EECE322-01: 자동제어공학개론

Introduction to Automatic Control

Chapter 7: Controller Design

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◆ The main objectives of this chapter are

1. Introduction

2. Stabilizing controllers

3. Two-degree-of-freedom control systems

1. Introduction

Controllers in the consideration

- Rational functions

$$C(s) = \frac{n[s]}{d[s]}$$

- $n[s],\ d[s]$: ploynomials of s

  - Proper functions

$$C(s) = \frac{n[s]}{d[s]}$$

- the order of $d[s] \geq$ the order of $n[s]$

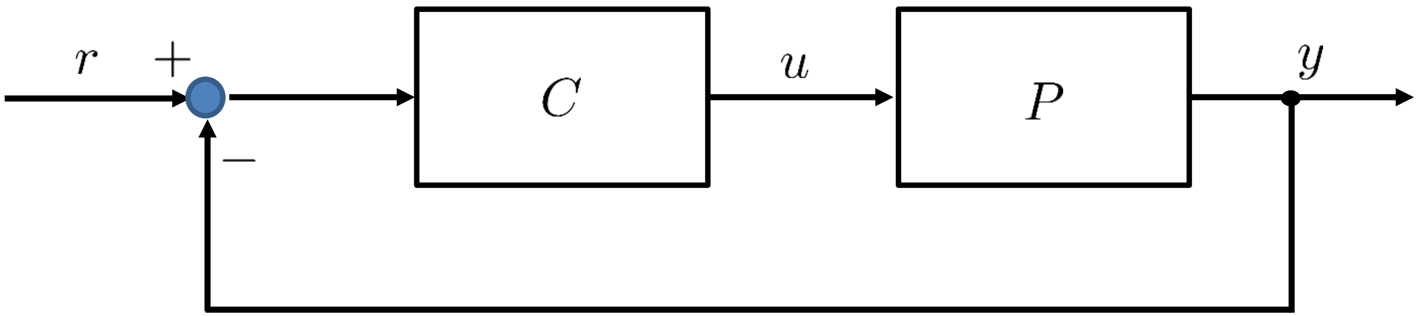
  - Strictly proper functions

$$C(s) = \frac{n[s]}{d[s]}$$

- the order of $d[s] >$ the order of $n[s]$

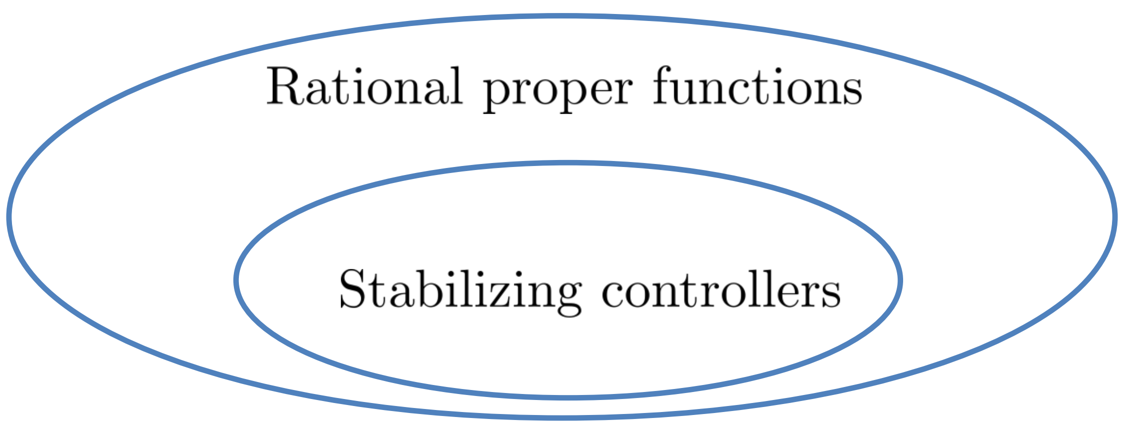
Motivation of stabilizing controllers

- Typical control problem



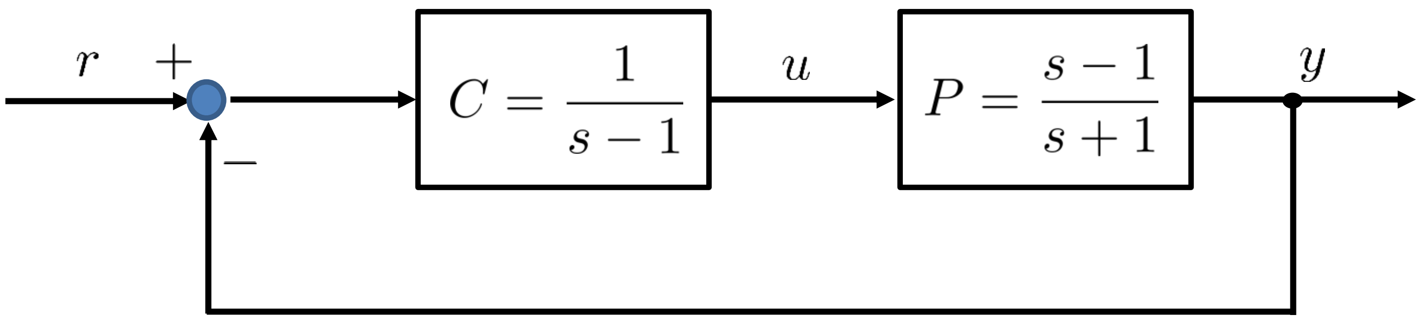
- Stabilizing controllers

Clarify the structure of all C that stabilize the closed-loop system



Example: undesirable controller

- Undesirable controller



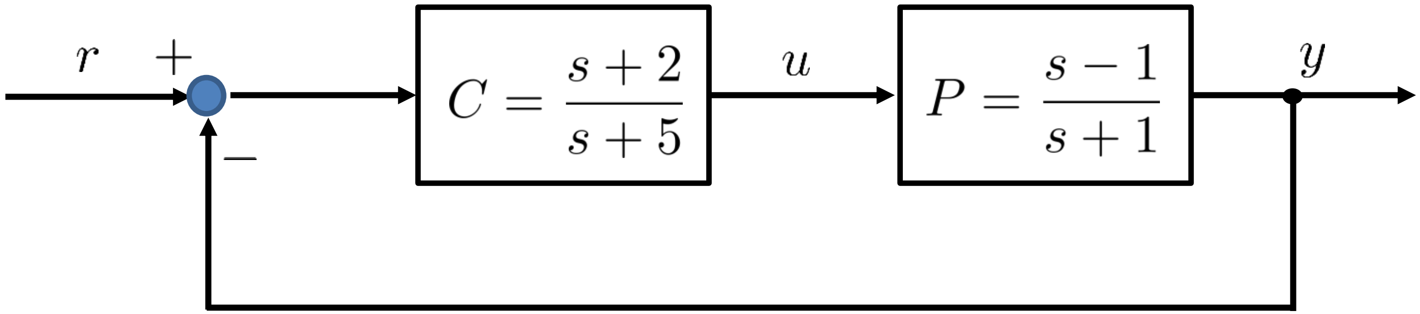
- $T_{yr}(s) = \dfrac{\dfrac{1}{s-1}\cdot\dfrac{s-1}{s+1}}{1+\dfrac{1}{s-1}\cdot\dfrac{s-1}{s+1}} = \dfrac{1}{s+2}$: stable

- $T_{ur}(s) = \dfrac{\dfrac{1}{s-1}}{1+\dfrac{1}{s-1}\cdot\dfrac{s-1}{s+1}} = \dfrac{s+1}{(s+2)(s-1)}$: unstable

$\rightarrow$ Unstable pole-zero cancellation is inadequate!!

