Eigenvalue and Eigenvector Homework

Olena Bormashenko

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For each of the matrices A below, do the following:

1. Find the characteristic polynomial of A, and use it to find all the eigenvalues of A.
2. State the algebraic multiplicity of each eigenvalue.
3. Find the fundamental eigenvectors of A, and state whether the matrix is diagonalizable.
4. If the matrix is indeed diagonalizable, check that $P^{-1}AP = D$ holds.

Here are the matrices in question:

1. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

   Solution:

   1. By definition, the characteristic polynomial of A is $|xI_n - A|$, and the notation for it is $p_A(x)$. Therefore,

      $$\begin{aligned} p_A(x) = |xI_n - A| &= \left| x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} x-1 & -2 \\ 0 & x-1 \end{vmatrix} \\ &= (x-1)(x-1) = \boxed{(x-1)^2} \end{aligned}$$

      To find the eigenvalues, set $p_A(x)$ to 0. Thus, we get

      $$0 = (x-1)^2 \Rightarrow x = 1$$

      Therefore, the only eigenvalue is $\boxed{\lambda = 1}$.

   2. The algebraic multiplicity of the eigenvalue λ is the highest power of $(x - \lambda)$ that appears in the polynomial $p_A(x)$. Here, there's only $\lambda = 1$, and the highest power of $(x - 1)$ appearing in $p_A(x)$ is 2. Therefore,

      $$\boxed{\text{The algebraic multiplicity of } \lambda = 1 \text{ is } 2}$$

3. Since there's only one eigenvalue, we only find the fundamental eigenvectors for it. We need to write E_1 as all linear combinations of some set of vectors. We know that E_λ is the set of solutions to $(\lambda I_n - A)\vec{x} = \vec{0}$, so E_1 is the set of solutions to $(I_2 - A)\vec{x} = \vec{0}$. Now,

$$I_2 - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$$

The system $(I_2 - A)\vec{x} = \vec{0}$ corresponds to the following augmented matrix (which row reduces very simply):

$$\left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1:1/2\times R_1} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

This system clearly corresponds to

$$\begin{aligned} c_1 &= c_1 \\ c_2 &= 0 \end{aligned}$$

Therefore,

$$E_1 = \{\vec{x} \mid A\vec{x} = \vec{x}\} = \left\{ \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \,\middle|\, c_1 \in \mathbb{R} \right\} = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \,\middle|\, c_1 \in \mathbb{R} \right\}$$

Thus,

$$\boxed{\text{The only fundamental eigenvector is } \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

(Of course, any scalar multiple of this is also correct.)
Furthermore, for an $n \times n$ matrix to be diagonalizable, it needs to have n different fundamental eigenvectors. Since here, we only have 1, and we would need 2, we conclude that $\boxed{\text{the matrix is not diagonalizable.}}$

4. Not applicable since the matrix is not diagonalizable.

2. $A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$.

Solution:

1. Like above, $p_A(x)$ is defined to be $|xI_n - A|$. Thus,

$$\begin{aligned} p_A(x) = |xI_n - A| &= \left| x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix} \right| \\ &= \begin{vmatrix} x-1 & 3 & -3 \\ 0 & x+5 & -6 \\ 0 & 3 & x-4 \end{vmatrix} = (x-1)((x+5)(x-4) - 3(-6)) \\ &= (x-1)(x^2 + x - 2) = \boxed{(x-1)^2(x+2)} \end{aligned}$$

To find the eigenvalues, set $p_A(x)$ to 0. Therefore,

$$0 = (x-1)^2(x+2) \Rightarrow x = 1, -2$$

Therefore, the eigenvalues are $\boxed{\lambda_1 = 1 \text{ and } \lambda_2 = -2}$.

2. The algebraic multiplicity of the eigenvalue λ is the highest power of $(x-\lambda)$ that appears in the polynomial $p_A(x)$. Using the fact that $p_A(x) = (x-1)^2(x+2)$, we get that

$$\boxed{\text{The algebraic multiplicity of } \lambda_1 = 1 \text{ is 2, and of } \lambda_2 = -2 \text{ is 1}}$$

3. Since there are two eigenvalues, we need to figure out the eigenvectors for them separately. For each λ, we will be writing E_λ as all linear combinations of some set of vectors.

