ELSEVIER

European Journal of Operational Research 92 (1996) 182-192

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

Theory and Methodology

Binary interactions and subset choice

Peter C. Fishburn [a,*], Irving H. LaValle [b]

[a] AT&T Bell Laboratories, 600 Maountain Avenue, Murray Hill, NJ 07974-0636, USA

[b] Tulane University, New Orleans, LA 70118, USA

Received August 1994; revised January 1995

Abstract

Subset evaluation and choice problems abound in practical decision settings. They are often analyzed with linear objective functions that value subsets as sums of utilities of items in the subsets. This simplifies assessment and computational tasks but runs a risk of substantial suboptimality because it disregards evaluative interdependencies among items.

This paper examines a binary-interaction model that accounts for preference interdependencies between items. Ordinal and cardinal versions of the model are axiomatized and compared to the simpler linear model as well as the general model that incorporates all orders of interdependence. Comparisons of computational complexity for standard subset-choice problems are made between the linear and binary-interaction models.

Keywords: Subset choice; Utility theory; Binary interdependence

1. Introduction

Subset evaluation and choice problems are both very common and notoriously challenging in regard to normative preference theory and combinatorial optimization. Their ubiquity is suggested by decisions on transportation-route selection and schedules, restaurant-menu composition, household and corporate budgets, student admissions, national research agency project funding, and optional accessories to offer buyers of a new car or computer. Their challenges for normative decision theory and optimization arise from problem size and evaluative interdependencies among items in potential choice sets. A 40-item set has more than a million million subsets, and the relatively modest problem of choosing a five-person committee from 20 candidates has more than 15 thousand options. Evaluative interdependencies are often ignored by using an additive objective function that values a subset as the sum of values assigned to its items. This may be satisfactory for some problems, but it is clearly inappropriate for others in which holistic subset values are strongly affected by substitutabilities or complementarities among items, or by a desire for representativeness from a diverse population of available items. There is a large literature on additive or non-interactive models, including Churchman and Ackoff (1954), Fishburn (1970, 1972a, 1992a), Gärdenfors (1976), Kannai and Peleg (1984), Pattanaik and Peleg (1984), Bossert (1989), Barberà, Sonnenschein and Zhou (1991) and Fishburn and LaValle (1993, 1994). Examples of approaches that allow interdependencies are described in Fishburn (1972b), Farquhar and Rao (1976), Oral, Kettani and Lang (1991), and Oral and Kettani (1992).

Our purpose here is to formulate and analyze the

* Corresponding author.

SSDI 0377-2217(95)00073-9

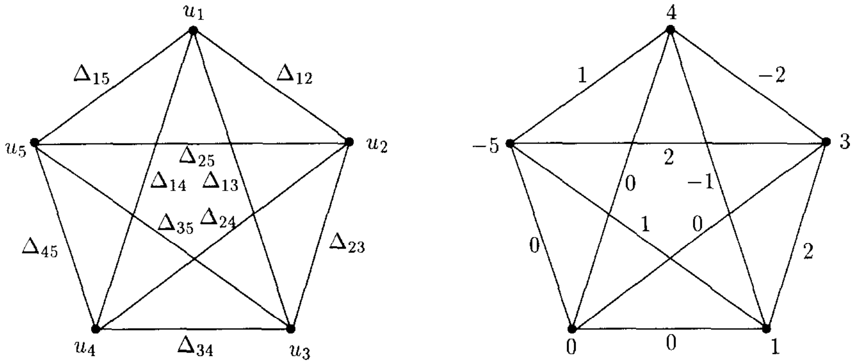

Fig. 1.

simplest model for subset evaluation and choice that can account for preference interdependencies among items. The model considers interdependencies only within pairs of items and will be referred to as a binary-interaction model. We discuss two versions of the model from the perspectives of preference theory and subset choice. The first version is an ordinal model based solely on preferences between subsets. The second uses the lottery formulation of expected utility with subsets as outcomes.

Both versions of the binary-interaction model are introduced by general subset-preference models that allow all orders of interaction among items. We then comment on the additive specialization of the general model that ignores interactions and presumes that the utility $u(A)$ of a subset A of items equals the sum of utilities for members of A:

$$u(A) = \sum_{x \in A} u(x). \tag{1}$$

This is followed by an account of the binary-interaction model in which $u(A)$ is presumed to equal $\sum_A u(x)$ plus an interaction term $\Delta(B)$ for every pair $B = \{x, y\}$ in A:

$$u(A) = \sum_{x \in A} u(x) + \sum_{B \in A_{(2)}} \Delta(B), \tag{2}$$

where $A_{(2)}$ denotes the set of unordered pairs of items in A.

Fig. 1 illustrates (1) and (2) by a valuation graph for items 1 through 5 with $u_i = u(i)$ and $\Delta_{ij} = \Delta(\{i, j\})$. Eq. (1) involves only the vertex valuations. Eq. (2) considers also the differential edge valuations, with $u(\{1,2\}) = u_1 + u_2 + \Delta_{12}$, $u(\{1,2,3\}) = u_1 + u_2 + u_3 + \Delta_{12} + \Delta_{13} + \Delta_{23}$, and so forth. The specific valuations on the right depend in part on the scaling convention, adopted throughout the paper, that

$$u(\varnothing) = 0. \tag{3}$$

This fixes an origin for utilities and gives meaning to positive and negative utility in relation to the utility of the empty set.

Suppose the five items are candidates for a committee of two or three people. The vertex valuations of Fig. 1 say that the evaluator favors 1, 2 and 3 on the committee, is neutral about 4, and does not want candidate 5 on the committee. Interaction term $\Delta_{12} = -2$ suggests that candidates 1 and 2 duplicate each other or would not work well together, and $\Delta_{23} = 2$ indicates that candidates 2 and 3 complement each other or would work well together. The best two-person committee on the basis of (1) would be $\{1, 2\}$, but it would be $\{2, 3\}$ on the basis of (2).

Fig. 1, or our binary-interaction approach, says nothing about interdependencies that appear only when three or more items come together. For example, any two of Susan, Karl and Paul might work well together on a committee but might also be counterproductive if all three were present. Although (2) does not take higher-order interactions into consideration, it is vastly more flexible than (1) and may capture the most salient interdependencies in its binary-interaction terms.

