EECE423-01: 현대제어이론

Modern Control Theory

Chapter 8: Controller Synthesis

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◆ The main topics of this chapter are

1. State Feedback

2. State Observers

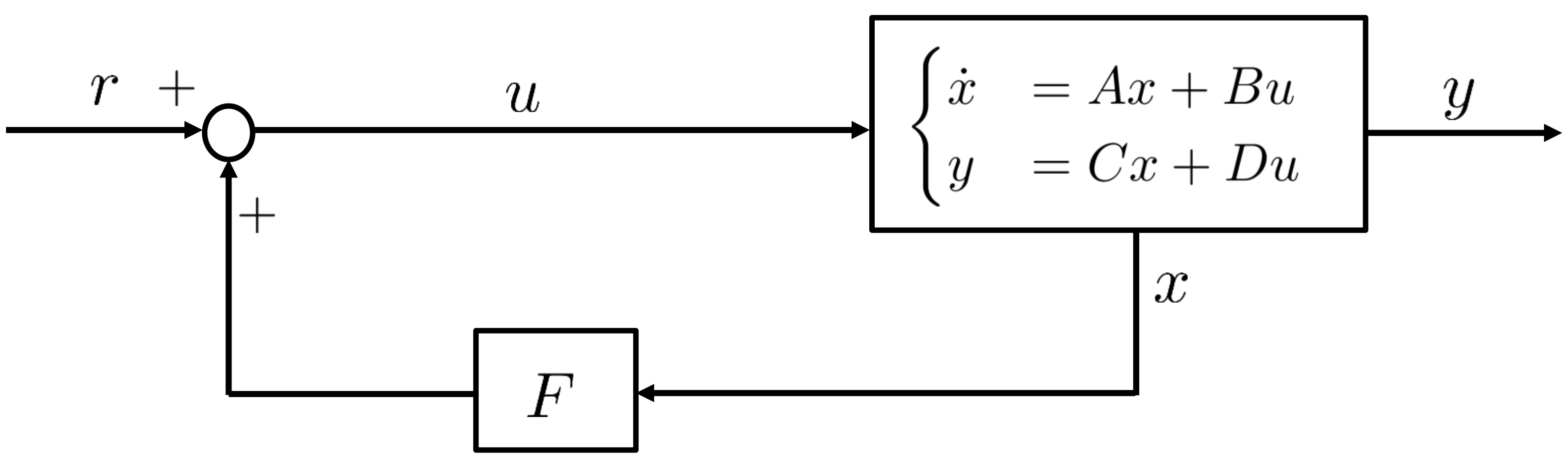
1. State Feedback

◆ Basic Structure

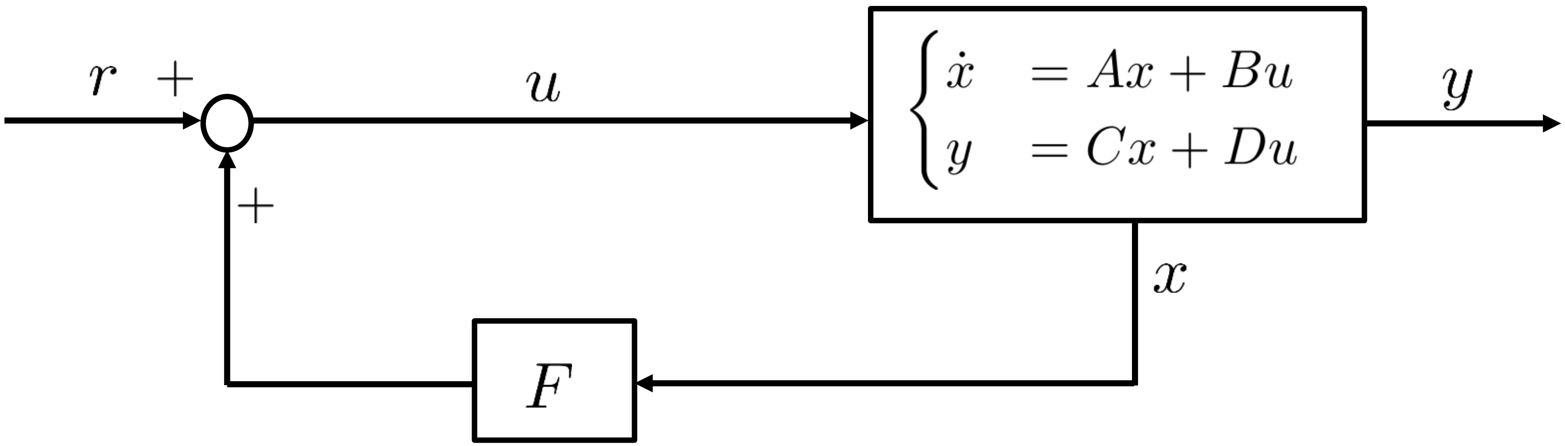
For the LTI system

$$\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}, \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^k,\ y(t) \in \mathbb{R}^m$$

let us consider the following structure:



◆ Closed-Loop Systems vs Open-Loop Systems



- Closed-loop systems

$$\begin{cases} \dot{x} &= (A + BF)x + Br \\ y &= (C + DF)x + Dr \end{cases}$$

- Open-loop systems

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

$\Rightarrow$ Taking an adequate feedback gain F could alter the pole locations

◆ Controllable Single-Input Systems

For the single-output LTI system $\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$,

(i.e., u is a scalar function ($k = 1$)) let us assume that it is controablle.

On the other hand, recall the controllable canonical form given by

$$\dot{x}_{\mathrm{c}} = A_{\mathrm{c}} x_{\mathrm{c}} + B_{\mathrm{c}} u$$

where $\quad A_{\mathrm{c}} := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\alpha_0 & \cdots & \cdots & \cdots & -\alpha_{n-1} \end{bmatrix}, \quad B_{\mathrm{c}} := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

$\det(sI - A) = s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0.$

Furthermore, consider the controllability matrix defined as

$$U_{\mathrm{c}} := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

which is an $n \times n$ nonsingular matrix.

Let $U_{\mathrm{c}}^{-1} =: \begin{bmatrix} \star \\ q \end{bmatrix}$, where q is the nth row of U_{c}^{-1}

and $\star$ means the remaining part of U_{c}^{-1}.

