EECE322-01: 자동제어공학개론

Introduction to Automatic Control

Chapter 3: Response of Control Systems

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◆ The main objectives of this chapter are

1. Basic concepts of linear systems

2. Laplace transform and transfer functions

3. Effects of pole locations and Block Diagrams

4. Time-domain specifications

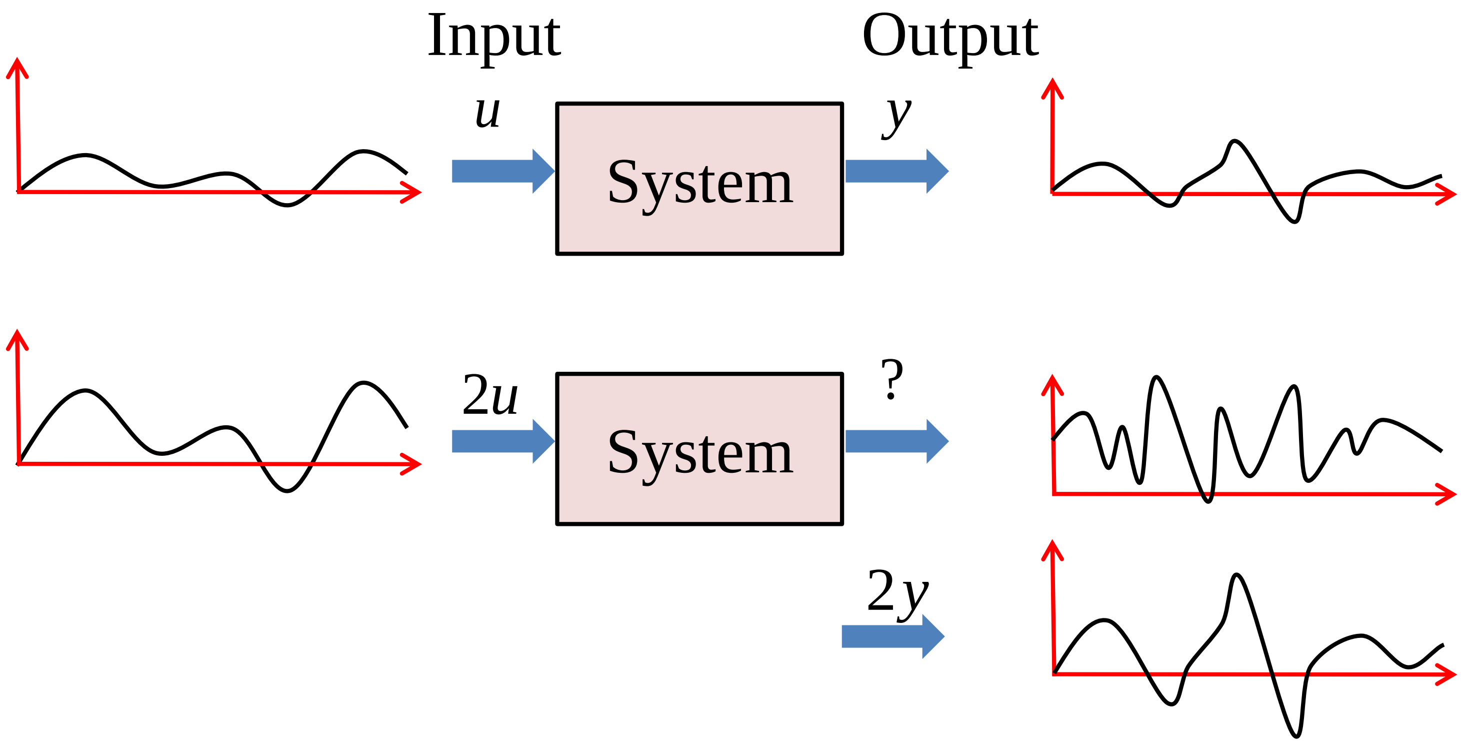
5. Effects of zeros and additional poles

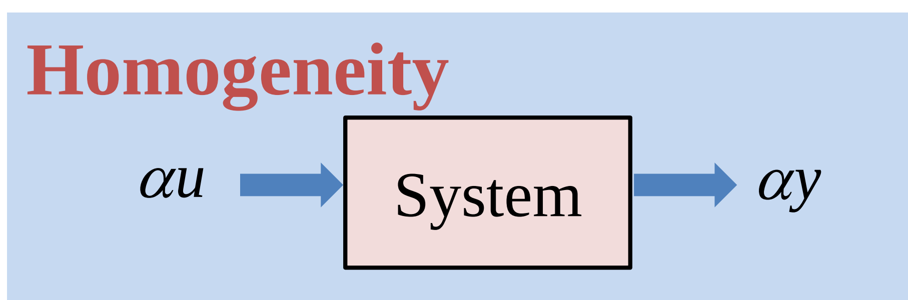
6. Stability

1. Basic Concepts of Linear Systems

Linearity (principle of superposition)

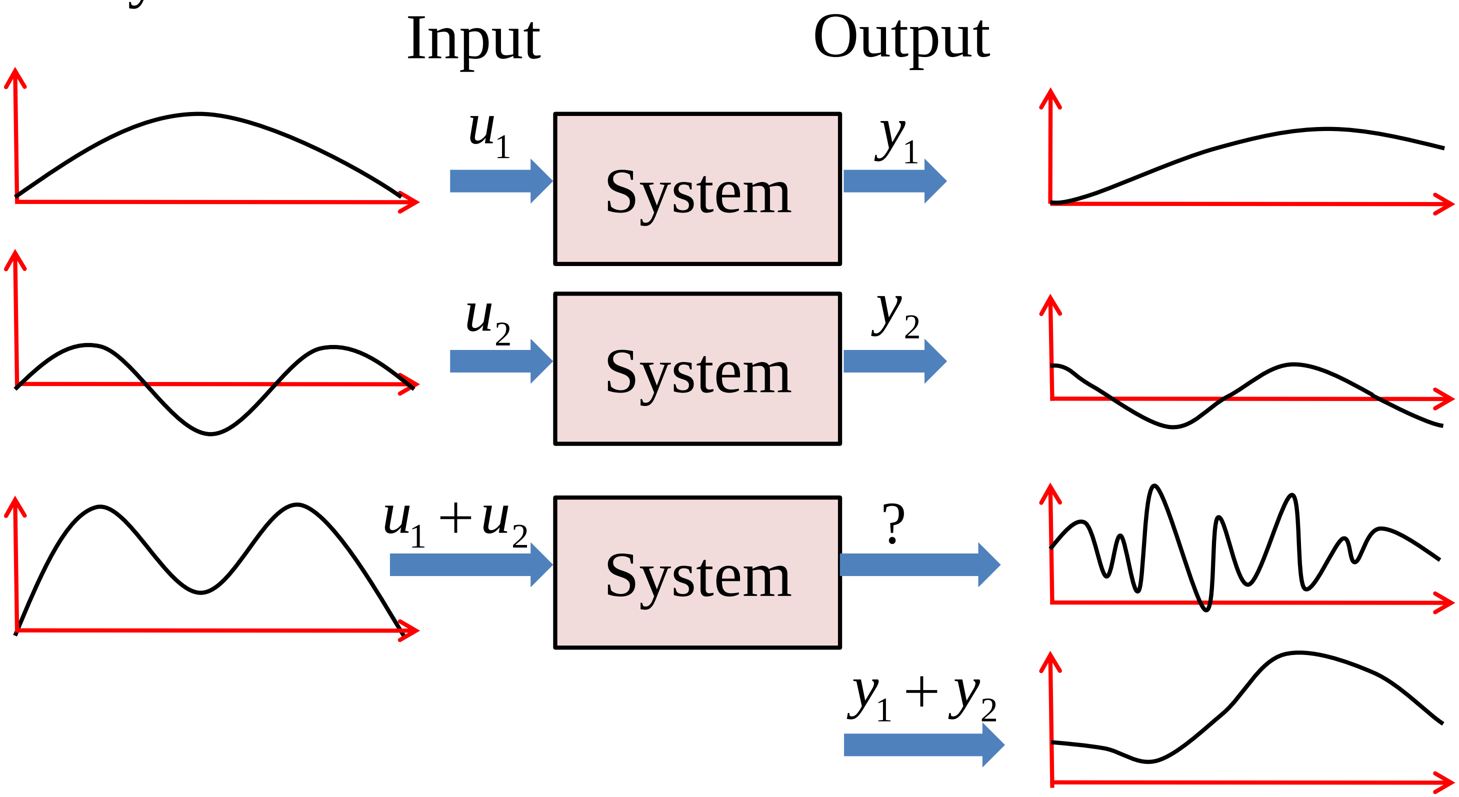
- Homogeneity

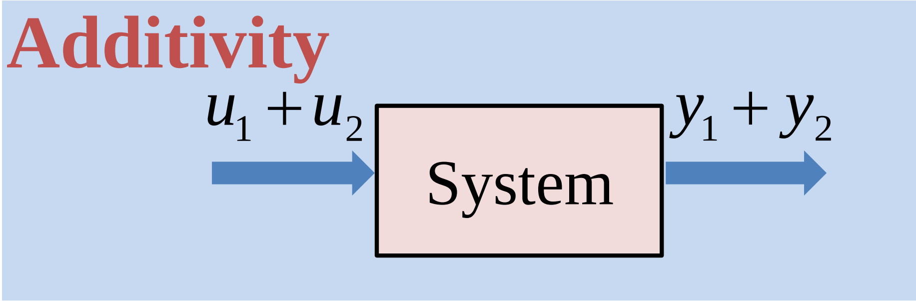




Linearity (principle of superposition)

- Additivity





Linearity (principle of superposition)

Linear system: a system which satisfies the principle of superposition

Principle of superposition: Additivity + Homogeneity

Principle of Superposition = **Homogeneity** + **Additivity**

Homogeneity

$$\alpha u \rightarrow \boxed{\text{System}} \rightarrow \alpha y$$

Additivity

$$u_1 + u_2 \rightarrow \boxed{\text{System}} \rightarrow y_1 + y_2$$

Alternatively,

Principle of Superposition

$$\alpha_1 u_1 + \alpha_2 u_2 \rightarrow \boxed{\text{System}} \rightarrow \alpha_1 y_1 + \alpha_2 y_2$$

Example of linearity

- Linear differential equation $\dot{y} + ky = u$

- Suppose $\dot{y}_1 + ky_1 = u_1, \quad \dot{y}_2 + ky_2 = u_2$

- Let $\bar{u} = \alpha_1 u_1 + \alpha_2 u_2.$

- Assume $\bar{y} = \alpha_1 y_1 + \alpha_2 y_2$

$$\Rightarrow \dot{\bar{y}} = \alpha_1 \dot{y}_1 + \alpha_2 \dot{y}_2 = \alpha_1(-ky_1 + u_1) + \alpha_2(-ky_2 + u_2)$$
$$= -k(\alpha_1 y_1 + \alpha_2 y_2) + (\alpha_1 u_1 + \alpha_2 u_2)$$
$$= -k\bar{y} + \bar{u}$$

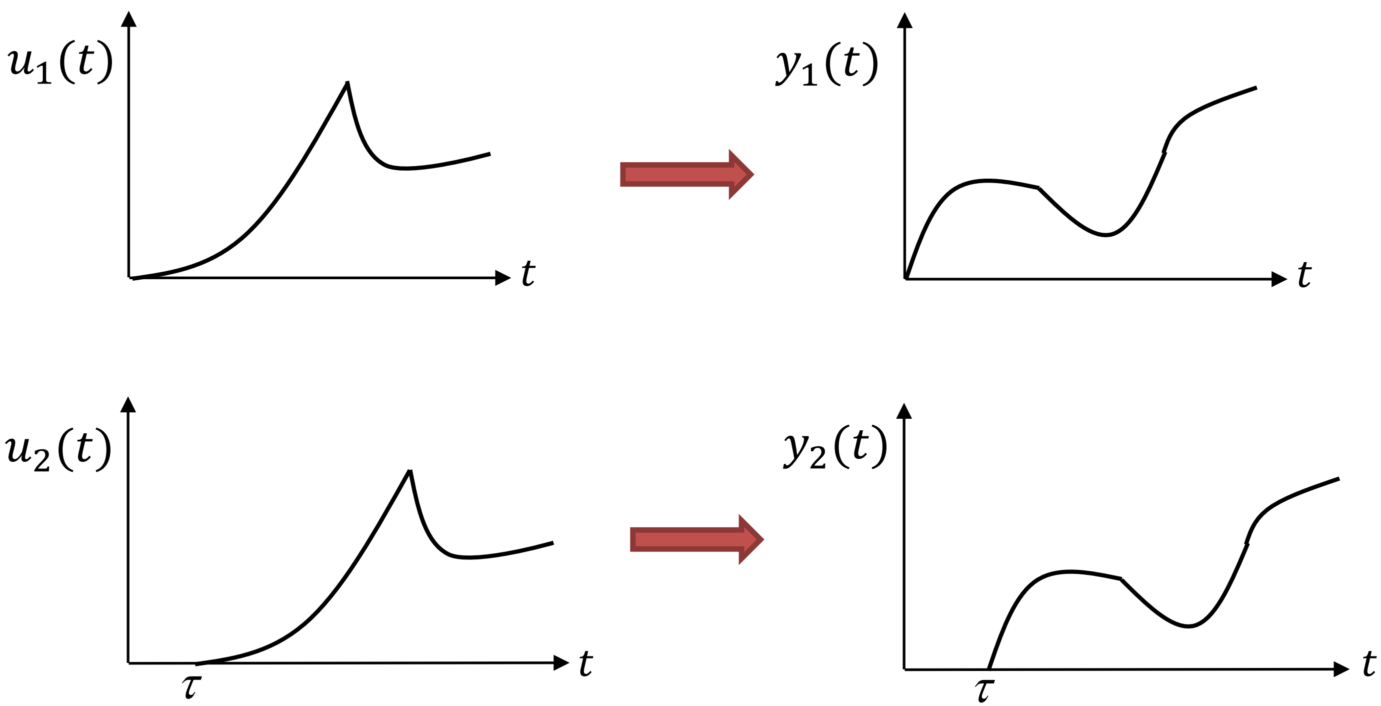
$$\Rightarrow \dot{\bar{y}} + k\bar{y} = \bar{u}$$

- Superposition holds for the linear first order differential equation.

Time invariance

- Time invariance (differential equation with constant coeffecients.)
- Consider $\dot{y}_1(t) + ky_1(t) = u_1(t)$
- What would be the solution for the input $u_2(t) = u_1(t-\tau)$?
- Assume $y_2(t) = y_1(t-\tau)$

$$\Rightarrow \dot{y}_2 = \dot{y}_1(t-\tau) = -ky_1(t-\tau) + u_1(t-\tau) = -ky_2(t) + u_2(t)$$

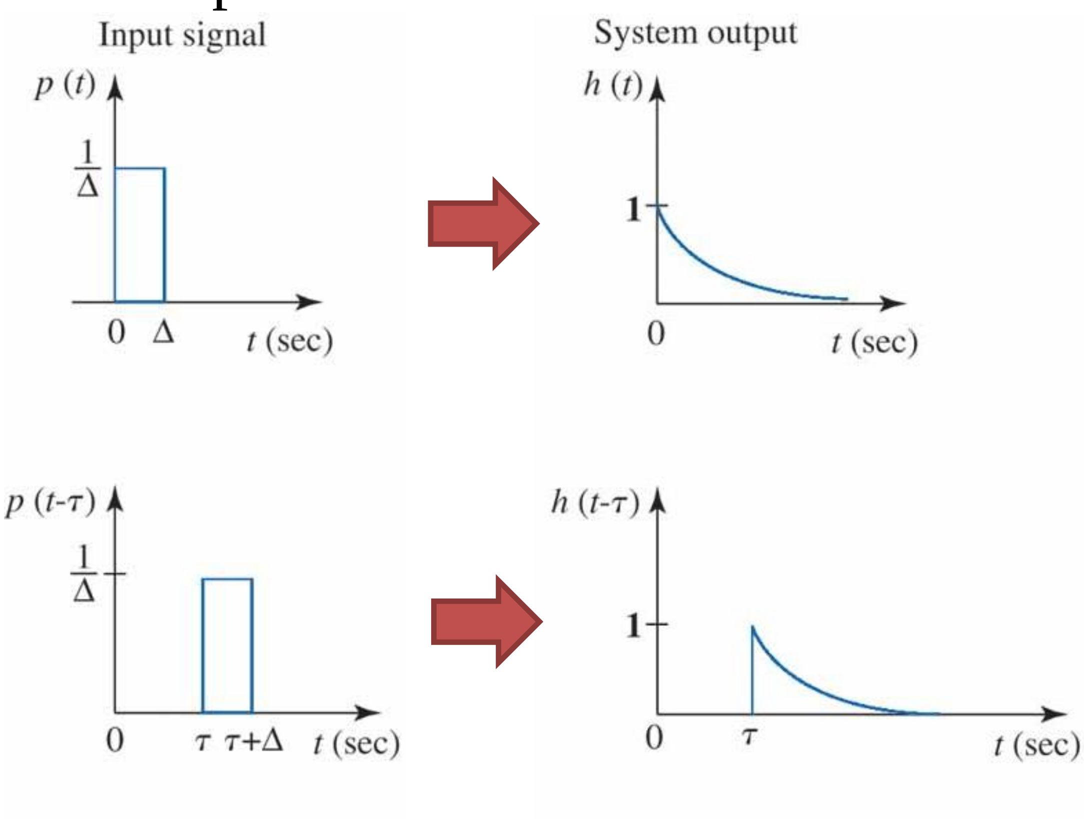


- If the system is time invariant, it follows that if the input is delayed by τ sec, then the output is also delayed by τ sec.

Response by convolution – basic concept

- By using the principle of superposition, we can find the response of the system through **the basic response with respect to a basic input**.
  - basic input: impulse and exponential.

- Response of LTI system w.r.t. short pulses
  suppose
  input: $u_1(t) = p(t)$
  →output: $y_1(t) = h(t)$

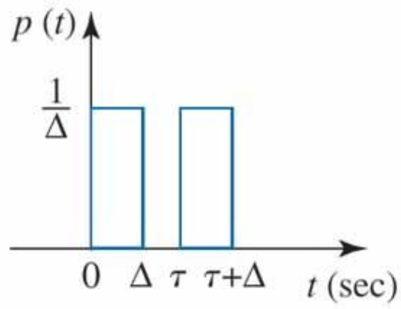


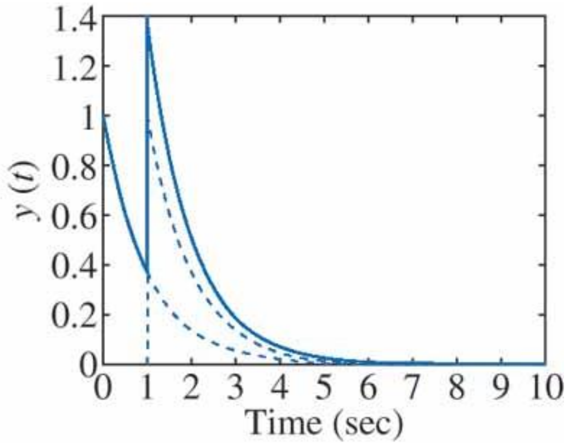
- Linearity
  input: $u_1(t) = u(0)\,p(t)$
  →output: $y_1(t) = u(0)h(t)$

- Time invariance
  input: $u_2(t) = p(t-\tau)$
  →output: $y_2(t) = h(t-\tau)$

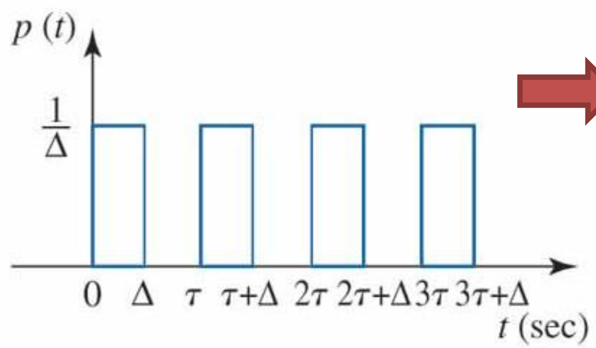
Response by convolution – extension to general input

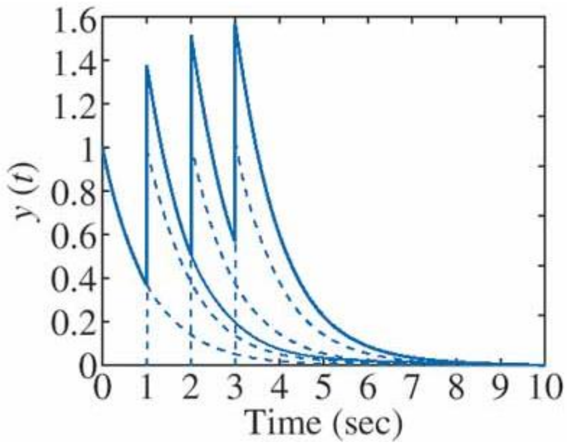
- Response to two pulses





- Response to four pulses

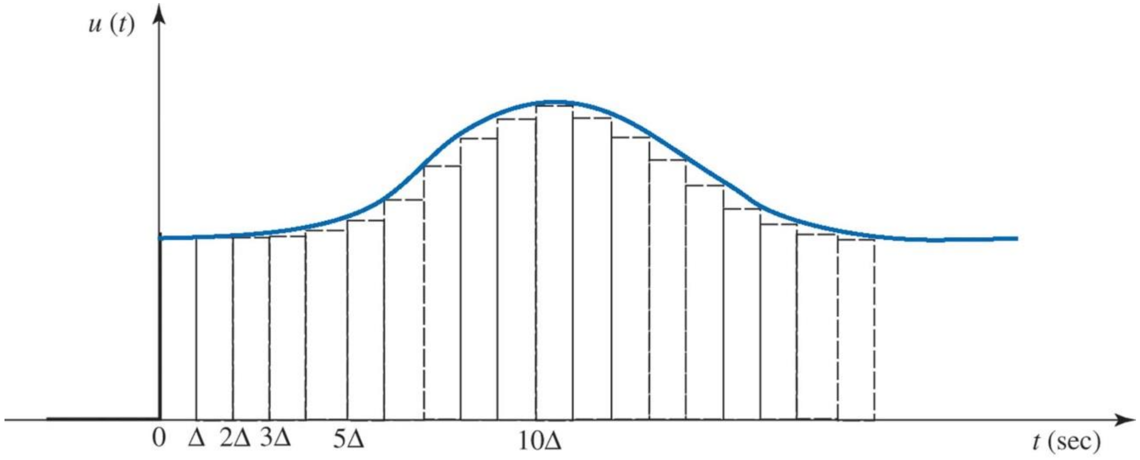




- How about arbitrary input signals?

