MULTIPLE-ATTRIBUTE DECISION MAKING WITH PARTIAL INFORMATION: THE COMPARATIVE HYPERVOLUME CRITERION

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ABSTRACT

A new approach is presented for analyzing multiple-attribute decision problems in which the set of actions is finite and the utility function is additive. The problem can be resolved if the decision makers (or group of decision makers) specifies a set of nonnegative weights for the various attributes or criteria, but we here assume that the decision maker(s) cannot provide a numerical value for each such weight. Ordinal information about these weights is therefore obtained from the decision maker(s), and this information is translated into a set of linear constraints which restrict the values of the weights. These constraints are then used to construct a polytope W of feasible weight vectors, and the subsets H_i (polytopes) of W over which each action a_i has the greatest utility are determined. With the Comparative Hypervolume Criterion we calculate for each action the ratio of the hypervolume of H_i to the hypervolume of W and suggest the choice of an action with the largest such ratio. Justification of this choice criterion is given, and a computational method for accurately approximating the hypervolume ratios is described. A simple example is provided to evaluate the efficiency of a computer code developed to implement the method.

INTRODUCTION

We consider finite action decision problems of the following nature: The decision maker(s) must choose one action from a finite set A of feasible actions. Each action $a_i \in A$, $i = 1, \ldots, m$, is evaluated with respect to a finite set $C = \{c_j | j = 1, \ldots, n\}$ of judgment criteria (attributes). Let s_{ij} be the raw score of action a_i with respect to attribute c_j; these scores may be on either ordinal or interval scales.

We suppose an action will be chosen on the basis of maximum utility, where the utility of action a_i is $u^*(a_i) = u(s_{i1}, \ldots, s_{in})$. We further assume that the utility function u is additive (see Fishburn [5-7]) so that

$$u^*(a_i) = u(s_{1i}, \ldots, s_{in}) = \sum_{j=1}^{n} u_j(s_{ij}). \tag{1}$$

It then follows from the basic results of additive utility theory that for the purposes of decision making we may replace the functions u_j of the raw scores s_{ij} by $w_j v_{ij}$, where v_{ij} is a relative value of the raw score s_{ij} based on the set of all possible scores s_{kj} with respect to attribute c_j, and each w_j is a positive weight. Furthermore, without loss of generality, we may choose the relative values so that for each attribute c_j, the most desired raw score s_{i^*j} of the m scores is mapped into the value $v_{i^*j} = 10$ while the least desired raw score $s_{i'j}$ of the m scores is mapped into the value $v_{i'j} = 0$. (This assumes that not all the values $u_j(s_{ij})$, $i = 1, \ldots, m$, are the same. If this were so, there would be no need to consider attribute c_j.) All the other raw scores s_{ij} for the various actions a_i are mapped into values between 0 and 10 inclusive. (Any finite values a and b, with $a < b$, could be used instead of 0 and 10, respectively; we choose the interval [0, 10] for its simplicity and intuitive appeal.) Also without loss of generality, we may choose the weights w_j so that $\sum_j w_j = 1$.

Thus we write the utility of each action a_i as

$$u^*(a_i) = \sum_{j=1}^{n} w_j v_{ij}. \tag{2}$$

Since the v_{ij} depend on the raw scores s_{ij} and the preferences of the decision maker(s) for these scores, we may asume that he (they) can provide the appropriate v_{ij} values without undue difficulty. We therefore define the m by n matrix V of the v_{ij} values: for each column j the values v_{ij} lie between 0 and 10 inclusive, with at least one of them being 0 and at least one of them being 10.

If we define the n by 1 column vector w of weights and let V_i be the i th row of V we may rewrite (2) as

$$u^*(a_i) = V_i w. \tag{3}$$

If the decision maker(s) provide(s) an appropriate weight vector $w = w^*$, then (2) or (3) should be used as the basis for selecting an action. In many cases, however, the decision maker(s) may not be willing to provide a particular w. In the case of an individual decision maker this may be simply due to the fact that he or she cannot articulate his or her preferences with such precision. Indeed, it may be somewhat unusual to find a decision maker who will specify a particular w. In the case of a group of decision makers, there may be considerable disagreement about the appropriate weight vector w. This could be the situation if, for example, they represent different sectors of society and the decision problem encompasses different criteria which are economic, environmental, and political in nature.

