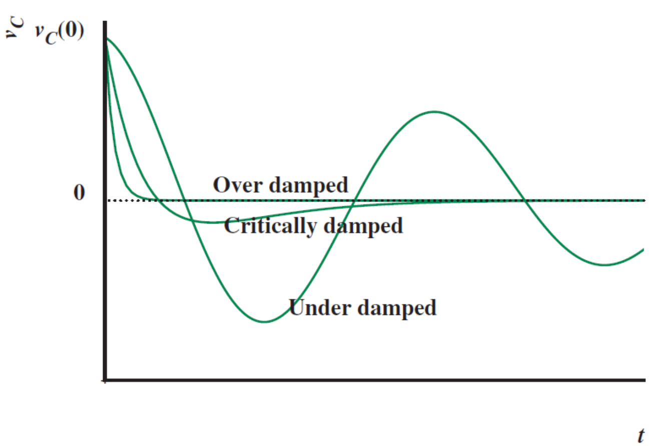
Dynamic Behavior

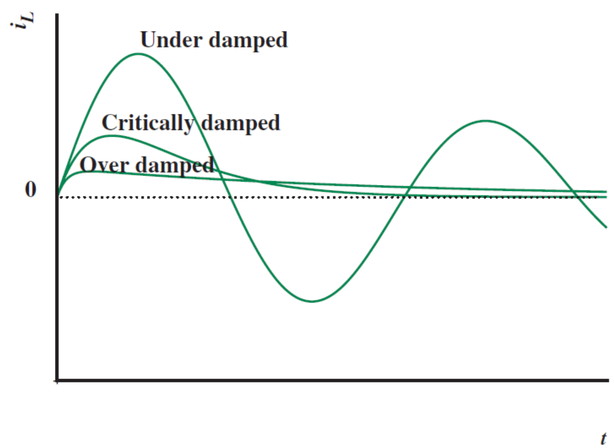
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \alpha = \frac{R}{2L}$$

The dynamic behavior of an RLC circuit is different for three cases:

$\boldsymbol{\alpha < \omega_0} \rightarrow$ **under-damped dynamics**
$\boldsymbol{\alpha = \omega_0} \rightarrow$ **critically damped dynamics**
$\boldsymbol{\alpha > \omega_0} \rightarrow$ **over-damped dynamics**





Case 1: Under-Damped

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \alpha = \frac{R}{2L}$$

Underdamped

$\alpha < \omega_0 \quad \rightarrow \quad R/2 < \sqrt{L/C}$ (i.e., R is small)

The roots $-\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$ are complex numbers, so let's define:

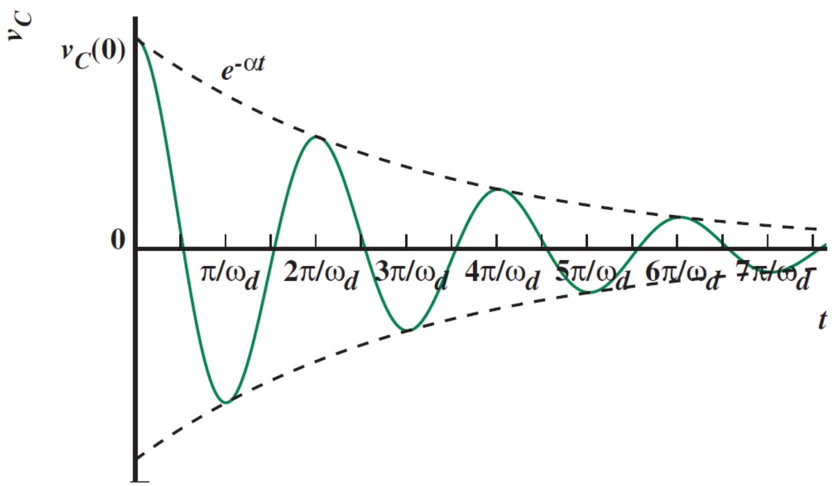
$$\omega_d \equiv \sqrt{\omega_0^2 - \alpha^2}$$

