EECE322-01: 자동제어공학개론

Introduction to Automatic Control

Chapter 5: Root Locus Analysis and Design

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◆ The main objectives of this chapter are

1. Basics for Root Locus
2. Rules for Root Locus
3. Dynamic Compensation

1. Basics for Root Locus

Motivations

- Poles of feedback systems are closely related to the responses (ex. impulse response, step response, and so on) of the feedback systems

→ Relation between one of the control parameters and the poles of the feedback system (i.e., the roots of the characteristic equation).

→ Relation between one of the control parameters and the system's dynamic response.

Basic concepts for root locus

- Closed-loop transfer function: $\dfrac{Y(s)}{R(s)} = \mathrm{T}(s) = \dfrac{D(s)G(s)}{1+D(s)G(s)H(s)}$

- Chacteristic equation: $1+D(s)G(s)H(s)=0$

$$\to a(s)+Kb(s)=0 \qquad (K := \text{parameter of interest})$$

$$\to 1+KL(s)=0 \;\; \left(L(s) := b(s)/a(s)\right) \; [*]$$

- Root locus: Plot of the locus of all possible roots of [*] as K varies from 0 to ∞ (from 0 to $-\infty$) [Root-locus method of Evans]

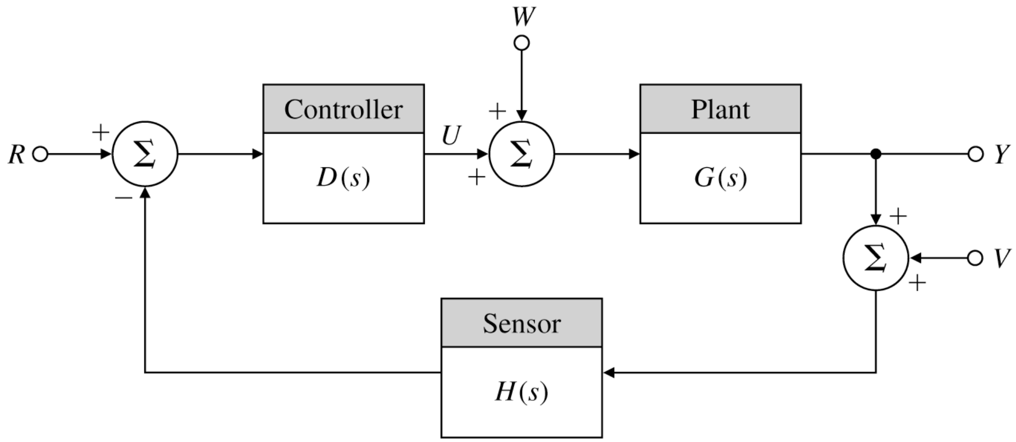


Figure 5.1 Basic closed loop block diagram

Basic concepts for root locus

- Recall: Characteristic equation

→ the roots (= the poles of the transfer function) of the characteristic equation determine the basic behavior of the system
(example: $L^{-1}\{\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}\} = e^{-t} - e^{-2t}$)

In many cases, the characteristic equation can be written as

$$1 + KL(s) = 0, \quad L(s) = \frac{b(s)}{a(s)}.$$

What happens to the closed loop system if we change the tuning variable K?
= Where will be the closed-loop system poles if we change K?

Basic concepts for root locus

Root locus: plot of the locus of all possible roots of $1+KL(s)=0$ as K changes

$$1+KL(s)=0, \quad L(s)=\frac{b(s)}{a(s)}$$

- Root locus method is useful when

- We want to select a constant K of the controller to have some nice closed-loop system.
- We want to investigate the effect of introducing some 'dynamic' controller.

- Typical case: L(s) is the open loop transfer function.
  → In this case one can determine the location of the closed loop system poles in terms of the open loop system poles and zeros.

Basic concepts for root locus

- Notation: $1+KL(s)=0, \quad L(s)=\dfrac{b(s)}{a(s)}$

$$b(s)=s^m+b_1s^{m-1}+\cdots+b_m=(s-z_1)(s-z_2)\cdots(s-z_m)=\prod_{i=1}^{m}(s-z_i),$$

$$a(s)=s^n+a_1s^{n-1}+\cdots+a_n=\prod_{i=1}^{n}(s-p_i). \qquad (n\ge m)$$

$$a(s)+Kb(s)=(s-r_1)(s-r_2)\cdots(s-r_n) \quad (r_i=r_i(K):\ \text{closed-loop pole})$$

- Characteristic equation: $1+KL(s)=0 \quad L(s)=\dfrac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}=\dfrac{b(s)}{a(s)}$

- Root locus forms: $1+KL(s)=0, \quad 1+K\dfrac{b(s)}{a(s)}=0$

$$a(s)+Kb(s)=0$$

$$L(s)=\frac{b(s)}{a(s)}=-\frac{1}{K}$$

Example of root locus

- Root Locus of a Motor Position Control

Find the root locus with respect to $A = K. \left(1 + DGH = 1 + \dfrac{A}{s(s+1)} = \dfrac{s^2 + s + A}{s(s+1)} \right)$

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{Y(s)}{U(s)} = G(s) = \frac{A}{s(s+c)},\ c = 1,\ D(s) = H(s) = 1 \text{ in Fig. 5.1}$$

Sol.) Closed-loop characteristic equation:

$$a(s) + Kb(s) = s^2 + s + A = (s^2 + s) + K \cdot 1 = 0$$

$$b(s) = 1, \quad m = 0, \quad z_i = \{\text{empty}\},$$

$$K = A,$$

$$a(s) = s^2 + s, \quad n = 2, \quad p_i = 0, -1$$

$$\rightarrow L(s) = \frac{1}{s(s+1)},$$

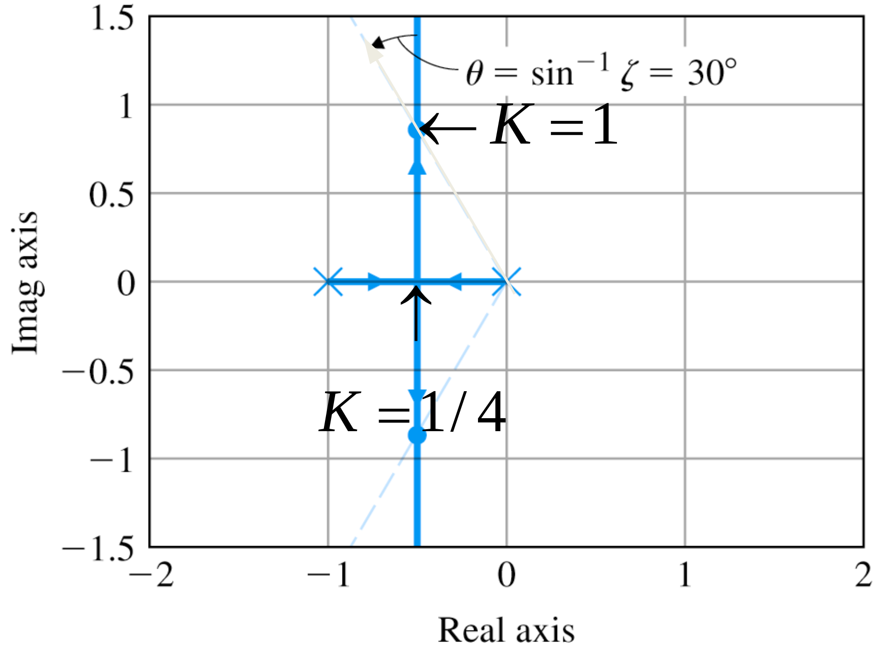


Figure 5.2 Root locus for $L(s) = 1/[s(s+1)]$

$$r_1, r_2 = -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2} \Rightarrow \begin{cases} r_1, r_2 = -1, 0 \to -\frac{1}{2}, \ \ 0 \le K \le \frac{1}{4} \\ r_1, r_2 = -\frac{1}{2}, \ \ K = \frac{1}{4} \\ r_1, r_2 = -\frac{1}{2} \pm j\,\frac{\sqrt{4K-1}}{2}, \ \ K > \frac{1}{4} \end{cases}$$

$$\varsigma = 0.5 \to \theta = 30^\circ \to \frac{\sqrt{4K-1}}{2} = \frac{\sqrt{3}}{2} \to K = 1$$

- Features

- 2 roots and 2 branches of root locus $\left(n = 2\right)$

- At $K = 0$, these branches begin at the poles of $L(s)$.

- Breakaway points: Points where roots move away from the real axis

- The roots move off to infinity. $\left(m = 0\right)$

Example of root locus

- Root Locus w.r.t. a Plant Open-Loop Pole

In Ex. 5.1, $A = D(s) = H(s) = 1$. Find the root locus with respect to $c = K$.

$$\left(1+G(s)=1+\frac{1}{s(s+c)}=\frac{s^2+cs+1}{s(s+c)}\right)$$

Closed-loop characteristic equation: $s^2+cs+1=0$ $\left(1+c\dfrac{s}{s^2+1}=0\right)$

$$L(s)=\frac{s}{s^2+1},\ b(s)=s,\ m=1,\ z_i=0,$$

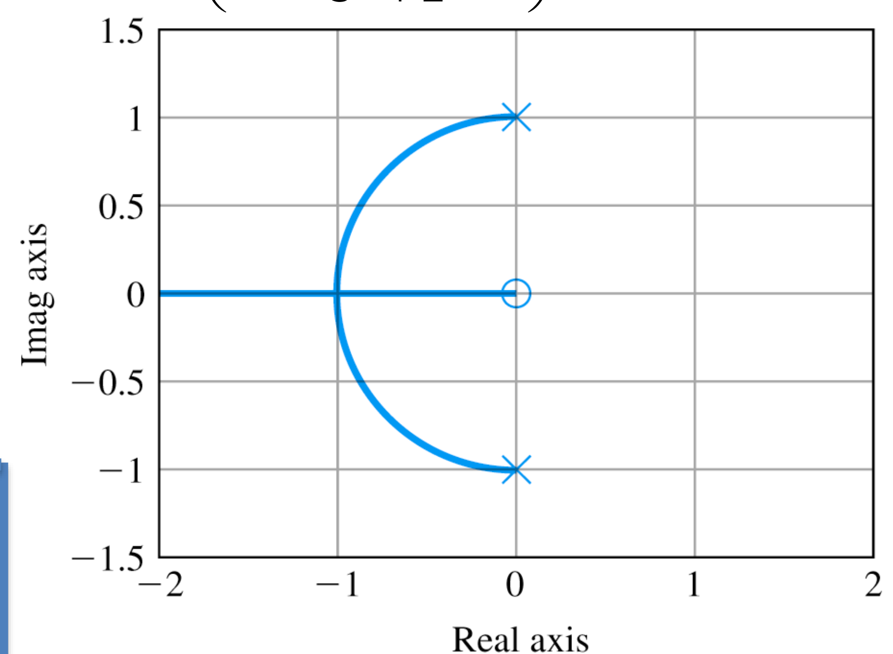
$$K=c,\ a(s)=s^2+1,\ n=2,\ p_i=\pm j$$

$$1+c\frac{s}{s^2+1}=0 \Rightarrow r_1,r_2=-\frac{c}{2}\pm\frac{\sqrt{c^2-4}}{2}$$

