EECE322-01: 자동제어공학개론

Introduction to Automatic Control

Chapter 2: Models of Control Systems

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◆ The main objectives of this chapter are

1. Introduction to Dynamic Models and Laplace Transform

2. Examples of Transfer Function for Various Systems

1. Introduction to Dynamic Models and Laplace Transform

Dynamic models

- **Model**:
  - a mathematical description of the process to be controlled.
  - Usually a set of differential equations.

- Getting models:
  - Using principles of the underlying physics.
  - Testing a prototype of the device: apply some inputs and measure the outputs, and then use the data to construct an analytic model (system identification).

- Modeling scope:
  - Depends on the problem.
  - The simplest model to describe some phenomenon is the best model.

Brief review of Laplace transform

Laplace transformation: $\mathcal{L}(f(t)) = F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}ds$

Properties of Laplace transformation:

1. Linearity:

$$\mathcal{L}(\alpha_1 f_1(t) + \alpha_2 f_2(t)) = \alpha_1 F_1(s) + \alpha_2 F_2(s)$$

2. Differential:

$$\mathcal{L}(\dot{f}(t)) = sF(s) - f(+0)$$

3. Integral:

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}F(s)$$

2. Examples of Transfer Function for Various Systems

Basics to linear motion of mechanical systems

- Equations of motion for mechanical systems
- Newton's law is the most fundamental law.

$$F = ma$$

- F: **the vector sum of all forces** applied to each body in a system, N or lb.
- a: **the vector acceleration** of each body with respect to an inertial reference frame (neither accelerating nor rotating w.r.t. the stars), m/sec^2 or ft/sec^2.
- m: **mass** of the body, kg or slug.

- Note: - one should define coordinates to account for the body's motion.

  - one should use the free-body diagram.

Example 1

- Cruise control model

- Write the equations of motion for the speed and forward motion. Engine imparts a force u as shown.

- Use MATLAB to find the response v for the case where u is step input of 500 N.

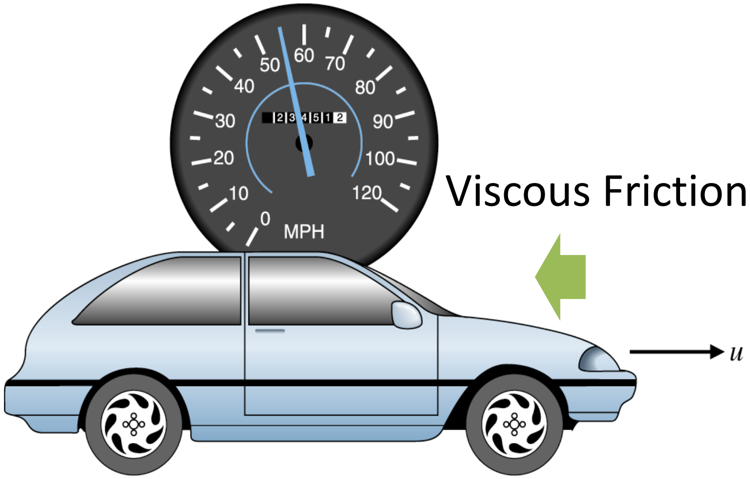


Figure 2.1 Cruise-control model

$$m = 1000\text{kg}, b = 50\text{N}\sec/m$$

$$\sum F_{\text{ext}} = m\ddot{x}$$

$$u - b\dot{x} = m\ddot{x}$$

$$\ddot{x} + \frac{b}{m}\dot{x} = \frac{u}{m}$$

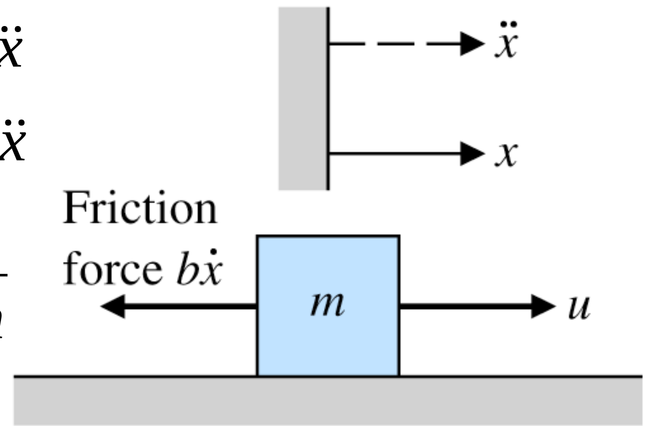


Figure 2.2 Free-body diagram for cruise control

(a) **Equations of motion**

$$u - b\dot{x} = m\ddot{x} \rightarrow \quad \ddot{x} + \frac{b}{m}\dot{x} = \frac{u}{m}$$

$$(\dot{x} := v) \quad \dot{v} + \frac{b}{m}v = \frac{u}{m}$$

Assume $v(t) = V_0 e^{st}$ for $u(t) = U_0 e^{st}$.

$$\left(s + \frac{b}{m}\right)V_0 e^{st} = \frac{1}{m}U_0 e^{st} \rightarrow \frac{V_0}{U_0} = \frac{1/m}{s + b/m}$$

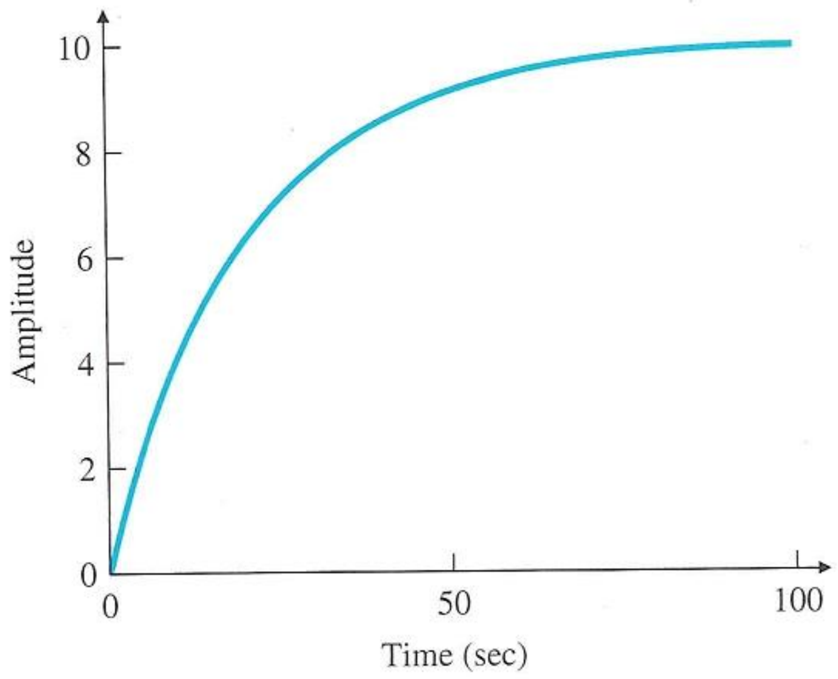
Laplace transform: $\dfrac{V(s)}{U(s)} = \dfrac{1/m}{s + b/m}$ (transfer function)

(b) **Time response**

$$u(t) = 500\ 1(t)\ \text{N},\ m = 1000\ \text{kg},\ b = 50\ \text{N}\cdot\text{sec/m}$$

num=1/*m*=1/1000, den= [1 *b*/*m*]= [1 50/1000]

```
num = 1/1000;               % 1/m
den = [1 50/1000];          % s + b/m
sys = tf(num*500,den);      % step gives the unit step response, so num*500
                              gives u=500N
step(sys);                  % plots the step response
```



Example 2

- A Two-Mass System: Suspension model
- Write the equations of motion (**quarter car**, vertical motion only)
  - mass=1580kg (including wheels-20kg each),
  - spring constant of suspension: k_s=130,000 N/m,
  - spring constant of the tire (or wheel): $k_w \approx$1,000,000 N/m,
    damping: b=9800 N·sec/m.