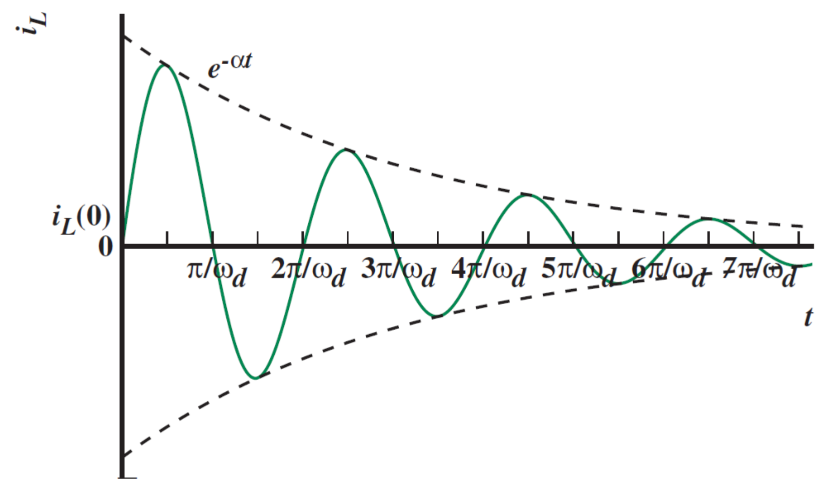
So the roots are:

$$s_{1,2} = -\alpha \pm j\omega_d$$

$$\begin{aligned} v_c(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} &= A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t} \\ &= e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right) \end{aligned}$$

Case 1: Under-Damped





$\alpha = R/2L$ is referred to as the damping (decay factor)

$\omega_0 = 1/\sqrt{LC}$ is called the undamped natural frequency (when $\alpha = 0$)

$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ called the damped natural frequency

Case 1: Under-Damped

- The Quality Factor Q of the circuit is defined as:

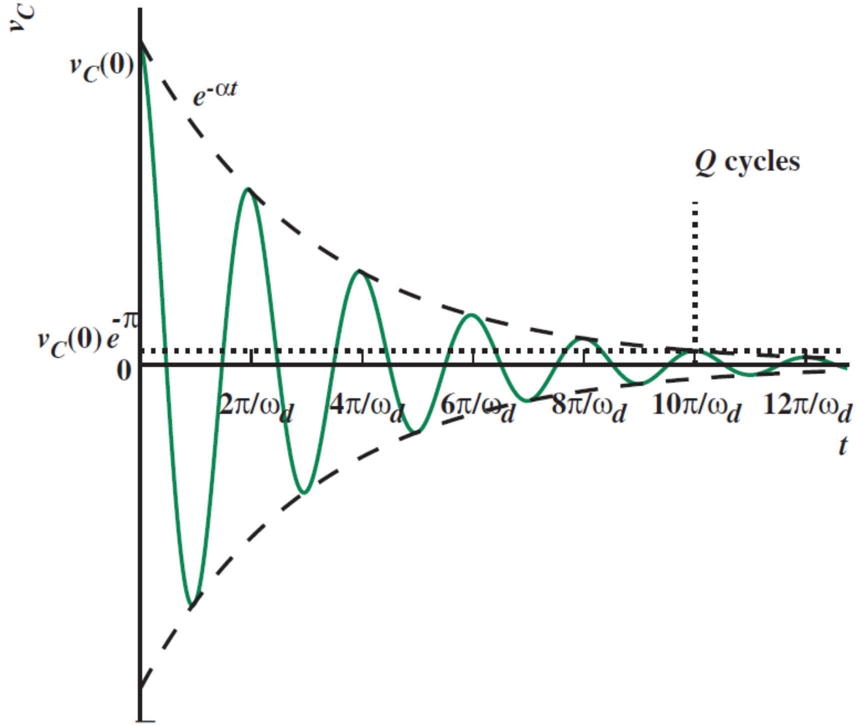
$$Q \equiv \frac{\omega_0}{2\alpha}$$
$$= \frac{1}{R}\sqrt{\frac{L}{C}} \text{ (for the series RLC circuit)}$$

- **What information does Q give us about the circuit?**

    1. If Q is large (α is small), the circuit oscillates for a long time

Case 1: Under-Damped

- One period of oscillation = $\frac{2\pi}{\omega_d}$



The amplitude after Q = $\frac{\omega_0}{2\alpha}$ periods of oscillations is?

$$e^{-2\pi Q\alpha/\omega_d} \approx e^{-\pi} = 4\% \text{ (for } \omega_d \approx \omega_0)$$