This immediately means that

$$qA^iB = 0 \ (i = 0, \ldots, n-2) \text{ and } qA^{n-1}B = 1.$$

By using q, if we also define $T := \begin{bmatrix} q \\ qA \\ \vdots \\ qA^{n-1} \end{bmatrix}$, we have the following result:

$$TU_{\mathrm{c}} = \begin{bmatrix} TB & TAB & \cdots & TA^{n-1}B \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & 1 & \times \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & \times & \times \\ 1 & \times & \cdots & \cdots & \times \end{bmatrix}.$$

This implies that $|TU_{\mathrm{c}}| = |T| \cdot |U_{\mathrm{c}}| \neq 0 \Rightarrow |T| \neq 0$.

In other words, T is a nonsinglular matrix.

Next, further note that

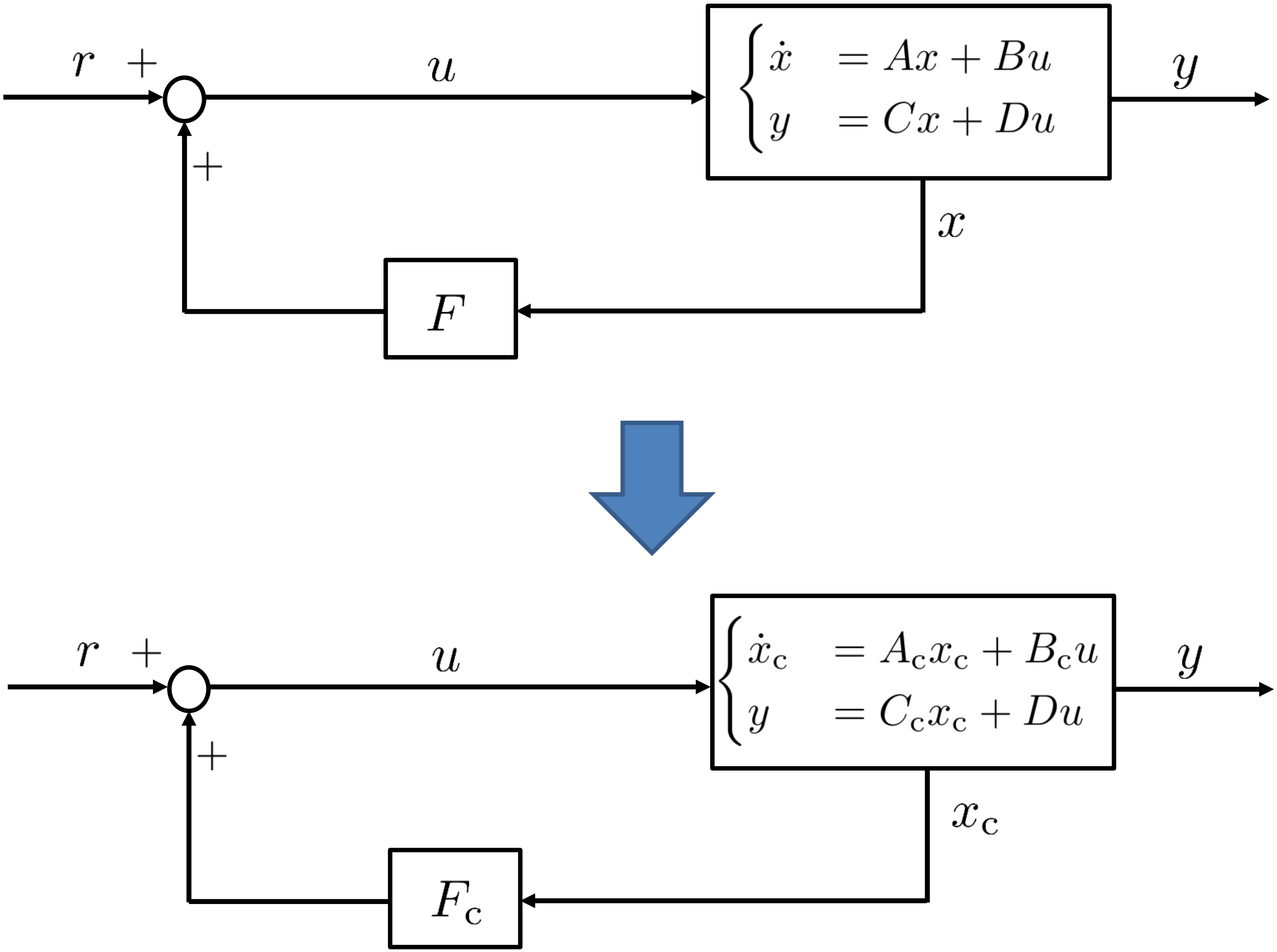
$$TA = \begin{bmatrix} qA \\ qA^2 \\ \vdots \\ qA^n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\alpha_0 & \cdots & \cdots & \cdots & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} q \\ qA \\ \vdots \\ qA^{n-1} \end{bmatrix} = A_{\mathrm{c}} T.$$

where the last row is established by using the Cayley–Hamilton Theorem.

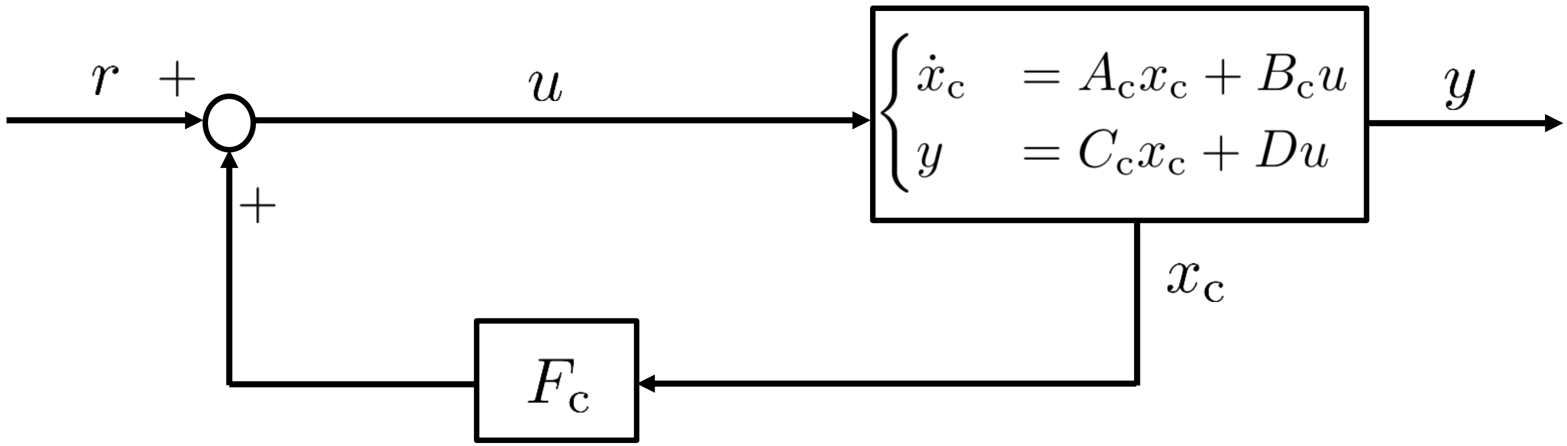
Hence, we obtain

$$TAT^{-1} = A_{\mathrm{c}} \text{ and } TB = \begin{bmatrix} qB \\ qAB \\ \vdots \\ qA^{n-1}B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = B_{\mathrm{c}}.$$

