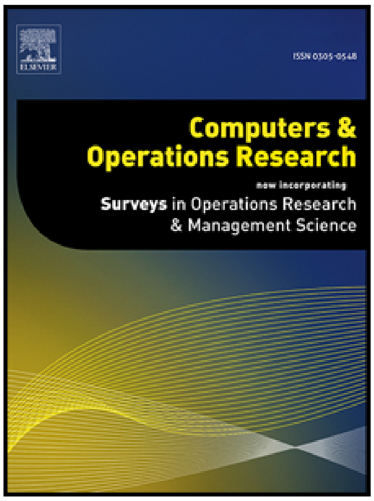

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Highlights

- An interactive algorithm for multi-objective integer programs is developed.
- The algorithm finds the most preferred point at a desired level of accuracy.
- The decision maker is assumed to have an underlying quasiconcave value function.
- Extensive computational experiments show the algorithm works very well.

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An Interactive Approximation Algorithm for Multi-objective Integer Programs

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Abstract: We develop an interactive algorithm that approximates the most preferred solution for any multi-objective integer program with a desired level of accuracy, provided that the decision maker's (DM's) preferences are consistent with a nondecreasing quasiconcave value function. Using pairwise comparisons of the DM, we construct convex cones and eliminate the inferior regions that are close to being dominated by the cones in addition to the regions dominated by the cones. The algorithm allows the DM to change the desired level of accuracy during the solution process. We test the performance of the algorithm on a set of multi-objective combinatorial optimization problems. It performs very well in terms of the quality of the solution found, the solution time, and the required preference information.

Keywords: multi-objective integer programming, approximation algorithm, interactive algorithm.

1. Introduction

Multi-objective integer programs (MOIPs) have many application areas in real life, such as facility location problems, scheduling problems, network design problems, routing problems, capital budgeting problems, and workforce planning problems. Since the decision maker (DM) has to deal with many conflicting criteria, MOIPs usually do not have a unique solution and are difficult to solve.

Several approaches have been developed to generate all nondominated points for MOIPs (see Özlen and Azizoğlu, 2009; Lokman and Köksalan, 2013; Kırlık and Sayın, 2014; Özlen et al., 2014; Dächert and Klamroth, 2015). Those methods work in a similar way and partition the solution space into a set of regions using bounds on the objectives. In the decomposition algorithm of Dächert and Klamroth (2015), they show that there exists a linear bound on the number of submodels to be solved for the three-criteria case.

Although the recently developed algorithms work efficiently for medium-sized problems, generating all nondominated points is not practical for many problems. The number of nondominated points increases substantially with the problem size (see Ehrgott and Gandibleux, 2000) and even if all those points are generated, the difficulty of comparing and choosing among a large number of points remains.

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The interactive approaches incorporate the preferences of the DM into the solution process and reduce the computational effort as well as the required information from the DM. The methods representing the preferences of the DM by an implicit value function aim to generate the optimal solution for MOIPs with respect to that value function (see Gonzales et al., 1985; Gabbani and Magazine, 1986; Karaivanova et al., 1993; Argyris et al., 2011). There are also approaches that are based on projecting a reference direction onto the nondominated set (see Narula and Vassilev, 1994; Alves and Clímaco, 2000). Based on the preferences of the DM, the aspiration levels for criteria are specified and a directional search is performed to find a new solution at each iteration. Alves and Clímaco (2007) present a review on the interactive methods for MOIPs.

Lokman et al. (2016) developed an interactive algorithm that finds the most preferred solution for MOIPs. Assuming that the pairwise comparisons of the DM are consistent with a quasiconcave value function, they generate convex cones and define the inferior regions with respect to the cones. Due to the increase in the number of nondominated points, the algorithm requires substantial computational effort for large-sized problems.

In this paper, we develop an algorithm to reduce the computational effort at the expense of a small deterioration in solution quality. Specifically, we approximate the most preferred solution of a MOIP for any desired level of accuracy by establishing regions that are possibly inferior. We first develop the theory to identify the regions that are possibly inferior or approximately cone dominated. We define a new distance metric based on the cone parameters and characterize the regions based on their "distance" from being dominated by the convex cones. In addition to the regions dominated by the cones, the algorithm eliminates approximately cone-dominated regions from further consideration and converges to a nondominated point with a performance guarantee. We make a weak assumption regarding the quasiconcavity of the value function. Differently from the existing studies, we generate a solution with a performance guarantee of minimal deviation from the most preferred solution. We test the performance of the algorithm on a set of multi-objective combinatorial optimization problems (MOCO), special MOIPs. The experiments on large problems show that the algorithm produces high-quality solutions. It requires a small number of pairwise comparisons and reasonable solution times.

We develop the requisite theory in Section 2 and present our algorithm in Section 3. We provide the results of the computational experiments in Section 4 and make concluding remarks in Section 5.

2. Definitions and some theory

A general MOIP with p objectives can be formulated as follows:

(P)

$$\text{"Max"} \left\{ z_1(\mathbf{x}), z_2(\mathbf{x}), ..., z_p(\mathbf{x}) \right\}$$

subject to

$\mathbf{x} \in \mathbf{X}$

In model (*P*), $\mathbf{x}$ denotes an *n*-dimensional integer decision vector and $z(\mathbf{x}) = (z_1(\mathbf{x}), z_2(\mathbf{x}), ..., z_p(\mathbf{x}))$ is the corresponding objective vector. $\mathbf{X}$ is the feasible set in the decision space and $\mathbf{Z} \in \Re^p$ is the image of $\mathbf{X}$ in the objective space. We assume $\mathbf{X}$ is bounded; and this is natural for practical problems. We use the quotation marks to indicate that the maximization of a vector is not a well-defined mathematical operation.

$\mathbf{z}(\mathbf{x}') \in \mathbf{Z}$ is said to *dominate* $\mathbf{z}(\mathbf{x}) \in \mathbf{Z}$, if $z_i(\mathbf{x}) \le z_i(\mathbf{x}')$ for all $i = 1, 2, ..., p$ and $z_i(\mathbf{x}) < z_i(\mathbf{x}')$ for at least one *i*. If there does not exist such an $\mathbf{x}'$, then $\mathbf{x}$ is said to be efficient and $\mathbf{z}(\mathbf{x})$ is said to be nondominated (see Steuer, 1986; p. 149).

The ideal point, $\mathbf{z}^{IP} = (z_1^{IP}, z_2^{IP}, ..., z_p^{IP})$, is composed of the best values of each objective over the feasible objective set $\mathbf{Z}$ while the nadir point, $\mathbf{z}^{NP} = (z_1^{NP}, z_2^{NP}, ..., z_p^{NP})$, is composed of the worst values of each objective over the set of nondominated points. That is, $z_i^{IP} = \max_{\mathbf{x} \in \mathbf{X}} (z_i(\mathbf{x})), \quad i = 1, ..., p$, and $z_i^{NP} = \min_{\mathbf{x} \in \mathbf{E}} (z_i(\mathbf{x})) \quad i = 1, ..., p$, where $\mathbf{E}$ is the set of all efficient solutions (see Ehrgott, 2005; p. 34). In addition to the various uses of the ideal and nadir points, they are useful in providing a perspective about the range of criterion values over the nondominated space.

If the DM has a nondecreasing value function, $f : \Re^p \to \Re^1$, $\mathbf{z}^m \in \mathbf{Z}$ is preferred to $\mathbf{z}^k \in \mathbf{Z}$ if and only if $f(\mathbf{z}^m) > f(\mathbf{z}^k)$. In this case, the most preferred point could be found by solving $\max_{\mathbf{x} \in \mathbf{X}} f(z(\mathbf{x}))$ (see Keeney and Raiffa, 1993; p. 68). The notation $\mathbf{z}^m \succ \mathbf{z}^k$ is used to denote that $\mathbf{z}^m$ is preferred to $\mathbf{z}^k$.

2.1. Theory of convex cones

Human preferences are considered to be represented well by nondecreasing quasiconcave value functions. These functions imply decreasing rates of marginal substitution which is a common behavioral pattern in a wide variety of decision situations. Using the properties of such functions and past preferences of a DM, Korhonen et al. (1984) defined convex cones to identify inferior regions.

Let $\mathbf{z}^m \in \Re^p \quad m = 1, 2, ..., q$ be q distinct points such that $\mathbf{z}^m \succ \mathbf{z}^k$ for some specific k, $m \ne k$. Korhonen et al. (1984) define the *q-point convex cone* that represents an inferior region as $\left\{ \mathbf{z} : \mathbf{z} = \mathbf{z}^k + \sum_{\substack{m=1 \\ m \ne k}}^{q} \mu_m (\mathbf{z}^k - \mathbf{z}^m), \quad \mu_m \ge 0 \;\; m \ne k \right\}$. They show that any point that is in this cone or is dominated by any point in the cone is at most as preferred as $\mathbf{z}^k$ and less preferred than $\mathbf{z}^m$, $m \ne k$. All such points are *cone-dominated points*.

There are many studies that use such cones to find the most preferred point of a DM for multi-objective choice problems for which all discrete alternatives are available at the outset (see for example, Köksalan et al., 1984; Köksalan, 1989; Köksalan and Taner, 1992; Fowler et al., 2010;

Dehnokhalaji et al., 2011). Ramesh et al. (1989) applied the theory in a branch and bound algorithm developed for MOIPs with two criteria.