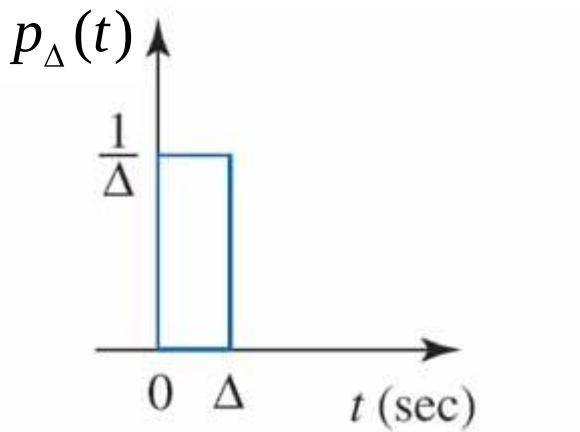
Approximation of input signal

- Representation of a general input signal as the sum of short pulses



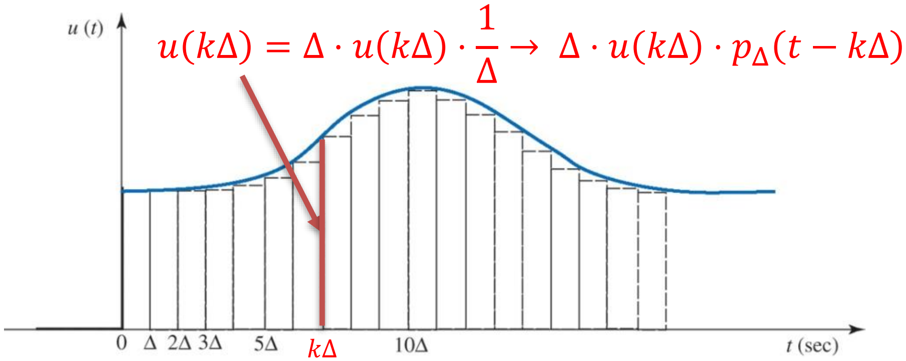
- For mathematical representation, define a short pulse $p_\Delta(t)$: rectangular pulse having unit area

$$p_\Delta(t) = \begin{cases} \dfrac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & \text{otherwise} \end{cases}$$



Derivation of convolution integral

- Suppose: input $p_\Delta(t)$ → output $h_\Delta(t)$



- The input pulse applied at $k\Delta$:

$$\Delta u(k\Delta)\, p_\Delta(t-k\Delta)$$

→ The response at t:

$$\Delta u(k\Delta) h_\Delta(t-k\Delta)$$

- Total response to the series of the short impulses at time t:

$$y(t) = \sum_{k=0}^{\infty} \Delta u(k\Delta) h_\Delta(t-k\Delta)$$

- Impulse and impulse response:

impulse: $\delta(t) := \lim_{\Delta\to 0} p_\Delta(t)$, impulse response: $h(t) := \lim_{\Delta\to 0} h_\Delta(t)$

- Total response in the limit (as $\Delta \to 0$): **convolution integral**

$$y(t) = \sum_{k=0}^{\infty} \Delta u(k\Delta) h_\Delta(t-k\Delta) \quad \Rightarrow \quad y(t) = \int_0^{\infty} u(\tau) h(t-\tau)\, d\tau$$

Approximation of input signal

- Impulse from physics: a very intense force for a very short time. (by Paul Dirac)

Impulse function $\delta(t)$: a function satisfying

$$\delta(t) = 0,\ t \neq 0, \ \text{ and } \int_{-\infty}^{\infty} \delta(t)dt = 1.$$

- Motion of a baseball hit by a bat.

Shifting property of impulse:

for a function $f(t)$ continuous at $t = \tau$.

$$\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = f(t).$$

- The impulse is so short and intense that no value of f matters except over the short range where the impulse occurs. The function f is represented as a sum of impulses.

Response by convolution

- Total response: summation of the basic response:
  - input: $u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau$
  - for a general linear system, we can express the impulse response as $h(t,\tau)$, the response at t to a unit impulse applied at τ.

$$y(t) = \int_{-\infty}^{\infty} u(\tau)h(t,\tau)d\tau$$

- Linear time invariant case: $h(t,\tau) \rightarrow h(t-\tau)$

$$y(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau.$$

Convolution integral

Example of convolution

- $\dot{y} + ky = u = \delta(t)$
  with an initial condition $y(0) = 0$ before the impulse.

- Integrate (just before 0 to just after 0)

$$\int_{0-}^{0+} \dot{y}dt + k\int_{0-}^{0+} ydt = \int_{0-}^{0+} \delta(t)dt$$

$$\Rightarrow\ y(0+) - y(0-) = 1 \quad \left(\because \int_{0-}^{0+} ydt = 0,\ y(0-) = 0\right)$$

$$\Rightarrow\ y(0+) = 1.$$

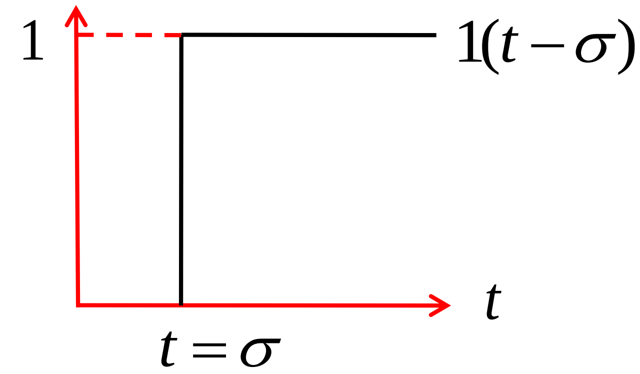
$$\Rightarrow\ \dot{y} + ky = 0,\ y(0+) = 1.$$

- Solution: $y(t) = e^{-kt},\ t > 0.$

Representation by unit step function

- Unit step function: for simplicity (or think physically)

Unit Step Function: $1(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$



- For system (with impulse input)

$$\dot{y} + ky = u = \delta(t),\ y(0) = 0 \text{ before the impulse.}$$

$$y(t) = h(t) = e^{-kt},\ \ t > 0. \Rightarrow y(t) = h(t) = e^{-kt}1(t).$$

- response for general input u(t) for system

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-k\tau}1(\tau)u(t-\tau)d\tau = \int_{0}^{\infty} e^{-k\tau}u(t-\tau)d\tau.$$

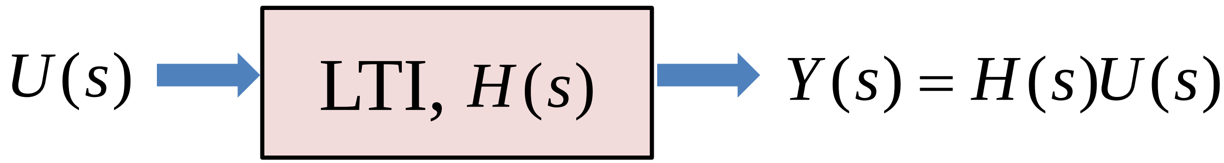
2. Laplace Transform and Transfer Functions

Derivation of transfer function

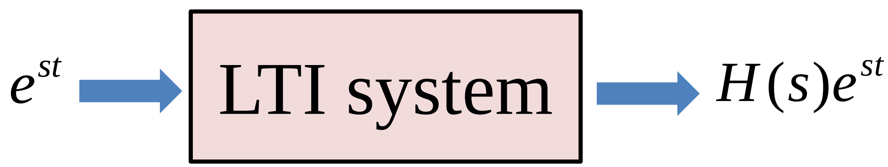
- Transfer function is

- the transfer gain from U(s) to Y(s).

- the Laplace transform of the unit impulse response.



- For LTI systems the response for e^{st} is $H(s)e^{st}$ $(s = \sigma + j\omega)$.



$$y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \cdot e^{st} = H(s)e^{st}.$$

- Causal system: a system is said to be causal if the output is not dependent on future inputs: $h(t) = 0$ for $t < 0$.

For causal systems, $y(t) = \int_{0}^{\infty} h(\tau)u(t-\tau)d\tau.$

Example of transfer function

- Example: Compute the transfer function of the system $\dot{y} + ky = u(t)$, and find the output y for $u = e^{st}$.

(1) $u(t) = \delta(t) \;\Rightarrow\; h(t) = e^{-kt}1(t)$

The transfer function $H(s)$ is defined as the Laplace transform of $h(t)$

$$\Rightarrow H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = \int_{0}^{\infty} e^{-(s+k)\tau}d\tau = -\frac{1}{s+k}e^{-(s+k)\tau}\Big|_{\tau=0}^{\infty} = \frac{1}{s+k}$$

(2) $u(t) = e^{st} \;\Rightarrow y(t) = H(s)e^{st},\;\; \dot{y}(t) = H(s)se^{st}$

$$\Rightarrow H(s)se^{st} + kH(s)e^{st} = e^{st} \;\Rightarrow (s+k)e^{st}H(s) = e^{st} \;\;\Rightarrow H(s) = \frac{1}{s+k}$$

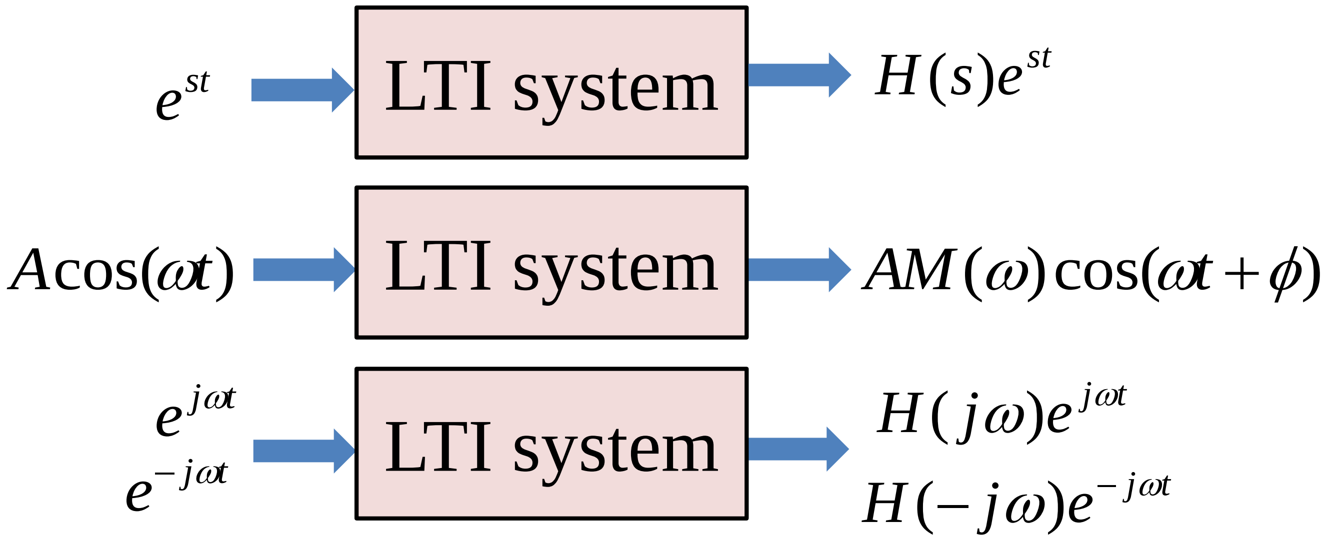
(3) $sY(s) + kY(s) = U(s) \Rightarrow \frac{Y(s)}{U(s)} = \frac{1}{s+k} = H(s)$

- You can integrate h(t) to get H(s), but it is easier to compute H(s) using the differential equation as shown above.

$$H(s) = \frac{1}{s+k},\;\; y = \frac{e^{st}}{s+k}.$$

Frequency response

- Frequency Response: response of the system w.r.t. sinusoidal inputs:



Euler's equation: $A\cos(\omega t)=\frac{A}{2}(e^{j\omega t}+e^{-j\omega t})$

Output to sinusoidal input:

$$y(t)=\frac{A}{2}[H(j\omega)e^{j\omega t}+H(-j\omega)e^{-j\omega t}].$$

Amplitude ratio and phase

- Frequency Response: response of the system w.r.t. sinusoidal inputs:

$$A\cos(\omega t) \longrightarrow \boxed{\text{LTI system}} \longrightarrow AM(\omega)\cos(\omega t+\phi)$$

$$y(t)=\frac{A}{2}[H(j\omega)e^{j\omega t}+H(-j\omega)e^{-j\omega t}].$$

- $H(j\omega)$ is a complex number: $H(j\omega)=M(\omega)e^{j\phi(\omega)}$

For input $u(t)=A\cos\omega t$,

$$y(t)=\frac{A}{2}[H(j\omega)e^{j\omega t}+H(-j\omega)e^{-j\omega t}].$$

$$\Rightarrow y(t)=\frac{A}{2}M(\omega)(e^{j(\omega t+\phi(\omega))}+e^{-j(\omega t+\phi(\omega))})=AM(\omega)\cos(\omega t+\phi(\omega)).$$

$M(\omega)=\left|H(j\omega)\right|$: Amplitude ratio

$\phi(\omega)=\angle H(j\omega)$: Phase

Example of frequency response

- Example: For the system $\dot{y}+ky=u(t)$, find the response to $u=A\cos(\omega t)$.

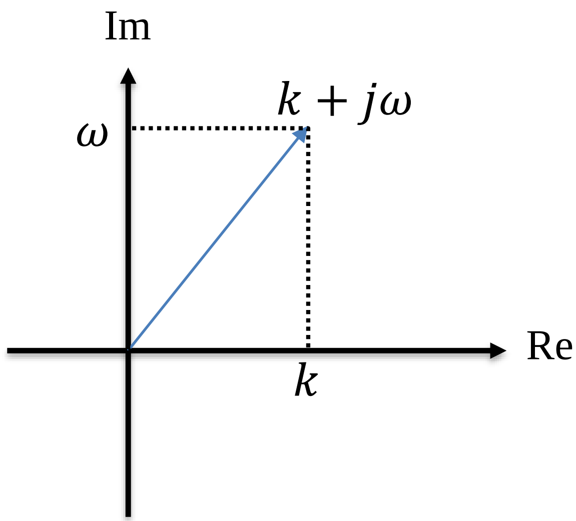
(a) Find the frequency response and plot the response for k=1.

$$\Rightarrow y(t) = AM(\omega)\cos(\omega t+\phi)$$

$$H(s)=\frac{1}{s+k}$$

$$M(\omega)=|H(j\omega)|=\left|\frac{1}{j\omega+k}\right|=\frac{1}{\sqrt{\omega^2+k^2}}$$

$$\phi(w)=\angle H(j\omega)=\angle\frac{1}{j\omega+k}=-\tan^{-1}(w/k)$$



$$M(\omega)=\frac{1}{\sqrt{\omega^2+k^2}},\phi=-\tan^{-1}(\omega/k).$$

- Bode plots

```
% MATLAB
k=1;
num=1;
den=[1 k];
sys=tf(num,den;
w=logspace(-2,2)
[mag,phase]=bode(sys,w);
loglog(w,squeeze(mag));
Semilogx(w,squeeze(phase));
```

