Math 216 Midterm 2 Study Guide

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Dimension, Linear Indepence

Subspaces and Spanning Sets

- **Notation:**
  1. $C(a, b) = \{\text{continuous functions on } (a, b)\}$
  2. $D(a, b) = \{ \text{ differentiable functions on } (a, b)\}$
  3. $D^n(a, b) = \{f | f \text{ has an nth derivative } \}$
  4. $C^n(a, b) = \{f | f^{[n]} \text{ is continous } \}$

- Say $\alpha = \{v_1, ...v_n\}$ is a basis for V, and $\vec{v} \in V$. Coordinates of $\vec{v}$ relative to the basis α are $[\vec{v}]_\alpha = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ where c is the set of contstants for $\vec{v} = c_1\vec{v_1} + ... + c_n\vec{v_n}$

- A subset W of a vector space BV is a subspace of V if W is a subspace under addition, scalar multiplication of V restricted to W. (That is, if W is closed under the same rules of scalar mulitplication and addition as V)

- Let W be a nonempty subset of a vector space V. W is a subspace of V iff, $\forall u, v \in W$ and $\forall c \in \mathbb{R}, u + w \in W, cu \in W$

- There are some other rules for vector spaces:
  1. The zero vector is unique
  2. the negative of a vector $v \in V$ is unique
  3. $0 \cdot \vec{v} = 0$
  4. $c \cdot \vec{0} = \vec{0}$
  5. $(-1)\vec{v} = -\vec{v}$

- If A is $m \times n$, solutions to system of homogenous linear equations $AX = 0$ is a subspace of $\mathbb{R}^n$

- A set of vectors $\{v_1, v_2, ..., v_n$ is **linearly independent** iff, for a system $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$ all constants c are zero. As a random note, no set containing the zero vector is independent.

- Vectors $v_1, ...v_n$ of a vector space V **span** V if $\text{Span}\{v_1, ...v_n\} = V$. In English, this means that $v1, ..., v_n$ span V if every vector in V is a linear combination of $v_1, ..., v_n$.

Dimension, Nullspace, Row Space, Column Space

- **Equivalent facts about nonsingular** $n \times n$ **matrices** (in particular some matrix A):
  A is nonsingular $\leftrightarrow$ rref$(A) = I \leftrightarrow rank(A) = n \leftrightarrow$
  A has existence property $\leftrightarrow A$ has uniqueness property $\leftrightarrow$
  A is invertible $\leftrightarrow A$ is a product of elementary matrices $\leftrightarrow$
  A represents a row reduction $\leftrightarrow det(A) \neq 0 \leftrightarrow$
  columns of A are linearly independent $\leftrightarrow$ rows of A are linearly independent $\leftrightarrow$
  $dim(CS(A)) = n \leftrightarrow dim(RS(A)) = n \leftrightarrow dim(NS(A)) = 0$

- If a vector space V has a basis of n vectors, the **dimension of** V **is** n. This is denoted as $\dim(V)$.

- $\dim(\mathbb{R}^n) = n, \dim M_m \times n(\mathbb{R}) = mn, \dim(P_n) = n + 1$

- The **basis vectors of a polynomial** of degree n are $\{x^n, x^{n-1}, ..., x, 1\}$. This fundamental set of solutions **spans** P_n.

- Suppose some $v_1, v_2, ...v_n$ in a vector space V. The vectors are **linearly dependent** iff *only one* if $v_1, v_2, ..., v_n$ is a linear combination of the others. To see if a set of vectors is linearly independent, solve for the constants for the homogenous equation; if a non-trivial solution exists, then the vectors are linearly dependent.

