Math 216 Midterm 3 Study Guide

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1 Matrices for Linear Transformations

- Suppose $T : V \to W$ is an LT; further suppose $v_1, ..., v_n$ form a basis α for V and $w_1, ..., w_m$ form a basis β for W. If we were to express $T(v_1), ..., T(v_n)$ in terms of $w_1, ..., w_m$, we get a series of equations of the form: $T(v_1) = a_{11}w_1 + a_21w_2 + ... + a_{m1}w_m$
  $T(v_2) = a_{12}w_1 + a_{22}w_2 + ... + a_{m2}w_m$
  $\vdots$
  $T(v_m) = a_{1n}w_1$

Chapter 9.3 ("Schur's Theorem and Symmetric Matrices")

- Recall tha B is similar to A if there exists an invertible $n \times n$ matrix such that

  $$B = P^{-1}AP$$

  If P is an orthogonal matrix, that is:

  $$P^{-1} = P^T \text{ and } B = P^T AP$$

  ...we say that B is **orthogonally simliar** to A.

- If P is an orthogonal matrix, B is an orthogonal basis for $\mathbb{R}^n$. [1]

- **Schur's Theorem:** Suppose A is an $n \times n$ matrix. If all the eigenvalues of A are real numbers, A is orthogonally similar to an upper triangle matrix.

- If C is a matrix whose entries are complex numbers, the **Hermitian conjugate** of C, notated as C^*, is given as $C^* = \bar{C}^T$

- An $n \times n$ matrix P with complex entries is a **unitary matrix** if $P^*P = I$. Unitary matrices are the analog of orthogonal matrices in a complex space.

- An $n \times n$ matrix B is unitaritly similar to an $n \times n$ matrix A if there exists a unitary P such that $B = P^T AP$.

- If A is $n \times n$ and symmetric with real entries, all eigenvalues of A are real.

- If A is symmetric with real entries, A is diagonalizable.

- If A is symmetric with real entries and $\vec{v_1}, \vec{v_2}$ are eigenvectors of A with different associated eigenvalues, v_1 is orthogonal to v_2.

- Commonly used steps for finding an orthogonal matrix P that diagonalizes an $n \times n$ matrix symmetrix matrix A with real entries:

  1. Find bases for eigenspaces of A
  2. Apply Gram-Schmidt process to basis of each eigenspace to obtain an orthonormal basis.
  3. P is a matrix made of columns from step 2. *YEET*

- If A Is an $n \times n$ symmetric matrix with real entries, then all the eigenvalues of a A are real, and all eigenspaces have real bases.

- In the *complex space* $\mathbb{C}$, we have to redefine the **inner product**. The **Hermitian dot product** on $\mathbb{C}$ is defined as:

  $$< \vec{v}, \vec{w} > = \sum v_i \bar{w}_i = \vec{v}^T \vec{w}$$

  The properties of the Hermitian dot product are:

  1. $< \vec{v}, \vec{w} >_H = < \vec{w}, \vec{v} >_H$
  2. $< \vec{u} + \vec{v}, \vec{w} >_H = < \vec{u}, \vec{w} > + < \vec{v}, \vec{v}, \vec{w} >_H$
  3. $< c\vec{v}, \vec{w} >_H = c < \vec{v}, \vec{w} >_H$

[1] Since P is change of basis matrix of $T(x) = Ax$

4. $< \vec{v}, \vec{v} >_H \geq, < \vec{v}, \vec{v} >_H = 0$ iff $\vec{v} = 0$ Functions satisfying these properties are called **Hermitian inner products.**
5. With the Hermitian inner product, we also have the **Hermitian transpose:**

   $$A^* = \bar{A}^T$$

   Note that Hermitian transpose $\leftrightarrow$ Hermitian conjugate $\leftrightarrow$ adjoint .
6. Transposes relate to symmetry by definition:

   $$< A\vec{v}, \vec{w} >_H = < \vec{w}, A\vec{v} >_H$$

   Real symmetric matrices are Hermitian as well.
7. if AA is real and symmetric with eigenvalues $\lambda_1 \neq \lambda_2$ and associated eigenvectors $\vec{v_1}, \vec{v_2}$, $\vec{v_1}, \vec{v_2}$ are **orthogonal**.
8. $< A\vec{v}, \vec{w} >_H = < \vec{v}, A^*\vec{w} >_H$
9. Recall that A, B are similar if $B = P^{-1}AP$, where P is a change of basis. We now say that: If A, B are similar by $BP^{-1}AP$ and P is orthogonal, then A, B are orthogonally similar, and $B = P^T AP$.
10. If A is diagonalizable with $D = P^{-1}AP$ (that is, columns of P are a basis of eigenvector) and if P is orthogonal, then we say A is **orthogonally diagonalizable**.
11. Every real, symmetric matrix is **orthogonally diagonalizable**.