   $\lambda_1 = 1$: From lecture, we have that E_λ is the set of solutions to $(\lambda I_n - A)\vec{x} = \vec{0}$. Therefore, E_1 is the set of solutions to $(I_3 - A)\vec{x} = \vec{0}$. Now,

$$I_3 - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -3 \\ 0 & 6 & -6 \\ 0 & 3 & -3 \end{bmatrix}$$

   Therefore, the system $(I_3 - A)\vec{x} = \vec{0}$ corresponds to the following augmented matrix:

$$\left[\begin{array}{ccc|c} 0 & 3 & -3 & 0 \\ 0 & 6 & -6 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

   Row reducing, this becomes

$$\begin{aligned} \left[\begin{array}{ccc|c} 0 & 3 & -3 & 0 \\ 0 & 6 & -6 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] &\xrightarrow{R_2:R_2-2R_1} \left[\begin{array}{ccc|c} 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \\ &\xrightarrow{R_3:R_3-R_1} \left[\begin{array}{ccc|c} 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &\xrightarrow{R_1:1/3\times R_1} \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{aligned}$$

   This clearly corresponds to the equation $c_2 - c_3 = 0$ (and two copies of the equation $0 = 0$), and hence results in

$$\begin{aligned} c_1 &= c_1 \\ c_2 &= c_3 \\ c_3 &= c_3 \end{aligned}$$

Therefore,

$$\begin{aligned} E_1 = \{\vec{x} \mid A\vec{x} = \vec{x}\} &= \left\{ \begin{bmatrix} c_1 \\ c_3 \\ c_3 \end{bmatrix} \;\middle|\; c_1, c_3 \in \mathbb{R} \right\} \\ &= \left\{ c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; c_1, c_3 \in \mathbb{R} \right\} \end{aligned}$$

Thus, the fundamental eigenvectors for $\lambda_1 = 1$ are $[1,0,0]^T$ and $[0,1,1]^T$.

$\lambda_2 = -2$: Just like above, E_λ is the set of solutions to $(\lambda I_n - A)\vec{x} = \vec{0}$. This means that E_{-2} is the set of solutions to $((-2)I_3 - A)\vec{x} = \vec{0}$. Calculating,

$$(-2)I_3 - A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 \\ 0 & 3 & -6 \\ 0 & 3 & -6 \end{bmatrix}$$

Therefore, the system $(I_3 - A)\vec{x} = \vec{0}$ corresponds to the following augmented matrix:

$$\left[\begin{array}{ccc|c} -3 & 3 & -3 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 3 & -6 & 0 \end{array}\right]$$

Row reducing, this becomes

$$\begin{aligned} \left[\begin{array}{ccc|c} -3 & 3 & -3 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 3 & -6 & 0 \end{array}\right] &\xrightarrow{R_3:R_3-R_2} \left[\begin{array}{ccc|c} -3 & 3 & -3 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &\xrightarrow{R_1:(-1/3)\times R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &\xrightarrow{R_1:1/3\times R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &\xrightarrow{R_1:R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{aligned}$$

This corresponds to the equations:

$$\begin{aligned} c_1 - c_3 &= 0 \\ c_2 - 2c_3 &= 0 \end{aligned}$$

and therefore becomes

$$\begin{aligned} c_1 &= c_3 \\ c_2 &= 2c_3 \\ c_3 &= c_3 \end{aligned}$$

Thus, we see that

$$\begin{aligned} E_{-2} = \{\vec{x} \mid A\vec{x} = (-2)\vec{x}\} &= \left\{ \begin{bmatrix} c_3 \\ 2c_3 \\ c_3 \end{bmatrix} \middle| \; c_3 \in \mathbb{R} \right\} \\ &= \left\{ c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \middle| \; c_3 \in \mathbb{R} \right\} \end{aligned}$$

This means that the only fundamental eigenvector for $\lambda_2 = -2$ is $[1, 2, 1]^T$. We therefore conclude that

$$\boxed{\text{The fundamental eigenvectors of } A \text{ are } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}$$