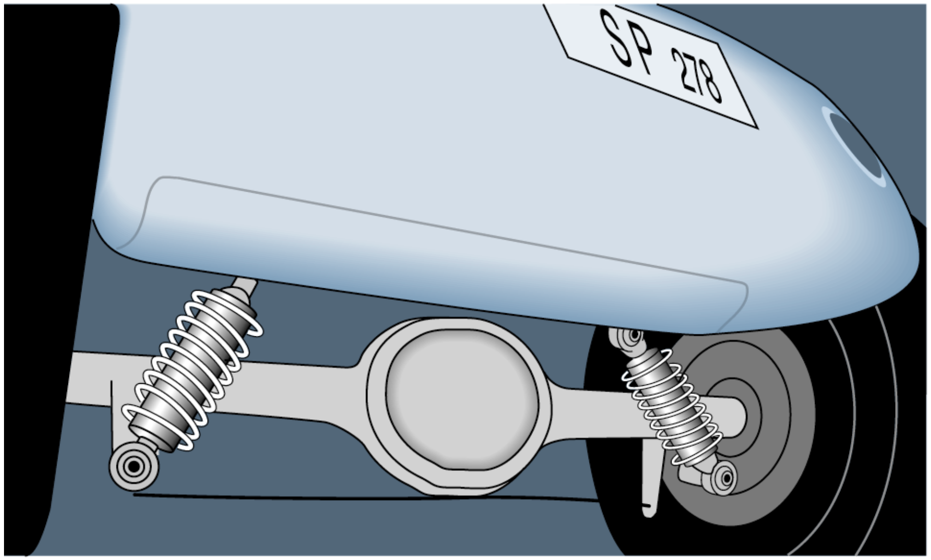


Figure 2.4 Automobile suspension

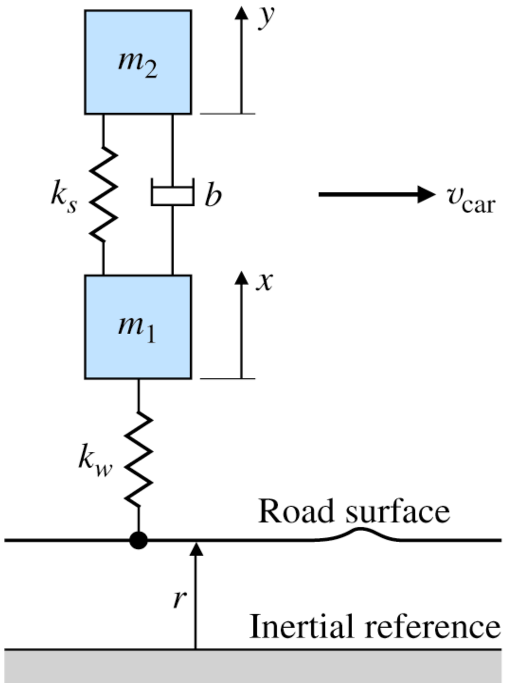


Figure 2.5 The quarter-car model

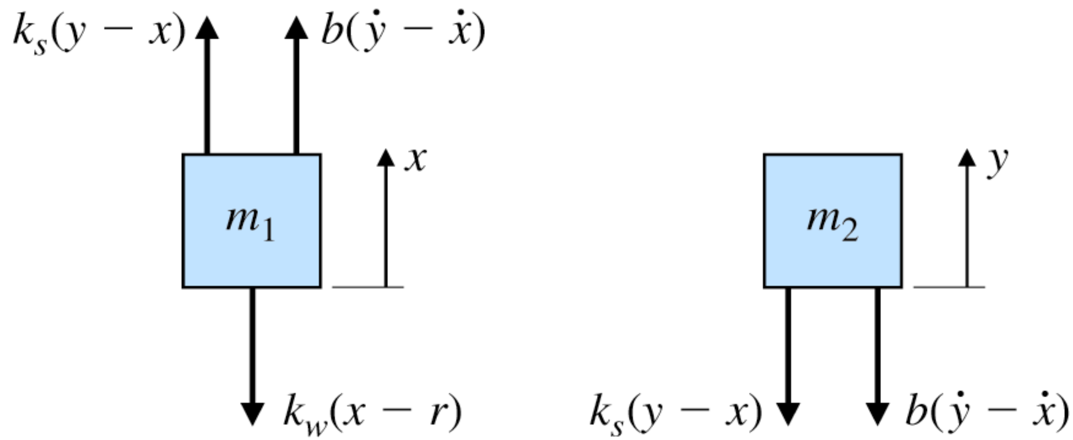


Figure 2.6 Free-body diagrams for suspension system

- Equations of motion: By applying Newton's law,

$$m_1:\ m_1\ddot{x} = k_s(y-x) + b(\dot{y}-\dot{x}\,) - k_w(x-r)$$

$$m_2:\ m_2\ddot{y} = -k_s(y-x) - b(\dot{y}-\dot{x})$$

$$\ddot{x} + \frac{b}{m_1}(\dot{x} - \dot{y}) + \frac{k_s}{m_1}(x - y) + \frac{k_w}{m_1} x = \frac{k_w}{m_1} r,$$

$$\ddot{y} + \frac{b}{m_2}(\dot{y} - \dot{x}) + \frac{k_s}{m_2}(y - x) = 0.$$

substituting s for d/dt

$$s^2 X(s) + s\frac{b}{m_1}(X(s) - Y(s)) + \frac{k_s}{m_1}(X(s) - Y(s)) + \frac{k_w}{m_1} X(s) = \frac{k_w}{m_1} R(s),$$

$$s^2 Y(s) + s\frac{b}{m_2}(Y(s) - X(s)) + \frac{k_s}{m_2}(Y(s) - X(s)) = 0.$$

$$\begin{bmatrix} s^2 + \frac{b}{m_1}s + \frac{k_s + k_w}{m_1} & -\frac{b}{m_1}s - \frac{k_s}{m_1} \\ -\frac{b}{m_2}s - \frac{k_s}{m_2} & s^2 + \frac{b}{m_2}s + \frac{k_s}{m_2} \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} \frac{k_w}{m_1} R(s) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} s^2 + \frac{b}{m_1}s + \frac{k_s + k_w}{m_1} & -\frac{b}{m_1}s - \frac{k_s}{m_1} \\ -\frac{b}{m_2}s - \frac{k_s}{m_2} & s^2 + \frac{b}{m_2}s + \frac{k_s}{m_2} \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} \frac{k_w}{m_1}R(s) \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = A^{-1} \begin{bmatrix} \frac{k_w}{m_1}R(s) \\ 0 \end{bmatrix} \quad A := \begin{bmatrix} s^2 + \frac{b}{m_1}s + \frac{k_s + k_w}{m_1} & -\frac{b}{m_1}s - \frac{k_s}{m_1} \\ -\frac{b}{m_2}s - \frac{k_s}{m_2} & s^2 + \frac{b}{m_2}s + \frac{k_s}{m_2} \end{bmatrix}$$

