EECE322-01: 자동제어공학개론

Introduction to Automatic Control

Chapter 6: Frequency Response Analysis

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◆ The main objectives of this chapter are

1. Frequency response
2. Bode plots
3. Nyquist plot and Nyquist stability criterion
4. Stability margins
5. Effect of time-delay

1. Frequency response

Re-analysis of frequency response

- Frequency response: Linear system's response to sinusoidal inputs. Assume zero initial condition !!

$A\sin(\omega t) \rightarrow$ LTI system $\rightarrow AM(\omega)\sin(\omega t+\phi)$ **at steady state!!**

System: $G(s)=\dfrac{Y(s)}{U(s)}$

e.g., $\ddot{y}+3\dot{y}+2y = A\sin\omega t$

$$y_p = p_1\sin\omega t + p_2\cos\omega t$$
$$y_h = h_1 e^{-t} + h_2 e^{-2t}$$

$$y = y_p + y_h \rightarrow y_p \ (t\rightarrow\infty)$$

Input: $u(t)=A\sin(\omega_0 t)1(t) \rightarrow U(s)=\frac{A\omega_0}{s^2+\omega_0^2}$

$$\left(y(0)=0\right) \rightarrow Y(s)=G(s)\frac{A\omega_0}{s^2+\omega_0^2}$$

Output: compute $y(t)$

Assuming the poles of $G(s)$ are distinct:

$$Y(s)=\frac{\alpha_1}{s-p_1}+\frac{\alpha_2}{s-p_2}+\cdots+\frac{\alpha_n}{s-p_n}+\frac{\alpha_0}{s+j\omega_0}+\frac{\alpha_0^*}{s-j\omega_0}$$

$$\rightarrow y(t)=\alpha_1 e^{p_1 t}+\alpha_2 e^{p_2 t}+\cdots+\alpha_n e^{p_n t}+\alpha_0 e^{-j\omega_0 t}+\alpha_0^* e^{j\omega_0 t}$$

$$Y(s)=\frac{\alpha_1}{s-p_1}+\frac{\alpha_2}{s-p_2}+\cdots+\frac{\alpha_n}{s-p_n}+\frac{\alpha_0}{s+j\omega_0}+\frac{\alpha_0^*}{s-j\omega_0}$$

$$\to y(t)=\alpha_1 e^{p_1 t}+\alpha_2 e^{p_2 t}+\cdots+\alpha_n e^{p_n t}+2|\alpha_0|\sin(\omega_0 t+\phi)$$

- Let $G(j\omega_0)=M(\omega_0)e^{j\phi(\omega_0)}$

$$\left(M(\omega_0)=|G(j\omega_0)|=|G(s)|_{s=j\omega_0},\ \phi(\omega_0)=\tan^{-1}\frac{\mathrm{Im}[G(j\omega_0)]}{\mathrm{Re}[G(j\omega_0)]}=\angle G(j\omega_0).\right)$$

- Then,

$$\alpha_0^*=\left[G(s)\frac{A\omega_0}{s^2+\omega_0^2}(s-j\omega_0)\right]_{s=j\omega_0}=\left[G(s)\frac{A\omega_0}{s+j\omega_0}\right]_{s=j\omega_0}=G(j\omega_0)\frac{A}{2j}$$

$$\alpha_0=(\alpha_0^*)^*=\left[G(s)\frac{A\omega_0}{s^2+\omega_0^2}(s+j\omega_0)\right]_{s=-j\omega_0}=\left[G(s)\frac{A\omega_0}{s-j\omega_0}\right]_{s=-j\omega_0}=G(-j\omega_0)\frac{A}{-2j}$$

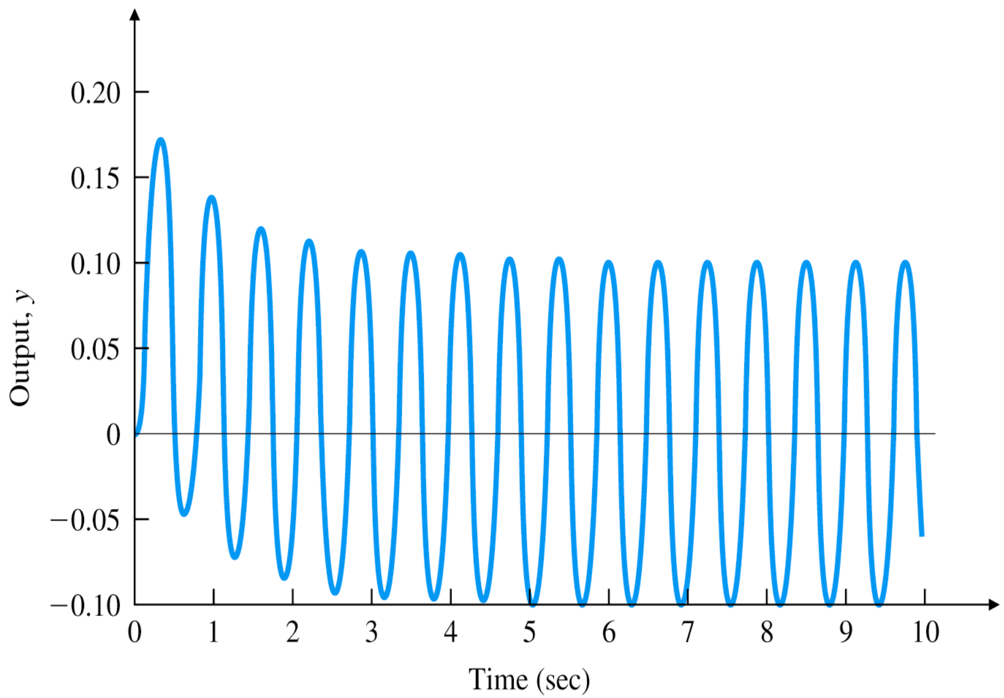
$$\frac{\alpha_0}{s+j\omega_0}+\frac{\alpha_0^*}{s-j\omega_0}\to G(-j\omega_0)\frac{A}{-2j}e^{-j\omega_0 t}+G(j\omega_0)\frac{A}{2j}e^{j\omega_0 t}$$

$$=\frac{jAM(\omega_0)}{2}e^{-j(\phi(\omega_0)+\omega_0 t)}-\frac{jAM(\omega_0)}{2}e^{j(\phi(\omega_0)+\omega t)}$$

$$=AM(\omega_0)\sin(\omega_0 t+\phi(\omega_0))$$

Example of frequency response

- Recall Example 3.5:



$$G(s) = \frac{1}{s+1}, \quad u(t) = A\sin(\omega_0 t)1(t)$$

- The natural response e^{-t} decays to zero as time grows.

- If the system is stable, the natural response disappears after several time constants.

Magnitude and phase

- Frequency response for a stable system with $G(s)$:

$$A\sin(\omega t) \longrightarrow \boxed{\text{LTI system}} \longrightarrow AM(\omega)\sin(\omega t+\phi)$$ **at steady state!!**

Response to sinusoid (SS): $y(t) = AM\sin(\omega_0 t+\phi)$,

$$M = |G(j\omega_0)| = |G(s)|_{s=j\omega_0}, \longleftarrow \textbf{Magnitude of } \boldsymbol{G(jw_0)}$$

$$\phi = \tan^{-1}\frac{\mathrm{Im}[G(j\omega_0)]}{\mathrm{Re}[G(j\omega_0)]} = \angle G(j\omega_0). \longleftarrow \textbf{Phase of } \boldsymbol{G(jw_0)}$$

- Polar form: $G(j\omega) = M(\omega)e^{j\phi(\omega)}$ $\left(M(\omega) = \text{ magnitude, } \phi(\omega) \ = \ \text{phase}\right)$

- The steady-state response w.r.t. sinusoidal function is the sinusoidal with the same frequency, magnitude multiplied by M, phase shift ϕ.

- Nonlinear or time-varying systems might contain other frequencies.