We may therefore distinguish two extreme situations that can apply to the state of knowledge of the decision maker(s) with respect to w:

(i) Nothing is known about w except that it lies in the set

$$\left\{ w \in E^n \middle| \sum_{j=1}^{n} w_j = 1, \text{ all } w_j \geqq 0 \right\}.$$

(ii) It is known that $w = w^*$.

We term case (i) multiple-attribute decision making with *no information* and case (ii) multiple-attribute decision making with *complete information*, and respectively refer to a multiple-attribute

problem with no information or complete information. We wish to consider the general case between these two extremes, and we term it multiple-attribute decision making with *partial information.* It will include the two cases above as specific extremes.

We therefore define the set W as the set of weight vectors that the decision maker(s) deem(s) feasible, i.e., may be an appropriate one in light of his subjective feelings (or in light of their range of agreement if there are several decision makers). In case (i) above, therefore, we have

$$W = \left\{ w \in E^n \middle| \sum_{j=1}^{n} w_j = 1, \text{ all } w_j \geqq 0 \right\} \equiv \overline{W},$$

while in case (ii) we have

$$W = \{w^*\}.$$

Having thus introduced the notion of multiple-attribute decision making with partial information, we shall, in the next section, discuss the construction and characterization of W. After that we will introduce the Comparative Hypervolume Criterion for choosing an action when the set W is not a unique point. The following section details the Monte Carlo method that has been developed to accurately approximate the numerical quantities used by the Comparative Hypervolume Criterion. The final section summarizes the numerical method and presents an illustrative example that has been used to partially evaluate the accuracy and efficiency of the Monte Carlo method and the computer program implementing it.

We note here that a similar approach has been presented in [3] for the case of no information and a minor extension of it. The numerical approach of [3] is only efficient for the case of no information, however, and cannot be used in the general case of partial information. The Monte Carlo technique presented herein is quite different and much more efficient.

THE SET W OF FEASIBLE WEIGHT VECTORS

We have seen above that the set W of feasible weight vectors is $\overline{W}$ in the case of no information and the singleton set $\{w^*\}$ in the case of complete information. In the general case we may suppose that W is defined by equality and inequality constraints involving the components $w_1, \ldots, w_n$ of w. In the present situation, however, we will confine ourselves to *linear* constraints involving the weights. These may arise very naturally as follows.

We can translate the statement "attribute c_j is at least as important as attribute c_k" on the part of the decision maker(s) into the mathematical statement $w_j \geqq w_k$. As a direct extension of this that is compatible with the foundations of additive utility [5], we let J and K be subsets of $N \equiv \{1, \ldots, n\}$ and translate the statement "the set of attributes with indices in J, taken together, are at least as important as the set of attributes with indices in K" into the mathematical statement

$$\sum_{j \in J} w_j \geqq \sum_{k \in K} w_k.$$

With similar reasoning, constraints of the form

$$\sum_{j \in J.} w_j \leqq b_J,$$

where b_J is a constant between zero and one, may be elicited from the feelings and statements of the decision maker(s). The important point is that the decision maker(s) agree(s) that weight vectors not satisfying such constraints are not to be considered.

In general, then, we shall assume that the partial information possessed by the decision maker(s) concerning w has been encoded in a set of linear constraints on $w_1, \ldots, w_n$. To these we add the normalizing constraint $\sum_j w_j = 1$. Without loss of generality we can convert all the constraints to inequality constraints; hence we may write W as

$$W = \{w \in E^n \mid Aw \leqq b,\ w \geqq 0\}, \tag{4}$$

where A is the s by n matrix of constraint coefficients and b is the s by 1 right-hand side vector. Thus W is a bounded polytope in E^n. $W \subset \overline{W}$, and in the case of no information, $W = \overline{W}$. In the case of complete information W is a single point.