$$s^2+1+C\cdot s=0$$

$$a(s)=s^2+1, b(s)=s, K=c$$

$$L(s)=\frac{b(s)}{a(s)} \Rightarrow 1+KL(s)=0 \Rightarrow L(s)=-\frac{1}{K}$$

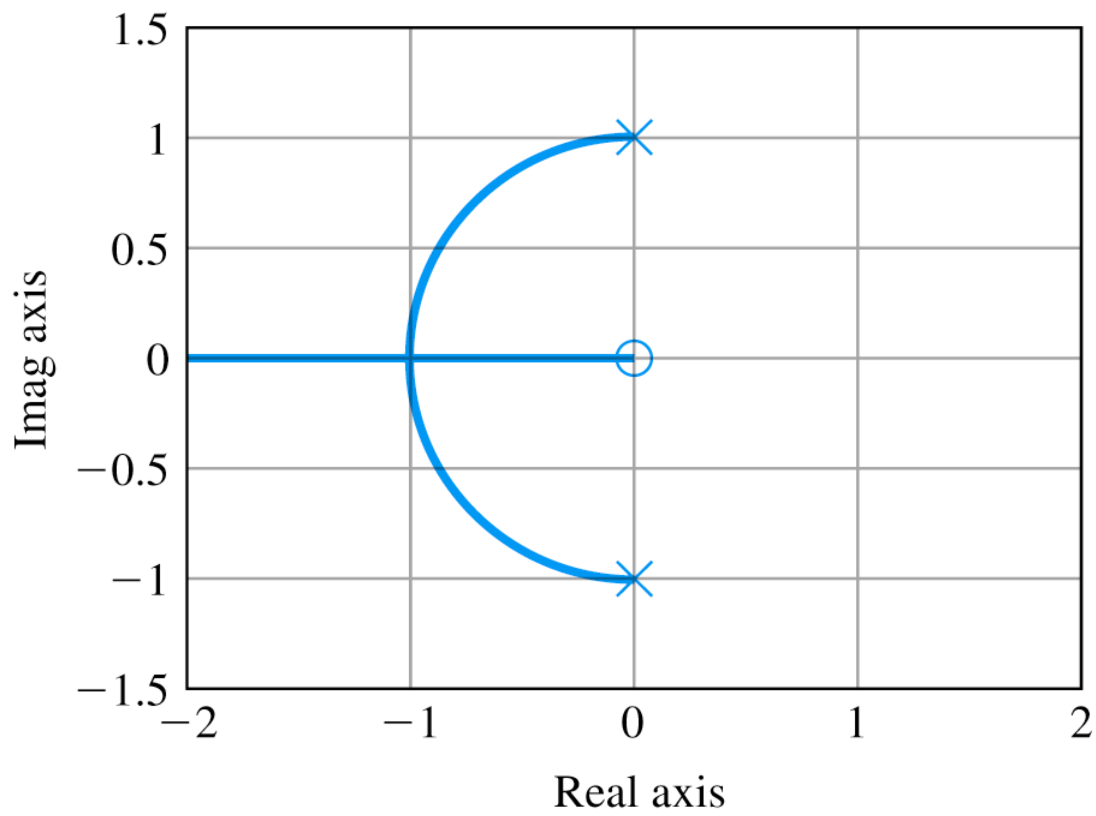


- 2 branches start from the poles of $L(s)$. $(n=2)$
- 1 root moves to a zero of $L(s)$. $(m=1)$
- Break-in point: the point where two or more roots come intor the real axis

$$s^2+cs+1=0 \quad \left(1+c\frac{s}{s^2+1}=0\right)$$

$$L(s)=\frac{s}{s^2+1},$$

$$b(s)=s, K=c,\ a(s)=s^2+1$$



2. Rules for root locus

Mathematical definitions of root locus

Definition: The root locus is the set of values of s for which $1+KL(s)=0$ is satisfied as the real parameter K varies from 0 to ∞ (from 0 to $-\infty$).

Definition: (Phase condition: $L(s) = -1/K,\ K \geq 0$)

The root locus of $L(s)$ is the set of points in the s-plane where the phase of $L(s)$ is $180°$.

$$\sum \psi_i - \sum \phi_i = 180° + 360°(l-1)$$

$\psi_i :=$ angle to the test point from the ith zero

$\phi_i :=$ angle to the test point from the ith pole

$$L(s) = -\frac{1}{K}$$

$$\Rightarrow |L(s)| = \frac{1}{K}$$

$$\Rightarrow \angle L(s) = 180°$$

$$\left(L(s) = \frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)} = -\frac{1}{K} \rightarrow \angle L(s) = \sum\left[\angle(s-z_i)\right] - \sum\left[\angle(s-p_i)\right] \right.$$

$$\left. = \sum \psi_i - \sum \phi_i = 180° + 360°(l-1)\ \left[0° + 360°(l-1) \text{ for } K<0\right] \right)$$

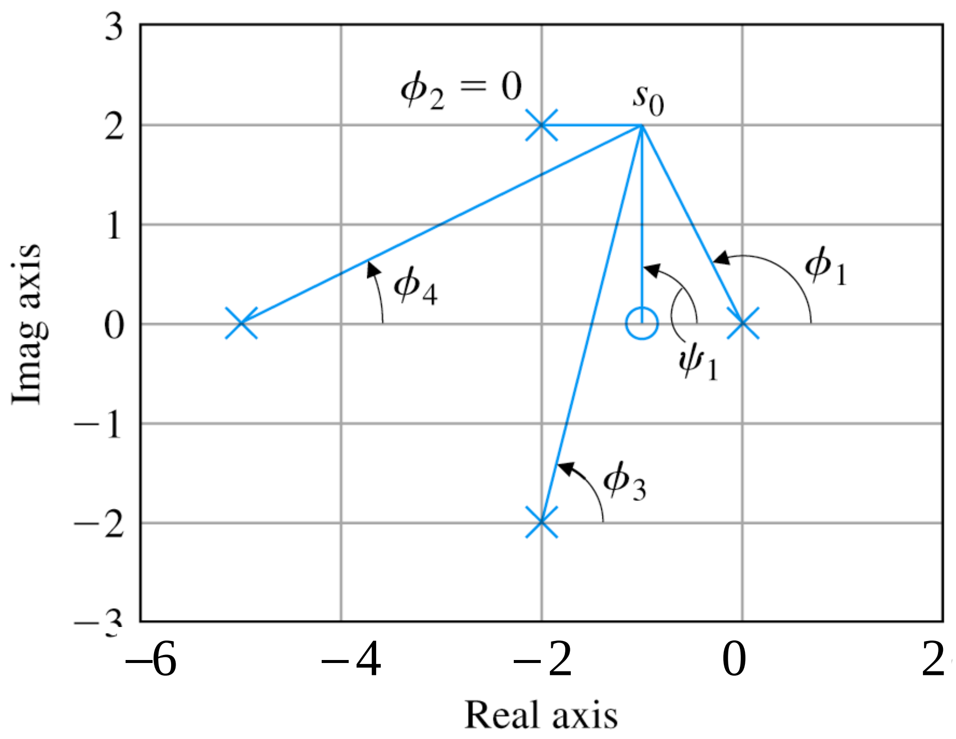
- Positive or 180° locus for $K \geq 0$, Negative or 0° locus for $K \leq 0$

Example

$$L(s)=\frac{s+1}{s(s+5)[(s+2)^2+4]} \qquad (1+L(s)=0)$$

$$\angle L(s_0)=\angle(s_0+1)-\angle s_0-\angle(s_0+5)-\angle\left[(s_0+2)^2+4\right]=180^\circ+360^\circ(l-1)$$

Test point: $s_0=-1+j2$



$$\begin{aligned}\angle L(s_0)&=\psi_1-\phi_1-\phi_2-\phi_3-\phi_4\\&=90^\circ-116.6^\circ-0^\circ-76^\circ-26.6^\circ\\&=-129.2^\circ\neq 180^\circ+360^\circ(l-1)\end{aligned}$$

$\rightarrow s_0=-1+2j$ is not on the root locus.

Rules for plotting a root locus

RULE 1: The n branches of the locus start at the poles of $L(s)$ and m of these branches end on the zeros of $L(s)$.

$$a(s) + Kb(s) = 0$$

$$\left(\begin{array}{l} K = 0 \Rightarrow a(s) = 0,\ \ L(s) = \dfrac{b(s)}{a(s)} \\ \\ K \to \infty \Rightarrow \dfrac{b(s)}{a(s)} = -\dfrac{1}{K} \to 0 \Rightarrow b(s) = 0,\ \text{or } s \to \infty\ \ (m < n) \end{array} \right)$$

Complement of Rule 1

- Zeros at infinity:

$$L(s)=\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}=-\frac{1}{K} \quad (m \le n)$$

- Finites zeros: $L(z_i)=0 \;\; i=1,2,\cdots,m$ (m finite zeros)

- Zeros at infinity ($m<n$):

$$\text{For } |s| >> 1, \;\; L(s)=\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)} \cong \frac{1}{s^{n\text{-}m}} \rightarrow 0 \text{ as } s \rightarrow \infty$$

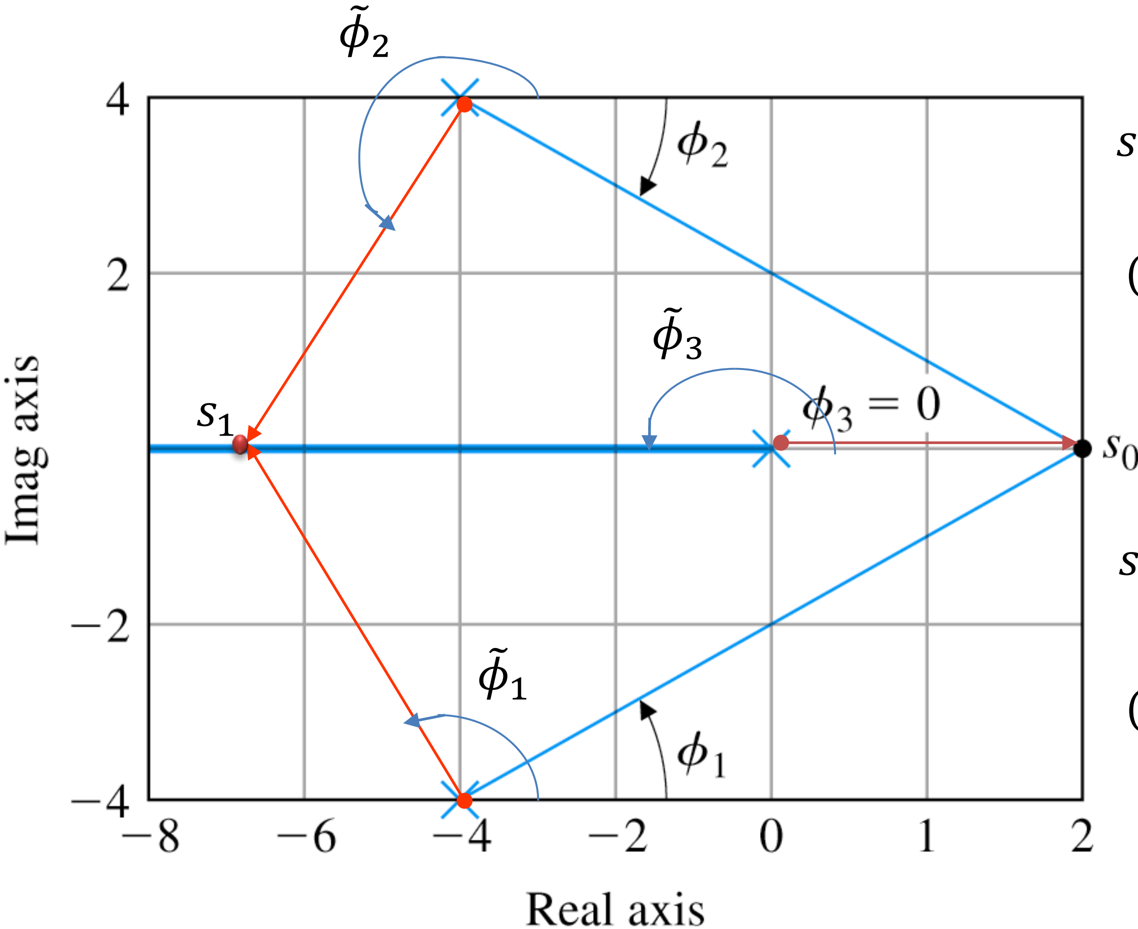
($n-m$ zeros at infinity)