Sections 2 and 3 of the paper focus on the ordinal approach, and Section 4 develops the lottery approach. In each case we identify conditions on preferences for the general model that allow subset utilities to be decomposed as in (1), or in (2). We also consider what can be said about preferences between subsets when such conditions are presumed but assessments are made only for individual items, as in (1), or for items and pairs of items, as in (2). This concern involves notions of dominance in the ordinal setting, but is essentially trivial in the lottery setting because of uniqueness properties.

We comment briefly on optimization problems and complexity issues in Section 5, and summarize our findings in Section 6.

Although the paper uses some abstract theory to make its points, we are motivated by wholly practical concerns. Primary among these is the belief that simple additive models, as in (1), can give misleading and possibly dangerous conclusions about subset values. We feel that consideration of binary interactions offers a powerful corrective even if there are significant higher-order interactions. We do not advocate going beyond binary interdependencies in most situations because of costs of complexity, difficulties of accurate evaluative assessments, and, in some cases, prohibitive problem size. Moreover, the binary-interaction model already introduces assessment and computational challenges far beyond those encountered in additive or linear approaches.

We conclude this introduction by fixing notation and assumptions that apply throughout the paper. The set of available items is denoted by X with cardinality $|X| = n$, $0 < n < \infty$. For each $A \subseteq X$, $A_{(k)}$ is the family of all k-item subsets of A. In particular, $X_{(0)} = \{\varnothing\}$, $X_{(1)} = \{\{x\} : x \in X\}$, $X_{(2)}$ is the set of all pairs in $X, \ldots$, and $X_{(n)} = \{X\}$. We usually write singleton $\{x\}$ as x, as in $u(x)$ instead of $u(\{x\})$.

The strict preference relation $\succ$, *is preferred to*, is our basic preference relation. The set on which $\succ$ applies will be clear in context. Its derived indifference relation $\sim$ and preference-or-indifference relation $\succsim$ are defined by

$$\alpha \sim \beta \quad \text{if neither } \alpha \succ \beta \text{ nor } \beta \succ \alpha,$$

$$\alpha \succsim \beta \quad \text{if } \alpha \succ \beta \text{ or } \alpha \sim \beta.$$

We assume that $\succ$ is a *weak order*, i.e. that it is asymmetric [if $\alpha \succ \beta$ then not$(\beta \succ \alpha)$] and negatively transitive [if not$(\alpha \succ \beta)$ and not$(\beta \succ \gamma)$ then not$(\alpha \succ \gamma)$]. This is tantamount to assuming that $\succ$ is asymmetric and that each of $\succ$ and $\sim$ is transitive.

2. The ordinal approach

The general ordinal model assumes that $\succ$ is a weak order on the set of all subsets of X. This implies that there exists a real-valued function u on the subsets such that, for all $A, B \subseteq X$,

$$A \succ B \Leftrightarrow u(A) > u(B). \tag{4}$$

As noted by (3), we take $u(\varnothing) = 0$ without loss of generality. In regard to uniqueness of the representation, u' also satisfies (4) and (3) if and only if $u'(A) = \tau[u(A)]$ for every $A \subseteq X$ and some strictly increasing $\tau : \mathbb{R} \to \mathbb{R}$ for which $\tau[0] = 0$.

It is instructive to rewrite u in a way that illustrates interactions. For each $A \subseteq X$ with $|A| = m \geqslant 2$, we define the interaction term $\Delta(A)$ for A on the basis of u by the following inclusion-exclusion formula:

$$\Delta(A) = u(A) - \sum_{B \in A_{(m-1)}} u(B) + \sum_{B \in A_{(m-2)}} u(B) - \cdots + (-1)^{m-1} \sum_{B \in A_{(1)}} u(B). \tag{5}$$

Thus $\Delta(\{x,y\}) = u(\{x,y\}) - u(x) - u(y)$, $\Delta(\{x,y,z\}) = u(\{x,y,z\}) - [u(\{x,y\}) + u(\{x,z\}) + u(\{y,z\})] + [u(x) + u(y) + u(z)]$, and so forth. Then, for every $A \subseteq X$ with $|A| = m \geqslant 2$, it follows (see Appendix A) that

$$u(A) = \sum_{x \in A} u(x) + \sum_{k=2}^{m} \sum_{B \in A_{(k)}} \Delta(B). \tag{6}$$

Hence $u(A)$ equals the sum of a utility term for each item in A and an interaction term for each subset in A with two or more items.

The additive specialization (1) holds when it is possible to define u for (4) so that Δ defined by (5) vanishes. And (2) holds when u can be defined for (4) so that $\Delta(A) = 0$ whenever $|A| \geqslant 3$. Cancellation or additivity conditions on subset preferences that are necessary and sufficient for these specializations are routinely developed from linear separation theory. We begin with (1).

For sequences $A_1, \ldots, A_J$ and $B_1, \ldots, B_J$ of subsets of X, let $(A_1, \ldots, A_J) \approx_1 (B_1, \ldots, B_J)$ mean that, for each $x \in X$, the number of A_j that contain x equals the number of B_j that contain x. An example is $(\{1,2,3\}, \{4,5\}, \varnothing, \varnothing) \approx_1 (\{1\}, \{5\}, \{3\}, \{2,4\})$.

Condition 1. For all $J \geqslant 2$ and all $A_1, \ldots, A_J$, $B_1, \ldots, B_J \subseteq X$, if $(A_1, \ldots, A_J) \approx_1 (B_1, \ldots, B_J)$ and $A_1 \succ B_1$, then $B_j \succ A_j$ for some $j \geqslant 2$.

This condition is obviously necessary for the existence of u on X that satisfies $A \succ B \Leftrightarrow \sum_A u(x) > \sum_B u(x)$ for all $A, B \subseteq X$. It is also sufficient for the additive representation, as can be seen from straightforward application of linear separation theory or a "theorem of the alternative": see, for example, Fishburn (1970, Chapter 4; 1972c; 1992c).

An explicit failure of Condition 1 is $\{1 \succ \varnothing, 2 \succ \varnothing, \varnothing \succsim \{1,2\}\}$. Here each of 1 and 2 is preferred to nothing ($\varnothing$), whereas the combination of 1 and 2 is not preferred to $\varnothing$. In other words, each of 1 and 2 alone is desirable, but together the combination is undesirable. Such preferences cannot be represented as in (1), but may be represented by the binary-interaction model.

