EECE423-01: 현대제어이론

Modern Control Theory

Chapter 5: Stability

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◆ The main topics of this chapter are

1. Matrix Norms

2. Lyapunov Stability

3. Lyapunov Stability Theorem

4. Input-Output Stability

Appendix: Stability for LTV systems

1. Matrix Norms

◆ Vector norms

1. The 1-norm of an n-dimensional vector:

$$\|v\|_1 := |v_1| + |v_2| + \cdots + |v_n|$$

2. The ∞-norm of an n-dimensional vector:

$$\|v\|_\infty := \max_{1 \le i \le n} |v_i|$$

3. The 2-norm of an n-dimensional vector:

$$\|v\|_2 := (v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2} = (v^T v)^{1/2}$$

◆ The 1-norm of a matrix

For an $m \times n$ matrix $A = [a_{ij}]$,

$$\|A\|_1 := \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1}$$

This can be further obtained by

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|$$

◆ The ∞-norm of a matrix

For an $m \times n$ matrix $A = [a_{ij}]$,

$$\|A\|_\infty := \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}$$

This can be further obtained by

$$\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|$$

◆ The 2-norm of a matrix

For an $m \times n$ matrix $A = [a_{ij}]$,

$$\|A\|_2 := \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

This can be further obtained by

$$\|A\|_2 = \sigma_{\max}(A)$$

where $\sigma_{\max}(A)$ denotes the largest singular value of A.

◆ Example

For $A = \begin{bmatrix} 3 & -3 \\ -2 & 5 \end{bmatrix}$, compute $\|A\|_1$, $\|A\|_\infty$ and $\|A\|_2$.

- $\|A\|_1 = \max\{3 + |-2|, |-3| + 5\} = 8$

- $\|A\|_\infty = \max\{3 + |-3|, |-2| + 5\} = 7$

- $\|A\|_2 = \max\left\{\sqrt{(47 + \sqrt{1885})/2}, \sqrt{(47 - \sqrt{1885})/2}\right\}$

$$= \sqrt{(47 + \sqrt{1885})/2}$$

◆ Equivalence of matrix norms

All matrix norms are ***equivalent*** in the sense that each one of them can be upper and lower bounded by any other times a multiplicative constant:

- $\frac{\|A\|_1}{\sqrt{m}} \le \|A\|_2 \le \sqrt{n}\|A\|_1, \quad \forall A \in \mathbb{R}^{m \times n}$

- $\frac{\|A\|_\infty}{\sqrt{n}} \le \|A\|_2 \le \sqrt{m}\|A\|_\infty, \quad \forall A \in \mathbb{R}^{m \times n}$

- $\frac{\|A\|_1}{m} \le \|A\|_\infty \le n\|A\|_1, \quad \forall A \in \mathbb{R}^{m \times n}$

◆ Properties of matrix norms

(1) Submultiplicative:

$$\|AB\|_p \leq \|A\|_p \|B\|_p \quad (p = 1, 2, \infty)$$

(2) There exists a vector $x^* \in \mathbb{R}^n$ such that

$$\|A\|_p = \frac{\|Ax^*\|_p}{\|x^*\|_p} \quad (p = 1, 2, \infty)$$

2. Lyapunov Stability

◆ Review of state solutions to LTI systems

Consider the following continous-time LTI system:

$$\begin{cases} \dfrac{dx(t)}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}, \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^k,\ y(t) \in \mathbb{R}^m$$

On the other hand, the unique solution to

$$\frac{dx(t)}{dt} = Ax(t), \quad x(0) = x_0 \in \mathbb{R}^n$$

is given by

$$x(t) = e^{At}x_0, \quad \forall t \geq 0$$

◆ Lyapunov stability

The LTI system $\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$ is said to be

1. *marginally stable* in the sense of Lyapunov if, for every initial

   condition $x(0) = x_0 \in \mathbb{R}^n$, the homogeneous state response

$$x(t) = e^{At} x_0, \quad \forall t \geq 0$$

   is uniformly bounded,

2. *asymptotically stable* in the sense of Lyapunov if, in addition,

   for every initial condition $x(0) = x_0 \in \mathbb{R}^n$, we have

$$x(t) \to 0 \text{ as } t \to \infty,$$

3. *exponentially stable* if, in addition, there exist constants $c, \lambda > 0$,

   such that, for every initial condtion $x(0) = x_0 \in \mathbb{R}^n$, we have

$$\|x(t)\| \leq ce^{-\lambda t}\|x(0)\|, \quad \forall t \geq 0, \qquad \text{or}$$

4. *unstable* if it is not marginally stable in the Lyapunov sense.

