Math 216 Midterm 3 Study Guide

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1 Change of Basis

- For some basis v and the standard basis s, the matrix $[I]_v^s$ is the change of basis matrix $s \to v$

2 Matrices for Linear Transformations

- Suppose $T : V \to W$ is an LT; further suppose $v_1, ..., v_n$ form a basis α for V and $w_1, ..., w_m$ form a basis β for W. If we were to express $T(v_1), ..., T(v_n)$ in terms of $w_1, ..., w_m$, we get a series of equations of the form:

$$T(v_1) = a_{11}w_1 + a_21w_2 + ... + a_{m1}w_mT(v_2) = a_{12}w_1 + a_{22}w_2 + ... + a_{m2}w_m \vdots T(v_m) = a_{1n}w_1$$

Chapter 9.3 ("Schur's Theorem and Symmetric Matrices")

- Recall tha B is similar to A if there exists an invertible $n \times n$ matrix such that

$$B = P^{-1}AP$$

If P is an orthogonal matrix, that is:

$$P^{-1} = P^T \text{ and } B = P^TAP$$

...we say that B is **orthogonally simliar** to A.

- If P is an orthogonal matrix, B is an orthogonal basis for $\mathbb{R}^n$. [1]
- **Schur's Theorem:** Suppose A is an $n \times n$ matrix. If all the eigenvalues of A are real numbers, A is orthogonally similar to an upper triangle matrix.
- If C is a matrix whose entries are complex numbers, the **Hermitian conjugate** of C, notated as C^*, is given as $C^* = \bar{C}^T$
- An $n \times n$ matrix P with complex entries is a **unitary matrix** if $P^*P = I$. Unitary matrices are the analog of orthogonal matrices in a complex space.
- An $n \times n$ matrix B is unitaritly similar to an $n \times n$ matrix A if there exists a unitary P such that $B = P^TAP$.
- If A is $n \times n$ and symmetric with real entries, all eigenvalues of A are real.
- If A is symmetric with real entries, A is diagonalizable.
- If A is symmetric with real entries and $\vec{v_1}, \vec{v_2}$ are eigenvectors of A with different associated eigenvalues, v_1 is orthogonal to v_2.
- Commonly used steps for finding an orthogonal matrix P that diagonalizes an $n \times n$ matrix symmetrix matrix A with real entries:
  1. Find bases for eigenspaces of A
  2. Apply Gram-Schmidt process to basis of each eigenspace to obtain an orthonormal basis.
  3. P is a matrix made of columns from step 2. *YEET*
- If A Is an $n \times n$ symmetric matrix with real entries, then all the eigenvalues of a A are real, and all eigenspaces have real bases.
- In the *complex space* $\mathbb{C}$, we have to redefine the **inner product**. The **Hermitian dot product** on $\mathbb{C}$ is defined as:

$$< \vec{v}, \vec{w} > = \sum v_i \bar{w}_i = \vec{v}^T \vec{w}$$

The **properties of the Hermitian dot product** are:

[1] Since P is change of basis matrix of $T(x) = Ax$

1. $< \vec{v}, \vec{w} >_H = < \vec{w}, \vec{v} >_H$
2. $< \vec{u} + \vec{v}, \vec{w} >_H = < \vec{u}, \vec{w} > + < \vec{v}, \vec{v}, \vec{w} >_H$
3. $< c\vec{v}, \vec{w} >_H = c < \vec{v}, \vec{w} >_H$
4. $< \vec{v}, \vec{v} >_H \geq, < \vec{v}, \vec{v} >_H = 0$ iff $\vec{v} = 0$

Functions satisfying these properties are called **Hermitian inner products.**

- With inner product spaces, we define the angle between two vectors as:
$$\theta = \cos^{-1} \frac{u \cdot v}{||u||||v||}$$
- One common (and important!) inner product is the L^2 inner product, which is defined as:
$$< f, g >= \int f \cdot g dx$$
- Magnitude with inner products is defined as
$$||v|| = \sqrt{< v, v >}$$
- With the Hermitian inner product, we also have the **Hermitian transpose:**
$$A^* = \bar{A}^T$$
Note that Hermitian transpose $\leftrightarrow$ Hermitian conjugate $\leftrightarrow$ adjoint .
- Transposes relate to symmetry by definition:
$$< A\vec{v}, \vec{w} >_H = < \vec{w}, A\vec{v} >_H$$
Real symmetric matrices are Hermitian as well.
- if AA is real and symmetric with eigenvalues $\lambda_1 \neq \lambda_2$ and associated eigenvectors $\vec{v_1}, \vec{v_2}$, $\vec{v_1}, \vec{v_2}$ are **orthogonal**.
- $< A\vec{v}, \vec{w} >_H = < \vec{v}, A^*\vec{w} >_H$
- Recall that A, B are similar if $B = P^{-1}AP$, where P is a change of basis. We now say that: If A, B are similar by $BP^{-1}AP$ and P is orthogonal, then A, B are orthogonally similar, and $B = P^T AP$.
- If A is diagonalizable with $D = P^{-1}AP$ (that is, columns of P are a basis of eigenvector) and if P is orthogonal, then we say A is **orthogonally diagonalizable**.
- Every real, symmetric matrix is **orthogonally diagonalizable**.