◆ State Feedback for Controllable Single-Input Systems



◆ Poles Placement



$$A_c := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\alpha_0 & \cdots & \cdots & \cdots & -\alpha_{n-1} \end{bmatrix}, \qquad B_c := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

If we let $F_\mathrm{c} := \begin{bmatrix} f_1 & \cdots & f_{n-1} \end{bmatrix}$,

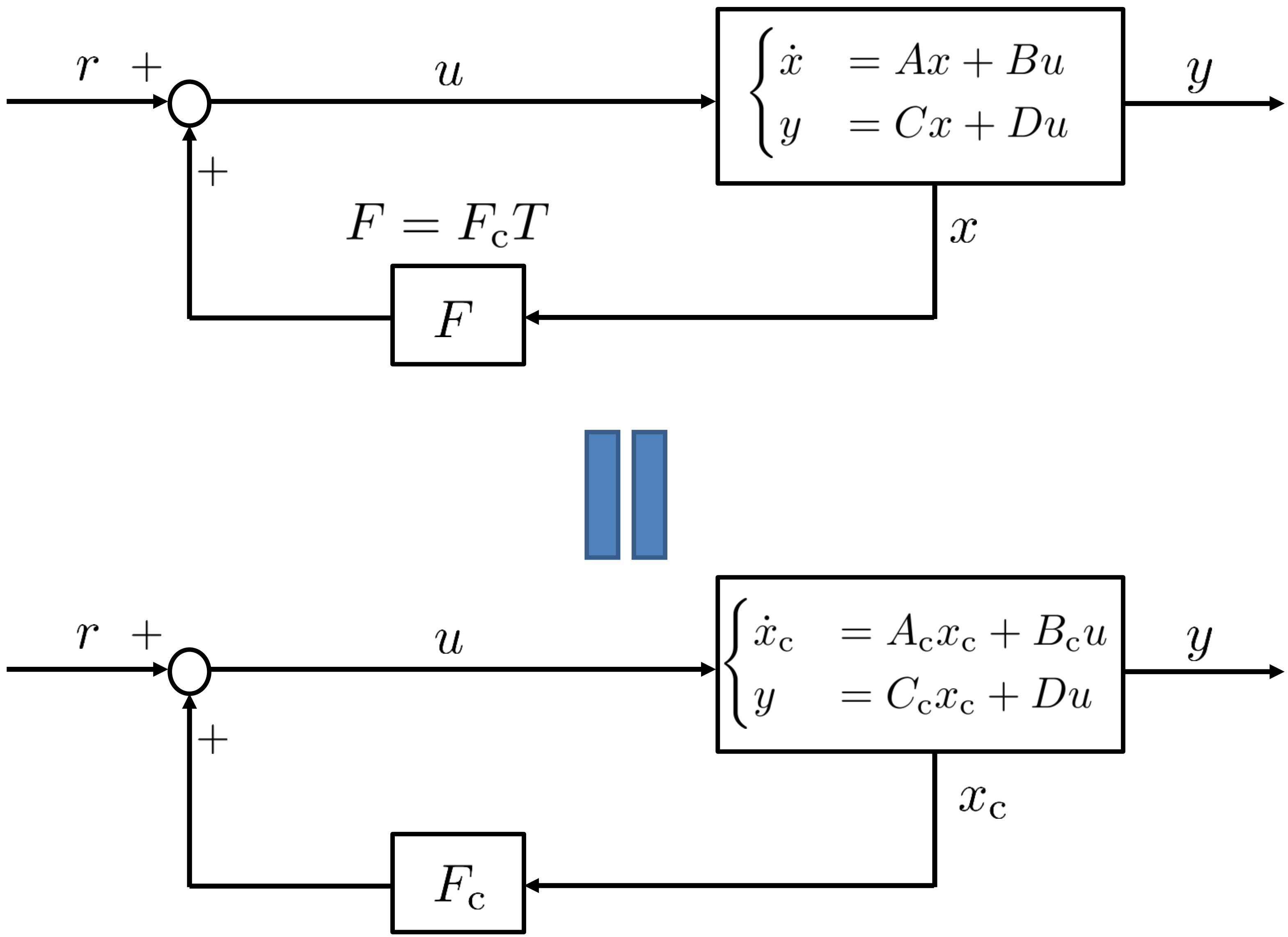
$$A_\mathrm{c} + B_\mathrm{c}F_\mathrm{c} := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\alpha_0 + f_0 & \cdots & \cdots & \cdots & -\alpha_{n-1} + f_{n-1} \end{bmatrix}$$

and

$$\det(sI - (A_\mathrm{c} + B_\mathrm{c}F_\mathrm{c})) = s^n + (\alpha_{n-1} - f_{n-1})s^{n-1} + \cdots + (\alpha_0 - f_0).$$

$\Rightarrow$ The n eigenvalues of $A_\mathrm{c} + B_\mathrm{c}F_\mathrm{c}$ can be assigned to arbitrary, real, or complex conjugate locations by selecting an adequate F_c.

◆ State Feedback



◆ Analysis for Multi-Input Systems

Let us consider the following mulit-output LTI system:

$$\begin{cases} \dfrac{dx(t)}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}, \qquad \text{with } x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^k,\ y(t) \in \mathbb{R}^m.$$

(i.e., u is a vector function ($k \geq 2$))

Assume that this system is controablle and $\operatorname{rank}(B) = k \leq n$.

In other words, the controllability matrix

$$U_{\mathrm{c}} := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

has rank n.

