EECE423-01: 현대제어이론

Modern Control Theory

Chapter 3: State-Space Representation

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◆ The main topics of this chapter are

1. State-Space Equations

2. Relations with Transfer Functions

3. Block Diagrams

1. State-Space Equations

◆ State-space equations

A continuous-time state-space linear system is defined as follows:

$$\begin{cases} \dfrac{dx(t)}{dt} & = A(t)x(t) + B(t)u(t) \quad : \text{state equation} \\ y(t) & = C(t)x(t) + D(t)u(t) \quad : \text{output equation} \end{cases}$$

$x(t) \in \mathbb{R}^n$: n-dimensional state vector

$u(t) \in \mathbb{R}^k$: k-dimensional input vector

$y(t) \in \mathbb{R}^m$: m-dimensional output vector

The above 2 equations describe an input-output relation between the input signal $u(\cdot)$ and the output signal $y(\cdot)$

◆ Terminology and notation

- When the input signal u takes scalar values ($k = 1$),

  $\rightarrow$ the system is called single input (SI).

- When the input signal u takes vector values ($k \geq 2$),

  $\rightarrow$ the system is called multi input (MI).

- When the output signal y takes scalar values ($m = 1$),

  $\rightarrow$ the system is called single output (SO).

- When the output signal y takes vector values ($m \geq 2$),

  $\rightarrow$ the system is called multi output (MO).

	$y(t) \in \mathbb{R}^m$ $(m = 1)$	$y(t) \in \mathbb{R}^m$ $(m \geq 2)$
$u(t) \in \mathbb{R}^k$ $(k = 1)$	single input single output (SISO)	single input multi output (SIMO)
$u(t) \in \mathbb{R}^k$ $(k \geq 2)$	multi input single output (MISO)	multi input multi output (MIMO)

◆ Classification of state-space linear systems

$$\begin{cases} \dfrac{dx(t)}{dt} & = A(t)x(t) + B(t)u(t) \quad : \text{ state equation} \\ y(t) & = C(t)x(t) + D(t)u(t) \quad : \text{ output equation} \end{cases}$$

- When there is no state equation ($n = 0$), i.e., $y(t) = D(t)u(t)$,
  the system is called memoryless.

- When the matrices $A(t), B(t), C(t), D(t)$ are time-variant,
  the system is called a linear time-varying (LTV) system.

- When the matrices $A(t), B(t), C(t), D(t)$ are constant $\forall t \geq 0$,
  the system is called a linear time-invariant (LTI) system.

◆ LTV system

$$\begin{cases} \dfrac{dx(t)}{dt} & = A(t)x(t) + B(t)u(t) \\ y(t) & = C(t)x(t) + D(t)u(t) \end{cases}$$

◆ LTI system

$$\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$$

The main contents of this course are confined to LTI systems

◆ Interpretations of state-space equations

$$\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$$

(a) state vector x describes enough about the system to determine its future behavior

(b) the output signal $y(t)$ at $t \geq t_1$ is uniformly determined according to the value of $x(t)$ at $t = t_1$ together with the input signal $u(t)$ at $t \geq t_1$

(c) even if the input signals $u(t)$ prior to $t = t_1$ are different, the output signals $y(t)$ at $t \geq t_1$ coincide each other, when the states $x(t)$ at $t = t_1$ have a common value with the same inputs $u(t)$ at $t \geq t_1$

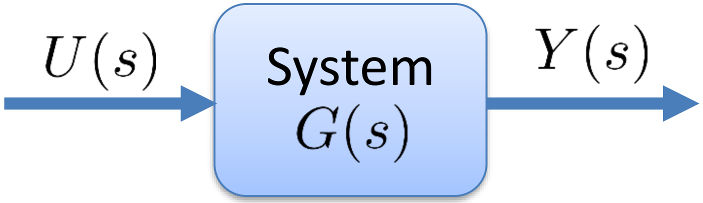
2. Relations with Transfer Functions

◆ Review of transfer function

- Laplace transform of signals

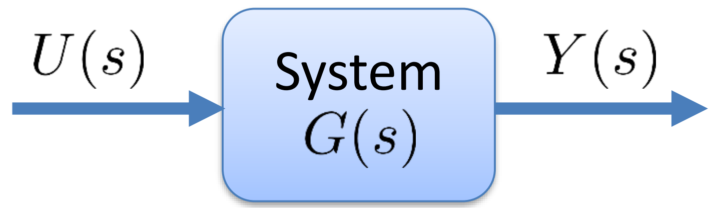
$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