Example: desirable controller

- Desirable controller



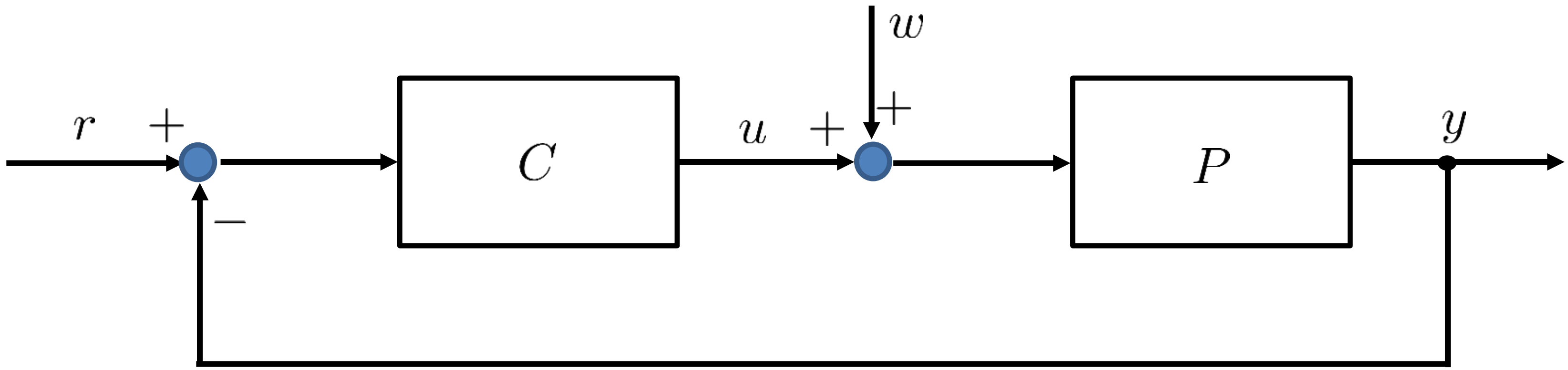
- $$T_{yr}(s) = \frac{\dfrac{s+2}{s+5}\cdot\dfrac{s-1}{s+1}}{1+\dfrac{s+2}{s+5}\cdot\dfrac{s-1}{s+1}} = \frac{(s+2)(s-1)}{(2s+1)(s+3)} : \text{stable}$$

- $$T_{ur}(s) = \frac{\dfrac{s+2}{s+5}}{1+\dfrac{s+2}{s+5}\cdot\dfrac{s-1}{s+1}} = \frac{(s+2)(s+1)}{(2s+1)(s+3)} : \text{stable}$$

$\rightarrow$ What is a generalized form of stabilizing controllers?

Internal stability

- Control problem with disturbance



- Internal stability

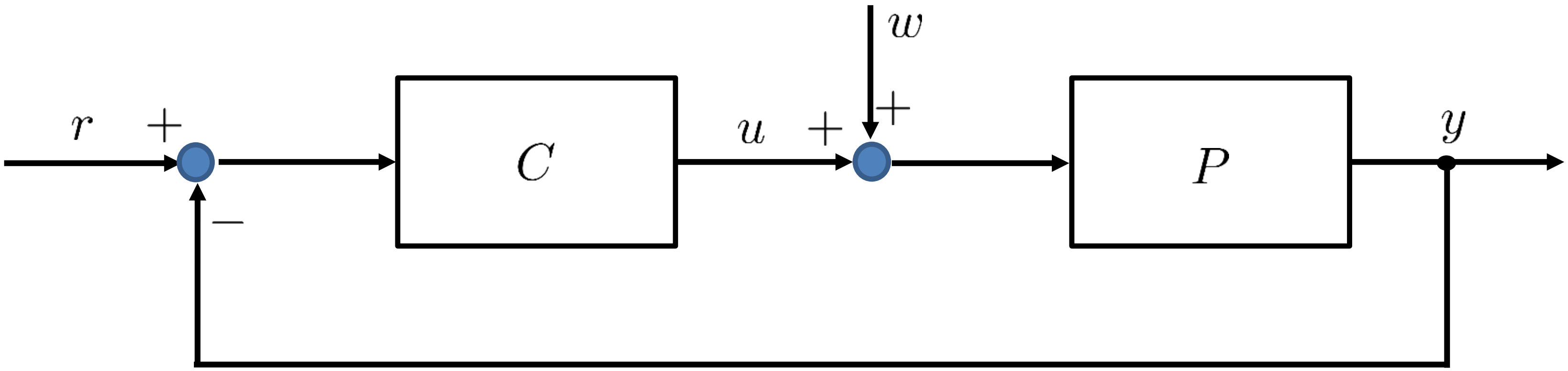
All the transfer functions from (r, w) to (u, y) are stable

$$T_{ur} = \frac{C}{1+PC} \quad T_{uw} = \frac{-PC}{1+PC}$$

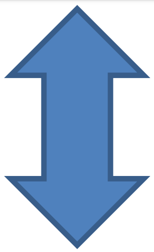
$$T_{yr} = \frac{PC}{1+PC} \quad T_{yw} = \frac{P}{1+PC}$$

$$\Rightarrow \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} \frac{C}{1+PC} & \frac{-PC}{1+PC} \\ \frac{PC}{1+PC} & \frac{P}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ w \end{bmatrix}$$

Structure of stabilizing controllers



The structure of all C that stabilize the closed-loop system

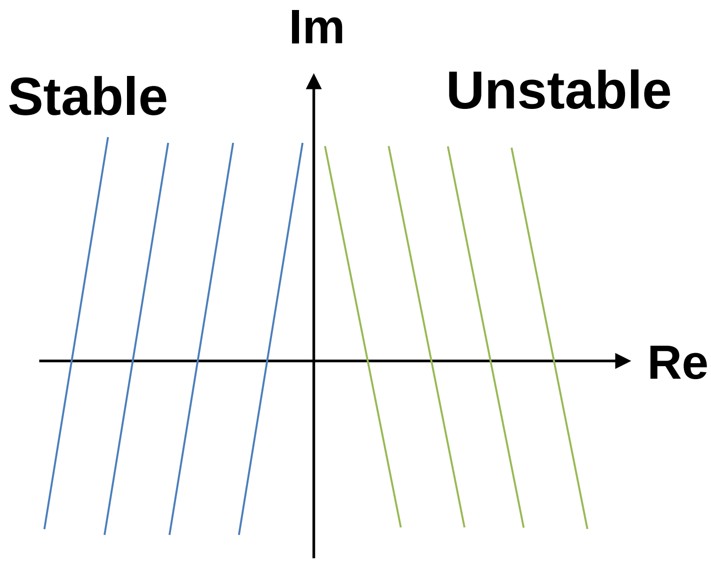


The structure of all C that stabilize the following transfer functions

$$\frac{C}{1+PC}, \quad \frac{-PC}{1+PC}, \quad \frac{PC}{1+PC}, \quad \frac{P}{1+PC}$$

2. Stabilizing controllers

Stable rational functions



$C_+ : \mathrm{Re}\{s\} \geq 0 \qquad C_- : \mathrm{Re}\{s\} < 0$

Pole p: $G(p) = \infty \qquad$ Zero z: $G(z) = 0$

RH_∞ : stable + proper

- Stable: $|G(s)| < \infty, \ \forall s \in C_+$ — No poles in C_+
- Proper: $|G(s)| < \infty, \ s \to \infty$ — No poles at ∞

Example

- Transfer functions in RH_∞

$$C(\text{constnat}), \quad \frac{1}{s+1}, \quad \frac{s-2}{s+1}, \quad \frac{s-1}{s+2}$$

$$\frac{\prod_{j=1}^{m}(s-z_j)}{\prod_{i=1}^{n}(s-p_i)} : n \geq m \text{ and } \mathrm{Re}[p_i] < 0 \text{ for all } p_i (i = 1, \ldots, n)$$

$$\frac{1}{s^3 + as^2 + bs + c} : a > 0,\ b > 0,\ c > 0,\ ab - c > 0$$

Motivation of coprime factorization over RH_∞

$$G(s) = \frac{s-1}{(s-2)^2} = \frac{n[s]}{d[s]}: \text{ ratio between polynomials}$$

$$= \frac{\dfrac{s-1}{(s+1)^2}}{\dfrac{(s-2)^2}{(s+1)^2}} = \frac{N(s)}{D(s)} \begin{array}{l} \in RH_\infty \\ \in RH_\infty \end{array} \quad : \text{ ratio between transfer functions in } RH_\infty$$

- $n, d \in \mathbb{Z}$: No common divisor $\Leftrightarrow \exists x, y \in \mathbb{Z},\ nx + dy = 1$

- $n[s], d[s] \in \mathbb{R}[s]$: No common divisor (i.e., zero)

$$\Leftrightarrow\ \exists x[s], y[s] \in \mathbb{R}[s],\ n[s]x[s] + d[s]y[s] = 1$$