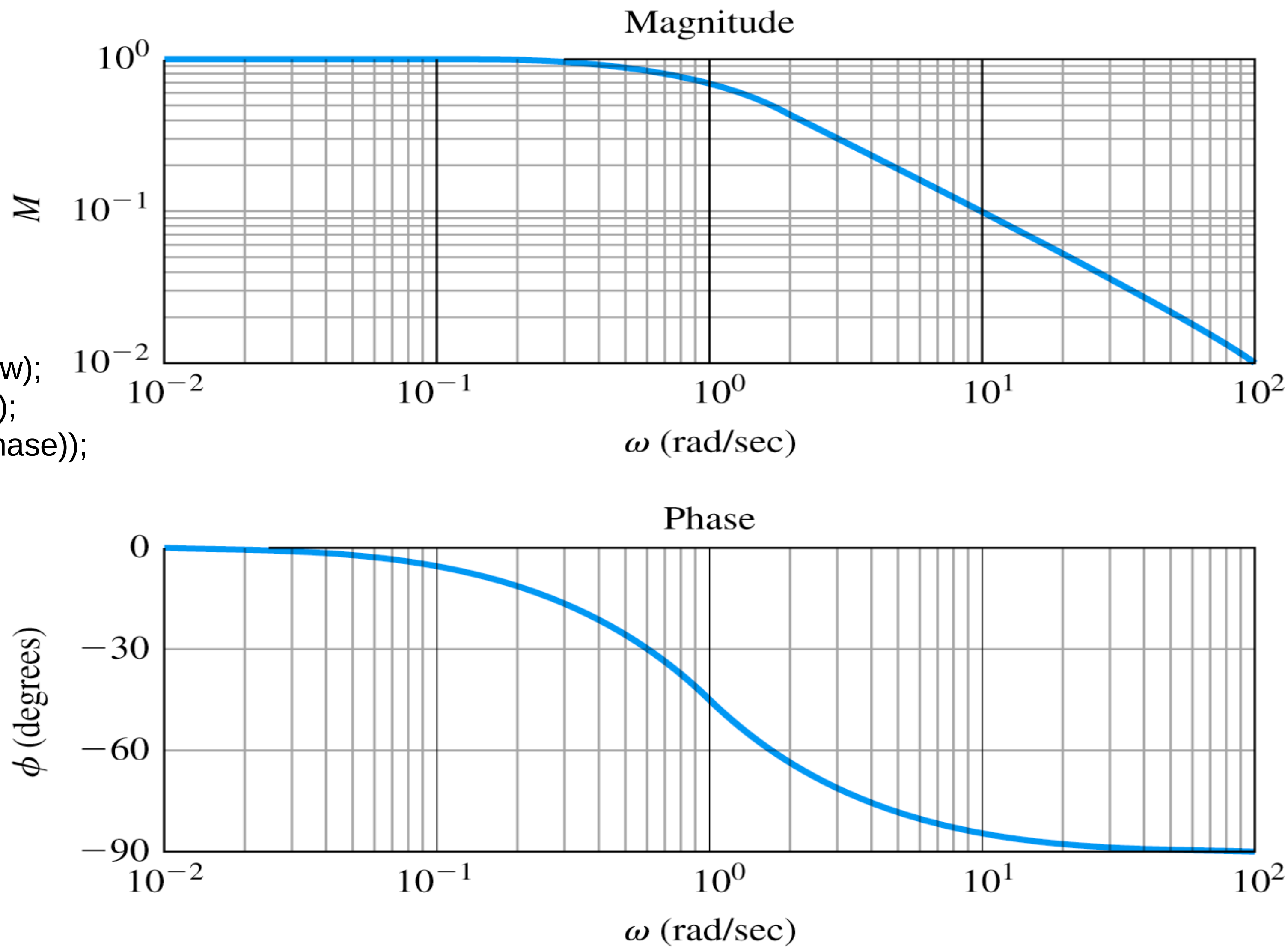


Figure 3.1 Frequency response for $k = 1$

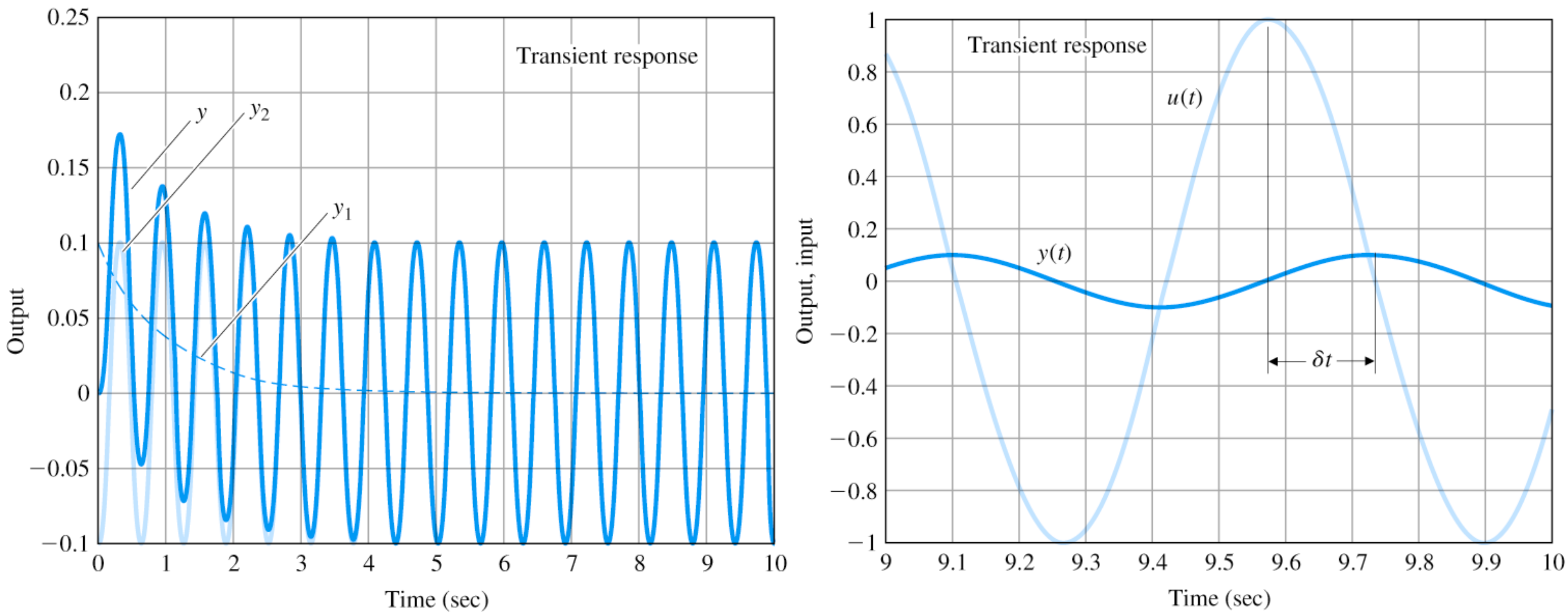
Example of frequency response

- Example: For the system $\dot{y} + ky = u(t),\ k = 1,$ find the complete response to $u(t) = \sin(10t) \left(= \cos\left(10t - \frac{\pi}{2}\right)\right)$

[1] $$y(t) = y_h(t) + y_p(t) = Ce^{-kt} + M(\omega)\cos\left(10t - \frac{\pi}{2} + \phi\right)$$
$$= Ce^{-kt} + \frac{1}{\sqrt{w^2 + k^2}}\sin\left(10t - \tan^{-1}\left(\frac{\omega}{k}\right)\right);\ \ \omega = 10, k = 1$$
$$= Ce^{-t} + \frac{1}{\sqrt{101}}\sin\left(10t - \tan^{-1}(10)\right);\ \text{assume that}\ y(0) = 0$$
$$= \frac{10}{\sqrt{101}}e^{-t} + \frac{1}{\sqrt{101}}\sin\left(10t - \tan^{-1}(10)\right)$$

[2] $$Y(s) = H(s) \cdot U(s) = \frac{1}{s+1} \cdot \frac{10}{s^2 + 100} \quad \Rightarrow y(t) = \mathcal{L}^{-1}\left(\frac{1}{s+1} \cdot \frac{10}{s^2 + 100}\right)$$

$$y(t) = \frac{10}{101}e^{-t} + \frac{1}{\sqrt{101}}\sin(10t + \varphi) = y_1(t) + y_2(t)$$
$$\varphi = \tan^{-1}(-10) = -84.2^o$$
$$(y_1(t) := \text{transient response} \xrightarrow{t\to\infty} 0,\ \ y_2(t) := \text{steady-state response})$$



- Output frequency: 10 rad/sec
- Steady-state phase difference:

$$\varphi(j10) = -10\delta t = -1.47 \text{ rad } = -84.2^\circ$$

- Steady-state amplitude ratio: $M(j10) = \dfrac{1}{\sqrt{101}} = 0.095$

Laplace transform and convolution integral

- The evaluation of convolution integral can be difficult → an indirect approach has been developed using Laplace transform.

Laplace transform of f(t): $F(s)=\int_{-\infty}^{\infty} f(t)e^{-st}dt.$

- Applying Laplace transform to the convolution integral yields

$$Y(s)=H(s)U(s),\ H(s)=\int_{-\infty}^{\infty} h(t)e^{-st}dt.$$

$$Y(s)=\int_{-\infty}^{\infty} y(t)e^{-st}dt=\int_{\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau\cdot e^{-st}dt=\int_{\infty}^{\infty}\int_{-\infty}^{\infty} u(t-\tau)e^{-st}dt\cdot h(\tau)\,d\tau$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(t-\tau)e^{-s(t-\tau)}dt\cdot h(\tau)e^{-s\tau}d\tau=\int_{\infty}^{\infty} h(\tau)e^{-s\tau}\cdot\int_{-\infty}^{\infty} u(\eta)e^{-s\eta}d\eta\,d\tau$$

$$=\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\cdot\int_{-\infty}^{\infty} u(\eta)\mathrm{e}^{-s\eta}d\eta=H(s)\cdot U(s)$$

Transfer Function: $H(s)=\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau.$

Characteristics of Laplace transform

- Laplace Transform: generalized version of the frequency response.

Laplace transform of *f*(*t*): $F(s)=\int_{-\infty}^{\infty} f(t)e^{-st}dt.$

- Laplace transform can be used to study the complete response of feedback system.

- Key property of Laplace transform:

$$Y(s)=H(s)U(s).$$

Very important!!

Procedure for obtaining system response

- Getting system response using Laplace transform:

STEP 1: Determine the transfer function: $H(s)$

$$H(s)= L\{\text{impulse response of the system}\}$$

STEP 2: Determine the Laplace transform of the input:

$$U(s)=L\{u(t)\}$$

STEP 3: Determine the Laplace transform of the output:

$$Y(s)=U(s)H(s)$$

STEP 4: Find the output by computing the inverse Laplace transform:

$$y(t)=L^{-1}\{Y(s)\}$$

L- Laplace transform

- One-sided(or unilateral) Laplace transform:

$$L_{-}\{f(t)\} = \int_{0-}^{\infty} f(t)e^{-st}dt. \quad \text{(two sided: } L\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-st}dt.\text{)}$$

- Uses ‘0-’.

- The impulse function can be applied at time t=0.

- Most cases we drop ‘0-’ and use ‘0’. (‘0-’ will be used if necessary)

- We will use L to mean L_{-}.

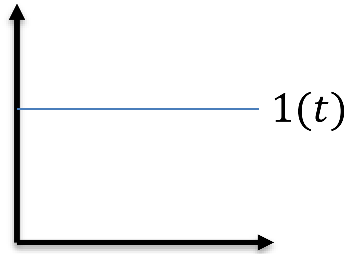
- Inverse Laplace transform is seldom used.

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F(s)e^{st}ds.$$

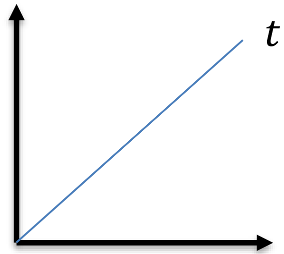
σ_c : selected value to the right of all the singularities of F(s).

Example of L- Laplace transform

- Step and Ramp:



$$\int_0^{\infty} 1(t)e^{-st}dt = \int_0^{\infty} 1 \cdot e^{-st}\,dt = \frac{1}{s}$$



$$\int_0^{\infty} t \cdot 1(t)e^{-st}dt = \int_0^{\infty} te^{-st}dt = \frac{1}{s^2}$$

$$L\{a1(t)\} = \frac{a}{s},\ L\{bt1(t)\} = \frac{b}{s^2}.$$

- Impulse Function:

$$\int_{-\infty}^{\infty} \delta(t)e^{-st}dt = e^{-st}\Big|_{t=0} = 1$$

$$L\{\delta(t)\} = 1.$$

Example of L- Laplace transform

- Sinusoid

$$L\{\sin \omega t\} = \int_0^\infty (\sin \omega t)e^{-st}dt$$

$$= \int_0^\infty \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right) e^{-st}dt$$

$$= \frac{1}{2j}\int_0^\infty \left(e^{(j\omega - s)t} - e^{-(j\omega + s)t} \right)dt = \frac{1}{2j}\left[\frac{e^{(j\omega - s)t}}{j\omega - s} - \frac{e^{-(j\omega + s)t}}{-(j\omega + s)} \right]_{t=0}^{\infty} = \frac{\omega}{s^2 + \omega^2}$$

$$L\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}$$

$$L\{\cos \omega t\} = \int_0^\infty (\cos \omega t)e^{-st}dt$$

$$L\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}$$

$$= \int_0^\infty \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2} \right) e^{-st}dt$$

$$= \frac{1}{2}\int_0^\infty \left(e^{(j\omega - s)t} + e^{-(j\omega + s)t} \right)dt = \frac{1}{2}\left[\frac{e^{(j\omega - s)t}}{j\omega - s} + \frac{e^{-(j\omega + s)t}}{-(j\omega + s)} \right]_{t=0}^{\infty} = \frac{s}{s^2 + \omega^2}$$

Laplace transform table

$F(s)$	$f(t), t \geq 0$
1	$\delta(t)$
$1/s$	$1(t)$
$\dfrac{m!}{s^{m+1}}$	t^m
$\dfrac{1}{s+a}$	e^{-at}
$\dfrac{1}{(s+a)^m}$	$\dfrac{1}{(m-1)!} t^{m-1} e^{-at}$
$\dfrac{a}{s^2+a^2}$	$\sin at$
$\dfrac{s}{s^2+a^2}$	$\cos at$
$\dfrac{s+a}{(s+a)^2+b^2}$	$e^{-at} \cos bt$
$\dfrac{b}{(s+a)^2+b^2}$	$e^{-at} \sin bt$

Properties of Laplace transform

- Superposition: $L\{\alpha f_1(t)+\beta f_2(t)\}=\alpha F_1(s)+\beta F_2(s).$

- Time Delay: $L\{f(t-\lambda)\}=e^{-s\lambda}F(s), \lambda>0.$

- Time Scaling: $L\{f(at)\}=\frac{1}{|a|}F(\frac{s}{a}).$

- Shift in Frequency: $L\{e^{-at}f(t)\}=F(s+a).$

- Differentiation: $L\left\{\dfrac{df(t)}{dt}\right\}=-f(0-)+sF(s).$

$$L\{f^{(m)}(t)\}=s^mF(s)-s^{m-1}f(0-)-s^{m-2}\dot{f}(0-)-\cdots-f^{(m-1)}(0-).$$

- Integration: $L\left\{\int_0^t f(\xi)d\xi\right\}=\dfrac{1}{s}F(s).$

- Convolution: $L\{f_1(t)*f_2(t)\}=F_1(s)F_2(s).$

- Time Product: $L\{f_1(t)f_2(t)\}=\frac{1}{2\pi j}F_1(s)*F_2(s).$

- Multiplication by Time: $L\{tf(t)\}=-\dfrac{d}{ds}F(s).$

Properties of Laplace transform - proof

- Superposition: $L\left\{\alpha f_1(t)+\beta f_2(t)\right\}=\alpha F_1(s)+\beta F_2(s).$

$$\int_0^\infty \big(\alpha f_1(t)+\beta f_2(t)\big)e^{-st}dt=\alpha\int_0^\infty f_1(t)e^{-st}dt+\beta\int_0^\infty f_2(t)e^{-st}dt$$
$$=\alpha F_1(s)+\beta F_2(s)$$

- Differentiation: $L\left\{\dfrac{df(t)}{dt}\right\}=-f(0-)+sF(s).$

$$L\big(f'(t)\big)=\int_0^\infty f'(t)e^{-st}dt=f(t)e^{-st}\Big|_{t=0}^{\infty}-\int_0^\infty f(t)\cdot(-s)e^{-st}dt$$
$$=0-f(0-)+s\int_{0-}^\infty f(t)e^{-st}dt=-f(0-)+sF(s)$$

$$\Downarrow$$

$$L\left\{f^{(m)}(t)\right\}=s^mF(s)-s^{m-1}f(0-)-s^{m-2}\dot{f}(0-)-\cdots-f^{(m-1)}(0-).$$

Properties of Laplace transform - proof

- Integration: $L\left\{\int_0^t f(\xi)d\xi\right\} = \frac{1}{s}F(s).$

$$L\left(\int_0^t f(\xi)d\xi\right) = \int_0^\infty \left(\int_0^t f(\xi)d\xi\right)e^{-st}dt$$

$$= -\frac{1}{s}e^{-st}\int_0^t f(\xi)d\xi\,\Big|_{t=0}^{\infty} + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$= 0 - 0 + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$= \frac{1}{s}F(s)$$

Inverse Laplace transform by partial fraction expansion

- Example: For $Y(s)=\dfrac{(s+2)(s+4)}{s(s+1)(s+3)}$, find $y(t)$.

-distinct poles: Y(s) can be represented by

$$Y(s)=\frac{C_1}{s}+\frac{C_2}{s+1}+\frac{C_3}{s+3}.$$

$$C_1 = sY(s)\Big|_{s=0} = \frac{(s+2)(s+4)}{(s+1)(s+3)}\Big|_{s=0} = \frac{8}{3}$$

$$C_2 = (s+1)Y(s)\Big|_{s=-1} = \frac{(s+2)(s+4)}{s(s+3)}\Big|_{s=-1} = \frac{-3}{2}$$

$$C_3 = (s+3)Y(s)\Big|_{s=-3} = \frac{(s+2)(s+4)}{s(s+1)}\Big|_{s=-3} = \frac{-1}{6}$$

$$y(t)=\frac{8}{3}L^{-1}\left(\frac{1}{s}\right)-\frac{3}{2}L^{-1}\left(\frac{1}{s+1}\right)-\frac{1}{6}L^{-1}\left(\frac{1}{s+3}\right)=\frac{8}{3}1(t)-\frac{3}{2}e^{-t}1(t)-\frac{1}{6}e^{-3t}1(t)$$

Inverse Laplace transform by partial fraction expansion

- For the rational function

$$F(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_n},$$

express F(s) as

$$F(s) = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}.$$

- pole: p_i, zero: z_i

- **If poles are distinct,** $F(s) = \frac{C_1}{s - p_1} + \frac{C_2}{s - p_2} + \cdots + \frac{C_n}{s - p_n}$

where $C_i = (s - p_i) F(s)\big|_{s = p_i}$.

$\rightarrow$ $f(t) = \sum_{i=1}^{n} C_i e^{p_i t} 1(t).$

Inverse Laplace transform by partial fraction expansion

- For the rational function

$$F(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_n},$$

express F(s) as

$$F(s) = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}.$$

- pole: p_i, zero: z_i

- **If poles are not distinct,** $m, l \le n$, multiple poles

$$F(s) = \frac{K \Pi_{i=1}^{m} (s - z_i)}{\Pi_{i=1}^{l} (s - p_i)^{k_i}} = \sum_{i=1}^{l} \sum_{j=1}^{k_i} \frac{\bar{C}_{ij}}{(s - p_i)^j} \xrightarrow{\mathcal{L}^{-1}} f(t) = \sum_{i=1}^{l} \sum_{j=1}^{k_i} C_{ij} t^{j-1} e^{p_i t}$$

The final value theorem

- Useful when compute the constant steady state value of a time function given its Laplace transform.

If all poles of sY(s) are in the OLHP, then

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s).$$

- It can be used only when the limit exists and is constant.

- DC gain: the steady state value of the output of a system w.r.t. the unit step input.

$$\text{DC gain} = \lim_{s\to 0} sG(s)\frac{1}{s} = \lim_{s\to 0} G(s).$$

The final value theorem - proof

- Proof

recall: $\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0^-) = \int_{0^-}^{\infty} e^{-st}\frac{dy}{dt}dt.$

$$\rightarrow \quad \lim_{s\to 0}[sY(s) - y(0)] = \lim_{s\to 0}\int_0^{\infty} e^{-st}\frac{dy}{dt}dt = \lim_{t\to\infty}[y(t) - y(0)].$$

$$\lim_{s\to 0}\int_0^{\infty} e^{-st}\frac{dy}{dt}dt = \int_0^{\infty}\lim_{s\to 0} e^{-st}\frac{dy}{dt}dt = \int_0^{\infty} dy = \lim_{t\to\infty}[y(t) - y(0)]$$

Another way: consider the case $Y(s) = \frac{C_1}{s} + \frac{C_2}{s - p_2} + \frac{C_3}{s - p_3}.$

$$\rightarrow \quad y(t) = C_1 1(t) + C_2 e^{p_2 t} 1(t) + C_3 e^{p_3 t} 1(t), \qquad \lim_{s\to 0} sY(s) = C_1$$

Example of the final value theorem

- Example: Find the final value of the system corresponding to

$$Y(s)=\frac{3(s+2)}{s(s^2+2s+10)}.$$

$$\to \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0}\frac{3(s+2)}{s^2+2s+10} = \frac{3}{5}$$

$\because$ The poles of $sY(s)$ are $p=-1\pm 3j$

- Example: Incorrect use of the final value theorem.

$$Y(s)=\frac{3}{s(s-2)}.$$

$$L^{-1}(Y(s)) = L^{-1}\left(\frac{3}{2}\left(-\frac{1}{s}+\frac{1}{s-2}\right)\right) = -\frac{3}{2}+\frac{3}{2}e^{2t}$$

$\Rightarrow$ Does not converge to 0 as time increases

Solution to differential equations

- Laplace transform can be used to solve differential equations.