Lokman et al. (2016) developed an interactive algorithm for MOIPs that iteratively constructs 2-point cones and reduces the solution space until the problem becomes infeasible. The algorithm guarantees to find the most preferred solution. They define their cones using the notation $C\left(\mathbf{z}^m;\mathbf{z}^k\right)=\left\{\mathbf{z}:\mathbf{z}=\mathbf{z}^k+\mu\left(\mathbf{z}^k-\mathbf{z}^m\right)\quad \mu\geq 0\right\}$ and the corresponding cone-dominated region as $D\left(\mathbf{z}^m;\mathbf{z}^k\right)=\left\{\mathbf{z}\in\Re^p:\mathbf{z}\ \leq\mathbf{z}',\ \mathbf{z}'\in C\left(\mathbf{z}^m;\mathbf{z}^k\right)\right\}$ where $\mathbf{z}^m$ and $\mathbf{z}^k$ are said to be *cone-generators* such that $\mathbf{z}^m\succ\mathbf{z}^k$. Let $S_{\leq}^{m,k}=\left\{i:z_i^k-z_i^m\leq 0\right\}$, $S_{>}^{m,k}=\left\{j:z_j^k-z_j^m>0\right\}$, and $S_{<}^{m,k}=\left\{i:z_i^k-z_i^m<0\right\}$. Verbally, $S_{\leq}^{m,k}$ denotes the set of the criteria in which $\mathbf{z}^m$ is greater than or equal to $\mathbf{z}^k$, and the other sets can be interpreted in a similar way. The cone-dominated region is defined as in Theorem 1.

Theorem 1 (Lokman et al., 2016). *Let f be a nondecreasing quasiconcave function defined in a p-dimensional Euclidean space, $\Re^p$. Consider two distinct nondominated points $\mathbf{z}^m$ and $\mathbf{z}^k$ such that $f\left(\mathbf{z}^k\right)<f\left(\mathbf{z}^m\right)$. Then, a point $\mathbf{z}$ is in or dominated by cone $C\left(\mathbf{z}^m;\mathbf{z}^k\right)$ if and only if the following conditions hold:*

(i) $\quad z_i\leq z_i^k \qquad \forall i\in S_{\leq}^{m,k}$

(ii) $\quad z_i\left(z_j^k-z_j^m\right)+z_j\left(z_i^m-z_i^k\right)\leq z_j^k z_i^m-z_i^k z_j^m \quad \forall i\in S_{\leq}^{m,k},\quad \forall j\in S_{>}^{m,k}$

Note that for $i\in S_{\leq}^{m,k}$ satisfying $z_i^m-z_i^k=0$, condition (ii) reduces to condition (i) since $z_j^k-z_j^m>0$. Hence, condition (ii) can be rewritten as

(ii') $\quad z_i\left(z_j^k-z_j^m\right)+z_j\left(z_i^m-z_i^k\right)\leq z_j^k z_i^m-z_i^k z_j^m \quad \forall i\in S_{<}^{m,k},\quad \forall j\in S_{>}^{m,k}$ without loss of generality.

Theorem 1 implies that $\mathbf{z}\notin D\left(\mathbf{z}^m;\mathbf{z}^k\right)$ if at least one of the following conditions holds:

(iii) There exists $i\in S_{\leq}^{m,k}$ satisfying $z_i^k<z_i$.

(iv) There exist $i\in S_{<}^{m,k}$ and $j\in S_{>}^{m,k}$ satisfying $z_j^k z_i^m-z_i^k z_j^m<z_i\left(z_j^k-z_j^m\right)+z_j\left(z_i^m-z_i^k\right)$.

Given $\mathbf{z}^1\succ\mathbf{z}^2$, we demonstrate these conditions for a two-objective problem in Figure 1 where constraints 1 and 2 correspond to conditions (iii) and (iv), respectively.

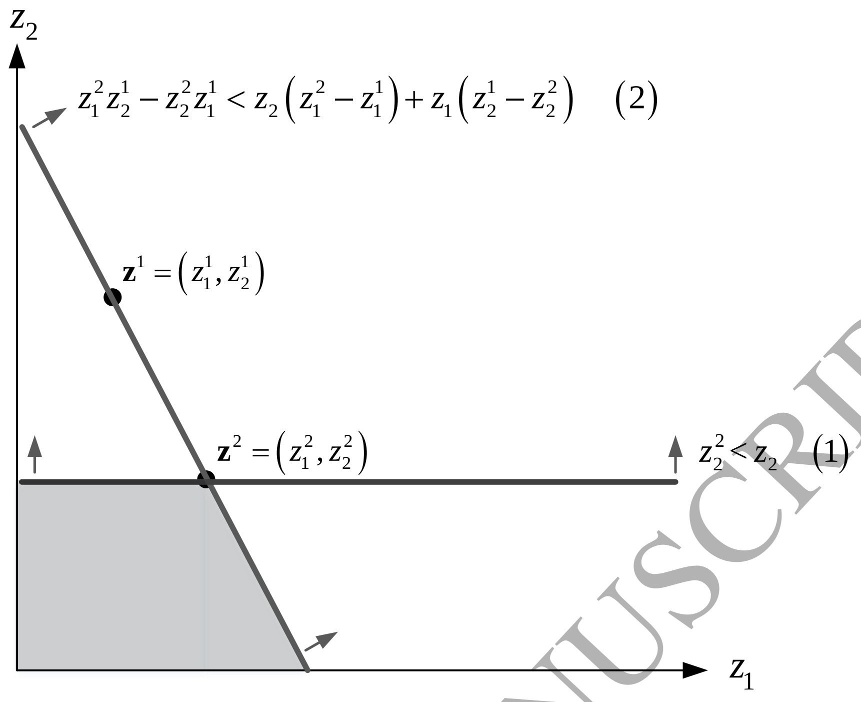


Figure 1. Cone dominated region (shaded) for a two objective problem, $D\left(\mathbf{z}^1;\mathbf{z}^2\right)$

2.2. Approximate cone dominance

Assuming that the DM's preferences are consistent with a quasiconcave value function, Prasad et al. (1997) define β-cone efficiency. Let β be the smallest nonnegative value for which $\mathbf{z}(1-\beta)$ is cone-dominated. β is a measure of how close a point is to the cone-dominated region and $\beta = 0$ indicates that $\mathbf{z}$ is cone-dominated. If $\beta > 0$, then $\mathbf{z}$ is said to be β-cone efficient. Dehnokhalaji et al. (2011) generalize the model of Prasad et al. (1997) and develop a procedure to find an approximate strict partial order for a given set of discrete alternatives. Given $\mathbf{z}^1 \succ \mathbf{z}^2$ and using the properties of a quasiconcave value function, they point out that the points in Region 4 are at most as good as $\mathbf{z}^2$ while the points in Region 3 are at least as good as $\mathbf{z}^2$ in Figure 2. For the points in Regions 1 and 2, the preferences are not known exactly. Therefore, they measure the distance of those points from Regions 3 and 4 and indicate the *possibly-better* or *possibly-worse* points than $\mathbf{z}^2$.

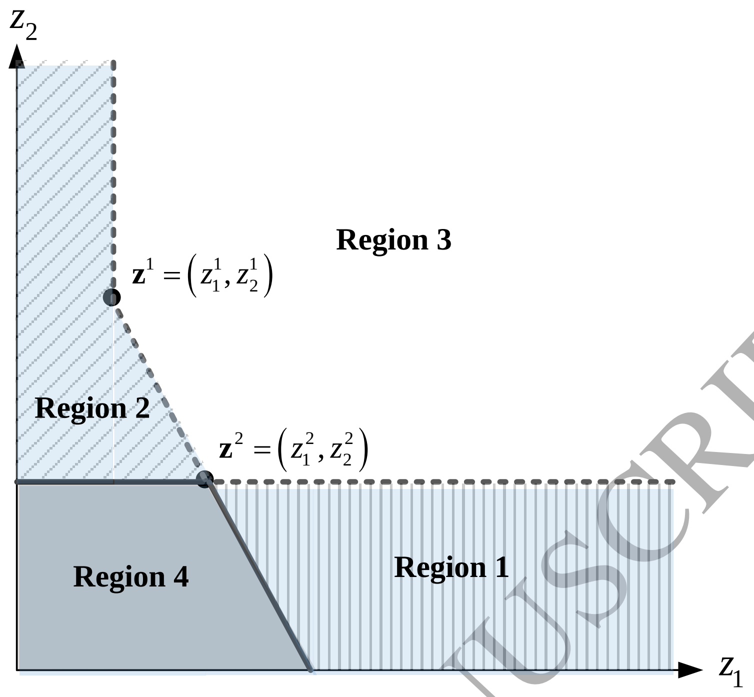


Figure 2. Two-point convex cone and classification of the regions ($\mathbf{z}^1 \succ \mathbf{z}^2$)

The ideas of cone dominance and β-cone efficiency have been developed for multi-objective choice problems. Since alternatives are explicitly known in choice problems, the alternatives that fall into these regions can be eliminated directly. The MOIP we address, on the other hand, is a multi-objective design problem for which the solutions are only implicitly known. Therefore, rather than eliminating alternatives, we need to characterize the regions containing solutions of interest to the DM. More specifically, we need to characterize the subset of the non-cone dominated regions that exclude the regions that are possibly worse than the known inferior solutions. We next give the necessary definitions and theory to characterize the desired regions using the desired precision parameter value, α.

Definition 1. *Consider two distinct nondominated points* $\mathbf{z}^m$ *and* $\mathbf{z}^k$ *such that* $f\left(\mathbf{z}^k\right) < f\left(\mathbf{z}^m\right)$. *Then, z is said to be α-cone-dominated with respect to* $C\left(\mathbf{z}^m;\mathbf{z}^k\right)$ *if and only if* $\alpha \geq 0$ *and*

(i') $\quad z_i \leq z_i^k \quad \forall i \in S_{\leq}^{m,k}$

(ii') $\quad z_i\left(z_j^k - z_j^m\right) + z_j\left(z_i^m - z_i^k\right) \leq \left(z_j^k z_i^m - z_i^k z_j^m\right)(1+\alpha) \quad \forall i \in S_{<}^{m,k}, \quad \forall j \in S_{>}^{m,k}$

As formalized in Definition 1 and illustrated in Figure 3, we identify α-cone-dominated regions by extending cone-dominated regions in the direction of possibly-worse regions based on the desired precision level. That is, we define a solution as α-cone-dominated based on its proximity to the cone-dominated regions as well as the characteristics of the region they lie in. Although a variation could be considered by extending the regions in all directions (i.e. by multiplying the right hand side value of constraints (i') by $(1+\alpha)$), this approach will eliminate the regions where the solutions are better than $\mathbf{z}^k$ in criteria $i \in S_{\leq}^{m,k}$ for which $\mathbf{z}^m$ is also better than or equal to $\mathbf{z}^k$, $S_{\leq}^{m,k} = \left\{ i : z_i^k - z_i^m \leq 0 \right\}$. These regions, in fact, correspond to the possibly-better regions where the preferred points may lie and eliminating such regions is likely to produce poor solutions.