The condition that allows u in (4) to be written as in (2) is similar to Condition 1 with strengthened hypotheses. With $(A_1, \ldots, A_J)$ and $(B_1, \ldots, B_J)$ as above, let $(A_1, \ldots, A_J) \approx_2 (B_1, \ldots, B_J)$ mean that $(A_1, \ldots, A_J) \approx_1 (B_1, \ldots, B_J)$ *and*, for every pair $C \in X_{(2)}$, the number of A_j that include C equals the number of B_j that include C.

Condition 2. For all $J \geqslant 4$ and all $A_1, \ldots, A_J$, $B_1, \ldots, B_J \subseteq X$, if $(A_1, \ldots, A_J) \approx_2 (B_1, \ldots, B_J)$ and $A_1 \succ B_1$, then $B_j \succ A_j$ for some $j \geqslant 2$.

The proof that Condition 2 is necessary and sufficient for the existence of u and Δ that satisfy

$$A \succ B \Leftrightarrow \sum_{x \in A} u(x) + \sum_{C \in A_{(2)}} \Delta(C) > \sum_{x \in B} u(x) + \sum_{C \in B_{(2)}} \Delta(C) \qquad (7)$$

for all $A, B \subseteq X$ is given in Appendix B and is analogous to that for Condition 1 and the simpler additive form. An explicit failure of Condition 2 is given by $(\varnothing, \{1,2\}, \{1,3\}, \{2,3\}) \approx_2 (\{1,2,3\}, 1, 3, 2)$ with $\varnothing \succ \{1,2,3\}$, $\{1,2\} \succsim 1$, $\{1,3\} \succsim 3$ and $\{2,3\} \succsim 2$. If (7) held we would have

$$0 > u(1) + u(2) + u(3) + \Delta(\{1,2\}) + \Delta(\{1,3\}) + \Delta(\{2,3\}),$$

$$u(1) + u(2) + \Delta(\{1,2\}) \geqslant u(1),$$

$$u(1) + u(3) + \Delta(\{1,3\}) \geqslant u(3),$$

$$u(2) + u(3) + \Delta(\{2,3\}) \geqslant u(2).$$

Summation of these inequalities gives the contradiction that $0 > 0$.

Suppose Condition 2 holds but preferences are assessed only on subsets with fewer than three items, i.e. on

$$X^* = \{\varnothing\} \cup X_{(1)} \cup X_{(2)}.$$

We are interested in the ability of the limited preference information to imply other preferences such as $A \succ B$ when at least one of A and B is not in X^*.

To focus this concern, consider a fixed weak order $\succ^*$ on X^* that is presumed to be the restriction to X^* of the decision maker's weak order $\succ$ on all subsets. Let U^* denote the set of all utility functions on X^* that have $u(\varnothing) = 0$ and satisfy

$$A \succ^* B \Leftrightarrow u(A) > u(B), \quad \text{for all } A, B \in X^*. \qquad (8)$$

For any particular $u \in U^*$, define Δ on $X_{(2)}$ by

$$\Delta(\{x,y\}) = u(\{x,y\}) - u(x) - u(y),$$

extend u to larger subsets by (2), and then define a weak order $\succ_u$ on all subsets by $A \succ_u B$ if $u(A) > u(B)$. This gives

$$A \succ_u B \Leftrightarrow \sum_{x \in A} u(x) + \sum_{C \in A_{(2)}} \Delta(C) > \sum_{x \in B} u(x) + \sum_{C \in B_{(2)}} \Delta(C)$$

for all $A, B \subseteq X$.

It follows by construction and the result for (7) that $\{\succ_u : u \in U^*\}$ is the set of all weak orders on subsets that are consistent with Condition 2 and the limited preference information of $\succ^*$. Hence $\succ$ is one of the $\succ_u$, although we do not know which one. However, if for arbitrary $A, B \subseteq X$, A ordinally dominates B in the sense that

$$A \succ_u B \quad \text{for all} \quad u \in U^*,$$

then it must be true that $A \succ B$. We pursue this further in the next section.

That a given $\succ^*$ on X^* can have different extensions $\succ_u$ that satisfy Condition 2 is related to the fact that the restriction to X^* of weak order $\succ$ on all subsets need not be sufficient to determine the sign of every $\Delta(\{x,y\})$. In other words, the X^* preference data for the ordinal binary-interaction model might not reveal whether the differential preference interaction of x and y is positive or negative. A given $u \in U^*$ for (8) does determine Δ on $X_{(2)}$ by the usual formula, but different $u \in U^*$ can generate rather different Δ functions.

The sign determinacy of a Δ term depends on the specific preferences on X^* as we now show by example. Consider the 12 linear orders on $\{\varnothing, 1, 2, \{1,2\}\}$ in which $1 \succ 2$. These orders divide into three groups according to the sign determinacy of $\Delta(\{1,2\}) = \Delta_{12}$ as follows:

$\Delta_{12} > 0$	$\Delta_{12} < 0$
$\{1,2\} \succ 1 \succ \varnothing \succ 2$	$1 \succ 2 \succ \{1,2\} \succ \varnothing$
$\{1,2\} \succ \varnothing \succ 1 \succ 2$	$1 \succ 2 \succ \varnothing \succ \{1,2\}$
$\varnothing \succ \{1,2\} \succ 1 \succ 2$	$1 \succ \varnothing \succ 2 \succ \{1,2\}$
$\varnothing \succ 1 \succ \{1,2\} \succ 2$	$1 \succ \{1,2\} \succ 2 \succ \varnothing$

Δ_{12} indeterminate
$\{1,2\} \succ 1 \succ 2 \succ \varnothing$
$1 \succ \{1,2\} \succ \varnothing \succ 2$
$1 \succ \varnothing \succ \{1,2\} \succ 2$
$\varnothing \succ 1 \succ 2 \succ \{1,2\}$

The first order in the first group has $u_1 + u_2 + \Delta_{12} > u_1 > 0 > u_2$ by (8), so $\Delta_{12} > -u_2 > 0$. The second order in the second group has $u_1 > u_2 > 0 > u_1 + u_2 + \Delta_{12}$, so $\Delta_{12} < 0$, and in fact $\Delta_{12} < -(u_1 + u_2)$. The first order in the indeterminate group has Δ_{12} positive, zero, or negative according to whether $u(\{1,2\})$ exceeds, equals, or is less than $u_1 + u_2$ respectively. Other cases have similar verifications.

