Midterm 2

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Differential Equations

Method of Undetermined Coefficients

- The solution to a differential equation is the sum of the particular solution and the homogenous solution:

$$y = y_h + y_p$$

- The goal is to find some y_p that works with some given differential equation, with $\lambda = r$ is a root of multiplicity m, k is highest power of x on the right side of the equation.
- If the root to an LDE of the form $a_n y^{(n)} + ... + a_1 y' + a_0 y = Ax^k e^{rx}$ is real, the particular solution will be $y_p = x^m(A_k x^k + ... + A_1 x + A_0)e^{rx}$
- If root to equation of form

$$a_n y^{(n)} + ... + a_1 y' + a_0 y = Ax^k e^{ax} \cos bx + Bx^k e^{ax} \sin bx$$

is imaginary, particular solution is given as:

$$y_p = x^m(A_k x^k + ... + A_1 x + A_0)e^{ax} \cos bx + x^m(B_k x^k + ... + B_1 x + B_0)e^{ax} \sin bx$$

- *Note: the method of undetermined coefficients is not an exact science!* There can be trial and error involved; a table of decent guesses is embedded at the end of this document.

Applications

- Sitautions with no external forces are usually given in the form

$$F = mu'' + fu' + ku = 0$$

where (in spring problems at least) k is the spring constant, f is the friction coefficient, and mass m. (Don't forget that pounds are a unit of force, *not* of mass!)

- In **unforced cases** (no external input to system), the quadratic equation can give us roots; the issue of imaginary roots arises (that is, $f^2 - 4km < 0$):
  1. If no friction ($f = 0$), roots are imaginary. These cases are kind of rare.
  2. If $f - 4km < 0$, $\lambda = -a \pm bi$; in this case, the oscillating thing in question takes a bit to stop moving. This system is **underdamped**.
  3. If $f - 4km > 0$, there will be two roots $r_1, r_2 < 0$, solutions will have the form $u = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ this system is **overdamped**.
  4. If $f^2 - 4km = 0$, we have one root $r = \frac{-f}{2m}$ and solutions $u = c_1 e^{rt} + c_2 t e^{rt}$. This system is **critically damped**, and is the case where the oscillating thing returns to rest within the shortest amount of time.

Linear Transformations

- A function f from a set X to a set Y is denoted as $f : X \to Y$; X is the domain of f, Y is called the **image set** or **codomain**. The subset $f(x)|x \in X$ of Y is called the **range**, which, in English, means "all the $f(x)$ that 'hit' something in Y."
- If V, W are vector spaces, a function $T : V \to W$ is called a **linear transformation** if, for all vectors $u, v \in V$ and all scalars c, the following two properties hold:
  1. $T(u + v) = T(u) + T(v)$
  2. $T(cv) = cT(v)$
- If $T : V \to V$, T is sometimes called a **linear operator**

- If A is an $m \times n$ matrix, $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(X) = AX$$

  is a linear transformation, but is more commonly called a **matrix transformation**

- The differential operator takes derivatives, and is a linear transformation denoted by

$$D : D(a, b) \to F(a, b)$$

- Int(f) denotes definite integral of a function f over a closed interval $[a, b]$

- For some **linear transformation**$T : V \to W$ the following properties hold:

  1. $T(0) = 0$
  2. $T(-v) = 0T(v)$ for any $v \in V$
  3. $T(u - v) = T(u) - T(v)$ for any $u, v \in V$
  4. $T(c_1v_1 + c_2v_2 + ... + c_kv_k) = c_1T(v_1) + c_2T(v_2) + ... + c_kT(v_k)$ for any scalars $c_1, c_2, ..., c_k$ and any vectors $v_1, v_2, ...v_k \in V$

- To determine what exactly a linear transformation does (assuming you have a set of input vectors by which the behavior of T may be observed):

  1. Assume, find, or steal some basis for the domain; this will determine all values of an LT.
  2. Determine the coefficients $c_1, ..., c_n$ for the basis vectors that yield a given output
  3. Determine how T combines the given input vectors to an output vector of form $[x_0, x_1, ..., x_n]$, solving again for coefficients

- The **kernel** of T, denoted ker(T), is defined as

$$\ker(T) = \{v \in V | T(v) = 0\}$$

  ...in English, the kernel is the set of all vectors that give an output of the zero vector. For matrix transformations, the kernel of the matrix transformation $T(X) = AX$ is the same as the nullspace of A:

$$\ker(T) = NS(A)$$

- To find a basis for the column space of a matrix, take the transpose (**T**), reduce (**R**), transpose (**T**), split into columns (**S**), which approximately spells **ToRToiSe**.

- Weirdly enough, if $T : V \to W$ is an LT where V is a finite-dimensional vector space, we have that

$$\dim(\ker(T) + \dim(\text{range}(T)) = \dim(V)$$

Trial solutions for the method of undetermined coefficients

	Form of $g(x)$	Guess for particular solution
1.	1 (any constant)	A
2.	$5x + 7$	$Ax + B$
3.	$3x^2 - 2$	$Ax^2 + Bx + C$
4.	$\sin 4x$	$A\cos 4x + B\sin 4x$
5.	$\cos 4x$	$A\cos 4x + B\sin 4x$
6.	e^{5x}	Ae^{5x}
7.	$(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
8.	x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
9.	$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
10.	$5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (E^2 + Fx + G)\sin 4x$
11.	$xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$
12.	$(5x + 7) + \sin 4x$	$(Ax + B) + (C\cos 4x + D\sin 4x)$