Recall that $D\left(\mathbf{z}^m;\mathbf{z}^k\right)$ represents the cone-dominated region for cone $C\left(\mathbf{z}^m;\mathbf{z}^k\right)$. We can define the α-cone-dominated region as $D_\alpha\left(\mathbf{z}^m;\mathbf{z}^k\right)$. By construction, $D\left(\mathbf{z}^m;\mathbf{z}^k\right) \subseteq D_\alpha\left(\mathbf{z}^m;\mathbf{z}^k\right)$. These relations can be seen for the two-objective case in Figure 3.

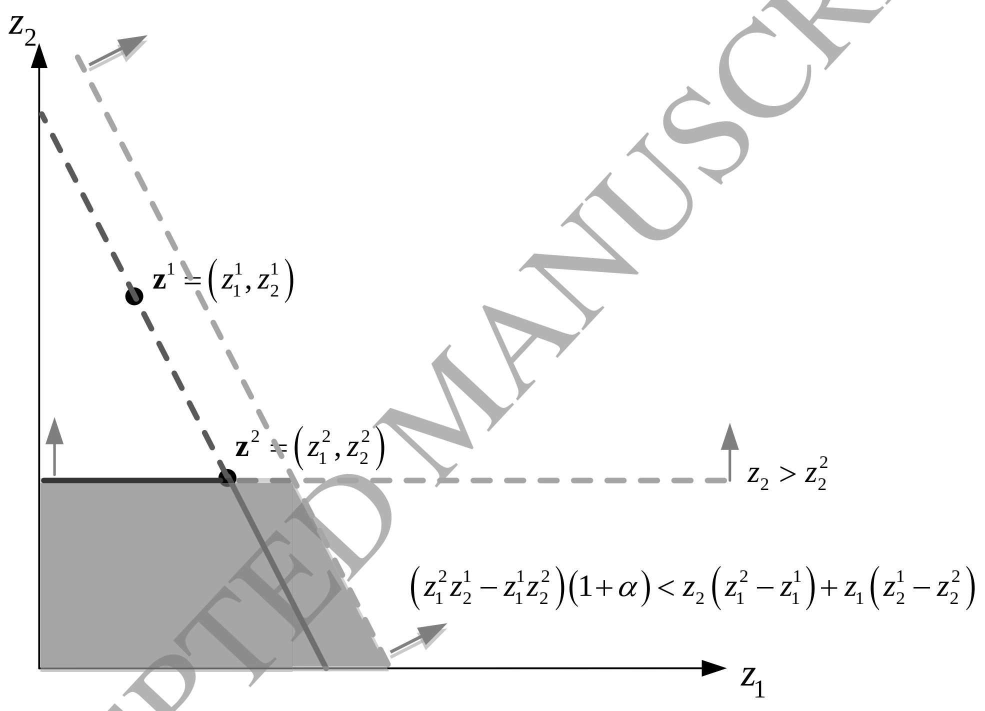


Figure 3. α-cone dominated region (shaded) for a two objective problem, $D_\alpha\left(\mathbf{z}^1;\mathbf{z}^2\right)$

In general, $\mathbf{z}$ is not α-cone-dominated, i.e., $\mathbf{z} \notin D_\alpha\left(\mathbf{z}^1;\mathbf{z}^2\right)$, if at least one of the following conditions holds:

(iii') There exists $i \in S_{\leq}^{m,k}$ satisfying $z_i^k < z_i$.

(iv') There exist $i \in S_{<}^{m,k}$ and $j \in S_{>}^{m,k}$ satisfying $\left(z_j^k z_i^m - z_i^k z_j^m\right)(1+\alpha) < z_i\left(z_j^k - z_j^m\right) + z_j\left(z_i^m - z_i^k\right)$.

For each $C(\mathbf{z}^m;\mathbf{z}^k)$, let $\Delta_\alpha(\mathbf{z}^m;\mathbf{z}^k)$ represent the region that is α-cone-dominated but not cone-dominated, i.e. $\Delta_\alpha(\mathbf{z}^m;\mathbf{z}^k) = D_\alpha(\mathbf{z}^m;\mathbf{z}^k) \setminus D(\mathbf{z}^m;\mathbf{z}^k)$. Using the definition of $D_\alpha(\mathbf{z}^m;\mathbf{z}^k)$ and $D(\mathbf{z}^m;\mathbf{z}^k)$, we characterize $\Delta_\alpha(\mathbf{z}^m;\mathbf{z}^k)$ in Proposition 1.

Proposition 1. *Consider two distinct nondominated points $\mathbf{z}^m$ and $\mathbf{z}^k$ such that $f(\mathbf{z}^k) < f(\mathbf{z}^m)$. $\mathbf{z} \in \Delta_\alpha(\mathbf{z}^m;\mathbf{z}^k)$ if and only if*

(a) $z_i \le z_i^k \quad \forall i \in S_{\le}^{m,k}$,

(b) $z_i(z_j^k - z_j^m) + z_j(z_i^m - z_i^k) \le (z_j^k z_i^m - z_i^k z_j^m)(1+\alpha) \quad \forall i \in S_{<}^{m,k}, \quad \forall j \in S_{>}^{m,k}$

(c) $z_i(z_j^k - z_j^m) + z_j(z_i^m - z_i^k) > (z_j^k z_i^m - z_i^k z_j^m)$ *for at least one* $i \in S_{<}^{m,k}$ *and* $j \in S_{>}^{m,k}$.

Proof. *The proof directly follows from the definition of $D_\alpha(\mathbf{z}^m;\mathbf{z}^k)$ (Definition 1) and Theorem 1.* ■

In order to measure the "distance" of a point from being dominated by $C(\mathbf{z}^m;\mathbf{z}^k)$, we use Tchebycheff metric i.e., $\left|\mathbf{z}^1 - \mathbf{z}^2\right| = \max_{i=1,\ldots,p}(z_i^1 - z_i^2)$ and derive a performance guarantee in Proposition 2.

Proposition 2. *Consider two distinct nondominated points $\mathbf{z}^m$ and $\mathbf{z}^k$ such that $f(\mathbf{z}^k) < f(\mathbf{z}^m)$. For any $\mathbf{z}(\alpha) \in \Delta_\alpha(\mathbf{z}^m;\mathbf{z}^k)$, there exists $\mathbf{z} \in D(\mathbf{z}^m;\mathbf{z}^k)$ satisfying*

$$\left|\mathbf{z}(\alpha) - \mathbf{z}\right| \le \max_{\forall i \in S_<^{m,k}, \forall j \in S_>^{m,k}} \left(\frac{\alpha(z_j^k z_i^m - z_i^k z_j^m)}{(z_i^m - z_i^k)} \right).$$

Proof. *Let $\mathbf{z}(\alpha) \in \Delta_\alpha(\mathbf{z}^m;\mathbf{z}^k)$. Let $\nu_{i,j}^{m,k}(\alpha) \ge 0$ be the slack in the inequality in condition (b) of Proposition 1:*

$$z_i(\alpha)(z_j^k - z_j^m) + z_j(\alpha)(z_i^m - z_i^k) = (z_j^k z_i^m - z_i^k z_j^m) + \alpha(z_j^k z_i^m - z_i^k z_j^m) - \nu_{i,j}^{m,k}(\alpha) \quad \forall i \in S_<^{m,k}, \forall j \in S_>^{m,k} \tag{3}$$

Since $z_i^m - z_i^k > 0$ for $i \in S_<^{m,k}$, multiplying and dividing the last two terms in (3) by $(z_i^m - z_i^k)$:

$$z_i(\alpha)(z_j^k - z_j^m) + z_j(\alpha)(z_i^m - z_i^k) = (z_j^k z_i^m - z_i^k z_j^m) + \frac{\left[\alpha(z_j^k z_i^m - z_i^k z_j^m) - \nu_{i,j}^{m,k}(\alpha)\right](z_i^m - z_i^k)}{(z_i^m - z_i^k)}$$

$\forall i \in S_<^{m,k}, \forall j \in S_>^{m,k}$.

