Math 122L Cheat Sheet

Pre-Calculus

Assorted

$x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$
$\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$
Sum of the squares of the first n positive integers: $\frac{n(n+1)(2n+1)}{6n^2}$
For every real number x,

$$\lim_{n\to\infty}\frac{x^n}{n!} = 0$$

A function f is **even** if $f(-x) = f(x)$
A function f is **odd** if $f(-x) = -f(x)$

$$\lim_{x\to\infty}\tan^{-1}x = \frac{\pi}{2}$$

$$\lim_{x\to-\infty}\tan^{-1}x = -\frac{\pi}{2}$$

Trig Functions

$\csc\theta = \frac{1}{\sin\theta}$
$\sec\theta = \frac{1}{\cos\theta}$
$1+\tan^2\theta = \sec^2\theta$
$1+\cot^2\theta = \csc^2\theta$
$\cos\theta = \cos(-\theta)$
$\tan(-theta) = -\tan(\theta)$
$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$
$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$
$\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$
$\sin(x+y) = \sin x\cos y + \cos x\sin y$
$\sin(x-y) = \sin x\cos y - \cos x\sin y$
$\cos(x-y) = \cos x\cos y - \sin x\sin y$
$\tan(x+y) = \frac{\tan x - \tan y}{1+\tan x\tan y}$

Trig Tables

x	sin	cos	tan
0	0	1	0
$\frac{pi}{6}$	0.5	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	0.5	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	*(undefined)*

Derivatives and Other Fun Things

L'Hospital's Rule

For some $f(x)$ and $g(x)$, if we have indeterminate forms of the type $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, $\infty-\infty$, ∞^0, 1^∞ we can evaluate it as

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

...if the limit on the right side exists.

The Definite Integral/Riemann Sums

The definite integral from a to b is defined as:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^n f(x^*)\Delta x$$

...where x^* is any point in the ith subinterval $[x_{i=1}, x_i]$ and $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ If the limit exists, the f is *integrable* on $[a,b]$.

Derivatives and Curves

The mean value theorem states that, for some differentiable f on the interval $[a,b]$, there is a number c such that $f'(c) = \frac{f(b)-f(a)}{b-a}$, or equivalently, $f(b)-f(a) = f('(c)(b-a)$
Concavityfor some $f(x)$ over $[a,b]$:

1. If $f''(x) > 0$ for all x in the interval, f is concave upward over interval.
2. If $f''(x) < 0$ for all x in the interval, f is concave downward over interval.

The **second derivative test** states that for some f'' continous near c:

1. $f'(c) = 0and(f''(x) > 0, f$ has a local minimum at c.
2. $f'(c) = 0and(f''(x) < 0, f$ has a local maximum at c.

Fundamental Theorem of Calculus (part 1): If f if continuous on $[a,b]$ and $a \le x \le b$, then $g(x) = \int_a^x f(t)dt$ is an antiderivative of f. **Fundamental Theorem of Calculus (part 2)**: Supposing f is continuous on $[a,b]$: 1

1. $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$
2. $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f.

Antiderivatives

$$\int \sec^2 u du = \tan u + C$$
$$\int \csc^2 u du = -\cot u + C$$
$$\int \tan u du = \ln|\sec u| + C$$
$$\int a^u du = \frac{a^u}{\ln a} + C$$
$$\int \sin^2 y du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$$
$$\int \tan^2 u du = \tan u - u + C$$
$$\int \sin^{-1} u du = u\sin^{-1}u + \sqrt{1-u^2}$$
$$\int \cos^{-1} u du = u\cos^{-1}u + \sqrt{1-u^2}$$
$$\int \tan^{-1} u du = u\tan^{-1}u - \frac{1}{2}\ln(1+u^2) + C$$
$$\int u dv = uv - \int v du$$
$$\int u\cos u du = \cos u + u\sin u$$
$$\int u\sin u du = \sin u - u\cos u$$

Derivatives

$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec^{-1}) = \frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(a^x) = a^x\ln a$$
$$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$$
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Average Value