For the $n \times nk$ controllability matrix defined as

$$U_{\mathrm{c}} := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

represent it by

$$U_{\mathrm{c}} = \begin{bmatrix} b_1 & \cdots & b_k & Ab_1 & \cdots & Ab_k & \cdots & A^{n-1}b_1 & \cdots & A^{n-1}b_k \end{bmatrix}$$

where b_i $(i = 1, \ldots, k)$ denotes the ith column of B.

Starting from the left and moving to the right,

select the first n independent columns.

Reorder these columns by taking first b_1, Ab_1, A^2b_1 etc.,

until all columns involving b_1 have been taken.

Then, take b_2, Ab_2, etc.; and lastly take b_k, Ab_k, etc.

Through the above procedure, we can obtain

$$\tilde{U}_{\mathrm{c}} := \begin{bmatrix} b_1 & Ab_1 & \cdots & A^{\mu_1 - 1} b_1 & \cdots & b_k & Ab_k & \cdots & A^{\mu_k - 1} b_k \end{bmatrix}$$

$$(\mu_1 + \cdots + \mu_k = n)$$

- All columns of B are always present since $\operatorname{rank}(B) = k$.
- $\mu_i \geq 1$ for all $i = 1, \ldots, k$.
- If $A^l b_j$ is present, then $A^{l-1} b_j$ must also be present.
- Column $A^{\mu_i} b_i$ is dependent on the previous ones:

$$A^{\mu_i} b_i = \sum_{j=1}^{k} \sum_{l=1}^{\min(\mu_i, \mu_j)} \alpha_{ijl} A^{l-1} b_j + \sum_{j=1}^{i-1} \beta_{ij} A^{\mu_i} b_j$$

First term: Independent columns in $B, AB, \ldots, A^{\mu_i - 1} B$

Second term: Independent columns in $A^{\mu_i} B$ to the left of $A^{\mu_i} b_i$

◆ Controllability Index

The k integers $\mu_i,\ i = 1, \ldots, k$ are the controllability idices of the system

$$\text{given by } \begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}, \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^k,\ y(t) \in \mathbb{R}^m.$$

Indeed, $\mu := \max\limits_{1 \le i \le k} \mu_i$ is called the controllability index of the system

and the followings are naturally established:

$$\sum_{i=1}^{k} \mu_i = n \text{ and } k\mu \geq n.$$

◆ Controllable Canonical Form

Next, consider $\tilde{U}_{\mathrm{c}}^{-1}$ and let $q_i \in \mathbb{R}^n$ $(i = 1, \ldots, k)$ denote its σ_ith row, i.e.,

$$\tilde{U}_{\mathrm{c}}^{-1} = \begin{bmatrix} \times \\ \vdots \\ \times \\ q_1 \\ \vdots \\ \times \\ \vdots \\ \times \\ q_k \end{bmatrix}, \qquad \text{where } \sigma_i := \sum_{j=1}^{i} \mu_j,\ i = 1, \ldots, k$$

$$(\text{i.e., } \sigma_1 = \mu_1,\ \sigma_2 = \mu_1 + \mu_2,\ \ldots,\ \sigma_k = \mu_1 + \cdots + \mu_k = n).$$

If we define

$$T := \begin{bmatrix} q_1^T & (q_1 A)^T & \cdots & (q_1 A^{\mu_1 - 1})^T & \cdots & q_k^T & (q_k A)^T & \cdots & (q_k A^{\mu_k - 1})^T \end{bmatrix}^T,$$

$$T\tilde{U}_{\mathrm{c}} = \begin{bmatrix} q_1 b_1 & \cdots & q_1 A^{\mu_1 - 1} b_1 & \cdots & q_1 b_k & \cdots & q_1 A^{\mu_k - 1} b_k \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ q_1 A^{\mu_1 - 1} b_1 & \cdots & \cdots & \cdots & \cdots & \cdots & q_1 A^{\mu_1 + \mu_k - 2} b_k \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ q_k b_1 & \cdots & q_k A^{\mu_1 - 1} b_1 & \cdots & q_k b_k & \cdots & q_k A^{\mu_k - 1} b_k \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ q_k A^{\mu_k - 1} b_1 & \cdots & \cdots & \cdots & \cdots & \cdots & q_k A^{2\mu_k - 2} b_k \end{bmatrix}$$

is an $n \times n$ nonsingular matrix (i.e., $|T\tilde{U}_{\mathrm{c}}| \neq 0$).

This implies that $|T\tilde{U}_{\mathrm{c}}| = |T| \cdot |\tilde{U}_{\mathrm{c}}| \neq 0 \Rightarrow |T| \neq 0$.

In other words, T is a nonsinglular matrix.

Supplementary for $|T| \neq 0$:

$$
\begin{aligned}
q_i \tilde{U}_{\mathrm{c}} &= \begin{bmatrix} q_i b_1 & \cdots & q_i A^{\mu_1 - 1} b_1 & \cdots & q_i A^{\mu_i - 1} b_i & \cdots & q_i b_k & \cdots & q_i A^{\mu_k - 1} b_k \end{bmatrix} \\
&= \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}, \; i = 1, \ldots, k
\end{aligned}
$$

where the 1 occurs at the σ_ith column. This can be also written as

$$
q_i A^{l-1} b_j = 0 \quad l = 1, \ldots, \mu_j, \quad i \neq j,
$$

$$
q_i A^{l-1} b_i = 0 \quad l = 1, \ldots, \mu_i - 1 \quad \text{and} \quad q_i A^{\mu_i - 1} b_i = 1, \quad i = j
$$

Next, further note that

$$
TA = \begin{bmatrix} q_1 A \\ q_1 A^2 \\ \vdots \\ q_1 A^{\mu_1} \\ \vdots \\ q_k A \\ q_k A^2 \\ \vdots \\ q_k A^{\mu_k} \end{bmatrix} = \begin{bmatrix} 0 & & 0 & \cdots & 0 \\ \vdots & I_{\mu_1 - 1} & \vdots & \ddots & \vdots \\ 0 & & 0 & \cdots & 0 \\ \times & \cdots \times & \times & \cdots & \times \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \\ \vdots & \ddots & \vdots & \vdots & I_{\mu_k - 1} \\ 0 & \cdots & 0 & 0 & \\ \times & \cdots & \times & \times & \cdots \times \end{bmatrix} \begin{bmatrix} q_1 \\ q_1 A \\ \vdots \\ q_1 A^{\mu_1 - 1} \\ \vdots \\ q_k \\ q_k A \\ \vdots \\ q_k A^{\mu_k - 1} \end{bmatrix} =: A_{\mathrm{c}} T, \text{ i.e.,}
$$