- Transfer function: input/output relation in the frequency-domain



$$G(s) = \frac{Y(s)}{U(s)}$$

◆ Transform of transfer function to state-space system



When a transfer function

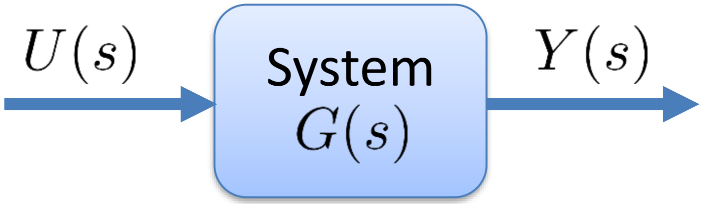
$$G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} + d$$

is given, we consider deriving its equivalent state-space equation

$$\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$$

This procedure is called *realization*.

◆ Method 1: Controllable canonical form



Let $G(s) = \dfrac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} + d$

If we define the state variable as

$$X(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} U(s)$$

Then, we obtain

$$Y(s) = (b_{n-1}s^{n-1} + \cdots + b_1 s + b_0)X(s) + dU(s)$$

This means

$$\boxed{\begin{aligned} y &= b_0 x + b_1 \frac{dx}{dt} + \cdots + b_{n-1} \frac{d^{n-1}x}{dt^{n-1}} + du \\ \frac{d^n x}{dt^n} &= -a_0 x - a_1 \frac{dx}{dt} - \cdots - a_{n-1} \frac{d^{n-1}x}{dt^{n-1}} + u \end{aligned}}$$

If we further define

$$x_1 := x,\ x_2 := \frac{dx}{dt},\ \cdots,\ x_n := \frac{d^{n-1}x}{dt^{n-1}}$$

it readily follows that

$$y = b_0 x_1 + b_1 x_2 + \cdots + b_{n-1} x_n + du$$

$$\begin{cases} \frac{dx_i}{dt} & = x_{i+1} \quad (i = 1, \ldots, n-1) \\ \frac{dx_n}{dt} & = -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + u \end{cases}$$

To put it another way, we have the following state-space equation:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & \cdots & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & \cdots & b_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + du$$

We call it *controllable canonical form*.

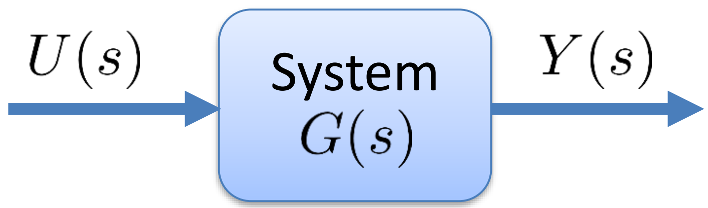
◆ Brief introduction to controllability

Controllability describes the ability of an external input to move the internal state of a system from any initial state to any other final state in a finite time interval.

$$\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$$

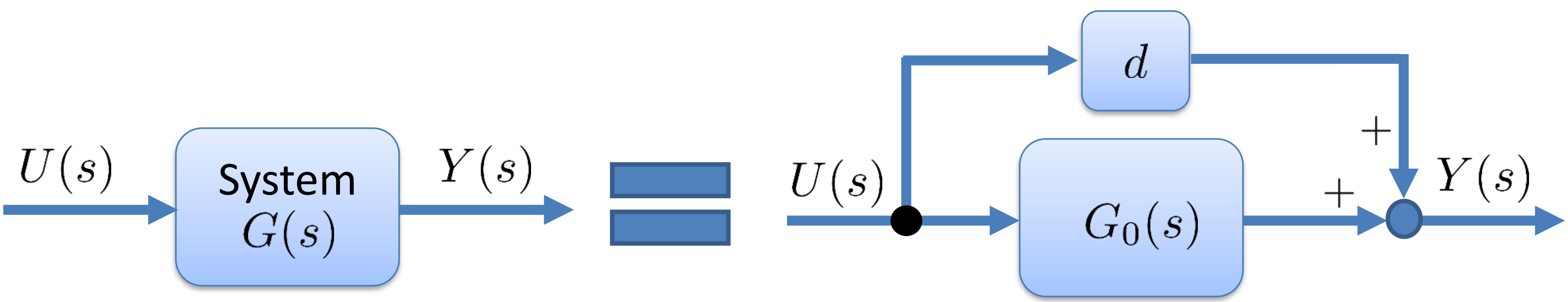
The details will be discussed in Chapter 6.