- Example: Find the solution to the differential equation

$$\ddot{y}(t) + y(t) = 0, y(0) = \alpha, \dot{y}(0) = \beta.$$

$$\rightarrow s^2 Y(s) - (sy(0) + \dot{y}(0)) + Y(s) = 0$$

$$\rightarrow s^2 Y(s) - (s\alpha + \beta) + Y(s) = 0$$

$$\rightarrow \quad Y(s) = \frac{\alpha s}{s^2 + 1} + \frac{\beta}{s^2 + 1}.$$

$$y(t) = (\alpha \cos t + \beta \sin t) 1(t)$$

Solutions to differential equations - example

- Example: Forced Differential Equation

$$\ddot{y}(t)+5\dot{y}(t)+4y(t)=3,\ y(0)=\alpha,\ \dot{y}(0)=\beta$$

Sol. $$s^2Y(s)-\alpha s-\beta+5[sY(s)-\alpha]+4Y(s)=\frac{3}{s}$$

$$Y(s)=\frac{s(\alpha s+\beta+5\alpha)+3}{s(s+1)(s+4)}$$

$$=\frac{\frac{3}{4}}{s}-\frac{\frac{3-\beta-4\alpha}{3}}{s+1}+\frac{\frac{3-4\beta-4\alpha}{12}}{s+4}$$

$$\xrightarrow{L^{-1}} y(t)=\left(\frac{3}{4}-\frac{3-\beta-4\alpha}{3}e^{-t}+\frac{3-4\beta-4\alpha}{12}e^{-4t}\right)1(t)$$

Solutions to differential equations - example

- Example: Forced Differential Equation with Zero I.C.

$$\ddot{y}(t)+5\dot{y}(t)+4y(t)=u(t),\ \dot{y}(0)=0,\ y(0)=0,\ u(t)=2e^{-2t}1(t)$$

Sol.)

$$s^2Y(s)+5sY(s)+4Y(s)=\frac{2}{s+2}$$

$$Y(s)=\frac{2}{(s+2)(s^2+5s+4)}=\frac{2}{(s+2)(s+1)(s+4)}$$

$$=-\frac{1}{s+2}+\frac{2/3}{s+1}+\frac{1/3}{s+4}\xrightarrow{L^{-1}}y(t)=\left(-1e^{-2t}+\frac{2}{3}e^{-t}+\frac{1}{3}e^{-4t}\right)1(t)$$

Poles and zeros

- A rational transfer function

$$H(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{N(s)}{D(s)}$$

can be described in the form

$$H(s) = K \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}.$$

K : **transfer function gain**

z_i : **zero**

p_i : **pole**

- Zero: $H(s)\big|_{s=z_i} = 0.$

  - can block some signal: $u = u_0 e^{z_1 t}, \;\; y(t) \equiv 0.$

  → will be clear if we use state space approach.

- Pole: $\left|H(s)\right|_{s=p_i} = \infty.$

  - related to the system's stability.

  - determines the natural (or unforced) behavior of the system, referred to as the modes of the system.

3. Effects of pole locations and Block Diagrams

Poles and zeros - review

- A rational transfer function

$$H(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{b(s)}{a(s)}$$

can be described in the form

$$H(s) = K \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}.$$

K : **transfer function gain**

z_i : **zero**

p_i : **pole**

- Pole: roots of a(s): $\left|H(s)\right|_{s=p_i} = \infty.$

- Zero: roots of b(s): $H(s)\big|_{s=z_i} = 0.$

- Impulse response

$$h(t) = L^{-1}\{H(s)\} = L^{-1}\left\{K \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}\right\} = \text{natural response}$$

- Poles identify the classes of signals contained in the impulse response.

Effect of pole locations – first order pole

- First order pole:

$$H(s)=\frac{1}{s+\sigma}\xrightarrow{L^{-1}}h(t)=e^{-\sigma t}1(t)$$

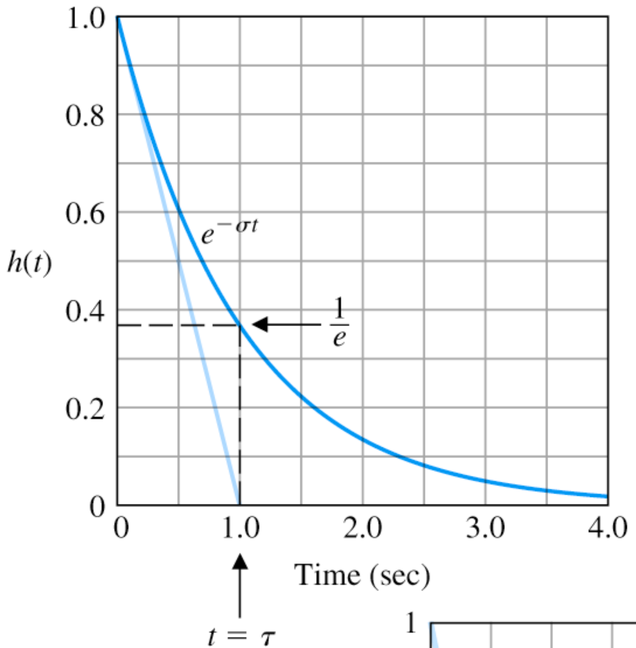
$$\left(\text{one pole at } s=-\sigma\right)$$

$$\sigma>0:\text{ stable}$$

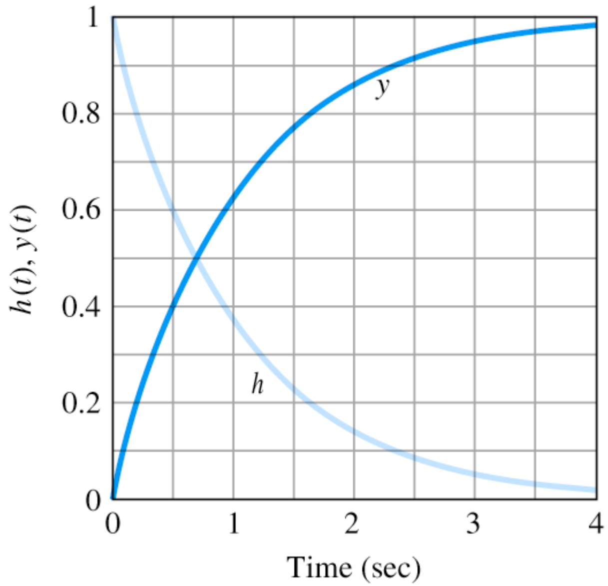
$$\sigma<0:\text{ unstable}$$

$$\tau=1/\sigma:=\text{ time constant}$$

$$\left(h(\tau)=h(1/\sigma)=e^{-\sigma(1/\sigma)}=e^{-1}=1/e\right)$$



- Step response for $H(s)=\dfrac{\sigma}{s+\sigma}$



Example – multiple poles

- Response versus pole locations, real roots

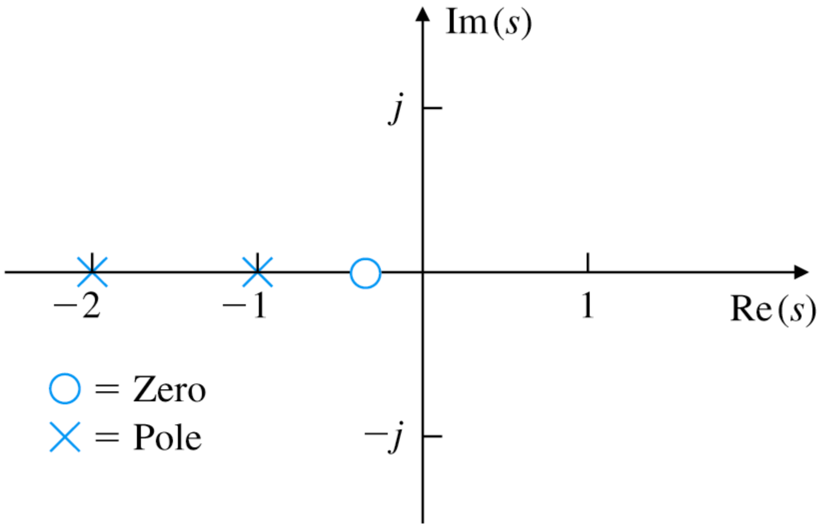
$$H(s) = \frac{2s+1}{s^2+3s+2} = \frac{2(s+1/2)}{(s+1)(s+2)}$$

$$= -\frac{1}{s+1} + \frac{3}{s+2}$$

poles : $-1(slow), -2(fast)$

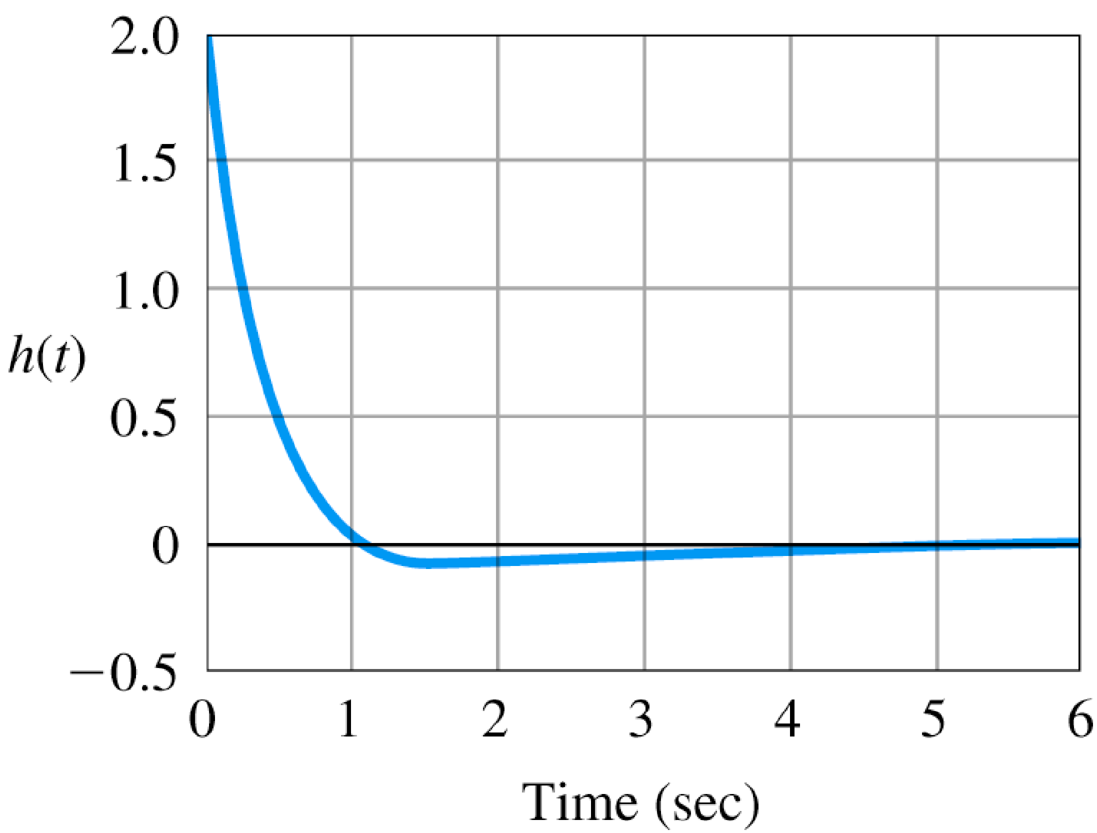
zeros : -0.5

$$\xrightarrow{L^{-1}} h(t) = \begin{cases} -e^{-t} + 3e^{-2t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



- Role of numerator: determines the size of the coefficient of each mode in the natural response

```
numH = [2  1];             % form numerator
denH = [1  3  2];          % form denominator
sysH = tf(numH,denH)       % define system from its numerator and denominator
Impulse(sysH)              % compute impulse response
```



Damping ration and natural frequency

- Complex poles: $s = -\sigma \pm j\omega_d$

$$a(s) = (s + \sigma - j\omega_d)(s + \sigma + j\omega_d) = (s + \sigma)^2 + \omega_d{}^2$$

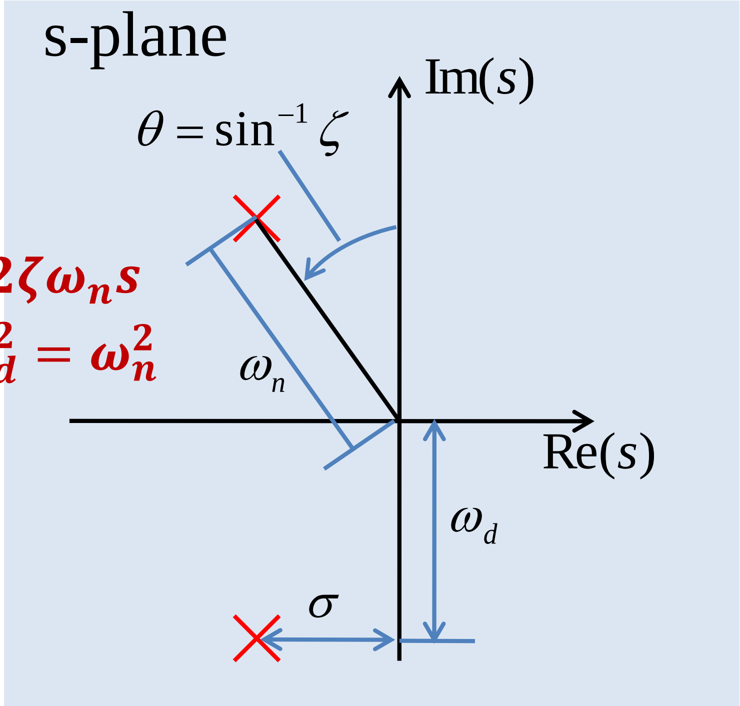
- Related transfer function:

$$H(s) = \frac{\omega_n{}^2}{(s+\sigma)^2 + \omega_d{}^2} = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

$$\sigma := \zeta\omega_n,\ \omega_d := \omega_n\sqrt{1-\zeta^2}$$

$$\left(\begin{array}{l} \zeta = \dfrac{\sigma}{\omega_n} := \text{damping ratio} \\ \omega_n := \text{undamped natural frequency} \\ \omega_d := \text{damped natural frequency} \end{array} \right)$$

$$\boldsymbol{2\sigma s = 2\zeta\omega_n s}$$
$$\boldsymbol{\sigma^2 + \omega_d^2 = \omega_n^2}$$



plot of a pair of complex poles

- Impulse response: $h(t) = \dfrac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t}(\sin\omega_d t)1(t).$

Response for complex poles – impulse response

- Impulse response of $H(s)=\dfrac{\omega_n^{\ 2}}{(s+\sigma)^2+\omega_d^{\ 2}}=\dfrac{\omega_n^{\ 2}}{s^2+2\zeta\omega_n s+\omega_n^{\ 2}}$

$$H(s)=\frac{\omega_n}{\sqrt{1-\zeta^2}}\frac{\omega_d}{(s+\sigma)^2+\omega_d^2}$$

$$\xrightarrow{L^{-1}} h(t)=\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\sigma t}\left(\sin\omega_d t\right)1(t)$$

$$=\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta(\omega_n t)}\left(\sin\left(\sqrt{1-\zeta^2}(\omega_n t)\right)\right)1(t)$$

- Normalization: $\tau := \omega_n t \rightarrow \omega_n = 1$

- The actual frequency decreases as the damping ratio increases.

- For very low damping, the response is oscillatory.

Response for complex poles – step response

- Step response of $H(s) = \dfrac{\omega_n{}^2}{(s+\sigma)^2 + \omega_d{}^2} = \dfrac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$

$$Y(s) = H(s)\frac{1}{s} = \frac{\omega_n{}^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n{}^2\right)}$$

$$= \frac{1}{s} - \frac{(s+\zeta\omega_n) + \dfrac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n{}^2\left(1-\zeta^2\right)}$$

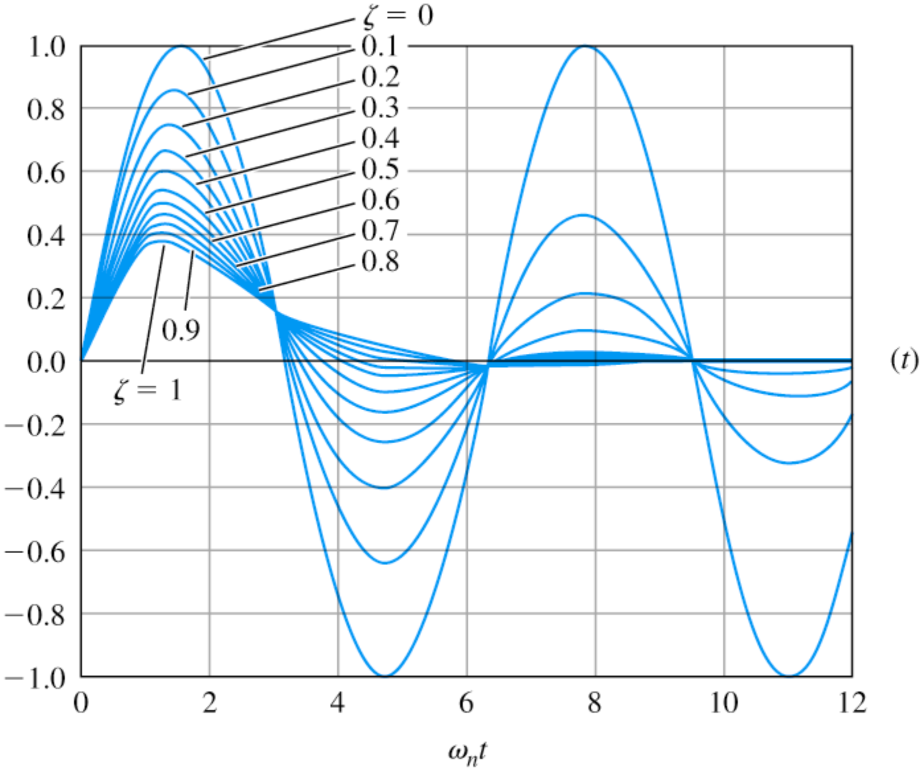
$$\xrightarrow{L^{-1}} y(t) = 1(t) - e^{-\zeta(\omega_n t)}\left(\cos\left(\sqrt{1-\zeta^2}(\omega_n t)\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}(\omega_n t)\right)\right)1(t)$$

$$= 1(t) - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta(\omega_n t)}\cos\left(\sqrt{1-\zeta^2}(\omega_n t) + \beta\right)1(t)$$

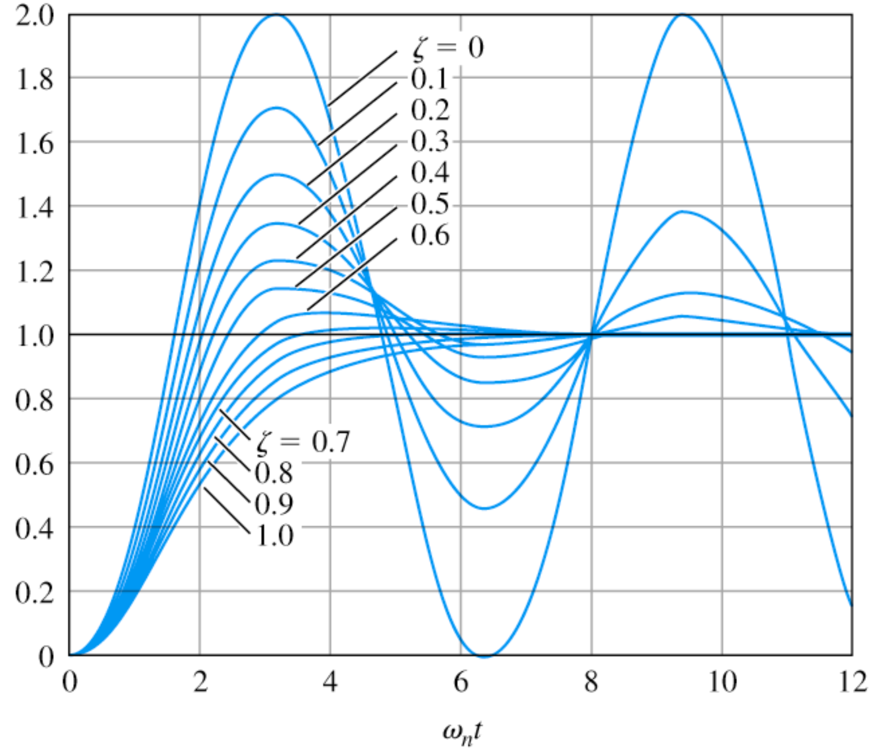
$$\left(\beta = \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}} = \tan^{-1}\frac{\sigma}{\omega_d}\right), \text{ Normalization: } \tau := \omega_n t \rightarrow \omega_n = 1$$