3. Ordinal dominance

We noted in the preceding section that a weak order $\succ$ on all subsets of X can be represented by a binary-interaction model as in (7) if, and only if, Condition 2 holds. Moreover, when Condition 2 is presumed, there exists a utility function on X^* for (8) whose extension by (2) satisfies (7). However, we will usually not know in practice which u on X^* is most suitable, because $\succ$ may be assessed only on X^* and have many different extensions that can satisfy (7).

The concept of ordinal dominance provides a limited resolution of the multiple-extensions problem by identifying pairs of subsets for which $u(A) > u(B)$ for every $u \in U^*$. We define it formally by noting first that (2) can be written as

$$u(A) = \sum_{C \in A_{(2)}} u(C) - (|A| - 2) \sum_{x \in A} u(x), \tag{9}$$

for all $A \subseteq X$, by replacing $\Delta(\{x,y\})$ in (2) by $u(\{x,y\}) - u(x) - u(y)$. We then define $\geqslant_D$ on the sets in X as follows:

$$A \geqslant_D B \text{ if } \sum_{C \in A_{(2)}} u(C) + (|B| - 2) \sum_{x \in B} u(x) \geqslant \sum_{C \in B_{(2)}} u(C) + (|A|) - 2) \sum_{x \in A} u(x)$$

for every u on X^* that has $u(\varnothing) = 0$ and satisfies (8).

Strict ordinal dominance $>_D$ and ordinal equivalence $=_D$ are defined by

$A >_D B$ if $A \geqslant_D B$ and not$(B \geqslant_D A)$

$A =_D B$ if $A \geqslant_D B$ and $B \geqslant_D A$.

It is obvious for $A, B \in X^*$ that $A >_D B \Leftrightarrow A \succ B$, and $A =_D B \Leftrightarrow A \sim B$, and it is easily checked that, for all $A, B \subseteq X$,

$$A >_D B \Leftrightarrow [u(A) > u(B) \quad \text{for every} \quad u \in U^*],$$

$$A =_D B \Leftrightarrow [u(A) = u(B) \quad \text{for every} \quad u \in U^*].$$

It follows that $=_D$ is an equivalence relation (reflexive, symmetric, transitive) and that $>_D$ is a strict partial order (asymmetric, transitive).

The extent to which ordinal dominance resolves preference between larger subsets of X under the presumption of Condition 2 clearly depends on the nature of $\succ$ on X^*. An example for $n = 4$ with $\succ$ on $\{1,2,3,4\}^*$ given by

$$12 \succ 1 \succ 2 \succ 13 \succ 23 \succ [\varnothing \sim 3 \sim 24] \succ 4 \succ 14 \succ 34 \tag{10}$$

yields, as we shall show in part,

$12 >_D 123 >_D 124 >_D 1234$
$123 >_D 234$
$\varnothing >_D 234$ [and $3 >_D 234$, $24 >_D 234$]
$34 >_D 134$

along with other $>_D$ relations that follow from these and $\succ$ on X^* with the use of transitivity. This allows us to conclude that $\{1,2\}$ is the uniquely most preferred subset of X. The worst subsets appear to be $\{1,3,4\}$, $\{2,3,4\}$ and $\{1,2,3,4\}$.

There is a simple test for ordinal dominance based on $\succ$ on X^* that involves utility only implicitly. For any two same-cardinality sequences of elements in X^*, say $(a_1,\dots,a_m)$ and $(b_1,\dots,b_m)$, we write $(a_1,\dots,a_m) \geqslant_M (b_1,\dots,b_m)$ and say that the first sequence *monotonically dominates* the second if, when the two are rearranged in decreasing order of preference as $(a_{(1)},\dots,a_{(m)})$ and $(b_{(1)},\dots,b_{(m)})$ with $a_{(1)} \succsim a_{(2)} \succsim \cdots \succsim a_{(m)}$ and $b_{(1)} \succsim b_{(2)} \succsim \cdots \succsim b_{(m)}$, it is true that

$$a_{(i)} \succsim b_{(i)} \quad \text{for} \quad i = 1,\dots,m.$$

To see how this applies to the ordinal dominance comparison of A vs. B, write $u(A)$ and $u(B)$ as in (9) and transpose negative terms in the resulting $u(A) \geqslant u(B)$ inequality to the other side so that, with possible repetitions of C_i and D_i terms from X^*, we have

$$u(A) \geqslant u(B) \Leftrightarrow u(C_1) + u(C_2) + \cdots + u(C_J) \geqslant u(D_1) + u(D_2) + \cdots + u(D_K).$$

If $J = K$, take $m = J$; if $J < K$, take $m = K$ and let $C_{J+1} = \cdots = C_K = \varnothing$; if $J > K$, take $m = J$ and let $D_{K+1} = \cdots = D_J = \varnothing$. We then have same-cardinality sequences $(C_1,\dots,C_m)$ and $(D_1,\dots,D_m)$. It follows from ordinal scaling for u that

$$A \geqslant_D B \Leftrightarrow (C_1,\dots,C_m) \geqslant_M (D_1,\dots,D_m).$$

Two applications for (10) illustrate the procedure. Suppose first that $A = 124$ and $B = 1234$. By (9),

$$u(A) = u(12) + u(14) + u(24) - [u(1) + u(2) + u(4)],$$
$$u(B) = u(12) + u(13) + u(14) + u(23) + u(24) + u(34) - 2[u(1) + u(2) + u(3) + u(4)].$$

Therefore $u(A) \geqslant u(B)$ if and only if

$$u(1) + u(2) + u(3) + u(3) + u(4) \geqslant u(13) + u(23) + u(34) + u(\varnothing) + u(\varnothing).$$

Monotone decreasing rearrangement of (1,2,3,3,4) gives (1,2,3,3,4), monotone decreasing rearrangement of $(13,23,34,\varnothing,\varnothing)$ gives $(13,23,\varnothing,\varnothing,34)$, and because

$$1 \succ 13, \quad 2 \succ 23, \quad 3 \succsim \varnothing, \quad 3 \succsim \varnothing \quad \text{and} \quad 4 \succ 34,$$

we have $(1,2,3,3,4) \geqslant_M (13,23,34,\varnothing,\varnothing)$. Since the converse monotonic dominance is obviously false, we conclude that $A >_D B$.