Case 1: Under-Damped

- The Quality Factor Q of the circuit is defined as:

$$Q \equiv \frac{\omega_0}{2\alpha}$$

$$= \frac{1}{R}\sqrt{\frac{L}{C}} \text{ (for the series RLC circuit)}$$

- **What information does Q give us about the circuit?**

1. If Q is large (α is small), the circuit oscillates for a long time

2. $Q = \frac{t_c \omega_0}{2}$ → knowing resonant frequency, then $Q \propto t_c$

Case 1: Under-Damped

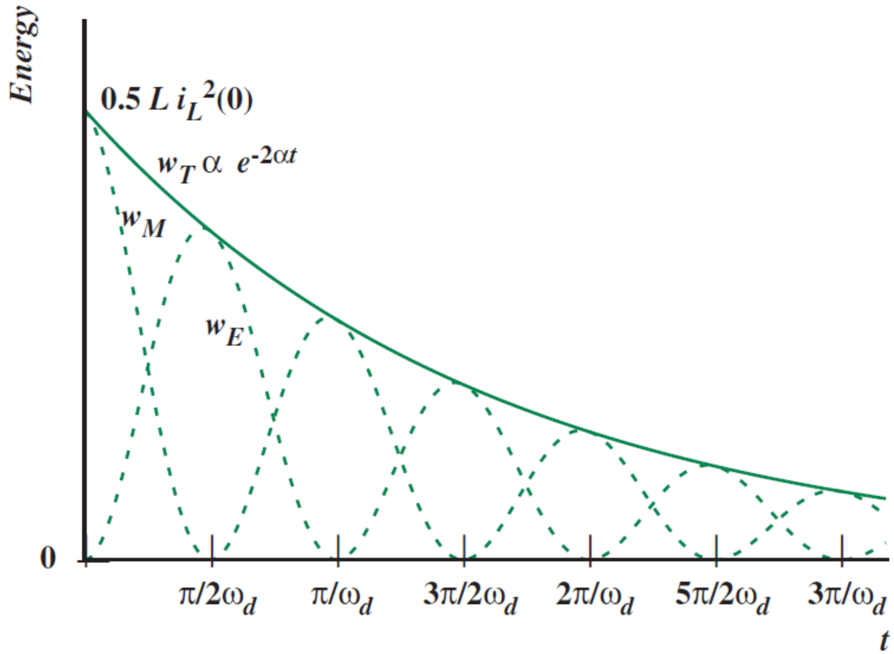
- How much energy is stored in C and L?
  - Energy stored in capacitor: $w_E = \frac{1}{2} C v_c^2(t)$
  - Energy stored in inductor: $w_M = \frac{1}{2} L i_L^2(t)$

$$w_E(t) \approx \left(\frac{1}{2} C v_C^2(0) + \frac{1}{2} L i_L^2(0)\right) e^{-2\alpha t} \cos^2\left(\omega_d t + \tan^{-1}\left(\sqrt{\frac{L}{C}} \frac{i_L(0)}{v_C(0)}\right)\right)$$

$$w_M(t) \approx \left(\frac{1}{2} C v_C^2(0) + \frac{1}{2} L i_L^2(0)\right) e^{-2\alpha t} \sin^2\left(\omega_d t + \tan^{-1}\left(\sqrt{\frac{L}{C}} \frac{i_L(0)}{v_C(0)}\right)\right)$$

The energy is transferred back and forth between C and L <u>TWO</u> times per cycle

Case 1: Under-Damped



The energy decay after Q cycles:

$$e^{-4\pi Q\alpha/\omega_d} \approx e^{-2\pi} = 0.2\%$$

The energy is transferred back and forth between C and L <u>TWO</u> times per cycle

Case 1: Under-Damped

- The Quality Factor Q of the circuit is defined as:

$$Q \equiv \frac{\omega_0}{2\alpha}$$
$$= \frac{1}{R}\sqrt{\frac{L}{C}} \text{ (for the series RLC circuit)}$$

- **What information does Q give us about the circuit?**

1. If Q is large (α is small), the circuit oscillates for a long time
2. $Q = \frac{t_c \omega_0}{2}$ → knowing resonant frequency, then $Q \propto t_c$
3. $Q = 2\pi \times \frac{energy\ stored\ in\ circuit}{energy\ dissipated\ per\ cycle} = \omega_0 \times \frac{energy\ stored\ in\ circuit}{power\ loss}$