- Number of finite zeros + Number of zeros at infinity $= n$

e.x. $$\frac{s+3}{s^5+2s^3+3s^2+s+1} \rightarrow \frac{1}{s^4} \;\; (s \rightarrow \infty)$$

Rules for plotting a root locus

RULE 2: The loci are on the real axis to the left of an odd number of poles and zeros.



$s_0: 0° - (0° + 360° - \phi_2 + \phi_1)$
$= -360°$
$(\because \phi_2 = \phi_1)$

$s_1: 0° - (\tilde{\phi}_3 + \tilde{\phi}_2 + \tilde{\phi}_1)$
$= -180°$
$(\because \tilde{\phi}_2 + \tilde{\phi}_1 = 360° \;\&\; \tilde{\phi}_3 = 180°)$

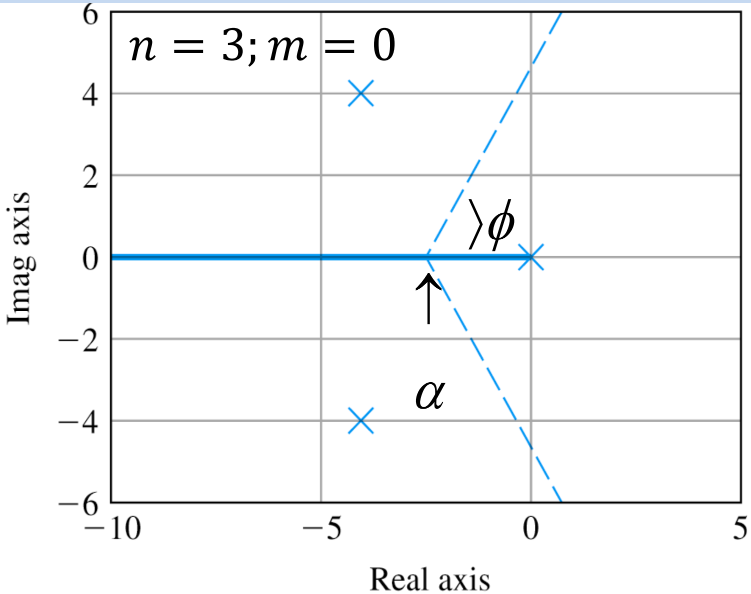
Rules for plotting a root locus

RULE 3: For large s and K, $n-m$ of the loci are asymptotic to the lines at angles ϕ_l radiating out from the points $s=\alpha$ on the real axis where

(Angles of asymptotes) $$\phi_l = \frac{180° + 360°(l-1)}{n-m}, \qquad l = 1, 2, \ldots, n-m$$

(Center of asymptotes) $$\alpha = \frac{\sum p_i - \sum z_i}{n-m}$$

$\left(n\right.$: the number of poles, m : the number of zeros$\left.\right)$



$$1 + K\frac{s^m + b_1 s^{m-1} + \cdots b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$
$$\cong 1 + K\frac{1}{(s-\alpha)^{n-m}} = 0$$

Complement of rule 3

As $K \to \infty$, $L(s) = -\dfrac{1}{K}$ is satisfied only if $L(s) = \dfrac{b(s)}{a(s)} = 0.\left(s = z_i \text{ or } s \to \infty.\right)$

$$1 + K\frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} \cong 1 + K\frac{1}{(s-\alpha)^{n-m}} = 0 \qquad \left(m < n\right)$$

$$\left(\frac{a(s)}{b(s)} = s^{n-m} + (a_1 - b_1)s^{n-m-1} + \cdots \cong (s-\alpha)^{n-m} \to a_1 - b_1 \cong -(n-m)\alpha, \text{ later}\right)$$

Search point: $s_0 = R\,\mathrm{e}^{j\phi}$

$$\to \frac{1}{(s-\alpha)^{n-m}} = -\frac{1}{K} \to (n-m)\phi_l = 180^\circ + 360^\circ(l-1)$$

$$\to \phi_l = \frac{180^\circ + 360^\circ\left(l-1\right)}{n-m},\ l = 1, 2, \ldots, n-m \text{ (angles of the asymptotes)}$$

- The lines of asymptotic locus comes from $s_0 = \alpha$ on the real axis.

$$s^n + a_1 s^{n-1} + \cdots + a_n = \left(s - p_1\right)\left(s - p_2\right)\cdots\left(s - p_n\right) \rightarrow -a_1 = \sum p_i$$

$$s^m + b_1 s^{m-1} + \cdots + b_m = \left(s - z_1\right)\left(s - z_2\right)\cdots\left(s - z_m\right) \rightarrow -b_1 = \sum z_i,$$

From the closed-loop characteristic equation,

$$s^n + a_1 s^{n-1} + \cdots + a_n + K\left(s^m + b_1 s^{m-1} + \cdots + b_m\right)$$

$$= \left(s - r_1\right)\left(s - r_2\right)\cdots\left(s - r_n\right) = 0$$

$(m < n-1) \rightarrow a_1 = -\sum r_i \rightarrow$ The center point of roots does not change with K.

$(m < n-1) \rightarrow -\sum r_i = -\sum p_i$

–Asymptotic behavior of roots: For large values of K,

m of the roots: approach the zeros z_i (i.e., $r_i \cong z_i,\ i = 1, \cdots, m$),

$n-m$ of the roots (i.e., $r_{m+1}, \cdots, r_n$): approach the branches of the asymptotic

system $1/(s-\alpha)^{n-m}$ whose poles add up to $(n-m)\alpha$.

$$1 + K\frac{1}{(s-\alpha)^{n-m}} = 0$$

$$\to (s-\alpha)^{n-m} + K = 0$$

$$\to s^{n-m}\underline{\qquad\quad} s^{n-m-1} + \cdots + K = 0$$

$$s^{n-m}\underline{-(n-m)\alpha}\, s^{n-m-1} + \cdots + (-\alpha)^{n-m} + K = \left(s - r_{m+1}\right)\left(s - r_{m+2}\right)\cdots\left(s - r_n\right)$$

$$\to -\sum_{i=m+1}^{n} r_i = -(n-m)\alpha$$

Therefore, $$-\sum_{i=1}^{n} r_i = -\sum_{i=1}^{m} z_i - \sum_{i=m+1}^{n} r_i = -\sum_{i=1}^{m} z_i - (n-m)\alpha$$

$$-\sum_{i=1}^{n} r_i = -\sum_{i=1}^{m} z_i - \sum_{i=m+1}^{n} r_i = -\sum_{i=1}^{m} z_i - (n-m)\alpha$$

When $n-1 > m,\ \ a_1 = -\sum_{i=1}^{n} r_i = -\sum_{i=1}^{n} p_i \rightarrow -\sum_{i=1}^{n} p_i = -\sum_{i=1}^{m} z_i - (n-m)\alpha$

$$\rightarrow \alpha = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m}$$

When $n-1 = m,$ we don't need to think about the asympototes,

since it is on the real axis.

Rules for plotting a root locus

RULE 4:

- Angle of departure from a pole p_j of multiplicity q:

$$q\phi_{l,\,\mathrm{dep}} = \sum \psi_i - \sum_{i \neq j} \phi_i - 180° - 360°(l-1)$$

$$\left(\sum \psi_i - \sum_{i \neq j} \phi_i - q\phi_{l,\,\mathrm{dep}} = 180° + 360°(l-1) \right), \quad \left(\psi_i = \angle(p_j - z_i), \;\; \phi_i = \angle(p_j - p_i) \right)$$

- Angle of arrival at a zero z_j of multiplicity q:

$$q\psi_{l,\,\mathrm{arr}} = \sum \phi_i - \sum_{i \neq j} \psi_i + 180° - 360°(l-1)$$

$$\left(\sum_{i \neq j} \psi_i + q\psi_{l,\,\mathrm{arr}} - \sum \phi_i = 180° - 360°(l-1) \right), \quad \left(\psi_i = \angle(z_j - z_i), \;\; \phi_i = \angle(z_j - p_i) \right)$$

Example

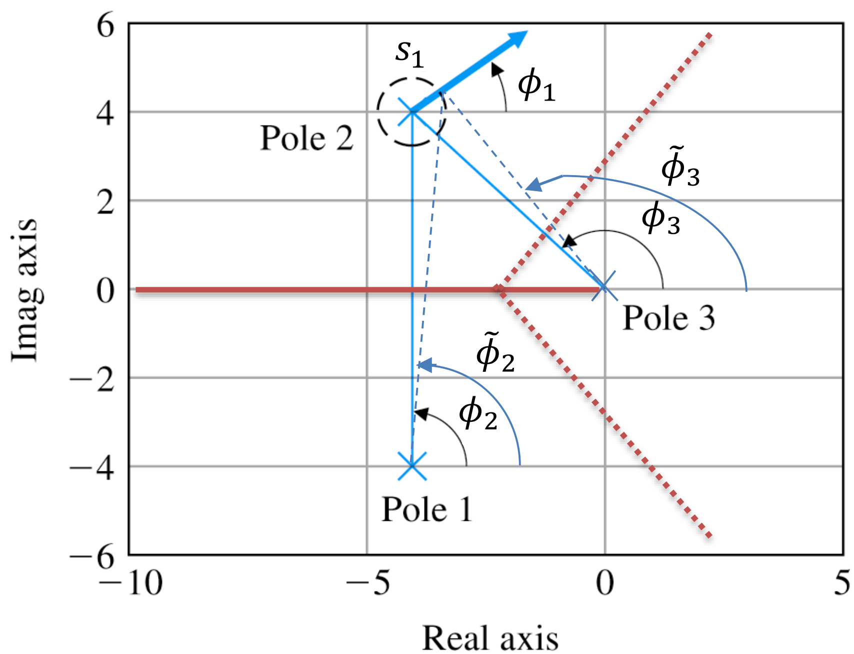


Figure 5.7 The departure and arrival angles are found by looking near a pole or zero.

$$0° - \tilde{\phi}_2 - \tilde{\phi}_3 - \phi_1 = 180° - 360° \times (l - 1)$$

$$\tilde{\phi}_2 \approx \phi_2 \;\&\; \tilde{\phi}_3 \approx \phi_3$$

$$\Rightarrow \phi_1 = -\phi_2 - \phi_3 - 180°$$

- Take a test point on the locus near the pole or zero, and push the test point to the pole or zero.

$$-\sum \phi_i = -\phi_1 - \phi_2 - \phi_3 = 180° - 360°(l-1)$$

$$\phi_1 = -\phi_2 - \phi_3 - 180° = -90° - 135° - 180° = -405° = -45°$$

Rules for plotting a root locus

RULE 5: The locus crosses the $j\omega$ axis where the Routh criterion shows a transition from roots in the LHP to the roots in RHP.