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

Substitution Rule

If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int^g(b)_{g(a)} f(u)du$$

Approximate Integration

Midpoint rule:

$$\sum_{i=1}^n f(\bar{x_n}\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $\bar{x_n} = \frac{1}{2}(x_{n-1}+x_n)$, that is, the midpoint of $[x_{n-1}, x_n]$ The **trapezoidal rule:** add a left-hand sum and a right-hand sum and divide both sides by two. **Error bounds:** Suppose $|f''(x)| \le K$ for $a \le x \le b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint rules, then $|E_T|\, leq \frac{K(b-a)^3}{12n^2}$ and $|E_M| \le \frac{K(b-a)^3}{24n^2}$
Simpson's Rule:
$S_n = \frac{\Delta}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$ where n is even and $\Delta x = \frac{b-a}{n}$
The **error bound for Simpson's rule** for $\left|f^{(4)}(x)\right| \le K$ for $a \le x \le b$ is given as

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

Improper Integrals

If $\int_a^b f(x)dx$ exists for every number $t \le a$, then $\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$, provided the limit exists as a finite number.
If $\int_t^b f(x)dx$ exists for every number $t \le q$, then $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$ ...provided this limit exists as a finite number. The integrals $\int_a^\infty f(x)dx$ and $\infty_-\infty^b f(x)dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.
If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$

Note that any real number can be used for a.
The improper integral $\int_1^\infty \frac{1}{x^p}dx$ is convergent if $p > 1$ and divergent if $p \le 1$.

Probability

The odds of a certain event happening (given n chances) are given as (winning Odds)(losing odds)$^{n-1}$

PDFs

(...PDF standing for probability density function...) Every continuous random variable X has a **probability density function** f. This means that the probability that X lies between a and b is found by integrating f from a to b:

$$P(a \le X \le b) = \int_x^b f(x)dx$$

For some PDF f,

$$\int_{-\infty}^\infty f(x)dx = 1$$

The **mean** of any probability density function f (which can be interpreted as the long-range average of the random variable X) is defined to be

$$\mu = \int_{-\infty}^\infty xf(x)dx$$

Average value is given as $\sum_{i=1}^n p_i x_i$
Expected value is given as $\sum_{\text{all } x} xp(x)$, where $p(x)$ is the PDF

Normal Distributions

Functions that can be modeled with a normal distribution have a PDF of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$$

...where σ represents the **standard deviation**. For any normal distribution,

$$\int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}dx = 1$$

Center of Mass

Density of a circle whose density changes with radius:

$$\int_a^b 2\pi xg(x)dx$$

Center of mass:

$$\bar{x} = \frac{\text{Sum of Moments of Mass}}{\text{Total Mass}}$$

...so, for a triangle with point masses of $5gm, 3gm$, and $1gm$ at $x = -10, x = 1, x = 2$:

$$\bar{x} = \frac{5(-10) + 3(1) + 1(2)}{5+3+1}$$

For a region R of constant density δ, the center of mass of R is given by the point $(\bar{x}, \bar{y}, \bar{z})$:

$$\bar{x} = \frac{\int x\delta f(x)dx}{\text{Total Mass}}$$
$$\bar{y} = \frac{\int y\delta f(y)dy}{\text{Total Mass}}$$
$$\bar{z} = \frac{\int z\delta f(z)dz}{\text{Total Mass}}$$

Series

Convergence and Divergence

If the sequence s_n is convergent and $\lim_{n\to\infty}s_n = s$ exists as a real number, then the series $\sum a_n$ is **convergent**, that is,

$$\sum_{n=1}^\infty a_n = s$$

If the sequence s_n is **divergent**, then the series is divergent.
If the series $\sum_{n=1}^\infty a_n$ is convergent, then $\lim_{n\to\infty}a_n = 0$ **Note that the converse of this is not true in general**; if $\lim_{n\to\infty}a = 0$, we cannot conclude that $\sum a_n$ is convergent.
The test for divergence states that if $\lim_{n\to\infty}a_n$ does not exist of if $\lim_{n\to\infty}a_n \ne 0$, then the series $\sum_{n=1}^\infty a_n$ is divergent.
Sum of a geometric series:

$$\sum_{n=1}^\infty ar^n = \frac{a}{1-r} \text{ for } |r| < 1$$

If $|r| \ge 1$, the series is divergent.
The **integral test** states that the for some continuous, positive f decreasing on $[1,\infty)$, and $a_n = f(n)$, then the series is convergent if and only if the improper integral $\int_1^\infty f(x)dx$ is convergent.
The p-series, $\sum_{m=1}^\infty \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$
Comparison test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