◆ Method 2: Observable canonical form



Let $G(s) = \dfrac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} + d$

We first decompose $G(s)$ as $G(s) = G_0(s) + d$ and consider $G_0(s)$



At the end of this section, we will return to the general case $G(s)$

Note that $Y(s) = G_0(s)U(s)$, i.e.,

$$s^n Y(s) + \cdots + a_0 Y(s) = b_{n-1} s^{n-1} U(s) + \cdots + b_0 U(s)$$

This means

$$\boxed{\begin{aligned} &a_0 y + a_1 \tfrac{dy}{dt} + \cdots + a_{n-1} \tfrac{d^{n-1} y}{dt^{n-1}} + \tfrac{d^n y}{dt^n} = b_0 u + b_1 \tfrac{du}{dt} + \cdots + b_{n-1} \tfrac{d^{n-1} u}{dt^{n-1}} \\ &a_0 y - b_0 u + \tfrac{d}{dt}(a_1 y - b_1 u + \tfrac{d}{dt}(\cdots + \tfrac{d}{dt}(a_{n-1} y - b_{n-1} u + \tfrac{dy}{dt}) = 0 \end{aligned}}$$

Here, we define $x_n := y$ and

$$x_{n-1} := a_{n-1} y - b_{n-1} u + \tfrac{dx_n}{dt}$$

$$\vdots$$

$$x_1 := a_1 y - b_1 u + \tfrac{dx_2}{dt}$$

$$0 = a_0 y - b_0 u + \tfrac{dx_1}{dt}$$

Then, it readily follows that

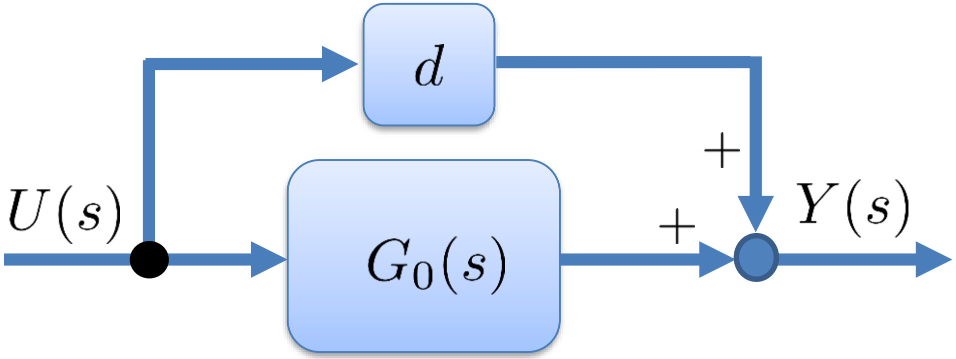
$$y = x_n$$

$$\begin{cases} \frac{dx_i}{dt} & = x_{i-1} - a_{i-1}x_n + b_{i-1}u \quad (i = 2, \ldots, n) \\ \frac{dx_1}{dt} & = -a_0 x_n + b_0 u \end{cases}$$

Here, if we return the general case $G(s) = G_0(s) + d$, we obtain

$$y = x_n + du$$

$$\begin{cases} \frac{dx_i}{dt} & = x_{i-1} - a_{i-1}x_n + b_{i-1}u \quad (i = 2, \ldots, n) \\ \frac{dx_1}{dt} & = -a_0 x_n + b_0 u \end{cases}$$



To put it another way, we have the following state-space equation:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -a_{n-2} \\ 0 & \cdots & 0 & 1 & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_0 \\ \vdots \\ b_{n-1} \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + du$$

We call it *observable canonical form*.

◆ Brief introduction to observability

Observability implies that the state at any instance can be determined by observing the output over a finite interval of time.

$$\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$$

The details will be discussed in Chapter 7.

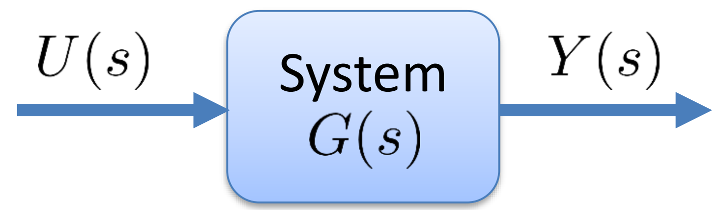
◆ Relation between controllable and observable canonical forms

Controllable canonical form	Observable canonical form
$\begin{cases} \dfrac{dx(t)}{dt} & = A_{\mathrm{c}}x(t) + B_{\mathrm{c}}u(t) \\ y(t) & = C_{\mathrm{c}}x(t) + D_{\mathrm{c}}u(t) \end{cases}$	$\begin{cases} \dfrac{dx(t)}{dt} & = A_{\mathrm{o}}x(t) + B_{\mathrm{o}}u(t) \\ y(t) & = C_{\mathrm{o}}x(t) + D_{\mathrm{o}}u(t) \end{cases}$

- $A_{\mathrm{c}}^T = A_{\mathrm{o}},\ B_{\mathrm{c}}^T = C_{\mathrm{o}},\ C_{\mathrm{c}}^T = B_{\mathrm{o}},\ D_{\mathrm{c}}^T = D_{\mathrm{o}}$
- They have the same transfer function

◆ Transform of state-space equation to transfer function

$$G : \begin{cases} \dfrac{dt}{dx} & = Ax + Bu \\ y & = Cx + Du \end{cases}$$



Let us assume that $x(0) = 0$.