$$\Rightarrow \frac{Y(s)}{R(s)} = \frac{\frac{k_w b}{m_1 m_2}\left(s + \frac{k_s}{b}\right)}{s^4 + \left(\frac{b}{m_1} + \frac{b}{m_2}\right)s^3 + \left(\frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_w}{m_1}\right)s^2 + \frac{k_w b}{m_1 m_2}s + \frac{k_w k_s}{m_1 m_2}}$$

$$\left(m_2 = (1580 - 4 \times 20)/4 = 375 \text{ kg},\ m_1 = 20 \text{ kg}\right)$$

$$(k_s = 1300 \text{ N/m},\ k_w \cong 1{,}000{,}000 \text{ N/m},\ b = 9800 \text{ N} \cdot \text{sec/m})$$

$$\frac{Y(s)}{R(s)} = \frac{1.31e06(s + 13.3)}{s^4 + (516.1)s^3 + (5.685e04)s^2 + (1.307e06)s + 1.733e07}$$

Basics to rotational motion of mechanical systems

- Newton's law for rotational motion (moment)

$$M = I\alpha$$

$M =$ sum of the external moments about the center of mass of the body, $\mathrm{N} \cdot \mathrm{m}$ or $\mathrm{lb} \cdot \mathrm{ft}$,

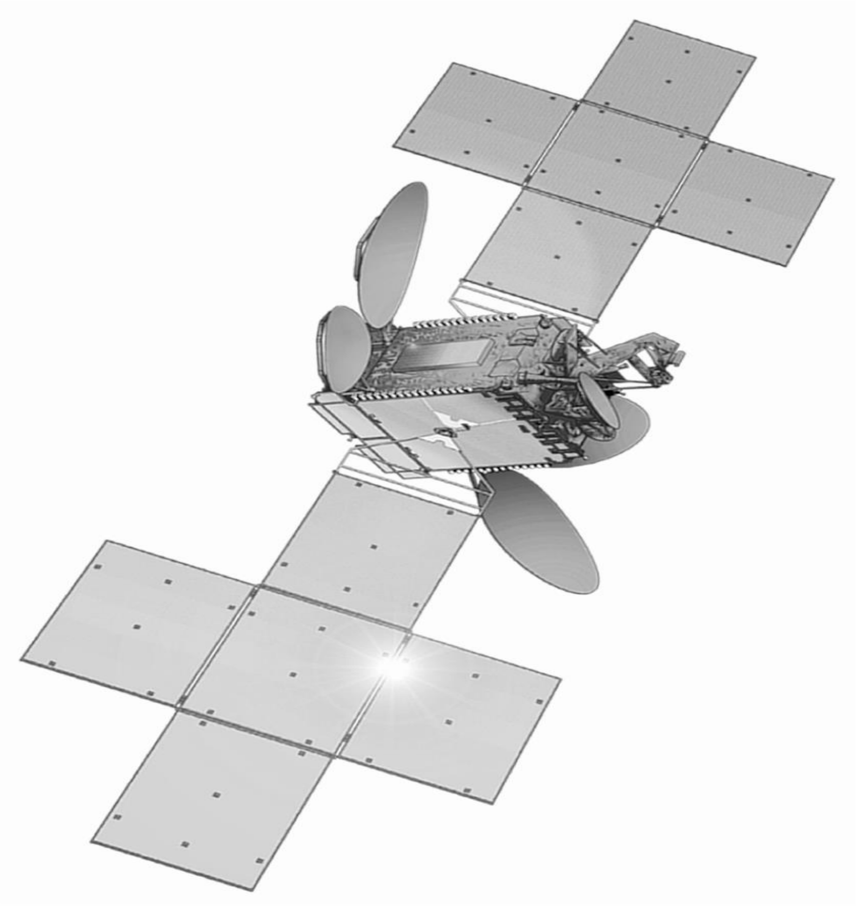
$I =$ body's mass moment of inertia about its center of mass, $\mathrm{kg} \cdot \mathrm{m}^2$ or $\mathrm{slug} \cdot \mathrm{ft}^2$,

$\alpha =$ angular acceleration of the body, $\mathrm{rad/sec}^2$.

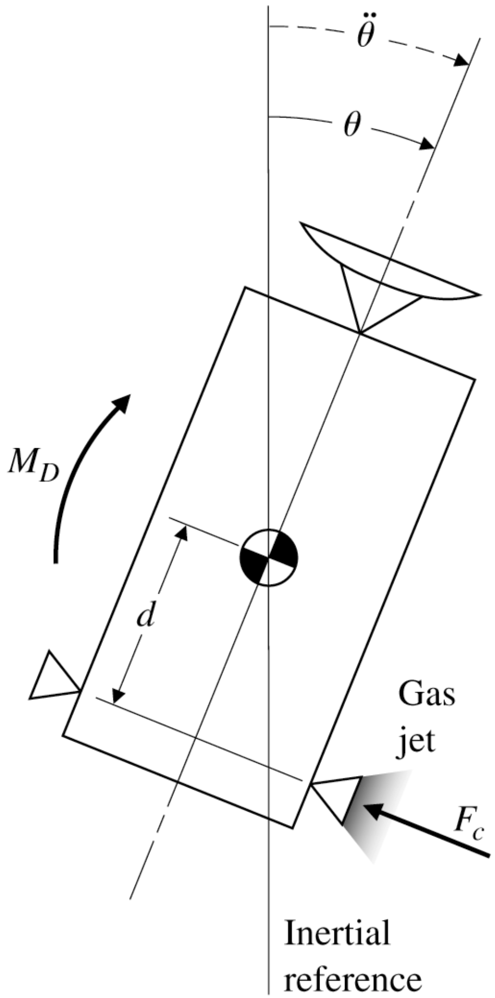
- Note: generalization of Newton's law to consider a system of particles.

Example 3

- Satellite Attitude Control Model:
  - Satellites need attitude control so that the antennas, sensors and solar panels are properly oriented.