◆ Summary of Lyapunov stability

1. The matrices B, C and D play no role in the definitions of Lyapunov stability; only A matters.

2. For *marginally stable* systems, the effect of initial conditions does not grow unbounded with time (but it may grow temporarily during a transient phase).

3. For *asymptotically stable* systems, the effect of initial conditions eventually disappears with time.

4. For *unstable* systems, the effect of initial conditions (may) grow over time (depending on the specific initial condtions and C).

◆ Eigenvalue conditions

By noting the Jordan canonical form of A (as well as e^{At}), we can conclude the followings:

The LTI system $\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$ is said to be

1. *marginally stable* if and only if all the eigenvalues of A have negative or zero real parts and all the Jordan blocks corresponding to eigenvalues with zero real parts are 1×1,

2. *asymptotically stable* if and only if all the eigenvalues of A have strictly negative real parts,

3. *exponentially stable* if and only if all the eigenvalues of A have strictly negative real parts,

4. *unstable* if and only if at least one eigenvalue of A has a positive real part or zero real part, but the corresponding Jordan blocks is larger than 1×1.

◆ Remarks on eigenvalue conditions

1. Asymptotic stability and exponential stabilty are equivalent concepts for LTI systems.

2. These conditions do not generalize to LTV systems, even if the eigenvalues of $A(t)$ do not depend on t. One can find matrix-valued signals $A(t)$ that are stability matrices for every $t \geq 0$, but the LTV system $\dot{x} = Ax(t)$ is not even stable.

◆ Examples

Let us consider the LTI homogeneous system

$$\frac{dx(t)}{dt} = Ax(t), \quad x(0) = x_0 \in \mathbb{R}^n.$$

(1) When $A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$, the system is marginally stable.

(2) When $A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$, the system is exponentially stable.

(3) When $A = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$, the system is unstable.

3. Lyapunov Stability Theorem

◆ Lyapunov stability theorem

- For the homogeneous LTI system

$$\frac{dx(t)}{dt} = Ax(t), \quad x(t) \in \mathbb{R}^n,$$

the following five conditions are equivalent:

1. The system is *asymptotically stable*.

2. The system is *exponentially stable*.

3. All the eigenvalues of A have strictly negative real parts.

4. For every symmetric positive defnite matrix Q, there exists a unique solution P to the following Lyapunov equation

$$A^T P + PA = -Q.$$

Moreover, P is a positive definite matrix.

Proof: The equivalence between condtions 1, 2 and 3 is readily followed if we note the Jordan canonocal form of A.

We first note that for a positive definite matrix S,

$$0 < \lambda_{\min}(S)\|x\|_2^2 \leq x^T S x \leq \lambda_{\max}(s)\|x\|_2^2, \quad \forall x \neq 0,$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and largest eigenvalues of $(\cdot)$, respectively.

(i) We prove that condtion 4 $\Rightarrow$ condtion 2.

$$\text{Let } v(t) := x^T(t)Px(t) \geq 0, \quad \forall t \geq 0.$$

Then,

$$\dot{v}(t) = \dot{x}(t)^T Px(t) + x^T(t)P\dot{x}(t) = x^T(t)(A^T P + PA)x(t)$$

$$= -x^T(t)Qx(t) \leq -\lambda_{\min}(Q)\|x(t)\|_2^2 \leq 0, \quad \forall t \geq 0.$$

$$(0 \leq \lambda_{\min}(Q)\|x\|_2^2 \leq x^T Qx \Rightarrow -x^T Qx \leq -\lambda_{\min}(Q)\|x\|_2^2)$$