Rearranging,

$$z_i(\alpha)\left(z_j^k - z_j^m\right) + \left[z_j(\alpha) - \frac{\alpha\left(z_j^k z_i^m - z_i^k z_j^m\right) - \nu_{i,j}^{m,k}(\alpha)}{\left(z_i^m - z_i^k\right)}\right]\left(z_i^m - z_i^k\right) = \left(z_j^k z_i^m - z_i^k z_j^m\right) \tag{4}$$
$$\forall i \in S_<^{m,k}, \quad \forall j \in S_>^{m,k}.$$

Let $d_j^{m,k}(\alpha) = \max\limits_{\forall i \in S_<^{m,k}} \dfrac{\alpha\left(z_j^k z_i^m - z_i^k z_j^m\right) - \nu_{i,j}^{m,k}(\alpha)}{\left(z_i^m - z_i^k\right)}$. *Then,*

$$z_i(\alpha)\left(z_j^k - z_j^m\right) + \left[z_j(\alpha) - d_j^{m,k}(\alpha)\right]\left(z_i^m - z_i^k\right) \le \left(z_j^k z_i^m - z_i^k z_j^m\right) \tag{5}$$
$$\forall i \in S_<^{m,k}, \quad \forall j \in S_>^{m,k}$$

holds. Let $\mathbf{z}$ *be such that* $z_i = z_i(\alpha)$ *for all* $i \in S_<^{m,k}$, *and* $z_j = z_j(\alpha) - d_j^{m,k}(\alpha)$ *for all* $j \in S_>^{m,k}$. *Substituting these in condition (a) of Proposition 1,*

$$z_i \le z_i^k \qquad \forall i \in S_\le^{m,k},$$

which corresponds to condition (i) of Theorem 1. Substituting $\mathbf{z}$ *in (5), we obtain:*

$$z_i\left(z_j^k - z_j^m\right) + z_j\left(z_i^m - z_i^k\right) \le \left(z_j^k z_i^m - z_i^k z_j^m\right) \qquad \forall i \in S_<^{m,k}, \quad \forall j \in S_>^{m,k}$$

that corresponds to condition (ii') of Theorem 1. Hence, $\mathbf{z} \in D\left(\mathbf{z}^m; \mathbf{z}^k\right)$ *and*

$$\left|\mathbf{z}(\alpha) - \mathbf{z}\right| = \max_{i=1,\dots,p}\left(z_i(\alpha) - z_i\right) = \max_{\forall j \in S_>^{m,k}} d_j^{m,k}(\alpha) \le \max_{\forall i \in S_<^{m,k}, \forall j \in S_>^{m,k}} \left(\frac{\alpha\left(z_j^k z_i^m - z_i^k z_j^m\right)}{\left(z_i^m - z_i^k\right)}\right)$$ *since* $\nu_{i,j}^{m,k}(\alpha) \ge 0$. ■

Note that the smaller the α value, the more precise the result is. The limiting case of $\alpha = 0$ corresponds to full precision and the approach becomes exact in this case.

Example. Let $\mathbf{z}^1 = (2,6)$, $\mathbf{z}^2 = (3,4)$, and assume $\mathbf{z}^1 \succ \mathbf{z}^2$. For $\alpha = 0.2$, employing Proposition 2, for each $\mathbf{z}(0.2) \in \Delta_{0.2}\left(\mathbf{z}^1; \mathbf{z}^2\right)$, there exists a corresponding inferior point $\mathbf{z}(0.2) \in D\left(\mathbf{z}^1; \mathbf{z}^2\right)$ at a distance satisfying $\left|\mathbf{z}(0.2) - \mathbf{z}\right| \le 1$ as shown in Figure 4.

This example shows that we establish an upper bound to the distance from being cone-dominated for each point that is not cone-dominated but is α-cone-dominated.

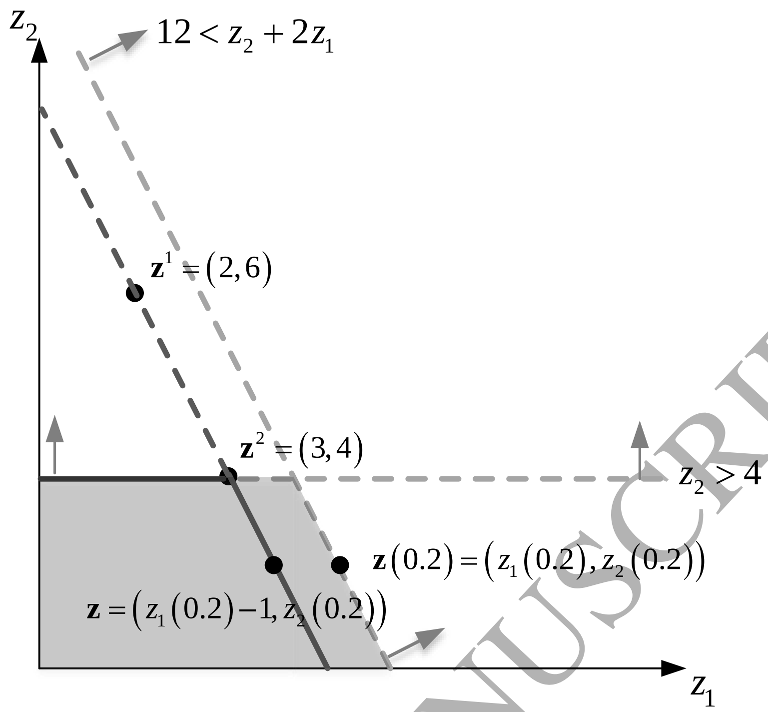


Figure 4. 0.2-cone dominated point and its corresponding inferior point

Interactive algorithms, typically, find a new solution in each iteration in search for improved solutions. In our case, given the previously-generated nondominated points and past preference information of the DM, we need to find new distinct nondominated points that are not α-cone-dominated. We next give a proposition that describes how such points can be generated, if they exist.

Proposition 3. *Let f be a nondecreasing quasiconcave function defined in a p-dimensional Euclidean space $\Re^p$. Consider distinct nondominated points $\mathbf{z}^k \in \mathbf{Z}$ $\quad k = 1, 2, ..., h-1, \quad h > 1$. Let $T = \{(\mathbf{z}^m, \mathbf{z}^k) : f(\mathbf{z}^k) < f(\mathbf{z}^m)\}$ be a set representing available preferences and $\mathbf{z}^{inc} = (z_1^{inc}, ..., z_p^{inc})$ be such that $f(\mathbf{z}^{inc}) = \underset{k=1,...,h-1}{\text{Max}} f(\mathbf{z}^k)$. Consider problem $(P^{h,\alpha})$:*

$\left(P^{h,\alpha}\right)$

$$\text{Max} \sum_{i=1}^{p} \lambda_i z_i\left(\mathbf{x}\right)$$

subject to

$$z_i^{inc} + \varepsilon - M_i^{inc}\left(1 - y_i\right) \le z_i\left(\mathbf{x}\right) \quad i = 1,..., p \tag{6}$$

$$\sum_{i=1}^{p} y_i = 1 \tag{7}$$

$$z_i^k + \varepsilon - M_i^{m,k}\left(1 - r_i^{m,k}\right) \le z_i\left(\mathbf{x}\right) \qquad \forall\left(\mathbf{z}^m, \mathbf{z}^k\right) \in T, \forall i \in S_{\le}^{m,k} \tag{8}$$

$$\left(z_j^k z_i^m - z_i^k z_j^m\right)\left(1+\alpha\right) + \varepsilon - M_{ij}^{m,k}\left(1 - t_{ij}^{m,k}\right) \le z_i\left(\mathbf{x}\right)\left(z_j^k - z_j^m\right) + z_j\left(\mathbf{x}\right)\left(z_i^m - z_i^k\right) \qquad \forall\left(\mathbf{z}^m, \mathbf{z}^k\right) \in T, \forall i \in S_{<}^{m,k}, \forall j \in S_{>}^{m,k} \tag{9}$$

$$\sum_{i \in S_{\le}^{m,k}} r_i^{m,k} + \sum_{\substack{i \in S_{<}^{m,k} \\ j \in S_{>}^{m,k}}} t_{ij}^{m,k} = 1 \qquad \forall\left(\mathbf{z}^m, \mathbf{z}^k\right) \in T \tag{10}$$

$$y_i \in \{0,1\} \qquad i = 1,..., p$$

$$r_i^{m,k} \in \{0,1\} \qquad \forall\left(\mathbf{z}^m, \mathbf{z}^k\right) \in T,\ \forall i \in S_{\le}^{m,k}, \forall j \in S_{>}^{m,k}$$

$$t_{ij}^{m,k} \in \{0,1\} \qquad \forall\left(\mathbf{z}^m, \mathbf{z}^k\right) \in T,\ \forall i \in S_{<}^{m,k}, \forall j \in S_{>}^{m,k}$$

$$\mathbf{x} \in \mathbf{X}$$

where $\lambda_i > 0$ is an arbitrary weight for objective i, ε is a small positive constant, and M_i^{inc}, M_i and M_{ij} are sufficiently large positive constants. If the solution space of $\left(P^{h,\alpha}\right)$ is not empty, then the optimal point, $\mathbf{z}^h$, is such that $\mathbf{z}^h \notin D_\alpha\left(\mathbf{z}^m;\mathbf{z}^k\right)$ for any $\left(\mathbf{z}^m,\mathbf{z}^k\right) \in T$ and $\mathbf{z}^h \ne \mathbf{z}^k$ $k = 1,...,h-1$.

Proof. *Let the solution space of $\left(P^{h,\alpha}\right)$ be not empty. Due to (6) and (7), $\varepsilon > 0$ and $\lambda_i > 0$ i=1,...,p, $\mathbf{z}^h$ is a nondominated point distinct from $\mathbf{z}^{inc}$. For each cone $C\left(\mathbf{z}^m;\mathbf{z}^k\right)$, if $r_i^{m,k} = 1$ for $i \in S_{\le}^{m,k}$, then constraint (8) reduces to $z_i^h \ge z_i^k + \varepsilon$, which is equivalent to condition (iii') for a sufficiently small $\varepsilon > 0$. Similarly, if $t_{ij}^{m,k} = 1$ for $i \in S_{<}^{m,k}$ and $j \in S_{>}^{m,k}$, then constraint (9) reduces to $z_i^h\left(z_j^k - z_j^m\right) + z_j^h\left(z_i^m - z_i^k\right) \ge \left(z_j^k z_i^m - z_i^k z_j^m\right)\left(1+\alpha\right) + \varepsilon$ which is equivalent to condition (iv') for a sufficiently small $\varepsilon > 0$. If $r_i^{m,k} = 0$ or $t_{ij}^{m,k} = 0$, the corresponding constraint becomes redundant. Constraint set (10) guarantees that $r_i^{m,k} = 1$ for some $i \in S_{\le}^{m,k}$ or $t_{ij}^{m,k} = 1$ for some $i \in S_{<}^{m,k}$ and $j \in S_{>}^{m,k}$, implying that conditions (iii') or (iv') hold for each $C\left(\mathbf{z}^m;\mathbf{z}^k\right)$. Then, $\mathbf{z}^h$ is not α-cone-dominated, i.e., $\mathbf{z}^h \notin D_\alpha\left(\mathbf{z}^m;\mathbf{z}^k\right)$ for any $\left(\mathbf{z}^m,\mathbf{z}^k\right) \in T$. Since $\mathbf{z}^k \in D_\alpha\left(\mathbf{z}^m;\mathbf{z}^k\right)$ for $\mathbf{z}^k \ne \mathbf{z}^{inc}$, $\mathbf{z}^h \ne \mathbf{z}^k$ $k = 1,...,h-1$.* ■