- $N(s), D(s) \in RH_\infty$: No common unstable zero (including ∞)

$$\Leftrightarrow \exists X(s), Y(s) \in RH_\infty,\ N(s)X(s) + D(s)Y(s) = 1$$

Coprime factorization

When there exist $N(s), X(s), D(s), Y(s) \in RH_\infty$ such that

$$P(s) = \frac{N(s)}{D(s)} \text{ and } N(s)X(s) + D(s)Y(s) = 1$$

we call $\dfrac{N(s)}{D(s)}$ the coprime factorization of $P(s)$

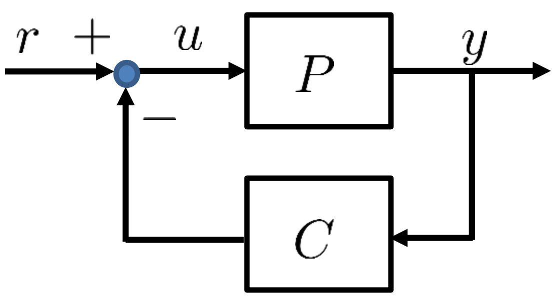
Computing coprime factorization

- Case 1: $P \in RH_\infty$

$$N = P,\ D = 1,\ X = 0,\ Y = 1$$

As a general solution, $X = Q \in RH_\infty,\ Y = 1 - PQ \in RH_\infty$

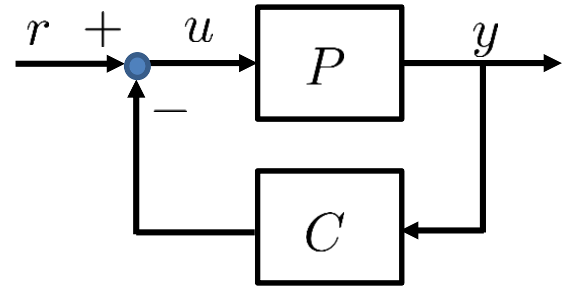
- Case 2: $C \in RH_\infty$ stabilizes P



$$N = \frac{P}{1+PC},\ D = \frac{1}{1+PC},\ X = C,\ Y = 1$$

$$Y = \frac{P}{1+PC}R = NR,\ U = \frac{1}{1+PC}R = DR$$

- Case 3: $C \notin RH_\infty$ stabilizes P



$$\frac{P}{1+PC} \cdot C + \frac{1}{1+PC} \cdot 1 = 1$$

$$\Downarrow$$

$$\frac{P}{1+PC} \cdot \frac{N_C}{D_C} + \frac{1}{1+PC} \cdot \frac{D_C}{D_C} = 1 \qquad \left(C = \frac{N_C}{D_C} \begin{array}{l} \in RH_\infty \\ \in RH_\infty \end{array} \right)$$

$$N = \frac{P}{1+PC} \cdot \frac{1}{D_C},\ D = \frac{1}{1+PC} \cdot \frac{1}{D_C},\ X = N_C,\ Y = D_C$$

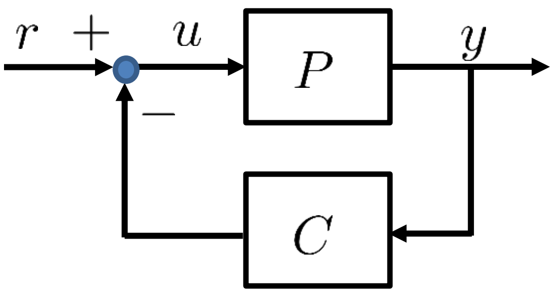
Summary of coprime factorization

$$P = \frac{N}{D}, \quad NX + DY = 1, \quad \{N, X, D, Y\} \in RH_\infty$$

	N	D	X	Y
$P \in RH_\infty$	P	1	0	1
$C \in RH_\infty$	$\frac{P}{1+PC}$	$\frac{1}{1+PC}$	C	1
$C = \frac{N_C}{D_C}$	$\frac{P}{1+PC} \cdot \frac{1}{D_C}$	$\frac{1}{1+PC} \cdot \frac{1}{D_C}$	N_C	D_C

Example

- Coprime factorization of $P = \dfrac{1}{s(s+2)} \notin RH_\infty$



By using $C = C_0(>0) \in RH_\infty$, P can be stabilized.

If we take $C = 1$,

$$N = \frac{P}{1+PC} = \frac{\frac{1}{s(s+2)}}{1+\frac{1}{s(s+2)}} = \frac{1}{s^2+2s+1} = \frac{1}{(s+1)^2} \in RH_\infty$$

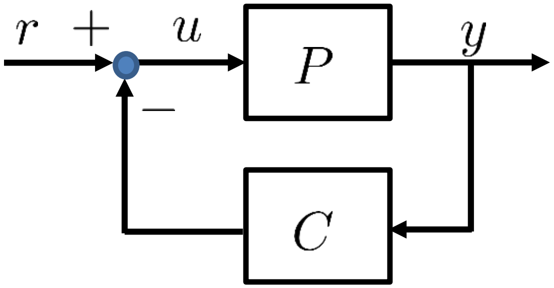
$$D = \frac{1}{1+PC} = \frac{1}{1+\frac{1}{s(s+2)}} = \frac{s^2+2s}{s^2+2s+1} = \frac{s^2+2s}{(s+1)^2} \in RH_\infty$$

$$X = C = 1 \in RH_\infty$$

$$Y = 1 \in RH_\infty$$

Example

- Coprime factorization of $P = \dfrac{1}{s} \notin RH_\infty$



By using $C = C_0 (> 0) \in RH_\infty$, P can be stabilized.

$$N = \frac{P}{1+PC} = \frac{\frac{1}{s}}{1+\frac{C_0}{s}} = \frac{1}{s+C_0} \in RH_\infty$$

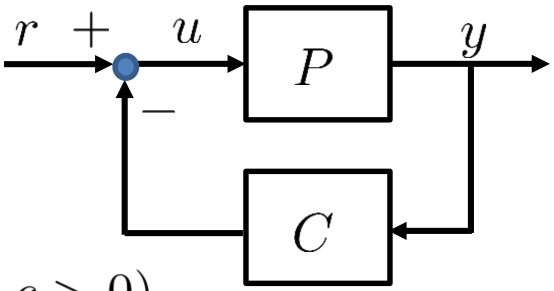
$$D = \frac{1}{1+PC} = \frac{1}{1+\frac{C_0}{s}} = \frac{s}{s+C_0} \in RH_\infty$$

$$X = C_0 \in RH_\infty$$

$$Y = 1 \in RH_\infty$$

Example

- Coprime factorization of $P = \dfrac{1}{s^2} \notin RH_\infty$



By using $C = \dfrac{bs+c}{s+a} \in RH_\infty$, P can be stabilized.

$(\Phi[s] = s^3 + as^2 + bs + c \Rightarrow a > 0,\ b > 0,\ c > 0,\ ab - c > 0)$

Let $a = b = 3,\ c = 1$

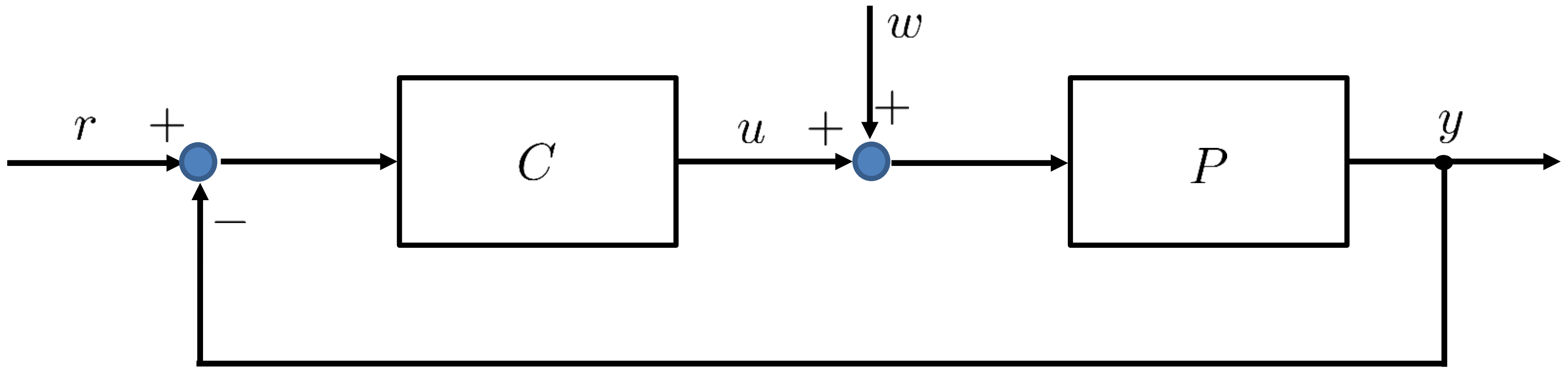
$$N = \frac{P}{1+PC} = \frac{s+a}{s^3 + as^2 + bs + c} = \frac{s+3}{(s+1)^3} \in RH_\infty$$

$$D = \frac{1}{1+PC} = \frac{s^2(s+a)}{s^3 + as^2 + bs + c} = \frac{s^2(s+3)}{(s+1)^3} \in RH_\infty$$

$$X = C = \frac{3s+1}{s+3} \in RH_\infty$$

$$Y = 1 \in RH_\infty$$

Internal stability and coprime factorization



All the transfer functions from (r, w) to (u, y) are stable

$$P = \frac{N}{D}, \quad NX + DY = 1 \qquad N,\ X,\ D,\ Y \in RH_\infty$$

$$C = \frac{N_C}{D_C}, \quad N_C X_C + D_C Y_C = 1 \qquad N_C,\ X_C,\ D_C,\ Y_C \in RH_\infty$$