Simulation results

Impulse response



Step response



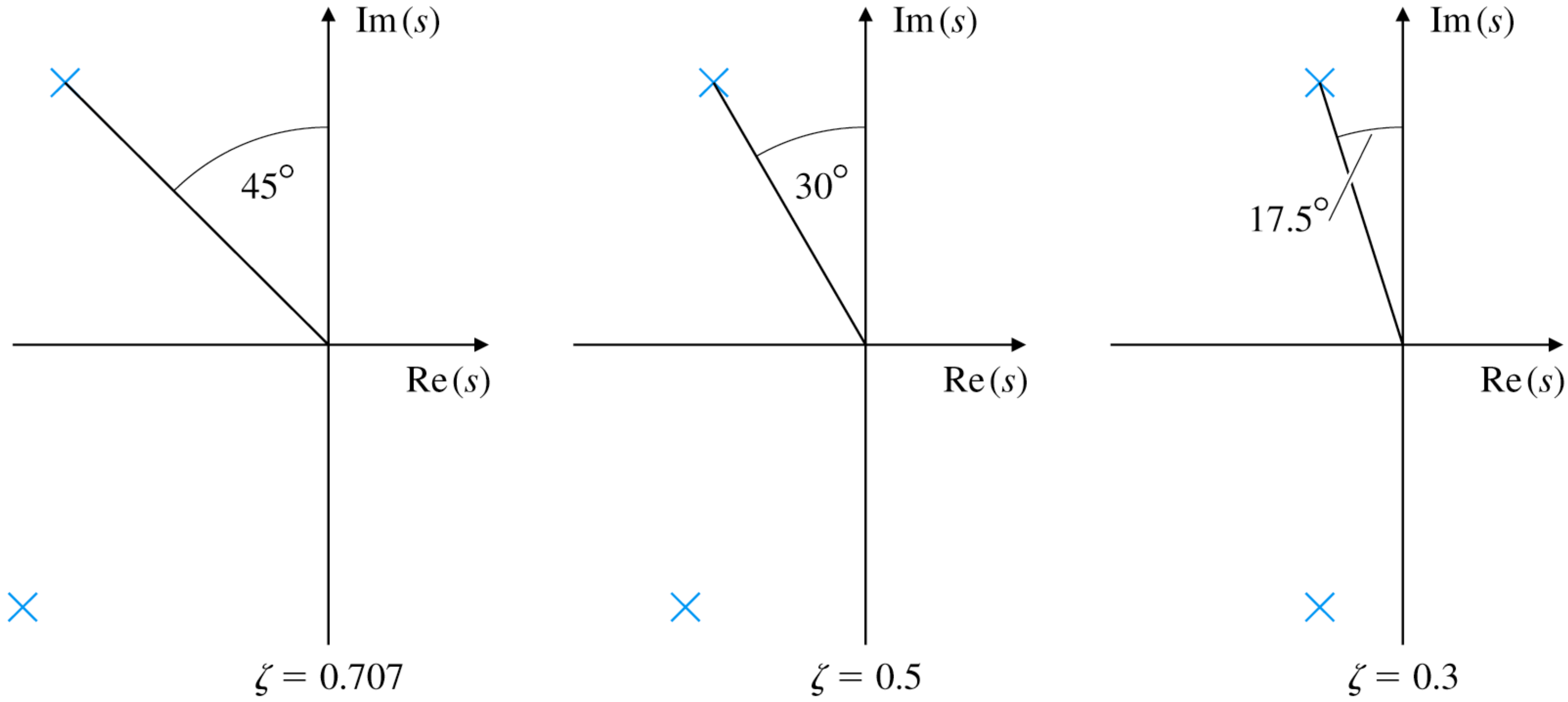
$$y(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta(\omega_n t)} \left(\sin\left(\sqrt{1-\zeta^2}(\omega_n t)\right)\right) 1(t)$$

$$\tau := \omega_n t \to \omega_n = 1$$

$$y(t) = 1(t) - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta(\omega_n t)} \cos\left(\sqrt{1-\zeta^2}(\omega_n t) + \beta\right) 1(t)$$

Pole locations and damping ratio

- Pole locations corresponding to damping ratio



Effect of pole locations – negative real part

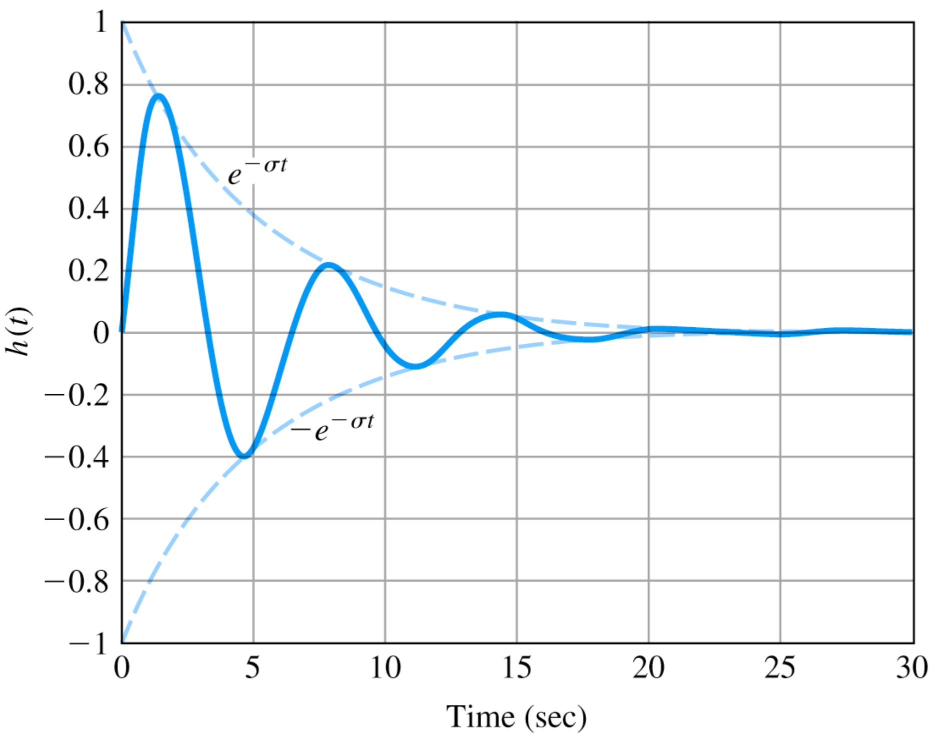
- The negative real part of the pole determines the decay rate of the exponential envelope.

$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} (\sin \omega_d t) 1(t), \ (\sigma = \zeta \omega_n)$$

- Stability of complex poles

$\left(\text{Complex poles at } s = -\sigma \pm j\omega_d \right)$

$$\begin{cases} \sigma < 0: \text{ unstable} \\ \sigma = 0: \text{ neutrally stable} \\ \sigma > 0: \text{ stable} \end{cases}$$



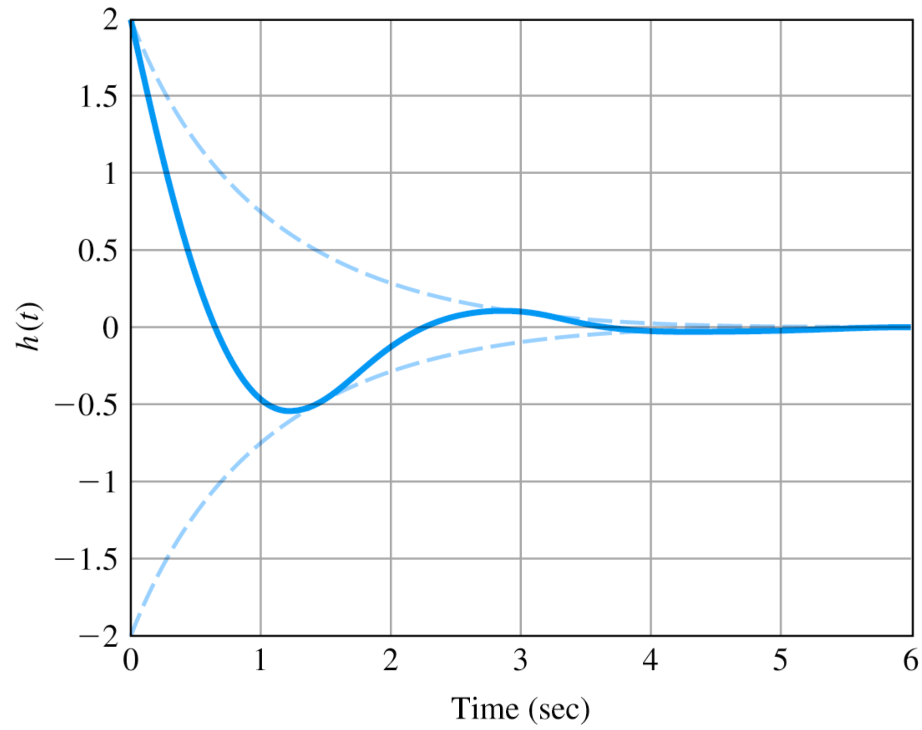
Example – effect of negative real part

- Oscillatory time response

$$H(s) = \frac{2s+1}{s^2+2s+5}$$

$$\omega_n^{\ 2} = 5 \Rightarrow \omega_n = \sqrt{5} = 2.24$$

$$2\varsigma\omega_n = 2 \Rightarrow \varsigma = \frac{1}{\sqrt{5}} = 0.447$$



$$H(s) = \frac{2s+1}{(s+1)^2+2^2}$$

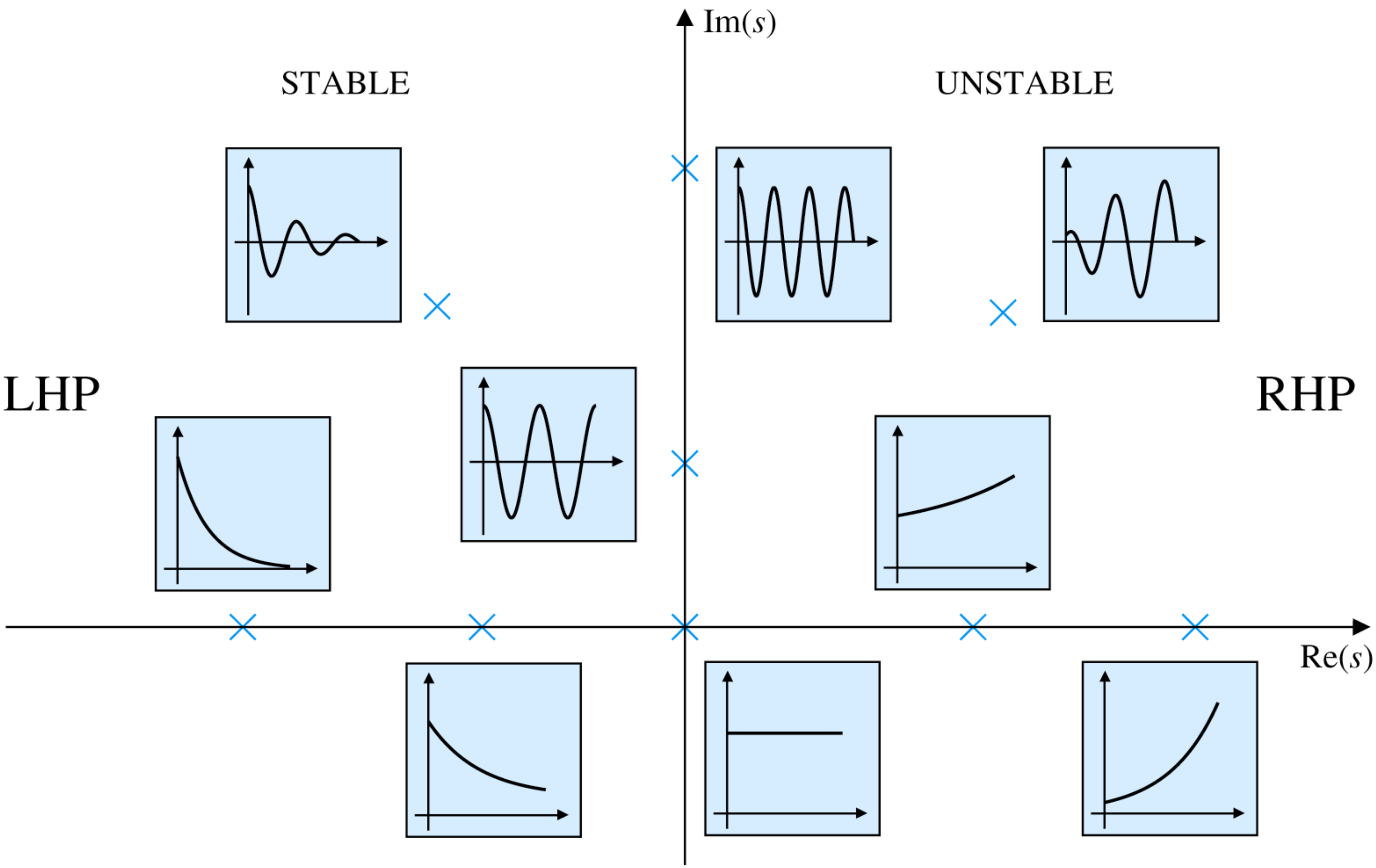
$$= 2\frac{s+1}{(s+1)^2+2^2} - \frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

$$\xrightarrow{L^{-1}} h(t) = \left(2e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t\right)1(t)$$

$$= \left(2\cos 2t - \frac{1}{2}\sin 2t\right)e^{-t}1(t)$$

Summary of effect of pole locations

- Pole locations determine "the shape" or "the way it behaves" for impulse response (as well as other responses) of the system.

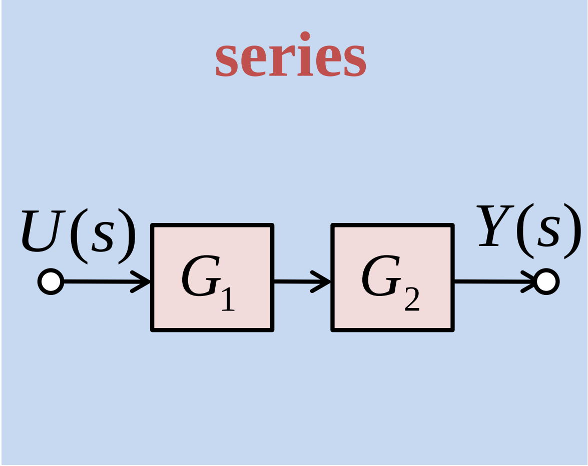


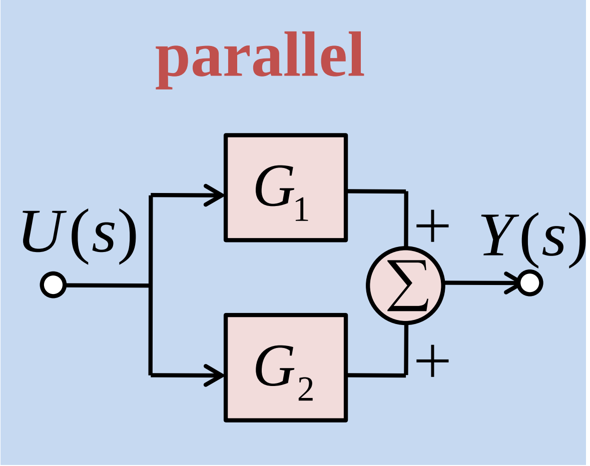
Block diagram

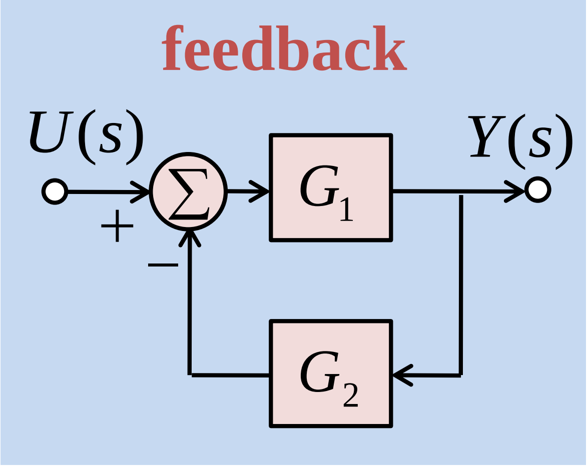
- In many cases, the control system is composed of systems called components (which can be dynamic systems) that interact with others

- Block diagrams can be used to illustrate the relationship between the components of given system.

- Important block diagrams: series, parallel, and feedback.

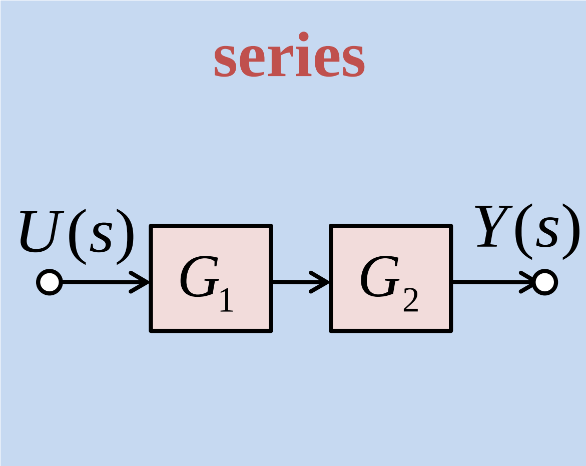




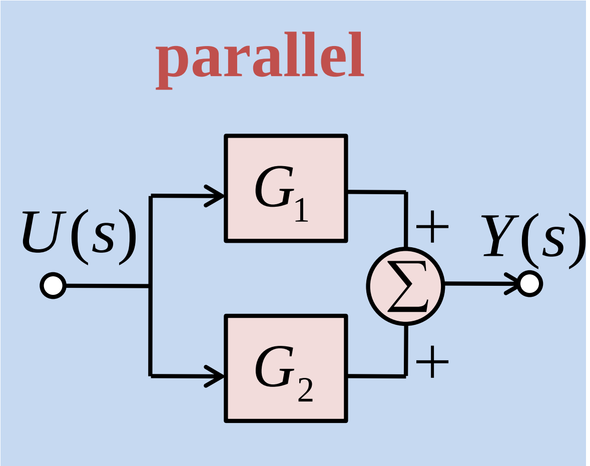


Transfer function corresponding to block diagram

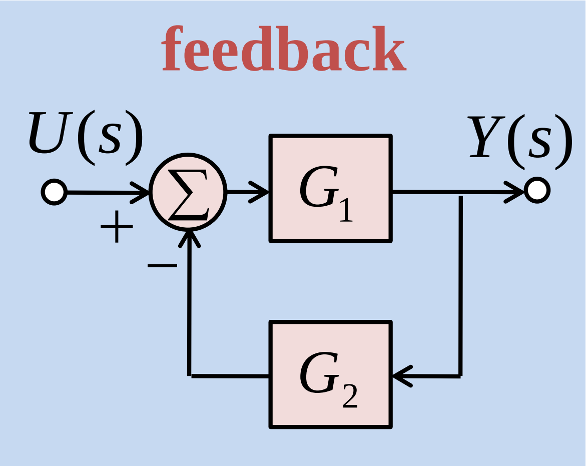
- Transfer function of elementary block diagram



$$\frac{Y(s)}{U(s)} = G_2(s)G_1(s)$$



$$\frac{Y(s)}{U(s)} = G_1(s) + G_2(s)$$

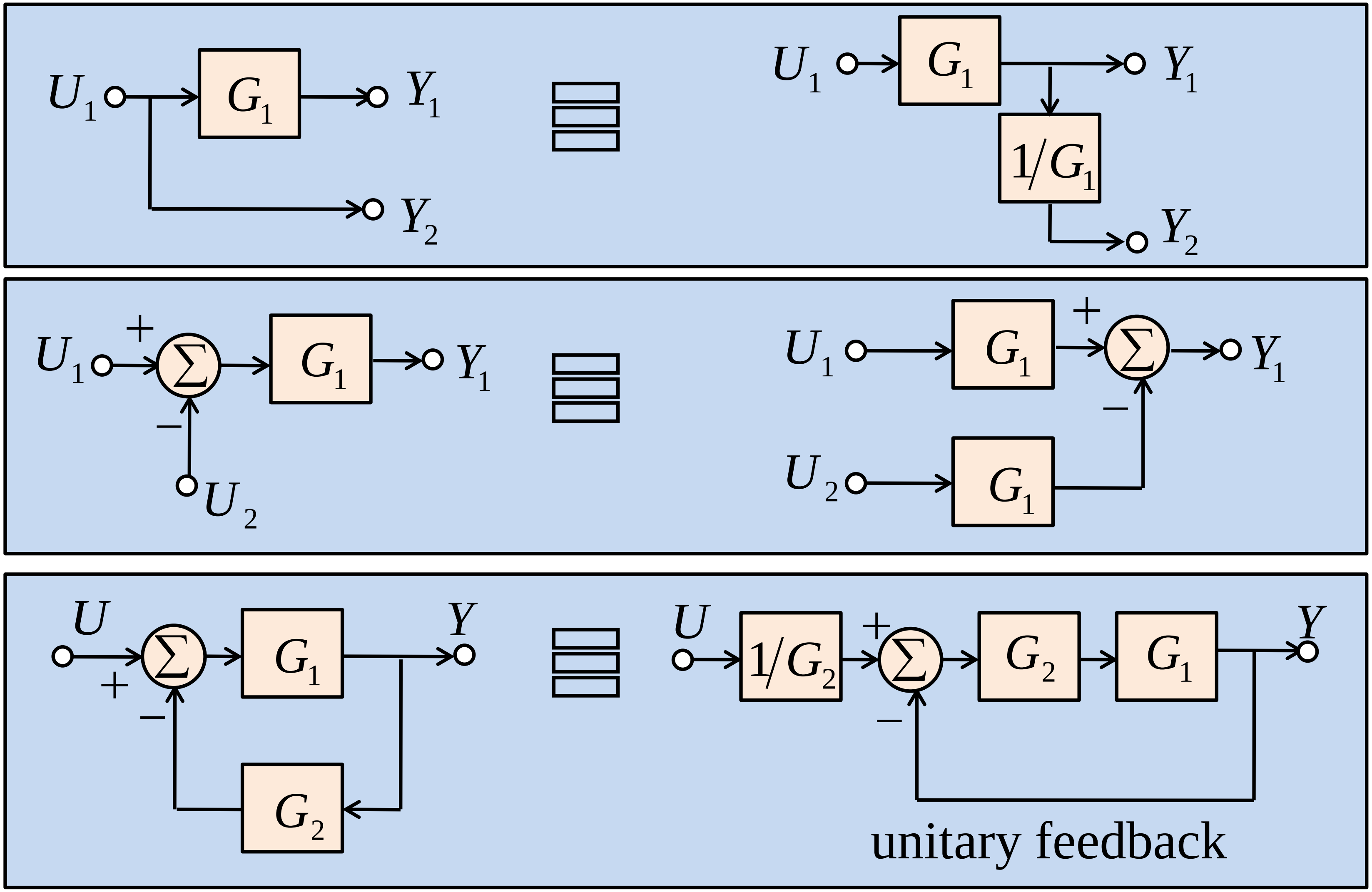


$$\frac{Y(s)}{U(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

The gain of a single-loop negative feedback system is given by [forward gain] divided by (1+ [loop gain])