In $\left(P^{h,\alpha}\right)$, if $z_i \in \mathbb{Z}^+ \; i=1,...,p$ for each $\mathbf{z} \in \mathbf{Z}$, setting $\varepsilon = 1$ is sufficient. In this case, we may also set $M_i^{inc} = z_i^{inc} + 1$ for all $i = 1,..., p$, $M_i^{m,k} = z_i^k + 1$ for all $\left(\mathbf{z}^m, \mathbf{z}^k\right) \in T$, $i \in S_{\le}^{m,k}$ and $j \in S_{>}^{m,k}$, and $M_{ij}^{m,k} = \left(z_j^k z_i^m - z_i^k z_j^m\right) + 1$ for all $\left(\mathbf{z}^m, \mathbf{z}^k\right) \in T$, $i \in S_{<}^{m,k}$ and $j \in S_{>}^{m,k}$. Then, (6), (8) and (9) simplify to:

$$\left(z_i^{inc} + 1\right) y_i \le z_i\left(\mathbf{x}\right) \quad i = 1,..., p \tag{6'}$$

$$\left(z_i^k + 1\right) r_i^{m,k} \le z_i\left(\mathbf{x}\right) \qquad \forall\left(\mathbf{z}^m, \mathbf{z}^k\right) \in T, \forall i \in S_{\le}^{m,k} \tag{8'}$$

$$\left[\left(z_j^k z_i^m - z_i^k z_j^m\right)\left(1+\alpha\right) + 1\right] t_{ij}^{m,k} \le z_i\left(\mathbf{x}\right)\left(z_j^k - z_j^m\right) + z_j\left(\mathbf{x}\right)\left(z_i^m - z_i^k\right) \quad \forall\left(\mathbf{z}^m, \mathbf{z}^k\right) \in T, \forall i \in S_{\le}^{m,k}, \forall j \in S_{>}^{m,k} \tag{9'}$$

Proposition 3 shows that if $\left(P^{h,\alpha}\right)$ is feasible, it generates a new non-*α*-cone-dominated challenger, $\mathbf{z}^h$, at iteration *h*. Since new constraints cut out parts of the bounded feasible set at each iteration, the remaining feasible set becomes empty in a finite number of iterations. Therefore, if we keep on updating $\left(P^{h,\alpha}\right)$, it becomes infeasible at some iteration indicating that there is no nondominated point outside the *α*-cone dominated regions.

3. The Algorithm

Based on our theory, we next develop an interactive algorithm that approximates the most preferred point of a MOIP using the desired precision parameter value, *α*. The algorithm iteratively generates new nondominated points that are not *α*-cone-dominated, and keeps the best-known point so far as the incumbent. At each iteration, the DM is asked to compare a new challenger with the incumbent, replacing the incumbent if necessary. Based on the responses of the DM, new approximate cones are constructed.

We next discuss the steps of the algorithm.

Step 0 (Initialization). Let α be the desired precision parameter value. Initialize $T = \varnothing$ and $h = 1$. Select a weight $\lambda_i > 0$ for each objective *i* arbitrarily and solve:

$$\text{Max} \sum_{i=1}^{p} \lambda_i z_i\left(\mathbf{x}\right)$$

subject to

$$\mathbf{x} \in \mathbf{X}$$

If the model is infeasible, stop. There does not exist any feasible point to the problem. Otherwise, denote the objective vector corresponding to the optimal solution as $\mathbf{z}^1$ and set $\mathbf{z}^{inc} = \mathbf{z}^1$.

Step 1 (Find a new point). Set $h \leftarrow h+1$. Solve $\left(P^{h,\alpha}\right)$. If the model is infeasible, go to Step 3. Otherwise, denote the objective vector corresponding to the optimal solution as the challenger, $\mathbf{z}^h$, and go to Step 2.

Step 2 (Comparison). Ask the DM to compare the incumbent, $\mathbf{z}^{inc}$, with $\mathbf{z}^h$. If $\mathbf{z}^{inc} \succ \mathbf{z}^h$, then $T = T \cup \left\{\left(\mathbf{z}^{inc}, \mathbf{z}^h\right)\right\}$. If $\mathbf{z}^h \succ \mathbf{z}^{inc}$, then $T = T \cup \left\{\bigcup_{k=1}^{h-1}\left(\mathbf{z}^h, \mathbf{z}^k\right)\right\}$ and $\mathbf{z}^{inc} = \mathbf{z}^h$. Go to Step 1.

Step 3. Stop. The incumbent point, $\mathbf{z}^{inc} = \left(z_1^{inc}, ..., z_p^{inc}\right)$, is the final point produced by the algorithm.

In order to efficiently search and reduce the feasible region, we utilize two of the mechanisms Lokman et al. (2016) develop. The aim is to construct less overlapping cones to reduce the feasible set faster. The first mechanism directly replaces the incumbent with the new challenger, if the incumbent has not been updated for several iterations. This helps generate substantially different cones by changing cone-generators totally. Since the original incumbent is still an eligible point, we maintain it as the super-incumbent. When another point qualifies as a potential super incumbent, we ask the DM to compare it with the current super incumbent, and make the preferred one the new super incumbent. At termination, we ask the DM to compare the super incumbent with the incumbent to determine the final point. The next mechanism we borrow from Lokman et al. (2016) is to reflect the past preferences of the DM in selecting the λ_i values in $\left(P^{h,\alpha}\right)$. The aim is to generate a $\boldsymbol{\lambda} = \left(\lambda_1, \lambda_2, ...\lambda_p\right)$ vector that is most consistent with the responses of the DM relative to a linear value function.

The number of binary variables and constraints in $\left(P^{h,\alpha}\right)$ depend on the number of objectives and the number of convex cones generated so far. At iteration *h*, $(h-1)$ new convex cones are generated if the incumbent is updated, and a single new convex cone is generated, otherwise. For each constructed convex cone, a number of binary variables, *NB*, and a number of constraints, *NC*, are included to $\left(P^{h,\alpha}\right)$ where $p \le NB = NC - 1 \le \left(\frac{p+1}{2}\right)^2$ if *p* is odd and $p \le NB = NC - 1 \le \left(\frac{p}{2}\right)\left(\frac{p}{2}+1\right)$ if *p* is even. Therefore, the number of binary variables and constraints grow quadratically with the number of objectives and linearly with the number of iterations. However, Lokman et al. (2016) discovered that many cones turned out to be redundant and established a procedure to identify and eliminate redundant cones. We also use the same procedure to eliminate redundant cones and hence improve computational efficiency.

In our algorithm, we fix the α value throughout the algorithm. There are other possible variations where the α value could be changed at different stages by the analyst or the DM. The way the α value is changed has different implications on the desired level of accuracy and developing such variations awaits further research.

If α is set to zero for all constructed cones, then the algorithm guarantees to find the most preferred point. If $\alpha > 0$ for at least one of those cones, then the point found by the algorithm may still be the most preferred point. The most preferred point will be missed only if it is located in one of the α-

cone-dominated regions as formally stated in Proposition 4. When this is the case, we can provide a worst case level of accuracy of the final point, which we derive in Proposition 5.

Proposition 4. *Let T be the preference set at the termination of the algorithm and* $\mathbf{z}^{inc} = \left(z_1^{inc}, ..., z_p^{inc}\right)$ *be the final point. Let* $\mathbf{z}^* \in \mathbf{Z}$ *be such that* $f\left(\mathbf{z}^*\right) = \underset{\mathbf{x} \in \mathbf{X}}{\text{Max}}\ f\left(\mathbf{z}\left(\mathbf{x}\right)\right)$, $\alpha > 0$, *and* $f\left(\mathbf{z}^*\right) > f\left(\mathbf{z}^{inc}\right)$. *Then,* $\mathbf{z}^* \in \Delta_\alpha\left(\mathbf{z}^m;\mathbf{z}^k\right)$ *for at least one* $\left(\mathbf{z}^m,\mathbf{z}^k\right) \in T$.

Proof. *Suppose* $\mathbf{z}^* \notin \Delta_\alpha\left(\mathbf{z}^m;\mathbf{z}^k\right)$ *for any* $\left(\mathbf{z}^m,\mathbf{z}^k\right) \in T$. *Then, for each* $\left(\mathbf{z}^m,\mathbf{z}^k\right) \in T$, *there are two possible cases:* $\mathbf{z}^* \in D\left(\mathbf{z}^m;\mathbf{z}^k\right)$ *or* $\mathbf{z}^* \notin D_\alpha\left(\mathbf{z}^m;\mathbf{z}^k\right)$. $\mathbf{z}^* \in D\left(\mathbf{z}^m;\mathbf{z}^k\right)$ *is not possible for any* $\left(\mathbf{z}^m,\mathbf{z}^k\right) \in T$ *since* $\mathbf{z}^*$ *cannot be inferior to any feasible point. Then,* $\mathbf{z}^* \notin D_\alpha\left(\mathbf{z}^m;\mathbf{z}^k\right)$ *must hold for all* $\left(\mathbf{z}^m,\mathbf{z}^k\right) \in T$. *In this case,* $\left(P^{h,\alpha}\right)$ *would be feasible and the algorithm would not terminate. This is a contradiction, therefore,* $\mathbf{z}^* \in \Delta_\alpha\left(\mathbf{z}^m;\mathbf{z}^k\right)$ *for at least one* $\left(\mathbf{z}^m,\mathbf{z}^k\right) \in T$. ■

Proposition 4 implies that the algorithm does not find the most preferred point if it is not cone-dominated but α-cone dominated. When this is the case, an upper bound on the Tchebycheff distance of the most preferred point to being dominated by one of the existing cones is given by Proposition 5.