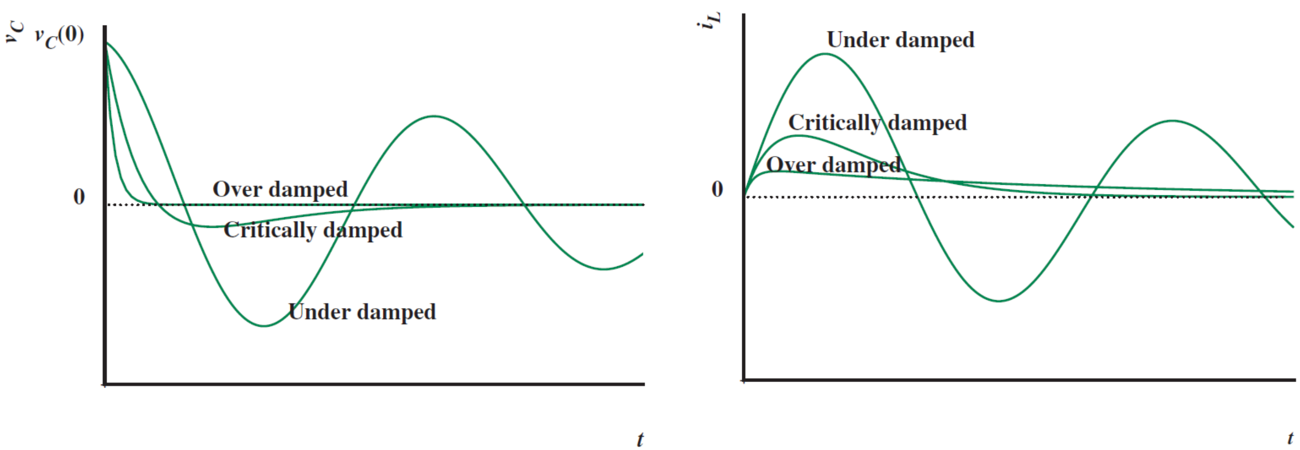
Case 2: Over-Damped

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \alpha = \frac{R}{2L}$$

The dynamic behavior of an RLC circuit is different for three cases:

$\boldsymbol{\alpha < \omega_0} \rightarrow$ **under-damped dynamics**
$\boldsymbol{\alpha = \omega_0} \rightarrow$ **critically damped dynamics**
$\boldsymbol{\alpha > \omega_0} \rightarrow$ **over-damped dynamics**



Case 2: Over-Damped

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \alpha = \frac{R}{2L}$$

Overdamped

$\alpha > \omega_0 \quad \rightarrow \quad R/2 > \sqrt{L/C}$ (i.e., R is large)

- The roots $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$ are both real numbers

Case 3: Critically-Damped

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \alpha = \frac{R}{2L}$$

Critically damped

$$\alpha = \omega_0 \quad \rightarrow \quad R/2 = \sqrt{L/C}$$

- **Now we have a repeated root:**

$s_1 = s_2 = -\alpha$

- $e^{s_1 t}$ and $e^{s_2 t}$ are no longer independent functions

When the characteristic equation has repeated roots, we haven't found the "full" general solution with a single exponential function. If there are two roots, in general there will be two linearly independent solutions. We need to find the second one.

Case 3: Critically-Damped

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \alpha = \frac{R}{2L}$$

Critically damped

$$\alpha = \omega_0 \quad \rightarrow \quad R/2 = \sqrt{L/C}$$

- **Now we have a repeated root:**

$$s_1 = s_2 = -\alpha$$

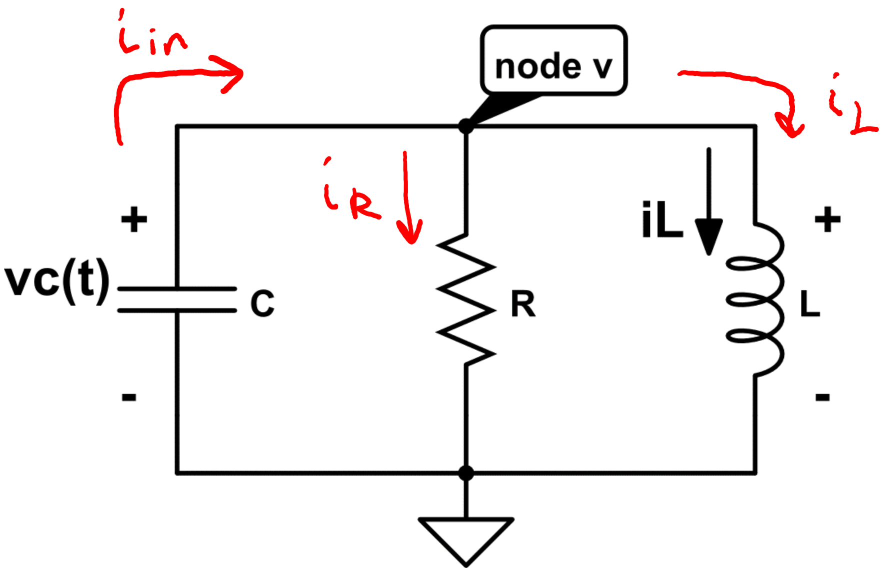
- $e^{s_1 t}$ and $e^{s_2 t}$ are no longer independent functions

