Math 216 Midterm 2 Study Guide

Jeffrey Wubbenhorst

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Dimension, Linear Indepence

Subspaces and Spanning Sets

- A subset W of a vector space BV is a subspace of V if W is a subspace under addition, scalar multiplication of V restricted to W. (That is, if W is closed under the same rules of scalar mulitplication and addition as V)
- Let W be a nonempty subset of a vector space V. W is a subspace of V iff, $\forall u, v \in W$ and $\forall c \in \mathbb{R}, u + w \in W, cu \in W$
- If A is $m \times n$, solutions to system of homogenous linear equations $AX = 0$ is a subspace of $\mathbb{R}^n$
- A set of vectors $\{v_1, v_2, ..., v_n$ is **linearly independent** iff, for a system $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$ all constants c are zero. As a random note, no set containing hte zero vector is independent.

Dimension, Nullspace, Row Space, Column Space

- If a vector space V has a basis of n vectors, the **dimension of** V **is** n. This is denoted as $\dim(V)$.
- $\dim(\mathbb{R}^n) = n, \dim M_m \times n(\mathbb{R}) = mn, \dim(P_n) = n + 1$
- Suppose some $v_1, v_2, ...v_n$ in a vector space V. The vectors are **linearly dependent** iff *only one* if $v_1, v_2, ..., v_n$ is a linear combination of the others. To see if a set of vectors is linearly independent, solve for the constants for the homogenous equation; if a non-trivial solution exists, then the vectors are linearly dependent.
- Vectors $v_1, v_2, ..., v_n$ of a vector space V are a basis for V if both of the following conditions are satisfied:
  1. $v_1, v_2, ..., v_n$ are **linearly independent**
  2. $v_1, v_2, ..., v_n$ **span** V
- Suppose some $v_1, v_2, ...v_n$ in a vector space V. Then $v_1, v_2, ..., v_n$ form a basis for V iff each vector in V is uniquely expressible as a linear combonation of $v_1, v_2, ..., v_n$
- If V is a bvector space adn $v_1, v_2, ..., v_n$ are vectors in V, then the set of all linear combinationfs of $v_1, v_2, ..., v_n$ is a subspace of V
- The subspace of some V consisting of all linear combinations of vectors $v_1, v_2, ..., v_n$ is referred to as the **subspace** of V spanned by $v_1, v_2, ..., v_n$
  ...in English, the span is the set of all the different places you could "go" if you combined the given vectors in every possible way. To see if something spans something else, solve for constants c; if there's no solution, then the set does not span.
- Suppose that V is a vector space of dimension n.
  1. If the vecrtors $v_1, v_2, ..., v_n$ are linearly independet, then $v_1, v_2, ..., v_n$ are linearly independent, then $v_1, v_2, ..., v_n$ form a basis for V
  2. If $v_1, v_2, ..., v_n$ span V, then $v_1, v_2, ..., v_n$ form a **basis**
- The solutions to the homogenous system $AX = 0$, where A is an $m \times n$ matrix form a subspace of $\mathbb{R}^n$. This vector space of solutions is the **nullspace** or **kernel** of A, denoted by $NS(A)$
- The **row space** of some matrix A is the span of the rows of A. If A, B are row-equivalent matrices, $RS(A) = RS(B)$
- Weirdly enough, for an $m \times n$ matrix , $\dim(RS(A)) + \dim(NSA(A)) = n$
- The **column space** of a matrix A is the subspace of $\mathbb{R}^m$ spanned by the columns of A and is denoted by $CS(A)$. To find the column space of a matrix, you:
  1. **T**ranspose
  2. **R**ow-reduce
  3. transpose back

4. take **B**asis vectors

...dumb mnemonic is **TR**i**B**e. (Or 'tribble' if you're into Star Trek.)

ird stuff For a given matrix $A, \dim(RS(A)) = \dim(CS(A))$; this common dimension is called the **rank** of A.

Differential Equations

Complex Solutions

- if $L(y) = 0$ is a CCLDE w/ real coefficientsr and if $y(x) = \mu(x) + iv(x)$ is a complex-valued solution, $\mu(x), v(x)$ are also solutions.
- If $L(y) = 0$ is a real CCLDE and $r, \bar{r}$ are a pair of roots of characteristic polynomial, then $e^{ax}\cos bx, e^{ax}s\sin bx$ are real, independent solutions, with same span as $e^{rx}, e^{\bar{r}x}$
- If $L(Y) = 0$ has a characteristic polynomial with roots $r_1, ..., r_n$, and sets of solutions formed by:
  1. For real roots r of multiplicity m, we include
  $$e^{rx}, xe^{rx}, ..., x^{m-1}e^{rx}$$
  2. For complex roots r of multiplicity m, we include
  $$e^{rx}\cos bx, e^{rx}\sin bx, xe^{rx}\cos bx, xe^{rx}\sin bx, ..., x^{m-1}e^{rx}\cos bx, x^{m-1}e^{rx}\sin bx$$

  ...then this set of functions is independent, and is a fundamental set of solutions.

Method of Undetermined Coefficients

- The solution to a differential equation is the sum of the particular solution and the homogenous solution:
$$y = y_h + y_p$$
- The goal is to find some y_p that works with some given differential equation, with $\lambda = r$ is a root of multiplicity m, k is highest power of x on the right side of the equation.
- If the root to an LDE of the form $a_n y^{(n)} + ... + a_1 y' + a_0 y = Ax^k e^{rx}$ is real, the particular solution will be $y_p = x^m(A_k x^k + ... + A_1 x + A_0)e^{rx}$
- If root to equation of form
$$a_n y^{(n)} + ... + a_1 y' + a_0 y = Ax^k e^{ax}\cos bx + Bx^k e^{ax}\sin bx$$
  is imaginary, particular solution is given as:
$$y_p = x^m(A_k x^k + ... + A_1 x + A_0)e^{ax}\cos bx + x^m(B_k x^k + ... + B_1 x + B_0)e^{ax}\sin bx$$
- *Note: the method of undetermined coefficients is not an exact science!* There can be trial and error involved; a table of decent guesses is embedded at the end of this document.