Communications satellite *(Courtesy Space Systems/Loral)*



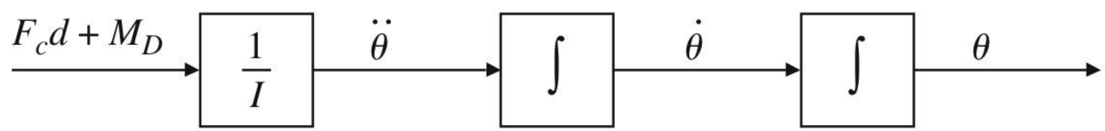
Satellite control schematic

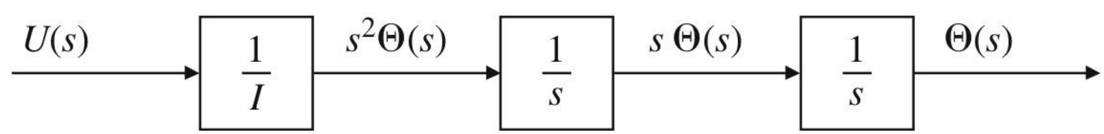
- Motion allowed only about the axis perpendicular to the page (1-axis attitude)

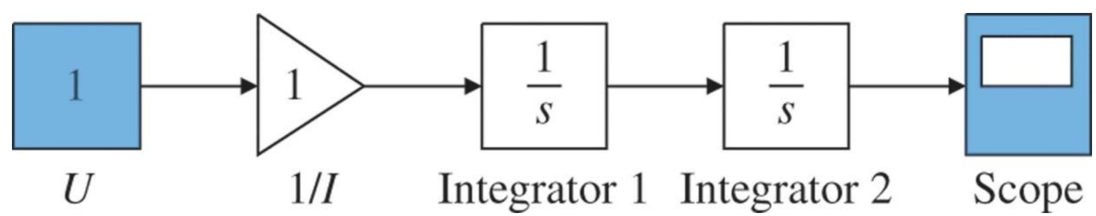
$$I\ddot{\theta} = F_c d + M_D (=: u) \quad \text{(double-integrator plant)}$$

$$\left(M_D : \text{disturbance moments}\right)$$

$$\frac{\Theta(s)}{U(s)} = \frac{1}{I}\frac{1}{s^2} \quad \left(\frac{1}{s^2}\ \text{plant}\right)\left(u = F_c d + M_D\right)$$







Example 4

- Flexible Read/Write for a Disk Drive:

  - Assume some flexibility between the read head and the drive motor.

$M_C =$ applied torque, $M_D =$ disturbance torque,

$b =$ damping contant

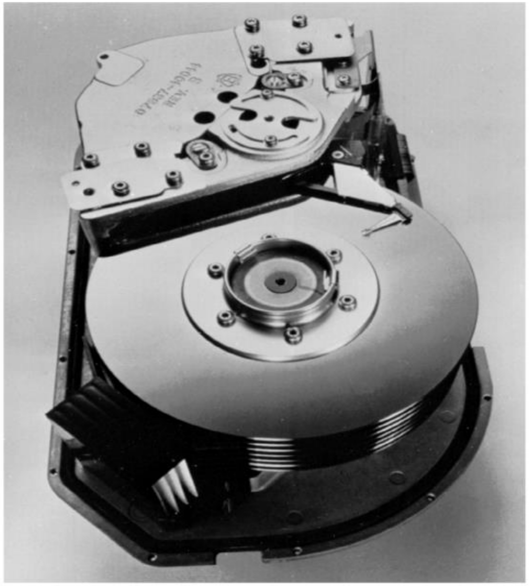
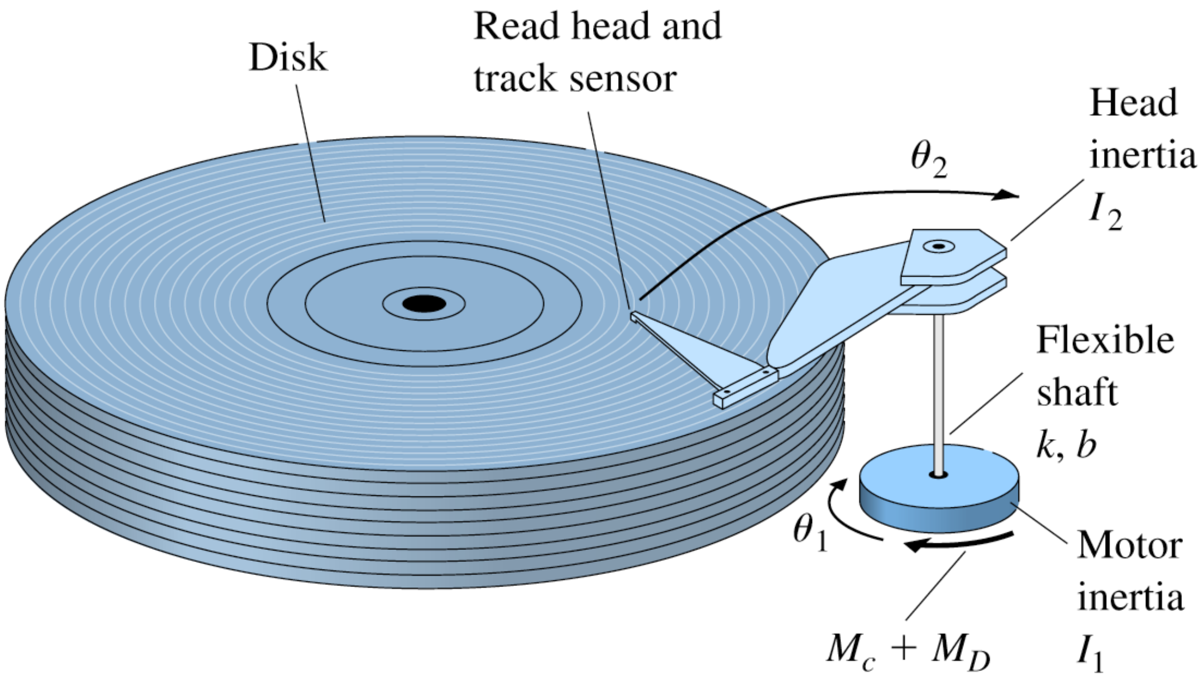
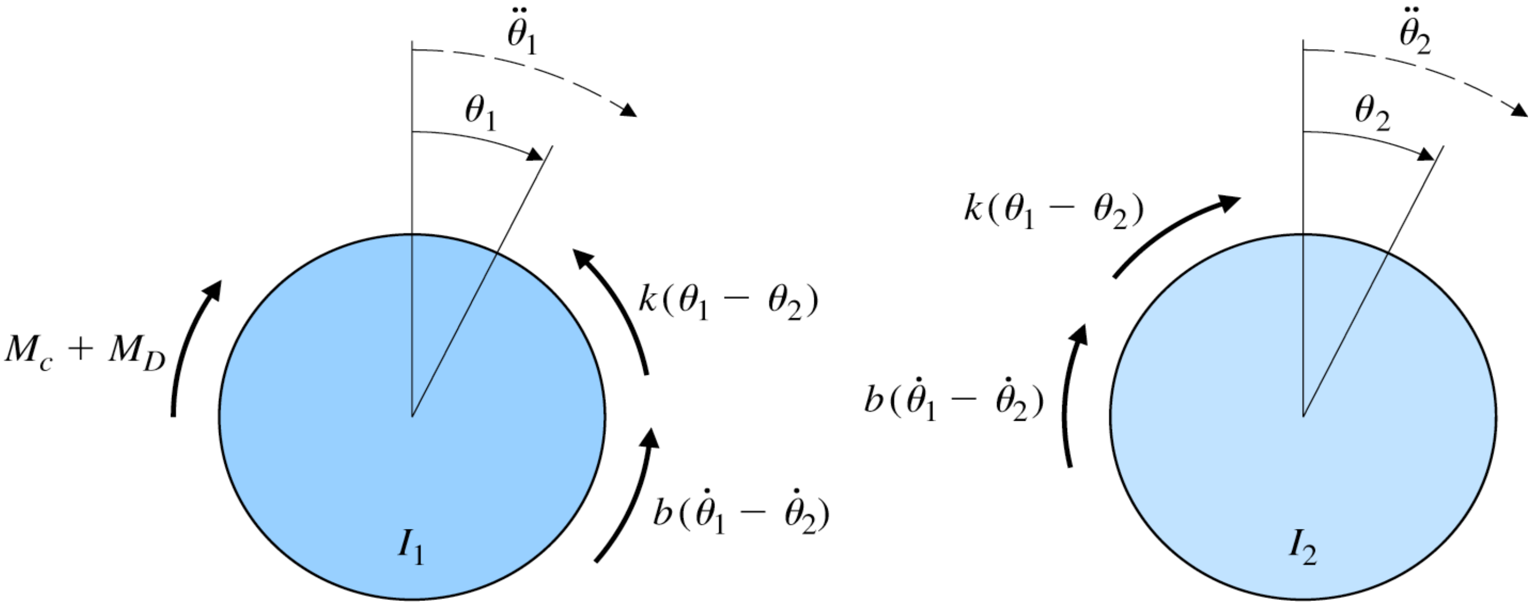