Then, applying Laplace transform to G leads to

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

Thus, the transfer function is described by

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D}$$

◆Relation between state-space equation and transfer function

$$G : \begin{cases} \dfrac{dt}{dx} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

Not uniformly determined

(e.g., controllable canonical form,
observable canonical form)

Uniformly determined

(e.g., $C(sI - A)^{-1}B + D$)

$$G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} + d$$

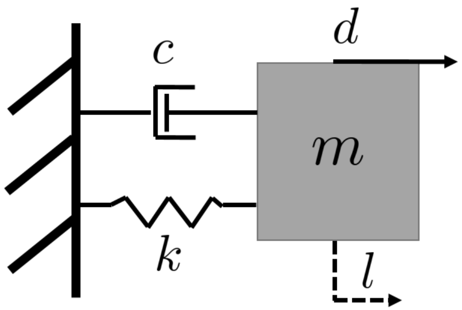
- Poles of $G(s)$ = Eigenvalues of A

◆ Zero-state equivalence

Two state-space systems are said to be *zero-state equivalent* if they realize the same transfer function. This means that they exhibit the same forced response to every input.

(ex. controllable canonical form and observable canonical form)

◆Example of mass-spring-damper system



$$m\ddot{l} = d - c\dot{l} - kl$$

$$\text{Let } m = 1,\ c = 3,\ k = 2$$

- Transfer function

$$\frac{L(s)}{D(s)} = \frac{1}{ms^2 + cs + k} = \frac{1}{s^2 + 3s + 2} \qquad \text{Poles: } s = -1, -2$$

- State-space equation

$$\frac{d}{dt}\begin{bmatrix} l \\ \dot{l} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} l \\ \dot{l} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} d = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

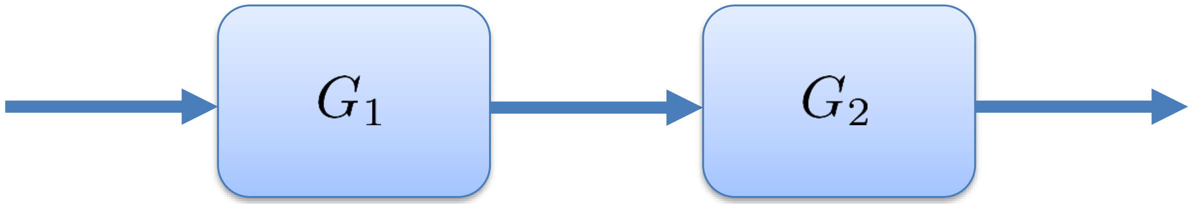
$$l = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} l \\ \dot{l} \end{bmatrix}$$