- Vectors $v_1, v_2, ..., v_n$ of a vector space V are a basis for V if both of the following conditions are satisfied:
  1. $v_1, v_2, ..., v_n$ are **linearly independent**
  2. $v_1, v_2, ..., v_n$ **span** V
- Suppose some $v_1, v_2, ...v_n$ in a vector space V. Then $v_1, v_2, ..., v_n$ form a basis for V iff each vector in V is uniquely expressible as a linear combonation of $v_1, v_2, ..., v_n$
- If V is a vector space and $v_1, v_2, ..., v_n$ are vectors in V, then the set of all linear combinations of $v_1, v_2, ..., v_n$ is a subspace of V
- The subspace of some V consisting of all linear combinations of vectors $v_1, v_2, ..., v_n$ is referred to as the **subspace** of V spanned by $v_1, v_2, ..., v_n$
  ...in English, the span is the set of all the different places you could "go" if you combined the given vectors in every possible way. To see if something spans something else, solve for constants c; if there's no solution, then the set does not span.
- Suppose that V is a vector space of dimension n.
  1. If the vectors $v_1, v_2, ..., v_n$ are linearly independent, then $v_1, v_2, ..., v_n$ are linearly independent, then $v_1, v_2, ..., v_n$ form a basis for V
  2. If $v_1, v_2, ..., v_n$ span V, then $v_1, v_2, ..., v_n$ form a **basis**
- The solutions to the homogenous system $AX = 0$, where A is an $m \times n$ matrix form a subspace of $\mathbb{R}^n$. This vector space of solutions is the **nullspace** or **kernel** of A, denoted by $NS(A)$
- The **row space** of some matrix A is the span of the rows of A. If A, B are row-equivalent matrices, $RS(A) = RS(B)$
- Weirdly enough, for an $m \times n$ matrix ,$\dim(RS(A)) + \dim(NS(A)) = n$
- The **column space** of a matrix A is the subspace of $\mathbb{R}^m$ spanned by the columns of A and is denoted by $CS(A)$. To find the column space of a matrix, you:
  1. **T**ranspose
  2. **R**ow-reduce
  3. transpose back
  4. take **B**asis vectors

  ...dumb mnemonic is **TR**i**B**e. (Or 'tribble' if you're into Star Trek.)
- For a given matrix $A, \dim(RS(A)) = \dim(CS(A))$; this common dimension is called the **rank** of A.

Wronskians

- Suppose some $f_1, f_2, ..., f_n$ are functions in $D^{n-1}(a, b)$. If the **Wronskian** $w(f_1(x), f_2(x), ..., f_n(x))$ of $f_1, f_2, ..., f_n$ is nonzero for some x in (a, b), then $f_1, f_2, ..., f_n$ are linearly independent elements of $D^{n-1}(a, b)$
- Let $y_1, ...y_n$ be solutions to the nth-order homogenous DE (satisfying the uniqueness/existence THM) $L(y) = 0$, then $(w(x) = 0$ for any $x_0 \to (\{y_1, ..., y_n\}$ is linearly dependent
- **Note:** The Wonskian can be computed for any convenient value of x. Try to find values that simplify the computations.
- *Note:* It *cannot* be said that $w(x) = 0 \Rightarrow \{f_1, ..., f_n\}$ are linearly dependent!
- A function $f : \mathbb{R}' \to \mathbb{R}'$ is **analytic** (also "real analytic") if its Taylor series converges to itself. For our purposes, this means that all sums, products, and fractions of $e^x, \cos x, \sin x$, and all polynomials will are analytic functions.
- If $f_1, ..., f_n$ are **analytic** and the **Wronskian** is identically zero (that is, zero everywhere), then $\{f_1, ..., f_n\}$ is **linearly dependent**.

Differential Equations

General

- The **general solution** gives us all possible solutions to a given differential equation
- **Existence and uniqueness theorem:**If $a, b > 0$ and f and $\partial f/\partial y$ are continuous on the rectangle $|x - x_0| < a$, and $|y - y_0| < b$, then there exists an $h > 0$ so that the IVP
  $$y' = f(x, y), y(x_0) = y_0$$
  has one *and only one* solution for $|x - x_0| \leq h$
- **Rational root theorem**: If $f(x) = a_n x^n + ... + a_1 x + a_0$ has *integer* coefficients zand if $r = p/q$ is a *rational* root, then p divides a_0 and a divides a_n. SO, a rational root p/q of $\lambda^3 - 5\lambda^2 + 6\lambda - 2 = 0$ must have p dividing 2 and q dividing 1, giving ± 1 and ± 2 as the only possible rational roots.