THE COMPARATIVE HYPERVOLUME CRITERION

Now that we have constructed W, how shall we use it in aiding the decision maker(s) to select an action a_i to implement? For every $w \in W$ we can compute, for each $a_i \in A$, the utility value $u^*(a_i) = V_i w$, and hence find the action or actions with highest utility value for this particular w. Let us call such an action an optimal one with respect to w and denote by $A(w)$ the set of such optimal actions with respect to w. Clearly, if there is an action a_i such that $a_i \in A(w)$ for all $w \in W$, then we can confidently claim that this a_i is the one to choose. Given A, b, and V, we can check, but not necessarily easily, for the existence of such a clearly optimal a_i (a necessary and sufficient condition for $a_i \in A(w)$ for all $w \in W$ is that $a_i \in A(w^k)$ for all extreme points w^k of W). But it will be rare indeed that such a clearly optimal action exists, so we must search further for a basis for choosing an action.

If there is an action a_i such that $a_i \in A(w)$ for "most" $w \in W$ (but not all $w \in W$), then it seems reasonably wise to select this action; it is the one most "likely" to lead to the highest utility of any of the actions in A. Here, "likely" is taken with respect to a uniform probability distribution of the $w \in W$. What we are now suggesting, therefore, is to measure for each $a_i \in A$ the relative frequency with which $a_i \in A(w)$. We then suggest as optimal an action with the largest such relative frequency. Stated another way, we define an optimal action to be one which maximizes the probability of yielding a utility value at least as high as that of all other actions for a randomly selected $w \in W$. For reasons which will shortly become apparent, we call this criterion the Comparative Hypervolume Criterion (CHC). Besides its intuitive appeal as illustrated above, its compatibility with Bernoullian utility constructs has been demonstrated in Ref. [1].

Let us now be more exact about the CHC. Define, for $i = 1, \ldots, m$,

$$H_i = \{w \in W \mid a_i \in A(w)\}, \tag{5}$$

where, to repeat,

$$A(w) = \{a_i \in A \mid V_i w \geqq V_k w,\ k = 1, \ldots, m\}. \tag{6}$$

Thus we have the equivalent characterization

$$H_i = \{w \in W \mid V_i w \geqq V_k w,\ k = 1, \ldots, m\}. \tag{7}$$

We then measure the relative frequency with which $a_i \in A(w)$ by

$$r_i \equiv \int_{H_i} dw \Big/ \int_W dw, \tag{8}$$

where the element of integration dw in (8) depends on the dimensionality of W (which will be discussed below). Thus, r_i is a ratio of hypervolumes. The Comparative Hypervolume Criterion then suggests the choice of an action a_i such that $r_i \geqq r_k$, $k = 1, \ldots, m$.

The computational difficulty faced by the CHC is the computation of the ratios $r_1, \ldots, r_m$. Although W has an explicit characterization as a polytope and the same is true of H_i by virture of (7), it is very difficult to carry out the integrations required by (8). We have developed a Monte Carlo simulation method to accurately approximate the ratios $r_1, \ldots, r_m$ needed by the CHC; this is detailed in the next section.

THE MONTE CARLO SIMULATION METHOD

In Ref. [3] is given a relatively efficient Monte Carlo method for evaluating the hypervolume ratios in the case of multiple-attribute decision making with no information. In this case $W = \overline{W}$, so a random point w within W can be easily generated by drawing n uniform random numbers and then normalizing them to sum to one. The quantities $V_i w$, $i = 1, \ldots, m$ are compu ed and compared, and a record is kept of the value or values of i that maximize(s) $V_i w$. If k_i s the count of the number of drawings for which action a_i yields the maximum value of $V_i w$ in a total sample size of g, then k_i/g is the appropriate estimate of r_i.

This approach might be extended to the case of partial information by discarding the point w drawn if $w \in W$, but this sampling will be highly inefficient if W is small relative to $\overline{W}$. Moreover, it will not work at all if the dimension of W (the number of independent vectors needed to span it) is less than $n-1$ (as will be the case if W is partially defined by an equality constraint other than $\sum w_j = 1$).

In Ref. [2] we outlined a general Monte Carlo approach for computing statistical measures for a linear function defined over a polytope. That general approach will be adapted and extended here in order to yield an efficient numerical procedure for approximating the hypervolume ratios. The basic idea is to change coordinate systems, taking a vertex of W as the new origin. The new coordinate system uses the ordinary Euclidean distance ρ and a set of angular coordinates. For a specific set of angular coordinate values, the integration with respect to ρ required in the numerator and denominator of (8) may be carried out explicitly. Integration with respect to the angular coordinates cannot be explicitly carried out, however, and is instead replaced by a Monte Carlo drawing of the possible sets of angular coordinates. These correspond to vectors randomly directed into W from the vertex chosen. As will be seen, the method requires the vertex chosen to be a nondegenerate one. Now we present the method in detail.