$$\text{Eigenvalues: } \lambda = -1, -2$$

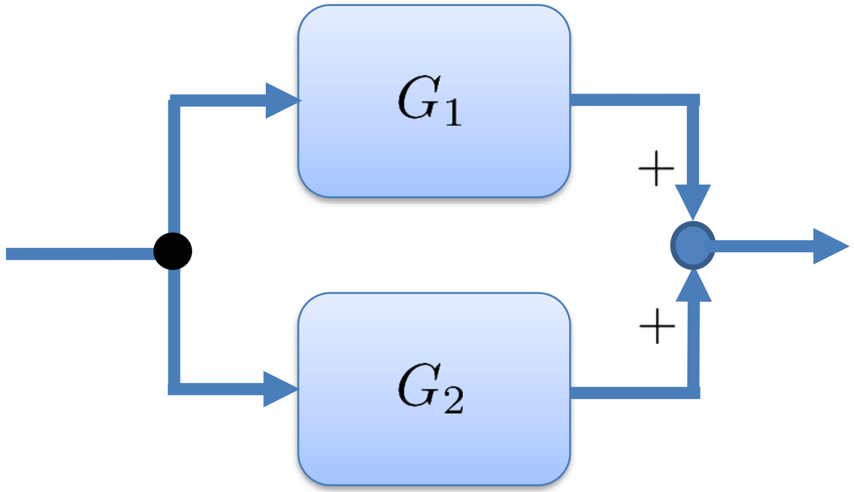
3. Block Diagrams

◆ Block diagram representations

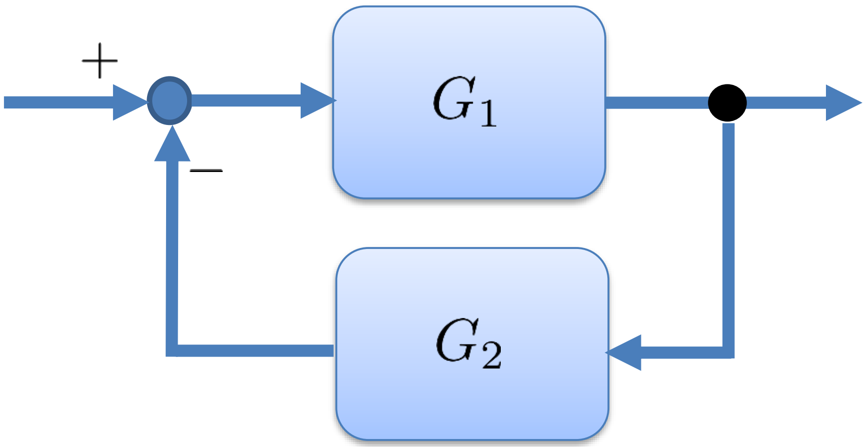
(a) Cascade



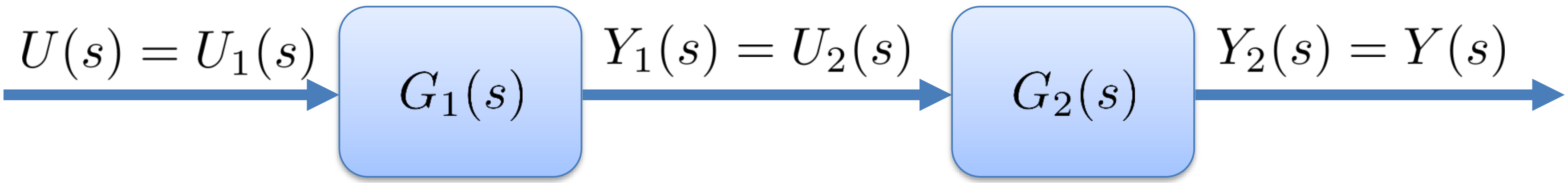
(b) Parallel



(c) Negative feedback



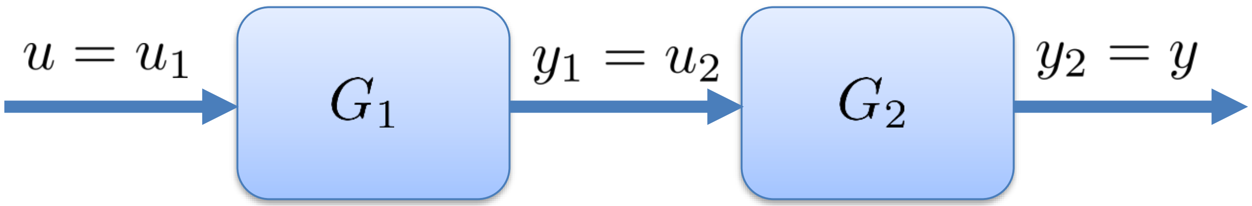
◆ Cascade: transfer function



- Transfer function: $\dfrac{Y(s)}{U(s)} = G_1(s)G_2(s)$

- How can we describe in the time-domain?

◆ Cascade: state-space equation



Let us assume that state-space equations are given by

$$G_1 : \begin{cases} \dfrac{dx_1}{dt} & = A_1 x_1 + B_1 u_1 \\ y_1 & = C_1 x_1 + D_1 u_1 \end{cases}, \qquad G_2 : \begin{cases} \dfrac{dx_2}{dt} & = A_2 x_2 + B_2 u_2 \\ y_2 & = C_2 x_2 + D_2 u_2 \end{cases}$$

Substituing $u_2 = y_1$ into G_2 leads to $\begin{cases} \dfrac{dx_2}{dt} &= A_2x_2 + B_1y_1 \\ y_2 &= C_2x_2 + D_1y_1 \end{cases}$

From $y_1 = C_1x_1 + D_1u_1$, we obtain

$$\begin{cases} \dfrac{dx_2}{dt} &= A_2x_2 + B_2(C_1x_1 + D_1u_1) \\ y_2 &= C_2x_2 + D_2(C_1x_1 + D_1u_1) \end{cases}$$