Suppose next that $A = 1$ and $B = 123$. Using (9),

$$u(A) \geqslant u(B) \Leftrightarrow u(1) + u(1) + u(2) + u(3) \geqslant u(12) + u(13) + u(23) + u(\varnothing).$$

The monotone decreasing rearrangements for the X^* terms on the two sides are $(1,1,2,3)$ and $(12,13,23,\varnothing)$. Because $12 \succ 1$ and $1 \succ 13$, neither sequence monotonically dominates the other, and we can conclude neither $A \geqslant_D B$ nor $B \geqslant_D A$.

A similar procedure holds with (1) in place of (2) or (9) when Condition 1 is presumed. In this setting, the A vs. B comparison pits the sequence of items in $A \setminus B$ against the sequence of items in $B \setminus A$, with $\varnothing$'s appended to the shorter sequence so that the two have the same number of terms. Consider the restriction of (10) to $\{\varnothing\} \cup X_{(1)}$, i.e.

$$1 \succ 2 \succ \varnothing \sim 3 \succ 4.$$

Because $3 \sim \varnothing$, we have $A \sim A \cup \{3\}$ whenever $3 \notin A$, assuming of course that Condition 1 holds. With 3 suppressed, monotonic dominance comparisons show that

$$12 \succ 124 \succ 14 \succ 24 \succ 4, \quad 1 \succ 14, \quad \text{and} \quad 2 \succ 24.$$

The indeterminate comparisons are 1 vs. 124, 2 vs. 124, 2 vs. 14, and $\varnothing$ vs. each of 124, 14 and 24.

4. A cardinal approach

Theories of cardinal utility and their implications for utility measurement are reviewed in Fishburn (1976) and Ellingsen (1994). We focus here on the expected utility approach in von Neumann and Morgenstern (1944) because it provides a widely accepted normative model for decision under risk as well as a means of precise utility measurement. We begin with a general model with subsets as outcomes and then consider its additive and binary-interaction specializations.

Let P denote the set of all lotteries on probability distributions $p, q, \ldots$ on the subsets of X. The convex combination of p with weight λ and q with weight $1-\lambda$, $0 \leqslant \lambda \leqslant 1$, is written as $\lambda p + (1-\lambda)q$ or (p, λ, q). The probability assigned to each $A \subseteq X$ by the convex combination is $\lambda p(A) + (1-\lambda)q(A)$. We assume that $\succ$ on P is a weak order that satisfies the independence and continuity axioms of expected utility (Herstein and Milnor 1953, Jensen 1967, Fishburn 1970, 1988). The axioms imply that there is a real-valued function u on P such that, for all $p, q \in P$ and all $0 \leqslant \lambda \leqslant 1$,

$$p \succ q \Leftrightarrow u(p) > u(q),$$

$$u(\lambda p + (1-\lambda)q) = \lambda u(p) + (1-\lambda)u(q),$$

with u unique up to positive affine transformations of the form $u \to \alpha u + \beta$ $(\alpha, \beta \in \mathbb{R}, \alpha > 0)$.

Let $u(A) = u(p)$ when $p(A) = 1$. Then $u(p)$ for $p \in P$ equals its expected utility over subsets of X:

$$u(p) = \sum_{A \subseteq X} p(A)u(A). \tag{11}$$

We fix an origin for u by specifying $u(\varnothing) = 0$. Under this scaling convention, u is unique up to multiplication by a positive constant $(u \to \alpha u, \alpha > 0)$.

Our special condition for (1) in this setting posits indifference between certain even-chance lotteries on disjoint subsets. In an expression like (A, λ, B) we identify A and B with the lotteries that assign probability 1 to these subsets.

Condition 3. For all disjoint $A, B \subseteq X$,

$$\left(A, \tfrac{1}{2}, B\right) \sim \left(A \cup B, \tfrac{1}{2}, \varnothing\right).$$

Fishburn (1992b) proves that Condition 3 holds if and only if u on subsets has an additive decomposition, i.e. $u(A) = \sum_A u(x)$, where $u(x) = u(\{x\})$. Hence, if u has been assessed for singletons along with $u(\varnothing) = 0$, then u is known for all subsets and for all lotteries since, by (11),

$$u(p) = \sum_{x \in X} u(x) \sum_{\{A : x \in A\}} p(A).$$

Effects of Condition 3 and related independence conditions on subset utilities in generalizations of expected utility theory are discussed in Fishburn and LaValle (1993).

Condition 3 will be false if there is noticeable preference interdependence among items. Its more accommodating companion for binary interactions is revealed by rewriting (9) as

$$\lambda_A u(A) + \sum_{x \in A} \lambda_A(|A| - 2)u(\{x\}) = \sum_{C \in A_{(2)}} \lambda_A u(C) + \left[1 - \lambda_A \binom{|A|}{2}\right] u(\varnothing) \tag{12}$$

for $|A| \geqslant 3$, where

$$\lambda_A = \frac{1}{1 + |A|(|A| - 2)}.$$

Note that $0 < \lambda_A < 1$ and $0 < \lambda_A\binom{|A|}{2} < 1$ when $|A| \geqslant 3$. The left side of (12) is the expected utility of the lottery that assigns probability λ_A to A and probability $\lambda_A(|A| - 2)$ to each singleton subset of A. We denote this lottery as P_A. The right side of (12) is the expected utility of the lottery that assigns probability $1 - \lambda_A\binom{|A|}{2}$ to $\varnothing$ and probability λ_A to each doubleton in A. We denote it by q_A.

Condition 4. For all $A \subseteq X$ with $|A| \geqslant 3$, $p_A \sim q_A$.

The formulation of Condition 4 shows that it is necessary for the binary-interaction model, i.e. for (2) or (9), in the expected utility setting. Conversely, if the condition holds, then $p_A \sim q_A$ and (11) imply (12), which is tantamount to (9) or (2). Hence Condition 4 is necessary and sufficient for the binary-interaction model in the present context.