This leads to

$$\dot{v}(t) \leq -\lambda_{\min}(Q)\|x(t)\|_2^2 \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}v(t) =: -\lambda v(t), \quad \forall t \geq 0$$

$$(0 \leq v(t) \leq \lambda_{\max}(P)\|x(t)\|_2^2 \Rightarrow -\lambda_{\max}(P)\|x(t)\|_2^2 \leq -v(t))$$

Here, if we let $u(t) := e^{\lambda t}v(t)$,

$$\dot{u}(t) = \lambda e^{\lambda t}v(t) + e^{\lambda t}\dot{v}(t) \leq \lambda e^{\lambda t}v(t) - \lambda e^{\lambda t}v(t) = 0$$

Thus,

$$e^{\lambda t}v(t) = u(t) \leq u(0) = v(0), \quad \forall t \geq 0$$

$$\Rightarrow v(t) \leq e^{-\lambda t}v(0), \quad \forall t \geq 0$$

It readily follows that

$$\|x(t)\|_2^2 \leq \frac{x^T(t)Px(t)}{\lambda_{\min}(P)} = \frac{v(t)}{\lambda_{\min}(P)}$$

$$\leq \frac{e^{-\lambda t}v(0)}{\lambda_{\min}(P)} \leq \frac{e^{-\lambda t}\lambda_{\max}(P)}{\lambda_{\min}(P)}\|x(0)\|_2^2, \quad \forall t \geq 0$$

$$(v(0) = x^T(0)Px(0) \leq \lambda_{\max}(P)\|x(0)\|_2^2)$$

Hence, we obtain the following *exponentially* stability:

$$\boxed{\|x(t)\|_2 \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}e^{-\frac{\lambda}{2}t}\|x(0)\|_2, \quad \forall t \geq 0.}$$

(ii) We prove that condtion 2 $\Rightarrow$ condtion 4.

Show that the unique solution to $A^T P + PA = -Q$ is given by

$$P := \int_0^\infty e^{A^T t} Q e^{At} dt.$$

(a) P is well-defined since $\|e^{A^T t} Q e^{At}\|_2$ converges to 0 exponentially fast as $t \to \infty$.

(b) $P^T = \int_0^\infty (e^{A^T t} Q e^{At})^T dt = \int_0^\infty e^{A^T t} Q e^{At} dt = P.$

$$x^T P x = \int_0^\infty x^T e^{A^T t} Q e^{At} x dt = 0 \Leftrightarrow e^{At} x = 0 \text{ (Q is postive definite)}$$
$$\Leftrightarrow x = 0 \text{ (e^{At} is nonsingular)}$$

$\Rightarrow$ P is a positive definite matrix.

(c) Because $\dfrac{d}{dt}(e^{A^T t}Qe^{At}) = A^T e^{A^T t}Qe^{At} + e^{A^T t}Qe^{At}A$,

$$
\begin{aligned}
A^T P + PA &= \int_0^\infty (A^T e^{A^T t}Qe^{At} + e^{A^T t}Qe^{At}A)dt \\
&= \int_0^\infty \frac{d}{dt}(e^{A^T t}Qe^{At})dt = \left[e^{A^T t}Qe^{At}\right]_{t=0}^\infty \\
&= (\lim_{t\to\infty} e^{A^T t}Qe^{At}) - Q = -Q.
\end{aligned}
$$

$\Rightarrow P$ is a solution to $A^T P + PA = -Q$.

(d) Assume that $\overline{P}$ is another solution such that

$$A^T\overline{P} + \overline{P}A = -Q \quad \text{(and} \quad A^TP + PA = -Q\text{)}$$

Then, we have

$$e^{A^Tt}A^T(P - \overline{P})e^{At} + e^{A^Tt}A^T(P - \overline{P})Ae^{At} = 0, \quad \forall t \geq 0.$$

Because

$$\frac{d}{dt}(e^{A^Tt}(P - \overline{P})e^{At}) = e^{A^Tt}A^T(P - \overline{P})e^{At} + e^{A^Tt}A^T(P - \overline{P})Ae^{At} = 0,$$

$e^{A^Tt}(P - \overline{P})e^{At}$ must remain constant for all times, and this quantity converges to 0 as $t \to \infty$. Since e^{At} is nonsingular, $P = \overline{P}$.