Chapter 6.1 (Theory of Systems of LDEs)

- If $a_{i,j}(x)$ and $g_i(x)$ are continuous on (a, b) containing x_0, for $a \leq i \leq n, 1 \leq j \leq n$, the IVP
$$Y' = A(x)Y + G(x), Y(x_0) = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$
has a unique solution on (a, b)
- The solutions to a homogenous system of first-order LDEs
$$Y' = A(x)Y$$
form a vector space (subspace) of dimension n - the basis for the subspace is a fundamental set of solutions
- A set of n linearly independent solutions $Y_1, Y_2, ..., Y_n$ to a homogenous system of n first-order LDEs is a fundamental set of solutions, often notated as M.
- The general solution to a homogenous system of 1st-order linear differential equations is written as
$$Y_H = MC$$
- If $Y_1, Y_2, ..., Y_n$ form a fundamental set of solutions to a system of homogenous first-order linear differential equations $Y' = A(x)Y$ ...then ever solution to this nonhomogenous system is of the form:
$$Y = Y_H + Y_P = c_1Y_1 + ... + c_nY_n + Y_p = MC + Y_p$$
- If the **Wronskian** is inequal to 0, functions are linearly independent. If the Wronskian is 0 for some $x_0 \in (a, b)$, then $Y_1, ..., Y_n$ is **linearly dependent**.
This further implies that, for the fundamental set of solutions over (a, b), $w(Y_1(x), Y_2(x), ..., Y_n(x)) \neq 0 \forall x \in (a, b)$

Chapter 6.2 (Constant Coefficient Homogenous Systems (Diagonalizable)

- A change of basis can simplify a system as well as a matrix. Diagonal systems are also easy to solve, and we like easy around here. [2]
- Diagonal systems are **decoupled**, which means that the variables we're solving for don't interact with each other, and can be solved for independently.
- If A, B are similar matrices with $B = P^{-1}AP$ where P is an invertible $n \times n$ matrix. If Z is a solution of $Y' = BY$, then PZ is a solution of $Y' = AY$.
  Also, if $Z_1, Z_2, ..., Z_n$ forms a fundamental solution to $Y' = BY$, then $PZ_1, PZ_2, ..., PZ_n$ forms a fundamental set of solutions to $Y' = AY$
- To rephrase this: if $M_B = [Z_1, Z_2, ..., Z_n]$ is a matrix of fundamental solutions for $Y' = BY$, then

$$M_A = PM_B = [PZ_1, PZ_2, ..., PZ_n]$$

  is a matreix of fundamental solutions to $Y' = AY$.
- Suppose a matrix A is diagonalizable and that there's some P such that $D = P^{-1}AP$ is the diagonal matrix:

$$D = \begin{bmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}$$

  ...which imples that the general solution to $Y' = AY$ is:

$$P \begin{bmatrix} c_1 e^{d_1 x} \\ c_2 e^{d_2 x} \\ \vdots \\ c_n e^{d_n x} \end{bmatrix}$$

  Note: you'll really, really want to be familiar with this. In the book, look at page 303.
- If $U(x) + iV(x)$ is a solution to $Y' = A(x)Y$, then $U(x)$ and $V(x)$ are solutions to $Y' = A(x)Y$.
- If some A is diagonalizable with

$$D = P^{-1}AP, A = PDP^{-1}$$

  ...we can rewrite

$$\vec{y}' = A\vec{y}$$

  as

$$\vec{y}' = PDP^{-1}$$

  and multiply to obtain a diagonal system $\vec{y}' = PDP^{-1}\vec{y}$ ...which eventualy becomes a decoupled system $\vec{Z}' = D\vec{Z}$, which, after many more steps, becomes a fundamental set of solutions: $\{e^{\lambda_1 x}\vec{e1}, ...e^{\lambda_n x}\vec{e_n}\}$
- Consider $\vec{y}' = A\vec{v}$, some diagonalizable A with eigenvalues $\lambda_1, \lambda_2, ...\lambda_n$ and eigenvectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$. Then

$$\{e^{\lambda_1 x} v_1, ..., e^{\lambda_n x}\vec{v_n}\}$$

  forms a fundamental set of solutions.

Diagonal matrices

- A matrix can be ***diagonalized*** iff there exists a basis β for $\mathbb{R}^n$ made of eigenvectors. The entries of the diagonal matrix are the eigenvalues.
- For a diagonal matrix D:

$$D = P^{-1}AP$$

[2] Hence Duke math.