Proposition 5. *Let T be the preference set at the termination of the algorithm and* $\mathbf{z}^{inc} = \left(z_1^{inc}, ..., z_p^{inc}\right)$ *be the final point. Let* $\mathbf{z}^* \in \mathbf{Z}$ *be such that* $f\left(\mathbf{z}^*\right) = \underset{\mathbf{x} \in \mathbf{X}}{\text{Max}}\ f\left(\mathbf{z}\left(\mathbf{x}\right)\right)$, $\alpha > 0$, *and* $f\left(\mathbf{z}^*\right) > f\left(\mathbf{z}^{inc}\right)$. *Then, there exists* $\mathbf{z} \in D\left(\mathbf{z}^m;\mathbf{z}^k\right)$ *for at least one* $\left(\mathbf{z}^m,\mathbf{z}^k\right) \in T$ *and*

$$\left|\mathbf{z}^* - \mathbf{z}\right| \le \max_{\left(\mathbf{z}^m,\mathbf{z}^k\right) \in T}\left[\max_{\forall i \in S_<^{m,k}, \forall j \in S_>^{m,k}}\left(\frac{\alpha\left(z_j^k z_i^m - z_i^k z_j^m\right)}{\left(z_i^m - z_i^k\right)}\right)\right].$$

Proof. *Based on Proposition 4,* $\mathbf{z}^* \in \Delta_\alpha\left(\mathbf{z}^m;\mathbf{z}^k\right)$ *for at least one* $\left(\mathbf{z}^m,\mathbf{z}^k\right) \in T$. *Let* $\left(\mathbf{z}^{m*},\mathbf{z}^{k*}\right) \in T$ *be one such pair. Then, there exists* $\mathbf{z} \in D\left(\mathbf{z}^{m*};\mathbf{z}^{k*}\right)$ *satisfying*

$$\left|\mathbf{z}^* - \mathbf{z}\right| \le \max_{\forall i \in S_<^{m,k}, \forall j \in S_>^{m,k}}\left(\frac{\alpha\left(z_j^{k*} z_i^{m*} - z_i^{k*} z_j^{m*}\right)}{\left(z_i^{m*} - z_i^{k*}\right)}\right) \qquad (11)$$

based on Proposition 2.

Since $\max_{\forall i \in S_<^{m,k}, \forall j \in S_>^{m,k}}\left(\frac{\alpha\left(z_j^{k*} z_i^{m*} - z_i^{k*} z_j^{m*}\right)}{\left(z_i^{m*} - z_i^{k*}\right)}\right) \le \max_{\left(\mathbf{z}^m,\mathbf{z}^k\right) \in T}\left[\max_{\forall i \in S_<^{m,k}, \forall j \in S_>^{m,k}}\left(\frac{\alpha\left(z_j^k z_i^m - z_i^k z_j^m\right)}{\left(z_i^m - z_i^k\right)}\right)\right]$,

$$\left|\mathbf{z}^* - \mathbf{z}\right| \le \max_{\left(\mathbf{z}^m, \mathbf{z}^k\right) \in T} \left[\max_{\forall i \in S_<^{m,k}, \forall j \in S_>^{m,k}} \left(\frac{\alpha\left(z_j^k z_i^m - z_i^k z_j^m\right)}{\left(z_i^m - z_i^k\right)} \right) \right]. \qquad \blacksquare$$

Proposition 5 presents a worst case level of accuracy for a given α value. If the DM decides to change the desired precision level during the algorithm, the algorithm can adapt to this change without repeating any previous calculations. All that is needed is for the analyst to modify the α-parameter for all existing convex cones. That is, only the constraint set (9) in $\left(P^{h,\alpha}\right)$ needs to be revised at that iteration. The algorithm always terminates satisfying the level of accuracy defined most recently.

We demonstrate the details of the algorithm on an example problem in the appendix.

4. Computational Experiments

We apply our algorithm to multi-objective assignment problems (MOAPs), and multi-objective knapsack problems (MOKPs), which we formulate as maximization problems. We use the random generation scheme of Lokman et al. (2016). We simulate the responses of the DM using three different underlying value functions:

(i) $f(\mathbf{z}) = \sum_{i=1}^{p} w_i z_i$,

(ii) $f(\mathbf{z}) = \sum_{i=1}^{p} -w_i^2 \left(z_i - z_i^{IP}\right)^2$

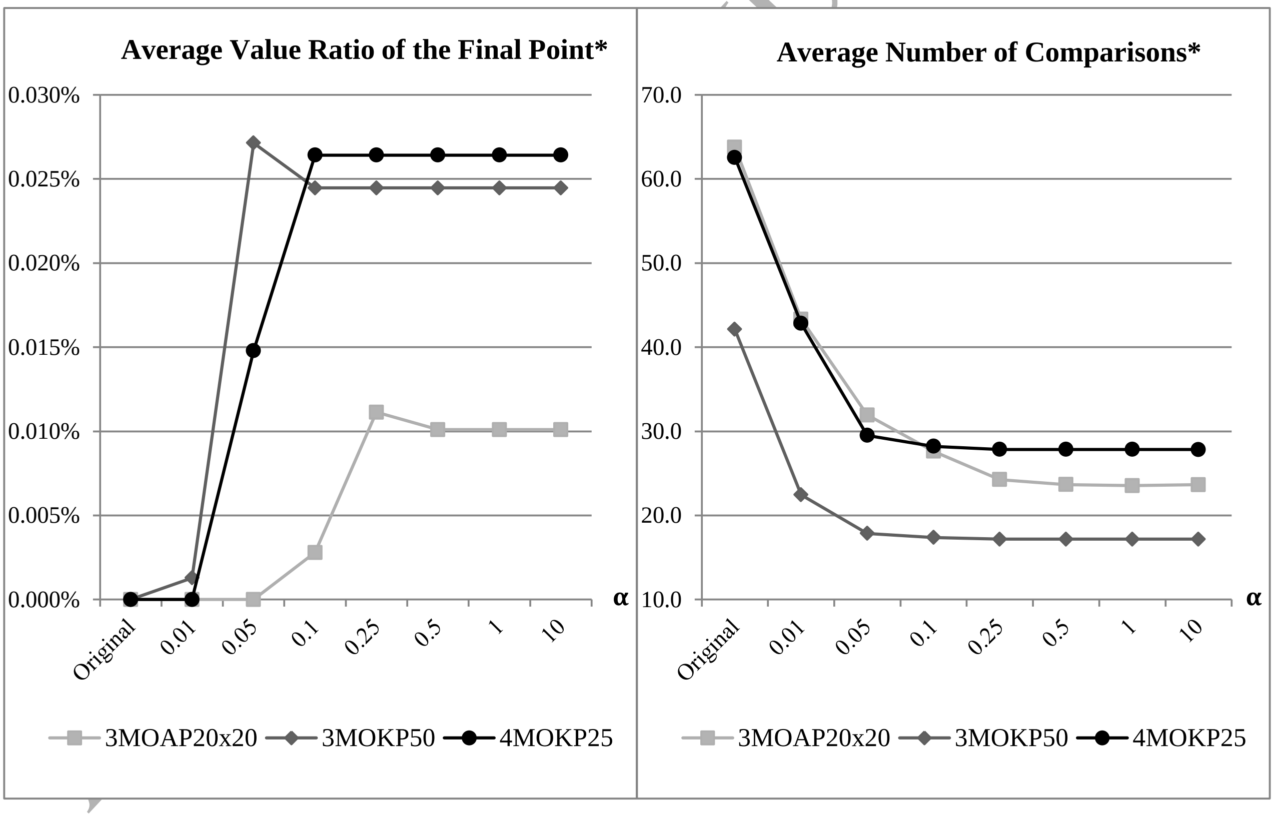
(iii) $f(\mathbf{z}) = \min_{i=1,\dots,p} \left(w_i \left(z_i - z_i^{IP}\right)\right)$

with three different weight vectors: $\mathbf{w}^1 = (0.7, 0.2, 0.1)$, $\mathbf{w}^2 = (0.1, 0.6, 0.3)$ and $\mathbf{w}^3 = (0.333, 0.333, 0.333)$ for three-criteria problems. For each instance, we make 10 replications and hence for each problem type, we solve 90 problems (3 types of value functions, 3 weight sets, 10 replications). For four-criteria problems, we use four different weight vectors: $\mathbf{w}^1 = (0.4, 0.3, 0.2, 0.1)$, $\mathbf{w}^2 = (0.1, 0.7, 0.1, 0.1)$, $\mathbf{w}^3 = (0.25, 0.25, 0.25, 0.25)$, and $\mathbf{w}^4 = (0.1, 0.1, 0.4, 0.4)$, and solve 60 problems for each instance (3 types of value functions, 4 weight sets, 5 replications). We coded the algorithm using Microsoft Visual Studio 2010 by C++ programming language with the callable library of CPLEX 12.3. The experiments are conducted on 64-bit Microsoft Windows 7 Professional installed on an Intel (R) Core (TM) i5-2410M CPU @ 2.30GHz computer with 4.00 GB RAM.

To set the value of the desired precision parameter, α, we conducted preliminary experiments on medium-sized problems: three-criteria 50-item MOKPs (3MOKP50s), three-criteria 20x20-size MOAPs (3MOAP20x20s), and four-criteria 25-item MOKPs (4MOKP25s), for which we had generated all nondominated points and hence had the most preferred solution at hand. $\overline{N}$ denotes the total number of nondominated points on the average. Figure 5 shows the number of comparisons and the quality of the final points as a function of α. We measure the quality of the final point using:

$$\text{Value Ratio }\left(\mathbf{z}^{inc}\right)=\frac{f\left(\mathbf{z}^*\right)-f\left(\mathbf{z}^{inc}\right)}{f\left(\mathbf{z}^*\right)-f\left(\mathbf{z}^{NP}\right)}$$

Although the problems and sizes are different, all results show a similar trend. As α increases, the number of comparisons decreases dramatically, while the quality of the incumbent point deteriorates only slightly. Based on these results, we set $\alpha = 0.05$, which seems to be a point at which results stabilize. Another important point is that the algorithm's results do not change much beyond $\alpha = 0.25$. This is expected, since the algorithm is designed to keep the solution quality at a certain level. In order to do that, it keeps all possibly-better regions and eliminates inferior regions regardless of the desired precision level. It only excludes a portion of the possibly-worse regions based on the value of α. Hence, after all points in possibly-worse regions are eliminated at a certain precision level, the algorithm only focuses on possibly-better regions and increasing the value of α beyond that level does not affect the performance of the algorithm.