Note: “w.r.t.” = “with respect to”

Example

- Example 6.2: Frequency-Response Characteristics of a Lead Compensator

$$D(s) = K\frac{Ts+1}{\alpha Ts+1},\quad \alpha<1\ \left(\text{zero}: -1/T,\ \text{pole}: -1/\alpha T\right)$$

$$K=1,\ T=1,\ \alpha=0.1$$

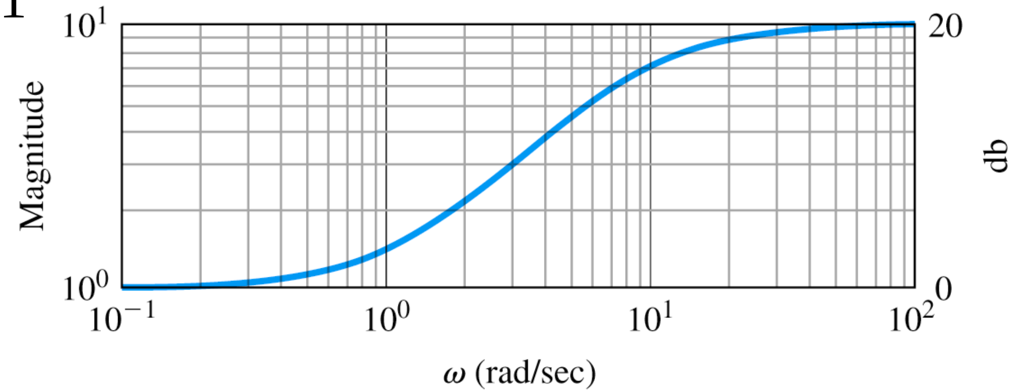
$$D(j\omega) = K\frac{Tj\omega+1}{\alpha Tj\omega+1}$$

$$M(\omega) = \left|D(j\omega)\right| = |K|\frac{\sqrt{1+(\omega T)^2}}{\sqrt{1+(\alpha\omega T)^2}}$$

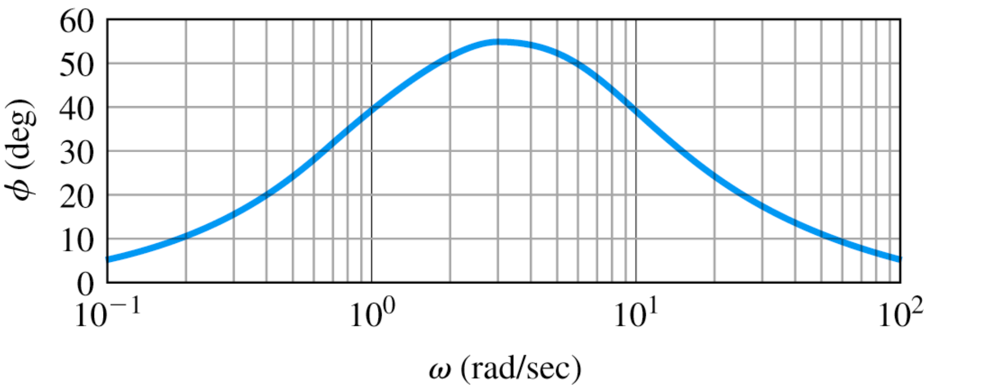
$$\phi(\omega) = \angle(1+j\omega T) - \angle(1+j\alpha\omega T)$$
$$= \tan^{-1}(\omega T) - \tan^{-1}(\alpha\omega T)$$

$$\begin{cases} \omega \ll 1 \Rightarrow M(\omega) \cong |K| = 1 \\ \omega \gg 1 \Rightarrow M(\omega) \cong |K/\alpha| = 10 \end{cases}$$

$$\begin{cases} \omega \ll 1 \text{ or } \omega \gg 1 \Rightarrow \phi(\omega) \cong 0, \\ \omega \ge 0 \end{cases}$$



(a)



(b)

Characteristics of frequency response

- Experimental determination of frequency response:
  - Excite the system with a sinusoid $\left(\text{e.g., } u(t) = \sin \omega t\right)$ varying in frequency.
  - Measure $M(\omega)$ and $\phi(\omega)$ in the steady state at each frequency.

- The dynamic response of the system can be determined from the knowledge of $M(\omega)$ and $\phi(\omega)$ of its transfer function.
  - Use Fourier series to compute the steady-state response for periodic input.
  - Is there a realtion between $(M(\omega),\ \phi(\omega))$ and the transient response?

Characteristics of frequency response

- There is some relation between the frequency response $(M(\omega), \phi(\omega))$ and the transient response.

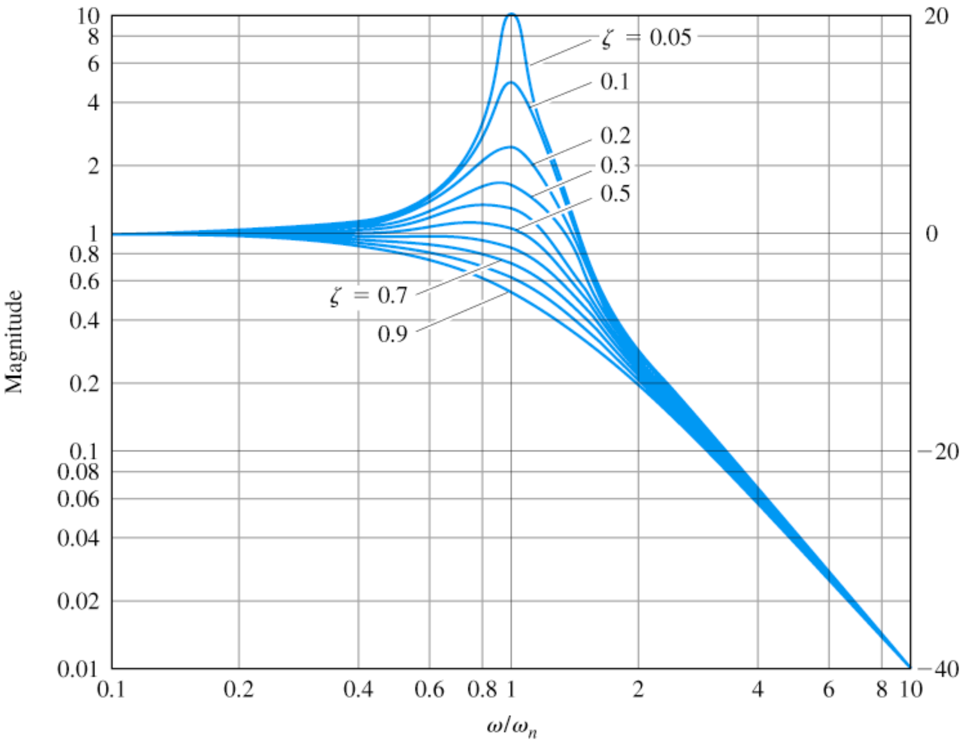
  - Damping can be determined from
    * the transient response overshoot
    * the peak in the magnitude of frequency response

  - The natural frequency ω_n is approx. equal to the bandwidth.
    → the rise time can be determined from the bandwidth.

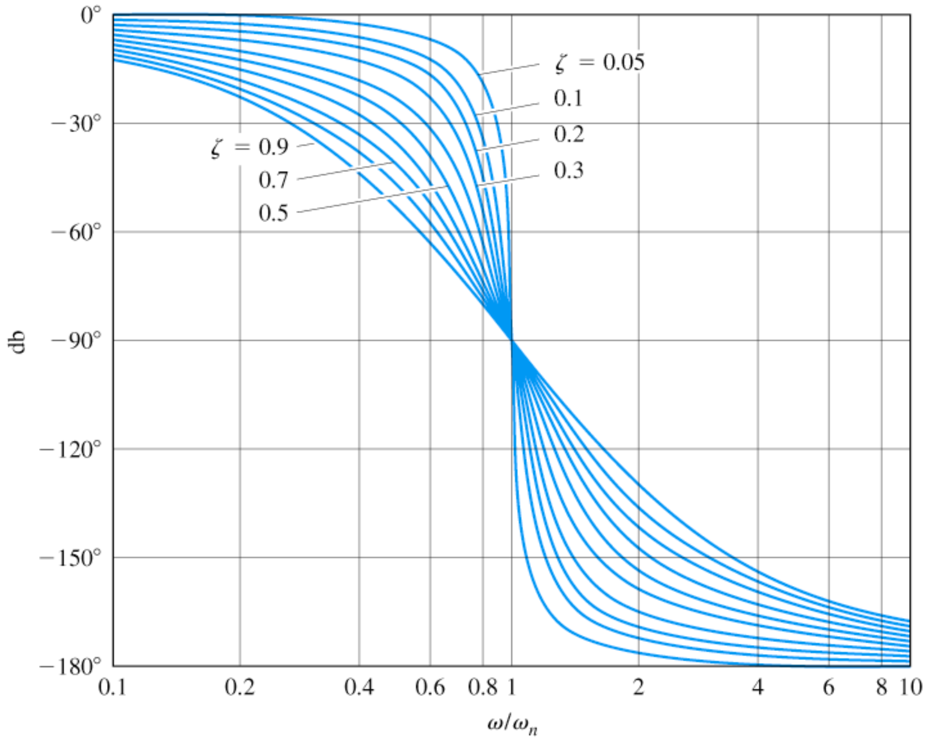
- The peak overshoot in frequency response $\cong 1/2\zeta$ for $\zeta < 0.5$
  → the peak overshoot in the step response can be determined from the peak overshoot in the freq. resp.