$\Rightarrow$ P is the unique solution to $A^TP + PA = -Q$.

◆ Characteristics of Lyapunov stability

- Without explicitly computing the solution of a LTI system, we can determine whether or not the LTI system is stable (by computing the eigenvalues of the matrix A).

- Lyapunov stability is concerned only with the effect of the initial conditions (i.e., the value of $x(t_0)$) on the response of the system (without considering the effect of the input $u(\cdot)$).

4. Input-Output Stability

◆ Review of output solutions to LTI systems

Consider the following continous-time LTI system:

$$\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}, \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^k,\ y(t) \in \mathbb{R}^m$$

The output $y(t)$ for zero initional condtions (i.e., $x(0) = 0$) is given by

$$y(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + D(t)u(t)$$

◆ Bounded input bounded output (BIBO) stability

The LTI system $\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$ is said to be

(*uniformly*) BIBO stable if there exists a finite constant c such that,

for every input $u(\cdot)$, the output $y(\cdot)$ with $x(0) = 0$ satisfies

$$\sup_{0 \le t < \infty} \|y(t)\| \le c \sup_{0 \le t < \infty} \|u(t)\|.$$

◆ Time-domain BIBO stability condition

- For the LTI system $\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$,

  the following two statements are equivalent:

  1. The LTI system is uniformly BIBO stable.
  2. For every entry $g_{ij}(t)$ of $Ce^{At}B$, we have

  $$\int_0^\infty |g_{ij}(t)|dt < \infty.$$

Proof: (i) statement 2 $\Rightarrow$ statement 1.

Let $\tilde{g}(t,\tau) := Ce^{A(t-\tau)}B$ and $\tilde{g}_{ij}(t,\tau)$ be the (i,j)th element of $\tilde{g}(t,\tau)$.

Then, we obtain the following relation:

$$\boxed{\begin{aligned} \|y(t)\| &\le \int_0^t \|\tilde{g}(t,\tau)\|\|u(\tau)\|d\tau + \|D\|\|u(t)\| \\ &\le \left(\int_0^t \|\tilde{g}(t,\tau)\|d\tau + \|D\|\right) \sup_{0\le\tau<\infty} \|u(\tau)\|, \quad \forall t \ge 0. \end{aligned}}$$

Here, if we note that

$$\int_0^t \|\tilde{g}(t,\tau)\|d\tau \le \int_0^t \sum_{i,j} |\tilde{g}_{ij}(t,\tau)|d\tau \le \int_0^\infty \sum_{i,j} |g_{ij}(t)|dt < \infty$$

the aforementioned relation further leads to

$$\|y(t)\| \le \left(\int_0^\infty \sum_{i,j} |g_{ij}(t)| dt + \|D\| \right) \sup_{0 \le \tau < \infty} \|u(\tau)\|, \quad \forall t \ge 0$$

This clearly means that

$$\boxed{\begin{array}{l} \displaystyle \sup_{0 \le \tau < \infty} \|y(\tau)\| \le c \sup_{0 \le \tau < \infty} \|u(\tau)\| \\ \text{where} \\ \displaystyle c := \int_0^\infty \sum_{i,j} |g_{ij}(t)| dt + \|D\|. \end{array}}$$

(ii) statement 1 $\Rightarrow$ statement 2.

We prove by showing that statement 2 is false $\Rightarrow$ statement 1 is false.

Suupose that 2 is false because

$$\int_0^\infty |g_{ij}(t)|dt$$

is unbounded for some i and j.

Consider the following switching input for some $T \geq 0$:

$$u_T(\tau) := \begin{cases} +e_j & \tilde{g}_{ij}(T,\tau) \geq 0 \\ -e_j & \tilde{g}_{ij}(T,\tau) < 0 \end{cases}$$

Then, the corresponding output $y(t)$ at $t = T$ is given by

$$y(T) = \int_0^T \tilde{g}(t, \tau) u_T(\tau) d\tau + D u_T(T)$$

and its ith entry (i.e., $y_i(T)$) coincides with

$$\int_0^T |g_{ij}(\tau)| d\tau \pm d_{ij}$$

Thus, it readily follows that

$$\boxed{\lim_{T \to \infty} y_i(T) = \int_0^\infty |g_{ij}(\tau)| d\tau \pm d_{ij}}$$

and $\sup_{0 \le t < \infty} \|y(t)\|$ is unbounded.

