Complex Variables Cheat Sheet

A complex number is written as $z = x + iy$ where x and y are real numbers and $i^2 = -1$. We write $\Re(z) = x$ (or $Re(z)$) for the *real part of* z and $\Im(z) = y$ (or $Im(z)$) for the *imaginary part of* z. The complex conjugate, denoted by z^* or $\bar{z}$ is defined a $\bar{z} = x - iy$. A complex number may be plotted in the 2D $x - y$ plane known as the *complex* (or Argand) plane. It can be viewed as a vector or point in 2D with coordinates (x, y). The length or *modulus* of $z = x + iy$ is the length of the vector,

$$|z| = \sqrt{x^2 + y^2}$$

Note that $|z|^2 = z\bar{z}$. From the geometry in the plane we see that

$$\begin{aligned} z &= r(\cos(\theta) + i\sin(\theta)), \\ r &= |z|, \quad \cos(\theta) = x/r, \quad \sin(\theta) = y/r \end{aligned}$$

To add, multiply or divide two complex numbers $z_1 = x_1 + iy_1$ and $x_2 + iy_2$ we use $i^2 = -1$,

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2), \\ z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2), \\ \frac{z_1}{z_2} &= \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{z_1\bar{z}_2}{|z_2|^2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{|z_2|^2} = \frac{(x_1x_2 + y_1y_2) + i(-x_1y_2 + y_1x_2)}{x_2^2 + y_2^2} \end{aligned}$$

From Taylor series

$$\begin{aligned} &\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots, \qquad \sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots, \\ &e^{ix} = 1 + ix + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \ldots = \Big(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots\Big) + i\Big(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots\Big) \end{aligned}$$

This leads to Euler's formula

$$e^{ix} = \cos(x) + i\sin(x)$$

and thus

$$\begin{aligned} z &= r\Big(\cos(\theta) + i\sin(\theta)\Big) = r\,e^{i\theta} \quad \text{(polar form)}, \\ r &= |z|, \quad \cos(\theta) = x/r, \quad \sin(\theta) = y/r \end{aligned}$$

We define e^z, $\cos(z)$, $\sin(z)$ etc. from the Taylor series. We have

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

From geometry we have the triangle inequality for complex numbers

$$\big|\,|z_1| - |z_2|\,\big| \le |z_1 + z_2| \le |z_1| + |z_2|$$