Example

- Frequency response of $G(s)=\dfrac{1}{(s/\omega_n)^2+2\zeta(s/\omega_n)+1}$



Magnitude $M(\omega)$

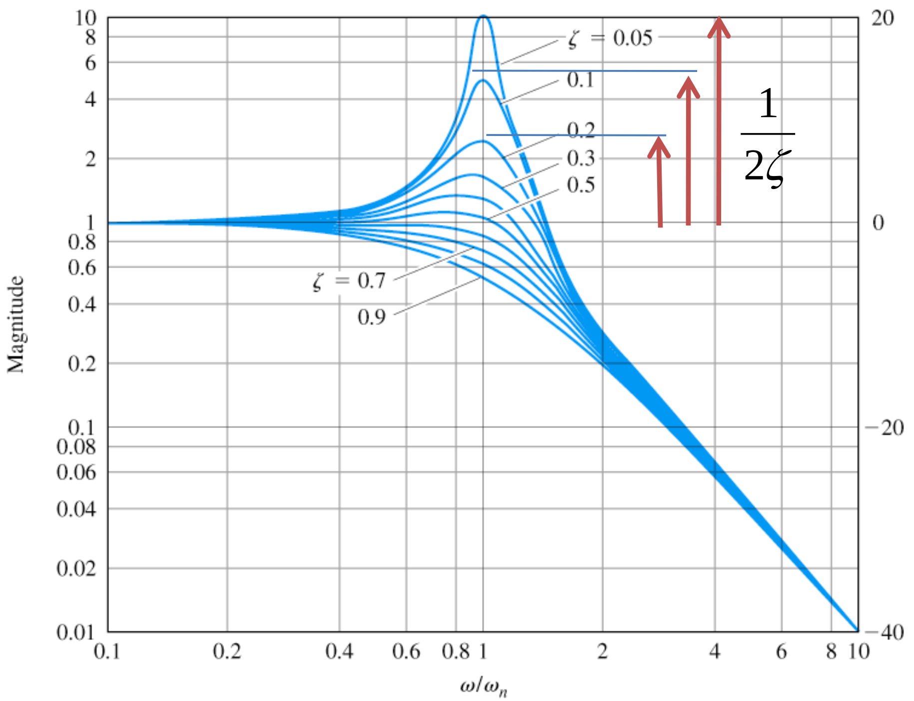


Phase $\phi(\omega)$

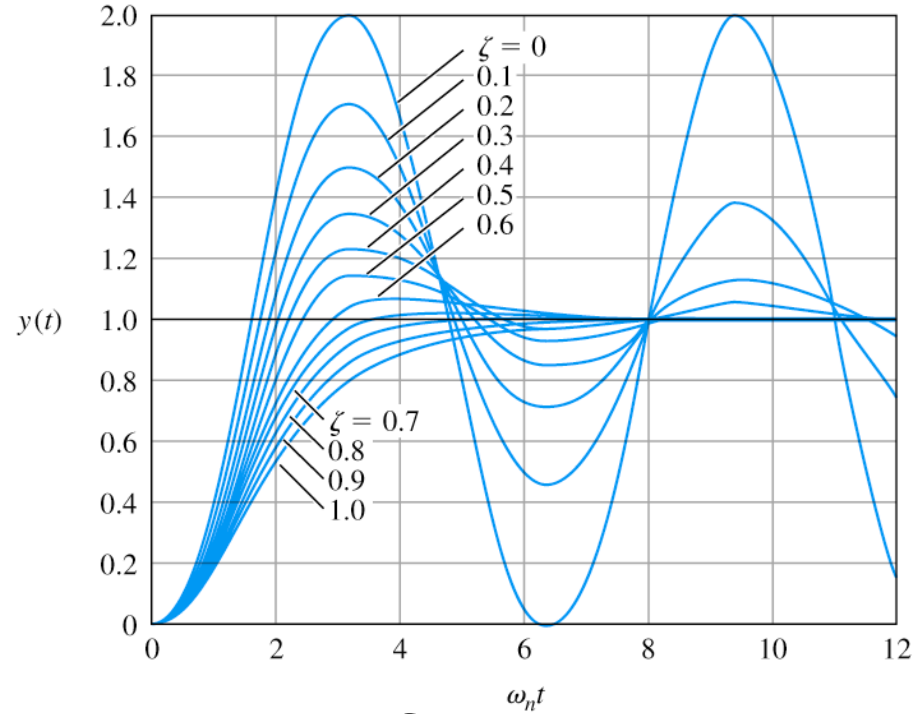
$$|G(j\omega_n)| = \frac{1}{\left|\left(\frac{j\omega_n}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega_n}{\omega_n}\right) + 1\right|} = \left|\frac{1}{2j\zeta}\right| = \frac{1}{2\zeta}$$

Interpretation from frequency response

- Frequency response vs Step response of $G(s)=\dfrac{1}{(s/\omega_n)^2+2\zeta(s/\omega_n)+1}$



Frequency response



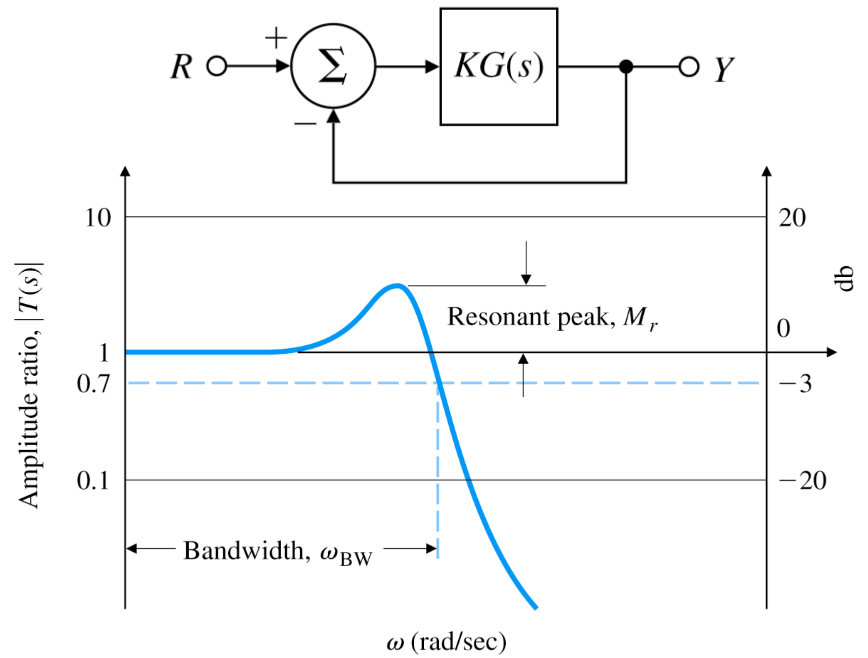
Step response

- Peak overshoot in frequency response: $\dfrac{1}{2\zeta}, \zeta < 0.5$

- Damping can be determined from the **overshoot in the step response** or from the **peak in the magnitude on the frequency response**.

Frequency response for closed-loop systems

- Bandwidth and resonance peak



- Closed-loop transfer function

$$\frac{Y(s)}{R(s)} := \mathrm{T}(s) = \frac{KG(s)}{1+KG(s)}$$

Typically, $\begin{cases} |\mathrm{T}| \cong 1 \text{ for low frequencies} \\ |\mathrm{T}| < 1 \text{ for high frequencies} \end{cases}$

- Resonance peak M_r : Max value of frequency response magnitude

- Bandwidth ω_{BW}: maximum frequency at which the output of the system tracks an input sinusoid in a satisfactory manner

$\rightarrow$ Frequency of the sinusoidal input r at which the output y is attenuated by a factor of 0.707(=1/$\sqrt{2}$) times the input r. (valid for low-pass filter)

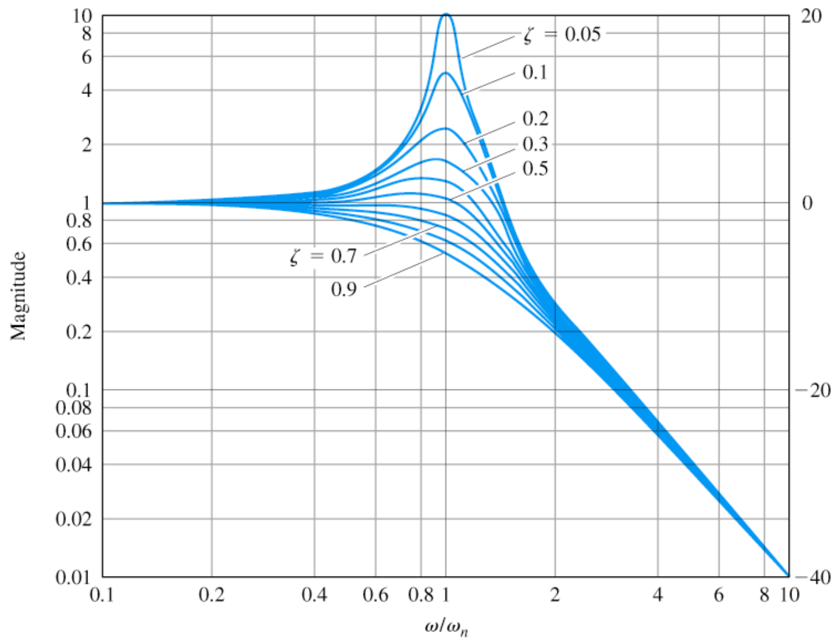
Frequency response for closed-loop systems