◆ BIBO stability vs Lyapunov stability

- When the LTI system $\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$

  is exponentially stable, then it must also be BIBO stable.

  Generally, the coverse is not true.

Proof: Let B_j be the jth column vector of B and C_i be the ith row vector of C.

Then, for every entry $g_{ij}(t)$ of $Ce^{At}B$, we have

$$|g_{ij}(t)| = |C_i e^{At} B_j| \leq \|C_i\| \|e^{At} B_j\|$$

If we define $x(0) := B_j$, then $x(t) = e^{At} B_j$.

Because this LTI system is asymptotically stable

(as well as expoenetially stable),

there exists a $\lambda < 0$ such that

$$|g_{ij}(t)| = |C_i e^{At} B_j| \leq \|C_i\| \|e^{At} B_j\| \leq e^{-\lambda t} \|C_j\| \|x(0)\|$$

Thus, we obtain

$$\boxed{\int_0^\infty |g_{ij}(t)| dt \leq \int_0^\infty e^{-\lambda t} \|C_j\| \|x(0)\| dt < \infty.}$$

Appendix: Stability for LTV Systems

◆ Review of state solutions to LTV systems

Consider the following continous-time LTV system:

$$\begin{cases} \dfrac{dx(t)}{dt} & = A(t)x(t) + B(t)u(t) \\ y(t) & = C(t)x(t) + D(t)u(t) \end{cases}, \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^k,\ y(t) \in \mathbb{R}^m$$

On the other hand, assume that the unique solution to

$$\frac{dx(t)}{dt} = A(t)x(t), \quad x(0) = x_0 \in \mathbb{R}^n$$

is given by

$$x(t) = \Phi(t,0)x_0, \quad \forall t \geq 0$$

◆ Lyapunov stability

The LTV system $\begin{cases} \dfrac{dx(t)}{dt} & = A(t)x(t) + B(t)u(t) \\ y(t) & = C(t)x(t) + D(t)u(t) \end{cases}$ is said to be

1. *marginally stable* in the sense of Lyapunov if, for every initial

   condition $x(0) = x_0 \in \mathbb{R}^n$, the homogenous state response

$$x(t) = \Phi(t, 0)x_0, \quad \forall t \geq 0$$

   is uniformly bounded,

2. *asymptotically stable* in the sense of Lyapunov if, in addition,

   for every initial condition $x(0) = x_0 \in \mathbb{R}^n$, we have

$$x(t) \to 0 \text{ as } t \to \infty,$$

3. *exponentially stable* if, in addition, there exist constants $c, \lambda > 0$,

   such that, for every initial condtion $x(0) = x_0 \in \mathbb{R}^n$, we have

$$\|x(t)\| \leq ce^{-\lambda t}\|x(0)\|, \quad \forall t \geq 0, \qquad \text{or}$$

4. *unstable* if it is not marginally stable in the Lyapunov sense.

◆ Remarks on Lyapunov stability

1. The matrices $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ play no role in the definitions of Lyapunov stability; only $A(\cdot)$ matters.

2. For *marginally stable* systems, the effect of initial conditions does not grow unbounded with time (but it may grow temporarily during a transient phase).

3. For *asymptotically stable* systems, the effect of initial conditions eventually disappears with time.

4. For *unstable* systems, the effect of initial conditions (may) grow over time (depending on the specific initional condtions and $C(\cdot)$).

◆ Review of output solutions to LTV systems

Consider the following continous-time LTV system:

$$\begin{cases} \dfrac{dx(t)}{dt} & = A(t)x(t) + B(t)u(t) \\ y(t) & = C(t)x(t) + D(t)u(t) \end{cases}, \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^k,\ y(t) \in \mathbb{R}^m$$

The output $y(t)$ for zero initional condtions (i.e., $x(0) = 0$) is given by

$$y(t) = \int_0^t C(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$

where $\Phi(t,\tau)$ denotes the system's state transition matrix.