* Values over 90 problems (3 types of value functions x 3 weight sets x 10 replications) for three-criteria problems and 60 problems (3 types of value functions x 4 weight sets x 5 replications) for four-criteria problems

Figure 5. Experiments on 3MOKP50s ($\overline{N}$ =520.9), 3MOAP20x20s ($\overline{N}$ =1908.5) and

4MOKP25s ($\overline{N}$ =178.2)

We compared the performance of our approximation algorithm for $\alpha = 0.05$ with that of the exact algorithm of Lokman et al. (2016). Table 1 presents the results. The approximation algorithm outperforms the exact algorithm substantially in terms of the solution time and produces either the most preferred point or a point that is very close to the most preferred point in terms of its value ratio. For example, when the value of α is set to 0.05 for three-criteria MOAPs, the solution time and the number of comparisons reduce by 80.74% and 45.30% on the average, respectively compared to the exact algorithm. While enjoying these savings, the approximation algorithm finds the most preferred point in 99.26% of the corresponding 270 instances and the average value ratio of the final point is 0.0006%. Similarly, for three-criteria MOKPs, the solution time improves by 86.65% and the number of comparisons decreases by 49.78% on the average when $\alpha = 0.05$. For these instances, the algorithm converges to the best point in 94.44% of the corresponding 270 instances and the average value ratio of the final point is 0.0370%. In Table 1, we also observe that the magnitude of the improvements get larger as the problem size increases, especially in computation times. For instance, while the solution time improves by 66.45% for 3MOAP10x10s, the improvement percentage increases to 88.28% for 3MOAP30x30s.

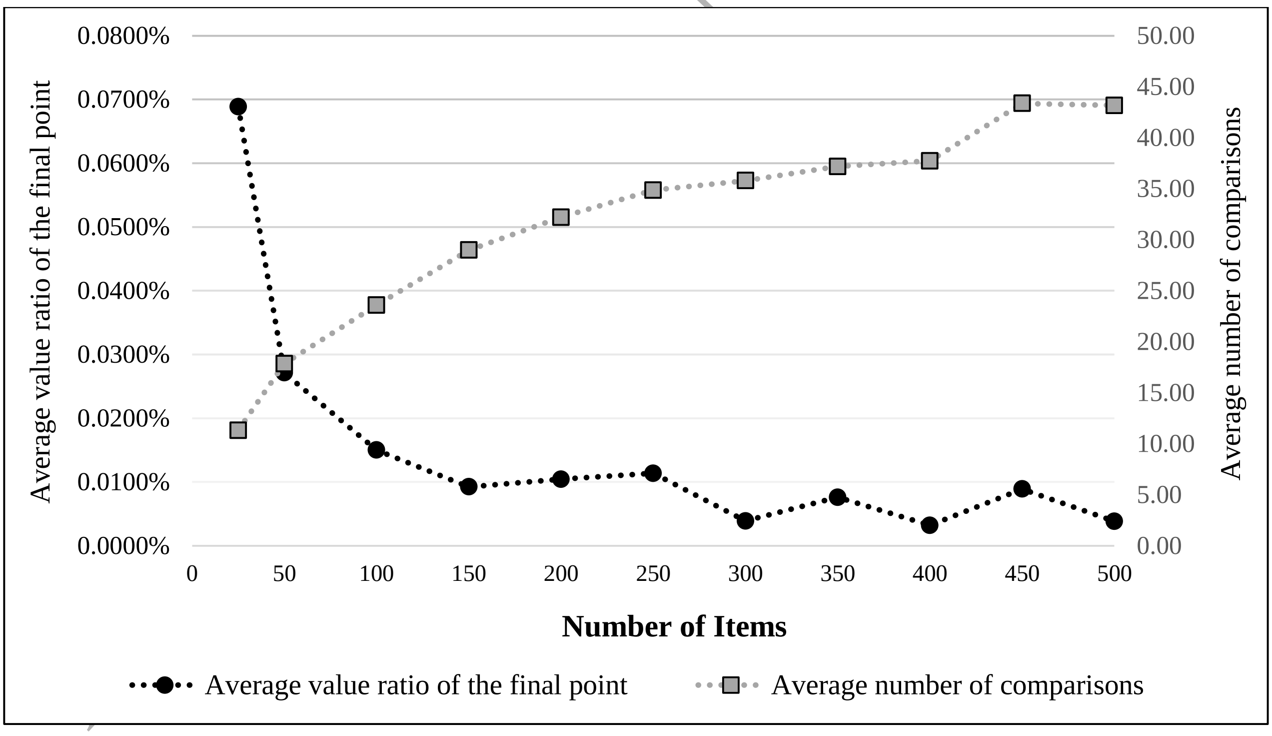
Table 1. Comparison of the approximation algorithm ($\alpha = 0.05$) with the exact algorithm on three-criteria problems

Problem	**Size**	**Total nond. Points**	**Method**	**Number of comp.s**	**Sol. time (sec.s)**	**Max. true rank**	**Best freq. %**	**Value ratio of the final point* %**	
								Avg.	**Max.**
MOAP	10x10	189.40	Exact	32.78	10.52	1.0	100.00	0.0000	0.0000
			Approx.	20.73	2.85	2.0	98.89	0.0016	0.1460
	20x20	1908.50	Exact	63.78	774.70	1.0	100.00	0.0000	0.0000
			Approx.	31.93	19.48	1.0	100.00	0.0000	0.0000
	30x30	5235.00	Exact	82.80	2299.05	1.0	100.00	0.0000	0.0000
			Approx.	34.78	45.02	2.0	98.89	0.0001	0.0107
MOKP	25	57.40	Exact	18.04	2.68	1.0	100.00	0.0000	0.0000
			Approx.	11.33	0.69	3.0	97.78	0.0689	4.4390
	50	520.90	Exact	42.13	64.04	1.0	100.00	0.0000	0.0000
			Approx.	17.86	2.03	9.0	92.22	0.0271	1.1870
	100	3768.20	Exact	71.03	1549.69	1.0	100.00	0.0000	0.0000
			Approx.	23.56	5.01	14.0	93.33	0.0150	0.6575

* Values over 90 problems (3 types of value functions x 3 weight sets x 10 replications)

In order to see how the performance of the algorithm changes with the problem size, we conduct further experiments on larger-sized problems. For comparison purposes, we found the most preferred point by maximizing the underlying value function using a commercial solver on the problems for which we could not generate the whole nondominated set within a time limit of 6 hours using the algorithm of Lokman and Köksalan (2013). We coded the algorithms on a similar computer using Microsoft Visual Studio 2010 by C++ programming language with the callable library of CPLEX 12.6. The experiments are conducted on 64-bit Microsoft Windows 7 Professional installed on an Intel (R) Core (TM) i7-4790 CPU @ 3.60GHz computer with 8.00 GB RAM. The results show that our approximation algorithm works well on large-sized problems as well. For instance, it converges to the most preferred point in 89 of the 90 instances of 3MOAP50x50s using 41.04 pairwise comparisons and 88.99 seconds of solution time on the average. The value ratio of the final point is between [0.0000%, 0.1097%] with an average value of 0.0012%.

For MOKPs, we summarize our results in Table 2 and demonstrate how the performance of the approximation algorithm changes with the number of items in Figure 6. The results on three-criteria MOKPs show that average number of comparisons and solution time do not increase dramatically with the problem size. As shown in the figure, the rate of the increase in the number of comparisons decreases while the average quality of the final point improves as the problem becomes larger.



* Values over 90 problems (3 types of value functions x 3 weight sets x 10 replications)

Figure 6. Experiments on MOKPs with three criteria, 3MOKPs

We also extend our computational study for the four-criteria case. We generated these problems as discussed at the beginning of this section. We try to understand how four-criteria results compare with those of three-criteria. Since the results for 4MOKP25s show a similar trend with those of Figure 5, we set $\alpha = 0.05$ and present our results in Table 2. The results show that increasing the number of objectives to four has a negative impact on both the solution times and the number of comparisons. Furthermore, the increase seems to be more than proportional as the number of items increases. That is, both the number of comparisons and the solution time ratios between three- and four-objective problems increase as the problem size increases. This is expected since in general the number of nondominated points increases exponentially with the problem size. Although the problems become more difficult to solve, the solution quality of the final points seems to be similar, indicating that the solution quality is robust to the number of criteria.

Table 2. Performance of the approximation algorithm ($\alpha = 0.05$) on 3MOKPs and 4MOKPs

Number of obj.s	Size	Number of comp.s	Sol. time (sec.s)	Best freq. %	Value ratio of the final point* %	
					Avg.	Max.
3	25	11.32	2.25	97.78%	0.0689	4.4390
	50	17.86	4.36	92.22%	0.0271	1.1870
	100	23.60	7.00	93.33%	0.0150	0.6576
	150	28.99	10.23	94.44%	0.0093	0.3005
	200	32.21	15.37	86.67%	0.0104	0.4010
	250	34.87	18.94	83.33%	0.0114	0.3391
	300	35.80	23.54	88.89%	0.0039	0.1493
	350	37.19	25.53	83.33%	0.0076	0.2098
	400	37.73	29.46	87.78%	0.0032	0.0914
	450	43.37	37.67	81.11%	0.0089	0.3137
	500	43.17	53.38	84.44%	0.0039	0.0732
4	25	29.52	8.52	98.33%	0.0148	0.8878
	50	63.60	45.31	93.33%	0.0250	1.0634
	100	155.13	394.65	83.33%	0.0507	0.7147
	150	249.80	2044.18	73.33%	0.0218	0.1693

* Values over 90 problems (3 types of value functions x 3 weight sets x 10 replications) for three-criteria problems and 60 problems (3 types of value functions x 4 weight sets x 5 replications) for four-criteria problems

5. Conclusions

We develop an efficient interactive algorithm for MOIPs that finds a highly-preferred point at a desired level of accuracy. The algorithm utilizes responses of the DM to pairwise comparisons in order to construct preference cones. We develop the requisite theory to define the admissible regions of the MOIPs characterizing the complement of the inferior regions and the regions that are close to being inferior for a desired precision parameter, α.