- Second order system with closed-loop transfer function

$$\mathrm{T}(s) = \frac{1}{(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1}$$

- the bandwidth is equal to the natural frequency (i.e., $\omega_n = \omega_{BW}$) for $\zeta = 0.7$.
- For other damping ratios, the bandwidth is approximately equal to the natural frequency with an error typically less than a factor of 2. (i.e., $\omega_{BW} \leq 2\omega_n$)

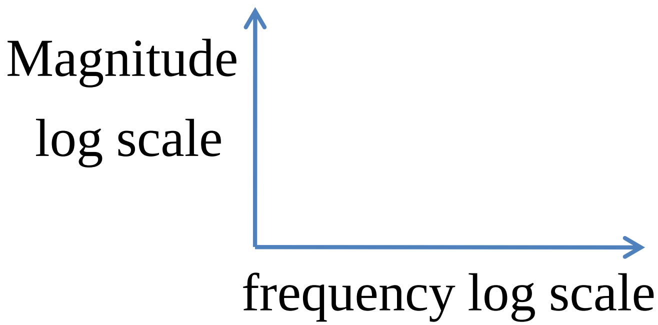
Closed-loop frequency response

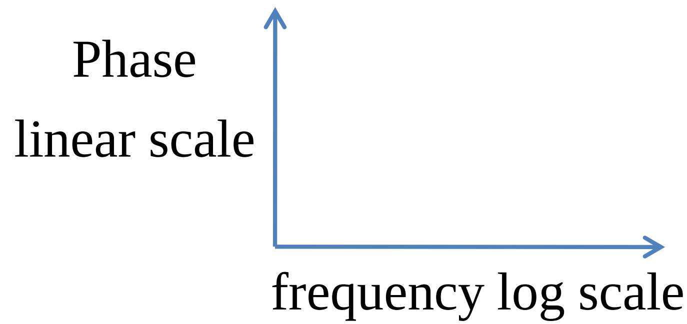


2. Bode plots

Bode plots

- Bode plot of G(s): plots of magnitude $|G(j\omega)|$ and phase $\angle G(j\omega)$





- Basic properties of complex numbers

- Polar form: $G(j\omega) = \dfrac{\vec{s}_1\vec{s}_2}{\vec{s}_3\vec{s}_4\vec{s}_5} = \dfrac{r_1 e^{j\theta_1} r_2 e^{j\theta_2}}{r_3 e^{j\theta_3} r_4 e^{j\theta_4} r_5 e^{j\theta_5}} = \left(\dfrac{r_1 r_2}{r_3 r_4 r_5}\right) e^{j(\theta_1+\theta_2-\theta_3-\theta_4-\theta_5)}$

$$\rightarrow \log_{10} G(j\omega) = \log_{10} M(\omega) e^{j\phi(\omega)} = \log_{10} M(\omega) + j\phi(\omega)\log_{10} e$$

$$\rightarrow \log_{10} |G(j\omega)| = \log_{10} r_1 + \log_{10} r_2 - \log_{10} r_3 - \log_{10} r_4 - \log_{10} r_5$$

$$\rightarrow \angle G(j\omega) = \phi(\omega) = \theta_1 + \theta_2 - \theta_3 - \theta_4 - \theta_5$$

Characteristics of Bode plots

- Decibel(db):
  -usually used in communications.
  -to measure the power gain in decibels.

$$\left|G\right|_{\text{db}} = 10\log_{10}\frac{P_2}{P_1}$$

$$\left|G\right|_{\text{db}} = 20\log_{10}\frac{V_2}{V_1} \quad \left(P_1 \propto V_1^2,\ P_2 \propto V_2^2\right)$$

- Advantages of Bode plots
  1. **Dynamic compensator design** can be done by using Bode plots
  2. Bode plots **can be determined experimentally**.
  3. Bode plots of systems in series simply add.
  4. Log scale permits a much wider range of frequencies to be displayed.

Sketching method of Bode plots

- Plotting Bode plots
  - it is convenient to write the transfer functions in Bode form:

$$KG(s) = K\frac{(s-z_1)(s-z_2)\cdots}{(s-p_1)(s-p_2)\cdots} \Rightarrow KG(j\omega) = K_0\frac{(j\omega\tau_1+1)(j\omega\tau_2+1)\cdots}{(j\omega\tau_a+1)(j\omega\tau_b+1)\cdots}$$

$$\angle(KG(j\omega)) = \angle(j\omega\tau_1+1)+\angle(j\omega\tau_2+1)+\cdots-\angle(j\omega\tau_a+1)-\angle(j\omega\tau_b+1)-\cdots$$

$$\left|KG(j\omega)\right|_{db} = \left|(j\omega\tau_1+1)\right|_{db}+\left|(j\omega\tau_2+1)\right|_{db}+\cdots-\left|(j\omega\tau_a+1)\right|_{db}-\left|(j\omega\tau_b+1)\right|_{db}-\cdots$$

$$|KG(j\omega)| = |K_0|\frac{|j\omega\tau_1+1||j\omega\tau_2+1|\cdots}{|j\omega\tau_a+1||j\omega\tau_b+1|\cdots}$$

Example

- Example: $KG(j\omega) = K_0 \dfrac{j\omega\tau_1 + 1}{(j\omega)^2 (j\omega\tau_a + 1)}.$

$$KG\left(j\omega\right) = K_0 \frac{j\omega\tau_1 + 1}{\left(j\omega\right)^2 \left(j\omega\tau_a + 1\right)}$$

$$\angle KG\left(j\omega\right) = \angle K_0 + \angle(j\omega\tau_1 + 1) - \angle(j\omega)^2 - \angle(j\omega\tau_a + 1)$$

$$\log\left|KG\left(j\omega\right)\right| = \log_{10}|K_0| + \log_{10}|j\omega\tau_1 + 1| - \log_{10}|(j\omega)^2| - \log_{10}|j\omega\tau_a + 1|$$

$$\left|KG\left(j\omega\right)\right|_{\text{db}} = 20(\log_{10}|K_0| + \log_{10}|j\omega\tau_1 + 1| - \log_{10}|(j\omega)^2| - \log_{10}|j\omega\tau_a + 1|)$$

Fundamentals for Bode plots

- Elements of Bode plots (three classes)

1. $K_0(j\omega)^n$ $\qquad$ $K_0 s^n$

2. $(j\omega\tau+1)^{\pm 1}$ $\qquad$ $(\tau s+1)^{\pm 1}$

3. $\left[\left(\frac{j\omega}{\omega_n}\right)^2+2\zeta\frac{j\omega}{\omega_n}+1\right]^{\pm 1}$ $\qquad$ $\left[\left(\frac{s}{\omega_n}\right)^2+2\zeta\left(\frac{s}{\omega_n}\right)+1\right]^{\pm 1}$

→ Draw Bode plots of each components and combine them (just add).

Case study for Bode plots

- Class 1: $K_0(j\omega)^n$, Singularities at the origin

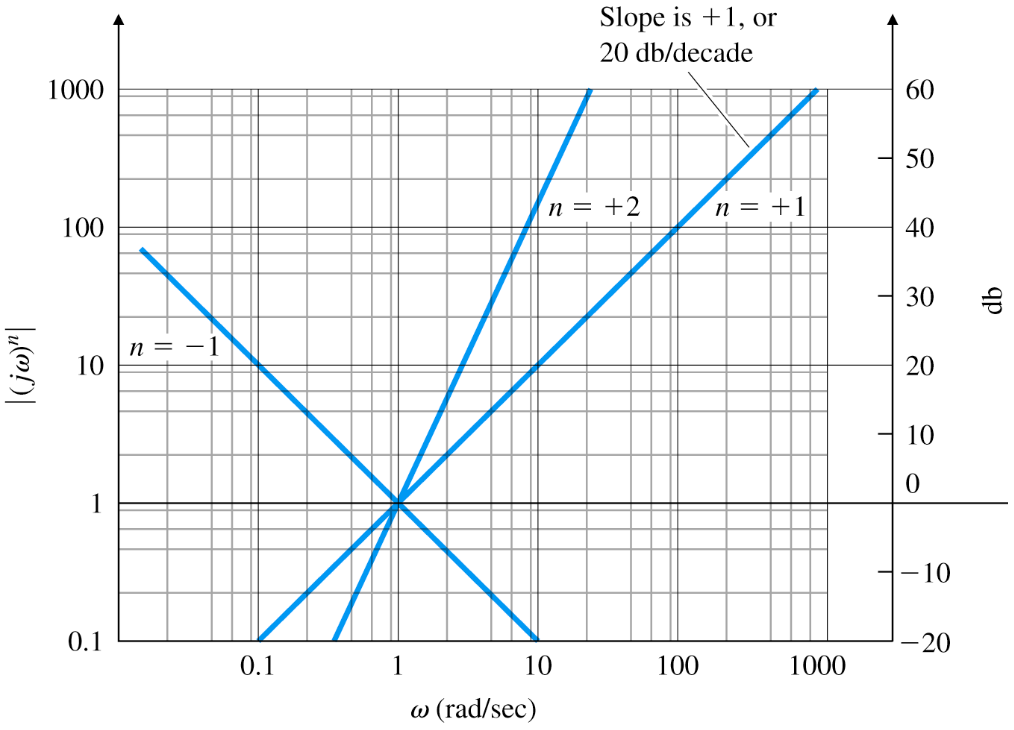
$$\log\left|K_0(j\omega)^n\right| = \log\left|K_0\right| + n\log\left|j\omega\right|$$