Modeling with Diff Eq

- **Continuously compounded interest, population, radioactive decay:** $t = Ce^{kt}$, where $C > 0$ and represents the quantity at $t = 0$, and k is constant (which will usually be solved for). **Exponential growth** if $k > 0$, **exponential decay** if $k < 0$
- **Temperature** changes are determined by the equation $\theta' = k(\theta_{environment} - \theta_{initial})$

Higher-order LDE

- If $p(\lambda)$ has all real roots, giving solutions of the forms $e^{r_i x}$ and $x^k e^{r_i x}$ then the solutions are **linearly independent**.
- For solutions to a differential equation $y_1, ..., y_n$ on an interval (a, b), if $w(y_1(x_0), ..., y_n(x_0)) = 0$ for any x_0 in (a, b), then $y_1, ..., y_n$ are linearly dependent on (a, b). *Note that this **only** holds if the number of functions is equal to the order of the differential equation.*
- If $r_1, ..., r_k$ are distinct real roots for a characteristic polynomial of form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

  then $e^{r_1}, ..., e^{r_k x}$ are linearly independent solutions to this DE, and form a fundamenmtal set of solutions.
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Complex Solutions

- ***Note:*** *there is an entire section on complex numbers at the end!*
- if $L(y) = 0$ is a CCLDE w/ real coefficientsr and if $y(x) = \mu(x) + iv(x)$ is a complex-valued solution, $\mu(x), v(x)$ are also solutions.
- If $L(y) = 0$ is a real CCLDE and $r, \bar{r}$ are a pair of roots of characteristic polynomial, then $e^{ax} \cos bx, e^{ax} s \sin bx$ are real, independent solutions, with same span as $e^{rx}, e^{\bar{r}x}$
- If $L(Y) = 0$ has a **characteristic polynomial** with roots $r_1, ..., r_n$, and sets of solutions formed by:
  1. For real roots r of multiplicity m, we include
  $$e^{rx}, xe^{rx}, ..., x^{m-1} e^{rx}$$
  2. For complex roots r of multiplicity m, we include
  $$e^{rx} \cos bx, e^{rx} \sin bx, xe^{rx} \cos bx, xe^{rx} \sin bx, ..., x^{m-1} e^{rx} \cos bx, x^{m-1} e^{rx} \sin bx$$

  ...then this set of functions is **independent**, and is a **fundamental set** of solutions.

Method of Undetermined Coefficients

- The solution to a differential equation is the sum of the particular solution and the homogenous solution:

$$y = y_h + y_p$$

- The goal is to find some y_p that works with some given differential equation, with $\lambda = r$ is a root of multiplicity m, k is highest power of x on the right side of the equation.
- If the root to an LDE of the form $a_n y^{(n)} + ... + a_1 y' + a_0 y = Ax^k e^{rx}$ is real, the particular solution will be

$$y_p = x^m (A_k x^k + ... + A_1 x + A_0) e^{rx}$$

- If root to equation of form
$$a_n y^{(n)} + ... + a_1 y' + a_0 y = Ax^k e^{ax} \cos bx + Bx^k e^{ax} \sin bx$$
  is imaginary, particular solution is given as:

$$y_p = x^m (A_k x^k + ... + A_1 x + A_0) e^{ax} \cos bx + x^m (B_k x^k + ... + B_1 x + B_0) e^{ax} \sin bx$$

- *Note: the method of undetermined coefficients is <u>not</u> an exact science!* There can be trial and error involved; a table of decent guesses is embedded at the end of this document.

Applications

- Sitautions with no external forces are usually given in the form

$$F = mu'' + fu' + ku = 0$$

where (in spring problems at least) k is the spring constant, f is the friction coefficient, and mass m. (Don't forget that pounds are a unit of force, *not* of mass!)