- If $n-m>2$, at least one branch of the locus crosses the imaginary axis.

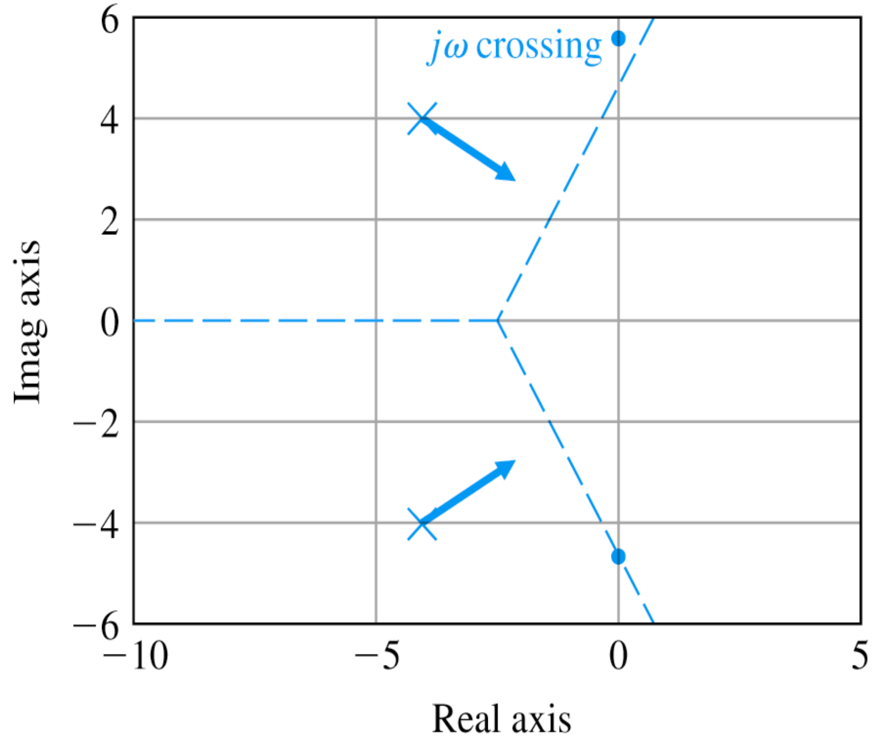
$$1+\frac{K}{s\left[(s+4)^2+16\right]}=0 \rightarrow s^3+8s^2+32s+K=0$$

$$\begin{array}{lcc} s^3: & 1 & 32 \\ s^2: & 8 & K \\ s^1: & \dfrac{8\cdot 32-K}{8} & 0 \\ s^0: & K & \end{array}$$

$$K=256 \rightarrow \left(8s^2+256=0\right)$$

$$\rightarrow (j\omega_0)^3+8(j\omega_0)^2+32(j\omega_0)+256=0$$

$$-\omega_0^3+32\omega_0=0 \rightarrow \omega_0=\pm\sqrt{32}=\pm 5.66$$



Rules for plotting a root locus

RULE 6: The loucs has multiple roots at a point on the locus

$$\text{oniy if } \left(b\frac{da}{ds} - a\frac{db}{ds} \right) = 0$$

Complement of Rule 6

If for $K = K_1$, the characteristic equation has $q(\geq 2)$ multiple poles at $s = r_1$:

$$a(s) + K_1 b(s) = (s - r_1)^q \; f(s)$$

Differentiate this equation:

$$\frac{da}{ds} + K_1 \frac{db}{ds} = q(s - r_1)^{q-1} \; f(s) + (s - r_1)^q \frac{df(s)}{ds} = (s - r_1)^{q-1} g(s)$$

$$\left[a(s) + K_1 b(s)\right]_{s=r_1} = 0 \rightarrow K_1 = -\left.\frac{a(s)}{b(s)}\right|_{s=r_1}$$

$$\rightarrow \left[\frac{da}{ds} - \frac{a(s)}{b(s)}\frac{db}{ds}\right]_{s=r_1} = 0 \rightarrow \left[b\frac{da}{ds} - a\frac{db}{ds}\right]_{s=r_1} = 0$$

Example

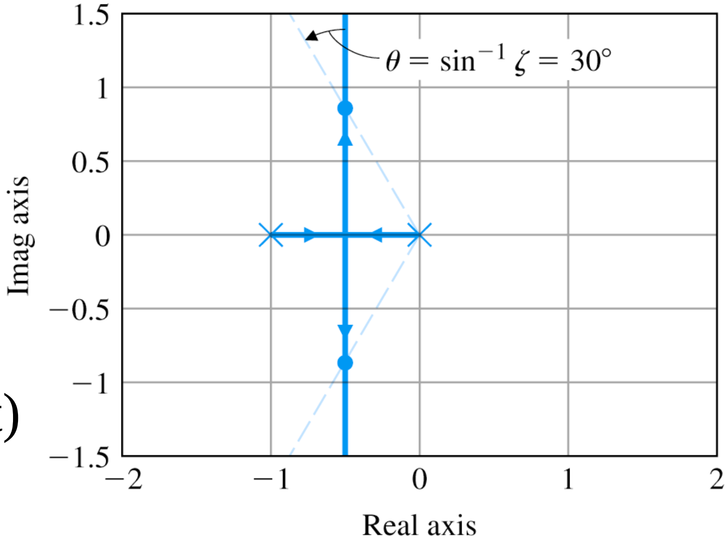
- A necessary condition for breakaway points or breakin points

• In Ex. 5.1,

$$1+G(s)=1+\frac{K}{s(s+1)} \rightarrow s^2+s+K=0$$

$$b(s)=1,\ \ a(s)=s^2+s$$

$$b\frac{da}{ds}-a\frac{db}{ds}=2s+1=0 \rightarrow s_0=-\frac{1}{2}\ \text{(breakaway point)}$$

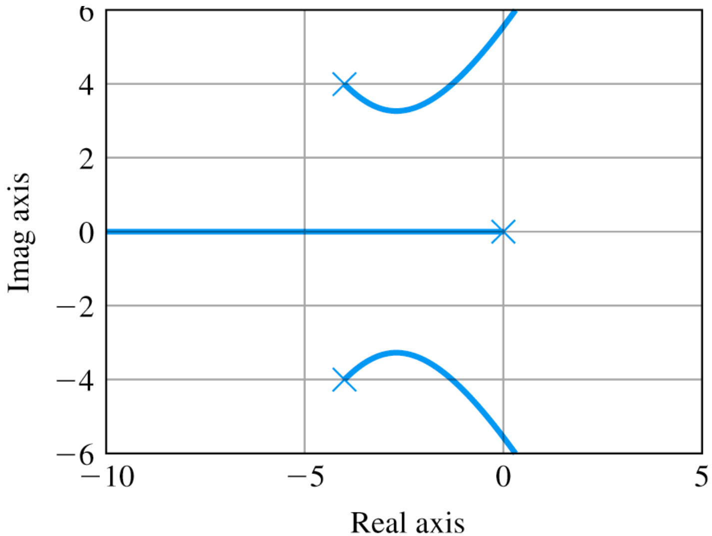


• For $L(s)=\dfrac{1}{s(s^2+8s+32)}$,

$$b\frac{da}{ds}-a\frac{db}{ds}=3s^2+16s+32=0$$

$$s_0=-2.67\pm 1.89j$$

- s_0 is not on the locus and is not a breakaway point.



Root locus for $L(s)=1/[s(s^2+8s+32)]$

Rules for plotting a root locus

- Continuation locus: Computation of the angles of arrival and departure from a point of multple roots

Let $K = K_1 + K_2$. Then, plot a new locus with parameter K_2.

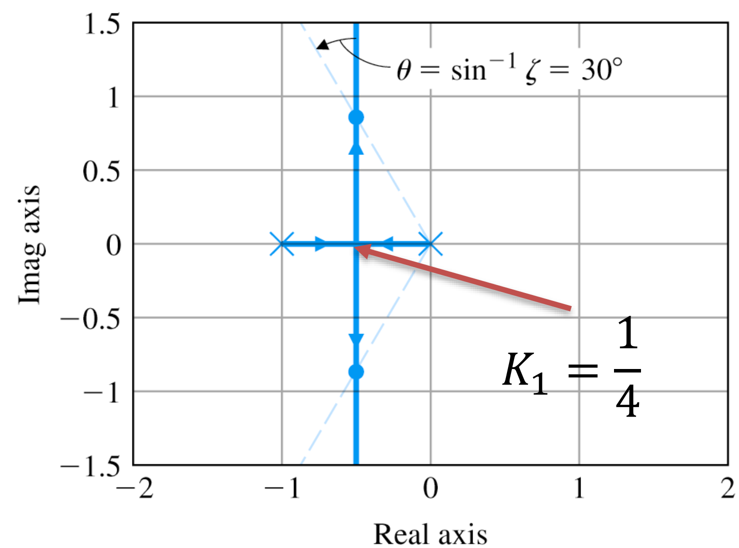
$$a(s) + Kb(s) = a(s) + \left(K_1 + K_2\right)b(s) = \left(a(s) + K_1 b(s)\right) + K_2 b(s)$$

- Multiple poles at $K = K_1 \rightarrow \begin{cases} K_2 > 0: \text{ angle of departure from multiple poles} \\ K_2 < 0: \text{ angle of arrival to multiple pole} \end{cases}$

Characteristic Equation: $s^2 + s + K = 0$

$$K_1 = \frac{1}{4},\ K = \frac{1}{4} + K_2 \rightarrow s^2 + s + \frac{1}{4} + K_2 = 0$$

$$\left(s + \frac{1}{2}\right)^2 + K_2 = 0,\quad 1 + K_2 \frac{1}{\left(s+\frac{1}{2}\right)^2} = 0$$