$$\left(Y = K + n\Omega,\ \ K = \log\left|K_0\right|, \Omega = \log\omega\right)$$

$$\rightarrow 20\log\left|K_0(j\omega)^n\right| = 20\log\left|K_0\right| + 20n\log\omega\ \ (\text{slope of } 20n \text{ db per decade})$$

$$\angle(j\omega)^n = n\times 90^\circ$$

$$\left(= \pm 90^\circ, \pm 180^\circ, \cdots \text{ for } n = \pm 1, \pm 2, \cdots\right)$$



Case study for Bode plots

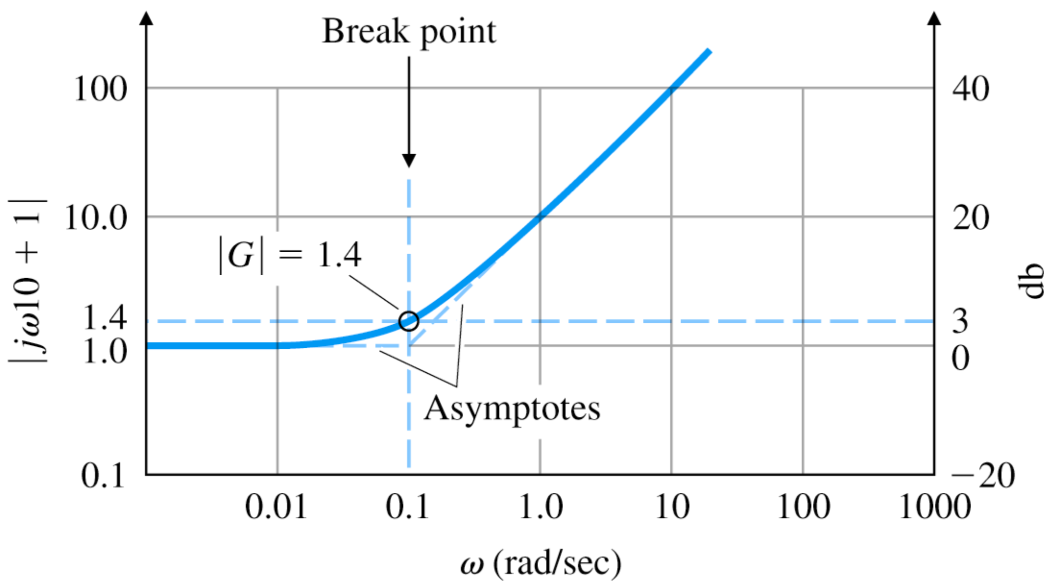
- Class 2: $(j\omega\tau+1)$, First-order term

$$\begin{cases} \omega\tau << 1 \Rightarrow j\omega\tau+1 \cong 1 \Rightarrow \log|j\omega\tau+1| \cong 0 \\ \omega\tau >> 1 \Rightarrow j\omega\tau+1 \cong j\omega\tau \Rightarrow \log|j\omega\tau+1| \cong \log|j\omega\tau| = \log|\tau| + \log|j\omega| \end{cases}$$

$$\left(\omega = 1/\tau = \text{break point} \leftarrow \log|\tau| + \log|j(1/\tau)| = 0\right)$$

- 2 asymptotes crosses at the break point, and the actual magnitude lies above that point by a factor of 1.4 (or + 3db).

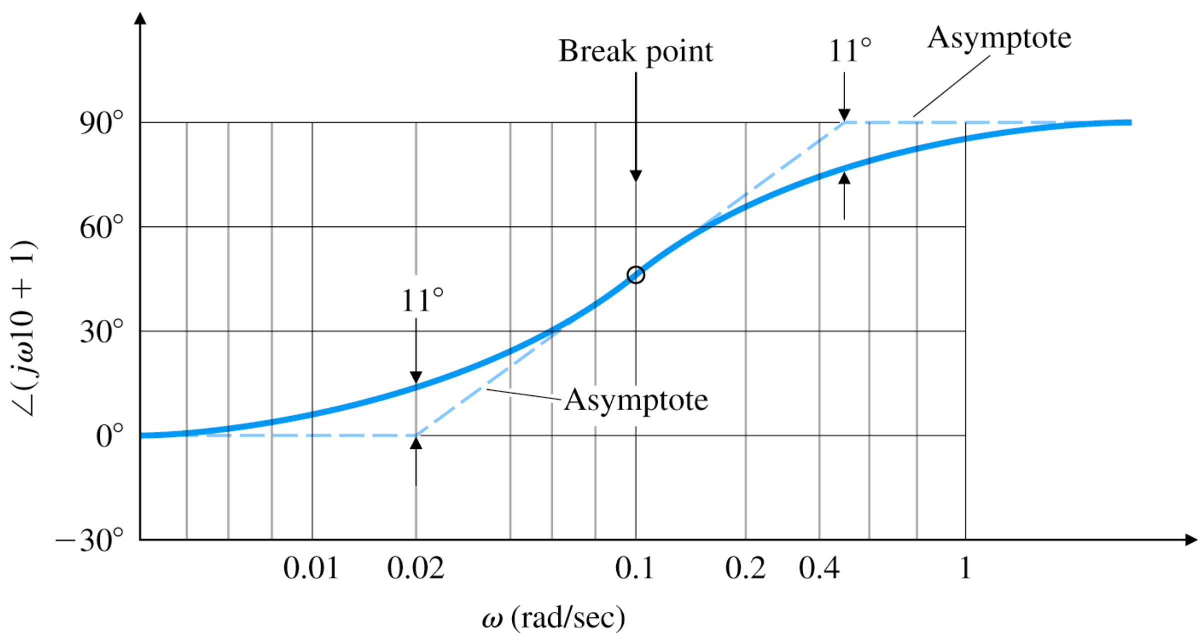
Magnitude plot



$$\begin{cases} \omega\tau << 1 \Rightarrow \angle(j\omega\tau+1) \cong \angle 1 = 0^\circ \\ \omega\tau >> 1 \Rightarrow \angle(j\omega\tau+1) \cong \angle j\omega\tau = 90^\circ \\ \omega\tau \cong 1 \Rightarrow \angle(j\omega\tau+1) \cong 45^\circ \ \ (\omega = 1/\tau = \text{break point}) \end{cases}$$

- For $\omega\tau \cong 1$, the $\angle(j\omega\tau+1)$ curve is tangent to an asymptote going from 90° at $\omega\tau = 0.2\left(\omega = 0.2\frac{1}{\tau} = \frac{1}{5}\frac{1}{\tau}\right)$ to 90° at $\omega\tau = 5\left(\omega = 5\frac{1}{\tau}\right)$.
- Actual phase curve deviates from the asymptotes by 11° at their intersections.