The limit comparison test:
Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms; if

$$\lim_{n\to\infty}\frac{a_n}{b_n} = c$$

...where c is a finite number and $c > 0$, then either both series converge or both series diverge.
The alternating series test:
If the series $\sum_{n=1}^\infty (-1)^{n-1}b_n$ (for $b_n > 0$) satisfies the conditions

- $b_{n+1} \le b_n$ for all n
- $\lim_{n\to\infty}b_n = 0$

...then the series is convergent.
Alternating series estimation theorem:
If $s = \sum(-1)^{n-1}b_n$ is the sum of alternating series that satisfies the alternating series theorem, then

$$|R_n| = |s - s_n| \le 4b_{n+1}$$

A series $\sum a_n$ is **absolutely convergent** if $\sum|a_n|$ is convergent. If a series is absolutely convergent, then it is convergent.
The **ratio test** for a series $\sum_{n=1}^\infty a_n$ where $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$

1. If $L < 1$, then the series is absolutely convergent
2. If $L > 1$ or $L = \infty$, then the series is divergent
3. If $L = 1$, the ratio test is inconclusive

Power Series

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

1. The series converges only when $x = a$
2. The series converges for all x
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

...the number R is called the **radius of convergence** of the power series.

Center of Mass

Density of a circle whose density changes with radius:

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Center of mass:

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...so, for a triangle with point masses of $5gm, 3gm$, and $1gm$ at $x = -10, x = 1, x = 2$:

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For a region R of constant density δ, the center of mass of R is given by the point $(\bar{x}, \bar{y}, \bar{z})$:

$$\bar{x} = \frac{\int x\delta f(x)dx}{\text{Total Mass}}$$

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$$\bar{z} = \frac{\int z\delta f(z)dz}{\text{Total Mass}}$$

Functions as Power Series

The radius of convergence R does not change when the power series is derived or integrated.

Taylor's Theorem

states that if f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

where $|x-a| < R$ then its coefficients are given by the formula $c_n = \frac{f^{(}n)(a)}{n!}$
A **Maclaurin series** is a Taylor series about $a = 0$.
If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Taylor's Inequality:

if $\left|f^{n+1}(x)\right| \leq M$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \leq d$. Find the largest possible value of M within the range of x values. Find the largest possible value of x when computing $|x-a|^{n+1}$

Important Maclaurin Series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(xn+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\ln(1+x) = \sum_{n=1} \infty (-1)^{n-1} \frac{x^n}{n}$$

Roots can be approximated using Taylor polynomials; for the cube root of x, $f(x) = x^{1/3}$; we can treat this just as we would treat any other Taylor polynomial (by finding the coefficients with derivations).

Fourier Series

Functions with periods $-\pi \leq x \leq \pi$.:

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

If f is a piecewise continuous on $[-L, L]$, the Fourier series is defined as:

$$a_0 + \sum_{n=1}^{\infty}\left[a_n cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

Coefficients are defined for $n \geq 1$ as

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

The **sum of the Fourier series** is equal to $f(x)$ at all numbers x where f is continuous. At the numbers x where f is discontinuous, the sum of the Fourier series is defined as $\frac{1}{2}\left[(x^+) + f(x^-)\right]$

Differential Equations and Friends

Basics

Differential Equations dictate the rate of change of a function. $\frac{dy}{dt} = k$ means the the function y changes at a constant rate k.
A **separable equation** is a first-order differential equation in which the expressions can be factored as a function of x times a function of y, that is, $\frac{dy}{dx} = g(x)f(y)$ **Euler's Method:** Approximate values for the solution of the initial-value problem $y = F(x, y), y(x_0) = y_0$, with step size h, at $x_n = x_{n-1}$ are given as

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}), n = 1, 2, 3...$$

The solution of the initial-value problem $\frac{dy}{dt} = ky$ where $y(0) = y_0$ is given as $y(t) = y_0e^{kt}$
Newton's Law of Cooling States that $\frac{PdT}{dt} = k(T - T_s)$, where k is a constant, T is the temperature and T_0 is the original tempe

Population Growth Models

The logistic differential equation (one of the simplest models for logistic growth) is given as

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where P is the population and M is the carrying capacity.
The solution to the logistic equation is given as $P(t) = \frac{M}{1+Ae^{-kt}}$ where $A = \frac{M-P_0}{P_0}$