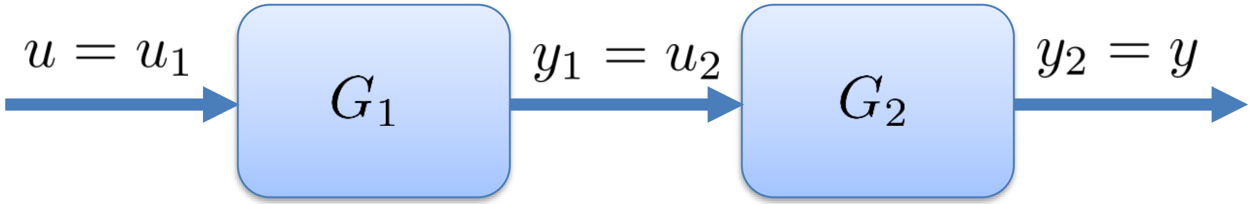
This together with $\begin{cases} \dfrac{dx_1}{dt} &= A_1x_1 + B_1u_1 \\ y_1 &= C_1x_1 + D_1u_1 \end{cases}$ derives the following:

$$
\begin{cases}
\dfrac{dx_1}{dt} & = A_1 x_1 + B_1 u_1 \\
\dfrac{dx_2}{dt} & = B_2 C_1 x_1 + A_2 x_2 + B_2 D_1 u_1 \\
y_2 & = D_2 C_1 x_1 + C_2 x_2 + D_2 D_1 u_1
\end{cases}
$$

If we let $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $u := u_1$ together with $y := y_2$, we have

$$
\begin{cases}
\dfrac{dx}{dt} & = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u \\
y & = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} x + D_2 D_1 u
\end{cases}
$$

◆ Example



$$G_1(s) = \frac{1}{s^2 + 3s + 2}, \qquad G_2(s) = \frac{s^2 + 7s + 10}{s^2 + 7s + 12}$$

Compute the state-space equation of the above system

◆ Solution 1

$$\text{For } G_1(s) = \frac{1}{s^2 + 3s + 2},$$

$$G_1 : \begin{cases} \dfrac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y & = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

$$\rightarrow A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D_1 = 0$$

For $G_2(s) = \dfrac{s^2 + 7s + 10}{s^2 + 7s + 12}$,

$$G_2 : \begin{cases} \dfrac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y & = \begin{bmatrix} -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u \end{cases}$$

$$\rightarrow A_2 = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_1 = \begin{bmatrix} -2 & 0 \end{bmatrix},\ D_1 = 1$$

From $\begin{cases} \dfrac{dx}{dt} &= \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u \\ y &= \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} x + D_2 D_1 u \end{cases},$

we have the following matrices for the state-space equation:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -12 & -7 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix},\ D = 0$$

- Controllable but not observable

◆ Solution 2

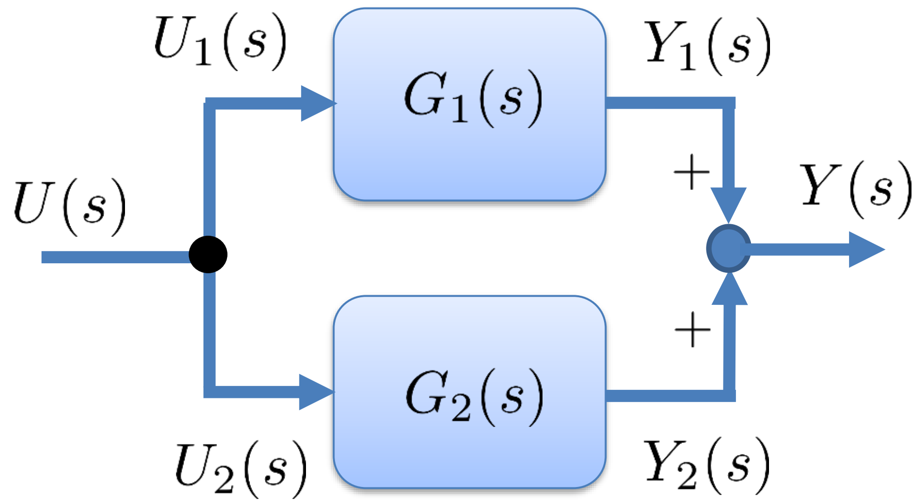
From $G(s) = G_1(s)G_2(s) = \dfrac{s+5}{s^3 + 8s^2 + 19s + 12}$,

we have the following matrices for the state-space equation:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -19 & -8 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 5 & 1 & 0 \end{bmatrix},\ D = 0$$