Phase plot



Example

- Example 6.3: Bode plot for real poles and zeros

$$KG(s)=\frac{2000(s+0.5)}{s(s+10)(s+50)} \rightarrow KG(j\omega)=\frac{2000\cdot 0.5\left(1+\frac{j\omega}{0.5}\right)}{(j\omega)\cdot 10\left(1+\frac{j\omega}{10}\right)\cdot 50\left(1+\frac{j\omega}{50}\right)}$$

$$KG(j\omega)=\frac{2[(j\omega/0.5)+1]}{j\omega[(j\omega/10)+1][(j\omega/50)+1]}=\frac{2}{j\omega}\cdot\frac{\left(1+\frac{j\omega}{0.5}\right)}{\left(1+\frac{j\omega}{10}\right)\cdot\left(1+\frac{j\omega}{50}\right)}$$

Low-frequency asymptote: $n=-1$,

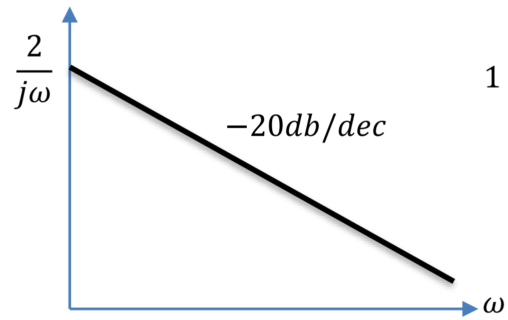
$$|G(j\omega)|\cong 2/\omega$$

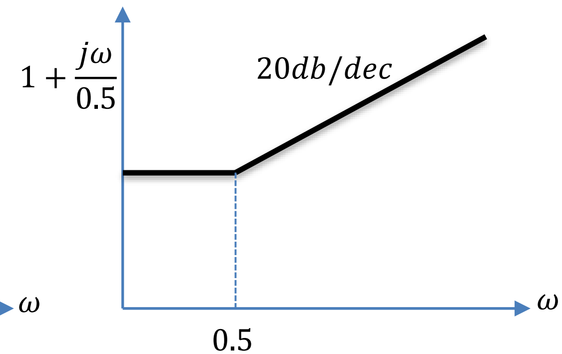
1st-order zero: $\omega=0.5$

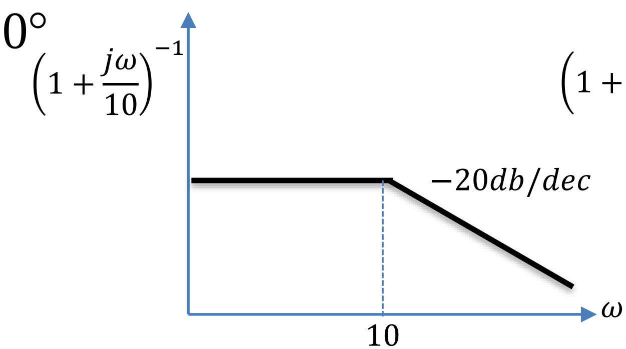
1st-order pole: $\omega=10$

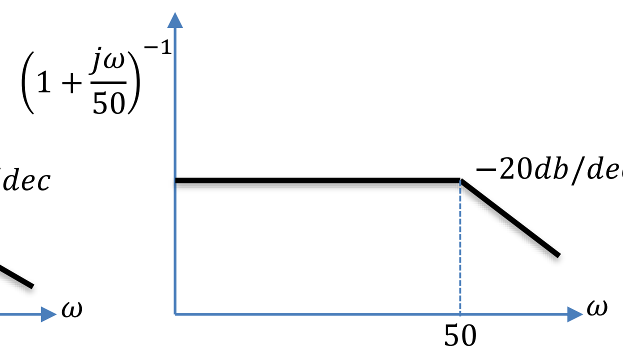
1st-order pole: $\omega=50$

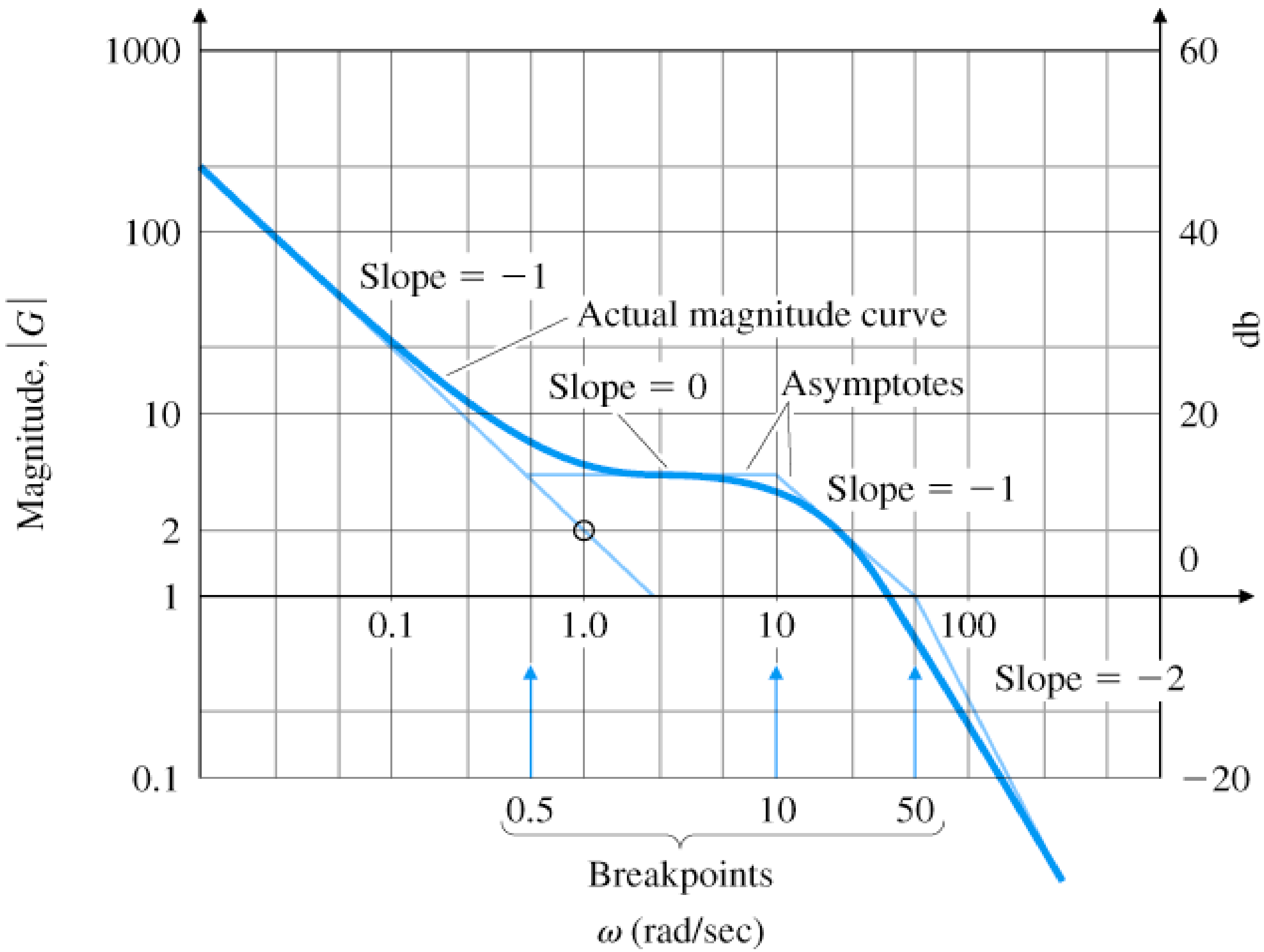
$$\phi=-90^\circ\rightarrow\cong 0^\circ\rightarrow\cong -90^\circ\rightarrow -180^\circ$$