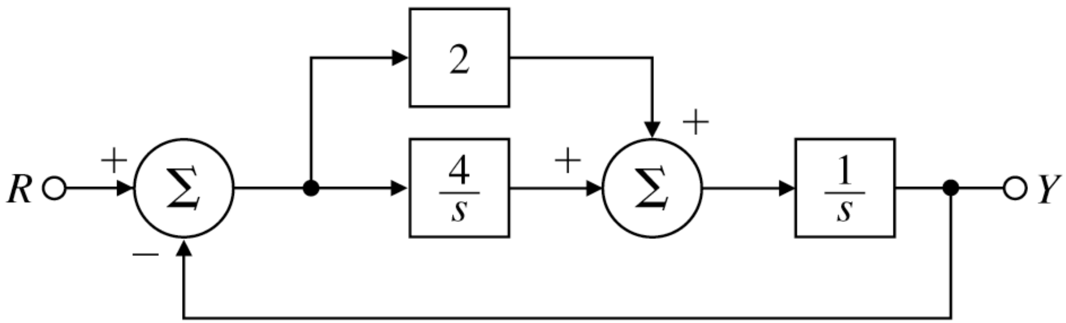
Equivalence of block diagram

- Transformation of block diagram may make the topology simple

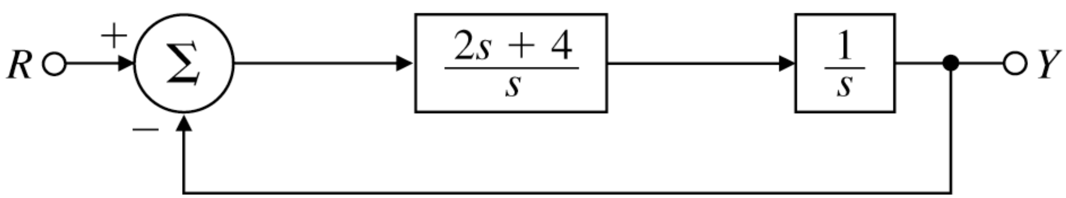


Example- computation of transfer function

- Transfer function from a simple block diagram



(a)

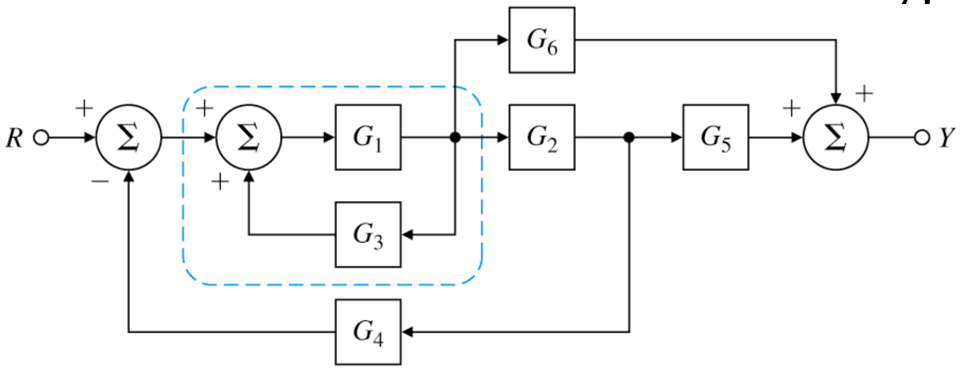


(b)

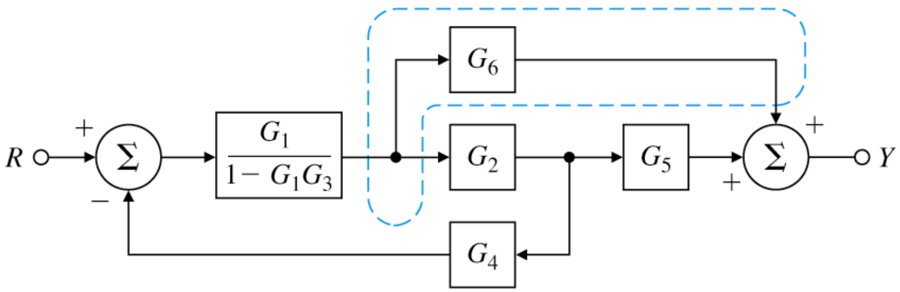
$$T(s)=\frac{Y(s)}{R(s)}=\frac{\frac{1}{s}\frac{2s+4}{s}}{1+\frac{1}{s}\frac{2s+4}{s}}=\frac{\frac{2s+4}{s^2}}{1+\frac{2s+4}{s^2}}=\frac{2s+4}{s^2+2s+4}$$

Example- computation of transfer function

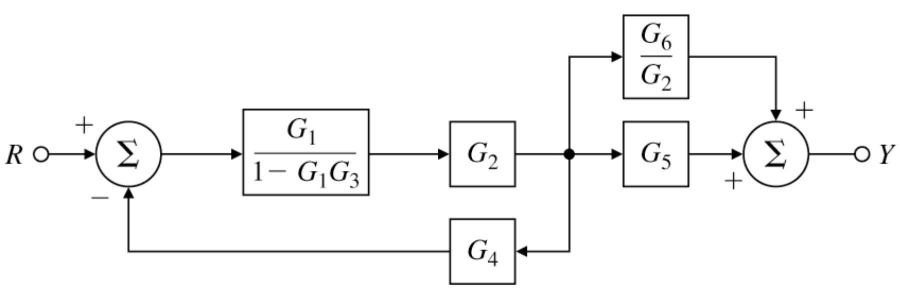
- Transfer function from a block diagram



(a)



(b)

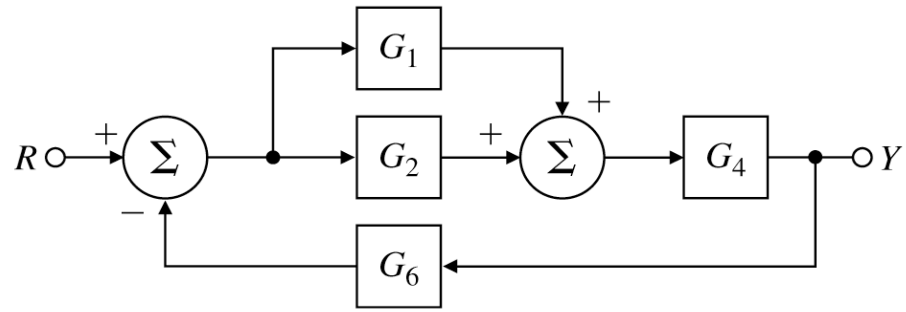


(c)

$$T(s)=\frac{Y(s)}{R(s)}=\frac{\dfrac{G_1G_2}{1-G_1G_3}}{1+\dfrac{G_1G_2G_4}{1-G_1G_3}}\left(G_5+\frac{G_6}{G_2}\right)$$

$$=\frac{G_1G_2G_5+G_1G_6}{1-G_1G_3+G_1G_2G_4}$$

Computation of transfer function using MATLAB

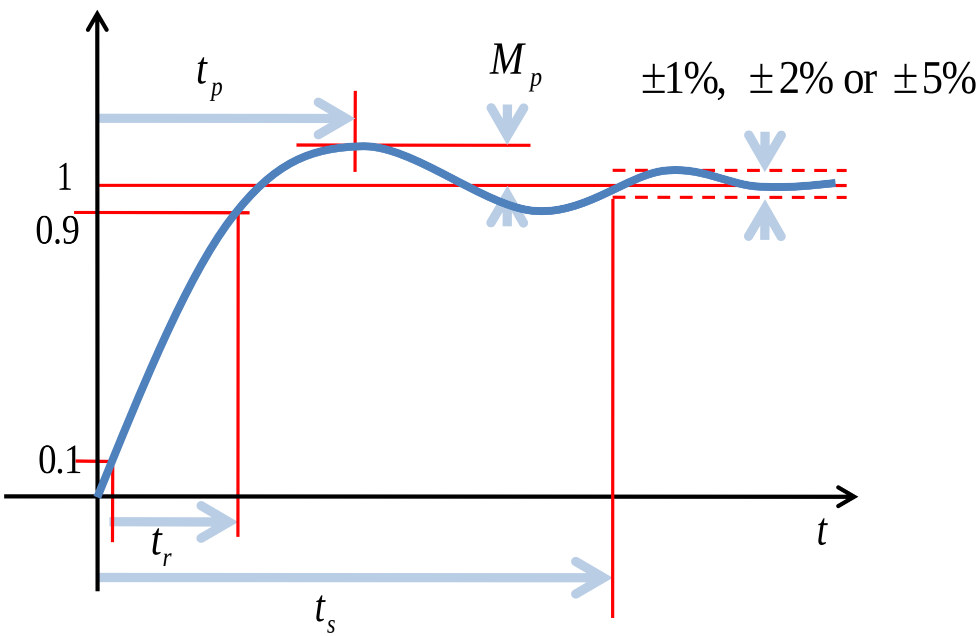



```
num1 = [2];
den1= [1];
sysG1 = tf(num1,den1);
num2 = [4]; den2 = [1 0];
sysG2 = tf(num2,den2);                     % define subsystem G2
% parallel combination of G1 and G2 to form subsystem G3
sysG3 =parallel(sysG1,sysG2);
%then we combine the result G3, with the G4 in series by
num4=[1];  den4=[1 0];
sysG4=tf(num4,den4);                       % form G4
sysG5=series(sysG3,sysG4);                 % series combination of G3 and G4
%complete the reduction of the system
num6 = [1];                                % form G6
den6= [1];
sysG6=tf(num6, den6);                      % define subsystem G6
[sysCL] = feedback(sysG5,sysG6,-1)         % feedback
```

4. Time-Domain Specifications

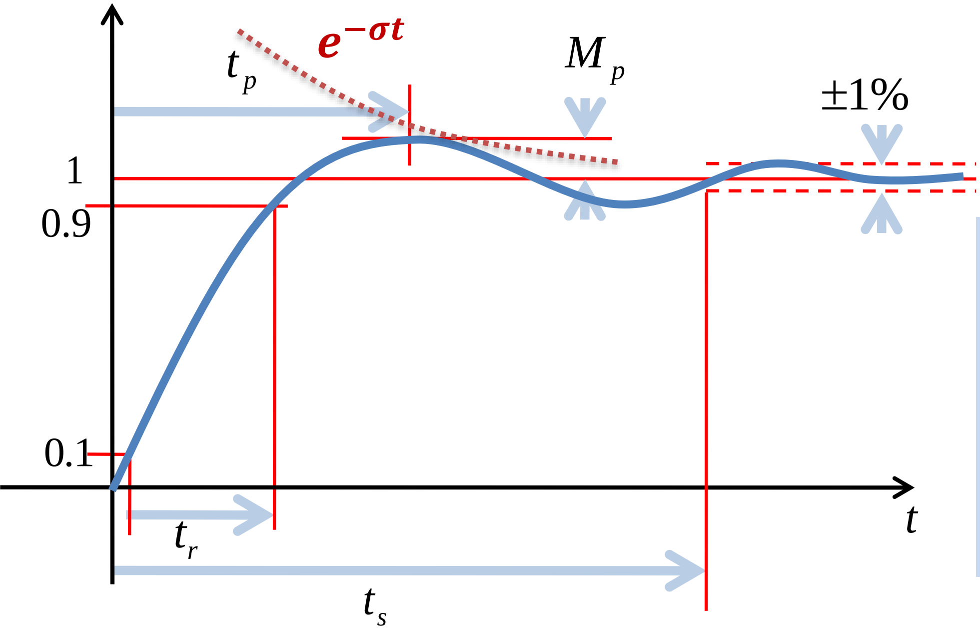
Introduction to time-domain specifications

- **Rise time** t_r: the time to reach the vicinity of its target point.
- **Settling time** t_s: the time the transients to decay.
- **Overshoot** M_p: the maximum amount the system overshoots its final value divided by its final value.
- **Peak time** t_p: the time to reach the maximum overshoot point.



Characteristics of second-order systems

- Time-domain specifications for **second-order systems**



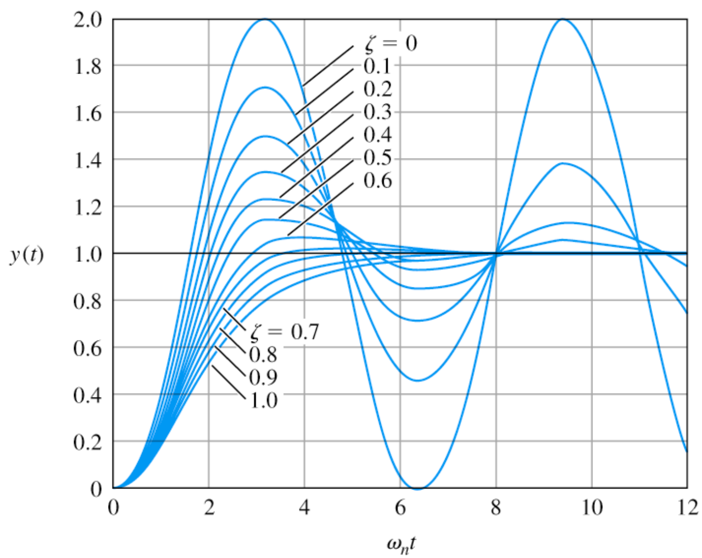
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Rise time: $t_r \cong 1.8/\omega_n$.

Settling time: $t_s = 4.6/\sigma$.

Overshoot: $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}, 0 \le \zeta < 1$.

Peak time: $t_p = \pi/\omega_d$.



$$y(t) = L^{-1}\left\{\frac{1}{s}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\}$$

$$= 1 - e^{-\sigma t}(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t).$$

$$\sigma = \zeta\omega_n, \omega_d = \omega_n\sqrt{1-\zeta^2}.$$

Time-domain specifications for second-order systems

Rise Time : All the curves rises in roughly the same time, and the rise time from y=0.1 to y=0.9 is $\omega_n t_r = 1.8$ for $\zeta = 0.5$.

$$\Rightarrow t_r \cong \frac{1.8}{\omega_n} \quad \text{(for 2nd-order systems with no zeros)}$$

Overshoot and Peak Time

- Step response: $H(s)\dfrac{1}{s}$

$$\xrightarrow{L^{-1}} y(t) = 1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)$$

$$= 1 - e^{-\sigma t}\sqrt{1 + \frac{\sigma^2}{\omega_d^2}}\cos(\omega_d t - \beta) \qquad \left(\beta = \tan^{-1}\left(\frac{\sigma}{\omega_d}\right)\right)$$

$$= 1 - e^{-\sigma t}\frac{1}{\sqrt{1-\varsigma^2}}\cos(\omega_d t - \beta)$$

$$\dot{y}(t) = \sigma e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right) - e^{-\sigma t}\left(-\omega_d \sin\omega_d t + \sigma\cos\omega_d t\right)$$

$$= e^{-\sigma t}\left(\frac{\sigma^2}{\omega_d}\sin\omega_d t + \omega_d \sin\omega_d t\right) = e^{-\sigma t}\left(\frac{\sigma^2}{\omega_d} + \omega_d\right)\sin\omega_d t = 0$$

$$\Rightarrow \sin\omega_d t = 0 \Rightarrow \omega_d t_p = \pi \Rightarrow t_p = \frac{\pi}{\omega_d}$$

$$y(t_p) := 1 + M_p = 1 - e^{-\sigma\pi/\omega_d}\left(\cos\pi + \frac{\sigma}{\omega_d}\sin\pi\right)$$

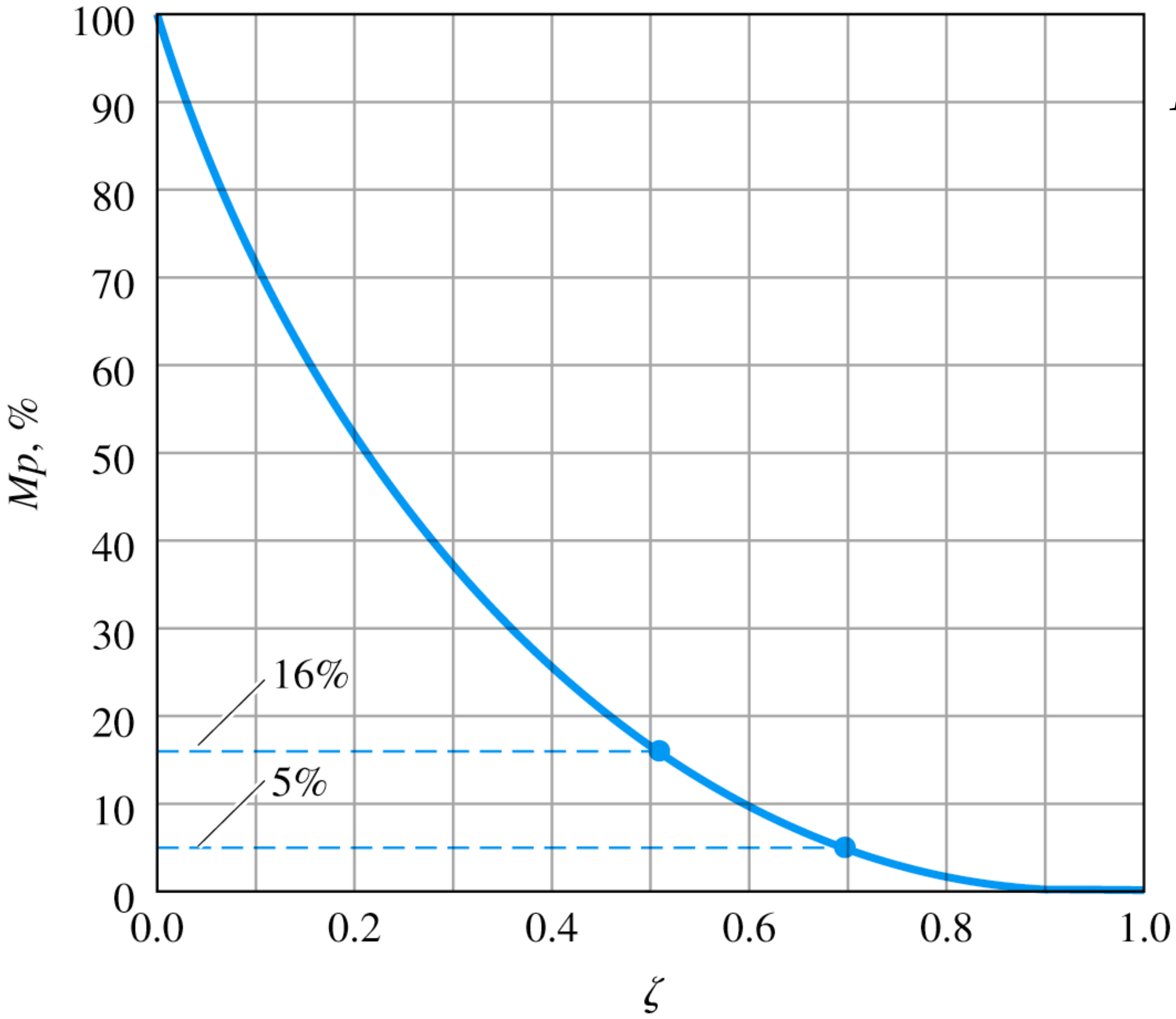
$$= 1 + e^{-\sigma\pi/\omega_d}$$

$$\Rightarrow M_p = e^{-\sigma\pi/\omega_d} = e^{-\pi\zeta/\sqrt{1-\zeta^2}}, \quad 0 \le \zeta < 1$$

$$\left(M_p = 0.16 \text{ for } \zeta = 0.5,\ M_p = 0.05 \text{ for } \zeta = 0.7\right)$$