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{C}{1+PC} & \dfrac{-PC}{1+PC} \\ \dfrac{PC}{1+PC} & \dfrac{P}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ w \end{bmatrix} = \begin{bmatrix} T_{ur} & T_{uw} \\ T_{yr} & T_{yw} \end{bmatrix} \begin{bmatrix} r \\ w \end{bmatrix}$$

$$T_{ur} = \frac{C}{1+PC} = \frac{N_C D}{NN_C + DD_C} \qquad T_{yr} = \frac{PC}{1+PC} = \frac{NN_C}{NN_C + DD_C}$$

$$T_{yw} = \frac{P}{1+PC} = \frac{ND_C}{NN_C + DD_C}$$

$$T_{uw} = \frac{-PC}{1+PC} = \frac{1}{1+PC} - 1 = \frac{D_C D}{NN_C + DD_C} - 1$$

Let $\Xi^{-1} := (NN_C + DD_C)^{-1}$

$$\text{Internal stability: } \begin{bmatrix} T_{ur} & T_{uw} \\ T_{yr} & T_{yw} \end{bmatrix} \in RH_\infty$$

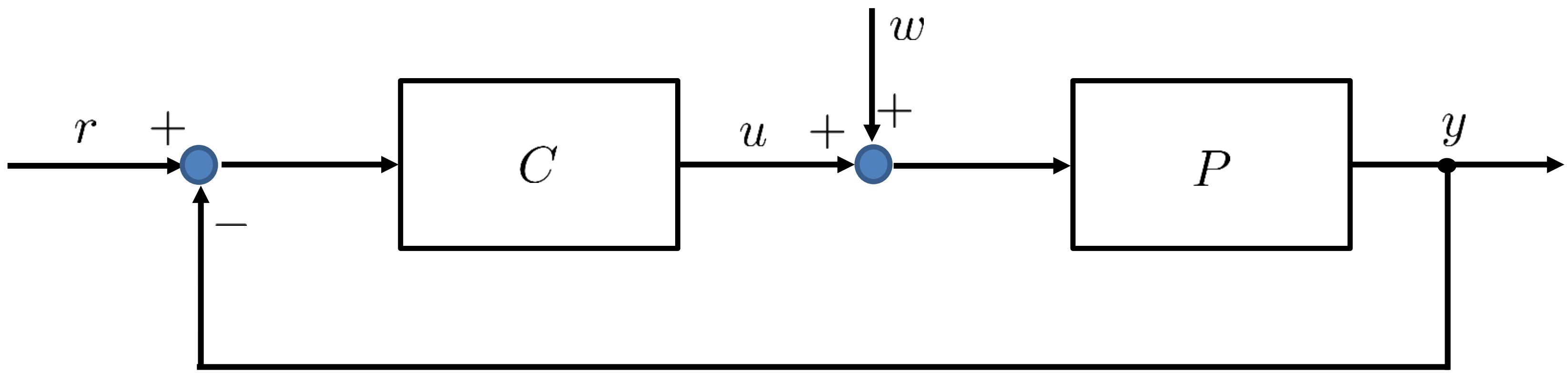
$$\Updownarrow$$

$$\frac{1}{\Xi} \begin{bmatrix} N_C D & D_C D \\ N_C N & D_C N \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \in RH_\infty$$

$$\Updownarrow$$

$$\frac{1}{\Xi} \begin{bmatrix} N_C D & D_C D \\ N_C N & D_C N \end{bmatrix} \in RH_\infty$$

Necessary and sufficient condition for internal stability



- This closed-loop system is internal stable if and only if $\Xi^{-1} \in RH_\infty$

- Proof:

  (a) Sufficient condition (<=) is obvious because $N, \ N_C, \ D, \ D_C \in RH_\infty$

(b) Necessary condition ($\Rightarrow$):

$$\text{Internal stability: } \begin{bmatrix} T_{ur} & T_{uw} \\ T_{yr} & T_{yw} \end{bmatrix} \in RH_\infty$$

$$\Updownarrow$$

$$\frac{1}{\Xi}\begin{bmatrix} N_C D & D_C D \\ N_C N & D_C N \end{bmatrix} \in RH_\infty$$

$$\Downarrow$$

$$\begin{bmatrix} Y & X \end{bmatrix} \frac{1}{\Xi}\begin{bmatrix} N_C D & D_C D \\ N_C N & D_C N \end{bmatrix} \begin{bmatrix} X_C \\ Y_C \end{bmatrix} \in RH_\infty$$

$$\Downarrow$$

$$\frac{1}{\Xi} \in RH_\infty$$

Parameter representation of stabilizing controllers

C is a stabilizing controller for $P = \dfrac{N}{D}$ if and only if

there exist $N_C,\ D_C \in RH_\infty$ such that $NN_C + DD_C = 1$ where $C = \dfrac{N_C}{D_C}$

- Proof:

(a) Sufficient condition ($\Leftarrow$) is obvious since $\Xi = NN_C + DD_C = 1$

(b) Necessary condition ($\Rightarrow$):

Let $\dfrac{\tilde{N}_C}{\tilde{D}_C}$ one of the coprime factorizations of C

For $\Xi = N\tilde{N}_C + D\tilde{D}_C,\ \Xi^{-1} \in RH_\infty$

If let $N_C = \dfrac{\tilde{N}_C}{\Xi}$ and $D_C = \dfrac{\tilde{D}_C}{\Xi}$, $NN_C + DD_C = 1$

General solution

$$\text{Finding all } C \text{ that stabilize } P \left(= \frac{N \ \in RH_\infty}{D \ \in RH_\infty} \right)$$

$\Updownarrow$

$$\text{Finding all } (N_C, D_C) \in RH_\infty \text{ such that } NN_C + DD_C = 1$$

- Bezout's identity: general solution to $NN_C + DD_C = 1$

$$\begin{aligned} NN_C + DD_C &= 1 \\ NX + DY &= 1 \end{aligned}$$

$N,\ X,\ D,\ Y \in RH_\infty$

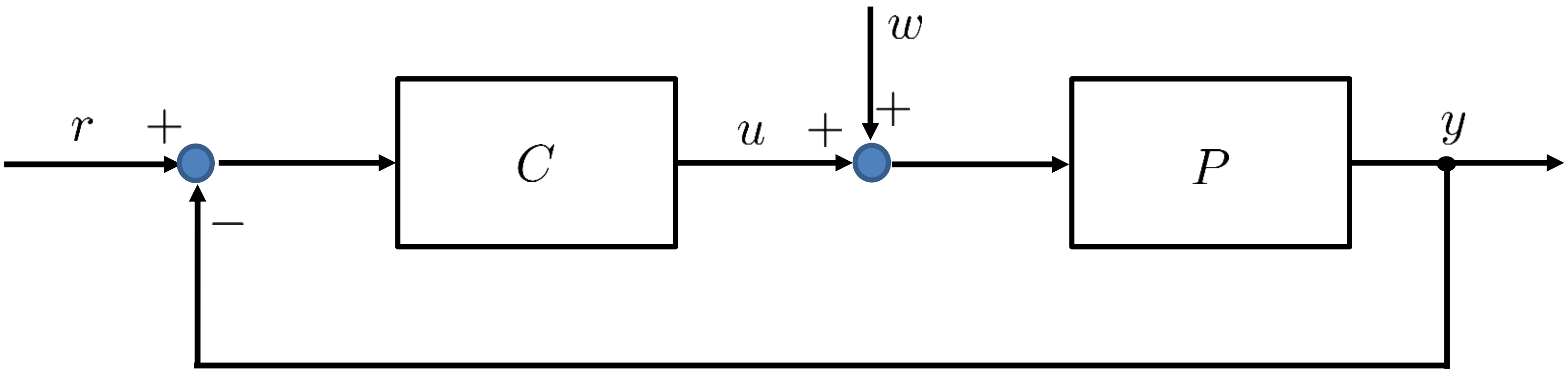
$N_C,\ D_C \in RH_\infty$

$\Rightarrow$

$$N(N_C - X) + D(D_C - Y) = 0$$

$$\begin{aligned} N_C &= X + DQ \\ D_C &= Y - NQ \end{aligned} \qquad \forall Q \in RH_\infty$$