Char. Eqn.: $s^2+s+K=0,\ K_1=\dfrac{1}{4},\ K=\dfrac{1}{4}+K_2 \rightarrow s^2+s+\dfrac{1}{4}+K_2=0$

$$\left(s+\frac{1}{2}\right)^2+K_2=0,\quad 1+K_2\frac{1}{\left(s+\frac{1}{2}\right)^2}=0$$

- Double pole at $s=-\dfrac{1}{2}$:

$2\phi_{dep}=-180°-360°(l-1)$

$\rightarrow \phi_{dep}=-90°-180°(l-1)$

$\rightarrow \phi_{dep}=\pm 90°$

(departure angles at breakaway)

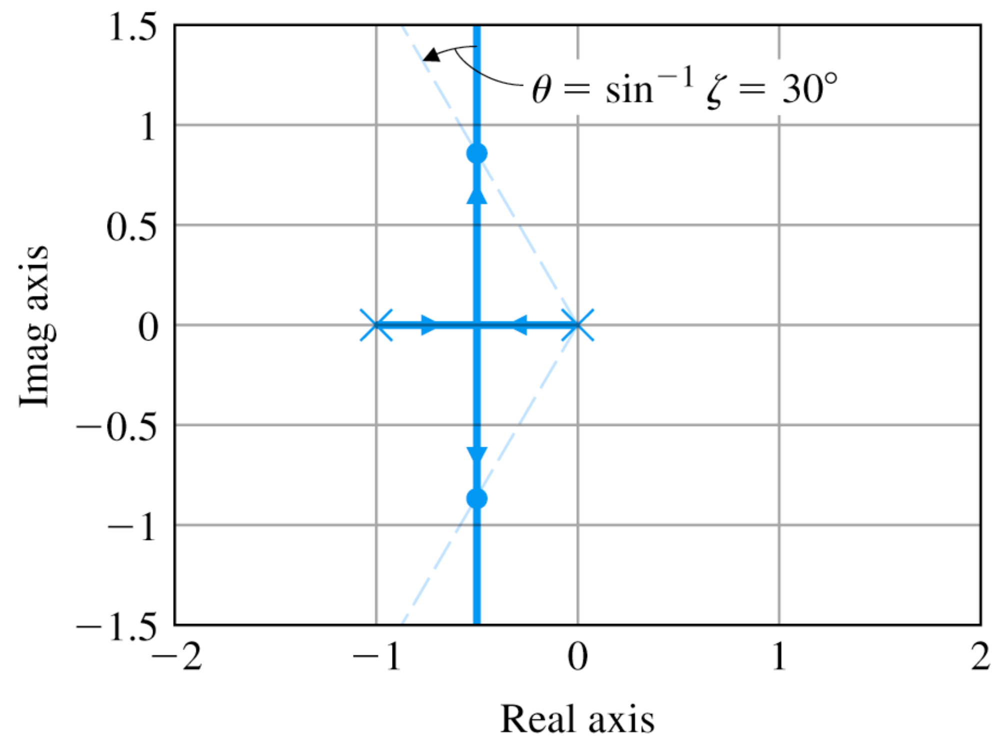
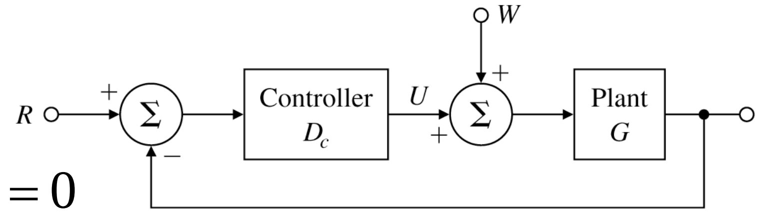


Figure 5.2 Root locus for $L(s)=1/[s(s+1)]$

Example

- Root Locus for Satellite Attitude Control with **P Control.**

$$\text{Plant: } G(s) = \frac{1}{s^2}$$



$$\text{P control} \rightarrow \text{Characteristic equation: } 1 + k_P \frac{1}{s^2} = 0$$

RULE 1: 2 branches which start at $s = 0$ and approach infinity.

RULE 2: No real axis segment

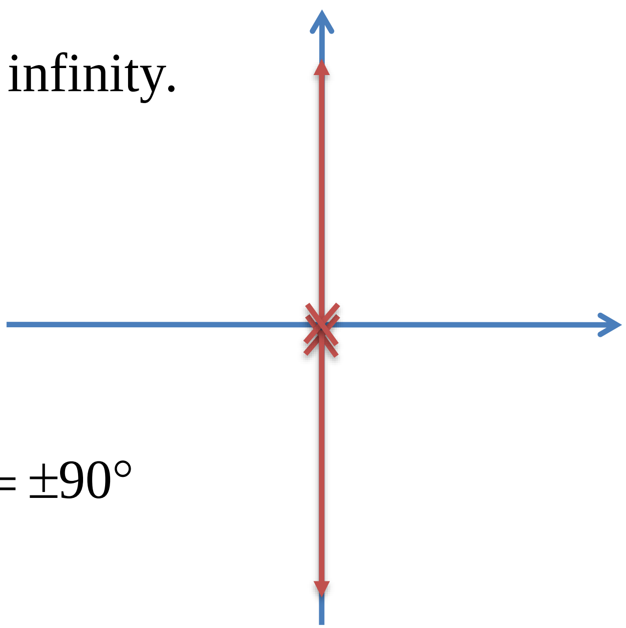
RULE 3: 2 asymptotes

$$\alpha = \frac{0}{2} = 0, \qquad \phi_l = \frac{180° + 360°(l-1)}{2} = \pm 90°$$

RULE 4: Departure angles at double poles at $s = 0$: $\phi = \pm 90°$

RULE 5: The loci remain on the imaginary axis.

RULE 6: The breakaway point is at $s = 0$.

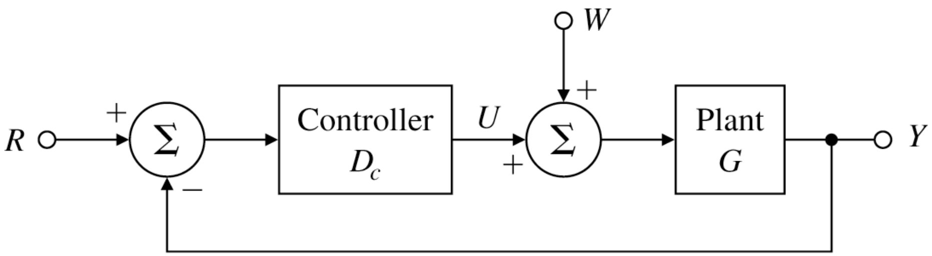


Example

- Root Locus for Satellite Attitude Control with **PD Control**

$$1+[k_P+k_D s]\frac{1}{s^2}=0 \quad (K=k_D,\ k_P/k_D=1)$$

$$\rightarrow 1+K\frac{s+1}{s^2}=0 \rightarrow s^2+Ks+K=0$$



$$\rightarrow 1+K\frac{s+1}{s^2}=0$$

RULE 1: 2 branches which start at $s=0$, one of which terminates on the zero at $s=-1$ and the other approaches infinity.

RULE 2: Real axis segment to the left of $s=-1$.

RULE 3: 1 asymptotes along the negative real axis

$$\alpha=\frac{0-(-1)}{2-1}=1$$

$$\phi_l=\frac{180°+360°(l-1)}{2-1}=180°$$

RULE 4: Departure angles at double poles at $s = 0$: $\phi = \pm 90°$

$$\left(q\phi_{l,\,\mathrm{dep}} = \sum \psi_i - \sum_{i \neq j} \phi_i - 180° - 360°\left(l - 1\right) \right)$$

RULE 5: Routh's Criterion:

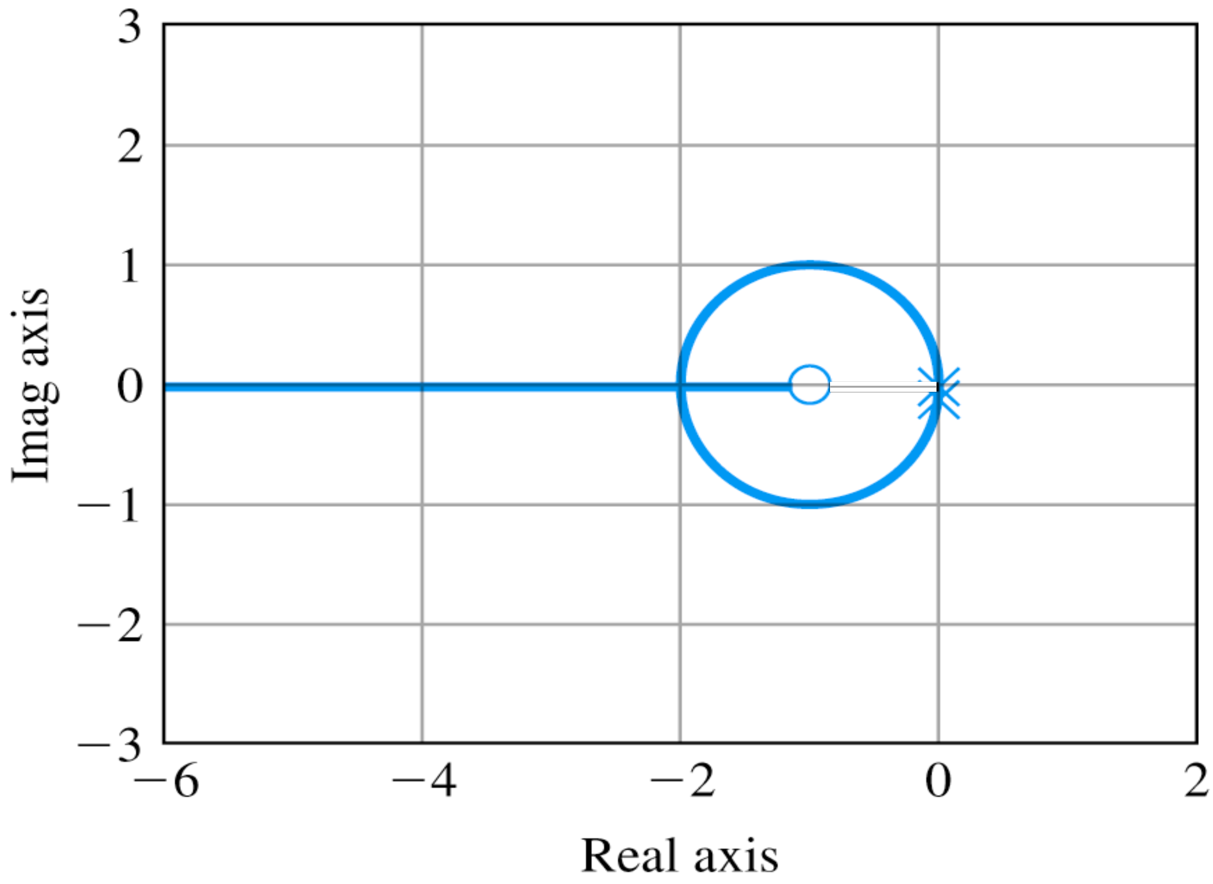
$$\begin{array}{lll} s^2 & 1 & K \\ s^1 & K & \\ s^0 & K & \end{array}$$

$\rightarrow$ No branch of the locus crosses the imaginary axis.