Overshoot and damping ratio

- Overshoot versus damping ratio



$$M_p = e^{-\sigma\pi/\omega_d}$$
$$= e^{-\pi\zeta/\sqrt{1-\zeta^2}}, \quad 0 \le \zeta < 1$$

Time-domain specifications for second-order systems

Settling Time (1 %, not correct)

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right) = 1 - e^{-\zeta\omega_n t}\frac{1}{\sqrt{1-\varsigma^2}}\cos(\omega_d t - \beta)$$

$$e^{-\zeta\omega_n t_s} = 0.01, \quad \zeta\omega_n t_s = 4.6$$

$$t_s = \frac{4.6}{\zeta\omega_n} = \frac{4.6}{\sigma}$$

They are roughly the same

Settling Time (1 %, correct)

$$e^{-\zeta\omega_n t_s}\frac{1}{\sqrt{1-\varsigma^2}} = 0.01$$

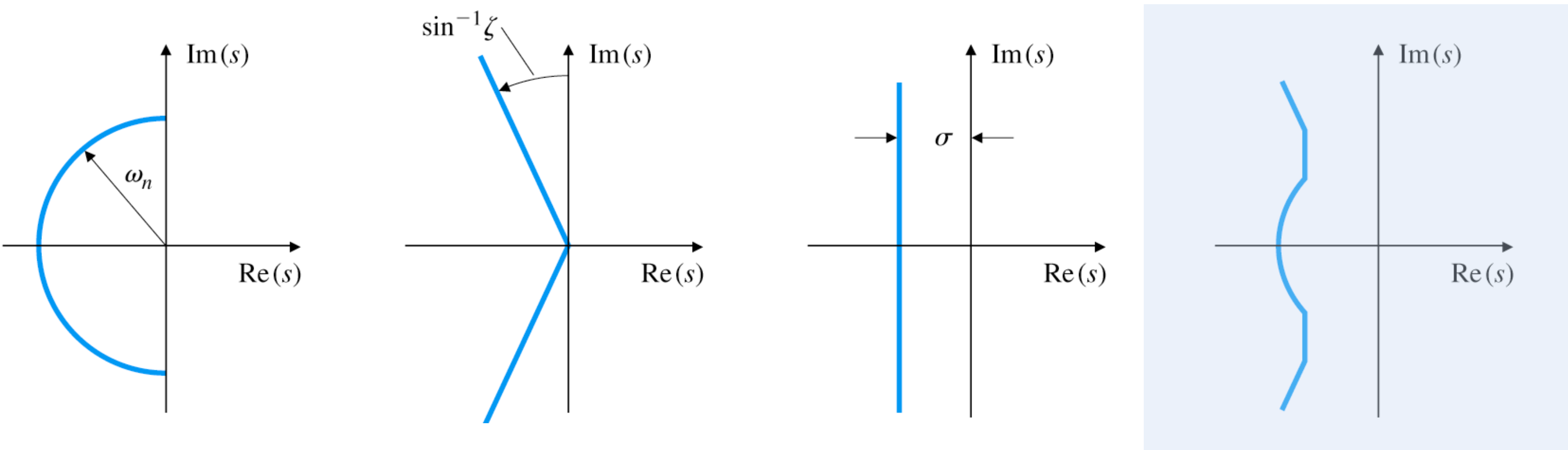
$$\zeta\omega_n t_s = -\ln\left(0.01\sqrt{1-\varsigma^2}\right)$$

$$t_s = \frac{-\ln\left(0.01\sqrt{1-\varsigma^2}\right)}{\zeta\omega_n} = \frac{-\ln\left(0.01\sqrt{1-\varsigma^2}\right)}{\sigma}$$

Summary of time-domain specifications

- Design synthesis (systems with no finite zeros and two complex poles): for specified values of t_r, M_p and t_s,

$$\omega_n \geq \frac{1.8}{t_r},\ \zeta \geq \zeta\left(M_p\right) \quad \left(\text{from Fig. 3.21}\right),\ \sigma \geq \frac{4.6}{t_s}$$

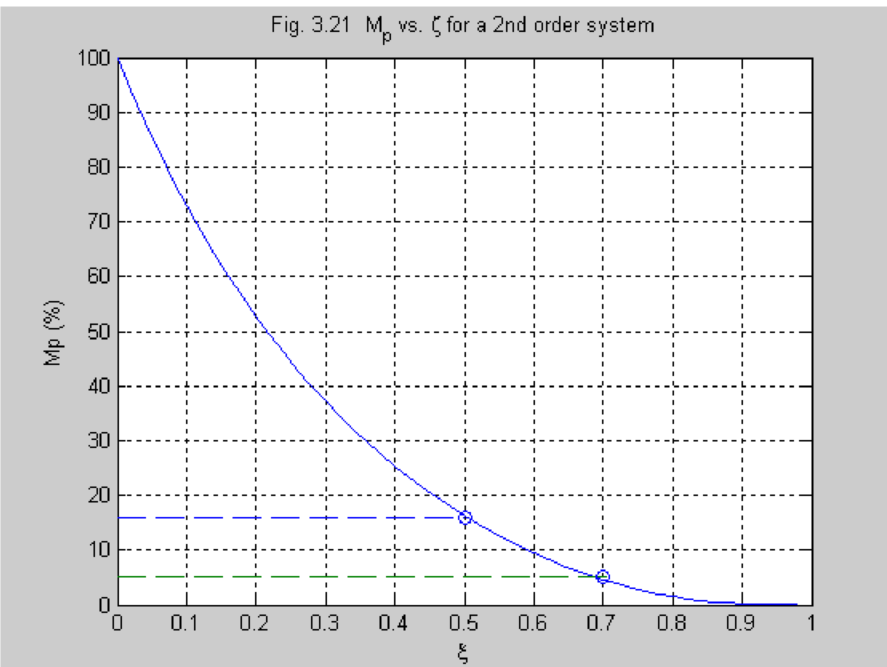


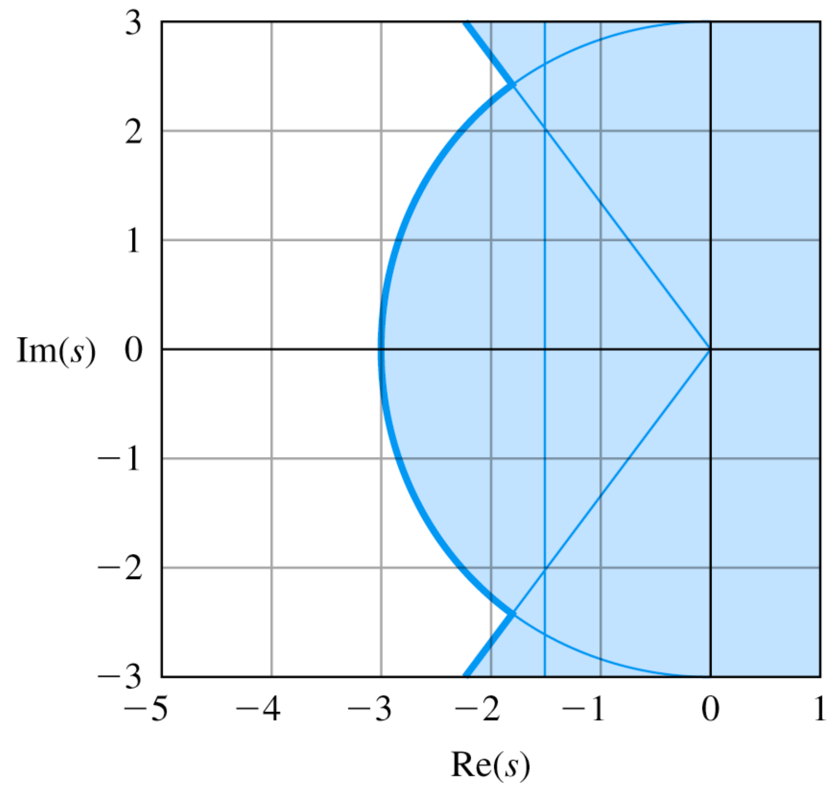
rise time | overshoot | settling time | composite

Example

System response requirements: $t_r \le 0.6$ sec, $\quad M_p \le 10$ %, $\quad t_s \le 3$ sec

$$\omega_n \ge \frac{1.8}{t_r} = 3.0 \text{ rad/sec},\ \zeta \ge 0.6 \ (\text{from Fig. 3.21}),\ \sigma \ge \frac{4.6}{t_s} = 1.5 \text{ sec}$$





5. Effects of Zeros and Additional Poles

Review of effects of poles and zeros

- Depends on the situation
  - In principle, poles determine the shape of basic functions (if we recall the partial fraction), and zeros determine the weighting of basic functions.

- For the simple second-order system:

$$\cdot \text{ Rise time too slow } \left(\omega_n \geq \frac{1.8}{t_r}\right) \rightarrow \text{ Raise the natural frequency.}$$

$$\cdot \text{ Too much overshoot } \left(\zeta \geq \zeta\left(M_p\right)\right) \rightarrow \text{ Increase damping.}$$

$$\cdot \text{ Transient too long } \left(\sigma \geq \frac{4.6}{t_s}\right) \rightarrow \text{ Move the poles to the left.}$$

$\rightarrow$ Used only as guidelines for more complicated systems.

Example – effect of zero near a pole

$$H(s) = \frac{2}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{2}{s+2}$$

$$H(s) = \frac{2(s+1.1)}{1.1(s+1)(s+2)} = \frac{0.18}{s+1} + \frac{1.64}{s+2}$$

$\rightarrow$ The component of the natural response corresponding to the pole near the zero is decreased.

Effects of zeros

- How zero affects the transient response?

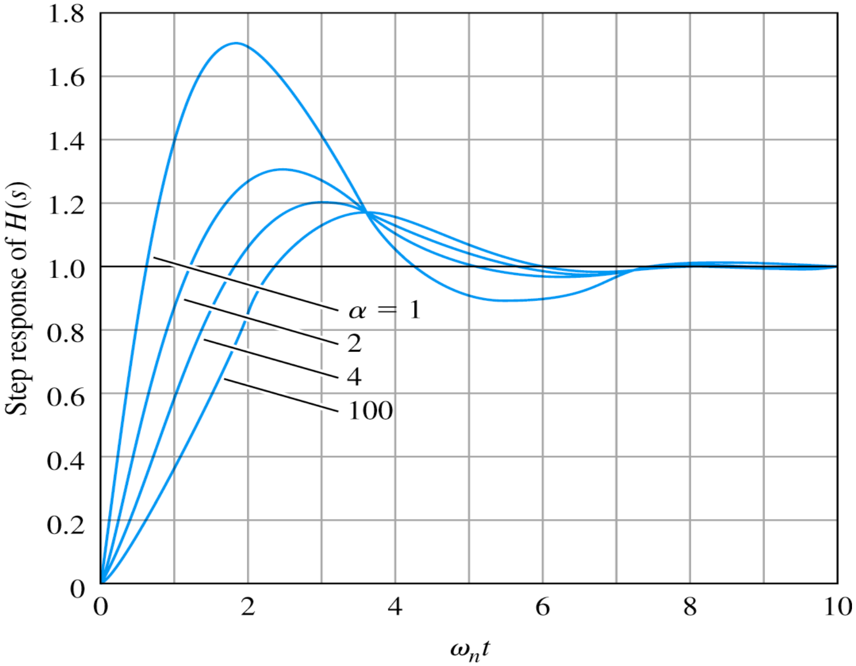
System with 2 poles and 1 zero: $H_1(s) = \dfrac{(s/\alpha\zeta\omega_n)+1}{(s/\omega_n)^2 + 2\zeta(s/\omega_n)+1}$

- zero at $-\alpha\zeta\omega_n = -\alpha\sigma$: $\begin{cases} \alpha >> 0 \text{ (little influence)} \\ \alpha \cong 1 \text{ (substantial influence)} \end{cases}$

$\rightarrow$ Increase in the overshoot M_p,

little influence on the settling time.

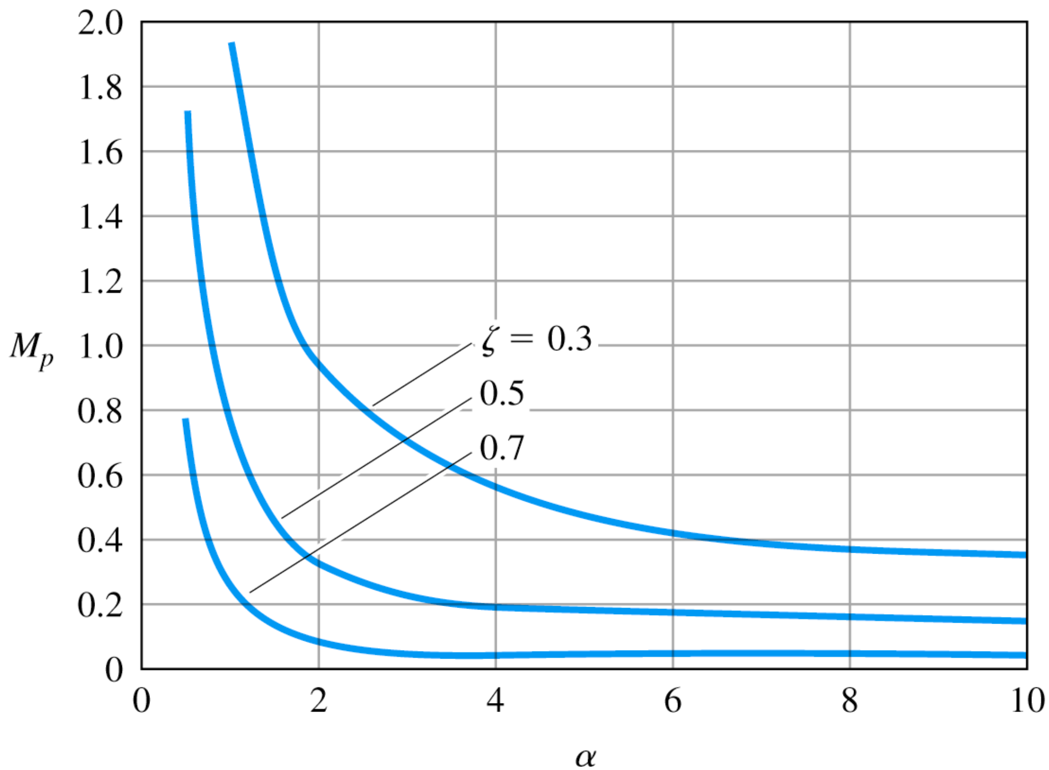
$\left(\begin{array}{l} \bullet \text{ For the system with 2 poles} \\ \text{and no zero with } \zeta = 0.5,\ M_p = 0.16. \end{array} \right)$



Effects of zeros

- How zero affect the transient response?

$$H(s)=\frac{s/\alpha\zeta+1}{s^2+2\zeta s+1}=\frac{1}{s^2+2\zeta s+1}+\frac{1}{\alpha\zeta}\frac{s}{s^2+2\zeta s+1}.$$



- The zero has little effect on the overshoot M_p if $\alpha > 3$.

Effects of zeros

- Step response:

$$Y(s)=\frac{s/\alpha\zeta+1}{s^2+2\zeta s+1}\frac{1}{s}=\frac{1}{s^2+2\zeta s+1}\frac{1}{s}+\frac{1}{\alpha\zeta}\frac{s}{s^2+2\zeta s+1}\frac{1}{s}$$

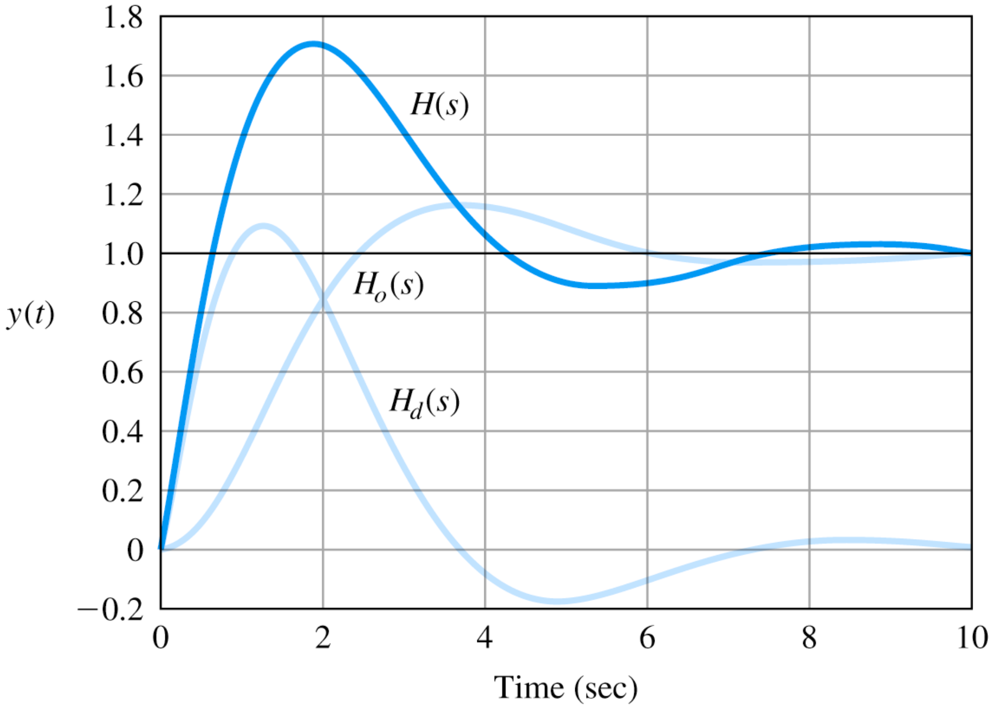
$$=Y_0(s)+Y_d(s)$$

$$y(t)=y_0(t)+y_d(t)$$

$$Y_d(s)=\frac{1}{\alpha\zeta}sY_0(s)$$

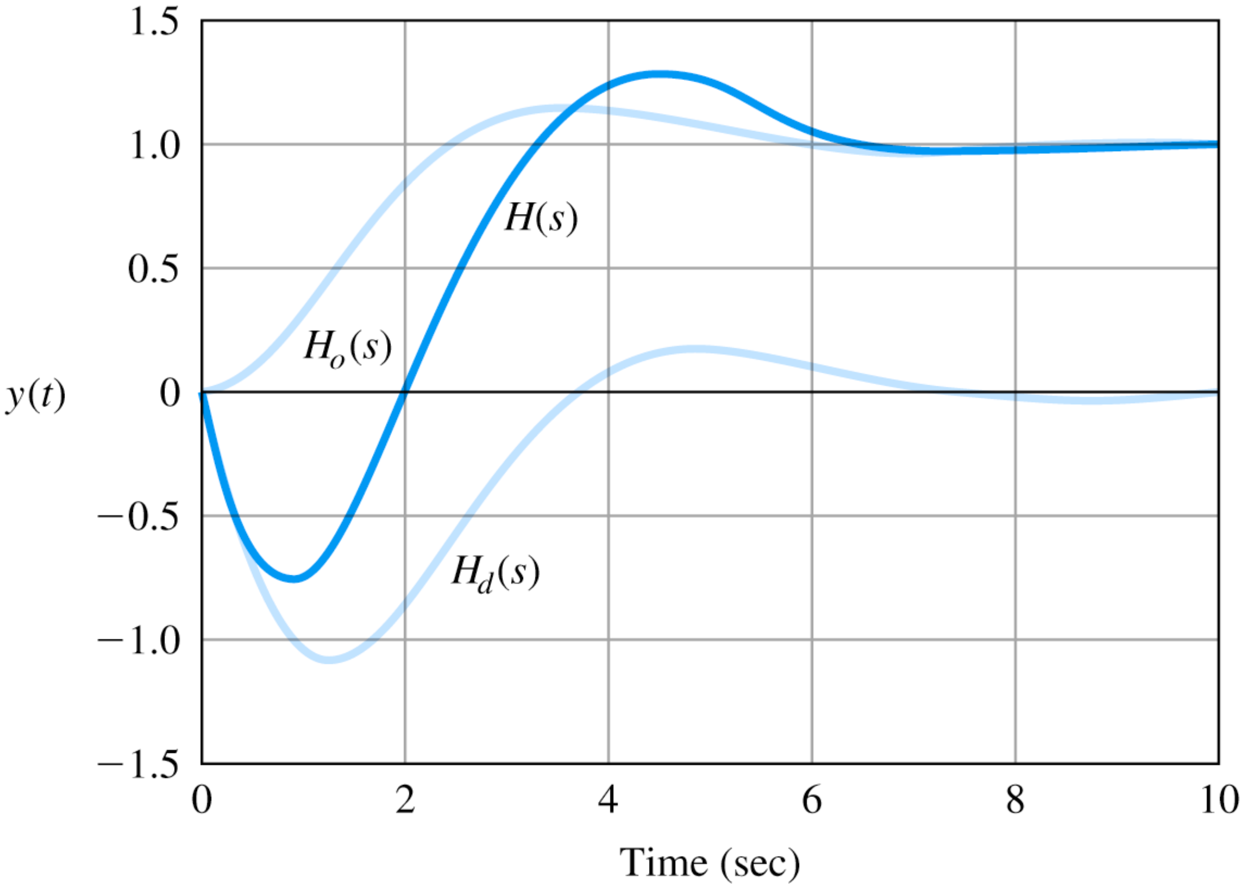
$$\xrightarrow{L^{-1}} y_d(t)=\frac{1}{\alpha\zeta}\frac{dy_0(t)}{dt}$$

$$\left(\alpha>>1\rightarrow h_d(t)\text{ is small}\right)$$



· $\alpha>0\rightarrow$ zero at $-\alpha\zeta$ is in the LHP (minimum-phase system)