All stabilizing controllers



- All stabilizing controllers

The structure of all stsabilizing controllers C is given by

$$C = \frac{X + DQ}{Y - NQ}, \quad \forall Q \in RH_\infty,$$

where $P = \dfrac{N}{D}, \quad NX + DY = 1, \quad \{N,\ X,\ D,\ Y\} \in RH_\infty$

Example

- All stabilizing controllers for $P = \dfrac{1}{(s+1)^2}$

$$P \in RH_\infty \Rightarrow N = P,\ D = 1,\ X = 0,\ Y = 1$$

$$C = \frac{X + DQ}{Y - NQ} = \frac{Q}{1 - PQ} = \frac{Q}{1 - \dfrac{1}{(s+1)^2}Q}, \quad \forall Q \in RH_\infty$$

- All stabilizing controllers for $P = \dfrac{1}{s}$

$P \notin RH_\infty$ and $C = C_0 > 0$ stabilizes P

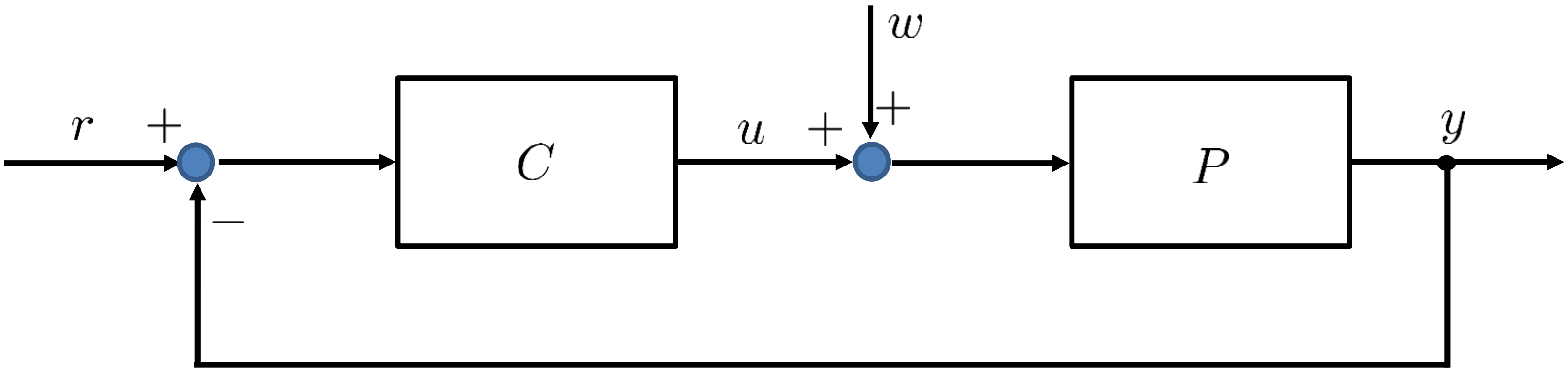
$$\Rightarrow N = \frac{1}{s + C_0},\ D = \frac{s}{s + C_0},\ X = C_0,\ Y = 1$$

$$C = \frac{X + DQ}{Y - NQ} = \frac{C_0 + \dfrac{s}{s + C_0}Q}{1 - \dfrac{1}{s + C_0}Q}, \quad \forall Q \in RH_\infty$$

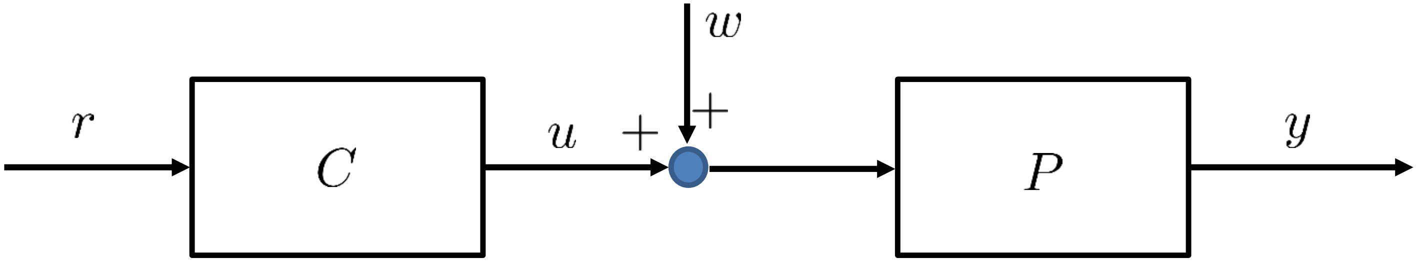
3. Two-degree-of-freedom control systems