Figure 2.8 Disk read/write mechanism (*Photo courtesy of Hewlett-Packard Company*)



Disk read/write head schematic for modeling

- Free body diagram



Free-body diagrams of the disk read/write head

- Dynamic equation:

$$I_1:\ \ I_1\ddot{\theta}_1 = M_C + M_D - k(\theta_1 - \theta_2) - b(\dot{\theta}_1 - \dot{\theta}_2)$$

$$I_2:\ \ I_2\ddot{\theta}_2 = k(\theta_1 - \theta_2) + b(\dot{\theta}_1 - \dot{\theta}_2)$$

- Dynamic equation

$$I_1\ddot{\theta}_1 + b\left(\dot{\theta}_1 - \dot{\theta}_2\right) + k\left(\theta_1 - \theta_2\right) = M_C + M_D$$

$$I_2\ddot{\theta}_2 + b\left(\dot{\theta}_2 - \dot{\theta}_1\right) + k\left(\theta_2 - \theta_1\right) = 0$$

Ignoring M_D and b:

$$\frac{\Theta_2(s)}{M_C(s)} = \frac{k}{I_1 I_2 s^2\left(s^2 + \frac{k}{I_1} + \frac{k}{I_2}\right)} \quad \text{(noncollocated)}$$

$$\frac{\Theta_1(s)}{M_C(s)} = \frac{I_2 s^2 + k}{I_1 I_2 s^2\left(s^2 + \frac{k}{I_1} + \frac{k}{I_2}\right)} \quad \text{(collocated)}$$

$$I_1 s^2 \Theta_1(s) + k\big(\Theta_1(s) - \Theta_2(s)\big) = M_c(s)$$
$$I_2 s^2 \Theta_2(s) + k\big(\Theta_2(s) - \Theta_1(s)\big) = 0$$

Example 5

- Rotational motion: Pendulum

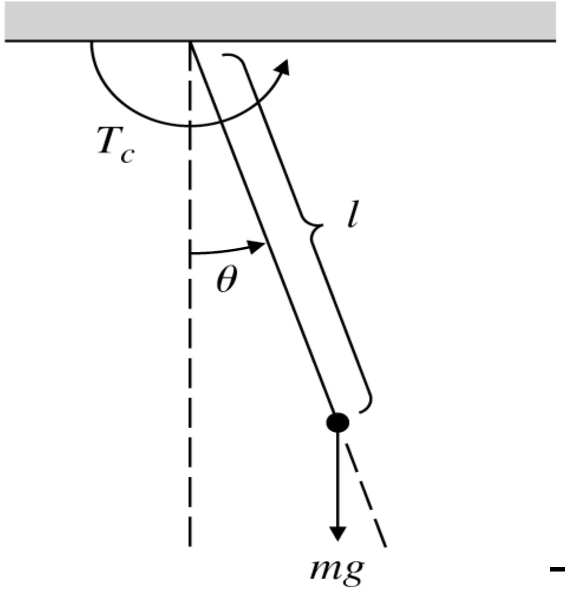


Figure 2.12 Pendulum

(a) Equations of motion

(moment of inertia: $I = ml^2$)

(T_c : applied torque)

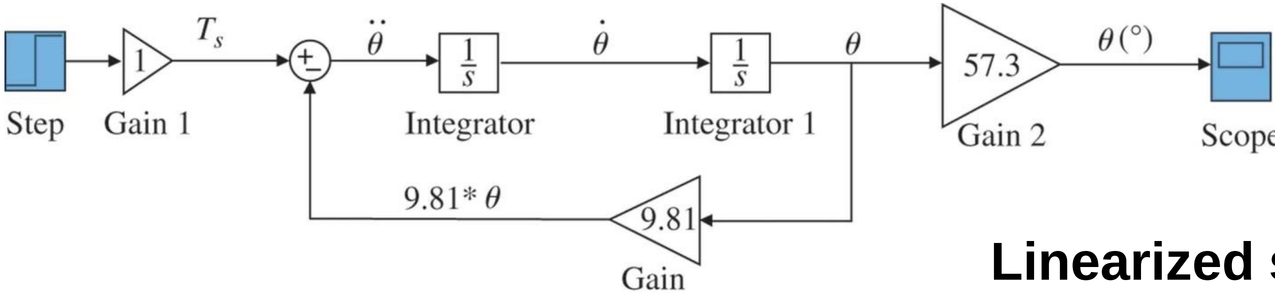
$$T_c - mgl\sin\theta = I\ddot{\theta}$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{T_c}{ml^2} \quad \text{(nonlinear)}$$

- Assuming that the motion is small enough, i.e., $\sin\theta \approx \theta$,

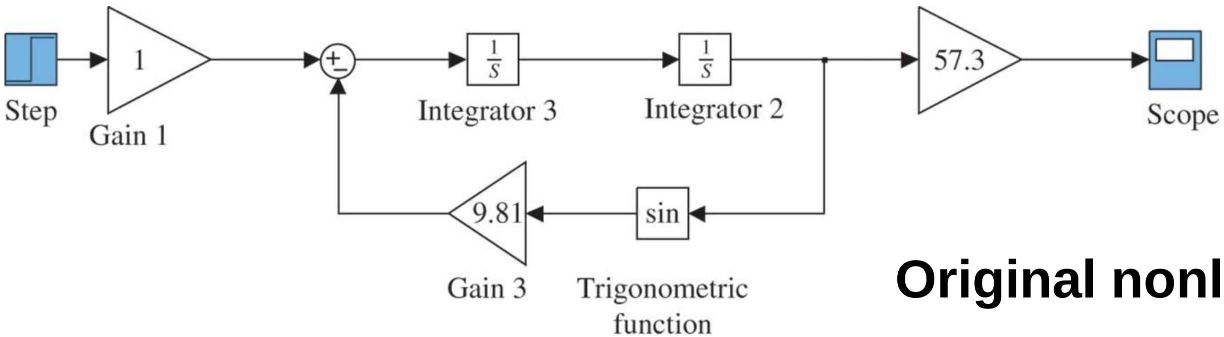
$$\ddot{\theta} + \frac{g}{l}\theta = \frac{T_c}{ml^2} \qquad \text{(linear)} \Rightarrow \frac{\Theta(s)}{T_c(s)} = \frac{\frac{1}{ml^2}}{s^2 + \frac{g}{l}}$$

$$\left(\ddot{\theta} + \frac{g}{l}\theta = 0 : \text{harmonic oscillator with natural frequency } \omega_n = \sqrt{\frac{g}{l}} \leftarrow s^2 + \frac{g}{l} = 0\right)$$



