1. Derive that impulse response and step response of the second order systems.

(Transfer function of the second order system: $H(s) = \dfrac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$)

Solution:

Impulse response: $y(t) = \mathcal{L}^{-1}\{\dfrac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}\} = \mathcal{L}^{-1}\{\dfrac{w_n^2}{(s+\zeta w_n)^2 + (\sqrt{1-\zeta^2}w_n)^2}\}$

$= \mathcal{L}^{-1}\{\dfrac{w_n}{\sqrt{1-\zeta^2}} \dfrac{\sqrt{1-\zeta^2}w_n}{(s+\zeta w_n)^2 + (\sqrt{1-\zeta^2}w_n)^2}\} = \dfrac{w_n}{\sqrt{1-\zeta^2}} e^{-\sigma t}(\sin w_d t)1(t)$

Step response: $y(t) = \mathcal{L}^{-1}\{\dfrac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}\dfrac{1}{s}\} = \mathcal{L}^{-1}\{\dfrac{w_n^2}{(s+\zeta w_n)^2 + (\sqrt{1-\zeta^2}w_n)^2}\dfrac{1}{s}\}$

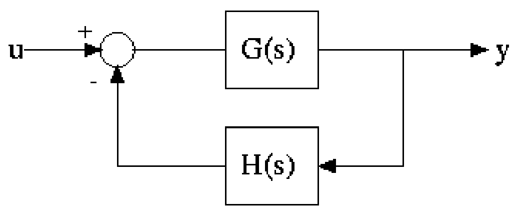
$= \mathcal{L}^{-1}\{\dfrac{1}{s} - \dfrac{s + 2\zeta w_n}{(s+\zeta w_n)^2 + (\sqrt{1-\zeta^2}w_n)^2}\}$

$= \mathcal{L}^{-1}\{\dfrac{1}{s} - \dfrac{s + \zeta w_n}{(s+\zeta w_n)^2 + (\sqrt{1-\zeta^2}w_n)^2} - \dfrac{\zeta}{\sqrt{1-\zeta^2}} \dfrac{\sqrt{1-\zeta^2}w_n}{(s+\zeta w_n)^2 + (\sqrt{1-\zeta^2}w_n)^2}\}$

$= (1 - e^{-\sigma t}(\sin w_d t + \dfrac{\zeta}{\sqrt{1-\zeta^2}} \cos w_d t))1(t)$

2. Derive the transfer function $T(s)$ of the following feedback system

$\Rightarrow T(s) = \dfrac{Y(s)}{U(s)} = ?$



Solution:

$$\begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

$$\begin{cases} (sI - A)X(s) = BU(s) \\ Y(s) = C(sI-A)^{-1}BU(s) + DU(s) \end{cases}$$

$$\therefore \quad \frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D$$

3. Transform the state-space equation to the transfer function

$$\begin{cases} \dfrac{dx}{dt} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Solution:

Let the input of the G(s) ; E(s)

$$\begin{cases} E(s) = U(s) - H(s)Y(s) \\ Y(s) = G(s)E(s) \end{cases}$$

transfer function: $\dfrac{Y(s)}{U(s)}$

From above two equations, $Y(s) = G(s)\{U(s) - H(s)Y(s)\}$

$\rightarrow \dfrac{Y(s)}{U(s)} = \dfrac{G(s)}{1 + G(s)H(s)}$

4. A single-input single-output system is given with usual realization with A, B, C, D, i.e.,

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

Note that transfer function of the system, $G(s)$, satisfies

$$G(s) = C(sI - A)^{-1}B + D$$

Prove that poles of $G(s)$ equals eigenvalues of A.
(In other word, $p \in \mathbb{C}$ is pole of $G(x)$ if and only if $p \in \mathbb{C}$ is an eigenvalue of A.)
Hint. Use well-defineness of the eigenvalue and the value of $G(s)$.

Solution:

i) Inverse of a square matrix M cannot be defined if and only if dimension of its nullspace is nonzero, i.e., there exists a vector v such that $Mv = 0$.
ii) Value of $G(s)$ cannot be defined if and only if $s \in \mathbb{C}$ is pole of $G(s)$.

By combine 'i), ii)', problem solved.

5. We will construct input $u(t)$ to stabilize following system.

$$\frac{dx}{dt} = Ax + Bu$$

where $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

1) Find eigenvalues and eigenvectors of A.Then, discuss about the stability of this system.

Solution:

$|\lambda I - A| = \lambda^2 - \lambda = \lambda(\lambda - 1) = 0 \;\Rightarrow\; \lambda_1 = 0,\; \lambda_2 = 1$

$\lambda_1 = 0 \;\Rightarrow\;$ eigenvector $v_1 = [10]^T$

$\lambda_2 = 1 \;\Rightarrow\;$ eigenvector $v_2 = [1 - 1]^T$

By problem 4, the eigenvalues of A are equivalent to the poles of this system's transfer function.

There is RHP pole ($s = 1$), so this system is unstable.

2) Then, we construct feedback system with $u(t) = -Kx(t)$, design such a $K = [k_1 \; k_2]$ with which the poles of this closed loop system is located in $-1 \pm j$.

Solution:

$$\frac{dx}{dt} = Ax + Bu = Ax + B(-Kx) = (A - BK)x = \begin{bmatrix} 0 & 1 \\ -k_1 & 1 - k_2 \end{bmatrix} x$$

$|\lambda I - (A - BK)| = \lambda^2 - (1 - k_2)\lambda + k_1 = 0$

We can determine $k_1,\; k_2$ by relation between roots and polynomial's coefficients.

Summation of 2 roots : $1 - k_2 = (-1 + j) + (-1 - j) = -2 \;\Rightarrow\; k_2 = 3$

Multiplication of 2 roots : $k_1 = (-1 + j)(-1 - j) = 2 \;\Rightarrow\; k_1 = 2$

$\therefore\; K = [k_1 \quad k_2] = [2 \quad 3]$