Since A is a 3×3 matrix, and we have 3 fundamental eigenvectors, we have a sufficient number. Therefore, $\boxed{\text{the matrix is diagonalizable.}}$

4. By definition, P is the matrix whose columns are the eigenvectors, and D is a diagonal matrix with the eigenvalues along the diagonal in the same order that we listed the eigenvectors. Therefore, here we have that

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Using the standard algorithm for the inverse, we calculate

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] &\xrightarrow{R_3 : R_3 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right] \\ &\xrightarrow{R_1 : R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right] \\ &\xrightarrow{R_2 : R_2 + 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right] \\ &\xrightarrow{R_3 : (-1) \times R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \end{aligned}$$

Therefore,

$$P^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Finally, checking that $P^{-1}AP = D$,

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = D \end{aligned}$$

as expected.

3. $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$.

Solution:

1. As usual, $p_A(x) = |xI_n - A|$. Thus,

$$\begin{aligned} p_A(x) = |xI_n - A| &= \left| x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \right| \\ &= \begin{vmatrix} x-3 & 1 & 0 \\ 1 & x-2 & 1 \\ 0 & 1 & x-3 \end{vmatrix} \end{aligned}$$

Expanding along the first row,

$$\begin{aligned} \begin{vmatrix} x-3 & 1 & 0 \\ 1 & x-2 & 1 \\ 0 & 1 & x-3 \end{vmatrix} &= (x-3) \begin{vmatrix} x-2 & 1 \\ 1 & x-3 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 0 & x-3 \end{vmatrix} \\ &= (x-3)((x-2)(x-3)-1) - (x-3) \\ &= (x-3)(x^2-5x+5-1) \\ &= (x-3)(x^2-5x+4) \\ &= (x-3)(x-1)(x-4) \end{aligned}$$

To find the eigenvalues, set $p_A(x)$ to 0. Therefore,

$$0 = (x-3)(x-1)(x-4) \Rightarrow x = 1, 3, 4$$

Hence, here the eigenvalues of A are $\boxed{\lambda_1 = 1, \lambda_2 = 3 \text{ and } \lambda_3 = 4}$.

2. The algebraic multiplicity of the eigenvalue λ is the highest power of $(x-\lambda)$ that appears in the polynomial $p_A(x)$. Since we know that $p_A(x) = (x-3)(x-1)(x-4)$,

$$\boxed{\text{The algebraic multiplicity of each of } \lambda_1 = 1, \lambda_2 = 3 \text{ and } \lambda_3 = 4 \text{ is } 1}$$

3. Let us figure out the eigenvectors for all the three eigenvalues separately. For each λ, we have to write E_λ as all linear combinations of some set of vectors.

   $\lambda_1 = 1$: As we had above, E_λ is the set of solutions to $(\lambda I_n - A)\vec{x} = \vec{0}$. Therefore, E_1 is the set of solutions to $(I_3 - A)\vec{x} = \vec{0}$. Calculating,

$$I_3 - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

   Hence, the system of equations given by $(2I_3 - A)\vec{x} = \vec{0}$ corresponds to the following augmented matrix:

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

   Row reducing,

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{\text{Swap } R_1,\ R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$
$$\xrightarrow{R_2:R_2+2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$
$$\xrightarrow{R_3:R_3+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\xrightarrow{R_2:(-1)\times R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\xrightarrow{R_1:R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

   This clearly corresponds to the following equations:

$$c_1 - c_3 = 0$$
$$c_2 - 2c_3 = 0$$

Solving, we get

$$\begin{aligned} c_1 &= c_3 \\ c_2 &= 2c_3 \\ c_3 &= c_3 \end{aligned}$$

$$\begin{aligned} E_1 = \{\vec{x} \mid A\vec{x} = \vec{x}\} &= \left\{ \begin{bmatrix} c_3 \\ 2c_3 \\ c_3 \end{bmatrix} \middle| \; c_3 \in \mathbb{R} \right\} \\ &= \left\{ c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \middle| \; c_3 \in \mathbb{R} \right\} \end{aligned}$$

Thus, the fundamental eigenvector for $\lambda_1 = 1$ is $[1, 2, 1]^T$.