Let d be the dimension of $W (d \leq n-1)$ and let γ be a nondegenerate vertex or extreme point of W. Thus, γ is formed by the intersection of exactly d linearly independent hyperplanes from the set of hyperplanes defining W. We choose γ as the origin in a spherical coordinate system utilizing the Euclidian distance ρ and $d-1$ angles. We will use ϕ to designate the vector of angles. For an arbitrary point $w \in W$ we have

$$\rho = ||w - \gamma|| = \left[\sum_{j=1}^{n} (w_j - \gamma_j)^2\right]^{1/2},$$

and the information needed to compute the appropriate value of ϕ for w (assuming $w \neq \gamma$) is contained in the unit vector $\delta = (w - \gamma)/||w - \gamma||$. Thus the denominator in (8), call it $S(W)$, may be expressed as

$$S(W) = J \iint_{W} \rho^{d-1} \, d\rho d\phi,$$

where J is a constant and $J\rho^{d-1}$ is the Jacobian of the coordinate transformation. An analogous expression, call it $S(H_i)$, holds for the numerator of (8) so that we have

$$r_i = S(H_i)/S(W) = \iint_{H_i} \rho^{d-1}\, d\rho d\phi \Big/ \iint_{W} \rho^{d-1}\, d\rho d\phi. \tag{9}$$

We shall shortly show that for any specific ϕ we can explicitly perform the indicated integration with respect to ρ. We cannot perform the integration with respect to ϕ, however, and instead will use Monte Carlo simulation to generate random values of ϕ.

With ϕ defined as above for any $w \in W$ (take $\delta = 0$ for $w = \gamma$), we may write $w \in W$ as $w = \gamma + \rho\delta$. Then define Δ as the set of all such unit vectors directed from γ into W, i.e.,

$$\Delta = \{\delta \mid \delta = (w - \gamma)/||w - \gamma||,\ w \in W,\ w \neq \gamma\}. \tag{10}$$

Thus for every ϕ such that $w = (\rho, \phi) \in W$ we can equivalently describe w as $\gamma + \rho\delta$ since there is a unique correspondence between ϕ and δ. Now rewrite $S(H_i)$ as

$$S(H_i) = \iint_{W} \Omega(H_i)\, \rho^{d-1}\, d\rho d\phi,$$

where $\Omega(H_i)$ is the indicator function of the set H_i. Now that both $S(H_i)$ and $S(W)$ are expressed in terms of integration over W, we will explicitly perform the integrations with respect to ρ.

Given an arbitrary $\delta \in \Delta$, denote by $\rho(\delta)$ the largest value of ρ such that $\gamma + \rho\delta \in W$. Thus, by the convexity of W, the limits of integration on ρ in $S(H_i)$ and $S(W)$ are 0 and $\rho(\delta)$, respectively. We can find $\rho(\delta)$ as follows: Letting A_l be the lth row of the s by n matrix A partially defining W through $Aw \leqq b$, $\rho(\delta)$ is the smallest distance to a plane $A_l w = b_l$ such that $A_l\gamma < b_l$. Of course $A_l(\gamma + \rho\delta) = b_l$ cannot occur for positive ρ if $A_l\delta \leqq 0$, so we must exclude this case. We thus find

$$\rho(\delta) = \min_l \{(b_l - A_l\gamma)/(A_l\delta) \mid A_l\gamma < b_l,\ A_l\delta > 0\}. \tag{11}$$

In the direction δ, the indicator function $\Omega(H_i)$ is 1 for ρ in the interval $[\underline{h}_i(\delta), \bar{h}_i(\delta)]$ and 0 for ρ outside this interval, where $\underline{h}_i(\delta)$ and $\bar{h}_i(\delta)$ are the smallest and largest nonnegative values of ρ, for $\rho \leqq \rho(\delta)$, respectively, such that $V_i(\gamma + \rho\delta) \geqq V_k(\gamma + \rho\delta)$, $k = 1, \ldots, m$. (If no nonnegative value of $\rho \leqq \rho(\delta)$ exists which satisfies this inequality, we take $\underline{h}_i(\delta) = \bar{h}_i(\delta) = 0$.) We will see below that it is a simple matter to determine all the intervals $[\underline{h}_i(\delta), \bar{h}_i(\delta)]$.