Feedback and feedforward

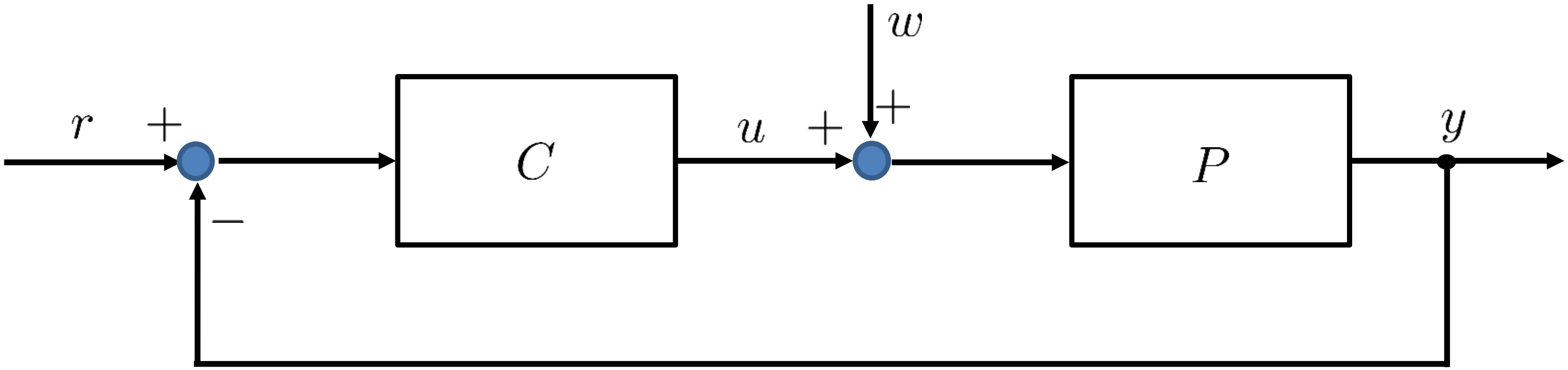
- Feedback (FB) control



- Feedforward (FF) control

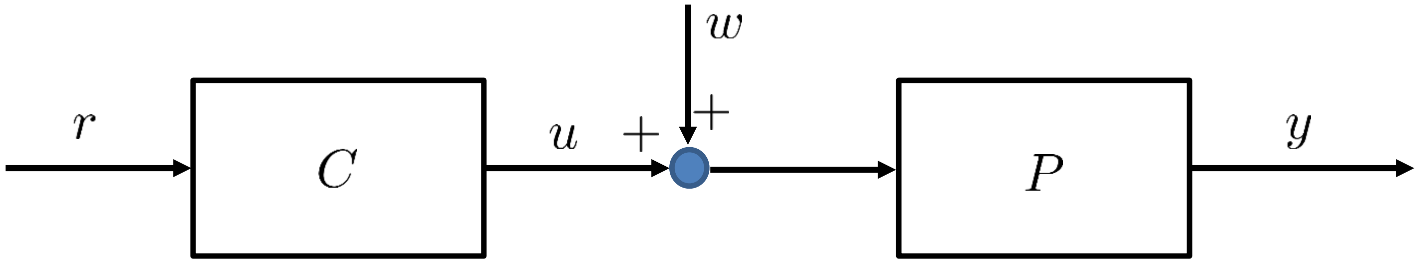


Characteristics of FB control



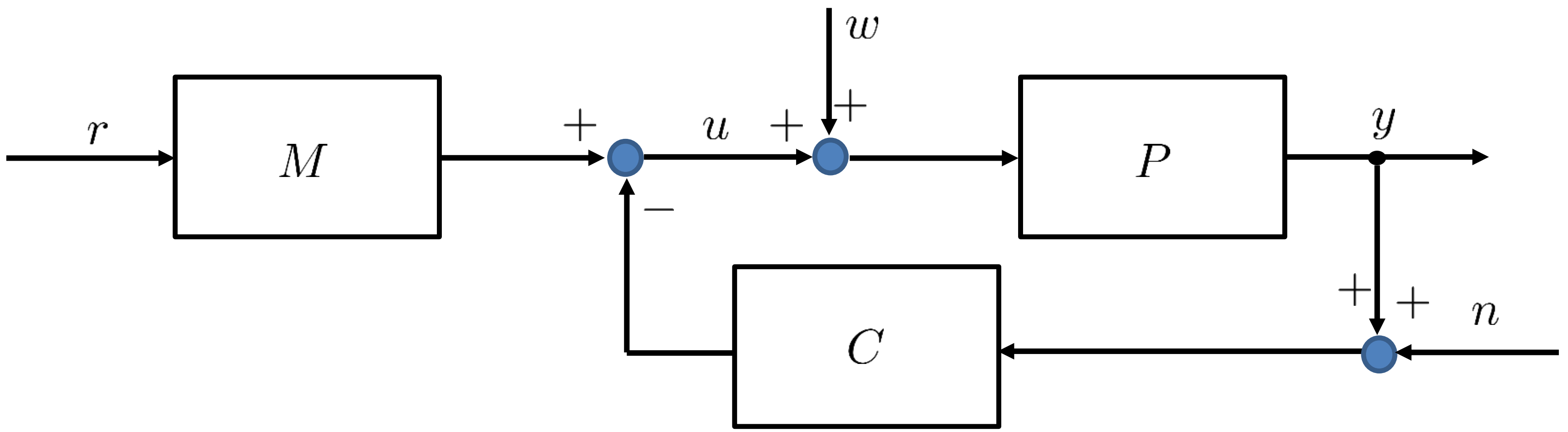
- Disturbance rejection
- Stabilization of unstable plant
- Robustness to model uncertainty (sensitivity)

Characteristics of FF control



- Simple and intuitive
- Improvement of reference tracking

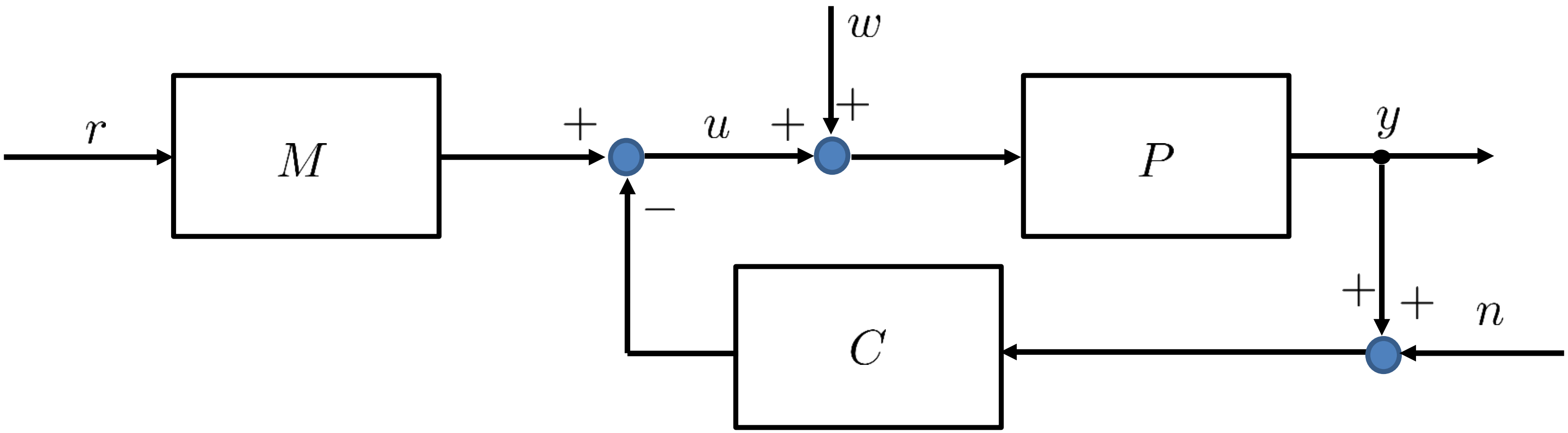
Structure of two-degree-of-freedom control systems



- Internal stability

> All the transfer functions from (r, w, n) to (u, y) are stable

Structure of two-degree-of-freedom control systems



- This closed-loop system is internal stable if and only if

$$\Xi^{-1} \in RH_\infty, \quad D_C M \in RH_\infty$$

where $\Xi = NN_C + DD_C, \quad P = \dfrac{N \ \in RH_\infty}{D \ \in RH_\infty}, \quad C = \dfrac{N_C \ \in RH_\infty}{D_C \ \in RH_\infty}$

- Proof:

(a) Necessary condition ($\Rightarrow$):

$$(w,n) \to (u,y) \in RH_\infty \Leftrightarrow \Xi^{-1} \in RH_\infty$$

$$r \to (u,y) \in RH_\infty \Leftrightarrow \begin{bmatrix} \dfrac{M}{1+PC} \\ \dfrac{PM}{1+PC} \end{bmatrix} = \frac{1}{\Xi}\begin{bmatrix} DD_C M \\ ND_C M \end{bmatrix} \in RH_\infty$$

$$\Downarrow$$

$$\begin{bmatrix} Y & X \end{bmatrix} \frac{1}{\Xi}\begin{bmatrix} DD_C M \\ ND_C M \end{bmatrix} = \frac{1}{\Xi} D_C M \in RH_\infty$$

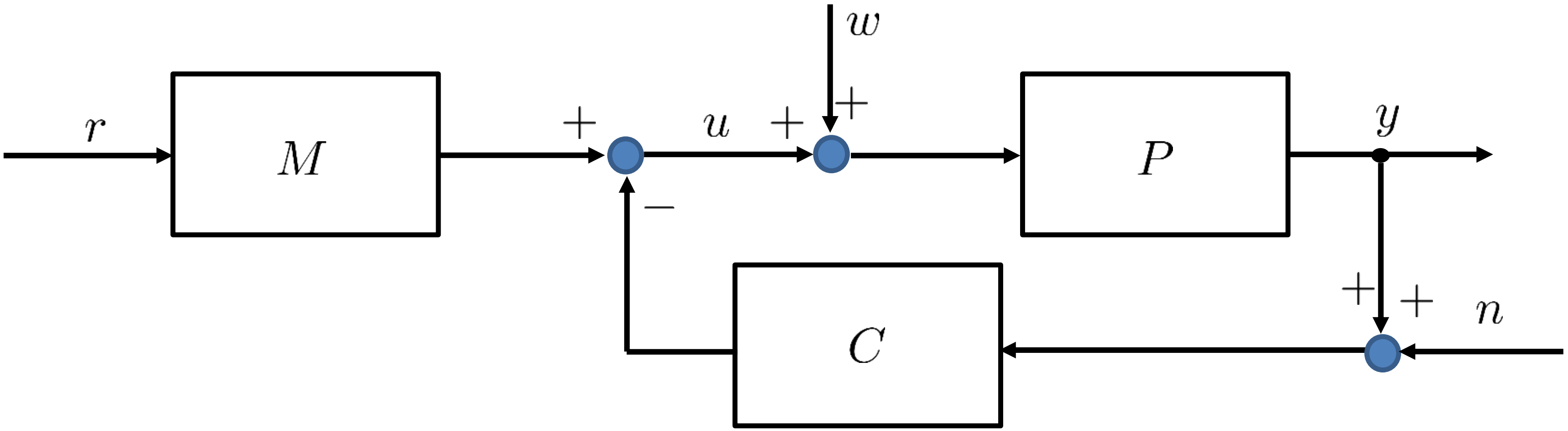
Because this relation holds for all $\Xi^{-1} \in RH_\infty$, $D_C M \in RH_\infty$

(b) Sufficient condition ($\Leftarrow$):

Because $\Xi^{-1} \in RH_\infty$, $(w,n) \to (u,y) \in RH_\infty$

Because $\Xi^{-1} \in RH_\infty$ and $D_C M \in RH_\infty$, $r \to (u,y) \in RH_\infty$