- In **unforced cases** (no external input to system), the quadratic equation can give us roots; the issue of imaginary roots arises (that is, $f^2 - 4km < 0$):
  1. If no friction ($f = 0$), roots are imaginary. These cases are kind of rare.
  2. If $f - 4km < 0$, $\lambda = -a \pm bi$; in this case, the oscillating thing in question takes a bit to stop moving. This system is **underdamped**.
  3. If $f - 4km > 0$, there will be two roots $r_1, r_2 < 0$, solutions will have the form $u = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ this system is **overdamped**.
  4. If $f^2 - 4km = 0$, we have one root $r = \frac{-f}{2m}$ and solutions $u = c_1 e^{rt} + c_2 t e^{rt}$. This system is **critically damped**, and is the case where the oscillating thing returns to rest within the shortest amount of time.

- In **forced cases**, there are a few possible situations:
  1. For instances of no friction (system is **undamped**), the equation will have the form:

  $$\mu'' + ku = h(t)$$

  We'll consider a case where $\omega_0 \neq \omega$, where $\omega = \sqrt{k/m}$; complete set of solutions can be obtained by:

  $$\mu_H = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

  $$\mu_P = \frac{a}{\omega_0^2 - \omega^2} \cos \omega t$$

  ...and $\mu = \mu_H + \mu_P$.
  2. **Resonance** occurs when $\omega = \omega_0$; in these situations, the amplitude increases as time increases. This has been known to cause some kinds of problems.

Linear Transformations

- A function f from a set X to a set Y is denoted as $f : X \to Y$; X is the domain of f, Y is called the **image set** or **codomain**. The subset $f(x)|x \in X$ of Y is called the **range**, which, in English, means "all the $f(x)$ that 'hit' something in Y."

- If V, W are vector spaces, a function $T : V \to W$ is called a **linear transformation** if, for all vectors $u, v \in V$ and all scalars c, the following two properties hold:
  1. $T(u + v) = T(u) + T(v)$
  2. $T(cv) = cT(v)$

- If $T : V \to V$, T is sometimes called a **linear operator**

- If A is an $m \times n$ matrix, $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(X) = AX$$

is a linear transformation, but is more commonly called a **matrix transformation**

- The differential operator takes derivatives, and is a linear transformation denoted by

$$D : D(a, b) \to F(a, b)$$

- Int(f) denotes definite integral of a function f over a closed interval $[a, b]$

- For some **linear transformation**$T : V \to W$ the following properties hold:
  1. $T(0) = 0$
  2. $T(-v) = -T(v)$ for any $v \in V$
  3. $T(u - v) = T(u) - T(v)$ for any $u, v \in V$
  4. $T(c_1 v_1 + c_2 v_2 + ... + c_k v_k) = c_1 T(v_1) + c_2 T(v_2) + ... + c_k T(v_k)$ for any scalars $c_1, c_2, ..., c_k$ and any vectors $v_1, v_2, ... v_k \in V$

- To determine what exactly a linear transformation does (assuming you have a set of input vectors by which the behavior of T may be observed):

  1. Assume, find, or steal some basis for the domain; this will determine all values of an LT.
  2. Determine the coefficients $c_1, ..., c_n$ for the basis vectors that yield a given output
  3. Determine how T combines the given input vectors to an output vector of form $[x_0, x_1, ..., x_n]$, solving again for coefficients

- The **kernel** of T, denoted $\ker(T)$, is defined as

  $$\ker(T) = \{v \in V | T(v) = 0\}$$

  ...in English, the kernel is the set of all vectors that give an output of the zero vector. For matrix transformations, the kernel of the matrix transformation $T(X) = AX$ is the same as the nullspace of A:

  $$\ker(T) = NS(A)$$

- To find a basis for the column space of a matrix, take the transpose (**T**), reduce (**R**), transpose (**T**), split into columns (**S**), which approximately spells **ToRToiSe**.

- Weirdly enough, if $T : V \to W$ is an LT where V is a finite-dimensional vector space, we have that

  $$\dim(\ker(T) + \dim(\text{range}(T)) = \dim(V)$$

Errata

- The **contrapositive** of a statement is the reverse of the premise and conclusion, and the negation of both. Original statement: "Duke won, and I'm happy." Contrapositive: "If I'm not happy, then Duke didn't win."