$A_{\mathrm{c}} = [A_{ij}],\ i, j = 1, \ldots, k$, where

$$
A_{ii} = \begin{bmatrix} 0 & & \\ \vdots & I_{\mu_i - 1} & \\ 0 & & \\ \times & \cdots & \times \end{bmatrix} \in \mathbb{R}^{\mu_i \times \mu_i},\ i = j, \quad A_{ij} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ \times & \cdots & \times \end{bmatrix} \in \mathbb{R}^{\mu_i \times \mu_j},\ i \neq j.
$$

On the other hand,

$$TB = \begin{bmatrix} q_1 B \\ q_1 AB \\ \vdots \\ q_1 A^{\mu_1 - 1} B \\ \vdots \\ q_k B \\ q_k AB \\ \vdots \\ q_k A^{\mu_k - 1} B \end{bmatrix} =: \begin{bmatrix} B_1 \\ \vdots \\ B_k \end{bmatrix} =: B_\mathrm{c}, \text{ i.e.,}$$

$$B_i := \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \times \cdots & \times \end{bmatrix} \in \mathbb{R}^{\mu_i \times k}, \ i = 1, \ldots, k,$$

where the 1 in the last row of B_i occurs at the ith column location.

Supplementary for B_i :

$$\begin{bmatrix} q_i B \\ \vdots \\ q_i A^{\mu_i - 1} B \end{bmatrix} = \begin{bmatrix} q_i b_1 & \cdots & q_i b_i & \cdots & q_i b_k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_i A^{\mu_i - 1} b_1 & \cdots & q_i A^{\mu_i - 1} b_i & \cdots & q_i A^{\mu_i - 1} b_k \end{bmatrix}$$

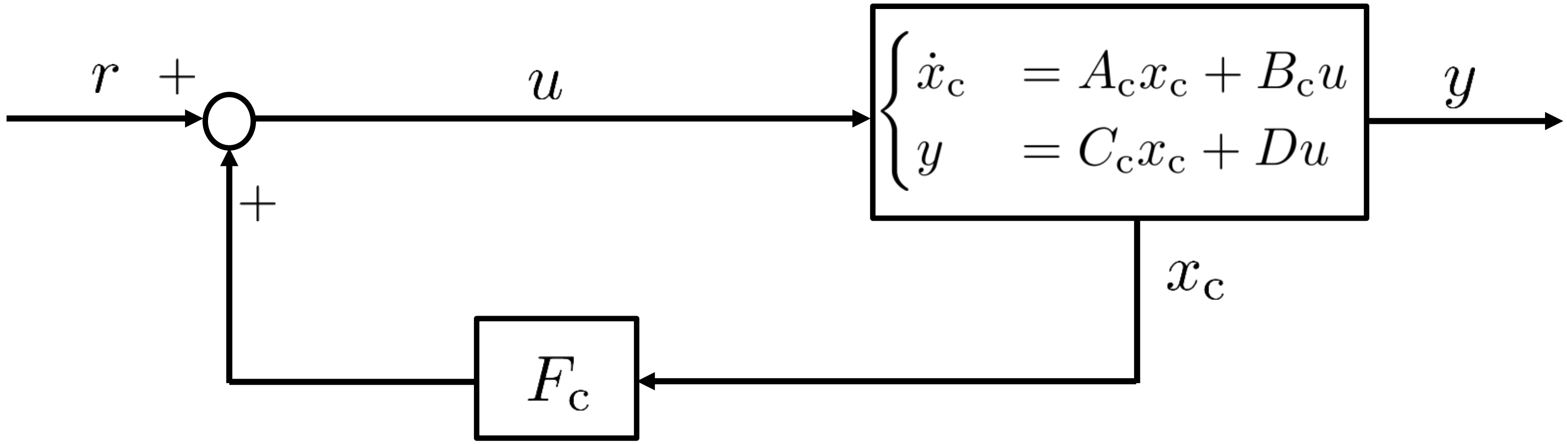
$$q_i A^{l-1} b_j = 0 \quad l = 1, \ldots, \mu_j, \quad i \neq j$$

$$q_i A^{l-1} b_i = 0 \quad l = 1, \ldots, \mu_i - 1 \quad \text{and} \quad q_i A^{\mu_i - 1} b_i = 1, \quad i = j$$

$$q_i A^{l-1} b_j = 0 \quad l = 1, \ldots, \mu_i, \quad i > j$$

$$q_i A^{l-1} b_j = 0 \quad l = 1, \ldots, \mu_i - 1, \quad i < j$$

◆ Poles Placement



$A_c = [A_{ij}],\ i, j = 1, \ldots, k,$ where

$$A_{ii} = \begin{bmatrix} 0 & & \\ \vdots & I_{\mu_i - 1} & \\ 0 & & \\ \times & \cdots & \times \end{bmatrix} \in \mathbb{R}^{\mu_i \times \mu_i},\ i = j,\quad A_{ij} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ \times & \cdots & \times \end{bmatrix} \in \mathbb{R}^{\mu_i \times \mu_j},\ i \neq j.$$

$$B_c = \begin{bmatrix} B_1 \\ \vdots \\ B_k \end{bmatrix}, \text{ where } B_i := \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \times \cdots & \times \end{bmatrix} \in \mathbb{R}^{\mu_i \times k},\ i = 1, \ldots, k.$$

If we let $F_{\mathrm{c}} := \begin{bmatrix} F_1 & \cdots & F_k \end{bmatrix}$, we can achive

$$A_{\mathrm{c}} + B_{\mathrm{c}} F_{\mathrm{c}} := \begin{bmatrix} \tilde{A}_{11} & & \\ & \ddots & \\ & & \tilde{A}_{kk} \end{bmatrix} = \mathrm{diag}(\tilde{A}_{11}, \dots, \tilde{A}_{kk})$$

$$\text{where } \tilde{A}_{ii} = \begin{bmatrix} 0 & & \\ \vdots & I_{\mu_i - 1} & \\ 0 & & \\ -\alpha_0^{[ii]} & \cdots & -\alpha_{\mu_i - 1}^{[ii]} \end{bmatrix} \in \mathbb{R}^{\mu_i \times \mu_i}, \; i = 1, \dots, k$$

and

the coefficients $-\alpha_0^{[ii]}, \dots, -\alpha_{\mu_i - 1}^{[ii]}$ can be arbitrary choosen.