All stabilizing controllers



- All stabilizing controllers

The structure of all stsabilizing controllers C and M are given by

$$C = \frac{X + DQ}{Y - NQ}, \quad \forall Q \in RH_\infty, \quad M = \frac{R}{Y - NQ}, \quad \forall R \in RH_\infty,$$

$$\text{where } P = \frac{N}{D}, \quad NX + DY = 1, \quad \{N,\ X,\ D,\ Y\} \in RH_\infty$$

- Proof:

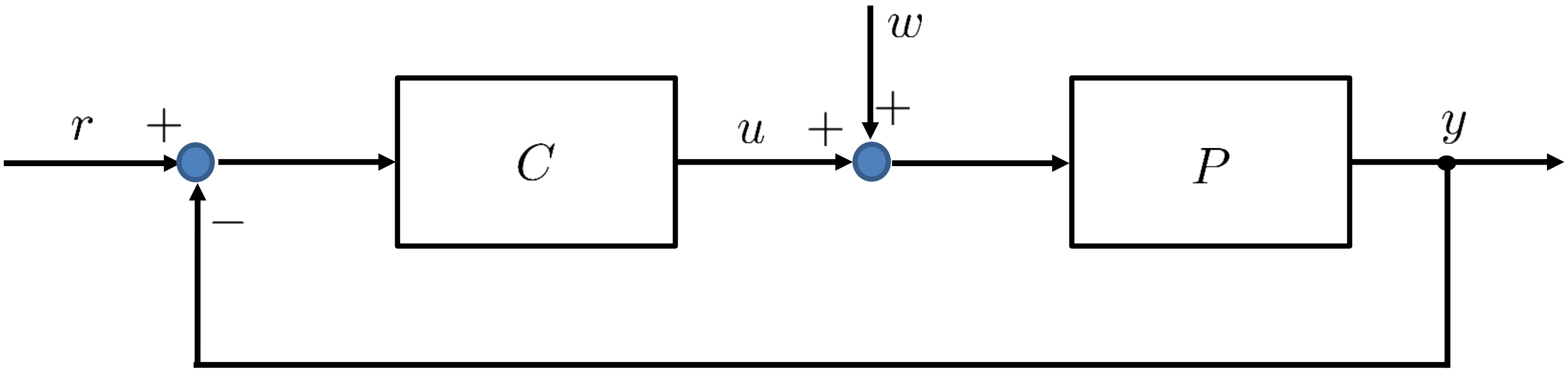
(a) For the structure of C, the derivation is essentially the same as one-degree-of freedom control system

(b) For the structure of M, $D_C M := R \in RH_\infty$

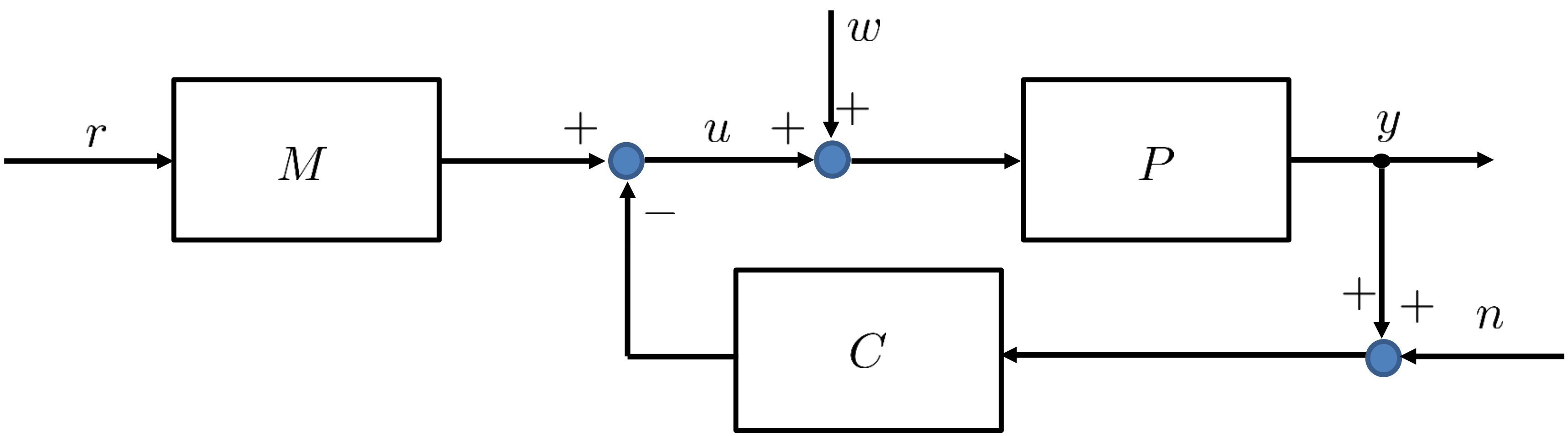
$$\Rightarrow M = \frac{R}{D_C} = \frac{R}{Y - NQ}$$

Remark: There could exist a case such that $M \notin RH_\infty$

Class of transfer function



- $T_{yr} = \dfrac{PC}{1+PC} = \dfrac{NN_C}{NN_C + DD_C} = N(X + DQ)$



- $T_{yr} = \dfrac{PM}{1+PC} = \dfrac{ND_CM}{NN_C + DD_C} = ND_CM = NR$

1 DOF vs 2 DOF

- Feasible transfer functions

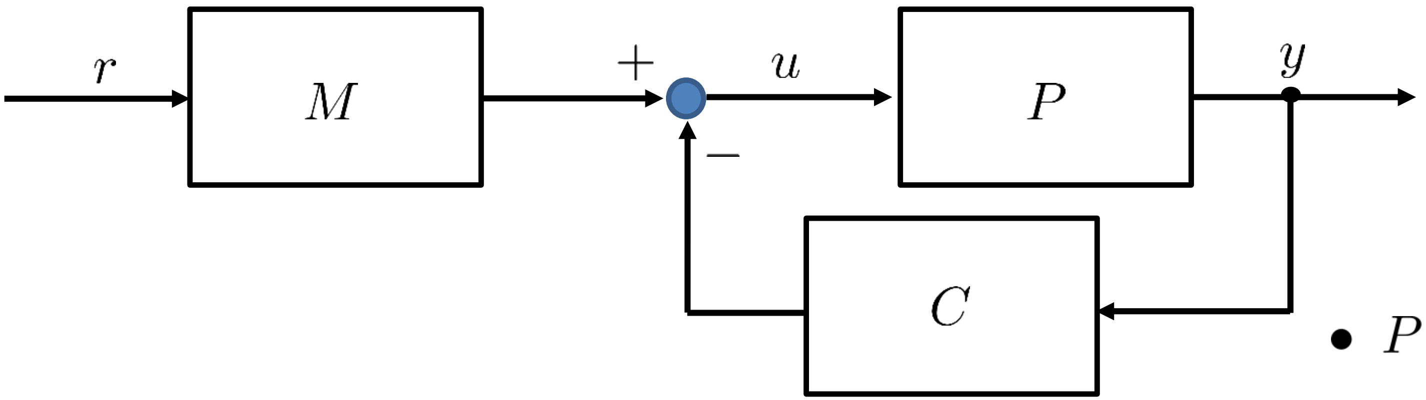
	1 DOF	2 DOF
T_{yr}	$N(X+DQ)$	NR
$S=\dfrac{1}{1+PC}$	$D(Y-NQ)$	$D(Y-NQ)$

T_{yr}: $r \to y$ $\qquad$ $S=\dfrac{1}{1+PC}$: sensitivity function

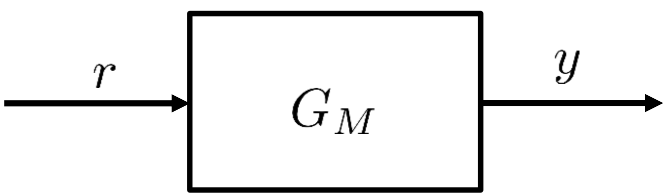
- Extension of feasible transfer functions by using 2 DOF

$$\{N(X+DQ) \mid \forall Q \in RH_\infty\} \subset \{NR \mid \forall R \in RH_\infty\}$$