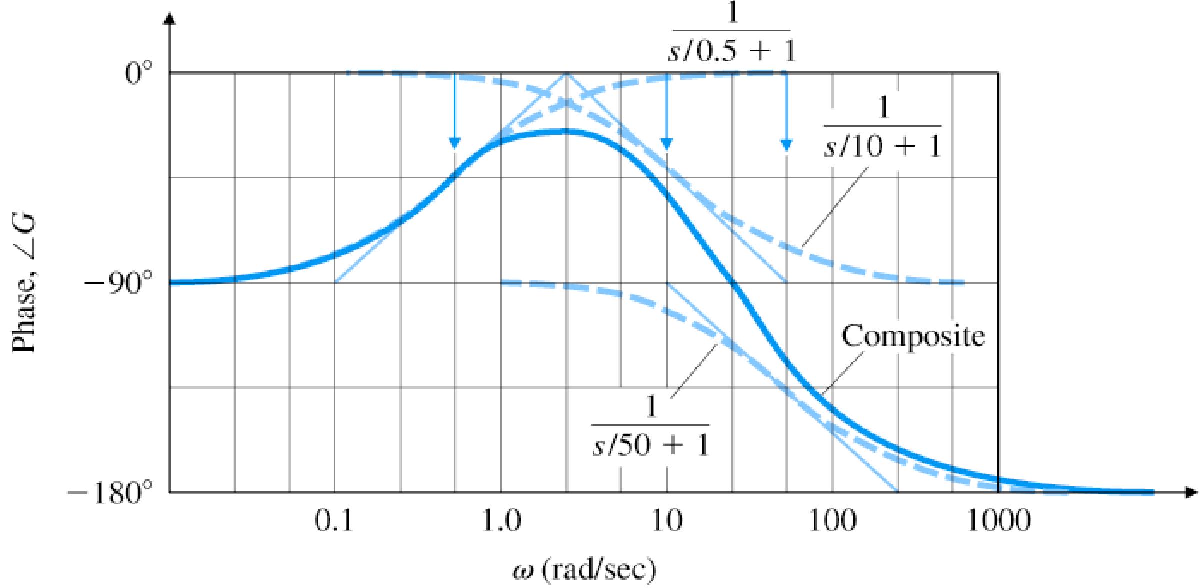




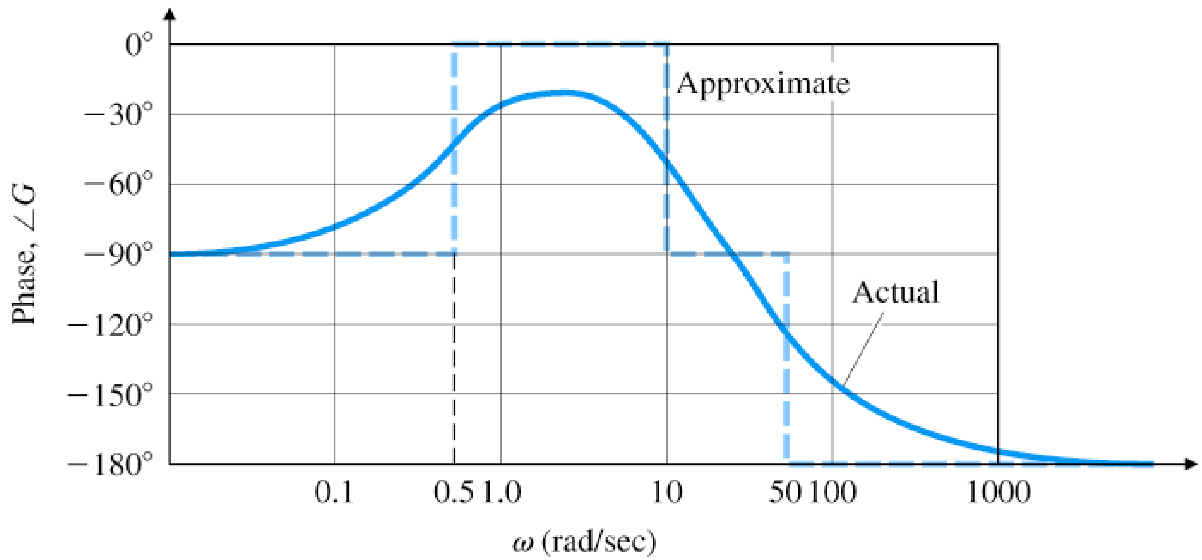



1000
100
10
2
1
0.1
60
40
20
0
−20
Magnitude, |G|
db
Slope = −1
Actual magnitude curve
Slope = 0
Asymptotes
Slope = −1
Slope = −2
0.1
1.0
10
100
0.5
10
50
Breakpoints
ω (rad/sec)
(a)

$$KG(j\omega) = \frac{2}{j\omega} \cdot \frac{\left(1 + \frac{j\omega}{0.5}\right)}{\left(1 + \frac{j\omega}{10}\right) \cdot \left(1 + \frac{j\omega}{50}\right)}$$



(b)



Case study for Bode plots

- Class 3: $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{\pm 1}$, Second-order term

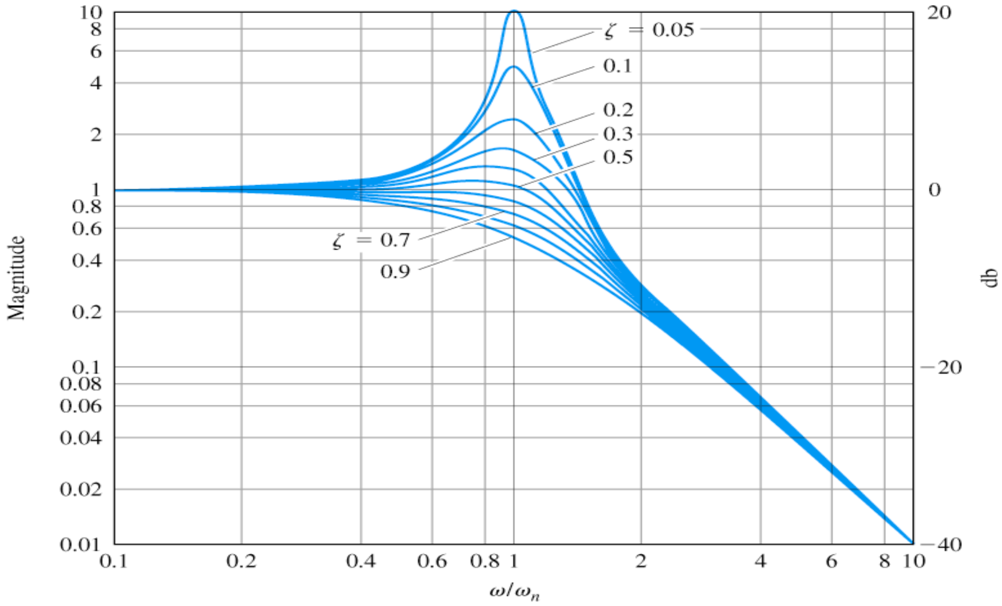
$$\begin{cases} \omega << \omega_n \Rightarrow G(j\omega) \cong 1 \Rightarrow \log\left|G(j\omega)\right| \cong 0 \\ \omega >> \omega_n \Rightarrow G(j\omega) \cong \left(j\omega/\omega_n\right)^{\pm 2} \Rightarrow \log\left|G(j\omega)\right| \cong \pm 2\log\left|1/\omega_n\right| \pm 2\log\left|\left(j\omega\right)\right| \end{cases}$$

- Break point: $\omega = \omega_n$

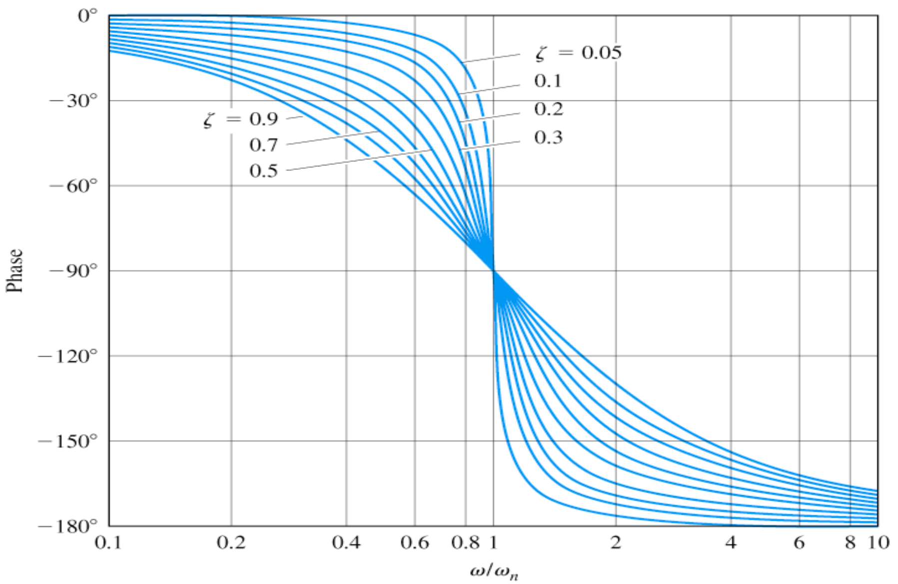
- The magnitude changes slope by a factor of ± 2 (or ± 40 db/ decade)

$$\left(\left|G(j\omega)\right| = \frac{1}{2\zeta} \text{ at } \omega = \omega_n \text{ for } n = -1\right)$$

- The phase changes by $\pm 180°$, and the transition through the break point region varies with the damping ratio ζ.