- **Quadratic equation:** $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Trig Identities

- STUFF HERE
- $\sin(\alpha + \beta) = \sin\alpha \cos\alpha + \sin\beta \cos\alpha$
- $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \sin\beta \cos\alpha$
- $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$
- $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

Complex stuff!!

add it yo

- $e^{i\theta} = \cos\theta + i \sin\theta$
- $(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1 \theta_2)}$
- **UNDERSTAND COMPLEX CONJUGATES**
- **Roots of unity:** Let $\alpha = e^{2\pi i/n}$. Then, $1, \alpha, \alpha^2, ..., \alpha^{n-1}$ are the nth roots of 1.
- ...applying this to any complex number: Let r be any such root. (That is, if $\alpha = me^{i\theta}, r = (m^{1/n})e^i\theta/n$. In this case, the nth roots of α are **NOTE THIS ISN"T DONE!!! 1**
- ...and, to that end, I am pirating someone else's work for this section. Thank you to RPI's Math 6800.

Complex Variables Cheat Sheet

A complex number is written as $z = x + iy$ where x and y are real numbers and $i^2 = -1$. We write $\Re(z) = x$ (or $Re(z)$) for the *real part of* z and $\Im(z) = y$ (or $Im(z)$) for the *imaginary part of* z. The complex conjugate, denoted by z^* or $\bar{z}$ is defined a $\bar{z} = x - iy$. A complex number may be plotted in the 2D $x - y$ plane known as the *complex* (or Argand) plane. It can be viewed as a vector or point in 2D with coordinates (x, y). The length or *modulus* of $z = x + iy$ is the length of the vector,

$$|z| = \sqrt{x^2 + y^2}$$

Note that $|z|^2 = z\bar{z}$. From the geometry in the plane we see that

$$\begin{gathered} z = r(\cos(\theta) + i\sin(\theta)), \\ r = |z|, \quad \cos(\theta) = x/r, \quad \sin(\theta) = y/r \end{gathered}$$

To add, multiply or divide two complex numbers $z_1 = x_1 + iy_1$ and $x_2 + iy_2$ we use $i^2 = -1$,

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2), \\ z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2), \\ \frac{z_1}{z_2} &= \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{z_1\bar{z}_2}{|z_2|^2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{|z_2|^2} = \frac{(x_1x_2 + y_1y_2) + i(-x_1y_2 + y_1x_2)}{x_2^2 + y_2^2} \end{aligned}$$

From Taylor series

$$\begin{aligned} &\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots, \qquad \sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots, \\ &e^{ix} = 1 + ix + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \dots = \Big(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots\Big) + i\Big(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots\Big) \end{aligned}$$

This leads to Euler's formula

$$e^{ix} = \cos(x) + i\sin(x)$$

and thus

$$\begin{gathered} z = r\Big(\cos(\theta) + i\sin(\theta)\Big) = r\,e^{i\theta} \quad \text{(polar form)}, \\ r = |z|, \quad \cos(\theta) = x/r, \quad \sin(\theta) = y/r \end{gathered}$$

We define e^z, $\cos(z)$, $\sin(z)$ etc. from the Taylor series. We have

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

From geometry we have the triangle inequality for complex numbers

$$\big|\,|z_1| - |z_2|\,\big| \le |z_1 + z_2| \le |z_1| + |z_2|$$

Trial solutions for the method of undetermined coefficients

	Form of $g(x)$	Guess for particular solution
1.	1 (any constant)	A
2.	$5x+7$	$Ax+B$
3.	$3x^2-2$	Ax^2+Bx+C
4.	$\sin 4x$	$A\cos 4x+B\sin 4x$
5.	$\cos 4x$	$A\cos 4x+B\sin 4x$
6.	e^{5x}	Ae^{5x}
7.	$(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
8.	x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
9.	$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x+Be^{3x}\sin 4x$
10.	$5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x+(E^2+Fx+G)\sin 4x$
11.	$xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x+(Cx+E)e^{3x}\sin 4x$
12.	$(5x+7)+\sin 4x$	$(Ax+B)+(C\cos 4x+D\sin 4x)$