$\cdot\ \alpha < 0 \rightarrow$ zero at $-\alpha\zeta$ is in the RHP (nonminimum-phase system)

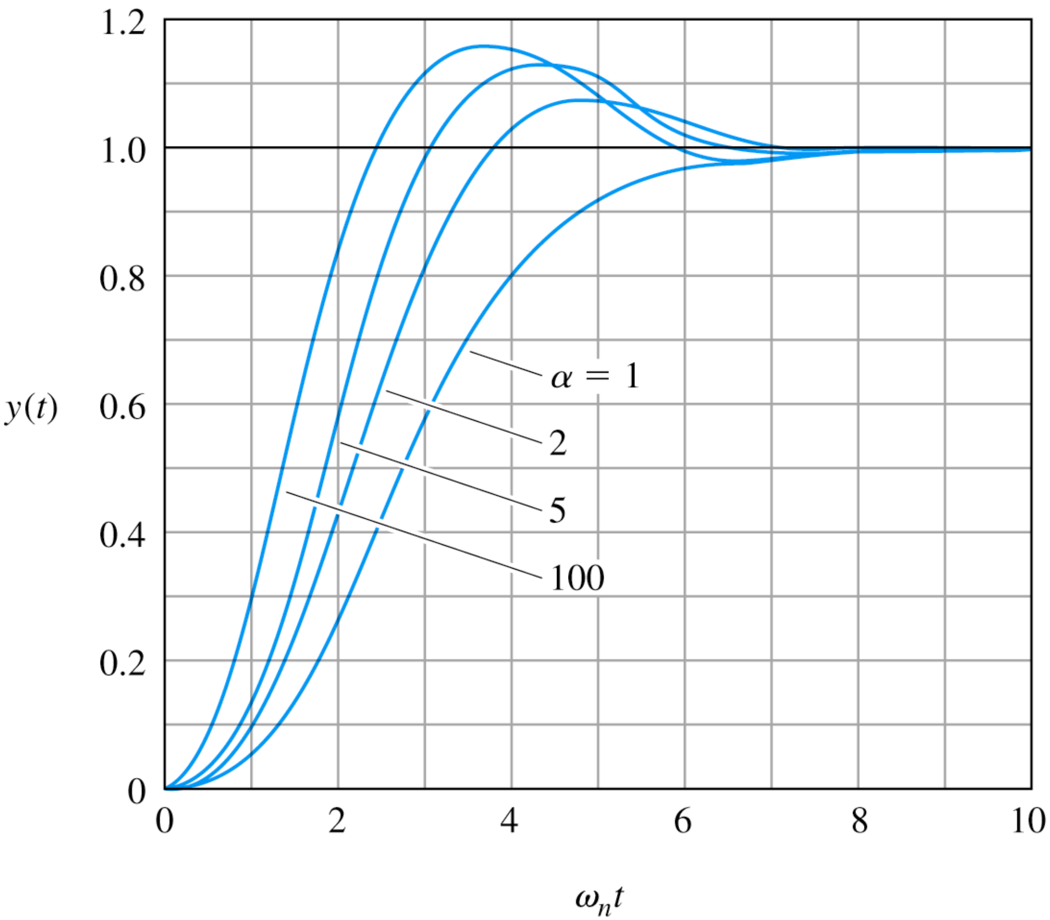


Effects of additional poles

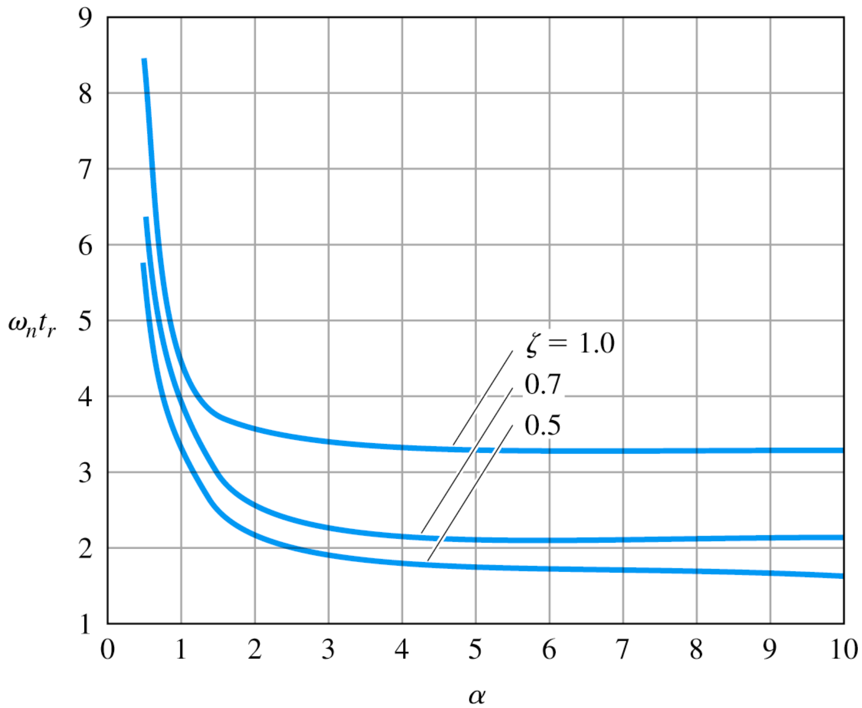
- Effect of extra pole

$$H(s) = \frac{1}{(s/\alpha\zeta\omega_n + 1)\left[(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1\right]}$$

$\rightarrow$ Increase in rise time



Normalized rise time



6. Stability

Stability

- One of the most important concepts in control engineering.
  - Quite difficult to identify in general.
  - Easy for linear time invariant systems.

A linear time-invariant system is said to be **stable** if all the roots of the transfer function denominator polynomial have negative real parts and **unstable** otherwise.

Roots of the transfer function denominator polynomial=pole
- stable if all the poles of the system are in OLHP.
- unstable if any pole of the system is in RHP, or CRHP.
- special case of unstable system: oscillatory system (called unstable because it is not stable).

Bounded input-bounded output stability

- A system is said to **bounded input-bounded output (BIBO) stable** if every bounded input results in a bounded output.

- Response by convolution:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau.$$

- if $u(t)$ is bounded, there is a constant M such that $|u| \le M < \infty.$

→ $$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau \right| \le \int_{-\infty}^{\infty} |h(\tau)||u(t-\tau)|d\tau \le M \int_{-\infty}^{\infty} |h(\tau)|d\tau.$$

→ The output is bounded if $\int_{-\infty}^{\infty} |h(\tau)|d\tau$ is bounded.

The system with impulse response $h(t)$ is BIBO-stable if and only if

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty.$$

Example of bounded input-bounded output stability

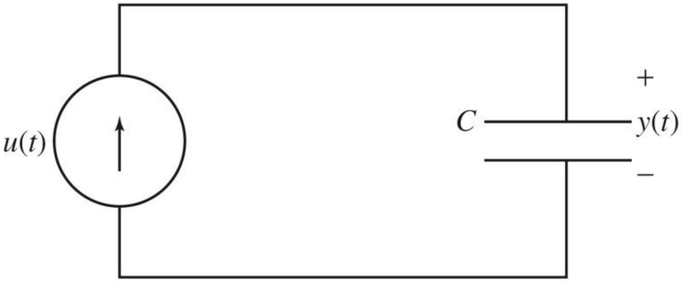
- Capacitor driven by a current source.

$$u(t) = C\frac{dy(t)}{dt},\ C = 1.$$

$$\rightarrow h(t) = 1(t)$$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{0}^{\infty} 1 d\tau$$

$\rightarrow$ Not bounded!!



- Transfer function $H(s) = \dfrac{1}{s}$: has a pole on the imaginary axis.

- In general, if an LTI system has any pole on the imaginary axis or in the RHP, the response will not be BIBO stable.

- If every pole is inside the LHP, then the response will be BIBO stable.

Stability of LTI systems

- Stability of a system with the transfer function
  - Characteristic equation: $s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n = 0.$
  - Assume that poles $\{p_i\}$are **distinct**:

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{K\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}, m \le n.$$

  - We allow $z_i = p_i$, for some i (before pole-zero cancellation).
  - The solution: $y(t) = \sum_{i=1}^{n} K_i e^{p_i t}$. K_i depends on the initial conditions and zeros.
  - The system is stable if and only if $e^{p_i t} \to 0, \forall p_i$.
  - Equivalently, the system is stable if and only if $\mathrm{Re}\{p_i\} < 0, \forall i$.
  - If any poles are repeated, $y(t) = \sum_{i=1}^{n} K_i(t) e^{p_i t}$, $K_i(t)$: polynomial of t.
  - For any case, the system is stable if and only if $\mathrm{Re}\{p_i\} < 0, \forall i$.
  - Neutrally stable if the system has non-repeated poles on the imaginary axis.

Stability analysis for LTI systems

- Stability and the characteristic equation: Is it possible to determine the stability of a system without obtaining the poles? YES.

- Characteristic equation: $a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n.$

(Necessary condition) If an LTI system is stable, then **all the coefficients of the characteristic polynomial are positive**.

$$
\begin{aligned}
a_1 &= -\sum \text{all roots} \\
a_2 &= +\sum \text{product of roots taken 2 at a time} \\
a_3 &= -\sum \text{product of roots taken 3 at a time} \\
&\ \ \vdots \\
a_n &= (-1)^n \text{product of all roots}
\end{aligned}
$$

Necessary and sufficient condition for stability

- Stability and the characteristic equation.

  - Characteristic equation: $a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n.$

- Routh's Stability Criterion

> (Necessary and sufficient condition) An LTI system is stable if and only if **all the elements in the first column of the Routh array are positive**.

Construction of Routh array

- Routh array:

Row	n	$s^n:$	1	a_2	a_4	$\dots$
Row	$n-1$	$s^{n-1}:$	a_1	a_3	a_5	$\dots$
Row	$n-2$	$s^{n-2}:$	b_1	b_2	b_3	$\dots$
Row	$n-3$	$s^{n-3}:$	c_1	c_2	c_3	$\dots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
Row	2	$s^2:$	$*$	$*$		
Row	1	$s:$	$*$			
Row	0	$s^0:$	$*$			

$$b_1 = -\frac{\det\begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1} = \frac{a_1a_2 - a_3}{a_1},$$

$$b_2 = -\frac{\det\begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1} = \frac{a_1a_4 - a_5}{a_1},$$

$$b_3 = -\frac{\det\begin{bmatrix} 1 & a_6 \\ a_1 & a_7 \end{bmatrix}}{a_1} = \frac{a_1a_6 - a_7}{a_1},$$

$$c_1 = -\frac{\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1} = \frac{b_1a_3 - a_1b_2}{b_1},$$

$$c_2 = -\frac{\det\begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1} = \frac{b_1a_5 - a_1b_3}{b_1},$$

$$c_3 = -\frac{\det\begin{bmatrix} a_1 & a_7 \\ b_1 & b_4 \end{bmatrix}}{b_1} = \frac{b_1a_7 - a_1b_4}{b_1}.$$

- The number of roots in the RHP equals the number of sign changes in the first column in the Routh array.

Example of Routh array

- Example : $a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$

s^6	1	3	1	4
s^5	4	2	4	0
s^4	2.5	0	4	0
s^3	2	-2.4	0	
s^2	3	4	0	
s^1	-76/15	0		
s^0	4			

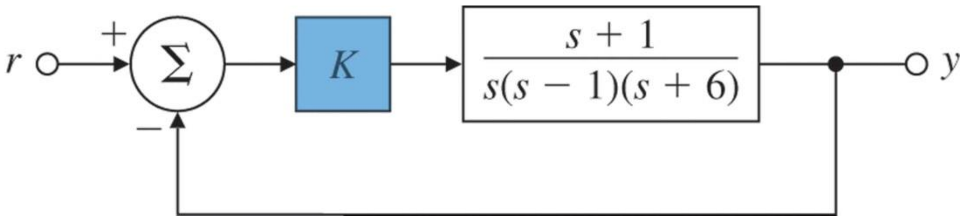
$\rightarrow$ 2 poles in the RHP

- The coefficients of any row may be multiplied or divided by a positive number without changing the signs of the first column.

Example of stability – one degree of freedom

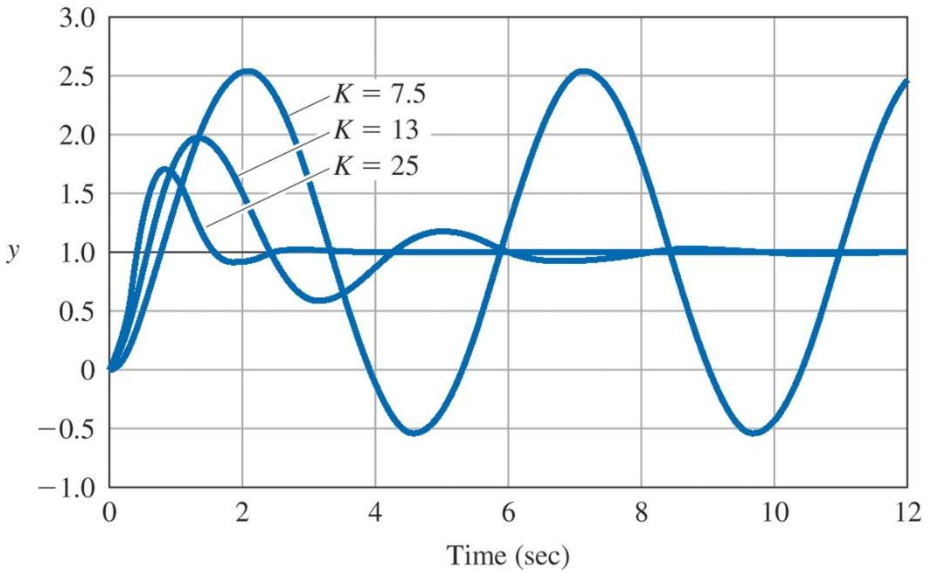
- Stability versus one parameter range.

Determine the range of K over which the system is stable.



$$s^3 + 5s^2 + (K-6)s + K = 0$$

s^3	1	$K-6$
s^2	5	K
s^1	$\dfrac{4K-30}{5}$	
s^0	K	



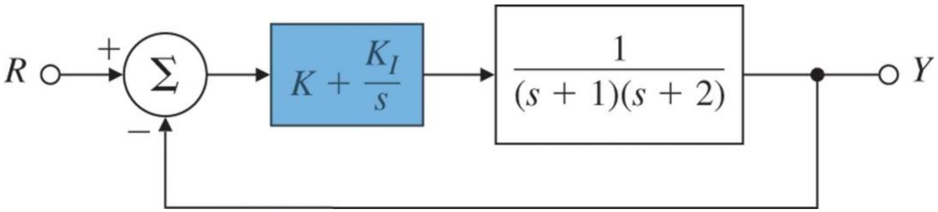
$$\frac{4K-30}{5} > 0 \quad \text{and} \quad K > 0$$

$K > 7.5$ and $K > 0$.

Example of stability – two degree of freedom

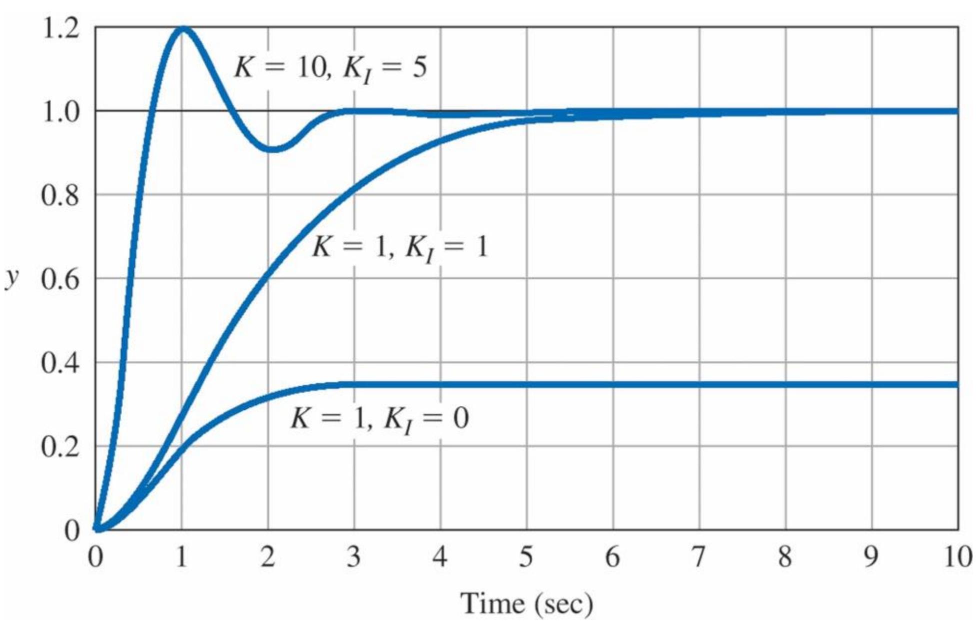
- Stability versus two parameter ranges.

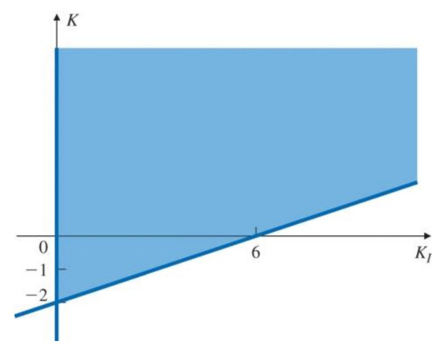
Determine the range of (K, K_I) over which the system is stable.



$$s^3 + 3s^2 + (2+K)s + K_I = 0$$

$$\begin{array}{lll} s^3 & 1 & 2+K \\ s^2 & 3 & K_I \\ s^1 & \frac{6+3K-K_I}{3} & \\ s^0 & K_I & \end{array}$$





$$K_I > 0 \text{ and } K > \frac{1}{3}K_I - 2.$$