- Controllable and observable

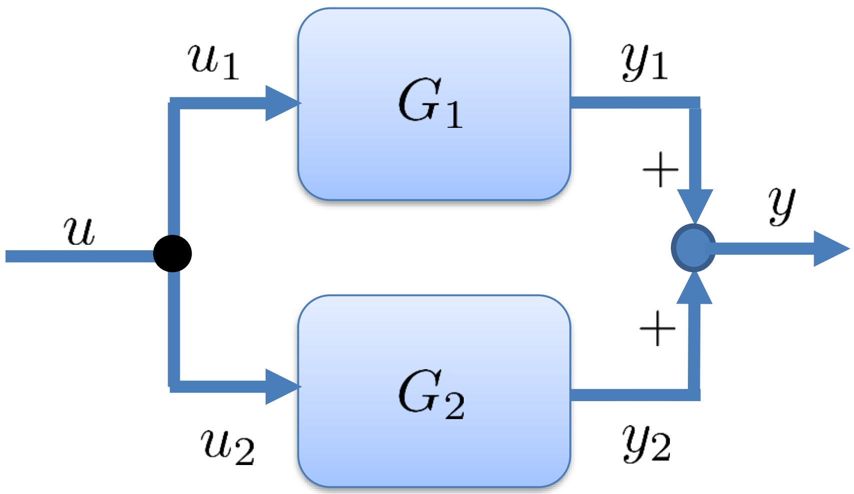
◆ Parallel: transfer function



- Transfer function: $\dfrac{Y(s)}{U(s)} = G_1(s) + G_2(s)$

- How can we describe in the time-domain?

◆ Parallel: state-space equation



Let us assume that state-space equations are given by

$$G_1 : \begin{cases} \dfrac{dx_1}{dt} & = A_1 x_1 + B_1 u_1 \\ y_1 & = C_1 x_1 + D_1 u_1 \end{cases}, \qquad G_2 : \begin{cases} \dfrac{dx_2}{dt} & = A_2 x_2 + B_2 u_2 \\ y_2 & = C_2 x_2 + D_2 u_2 \end{cases}$$

It immediately follows from $u = u_1 = u_2$ that

$$\begin{cases} \dfrac{dx_1}{dt} &= A_1 x_1 + B_1 u \\ y_1 &= C_1 x_1 + D_1 u \end{cases}, \qquad \begin{cases} \dfrac{dx_2}{dt} &= A_2 x_2 + B_2 u \\ y_2 &= C_2 x_2 + D_2 u \end{cases}$$

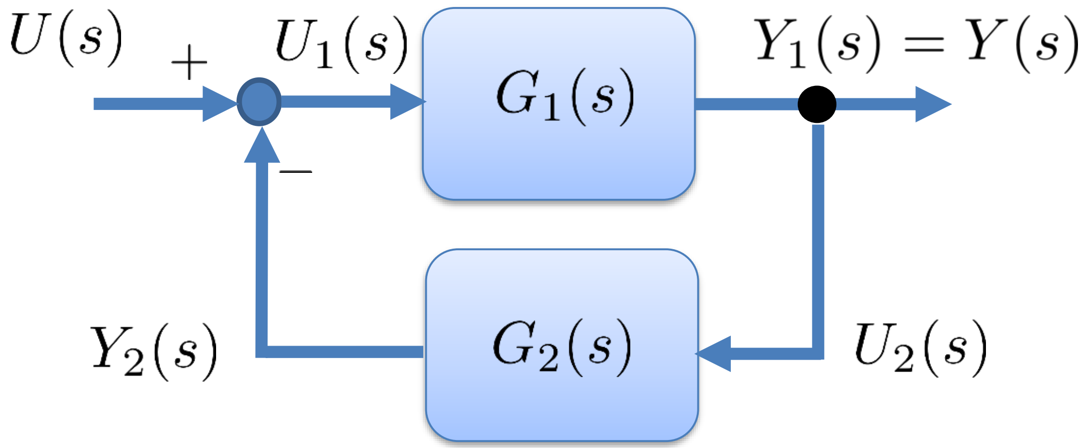
If we note $y = y_1 + y_2$, we readily see that

$$\begin{cases} \dfrac{dx_1}{dt} &= A_1 x_1 + B_1 u \\ \dfrac{dx_2}{dt} &= A_2 x_2 + B_2 u \\ y &= C_1 x_1 + C_2 x_2 + (D_1 + D_2) u \end{cases}$$

By defining $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we obtain the following:

$$\begin{cases} \dfrac{dx}{dt} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ y &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} x + (D_1 + D_2) u \end{cases}$$