Performing the integrations with respect to ρ in (9) now, we obtain

$$r_i = \frac{\int \left[\left(\bar{h}_i(\delta)\right)^d - \left(\underline{h}_i(\delta)\right)^d\right] d\phi}{\int \left[\rho(\delta)\right]^d d\phi}, \tag{12}$$

where the integrations with respect to ϕ must be performed over all ϕ such that the corresponding δ lies in Δ. We cannot perform these integrations easily, so we shall instead approximate (12) by randomly generating direction vectors $\delta \in \Delta$. For each such δ^k so generated we can easily compute the quantities $\rho(\delta^k)$, $\underline{h}_i(\delta^k)$, and $\bar{h}_i(\delta^k)$, $i = 1, \ldots, m$. With a sample of direction vectors δ of size g, we then approximate (12) by

$$\bar{r}_i = \frac{\sum_{k=1}^{g} \left[\left(\bar{h}_i(\delta^k)\right)^d - \left(\underline{h}_i(\delta^k)\right)^d\right]}{\sum_{k=1}^{g} \left[\rho(\delta^k)\right]^d}. \tag{13}$$

Equation (13) represents a sampling formulation of the hypervolume ratio r_i of (8); it is a statistical approximation of r_i with a sampling error that is a function of the sample size g. Thus, for g sufficiently large, all $\bar{r}_i$ will approximate their respective r_i accurately enough to be used in their place in connection with the CHC.

We have thus far failed to treat three important points: (a) determination of the dimension d of W, (b) explicit computation of the $\underline{h}_i(\delta)$ and $\bar{h}_i(\delta)$, and (c) generation of random direction vectors $\delta \in \Delta$. The last of these depends on the fact that γ is a nondegenerate extreme point of W, and the first is aided considerably by this fact. Being nondegenerate, γ is formed by the intersection of exactly d linearly independent hyperplanes from the set of hyperplanes defining W. As demonstrated elsewhere [1], the edges of W at γ, i.e., the intersection of the d linearly independent hyperplanes taken $(d\text{-}1)$ at a time, comprise a set of d linearly independent vectors along which lie the extreme points of W adjacent to γ. Thus there are exactly d adjacent extreme points, and since γ is nondegenerate (in the linear programming sense also), each of these adjacent extreme points can be generated by part of one simplex pivot. Denote these adjacent extreme points by $\gamma^1, \ldots, \gamma^d$. Note that the number of them serves to define the dimension d, thus taking care of point (a) above.

Define the unit vectors $q^1, \ldots, q^d$ by

$$q^l = (\gamma^l - \gamma)/||\gamma^l - \gamma||, \quad l = 1, \ldots, d. \tag{14}$$

This set of linearly independent direction vectors spans Δ and will be used to generate random vectors $\delta \in \Delta$. Let q be a convex combination of $q^1, \ldots, q^d$, i.e., $q = \sum_{l=1}^{d} t_l q^l$, where $\sum_{l=1}^{d} t_l = 1$ and all $t_l \geqq 0$. Thus $q/||q||$ is an element of Δ, and in fact we have

$$\Delta = \{q/||q|| \mid q = \sum_{l=1}^{d} t_l q^l, \ \sum_{l=1}^{d} t_l = 1, \ \text{all } t_l \geqq 0\}. \tag{15}$$

Since $q^1, \ldots, q^d$ are linearly independent, each unique convex combination of them generates a unique vector q and hence a unique vector $\delta = q/||q||$ which is an element of Δ. Thus, in order to generate a random element of Δ we might simply generate d uniform random numbers, normalize them to form a random convex combination, and then use them in expression (15) to obtain a random element $\delta \in \Delta$. Unfortunately, this straightforward approach leads to a biased choice of $\delta \in \Delta$. It is not difficult to show, however, that if all pairs of unit vectors (q^k, q^l), $l \leqslant k < l \leqslant d$, form acute angles, then an unbiased choice results if we use