Because $\det(sI - (A_{\mathrm{c}} + B_{\mathrm{c}} F_{\mathrm{c}})) = \det(sI - \tilde{A}_{11}) \cdots \det(sI - \tilde{A}_{kk})$,

the n eigenvalues of $A_{\mathrm{c}} + B_{\mathrm{c}} F_{\mathrm{c}}$ can be assigned to arbitrary,

real, or complex conjugate locations by selecting an adequate F_{c}.

◆ Example

Let us consider the following system:

$$A_{\mathrm{c}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\alpha_0^{[11]} & -\alpha_1^{[11]} & -\alpha_2^{[11]} & -\alpha_0^{[12]} & -\alpha_1^{[12]} & -\alpha_2^{[12]} & -\alpha_3^{[12]} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\alpha_0^{[21]} & -\alpha_1^{[21]} & -\alpha_2^{[21]} & -\alpha_0^{[22]} & -\alpha_1^{[22]} & -\alpha_2^{[22]} & -\alpha_3^{[22]} \end{bmatrix} \quad \text{and } B_{\mathrm{c}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & b_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We take F_{c} by

$$F_{\mathrm{c}} = \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} \alpha_0^{[11]} - d_0^{[11]} & \alpha_1^{[11]} - d_1^{[11]} & \alpha_2^{[11]} - d_2^{[11]} & \alpha_0^{[12]} & \alpha_1^{[12]} & \alpha_2^{[12]} & \alpha_3^{[12]} \\ \alpha_0^{[21]} & \alpha_1^{[21]} & \alpha_2^{[21]} & \alpha_0^{[22]} - d_0^{[22]} & \alpha_1^{[22]} - d_1^{[22]} & \alpha_2^{[22]} - d_2^{[22]} & \alpha_3^{[22]} - d_3^{[22]} \end{bmatrix}$$

Then, $A_{\mathrm{c}} + B_{\mathrm{c}}F_{\mathrm{c}}$ has the following form:

$$A_{\mathrm{c}} + B_{\mathrm{c}}F_{\mathrm{c}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -d_0^{[11]} & -d_1^{[11]} & -d_2^{[11]} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -d_0^{[22]} & -d_1^{[22]} & -d_2^{[22]} & -d_3^{[22]} \end{bmatrix}$$

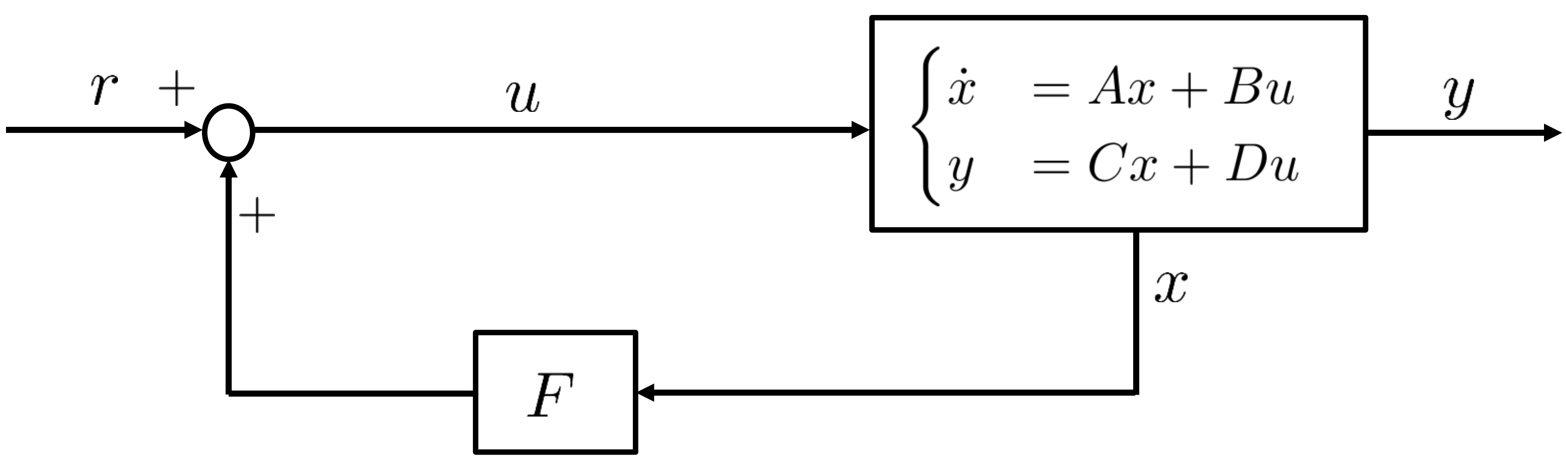
$$\begin{aligned} \det(sI - (A_{\mathrm{c}} + B_{\mathrm{c}}F_{\mathrm{c}})) = {} & (s^3 + d_2^{[11]}s^2 + d_1^{[11]}s + d_0^{[11]}) \\ & \times (s^4 + d_3^{[22]}s^3 + d_2^{[22]}s^2 + d_1^{[22]}s + d_0^{[22]}) \end{aligned}$$

◆ Case for Uncontrollable Systems

For the uncontrollable LTI system

$$\begin{cases} \dfrac{dx(t)}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}, \qquad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^k,\ y(t) \in \mathbb{R}^m$$

let us consider the following structure:



The corresponding closed-loop system is given by

$$\begin{cases} \dot{x} &= (A+BF)x + Br \\ y &= (C+DF)x + Dr \end{cases}$$

Because this system is uncontrollable,

there eixsts a nonsingular matrix T such that

$$T(A+BF)T^{-1} = TAT^{-1} + TBFT^{-1} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} FT^{-1}$$

$$=: \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{A}_{11} + \tilde{B}_1\tilde{F}_1 & \tilde{A}_{12} + \tilde{B}_1\tilde{F}_2 \\ 0 & \tilde{A}_{22} \end{bmatrix}$$

where the pair $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable.

$$\boxed{T(A+BF)T^{-1} = \begin{bmatrix} \tilde{A}_{11} + \tilde{B}_1\tilde{F}_1 & \tilde{A}_{12} + \tilde{B}_1\tilde{F}_2 \\ 0 & \tilde{A}_{22} \end{bmatrix}}$$

- The eigenvalues of $A + BF$ are the same as the eigenvalues of $T(A + BF)T^{-1}$.

- The eigenvalues of $\tilde{A}_{11} + \tilde{B}_1\tilde{F}_1$ can be arbitrary located by selecting an adequate $\tilde{F}_1$ since the pair $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable.