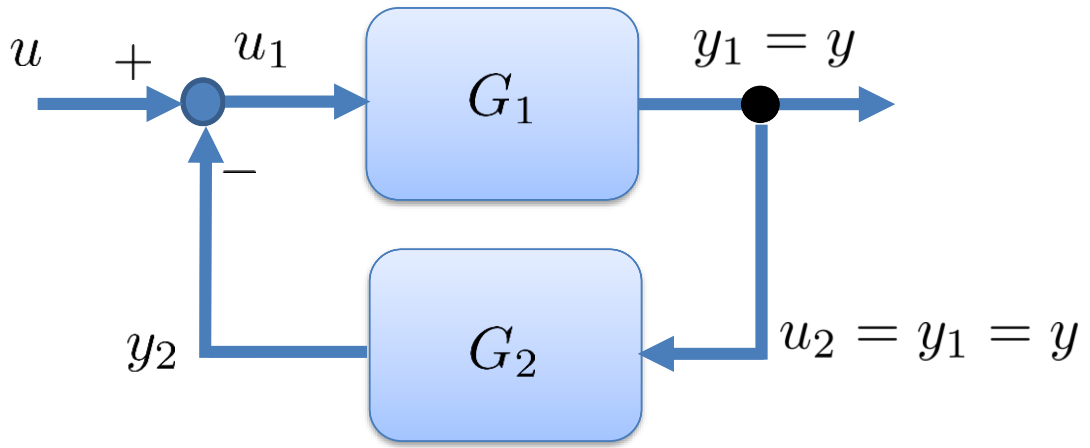
◆ Negative feedback: transfer function



- Transfer function: $\dfrac{Y(s)}{U(s)} = \dfrac{G_1(s)}{1 + G_2(s)G_1(s)}$

- How can we describe in the time-domain?

◆ Negative feedback: state-space equation



Let us assume that state-space equations are given by

$$G_1 : \begin{cases} \dfrac{dx_1}{dt} & = A_1 x_1 + B_1 u_1 \\ y_1 & = C_1 x_1 + D_1 u_1 \end{cases}, \qquad G_2 : \begin{cases} \dfrac{dx_2}{dt} & = A_2 x_2 + B_2 u_2 \\ y_2 & = C_2 x_2 + D_2 u_2 \end{cases}$$

Substituting $y_1 = u_2$ into G_1 leads to $\begin{cases} \dfrac{dx_1}{dt} & = A_1x_1 + B_1u_1 \\ u_2 & = C_1x_1 + D_1u_1 \end{cases}$

By substituting this into G_2, we obtain

$$\begin{cases} \dfrac{dx_2}{dt} & = A_2x_2 + B_2u_2 = A_2x_2 + B_2C_1x_1 + B_2D_1u_1 \\ y_2 & = C_2x_2 + D_2u_2 = C_2x_2 + D_2C_1x_1 + D_2D_1u_1 \end{cases}$$

On the other hand,

$$\begin{aligned} & u_1 = u - y_2 = u - C_2x_2 - D_2C_1x_1 - D_2D_1u_1 \\ \rightarrow\ & (I + D_2D_1)u_1 = u - C_2x_2 - D_2C_1x_1 \end{aligned}$$

Here, we should assume that $|(I + D_2 D_1)| \neq 0$.

By defining $E := (I + D_2 D_1)^{-1}$, we obtain

$$u_1 = -ED_2C_1x_1 - EC_2x_2 + Eu$$

Substituting this into $\dfrac{dx_1}{dt} = Ax_1 + B_1u_1$ and

$$\frac{dx_2}{dt} = A_2x_2 + B_2C_1x_1 + B_2D_1u_1 \text{ with } y = y_1 = C_1x_1 + D_1u_1$$

leads to the following:

$$
\begin{cases}
\dfrac{dx_1}{dt} & = (A_1 - B_1 E D_2 C_1) x_1 - B_1 E C_2 x_2 + B_1 E u \\
\dfrac{dx_2}{dt} & = B_2 (I - D_1 E D_2) C_1 x_1 + (A_2 - B_2 D_1 E C_2) x_2 + B_2 D_1 E u \\
y & = (I - D_1 E D_2) C_1 x_1 - D_1 E C_2 x_2 + D_1 E u
\end{cases}
$$

By defining $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we obtain the following:

$$
\begin{cases}
\dfrac{dx}{dt} & = \begin{bmatrix} A_1 - B_1 E D_2 C_1 & -B_1 E C_2 \\ B_2 (I - D_1 E D_2) C_1 & A_2 - B_2 D_1 E C_2 \end{bmatrix} x + \begin{bmatrix} B_1 E \\ B_2 D_1 E \end{bmatrix} u \\
y & = \begin{bmatrix} (I - D_1 E D_2) C_1 & -D_1 E C_2 \end{bmatrix} x + D_1 E u
\end{cases}
$$