A potential disadvantage of the present version of the binary-interaction model in comparison to its ordinal counterpart is the effort needed to assess the $u(x)$ and $u(\{x, y\})$, say by lottery-indifference methods. This may be more than offset by the rigid scaling properties of expected utility. Given Condition 4, u on X^* with $u(\varnothing) = 0$ is uniquely determined (up to the choice of scale unit) and u on larger subsets is uniquely specified by (2) or (9). There is therefore no ambiguity about subset preferences throughout X of the type that motivated our consideration of ordinal dominance in the preceding section.

5. Subset choice problems

It seems clear that the binary-interaction model is superior to the basic additive model for estimating accurately utilities of subsets. However, it also requires substantially more assessment, perhaps by a factor of about $n/2$, since the additive model asks for the utilities of n items and the interaction model is concerned with utilities of n items and $\binom{n}{2}$ two-item subsets.

The additive and binary-interaction models may also be compared in regard to computations needed to solve standard problems of subset choice. We mention three such problems, assuming prior assessment of the $u(x)$ for the additive model and of the $u(x)$ and $u(\{x,y\})$ [or $\Delta(\{x,y\}) = u(\{x,y\}) - u(x) - u(y)$] for the interaction model. Utilities for subsets are presumed to be computed by (1) for the additive model and by (2) for the binary-interaction model.

Problem 1. Determine $A \subseteq X$ that maximizes $u(A)$.

Problem 2. Given k, determine $A \in X_{(k)}$ that maximizes $u(A)$ over $X_{(k)}$.

Problem 3. Given $b > 0$ and $c(x) > 0$ for each $x \in X$, determine $A \subseteq X$ that maximizes $u(A)$ subject to $\sum_A c(x) \leqslant b$.

This yields a total of six problems, three for (1) and three for (2). The theory of computational complexity (Cook 1971, Garey and Johnson 1979) shows that four of the six are computationally hard, in the class of NP-complete problems, although some have efficient polynomial-time approximation algorithms (Johnson 1974) or interesting special cases that can be solved in polynomial time (Balas, Chvátal and Nešetřil 1987).

The two not in the NP-complete class are Problems 1 and 2 for the additive model (1). Given (1), Problem 1 can be done in linear time (compare each $u(x)$ with 0), and the simple bisection-ranking procedure shows that Problem 2 requires no more than $n \log_2 n$ comparisons between $u(x)$ values.

Problem 3 with (1) is the classic NP-complete knapsack problem, and specialization in which $u(x) = c(x)$ for every $x \in X$ is the NP-complete subset-sum problem. Further information on this type of problem is available in Johnson (1974) and Sahni (1973).

All three problems for (2) are hard. This is obvious for Problem 3 by our preceding remarks. In the other cases it is useful to view the data for (2) as vertex and edge evaluations for the complete graph on n vertices, as in Fig. 1. We can then invoke well known results of complexity theory for graphs. For example, even if we set $u(x) = 0$ for all x, leaving only the Δ edge evaluations, Problem 2 for $2 < k < n$ is as hard as the classic maximum clique problem, which is NP-complete (Garey and Johnson 1979). Special cases of Problem 1 are easy when (2) applies, but the general problem with positive and negative evaluations on vertices and edges is hard. For example, if every vertex has utility 1, and Δ for each edge is either 0 or $-n^2$, then Problem 1 is the NP-complete maximum independent set problem.

Despite the fact that subset choice problems tend to be much easier with (1) than (2), the use of (1) runs a substantial risk of giving a solution that is far from optimal because of its disregard of interdependencies. Moreover, methods have been devised to alleviate computational difficulties for (2) although its worst-case situations may be hard. Our problems for (2) can be phrased in the notation of Fig. 1 as

$$\text{maximize} \sum_{i=1}^{n} \alpha_i u_i + \sum_{1 \leqslant i < j \leqslant n} \alpha_i \alpha_j \Delta_{ij}$$

$$\text{subject to } (\alpha_1, \ldots, \alpha_n) \in C,$$

where C specifies the constraints, including $\alpha_i \in \{0,1\}$, of the specific problem. One way to deal with the quadratic expression in the objective function involves linearization models (Glover 1975, Oral and Kettani 1992) in which the problem is transformed into a mixed-integer linear program. The method of Oral and Kettani (1992), which handles cubic terms $(\alpha_i \alpha_j \alpha_k \Delta_{ijk})$ as well as the quadratic expression, requires n new continuous variables and n auxiliary linear constraints for the quadratic case. An alternative approach that avoids linearization is explored in Kettani and Oral (1993). This approach applies 0-1 variables to subset utilities and uses special techniques to reduce the number of such variables that require consideration. We anticipate further developments along these lines that may bring the quadratic and perhaps higher-order interaction models within the realm of computational tractability for reasonably large values of n.

Computational complexity and difficulties by themselves may be issues of decreasing practical importance as computing continues to increase in power and decline in cost. A more salient issue may be assessment complexity. As suggested by the first paragraph of this section, assessment complexity of the at-most-$j$ary-interactional model is polynomial of degree j for each j in $\{2,3,\ldots\}$. Our decision to focus only on binary interactions reflects both the difficulties of accurate assessment and the belief that a significant proportion of interdependent effects can be captured by the binary-interaction model.

6. Summary

Subset evaluation and choice problems abound in many decision contexts. Because of assessment ease and computational simplicity, it is tempting to ignore preference interdependencies and use an objective function that values each subset as the sum of utilities of its items. The conditions that justify this approach are very strong, and it can lead to highly suboptimal decisions.

This paper explores an alternative model that requires substantially more assessment effort and is computationally more demanding, but is also much more likely to yield good decisions. Our model considers binary interdependencies between items as well as the main effects of items viewed separately. Two versions of the binary-interaction model were discussed. The ordinal version is based solely on a preference order of items, pairs of items, and the empty set. Its conclusions for preference between larger subsets, which depend on dominance comparisons, may be informative but can leave many preferences unresolved. The expected utility version focuses on the same small subsets for assessment and can entail greater assessment effort, but it provides unambiguous preference conclusions between other subsets under the presumption that preference interdependencies are accounted for by binary interactions.

We note in closing that higher-order interdependencies, of order at most j with $j \geqslant 3$, could be modeled by starting from (6) with m replaced by $j \leqslant m$. Conditions 2 and 4 could then be extended in natural ways for $j = 3,4,\ldots$. Our reasons for focusing only on binary interactions have been stated above, but should not dissuade others from considering higher-order interactions for situations in which it is felt that the additional effort required will be offset by more accurate evaluations of larger subsets.