For our RLC circuit in the critically damped case, we have the expression:

$$v_c(t) = A_1 e^{-\alpha t} + A_2 t e^{-\alpha t}$$

In-Class Exercise: Parallel RLC Circuit

Write the ODE, the general expression for $v_c(t)$, and the solutions to the characteristic equation for the undriven parallel RLC circuit below:

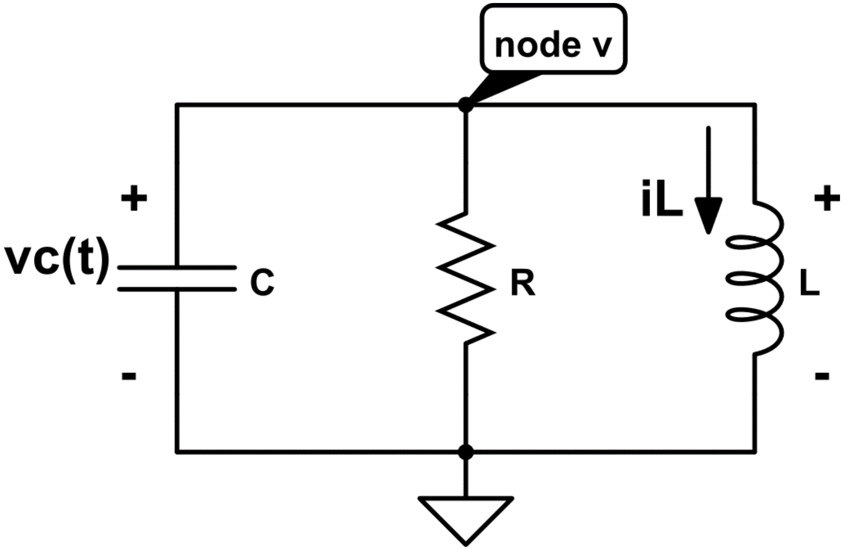


$$i_c - \frac{v}{R} - i_L = 0$$

$$-C\frac{dv}{dt} - \frac{v}{R} - \frac{1}{L}\int_{-\infty}^{t} v(t')\,dt' = 0$$

In-Class Exercise: Parallel RLC Circuit

Write the ODE, the general expression for $v_c(t)$, and the solutions to the characteristic equation for the undriven parallel RLC circuit below:



- Node KCL:

$$-C\frac{dv(t)}{dt}-\frac{v(t)}{R}-\frac{1}{L}\int_{-\infty}^{t}v(t')dt'=0$$

- Take the derivative:

$$\frac{d^2v(t)}{dt^2}+\frac{1}{RC}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=0$$

- By substituting Ae^{st}, we get the characteristic equation:

$$s^2+\frac{1}{RC}s+\frac{1}{LC}=0$$

Undriven RLC Circuit: 2nd-Order Transients

- We can also write the characteristic equation as:

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

- Where we have defined:

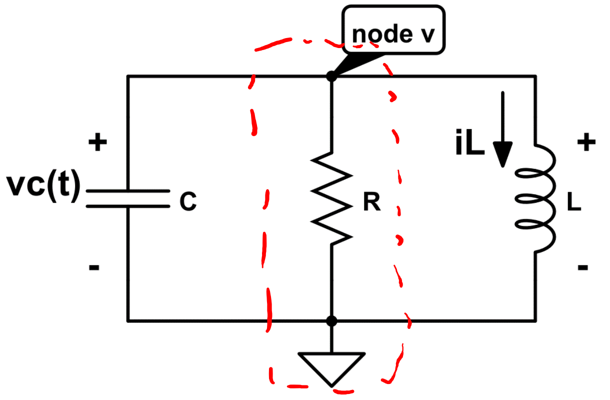
$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- The roots of the characteristic equation are:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$v_1(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