$$\delta = \sum_{l=1}^{d} t_l q^l / || \sum_{l=1}^{d} t_l q^l ||, \tag{16}$$

where the t_l are independent uniform random numbers on (0, 1) and a sequence $(t_1, \ldots, t_d)$ is discarded if $||\Sigma_l t_l q^l|| > 1$. If, however, some pairs of unit vectors (q^k, q^l) form obtuse angles, then it is necessary to further restrict the sequences $(t_1, \ldots, t_d)$ used to those for which $||\Sigma_l t_l q^l|| \leqslant u$, where $u = (1 - \cos^2\beta)^{1/2}$ and β is the largest angle between pairs of unit vectors (q^k, q^l). This takes care of point (c).

Now we deal with point (b). Given the nondegenerate extreme point γ, a randomly generated direction vector $\delta \in \Delta$, and $\rho(\delta)$, there exists at least one a_k such that $V_k\gamma \geqq V_i\gamma$ for $i = 1, \ldots, m$. Without loss of generality we will assume that a_1 has this property, i.e., is optimal for the weight vector γ. Then $h_1(\delta) = 0$. Now suppose a_1 is the unique optimal action at γ; i.e., $V_1\gamma > V_i\gamma$ for $i > 1$. Then either (a) $V_1(\gamma + \rho\delta) > V_i(\gamma + \rho\delta)$ for $i > 1$ and all $\rho \in [0, \rho(\delta)]$, in which case $\bar{h}_1(\delta) = \rho(\delta)$, or (b) there exists a smallest scalar $z \in (0, \rho(\delta)]$ such that $V_1(\gamma + z\delta) = V_i(\gamma + z\delta)$ for some $i > 1$. In case (a) we take $\underline{h}_k(\delta) = \bar{h}_k(\delta) = 0$ for $k > 1$. In case (b) we clearly have $\bar{h}_1(\delta) = z$, where z is explicitly given by

$$z = \min_{k>1}\{(V_1 - V_k)\gamma/(V_k - V_1)\delta \mid (V_1 - V_k)\gamma/(V_k - V_1)\delta > 0\}. \tag{17}$$

If z from (17) exceeds $\rho(\delta)$, then case (a) applies. Suppose case (b) applies and $i = 2$. Then we have $\underline{h}_2(\delta) = z = \bar{h}_1(\delta)$, and the above method for finding $\bar{h}_1(\delta)$ can now be used to find $\bar{h}_2(\delta)$. In a straightforward and efficient way, therefore, all the intervals $\left[\underline{h}_i(\delta),\ \bar{h}_i(\delta)\right]$, $i = 1, \ldots, m$, may be determined.

We have thus far ignored the case in which a_1 is *not* the unique optimal action at γ, i.e., in which $V_1\gamma = V_i\gamma$ for at least one $i > 1$. To see whether we want a_1 or one of the other equally good actions at γ we have merely to compare the values of $V_1(\gamma + \epsilon\delta)$ and $V_i(\gamma + \epsilon\delta)$ for a sufficiently small positive ϵ and use the action with highest such value. Note that while this form of tie among several actions may occur at γ, it will not occur in practice at the point $\gamma + \bar{h}_1(\delta)$ because δ is chosen randomly and hence the probability of such a tie is effectively zero.

One more point deserves mention here. The Monte Carlo method requires that γ be a nondegenerate extreme point of W. We can clearly increase the likelihood of finding such a nondegenerate extreme point to use by eliminating obviously redundant constraints from the definition of W and keeping any equality constraints defining W as equalities instead of two inequalities. Despite these precautions, it is conceivable that W may not possess any nondegenerate extreme points, or that a reasonable amount of searching does not identify one. In such a case it is computationally reasonable to modify a degenerate extreme point by making the basic variables which are zero slightly positive. W will thus be modified and will have at least one new extreme point, a nondegenerate one. The Monte Carlo method can then be used, and, since this type of perturbation can be made arbitrarily small, the errors introduced by it can also be made arbitrarily small.

We have now treated all aspects of the Monte Carlo method developed to approximate the hypervolume ratios $r_1, \ldots, r_m$ by $\bar{r}_1, \ldots, \bar{r}_m$. In the next section we will give a concise step-by-step summary of the method and then present an illustrative example whose obvious solution serves as a check on the Monte Carlo method and the computer program written to implement it.