- The eigenvalues of the uncontrollable part $(\tilde{A}_{22}, 0)$ cannot be changed.

- If an eigenvalue of $\tilde{A}_{22}$ is unstable, this system cannot be stabilized.

◆ Summary for Controllability

For the LTI system described by

$$\begin{cases} \dfrac{dx(t)}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}, \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^k,\ y(t) \in \mathbb{R}^m,$$

there exists $F \in \mathbb{R}^{k \times n}$ so that the eigenvalues of $A + BF$

are assigned to arbitrary real or complex conjugate locations

if and only if the pair (A, B) is controllable.

Proof: The sufficient condition has been already confirmed through the aforementioned poles placement procedure.

The necessary condition has been also already confirmed by the arguments associated with the case for uncontrollable systems, which show that if the pair (A, B) is uncontrollable the poles of $A + BF$ cannot be arbitrary assigned by selecting an adequate F.

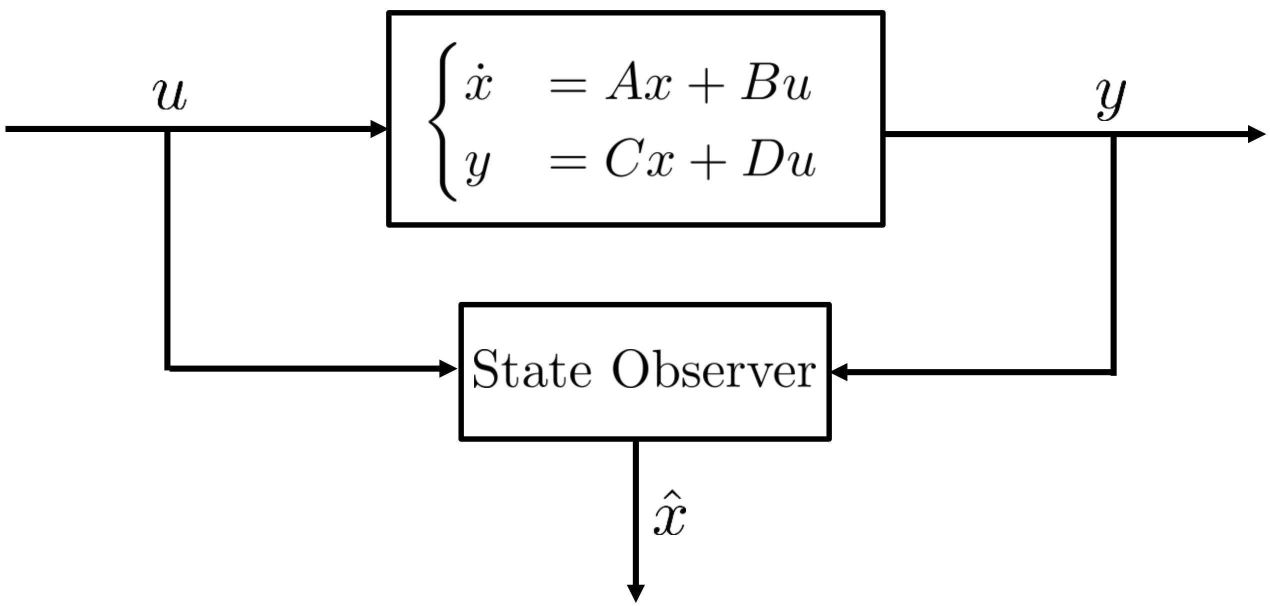
2. State Observers

◆ Basic Structure

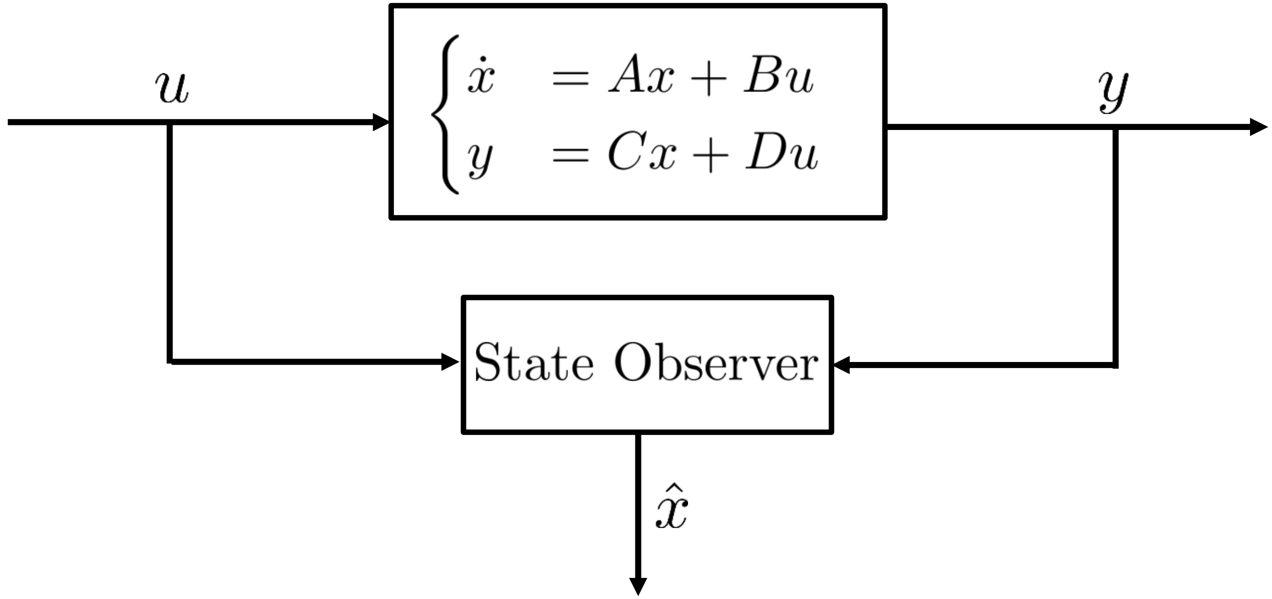
For the LTI system

$$\begin{cases} \dfrac{dx(t)}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}, \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^k,\ y(t) \in \mathbb{R}^m$$

let us consider the following structure:



◆ Mathematical Description of State Observers



Statae Observer:

$$\hat{y} = C\hat{x} + Du$$

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu - K(y - \hat{y}) \\ &= A\hat{x} + Bu - K(Cx + Du - C\hat{x} - Du) \end{aligned}$$

◆ Error Dynamics

Let us define $e(t) := x(t) - \hat{x}(t)$.