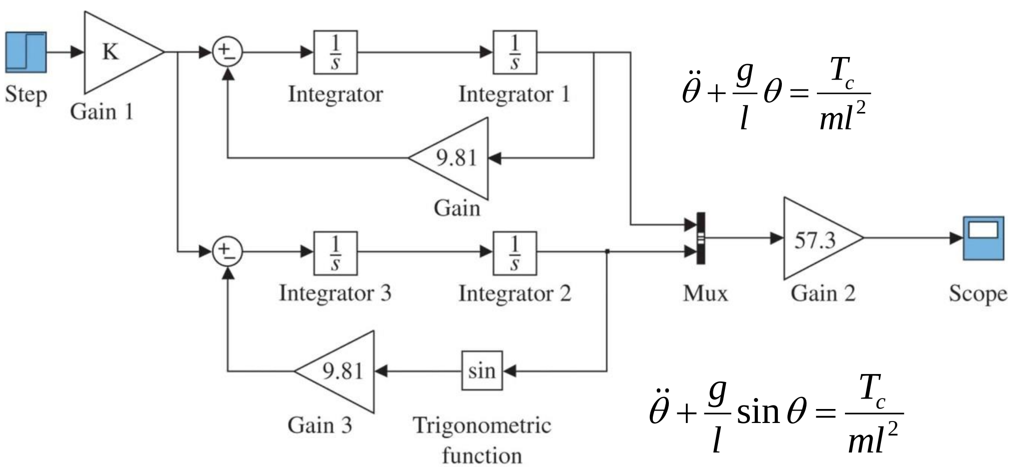
$$\ddot{\theta}+\frac{g}{l}\theta=\frac{T_c}{ml^2}$$

Linearized system



$$\ddot{\theta}+\frac{g}{l}\sin\theta=\frac{T_c}{ml^2}$$

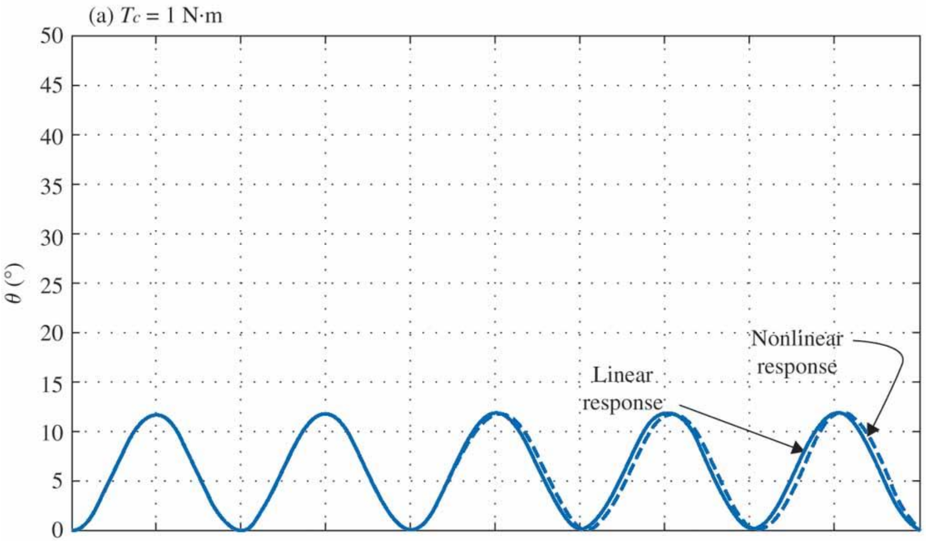
Original nonlinear system



$$\ddot{\theta}+\frac{g}{l}\theta=\frac{T_c}{ml^2}$$

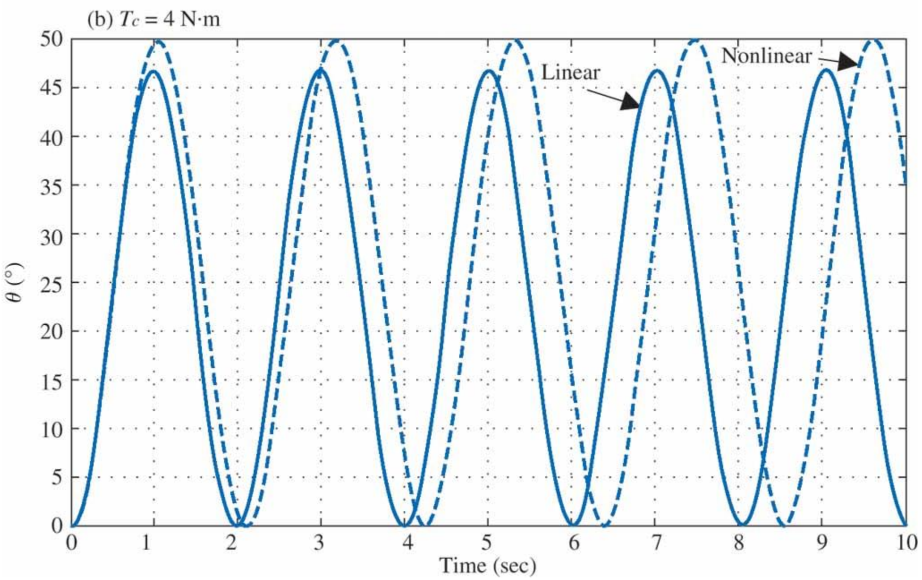
$$\ddot{\theta}+\frac{g}{l}\sin\theta=\frac{T_c}{ml^2}$$

Comparison between linearized & nonlinear systems



Input torque $T_C = 1\mathrm{N}\cdot\mathrm{m}$
⇒The gap between two response is small