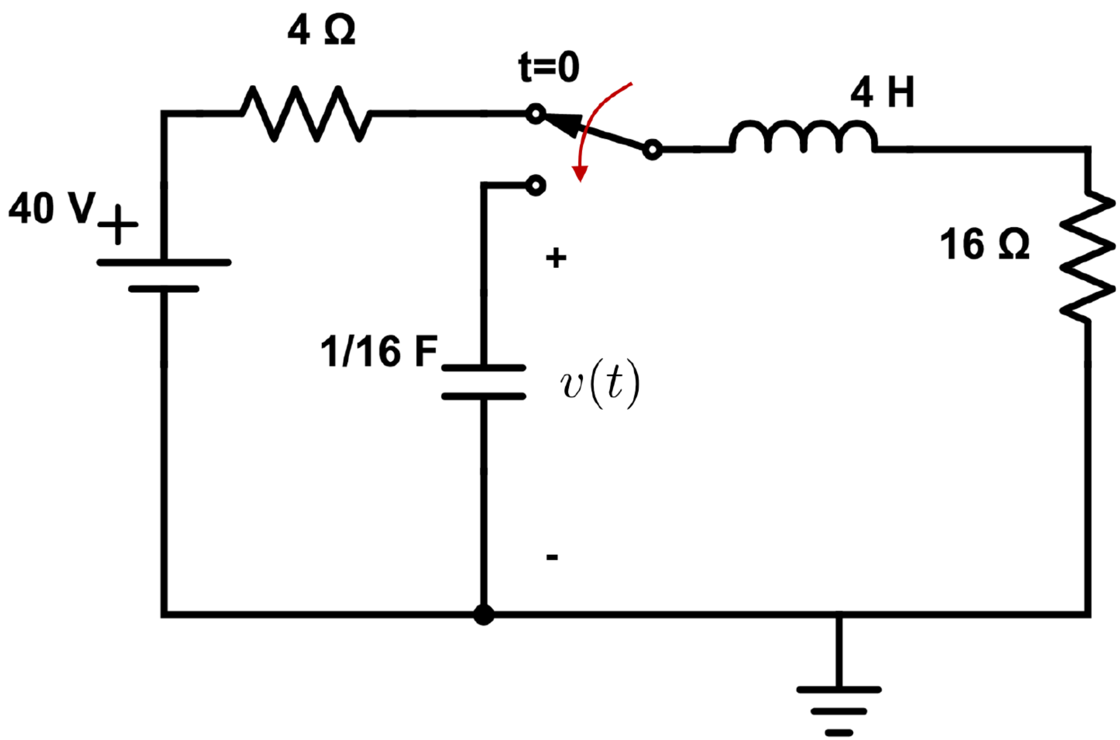


In-Class Exercise

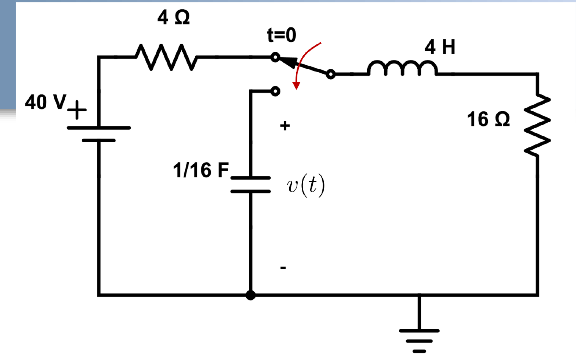
The switch has been at position 1 for all $t < 0$. At time $t = 0$, the switch flips to position 2. Assume v(0) = 0.

Find i(0).

Determine v(t) for t > 0.



In-Class Exercise



The switch has been at position 1 for all t < 0. At time t = 0, the switch flips to position 2. Assume v(0) = 0.

Find i(0).Determine v(t) for t > 0.

$\frac{40V}{20\Omega} = 2A$ $\alpha = \frac{R}{2L} = 2$; $\omega_0 = \frac{1}{\sqrt{LC}} = 2$

$V_c(t) = A_1 e^{-2t} + A_2 t e^{-2t}$ $A_1 = 0.$

$V_c(t) = A_2 t e^{-2t} \rightarrow A_2 = \frac{dV_c(0)}{dt} = \frac{i_c(0)}{C} = \frac{-i_L(0)}{C} = \frac{-2A}{C}$