Model matching problem



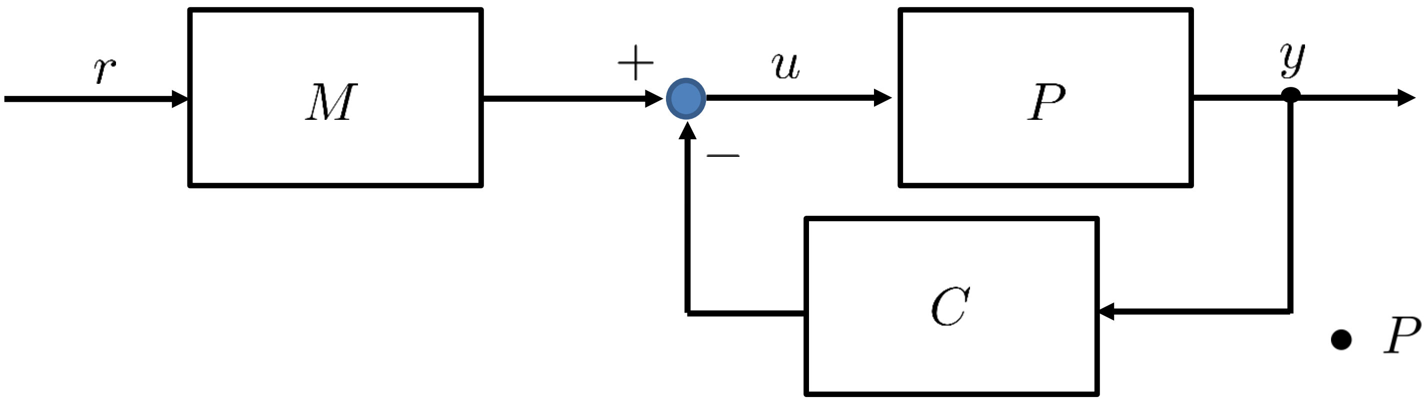
- P: given



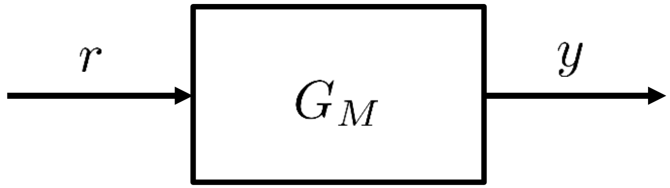
- $G_M \in RH_\infty$: given reference model

Find M and C such that $T_{yr} = G_M$

Stabilizing controllers for model matching problem



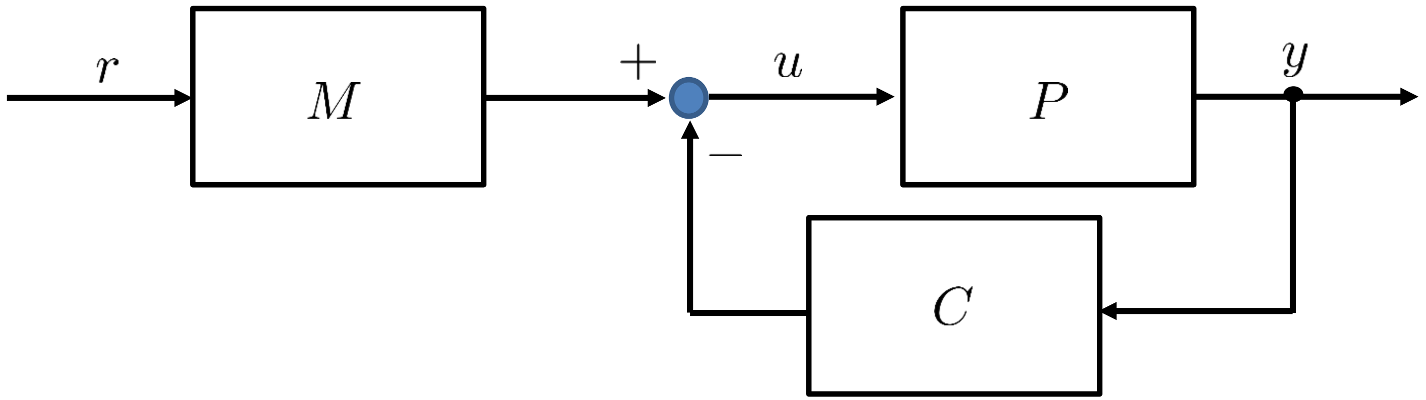
- P: given



- $G_M \in RH_\infty$: given reference model

Stabilizing controllers C and M by which $T_{yr} = G_M$ exist if and only if $\dfrac{G_M}{P} \in RH_\infty$ ($\Leftrightarrow \exists R \in RH_\infty,$ s.t. $G_M = NR$)

Controller design procedure

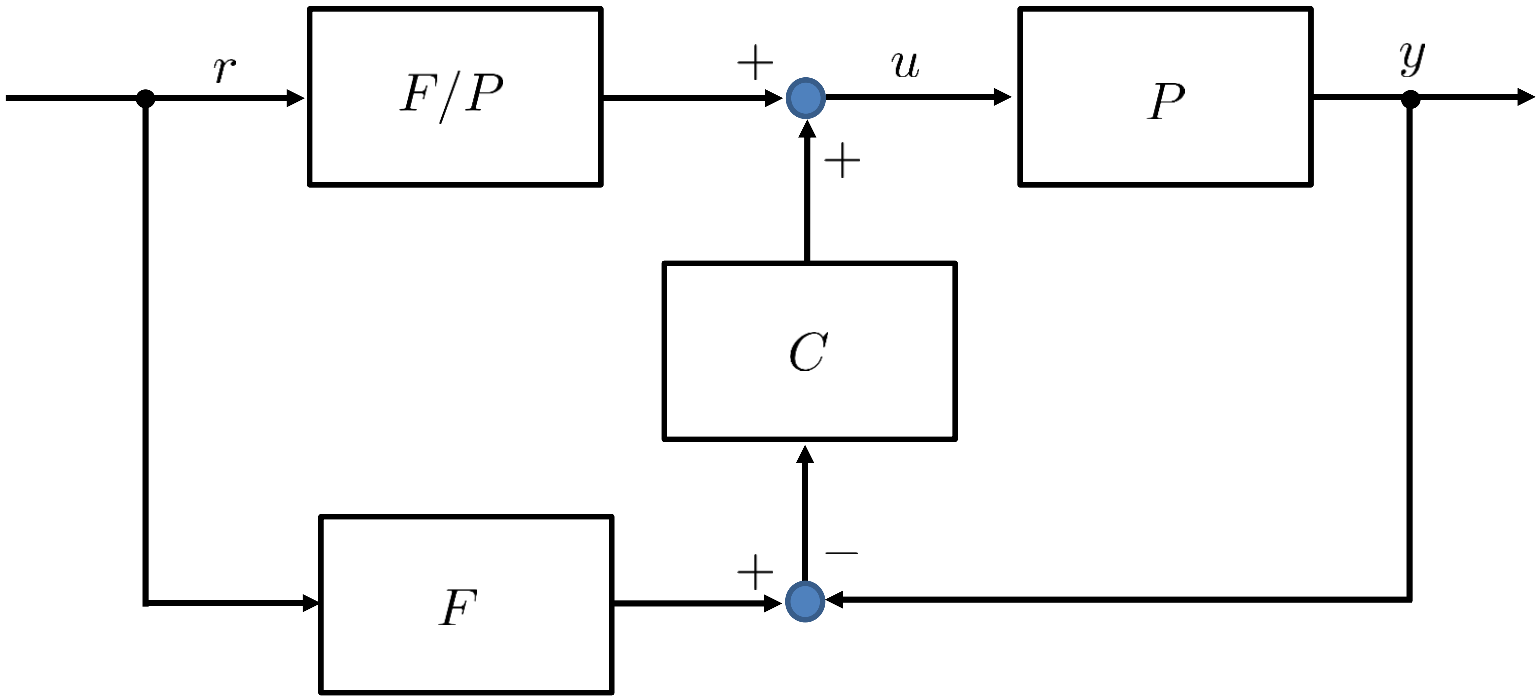


1. Check $\dfrac{G_M}{P} \in RH_\infty$

2. Design of stabilizing controller $C \left(= \dfrac{X + DQ}{Y - NQ} \right)$

3. $M = \dfrac{G_M}{N(Y - NQ)} \left(= \dfrac{R}{Y - NQ} \right)$

Another procedure for controller design



1. Check $\dfrac{G_M}{P} \in RH_\infty$

2. $F = G_M$

3. Design of stabilizing controller C

Example

- $P = \dfrac{1}{s-2}, \quad G_M = \dfrac{16}{s^2 + 64s + 16}$

1. $\dfrac{G_M}{P} = \dfrac{16(s-2)}{s^2 + 64s + 16} \in RH_\infty$

2. $F = G_M = \dfrac{16}{s^2 + 64s + 16}$

3. Design of arbitrary C that stabilizes P

$$\left(C(s) = \frac{3 + \dfrac{s-2}{s+1} Q(s)}{1 - \dfrac{1}{s+1} Q(s)} \right)$$