$\lambda_2 = 3$: Just like above, E_λ is the set of solutions to $(\lambda I_n - A)\vec{x} = \vec{0}$. This means that E_3 is the set of solutions to $(3I_3 - A)\vec{x} = \vec{0}$. Calculating,

$$3I_3 - A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Therefore, the system $(3I_3 - A)\vec{x} = \vec{0}$ corresponds to the following augmented matrix:

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Row reducing, this becomes

$$\begin{aligned} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] &\xrightarrow{R_3 : R_3 - R_1} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &\xrightarrow{\text{Swap } R_1, R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &\xrightarrow{R_1 : R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{aligned}$$

This corresponds to the system:

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_2 &= 0 \end{aligned}$$

and therefore becomes

$$\begin{aligned} c_1 &= -c_3 \\ c_2 &= 0 \\ c_3 &= c_3 \end{aligned}$$

Thus, we get that

$$\begin{aligned} E_3 = \{\vec{x} \mid A\vec{x} = 3\vec{x}\} &= \left\{ \begin{bmatrix} -c_3 \\ 0 \\ c_3 \end{bmatrix} \middle| \ c_3 \in \mathbb{R} \right\} \\ &= \left\{ c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \middle| \ c_3 \in \mathbb{R} \right\} \end{aligned}$$

This means that the only fundamental eigenvector for $\lambda_2 = 3$ is $[-1, 0, 1]^T$.

$\lambda_3 = 4$: As before, E_4 is the set of solutions to $(4I_3 - A)\vec{x} = \vec{0}$. Calculating,

$$4I_3 - A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Therefore, the system $(4I_3 - A)\vec{x} = \vec{0}$ corresponds to the following augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Row reducing to solve this,

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] &\xrightarrow{R_2 : R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \\ &\xrightarrow{R_3 : R_3 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &\xrightarrow{R_1 : R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{aligned}$$

This clearly corresponds to the following equations:

$$\begin{aligned} c_1 - c_3 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

Solving, we get

$$\begin{aligned} c_1 &= c_3 \\ c_2 &= -c_3 \\ c_3 &= c_3 \end{aligned}$$

Therefore,

$$\begin{aligned} E_4 = \{\vec{x} \mid A\vec{x} = 4\vec{x}\} &= \left\{ \begin{bmatrix} c_3 \\ -c_3 \\ c_3 \end{bmatrix} \middle| \; c_3 \in \mathbb{R} \right\} \\ &= \left\{ c_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \middle| \; c_3 \in \mathbb{R} \right\} \end{aligned}$$

Thus, the fundamental eigenvector for $\lambda_3 = 4$ is $[1, -1, 1]^T$.
We therefore conclude that

$$\boxed{\text{The fundamental eigenvectors of } A \text{ are } \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$$

Since A is a 3×3 matrix, and we have 3 fundamental eigenvectors, this suffices. Therefore, $\boxed{A \text{ is diagonalizable.}}$

4. By definition, P is the matrix whose columns are the eigenvectors, and D is a diagonal matrix with the eigenvalues along the diagonal in the same order that we listed the eigenvectors. Thus, from the derivations above, we get

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Finding the inverse, we see

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] &\xrightarrow{R_2 : R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -3 & -2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \\ &\xrightarrow{R_3 : R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -3 & -2 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right] \\ &\xrightarrow{\text{Swap } R_1, R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \\ 0 & 2 & -3 & -2 & 1 & 0 \end{array}\right] \\ &\xrightarrow{R_2 : (1/2) \times R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 0 & 1/2 \\ 0 & 2 & -3 & -2 & 1 & 0 \end{array}\right] \end{aligned}$$

$$\xrightarrow{R_1:R_1+R_2}\left[\begin{array}{ccc|ccc}1&0&1&1/2&0&1/2\\0&1&0&-1/2&0&1/2\\0&2&-3&-2&1&0\end{array}\right]$$

$$\xrightarrow{R_3:R_3-2R_2}\left[\begin{array}{ccc|ccc}1&0&1&1/2&0&1/2\\0&1&0&-1/2&0&1/2\\0&0&-3&-1&1&-1\end{array}\right]$$

$$\xrightarrow{R_3:(-1/3)\times R_3}\left[\begin{array}{ccc|ccc}1&0&1&1/2&0&1/2\\0&1&0&-1/2&0&1/2\\0&0&1&1/3&-1/3&1/3\end{array}\right]$$

$$\xrightarrow{R_1:R_1-R_3}\left[\begin{array}{ccc|ccc}1&0&0&1/6&1/3&1/6\\0&1&0&-1/2&0&1/2\\0&0&1&1/3&-1/3&1/3\end{array}\right]$$

