Basic Trigonometry

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1 Fundementals

1.1 Trigonometric Functions

There are three basic trigonometric functions, sine, cosine and tangent, whose definitions can be easily observed in a right triangle. Where o = the opposite side, a = the adjacent side and h = the hypotenuse,

$$\sin\theta = \frac{o}{h}$$

$$\cos\theta = \frac{a}{h}$$

$$\tan\theta = \frac{o}{a} = \frac{sin(\theta)}{cos(\theta)}$$

This is easily remembered using the acronynm SOHCAHTOA.

The other 3 trigonometric functions, cosecant, secant and cotangent, are defined in terms of the first three.

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\csc\theta}{\sec\theta} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

*With much material stolen from Mildorf04

1.2 Trigonometric Values

The following values, derived by analyzing 30-60-90 and 45-45-90 triangles, are the basis for most of the trigonometry encountered in contest math. It is quite important that these values be memorized completely.

$\sin 0 = 0 \quad \cos 0 = 1 \quad \tan 0 = 0$
$\sin \frac{\pi}{6} = \frac{1}{2} \quad \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \quad \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$
$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \tan \frac{\pi}{4} = 1$
$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad \cos \frac{\pi}{3} = \frac{1}{2} \quad \tan \frac{\pi}{3} = \sqrt{3}$
$\sin \frac{\pi}{2} = 1 \quad \cos \frac{\pi}{2} = 0 \quad \tan \frac{\pi}{2} = \infty$

1.3 Simple Trigonometric Identities

The basic identities given here allow any trigonometric functions to be evaluated, as long as $\theta | \frac{\pi}{6}$.

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$
$\sin(-\theta) = -\sin\theta \quad \cos(-\theta) = \cos\theta$
$\sin(\pi - \theta) = \sin\theta \quad \cos(\pi - \theta) = -\cos\theta$
$\sin^2\theta + \cos^2\theta = 1$

2 Advanced Computational Concepts

2.1 Addition and Subtraction Formulae

To derive the values of the trigonometric functions for different θ, it is neccessary to use addition and subtraction formulae. These formulae are extremely important, and are well worth memorizing.

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\tan(\theta + \phi) = \frac{\tan(\theta) + \tan(\phi)}{1 - \tan\theta\tan\phi}$$

Using the Simple Trigonometric Identities above, Subtraction Formulae can also be derived. These are listed here for reference.

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

$$\tan(\theta - \phi) = \frac{\tan(\theta) - \tan(\phi)}{1 + \tan\theta\tan\phi}$$

2.2 Double-Angle and Half-Angle Formulae

The Double-Angle formulae are created directly from the Addition Formulae above, setting $\theta = \phi$.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

These formulae are used to derive the half-angle formulae.

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos^2(2\theta)}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos^2(2\theta)}{2}}$$

$$\tan\frac{\theta}{2} = \pm\frac{\sin\theta}{1 + \cos\theta}$$

2.3 Sum-Product Formulae

These formulae are usually not needed for trigonometry problems, but they can often make life a lot easier.

$$\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2}$$

$$\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2}$$

$$\sin(\alpha)\cos(\beta) = \frac{\sin(\beta + \alpha) - \sin(\beta - \alpha)}{2}$$

3 Trigonometric Theory

3.1 Trigonometric Laws

In the statement of these laws, a, b, and c are the sides of a triangle, and A, B, and C are the angles opposite those sides.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin B} = 2R$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

3.2 Identities

This section containse some identities that often are useful with contest math.

In a triangle ABC,

$$\tan A \tan B \tan C = \tan A + \tan B + \tan C$$

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$$

3.3 Rocco's Method

Rocco Repeski (TJ Class of 2004) developed this method for solving trig problems. This should only be used as an absolute last resort, given that it doesn't usually work; but if time is running out and no other solution method appears, it's better than nothing.

The theory behind Rocco's method is that any trig functions can be changed to a known trig function. For example, what is $\cos 50^o$? You probably have no idea! Rocco's method says that $\cos 50^o = \cos 45^{o-}$, because the the cosine will get slightly smaller going from 45^o to 50^o. Hopefully, by the end of the problem, the +s and -s will cancel out, and a nice, pretty answer will result. This rarely works, but at the worst provides a reasonably close guess.

For an example of a problem solvable using Rocco's Method, see the final question in the problem set.

4 Problems

1. (Traditional) $\cos x + \sin x = .5$. Solve for $\sin 2x$.
2. (Schafer06) Compute $\sum_{i=1}^{90} \sin i^o + \sum_{i=91}^{180} \cos i^o$.
3. (Traditional) $\tan 20^o \tan 40^o \tan 80^o = \tan x^o$. Solve for x.

5 Hints

All of the these hints are written in order to avoid giving away the solution, but it is still better to try each problem for a reasonable amount of time before using these hints.

1. Consider simple trigonometric identities. What can be done to the given to tranform it to an identity?

2. Clearly, adding all of the numbers up isn't possible. What can be done to simplify the problem?

3. Seperate out $\tan\theta$ into $\frac{\sin\theta}{\cos\theta}$. If that fails, try Rocco's Method (this is one of the few problems where it works!)

6 Solutions

It is highly recommended that you not look at the solutions until you have put forth a strong effort to solve the problems yourself.

1. $-.75$. Squaring both sides, $(\cos x + \sin x)^2 = \cos^2 x + 2\cos x \sin x + \sin^2 x = .25$. As $2 \sin x \cos x = \sin 2x$, $1 + 2\cos x \sin x = .25$, and $\sin 2x = -.75$.

2. 0. Remember that $\cos(90^o + \theta) = \sin -\theta = -\sin\theta$. Thus, the expression given is equivelent to $\displaystyle\sum_{i=1}^{90} \sin i^o + \sum_{i=1}^{90} -\sin i^o = 0$.

3. 60.

   Method 1: Actual Trig. Expand the expression: $\tan 20^o \tan 40^o \tan 80^o = \frac{\sin 20^o \sin 40^o \sin 80^o}{\cos 20^o \cos 40^o \cos 80^o}$. Using the Product-Sum formulae, this expression is shown to equal $\frac{(\cos 20^o - \cos 60^o)\sin 80^o}{(\cos 20^o + \cos 60^o)\cos 80^o}$. Multiplying out again and using the Product-Sum formulae, this equals $\frac{\cos 20^o \sin 80^o - \cos 60^o \sin 80^o}{\cos 20^o \cos 80^o + \cos 60^o \cos 80^o} = \frac{(\frac{\sin 100^o - \sin(-60^o)}{2}) - (\frac{\sin 80^o}{2})}{(\frac{\cos 100^o + \cos 60^o}{2}) + (\frac{\cos 80^o}{2})}$, and as $\cos 100^o = -\cos 80^o$ and $\sin 100^o = \sin 80^o$, $\frac{(\frac{\sin 60^o}{2}) + (\frac{\sin 100^o - \sin 80^o}{2})}{(\frac{\cos 60^o}{2}) + (\frac{\cos 100^o + \cos 80^o}{2})} = \frac{\sin 60^o}{\cos 60^o} = \tan 60^o$.

   Method 2: Rocco's Method. Simplifying, $\tan 20^o \tan 40^o \tan 80^o = \tan 30^{o-} \tan 60^{o-} \tan 60^{o++} = \tan 60^o$. This really shouldn't work, but it does.