COMPUTATIONAL SUMMARY AND AN EXAMPLE

The computational method for computing $\bar{r}_1, \ldots, \bar{r}_m$ by (13) may be summarized as follows:

(i) Select a nondegenerate vertex of W, and label it γ. This can be accomplished by generating an arbitrary linear objective function to be maximized over W, and, utilizing the simplex method, continuing to pivot until a suitable vertex is obtained.

(ii) Determine the vertices $\gamma^1, \ldots, \gamma^d$ adjacent to γ and then the unit vectors $q^1, \ldots, q^d$. Again using the simplex method, each γ^l is obtained by pivoting into the basis represented by γ an appropriate nonbasic variable. Slack variables must be considered too. The dimension of W, d, is determined by the number of adjacent vertices.

For $k = 1, \ldots, g$, carry out steps (iii) through (vi) (in which the dependence on k is suppressed).

(iii) Generate sets of d uniform random numbers $(t_1, \ldots, t_d)$ until a set is found such that $||q|| \leqslant 1$ or $||q|| \leqslant u$ (see discussion following (16)), where $q = \Sigma_l t_l q^l$. Then generate the direction vector $\delta = q/||q||$.

(iv) Determine the upper integration limit $\rho(\delta)$ from (11).

(v) Determine the intervals $[h_i(\delta), \bar{h}_i(\delta)]$, $i = 1, \cdots, m$.

(vi) Compute the kth terms in the summations in the numerator and denominator of (13) and add them to the previous subtotals.

(vii) Compute all $\bar{r}_i$ via (13).

Once the $\bar{r}_i$ are available as accurate approximations of the r_i, the Comparative Hypervolume Criterion suggests the choice of the action a_i with largest value of $\bar{r}_i$.

A computer code was written in FORTRAN to implement the numerical method developed above, and the following simple example was used to get some idea of the code's accuracy and running time. The example had $i = 3$ and $n = 52$. V_i (of size 52) had a 10 in column i and a 0 in the other 51 columns, $i = 1, 2, 3$. $W = \bar{W}$ was used. By virtue of the symmetry present it is easily seen that $r_1 = r_2 = r_3 = 1/3$.

Table 1 shows the numerical values obtained for the $\bar{r}_i$ for the first g random vectors δ, $g = 100, 200, 400, 900, 2000$. The results are quite accurate for the larger values of g and are reasonably accurate even for small values of g. The cpu time required for each sample of 100 direction vectors averaged about 1.25 seconds on an IBM 370/158 computer.

TABLE 1. *Computed Hypervolume Ratios for the Example*

Sample Size g	Computed Ratios		
	$\bar{r}_1$	$\bar{r}_2$	$\bar{r}_3$
100	0.2821	0.3883	0.3295
200	0.3244	0.3424	0.3332
400	0.3362	0.3299	0.3340
900	0.3230	0.3449	0.3321
2000	0.3343	0.3352	0.3305

DISCUSSION

The Comparative Hypervolume Criterion has been presented as a rationale and method by which a decision maker or group of decision makers can select one action from a finite set of actions when the selection must account for multiple criteria or attributes. It does not require that the decision maker(s) specify exact values by which to weight the various attributes, and this could be an important consideration in practical decision making situations. If the comparative hypervolume ratio r_i for a particular action a_i is significantly greater than one-half, then the decision maker(s) should be able to adopt it with confidence. If, on the other hand, there is no action with a fairly large ratio, then more caution is called for. Since r_i is the proportion of weight vectors for which a_i has a utility value at least as high as *all* other actions, it is possible that *no* action will have a very large ratio. This might well be the case, for instance, if several different actions have rows V_i of V which are fairly similar. It would then be appropriate to use the CHC to make comparisons among the actions in specific subsets of A. For example, some judicious pairwise comparisons might enable the decision maker(s) to eliminate a number of actions. The CHC might then point rather definitively to one of the remaining ones.

Thus, even if it is not used as the sole basis for selecting an action, use of the CHC can greatly aid the decision maker(s) in excluding some possible choices or in evaluating the effect of various constraints placed on the attribute weights. To aid decision makers in these ways, the computer code could easily be operated in an interactive mode.

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