Therefore,

$$P^{-1}=\begin{bmatrix}1/6&1/3&1/6\\-1/2&0&1/2\\1/3&-1/3&1/3\end{bmatrix}=\frac{1}{6}\begin{bmatrix}1&2&1\\-3&0&3\\2&-2&2\end{bmatrix}$$

Finally, checking whether the equality $P^{-1}AP=D$ holds,

$$\begin{aligned}P^{-1}AP&=\frac{1}{6}\begin{bmatrix}1&2&1\\-3&0&3\\2&-2&2\end{bmatrix}\begin{bmatrix}3&-1&0\\-1&2&-1\\0&-1&3\end{bmatrix}\begin{bmatrix}1&-1&1\\2&0&-1\\1&1&1\end{bmatrix}\\&=\frac{1}{6}\begin{bmatrix}1&2&1\\-9&0&9\\8&-8&8\end{bmatrix}\begin{bmatrix}1&-1&1\\2&0&-1\\1&1&1\end{bmatrix}\\&=\frac{1}{6}\begin{bmatrix}6&0&0\\0&18&0\\0&0&24\end{bmatrix}=\begin{bmatrix}1&0&0\\0&3&0\\0&0&4\end{bmatrix}=D\end{aligned}$$

as expected.

4. $A=\begin{bmatrix}-1&2&3\\0&-1&-1\\0&0&-1\end{bmatrix}$.

1. Calculating $p_A(x)=|xI_n-A|$, we get

$$\begin{aligned}p_A(x)=|xI_n-A|&=\left|x\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}-1&2&3\\0&-1&-1\\0&0&-1\end{bmatrix}\right|\\&=\begin{vmatrix}x+1&-2&-3\\0&x+1&1\\0&0&x+1\end{vmatrix}\\&=\boxed{(x+1)^3}\end{aligned}$$

To find the eigenvalues, set $p_A(x)$ to 0. Thus, we get

$$0 = (x+1)^2 \Rightarrow x = -1$$

Therefore, the only eigenvalue is $\boxed{\lambda = -1}$.

2. The algebraic multiplicity of the eigenvalue λ is the highest power of $(x-\lambda)$ that appears in the polynomial $p_A(x)$. Since $p_A(x) = (x+1)^3$, and we only have the one eigenvalue $\lambda = -1$, we conclude

$$\boxed{\text{The algebraic multiplicity of } \lambda = -1 \text{ is } 3}$$

3. Let us find the fundamental eigenvectors for our one eigenvalue $\lambda = -1$. We need to write E_{-1} as all linear combinations of some set of vectors. Since E_λ is the set of solutions to $(\lambda I_n - A)\vec{x} = \vec{0}$, E_{-1} is the set of solutions to $((-1)I_3 - A)\vec{x} = \vec{0}$. Now,

$$(-1)I_3 - A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Writing this as an augmented matrix and row reducing,

$$\left[\begin{array}{ccc|c} 0 & -2 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 : R_1 + 3R_2} \left[\begin{array}{ccc|c} 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 : (-1/2) \times R_1} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system clearly corresponds to

$$\begin{aligned} c_1 &= c_1 \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned}$$

Therefore,

$$E_{-1} = \{\vec{x} \mid A\vec{x} = \vec{x}\} = \left\{ \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix} \middle| \; c_1 \in \mathbb{R} \right\} = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \middle| \; c_1 \in \mathbb{R} \right\}$$

Thus,

$$\boxed{\text{The only fundamental eigenvector is } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

Furthermore, for an $n \times n$ matrix to be diagonalizable, it needs to have n different fundamental eigenvectors. Since here, we only have 1, and we would need 3, we conclude that $\boxed{\text{the matrix is not diagonalizable.}}$

4. Not applicable since the matrix is not diagonalizable.