RULE 6: Multiple roots at points on the locus, where

$$b = s + 1, \quad \frac{db}{ds} = 1, \quad a = s^2, \quad \frac{da}{ds} = 2s$$

$$b\frac{da}{ds} - a\frac{db}{ds} = (s+1)2s - s^2 = 0 \quad \rightarrow \quad s^2 + 2s = 0 \quad \therefore s_i = 0, -2.$$



Root locus for $L(s) = G(s) = (s + 1)/s^2$

The addition of the zero has pulled the locus into the LHP.

Example

- Root Locus of the Satellite Control with Modified PD or Lead Compensation

$$1+K\frac{s+1}{s^2(s+12)}=0 \qquad \left(\text{Lead compensator: } K\frac{s+1}{s+12}\right)$$

RULE 1: 3 branches, 2 starting at $s=0$ and one starting at $s=-12$

RULE 2: Real axis segment between $-12 \le s \le -1$

RULE 3: 2 asymptotes centered at $\alpha = \dfrac{-12-(-1)}{3-1} = -11/2 = -5.5$

and at angles $\phi_l = \dfrac{180° + 360°(l-1)}{2(=3-1)} = \pm 90°$

RULE 4: Departure angles at the pole at $s=0$: $\phi = \pm 90°$

$$\left(q\phi_{l,\,\mathrm{dep}} = \sum \psi_i - \sum_{i\neq j}\phi_i - 180° - 360°(l-1)\right)$$

RULE 5: Routh's Criterion:

$\rightarrow$ No branch of the locus crosses the imaginary axis.

RULE 6: Multiple roots at points on the locus.

$$b = s+1,\quad \frac{db}{ds}=1,\quad a = s^3+12s^2,\quad \frac{da}{ds}=3s^2+24s$$

$$b\frac{da}{ds}-a\frac{db}{ds}=(s+1)(3s^2+24s)-(s^3+12s^2)=0$$

$$2s^3+15s^2+24s=0$$

$$s_i = 0, -2.31, -5.18$$

$\rightarrow$ The locus near the origin is similar to the PD control case.

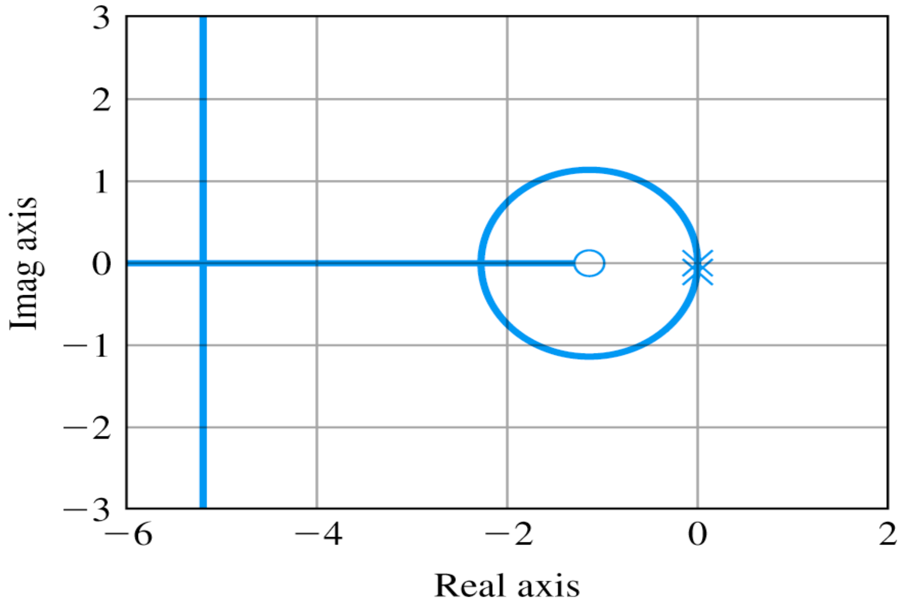


Figure 5.11 Root locus for $L(s) = (s+1)/s^2(s+12)$

Example

- Example: Root Locus of the Satellite Control with Lead having a Relatively Small Value for the Pole

$$1 + K\frac{s+1}{s^2(s+4)} = 0 \qquad \left(\text{Lead compensator: } K\frac{s+1}{s+4}\right)$$

RULE 1: 3 branches, 2 starting at $s = 0$ and one starting at $s = -4$

RULE 2: Real axis segment between $-4 \le s \le -1$

RULE 3: 2 asymptotes centered at $\alpha = \dfrac{-4-(-1)}{3-1} = -3/2 = -1.5$

and at angles $\phi_l = \dfrac{180° + 360°(l-1)}{2} = \pm 90°$

RULE 4: Departure angles at the pole

at $s = 0$: $\phi = \pm 90°$

RULE 5: Routh's Criterion:

$\rightarrow$ No branch of the locus crosses the imaginary axis.

RULE 6: Multiple roots at points on the locus.

$$b = s+1,\ \frac{db}{ds} = 1,\ a = s^3 + 4s^2,\ \frac{da}{ds} = 3s^2 + 8s$$

$$b\frac{da}{ds} - a\frac{db}{ds} = (s+1)(3s^2+8s) - (s^3+4s^2) = 0$$

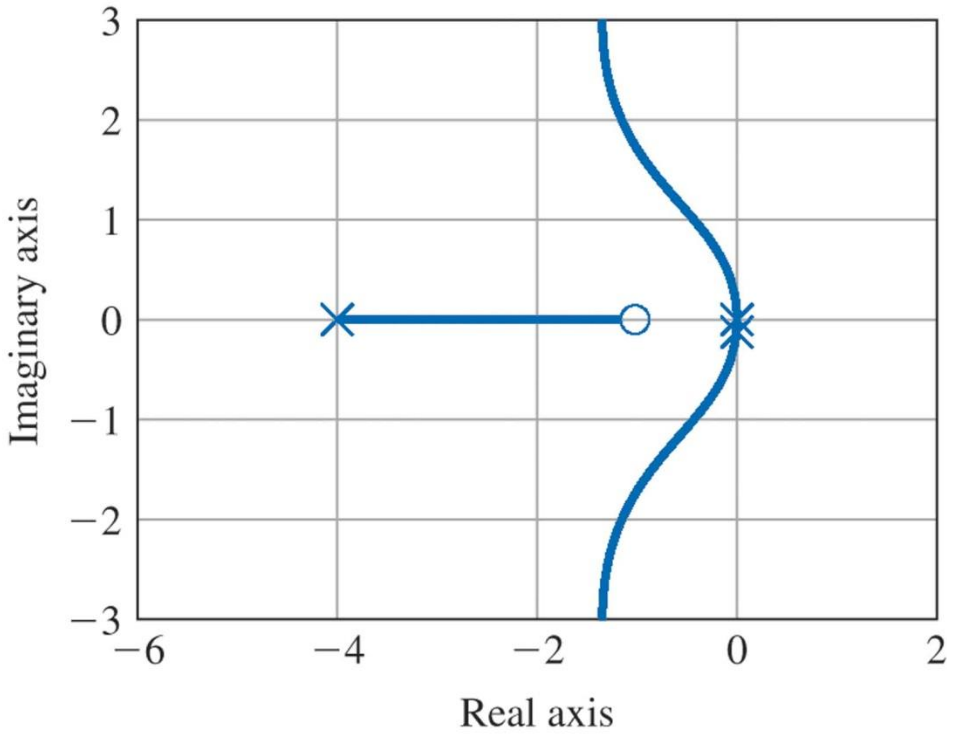
$$2s^3 + 7s^2 + 8s = 0$$

$$s_i = 0, -1.75 \pm j0.97$$

$$-1.75 \pm j0.97 :\ \text{not on RL}$$

$\rightarrow$ No breakin or breakaway points

- At $p = 9$, the locus breaks in at $s = -3$ in a triple multiple root.



Example

- Root Locus for the Satellite with a Transition Value for the Pole

$$1 + K\frac{s+1}{s^2(s+9)} = 0 \qquad \left(\text{Lead compensator: } K\frac{s+1}{s+9}\right)$$

RULE 1: 3 branches, 2 starting at $s = 0$ and one starting at $s = -9$

RULE 2: Real axis segment between $-9 \le s \le -1$

RULE 3: 2 asymptotes centered at $\alpha = \dfrac{-9-(1)}{3-1} = -4$

and at angles $\phi_l = \dfrac{180° + 360°\left(l-1\right)}{2} = \pm 90°$

RULE 4: Departure angles at the pole

at $s = 0$: $\phi = \pm 90°$

RULE 5: Routh's Criterion: No branch of the locus crosses the imaginary axis.

RULE 6: Multiple roots at points on the locus.

$$b = s+1, \quad \frac{db}{ds} = 1, \quad a = s^3 + 9s^2, \quad \frac{da}{ds} = 3s^2 + 18s$$

$$b\frac{da}{ds} - a\frac{db}{ds} = (s+1)(3s^2+18s) - (s^3+9s^2) = 0$$

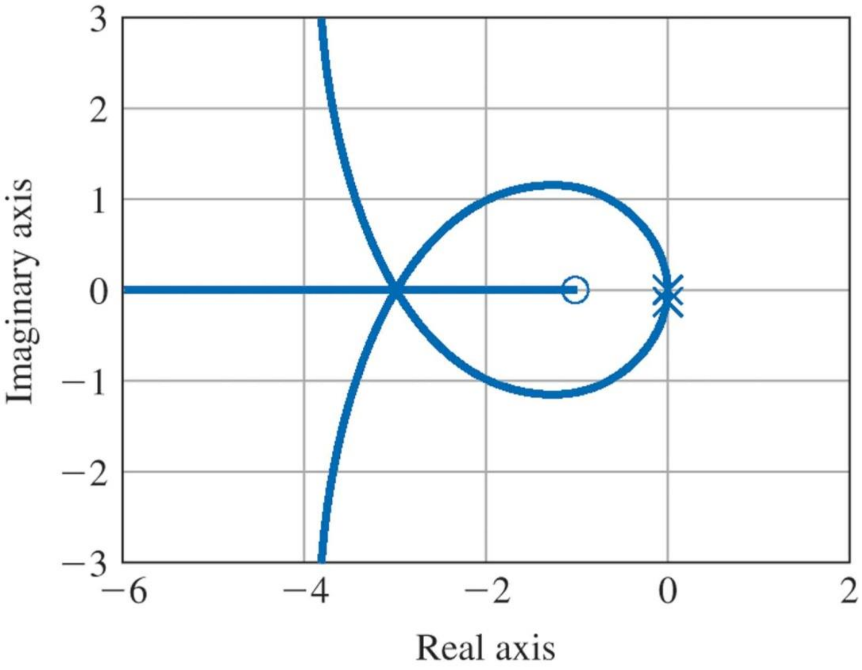
$$2s^3 + 12s^2 + 18s = 0 \qquad (18 = 2p)$$