Input torque increases

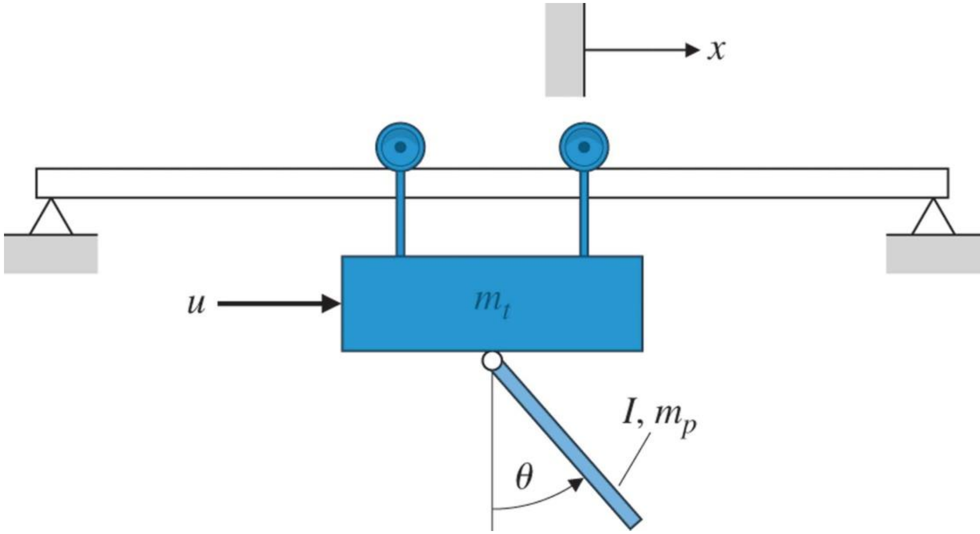


Input torque $T_C = 4N\cdot\mathrm{m}$
⇒The gap between two response is large

Example 5

- Rotational and Translational Motion: Hanging Crane

- Dynamic equation around $\theta \approx 0$



$$(I + m_p l^2)\ddot{\theta} + m_p g l \theta = -m_p l \ddot{x}$$

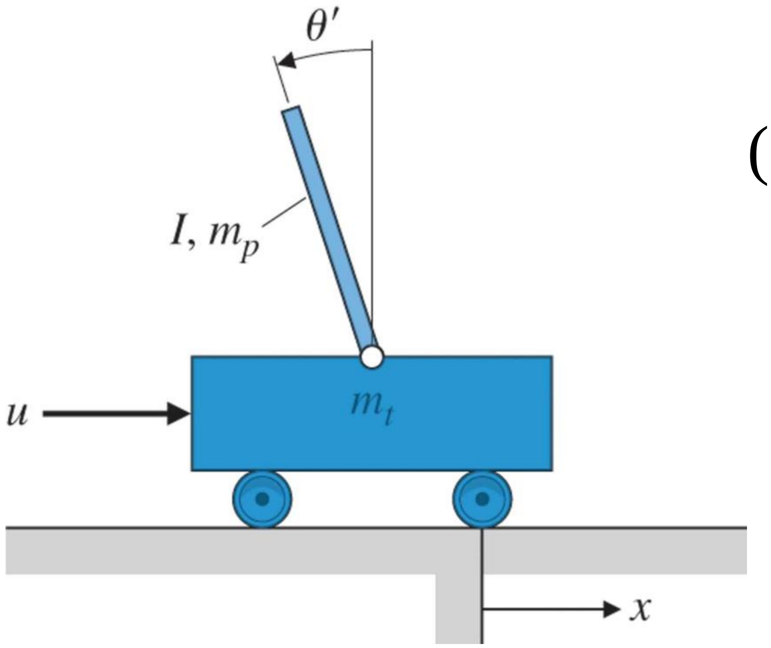
$$(m_t + m_p)\ddot{x} + b\dot{x} + m_p l \ddot{\theta} = u$$

- Transfer function neglecting friction:

$$\frac{\theta(s)}{U(s)} = \frac{-m_p l}{((I + m_p l^2)(m_t + m_p) - m_p^2 l^2)s^2 + m_p g l (m_t + m_p)}$$

Example 6

- Dynamic equation around $\theta \approx \pi \rightarrow \theta = \pi + \theta'$



$$(I + m_p l^2)\ddot{\theta}' - m_p g l \theta' = m_p l \ddot{x}$$

$$(m_t + m_p)\ddot{x} + b\dot{x} - m_p l \ddot{\theta}' = u$$

- Transfer function neglecting friction:

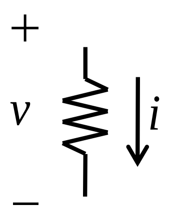
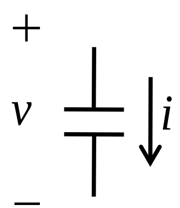
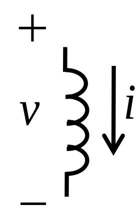
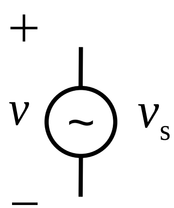
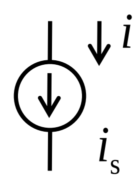
$$\frac{\theta'(s)}{U(s)} = \frac{m_p l}{((I + m_p l^2) - m_p^2 l^2)s^2 - m_p g l(m_t + m_p)}$$

- Segway

Basics to electric circuits

- Electric circuits:
  - Easy manipulation and processing.
  - Still analog circuits are used (power amplifier, etc.)

- Elements

	Resister	Capacitor	Inductor	Voltage source	Current source
Symbol					
Equation	$v = Ri$	$i = C\dfrac{dv}{dt}$	$v = L\dfrac{di}{dt}$	$v = v_s$	$i = i_s$

Modeling of electric circuits

- Basic equations of electric circuits: Kirchhoff's laws

- Kirchhoff's laws

  \- **Kirchhoff's current law (KCL):** The algebraic sum of currents leaving a junction or node equals the algebraic sum of currents entering that node.

  \- **Kirchhoff's voltage law (KVL):** The algebraic sum of all voltages taken around a closed path in a circuit is zero.

- There are many well organized tools for circuit analysis.

- Node analysis

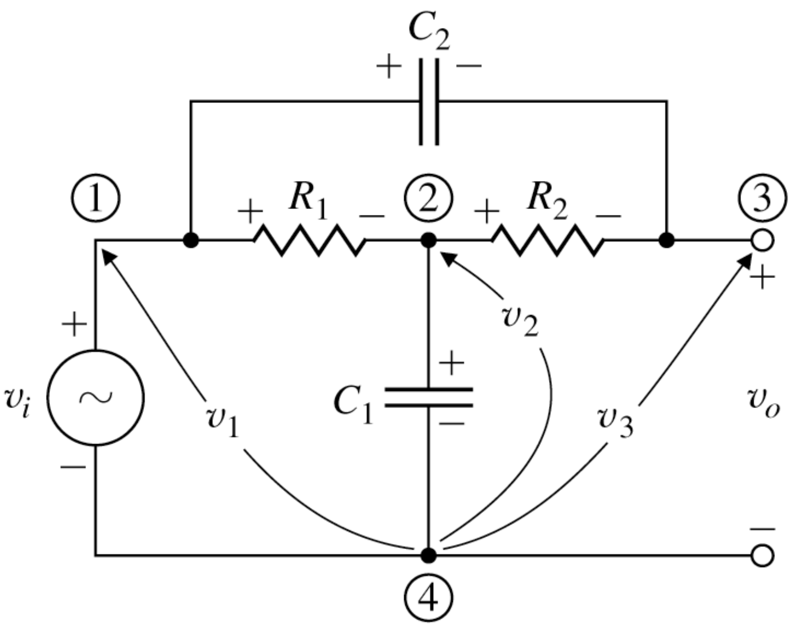
  \- One node is selected as a reference.