Acknowledgements

We thank David Johnson for his help on matters of computational complexity, and Muhittin Oral for his advice on practical methods to alleviate computational difficulties.

Appendix A

We use the well-known identity

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i} = 0, \quad k = 1,2,\ldots$$

to verify (6). Suppose $A \subseteq X$, $|A| = m \geqslant 2$. We claim that

$$\begin{aligned}
u(A) = {}& \\
& u(A) - \sum_{B\in A_{(m-1)}} u(B) + \sum_{B\in A_{(m-2)}} u(B) \\
& - \sum_{B\in A_{(m-3)}} u(B) + \cdots + (-1)^{m-1} \sum_{B\in A_{(1)}} u(B) \\
& + \sum_{B\in A_{(m-1)}} \Bigg[u(B) - \sum_{C\in B_{(m-2)}} u(C) + \sum_{C\in B_{(m-3)}} u(C) \\
& \qquad + \cdots + (-1)^{m-2} \sum_{C\in B_{(1)}} u(C) \Bigg] \\
& + \sum_{B\in A_{(m-2)}} \Bigg[u(B) - \sum_{C\in B_{(m-3)}} u(C) + \cdots \\
& \qquad + (-1)^{m-3} \sum_{C\in B_{(1)}} u(C) \Bigg] \\
& \vdots
\end{aligned}$$

$$\vdots$$

$$+ \sum_{B \in A_{(1)}} u(B).$$

Suppose $C \subseteq A$, $|C| = m-j$. The absolute value of the coefficient of $u(C)$ in row i of this display, $1 \leqslant i \leqslant j+1$, is the number of $B \subseteq A$ with $|B| = m-i+1$ for which $C \subseteq B$. This number is $\binom{j}{j+1-i}$, i.e. the number of ways we can choose $j+1-i$ items from the j not in C but in A to get B, $C \subseteq B \subseteq A$ and $|B| = m-i+1$. Hence, for $j \geqslant 1$, the total coefficient of $u(C)$ with $|C| = m-j$ is $\pm\binom{j}{j} \mp \binom{j}{j-1} \pm \binom{j}{j-2} \cdots + \binom{j}{0} = 0$. This shows that the displayed equation reduces to $u(A) = u(A)$.

The first row of that equation is $\Delta(A)$, according to (5). The other rows are, by (5),

$$\sum_{B \in A_{(m-1)}} \Delta(B), \sum_{B \in A_{(m-2)}} \Delta(B), \ldots, \sum_{B \in A_{(2)}} \Delta(B),$$

and, for the final row, $\sum_{x \in A} u(x)$. Hence

$$u(A) = \sum_{k=2}^{m} \sum_{B \in A_{(k)}} \Delta(B) + \sum_{x \in A} u(x),$$

which is (6).

Appendix B

Assume that $\succ$ on the subsets of X is a weak order. Suppose (7) holds, $(A_1, \ldots, A_J) \approx_2 (B_1, \ldots, B_J)$ with $J \geqslant 4$, and $A_1 \succ B_1$. Then, if Condition 2 fails, we also have $A_j \succsim B_j$ for $j = 2, \ldots, J$: hence, by (7),

$$\sum_{x \in A_1} u(x) + \sum_{C \in A_{1(2)}} \Delta(C) > \sum_{x \in B_1} u(x) + \sum_{C \in B_{1(2)}} \Delta(C),$$

$$\sum_{x \in A_j} u(x) + \sum_{C \in A_{j(2)}} \Delta(C) \geqslant \sum_{x \in B_j} u(x) + \sum_{C \in B_{j(2)}} \Delta(C),$$

$$j = 2, \ldots, J.$$

Because $(A_1, \ldots, A_J) \approx_2 (B_1, \ldots, B_J)$, there is an exact identity match between the $u(x)$ terms on the left sides and the right sides of these inequalities, and there is an exact identity match between the $\Delta(C)$ terms on the two sides. Therefore addition of the J inequalities gives the contradiction that $0 > 0$. We conclude that if (7) holds then Condition 2 also holds.

To prove that Condition 2 implies the existence of u and Δ that satisfy (7), let $N = n + \binom{n}{2}$ and enumerate the singleton and doubleton subsets of x as $C_1, C_2, \ldots, C_N$. For each $A \subseteq X$ let $A' \in \{0,1\}^N$ be the characteristic vector of A defined by

$$A'_i = \begin{cases} 1 & \text{if } C_i \subseteq A, \\ 0 & \text{otherwise.} \end{cases}$$

Also let $\rho = (\rho(C_1), \rho(C_2), \ldots, \rho(C_N)) \in \mathbb{R}^N$. With the understanding that we will take $u(x) = \rho(\{x\})$ and $\Delta(\{x,y\}) = \rho(\{x,y\})$, we rewrite the sum in (7) for A as a scalar product:

$$\sum_{x \in A} u(x) + \sum_{C \in A_{(2)}} \Delta(C) = A' \bullet \rho.$$

Then (7) can be expressed as

$$(A' - B') \bullet \rho > 0$$

for all $A, B \subseteq X$ for which $A \succ B$,

$$(A' - B') \bullet \rho = 0$$

for all $A, B \subseteq X$ for which $A \sim B$.
This is a finite system of linear inequalities and equalities, with rational coefficients because $A' - B'$ lies in $\{1, 0, -1\}^N$ for all $A' - B'$. We refer to it as system S. If $\succ$ is empty then system S has solution $\rho \equiv 0$. And Condition 2 holds trivially.

Assume henceforth that $\succ$ is not empty. Let $D = A' - B'$. Suppose S has $K \geqslant 1$ instances of $A \succ B$ with corresponding D vectors $D^1, D^2, \ldots, D^K$, and $M - K$ non-trivial ($A \neq B$) instances of $A \sim B$ with corresponding D vectors $D^{K+1}, \ldots, D^M$. Then system S can be written as

$$D^k \bullet \rho > 0 \quad \text{for } k = 1, \ldots, K,$$

$$D^k \bullet \rho = 0 \quad \text{for } k = K+1, \ldots, M.$$

According to Lemma 5.3 in Fishburn (1972c), which is a typical Theorem of the Alternative for linear systems with rational coefficient vectors, *either* system S has a ρ solution *or* (exclusive) there are non-negative integers $r_1, r_2, \ldots, r_K$, at least one of which is positive, and integers $r_{K+1}, \ldots, r_M$ such that

$$\sum_{k=1}^{M} r_k D_i^k = 0 \quad \text{for } i = 1, \ldots, N.$$

We can in fact presume that $r_k > 0$ for all k, for if $r_k = 0$, we simply omit it, and if $r_k < 0$ for $D = A' - B'$ with $A \sim B$ then we replace it and D by $-r_k > 0$ and $-D = B' - A'$ with $B \sim A$.