Applications

- Sitautions with no external forces are usually given in the form
$$F = mu'' + fu' + ku = 0$$
  where (in spring problems at least) k is the spring constant, f is the friction coefficient, and mass m. (Don't forget that pounds are a unit of force, *not* of mass!)
- In **unforced cases** (no external input to system), the quadratic equation can give us roots; the issue of imaginary roots arises (that is, $f^2 - 4km < 0$):
  1. If no friction ($f = 0$), roots are imaginary. These cases are kind of rare.
  2. If $f - 4km < 0$, $\lambda = -a \pm bi$; in this case, the oscillating thing in question takes a bit to stop moving. This system is **underdamped**.
  3. If $f - 4km > 0$, there will be two roots $r_1, r_2 < 0$, solutions will have the form $u = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ this system is **overdamped**.
  4. If $f^2 - 4km = 0$, we have one root $r = \frac{-f}{2m}$ and solutions $u = c_1 e^{rt} + c_2 t e^{rt}$. This system is **critically damped**, and is the case where the oscillating thing returns to rest within the shortest amount of time.
- In **forced cases**, there are a few possible situations:

1. For instances of no friction (system is **undamped**), the equation will have the form:

$$\mu'' + ku = h(t)$$

We'll consider a case where $\omega_0 \neq \omega$, wher $\omega = \sqrt{k/m}$; complete set of solutions can be obtained by:

$$\mu_H = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

$$\mu_P = \frac{a}{\omega_0^2 - \omega^2} \cos \omega t$$

...and $\mu = \mu_H + \mu_P$.

2. **Resonance** occurs when $\omega = \omega_0$; in these situations, the amplitude increases as time increases. This has been known to cause some kinds of problems.

Linear Transformations

- A function f from a set X to a set Y is denoted as $f : X \to Y$; X is the domain of f, Y is called the **image set** or **codomain**. The subset $f(x)|x \in X$ of Y is called the **range**, which, in English, means "all the $f(x)$ that 'hit' something in Y."
- If V, W are vector spaces, a function $T : V \to W$ is called a **linear transformation** if, for all vectors $u, v \in V$ and all scalars c, the following two properties hold:
  1. $T(u + v) = T(u) + T(v)$
  2. $T(cv) = cT(v)$
- If $T : V \to V$, T is sometimes called a **linear operator**
- If A is an $m \times n$ matrix, $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

  $$T(X) = AX$$

  is a linear transformation, but is more commonly called a **matrix transformation**
- The differential operator takes derivatives, and is a linear transformation denoted by

  $$D : D(a, b) \to F(a, b)$$

- $\mathrm{Int}(f)$ denotes definite integral of a function f over a closed interval $[a, b]$
- For some **linear transformation**$T : V \to W$ the following properties hold:
  1. $T(0) = 0$
  2. $T(-v) = 0T(v)$ for any $v \in V$
  3. $T(u - v) = T(u) - T(v)$ for any $u, v \in V$
  4. $T(c_1 v_1 + c_2 v_2 + ... + c_k v_k) = c_1 T(v_1) + c_2 T(v_2) + ... + c_k T(v_k)$ for any scalars $c_1, c_2, ..., c_k$ and any vectors $v_1, v_2, ...v_k \in V$
- To determine what exactly a linear transformation does (assuming you have a set of input vectors by which the behavior of T may be observed):
  1. Assume, find, or steal some basis for the domain; this will determine all values of an LT.
  2. Determine the coefficients $c_1, ..., c_n$ for the basis vectors that yield a given output
  3. Determine how T combines the given input vectors to an output vector of form $[x_0, x_1, ..., x_n]$, solving again for coefficients
- The **kernel** of T, denoted $\ker(T)$, is defined as

  $$\ker(T) = \{v \in V | T(v) = 0\}$$

  ...in English, the kernel is the set of all vectors that give an output of the zero vector. For matrix transformations, the kernel of the matrix transformation $T(X) = AX$ is the same as the nullspace of A:

  $$\ker(T) = NS(A)$$

- To find a basis for the column space of a matrix, take the transpose (**T**), reduce (**R**), transpose (**T**), split into columns (**S**), which approximately spells **ToRToiSe**.
- Weirdly enough, if $T : V \to W$ is an LT where V is a finite-dimensional vector space, we have that

  $$\dim(\ker(T) + \dim(\mathrm{range}(T)) = \dim(V)$$

Trial solutions for the method of undetermined coefficients

	Form of $g(x)$	Guess for particular solution
1.	1 (any constant)	A
2.	$5x+7$	$Ax+B$
3.	$3x^2-2$	Ax^2+Bx+C
4.	$\sin 4x$	$A\cos 4x + B\sin 4x$
5.	$\cos 4x$	$A\cos 4x + B\sin 4x$
6.	e^{5x}	Ae^{5x}
7.	$(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
8.	x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
9.	$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
10.	$5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x + (E^2+Fx+G)\sin 4x$
11.	$xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x + (Cx+E)e^{3x}\sin 4x$
12.	$(5x+7)+\sin 4x$	$(Ax+B)+(C\cos 4x + D\sin 4x)$

Errata

- The **contrapositive** of a statement is the reverse of the premise and conclusion, and the negation of both. Original statement: "Duke won, and I'm happy." Contrapositive: "If I'm not happy, then Duke didn't win."

Trig Identities

- STUFF HERE

Complex stuff!!

add it yo