  \- Apply KCL at the other nodes.

  \- For circuits containing voltage sources, use KVL.

Example 7

- Equations for the Bridged Tee Circuit



Bridged tee circuit

- Select node 4 as the reference node.

At node 1: $v_1 = v_i$

At node 2 (KCL):

$$-\frac{v_1 - v_2}{R_1} + \frac{v_2 - v_3}{R_2} + C_1 \frac{dv_2}{dt} = 0$$

At node 3 (KCL):

$$\frac{v_3 - v_2}{R_2} + C_2 \frac{d\left(v_3 - v_1\right)}{dt} = 0$$

Transfer function of Bridged Tee Circuit ($v_i \rightarrow v_o$)

$$\frac{V_o(s)}{V_{i(s)}} = \frac{C_2 R_2 s(R_1 + R_2 + R_1 R_2 C_1 s) + R_2}{C_2 R_2 s(R_1 + R_2 + R_1 R_2 C_1 s) + R_2 + R_1 R_2 C_1 s}$$

Modeling of electromechanical systems

- Interaction between electric field and magnetic field.

- Law of motors (Fleming's left-hand rule)

F : force on the conductor (newtons)

B : magnetic field (telsa)

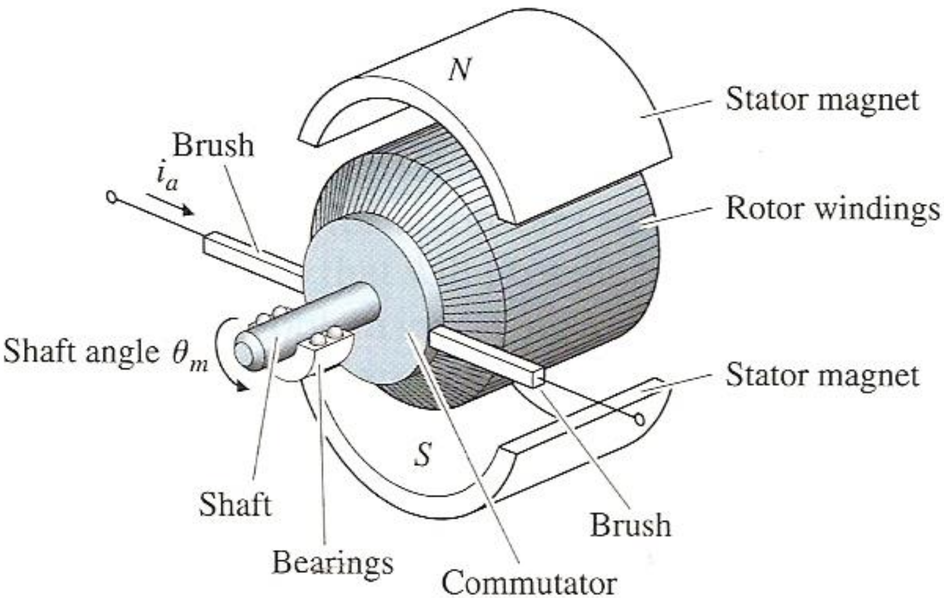
l : length of the conductor (meters)

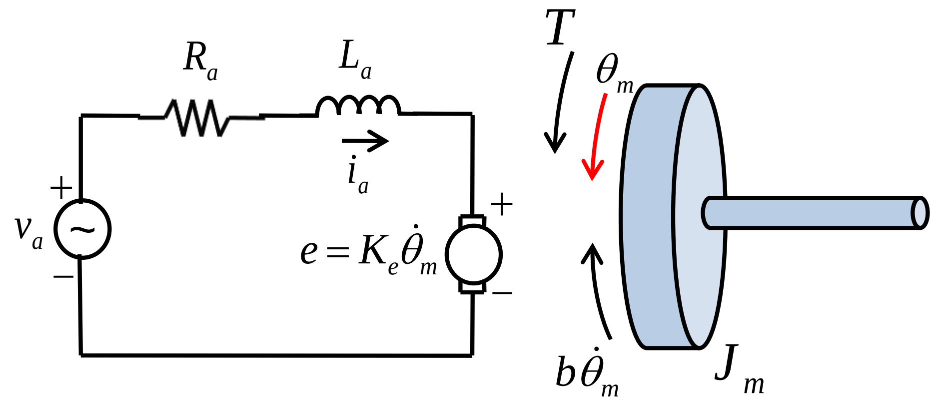
i : current in the conductor (amperes)

- If the conductor with current is at right angle in a magnetic field, there is a force on the conductor at right angles to the plane of i and B.

Example 8

- DC motor





- motor torque: $T = K_t i_a$

- back emf: $e = K_e\dot{\theta}_m$

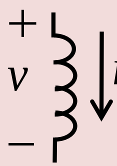
- Newton's law:

$$J_m\ddot{\theta}_m + b\dot{\theta}_m = K_t i_a.$$

- Electrical equation:

$$L_a\frac{di_a}{dt} + R_a i_a = v_a - K_e\dot{\theta}_m.$$

Inductor

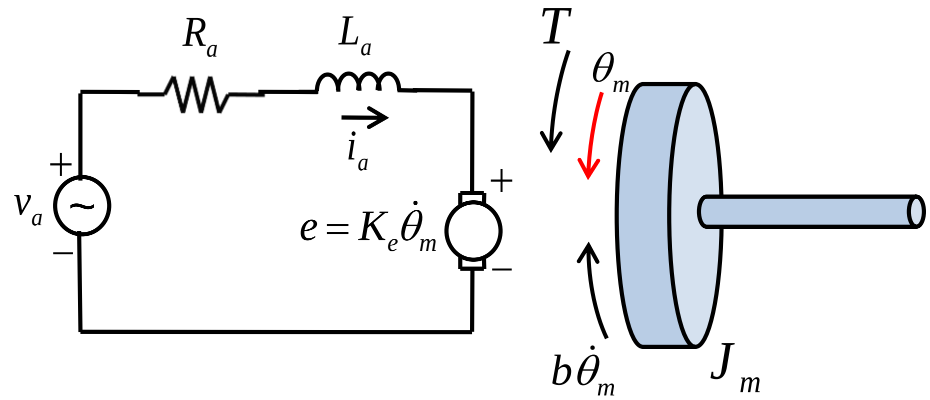


$$v = L\frac{di}{dt}$$

- Newton's law:

$$J_m \ddot{\theta}_m + b\dot{\theta}_m = K_t i_a.$$

$$\Rightarrow \; J_m s^2 \Theta_m(s) + bs\Theta_m(s) = K_t I_a(s)$$



- Electrical equation:

$$L_a \frac{di_a}{dt} + R_a i_a = v_a - K_e \dot{\theta}_m.$$

$$\Rightarrow \; L_a s I_a(s) + R_a I_a(s) = V_a(s) - K_e s\Theta_m(s)$$

Transfer function of DC motor

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{K_t}{s((J_m s + b)(L_a s + R_a) + K_t K_e)}.$$

voltage→ angle

$$\frac{\Omega_m(s)}{V_{a(s)}} = \frac{K_t}{(J_m s + b)(L_a s + R_a) + K_t K_e}$$

voltage→ angular velocity