$$s(s+3)^2 = 0$$

$$s_i = 0, -3, -3$$

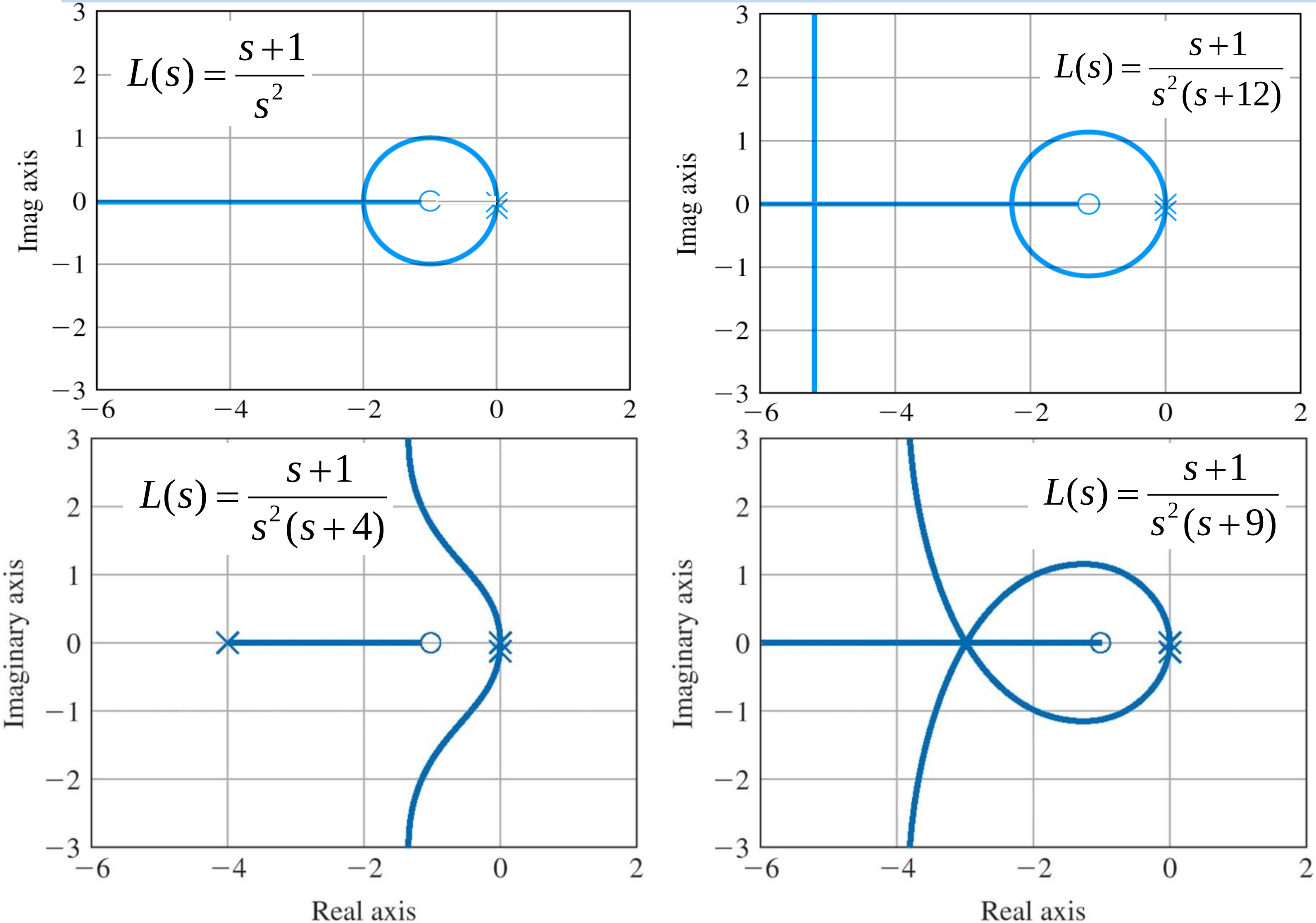
$\rightarrow$ Angles of breakin: $-180°$, $\pm 60°$

Angles of breakaway: $0°$, $\pm 120°$



Comparison

An additional pole moving in from the far left tends to push the locus branches to the right as it approaches a given locus.



Selecting the parameter values by root locus

- Positive root locus: Plot of all possible location for roots to the equation $1+KL(s)=0$ for some positive value of K.

- Select a particular value of K that will meet the specifications for static and dynamic response.

- Magnitude condition:

For s on the root locus, the gain is given by $K=-\dfrac{1}{L(s)}$.

For s on the positive root locus, $K=\dfrac{1}{|L(s)|}$.

- MATLAB: [K, p]=rlocfind(sys)

Example

- Example: For $L(s)=\dfrac{1}{s\left[(s+4)^2+16\right]}$, Find K that makes $\zeta=0.5$.

$\rightarrow$ Find the value of K when a root is s_0.

$$\left(\phi_l=\frac{180^\circ+360^\circ(l-1)}{3-0}=\pm60^\circ,180^\circ,\ \alpha=\frac{-8}{3-0}=-\frac{8}{3}\right)$$

$$L(s)=\frac{1}{s_0(s_0-s_2)(s_0-s_3)}$$

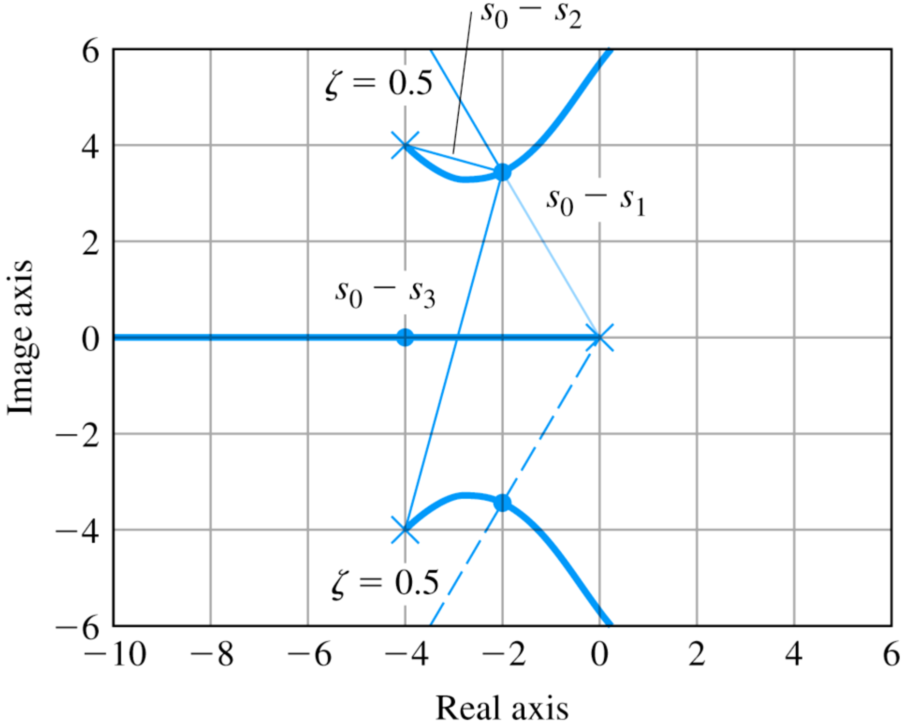
$$K=\frac{1}{\left|L(s_0)\right|}=\left|s_0\right|\left|s_0-s_2\right|\left|s_0-s_3\right|$$

$$\left|s_0\right|\cong 4.0,$$

$$\left|s_0-s_2\right|\cong 2.1$$

$$\left|s_0-s_3\right|\cong 7.7$$

$$K\cong 4.0(2.1)(7.7)=65\rightarrow\zeta=0.5$$

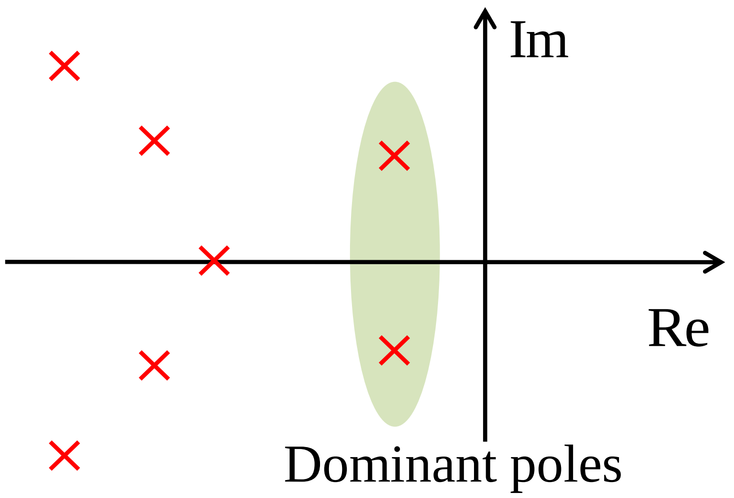


3. Dynamic Compensation

Dominant poles

- Dominant poles: the pole(s) whose real parts are closer to the imaginary axis than other poles. Typically complex poles.

- dominant poles determine the behavior because they don't decay to zero before the other poles do.



Two dynamic compensation

- Two Compensation Schemes

  - Lead compensation: $D(s) = K\dfrac{s+z}{s+p}, \ z < p$

    · $\approx$ PD control $\left(k_P + k_D s\right)$.

    · speeds up response by lowering the rise time and decreases the transient overshoot.

  - Lag compensation: $D(s) = K\dfrac{s+z}{s+p}, \ z > p$

    · $\approx$ PI control $\left(k_P + k_I\dfrac{1}{s}\right)$.

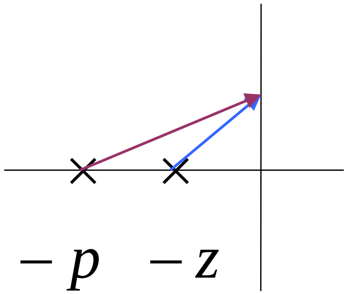
    · improves the steady-state accuracy.