Then, we can obtain the following relation:

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu - (A\hat{x} + Bu - K(y - \hat{y}))$$

$$= A(x - \hat{x}) + K(y - \hat{y}) = A(x - \hat{x}) + K(Cx + Du - C\hat{x} - Du)$$

$$= A(x - \hat{x}) + K(Cx - C\hat{x}) = (A + KC)e.$$

Thus, we see that

$$e(t) = e^{(A+KC)t}e(0).$$

If $A + KC$ is stable, $e(t) \to 0$ $(t \to \infty)$ for all $e(0) \in \mathbb{R}^n$.

◆ Case for Observable Systems

For the LTI system described by

$$\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}, \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^k,\ y(t) \in \mathbb{R}^m,$$

there exists $K \in \mathbb{R}^{n\times m}$ so that the eigenvalues of $A + KC$

are assigned to arbitrary real or complex conjugate locations

if the pair (C, A) is observable.

Proof: The eigenvalues of $A + KC$ are the same as the eigenvalues of $(A + KC)^T = A^T + C^T K^T$.

Note that the observablitly of the pair (C, A) is the same as the controllability of the pair (A^T, C^T).

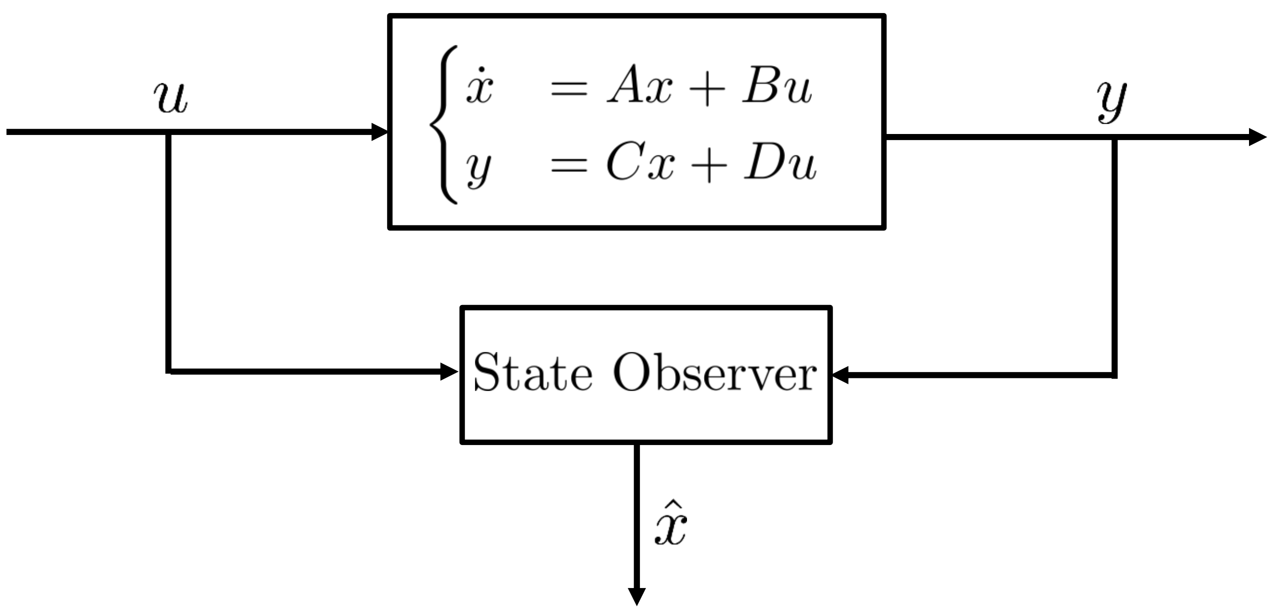
Hence, we can arbitrary assign the poles of $A^T + C^T K^T$ by selecting an adequate K^T since the pair (A^T, C^T) is controllable.

◆ Case for Unobservable Systems

For the unobservable LTI system

$$\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}, \qquad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^k,\ y(t) \in \mathbb{R}^m,$$

let us consider the following structure:



The corresponding error dynamics is given by

$$\dot{e} = (A + KC)e.$$

Because this system is unobservable,

there eixsts a nonsingular matrix T such that

$$T(A+KC)T^{-1} = TAT^{-1} + TKCT^{-1} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} + TK \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix}$$

$$=: \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} + \begin{bmatrix} \tilde{K}_1 \\ \tilde{K}_2 \end{bmatrix} \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{A}_{11} + \tilde{K}_1\tilde{C}_1 & 0 \\ \tilde{A}_{21} + \tilde{K}_2\tilde{C}_1 & \tilde{A}_{22} \end{bmatrix}$$

where the pair $(\tilde{C}_1, \tilde{A}_{11})$ is observable.

$$\boxed{T(A+KC)T^{-1} = \begin{bmatrix} \tilde{A}_{11} + \tilde{K}_1\tilde{C}_1 & 0 \\ \tilde{A}_{21} + \tilde{K}_2\tilde{C}_1 & \tilde{A}_{22} \end{bmatrix}}$$

- The eigenvalues of $A+KC$ are the same as the eigenvalues of $T(A+KC)T^{-1}$.

- The eigenvalues of $\tilde{A}_{11} + \tilde{K}_1\tilde{C}_1$ can be arbitrary located by selecting an adequate $\tilde{K}_1$ since the pair $(\tilde{C}_1, \tilde{A}_{11})$ is observable,
  i.e., the pair $(\tilde{A}_{11}^T, \tilde{C}_1^T)$ is controllable.

- The eigenvalues of the unobservable part $(\tilde{A}_{22}, 0)$ cannot be changed.

- If an eigenvalue of $\tilde{A}_{22}$ is unstable, $e(t)$ does not converge to 0 as $t \to \infty$.

◆ Summary for Observability

For the LTI system described by

$$\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}, \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^k,\ y(t) \in \mathbb{R}^m,$$

there exists $K \in \mathbb{R}^{n \times m}$ so that the eigenvalues of $A + KC$

are assigned to arbitrary real or complex conjugate locations

if and only if the pair (C, A) is observable.

Proof: The sufficient condition has been already confirmed through the aforementioned poles placement procedure.

The necessary condition has been also already confirmed by the arguments associated with the case for unobservable systems, which show that if the pair (C, A) is unobservable the poles of $A + KC$ cannot be arbitrary assigned by selecting an adequate K.