We conduct an extensive computational study on MOCO problems with different sizes. The results show that the algorithm finds the most preferred point in a vast majority of the instances and produces highly preferred solutions in the remaining instances. Based on our experiments on large problems, the algorithm is promising for large-scale problems from practice considering the required information from the DM and solution times.

The value of the precision parameter, α, is initialized at the beginning and kept constant throughout the algorithm. As an extension, it is possible to start with $\alpha = 0$, and then increase its value whenever the incumbent does not improve for a number of iterations, with the expectation that the current solution is highly satisfactory. It is also possible to start with a large precision level, and decrease it, if the incumbent does not improve for a number of iterations and the DM is not satisfied with the incumbent point. For both cases, the algorithm could be continued without a restart, only modifying the previously-constructed convex cones to account for this change. Another variation may be to define different precision parameters for different cones to standardize the additional regions covered by the cones. These and other possible variations are a subject of future research.

Appendix: An Example

In order to demonstrate the progress of the algorithm, we present an example three-criteria MOIP for which nondominated points are assumed to be available to us, for the sake of simplicity. We provide these nondominated points in Table 3. We will use this information only for the purpose of determining the most preferred solution of the corresponding MOIP problem.

Assume the preferences of the DM are consistent with a linear value function $f(\mathbf{z}) = 0.44z_1 + 0.36z_2 + 0.20z_3$ (a special nondecreasing quasiconcave function). Note that the most preferred point is $\mathbf{z}^* = (95, 50, 25)$ in Table 3 but we pretend this information is not available to us. Table 4 demonstrates the progress of the algorithm.

Table 3. The entire set of nondominated points for example problem

	$\mathbf{z}=(z_1,z_2,z_3)$	$f(\mathbf{z})$
1	(20,40,120)	47.20
2	(40,100,50)	63.60
3	(50,50,100)	60.00
4	(60,80,40)	63.20
5	(70,45,35)	54.00
6	(80,40,30)	55.60
7	(95,50,25)	**64.80**
8	(100,20,10)	53.20

Table 4. Demonstration of the algorithm on example problem

Iteration 1. $(h=1)$	**Step 0 (Initialization).** Set α=0.1. Initialize $T=\varnothing$ and select a weight $\lambda_i=1$ for each objective *i* and solve: $(P^{1,0.1})$ Max $z_1(\mathbf{x})+z_2(\mathbf{x})+z_3(\mathbf{x})$ subject to $\mathbf{x}\in\mathbf{X}$ $\mathbf{z}^1=(50,50,100)$ is the point corresponding to the optimal solution; initialize $\mathbf{z}^{inc}=\mathbf{z}^1$.
Iteration 2. $(h=2)$	**Step 1 (Find a new point).** Solve: $(P^{2,0.1})$ Max $z_1(\mathbf{x})+z_2(\mathbf{x})+z_3(\mathbf{x})$ subject to $51y_1\le z_1(\mathbf{x})$ $51y_2\le z_2(\mathbf{x})$ $101y_3\le z_3(\mathbf{x})$ $y_1+y_2+y_3=1$ $y_1,y_2,y_3\in\{0,1\}$ $\mathbf{x}\in\mathbf{X}$ $\mathbf{z}^2=(40,100,50)$ is the point corresponding to the optimal solution. **Step 2 (Comparison).** Ask the DM to compare $\mathbf{z}^2$ with $\mathbf{z}^{inc}$. Since $f(\mathbf{z}^{inc})<f(\mathbf{z}^2)$, the DM prefers $\mathbf{z}^2$ to $\mathbf{z}^{inc}$. Set $\mathbf{z}^{inc}=\mathbf{z}^2$ and $T=T\cup\{(\mathbf{z}^2,\mathbf{z}^1)\}$. Go to Step 1.

Table 4. Demonstration of the algorithm on example problem (continued)

Iteration 3. $(h=3)$	**Step 1 (Find a new point).** Solve: $(P^{3,0.1})$ Max $z_1(\mathbf{x})+z_2(\mathbf{x})+z_3(\mathbf{x})$ subject to $41y_1 \le z_1(\mathbf{x})$ $101y_2 \le z_2(\mathbf{x})$ $51y_3 \le z_3(\mathbf{x})$ $y_1+y_2+y_3=1$ $51r_1^{2,1} \le z_2(\mathbf{x})$ $3301t_{21}^{2,1} \le 10z_2(\mathbf{x})+50z_1(\mathbf{x})$ $8251t_{23}^{2,1} \le 50z_2(\mathbf{x})+50z_3(\mathbf{x})$ $r_1^{2,1}+t_{21}^{2,1}+t_{23}^{2,1}=1$ $r_1^{2,1},\ t_{21}^{2,1}, t_{23}^{2,1}, y_1, y_2, y_3 \in \{0,1\}$ $\mathbf{x} \in \mathbf{X}$ $\mathbf{z}^3=(60,80,40)$ is the point corresponding to the optimal solution. **Step 2 (Comparison).** Ask the DM to compare $\mathbf{z}^3$ with $\mathbf{z}^{inc}$. Since $f(\mathbf{z}^3)<f(\mathbf{z}^{inc})$, the DM prefers $\mathbf{z}^{inc}$ to $\mathbf{z}^3$. Set $T=T\cup\{(\mathbf{z}^2,\mathbf{z}^3)\}$ and go to Step 1.
Iteration 4. $(h=4)$	**Step 1 (Find a new point).** Solve: $(P^{4,0.1})$ Max $z_1(\mathbf{x})+z_2(\mathbf{x})+z_3(\mathbf{x})$ subject to $41y_1 \le z_1(\mathbf{x})$ $101y_2 \le z_2(\mathbf{x})$ $51y_3 \le z_3(\mathbf{x})$ $y_1+y_2+y_3=1$ $51r_1^{2,1} \le z_2(\mathbf{x})$ $3301t_{21}^{2,1} \le 10z_2(\mathbf{x})+50z_1(\mathbf{x})$ $8251t_{23}^{2,1} \le 50z_2(\mathbf{x})+50z_3(\mathbf{x})$ $81r_2^{2,3} \le z_2(\mathbf{x})$ $41r_3^{2,3} \le z_3(\mathbf{x})$ $3081t_{21}^{2,3} \le 20z_2(\mathbf{x})+20z_1(\mathbf{x})$ $1541t_{31}^{2,3} \le 20z_3(\mathbf{x})+10z_1(\mathbf{x})$ $r_2^{2,3}+r_3^{2,3}+t_{21}^{2,3}+t_{31}^{2,3}=1$ $r_1^{2,1},\ t_{21}^{2,1}, t_{23}^{2,1}, r_2^{2,3},\ r_3^{2,3}, t_{21}^{2,3}, t_{31}^{2,3} \in \{0,1\}$ $y_1, y_2, y_3 \in \{0,1\}$ $\mathbf{x} \in \mathbf{X}$ $(P^{4,0.1})$ is infeasible. Go to Step 3. **Step 3.** Stop. $\mathbf{z}^{inc}=(40,100,50)$ is the final point produced by the algorithm.

The algorithm stops with two pairwise comparisons based on which the approximately cone-dominated regions are excluded. It yields a final point for which the worst-case level of accuracy could be calculated as follows:

Suppose $f(\mathbf{z}^*) > f(\mathbf{z}^{inc})$, that is, the algorithm could not find the most preferred solution. Based on Proposition 3, $\mathbf{z}^* \in \Delta_\alpha(\mathbf{z}^2;\mathbf{z}^1)$ or $\mathbf{z}^* \in \Delta_\alpha(\mathbf{z}^2;\mathbf{z}^3)$.

If $\mathbf{z}^* \in \Delta_\alpha(\mathbf{z}^2;\mathbf{z}^1)$, then there exists an inferior point $\mathbf{z} \in D(\mathbf{z}^2;\mathbf{z}^1)$ such that

$$|\mathbf{z}^* - \mathbf{z}| \le \max\left(\frac{3000\alpha}{50}, \frac{7500\alpha}{50}\right) = 6.$$

Similarly, if $\mathbf{z}^* \in \Delta_\alpha(\mathbf{z}^2;\mathbf{z}^3)$, there exists an inferior point $\mathbf{z} \in D(\mathbf{z}^2;\mathbf{z}^3)$ such that

$$|\mathbf{z}^* - \mathbf{z}| \le \max\left(\frac{2800\alpha}{20}; \frac{1400\alpha}{10}\right) = 14.$$

Hence, the algorithm guarantees that there exists an inferior point $\mathbf{z}$ that is at most 14 units away from the most preferred point, $\mathbf{z}^*$, in terms of Tchebycheff distance. In other words, $\mathbf{z}^*$ is not better than a solution, $\mathbf{z}$, that is inferior to $\mathbf{z}^{inc} = (40,100,50)$ in any of its criterion values due to Proposition 4. Note that this is an upper bound value. We do know the exact values for this example problem since we have all nondominated points available at the outset. In this example, $\mathbf{z}^* \in \Delta_\alpha(\mathbf{z}^2;\mathbf{z}^3)$. Using the derivation of Proposition 2, we find $\mathbf{z} = (90,50,25) \in C(\mathbf{z}^2;\mathbf{z}^3)$ and $|\mathbf{z}^* - \mathbf{z}| = 5$. For this example problem, our upper bound of 14 units is quite conservative.

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