Suppose the latter alternative holds. Then Condition 2 fails. To see this, let $r_k > 0$ correspond to $A_k \succ B_k$ or $A_k \sim B_k$, with $A_1 \succ B_1$ for definiteness. Form sequences of A's and B's as follows. The A sequence has r_1 A_1's, followed by r_2 A_2's, ..., and ends with r_M A_M's. The B sequence has r_1 B_1's, followed by r_2 B_2's, ..., and ends with r_M B_M's. If $\sum r_k = J < 4$, add $\varnothing \sim \varnothing$ at the ends to get $J \geqslant 4$. Re-index the terms so that the A sequence is $A_1, A_2, \ldots, A_J$ and the B sequence is $B_1, B_2, \ldots, B_J$. We then have $A_1 \succ B_1$ and $A_j \succsim B_j$ for $j = 2, \ldots, J$. However, $\sum r_k D_i^k = 0$ for $i = 1, \ldots, N$ implies that $(A_1, \ldots, A_J) \approx_2 (B_1, \ldots, B_J)$, and therefore Condition 2 fails.

Consequently, if Condition 2 holds, then system S must have a ρ solution. Suppose Condition 2 holds. Let ρ be a solution, and let $u(x) = \rho(\{x\})$ and $\Delta(\{x, y\}) = \rho(\{x, y\})$. Then (7) holds and the proof is complete.

References

Balas, E., Chvátal, V., and Nešetřil, J. (1987), "On the maximum weight clique problem", *Mathematics of Operations Research* 12, 522–535.

Barberà, S., Sonnenschein, H., and Zhou, L. (1991), "Voting by committees", *Econometrica* 59, 595–609.

Bossert, W. (1987), "On the extension of preferences over a set to the power set: An axiomatic characterization of a quasi-ordering", *Journal of Economic Theory* 49, 84–92.

Churchman, C. W., and Ackoff, R. L. (1954), "An approximate measure of value", *Operations Research* 2, 172–187.

Cook, S. A. (1971), "The complexity of theorem-proving procedures", in: *Proceedings of the Third Annual ACM Symposium on the Theory of Computing*, Association for Computing Machinery, New York, pp. 151–158.

Ellingsen, T. (1994), "Cardinal utility: A history of hedonimetry", in: M. Allais and O. Hagen (eds.), *Cardinalism*, Kluwer Academic, Dordrecht, pp. 105–165.

Farquhar, P.H. and Rao, V.R. (1976), "A balance model for evaluating subsets of multiattributed items", *Management Science* 22, 528–539.

Fishburn, P.C. (1970), *Utility Theory for Decision Making*, Wiley, New York.

Fishburn, P.C. (1972a), "Even-chance lotteries in social choice theory", *Theory and Decision* 3, 18–40.

Fishburn, P.C. (1972b), "Interdependent preferences on finite sets", *Journal of Mathematical Psychology* 9, 225–236.

Fishburn, P.C. (1972c), *Mathematics of Decision Theory*, Mouton, The Hague.

Fishburn, P.C. (1976), "Cardinal utility: An interpretive essay", *International Review of Economics and Business* 23, 1102–1114.

Fishburn, P.C. (1988), *Non-linear Preference and Utility Theory*, Johns Hopkins University Press, Baltimore.

Fishburn, P.C. (1992a), "Signed orders and power set extensions", *Journal of Economic Theory* 56, 1–19.

Fishburn, P.C. (1992b), "Utility as an additive set function", *Mathematics of Operations Research* 17, 910–920.

Fishburn, P.C. (1992c), "A general axiomatization of additive measurement with applications", *Naval Research Logistics* 39, 741–755.

Fishburn, P.C., and LaValle, I.H. (1993), "Subset preferences in linear and non-linear utility theory", *Journal of Mathematical Psychology* 37, 611–623.

Fishburn, P.C., and LaValle, I.H. (1994), "Signed orders in linear and non-linear utility theory", preprint, AT&T Bell Laboratories, Murray Hill, NJ.

Gärdenfors, P. (1976), "Manipulation of social choice functions", *Journal of Economic Theory* 13, 217–228.

Garey, M., and Johnson, D. (1979), *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, San Francisco.

Glover, F. (1975), "Improved linear integer programming formulations of non-linear integer problems", *Management Science* 22, 455–460.

Herstein, I.N., and Milnor, J. (1953), "An axiomatic approach to measurable utility", *Econometrica* 21, 291–297.

Jensen, N. E. (1967), "An introduction to Bernoullian utility theory. I. Utility functions", *Swedish Journal of Economics* 69, 163–183.

Johnson, D.S. (1974), "Approximation algorithms for combinatorial problems", *Journal of Computer and System Sciences* 9, 256–278.

Kannai, Y., and Peleg, B. (1984), "A note on the extension of an order on a set to the power set", *Journal of Economic Theory* 32, 172–175.

Kettani, O., and Oral, M. (1993), "Reformulating quadratic assignment problems for efficient optimization", *IIE Transactions* 25, 97–107.

Oral, M., and Kettani, O. (1992), "A linearization procedure for quadratic and cubic mixed-integer problems", *Operations Research* 40, Supplement 1, S109–S116.

Oral, M., Kettani, O., and Lang, P. (1991), "A methodology for collective evaluation and selection of industrial R&D projects", *Management Science* 37, 871–885.

Pattanaik, P.K., and Peleg, B. (1984), "An axiomatic characterization of the lexicographic maximin extension of an ordering over a set to the power set", *Social Choice and Welfare* 1, 113–122.

Sahni, S.K. (1973), "On the knapsack and other computationally related problems", Ph.D. Dissertation, Cornell University, Ithaca, NY.

von Neumann, J., and Morgenstern, O. (1944), *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, NJ.