◆ Bounded input bounded output (BIBO) stability

The LTV system $\begin{cases} \dfrac{dx(t)}{dt} & = A(t)x(t) + B(t)u(t) \\ y(t) & = C(t)x(t) + D(t)u(t) \end{cases}$ is said to be

(*uniformly*) BIBO stable if there exists a finite constant c such that,

for every input $u(\cdot)$, the output $y(\cdot)$ with $x(0) = 0$ satisfies

$$\sup_{0 \le t < \infty} \|y(t)\| \le c \sup_{0 \le t < \infty} \|u(t)\|.$$

◆ Time-domain BIBO stability condition for LTV systems

- For the LTV system

$$\begin{cases} \dfrac{dx(t)}{dt} & = A(t)x(t) + B(t)u(t) \\ y(t) & = C(t)x(t) + D(t)u(t) \end{cases},$$

the following two statements are equivalent:

1. The LTV system is uniformly BIBO stable.

2. Every entry of $D(\cdot)$ is uniformly bounded and

$$\sup_{t \geq 0} \int_0^t |g_{ij}(t,\tau)| d\tau < \infty,$$

for every entry $g_{ij}(t,\tau)$ of $C(t)\Phi(t,\tau)B(\tau)$.

Proof: We briefly sketch the proof.

(i) statement 2 $\Rightarrow$ statement 1.

$$\|y(t)\| \leq \int_0^t \|C(t)\Phi(t,\tau)B(\tau)\|\|u(\tau)\|d\tau + \|D(t)\|\|u(t)\|$$

$$\leq \left(\sup_{t\geq 0}\int_0^t \|C(t)\Phi(t,\tau)B(\tau)\|d\tau + \sup_{0\leq t<\infty}\|D(t)\|\right)\sup_{0\leq t<\infty}\|u(t)\|, \quad \forall t \geq 0.$$

If we define $c := \left(\sup_{t\geq 0}\int_0^t \|C(t)\Phi(t,\tau)B(\tau)\|d\tau + \sup_{0\leq t<\infty}\|D(t)\|\right)$,

we can show g is finite by using $\sup_{t\geq 0}\int_0^t |g_{ij}(t,\tau)|d\tau < \infty$

together with $\sup_{0\leq t<\infty}\|D(t)\| < \infty$.

(ii) statement 1 $\Rightarrow$ statement 2.

We show that statement 2 is false $\Rightarrow$ statement 1 is false.

Suupose that 2 is false because the entry $d_{ij}(t)$ of $D(t)$ is unbounded at $t = T$.

Consider the following step input

$$u_T(t) := \begin{cases} 0 & 0 \leq t < T \\ e_j & t \geq T \end{cases}$$

where $e_j \in \mathbb{R}^k$ is the jth standard basis of $\mathbb{R}^k$.

Then, $\sup_{0 \leq t < \infty} \|u_T(t)\| = 1,\ y(T) = D(T)u_T(T) = D(T)e_j$ together with

$$\sup_{0 \leq t < \infty} \|y(t)\| \geq \|y(T)\| = \|D(T)u_T(T)\| = \|D(T)e_i\| \geq |d_{ij}(t)|e_j.$$

$\Rightarrow \sup_{0 \leq t < \infty} \|y(t)\|$ is unbounded.

Suupose that 2 is false because

$$\int_0^T |g_{ij}(T,\tau)|d\tau$$

is unbounded for some i and j together with $T \geq 0$.

Consider the following switching input

$$u_T(\tau) := \begin{cases} +e_j & g_{ij}(T,\tau) \geq 0 \\ -e_j & g_{ij}(T,\tau) < 0 \end{cases}$$

Then, the corresponding output $y(t)$ at $t = T$ is given by

$$y(T) = \int_0^T C(T)\Phi(T,\tau)B(\tau)u(\tau)d\tau + D(T)u(T)$$

and its ith entry coincides with

$$\int_0^T |g_{ij}(T,\tau)|d\tau \pm d_{ij}(T)$$

$\Rightarrow \sup\limits_{0\leq t<\infty} \|y(t)\|$ is not bounded.