Characteristics of compensations

- Lead compensation: $D(s) = K\dfrac{s+z}{s+p},\ z < p$

$$\phi(j\omega) = \angle D(j\omega) = \angle\left(K\frac{j\omega-(-z)}{j\omega-(-p)}\right) = \tan^{-1}\left(\frac{\omega}{z}\right) - \tan^{-1}\left(\frac{\omega}{p}\right) > 0$$

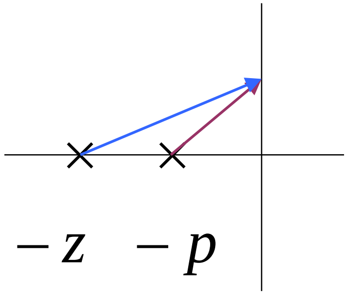
$\rightarrow$ phase lead



- Lag compensation: $D(s) = K\dfrac{s+z}{s+p},\ z > p$

$$\phi(j\omega) = \angle D(j\omega) = \tan^{-1}\left(\frac{\omega}{z}\right) - \tan^{-1}\left(\frac{\omega}{p}\right) < 0$$

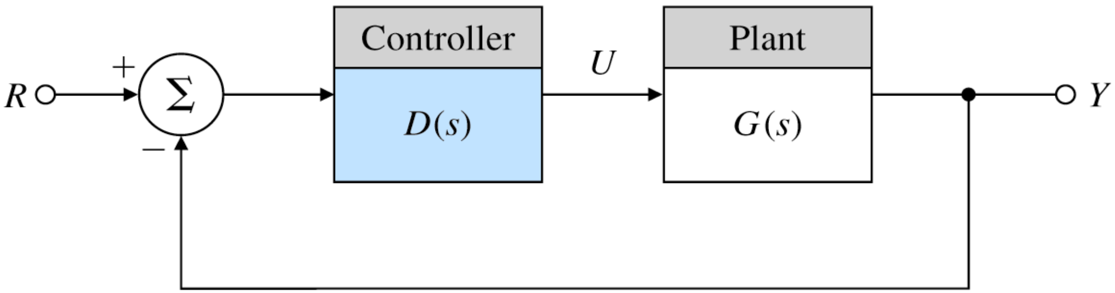
$\rightarrow$ phase lag



Characteristic equation:

$$1 + D(s)G(s) = 0$$

$$1 + KL(s) = 0$$



Feedback system with compensation

Properties and example of Lead compensation

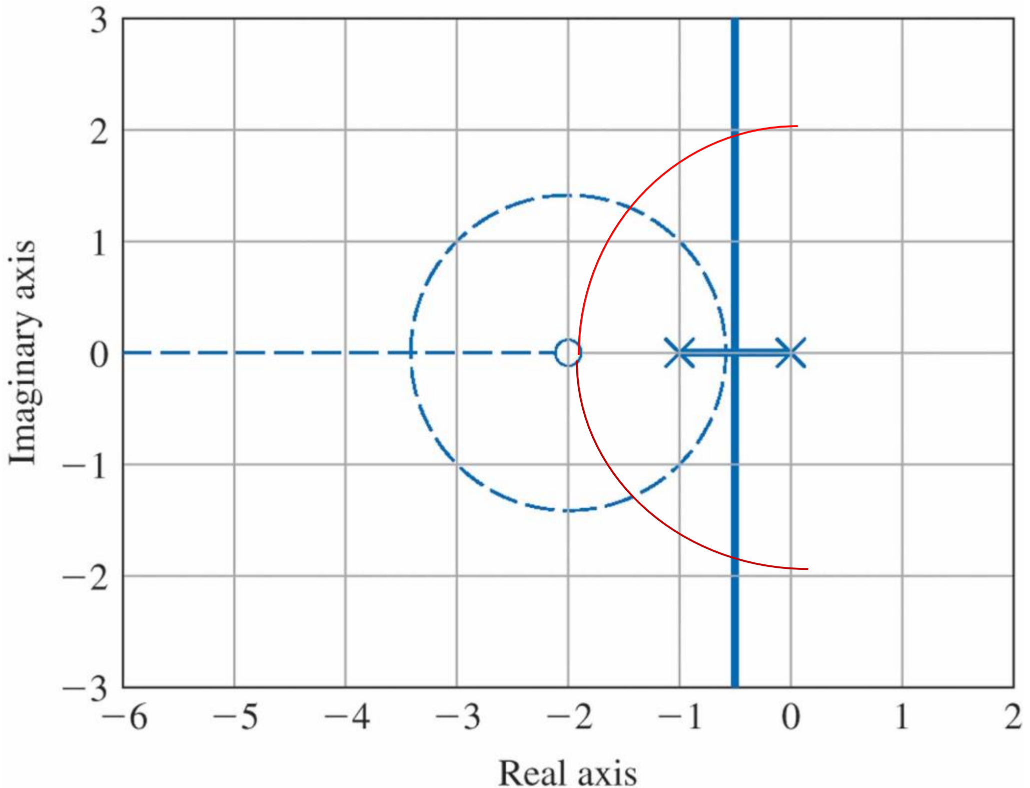
- Lead Compensation

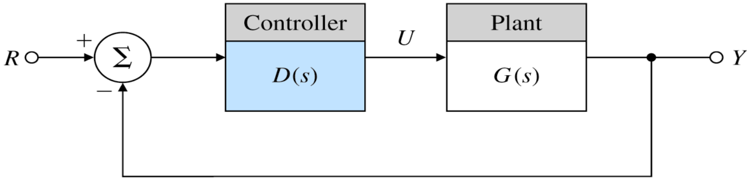
- Stabilizing effect of lead compensation

- Position control system for the plant $G(s) = \dfrac{1}{s(s+1)}$

P control: $D(s) = K$ -solid $1 + K\dfrac{1}{s(s+1)}$

PD control: $D(s) = K(s+2)$ -dashed $1 + K\dfrac{s+2}{s(s+1)}$





Feedback system with compensation

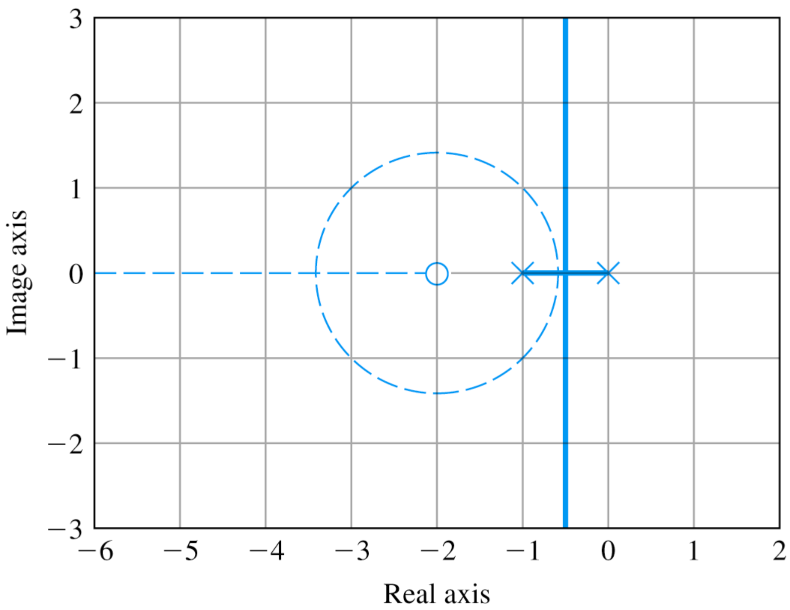
Required speed-of-response specification: $\omega_n \cong 2$

$\rightarrow$ Recall $\omega_n =$ the distance of second order pole.

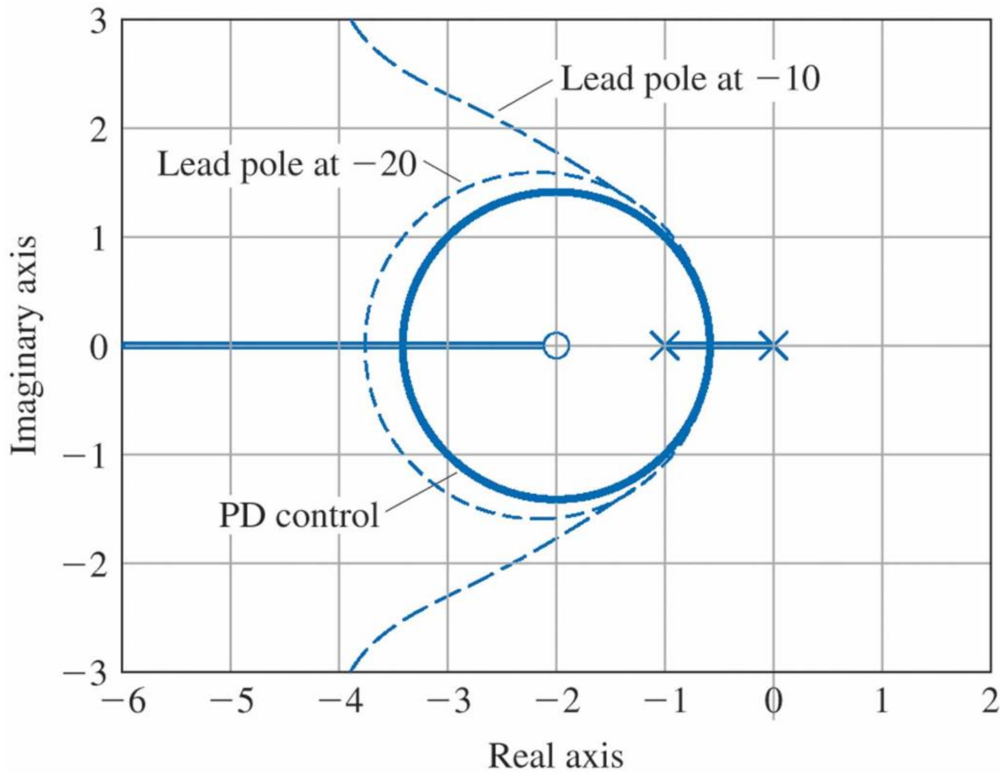
- The effect of zero is to move the locus to the left(more stable part).
- $\omega_n \cong 2$ with P control results in very low value of damping ratio.
- $\omega_n \cong 2$ with PD control results in reasonable value of damping ratio($\varsigma \geq 0.5$).
- Pure derivative control is not practical due to amplification of sensor noise.

$\rightarrow$ Consider the lead compensation:

$$D(s) = K\frac{s+2}{s+p}, \qquad p = 10, 20$$



P control [solid], PD control [dashed]



$G(s) = 1/[s(s+1)]$

PD control: $D(s) = K(s+2)$

Lead Comp.: $D(s) = K(s+2)/(s+20)$

Lead Comp.: $D(s) = K(s+2)/(s+10)$

Properties and example of Lag compensation

- Lag Compensation

  \- Satisfactory dynamic response can be obtained by lead compensation, but the low-frequency gain (K_p, K_v, etc.) may be too low.

  $\rightarrow$ Increase these constants in such a way not to disturb the dynamic response.

  $\rightarrow$ Employ $D(s)$ with a large gain at $s=0$ and nearly unity at the higher frequency ω_n.

  \- Use lag compensation: $D(s) = K\dfrac{s+z}{s+p},\ z > p$

$$z,\, p \ll \omega_n$$

$$D(0) = \frac{z}{p} = 3 \mapsto 10$$

In Ex. 5.11, $G(s)=\dfrac{1}{s(s+1)}, D(s)=\dfrac{91(s+2)}{s+13}$

Velocity constant: $K_v=\lim_{s\to 0} sKDG=\lim_{s\to 0} s(91)\dfrac{s+2}{s+13}\dfrac{1}{s(s+1)}=14$

Want to keep the transient response of the lead compensation and to make K_v larger ($K_v=70$).

Use a lag compensator.

$$D_2(s)=\frac{s+z}{s+p}, \quad z>p,$$

and set z and p small (e.g. $-z=-0.05,\ -p=-0.01$ to make K_v =70).

$$\rightarrow \qquad D_2(s)=\frac{s+0.05}{s+0.01}$$

- The lag compensator changes the root locus locally near the origin, but not globally.
- A closed-loop root remains very near the lag-compensation zero.

→ Very slowly decaying transient, but small magnitude.

→ May influence the settling time. (move pole/zero to the left if possible.)