Magnitude plot of $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{-1}$



Phase plot of $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{-1}$

Example

- Example 6.4: Bode plot with complex poles

$$KG(s)=\frac{10}{s(s^2+0.4s+4)}$$

$$KG(s)=\frac{10}{4}\frac{1}{s(s^2/4+2(0.1)s/2+1)}$$

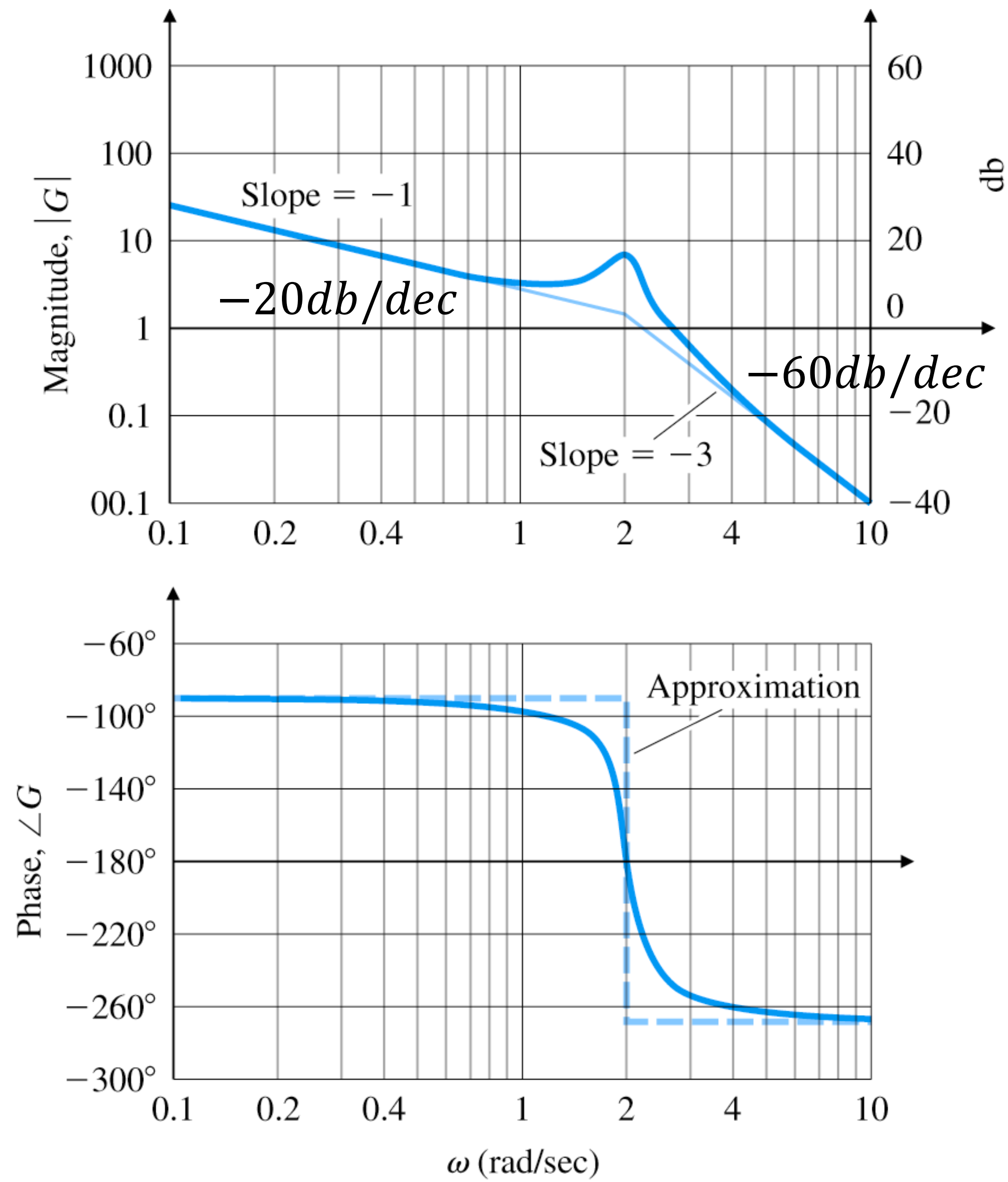
Low-frequency asymptote: $n=-1$,

$$|G(j\omega)|\cong 2.5/\omega$$

2nd-order pole: $\omega_n=2,\ \zeta=0.1$

$\rightarrow 1/2\zeta=1/0.2=5$

$\phi=-90^\circ\rightarrow-180^\circ\rightarrow-270^\circ$



Example

- Example 6.5: Bode plot with complex poles and zeros

$$KG(s) = \frac{0.01(s^2 + 0.01s + 1)}{s^2[(s^2/4) + 0.02(s/2) + 1]}$$

Low-frequency asymptote: $n = -2$,

$|G(j\omega)| \cong 0.01/\omega^2$

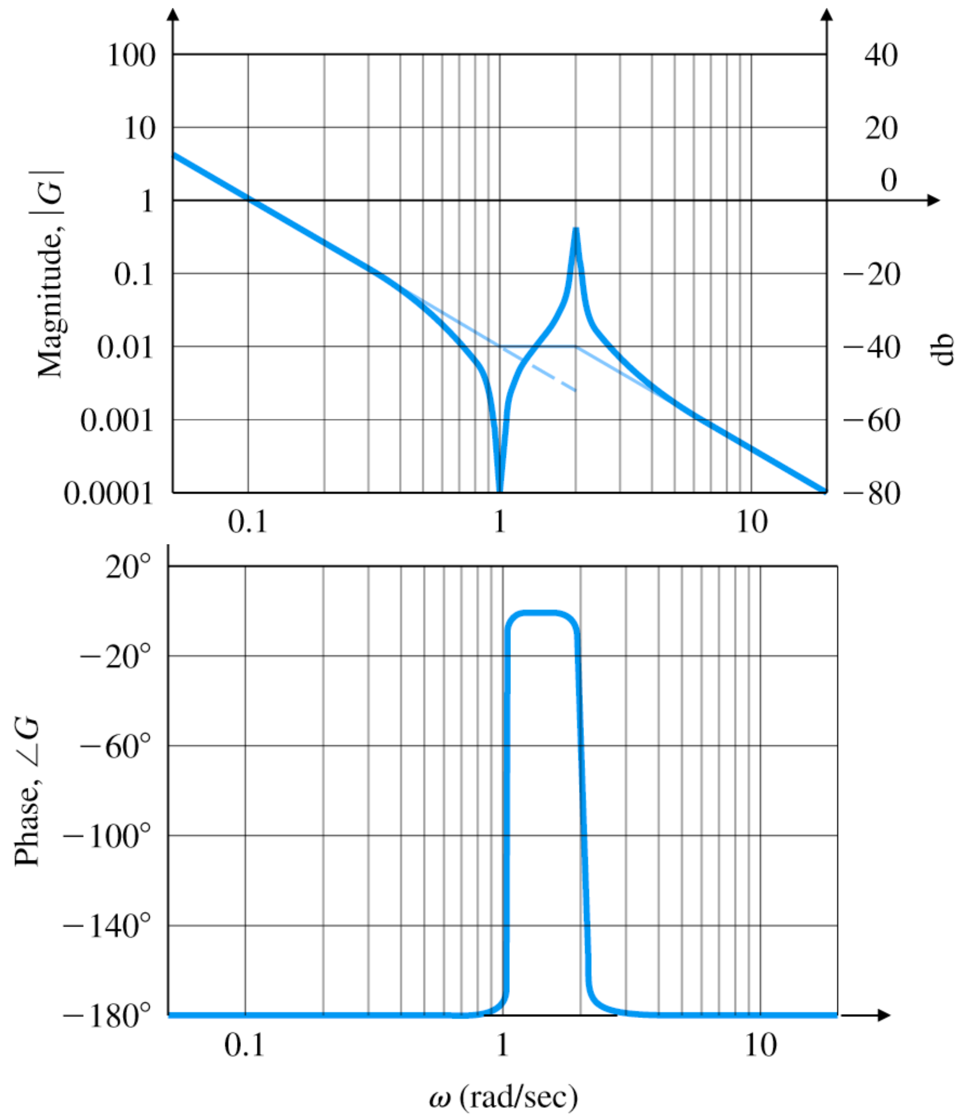
2nd-order zero: $\omega_n = 1,\ 2\zeta = 0.01$

$\rightarrow 1/2\zeta = 1/0.01 = 100$

2nd-order pole: $\omega_n = 2,\ 2\zeta = 0.02$

$\rightarrow 1/2\zeta = 1/0.02 = 50$

$\phi = -180° \rightarrow \cong 0° \rightarrow -180°$



Summary for sketching Bode plots

1. Manipulate the transfer function into the Bode form.
2. Plot the lowest frequency portion of the asymptote using $K_0(j\omega)^n$:
   asymptote through the point K_0 at $\omega = 1$ with a slope of *n*.
3. Complete the composite magnitude asymptotes: Extend the low-frequency asymptote untill the first break point. Then step the slope at each break point frequency: ± 1 for a first order term and ± 2 for a second order term.
4. Increase [Decrease] the asymptote value by +3 db [-3 db] at the first-order numerator [denominator] break points.
   At the second-order break points, sketch the resonant peak [valley] using the relation $\left|G(j\omega)\right| = \frac{1}{2\zeta}$ at $\omega = \omega_n$ at denominater [$\left|G(j\omega)\right| = 2\zeta$ at $\omega = \omega_n$ at numerator] break points.
5. Plot the low-frequency asymptotes of the phase curve: $\phi = n \times 90°$.
6. Sketch the approximate phase curve at each break point in order of ascending frequency: $\pm 90°$ for a first order term and $\pm 180°$ for a second order term.
7. Sketch in each individual phase curve.
8. Graphically add each phase curve.

Minimum phase and nonminimum phase

- Minimum phase versus nonminimum phase

- Minimum phase systems: Systems with all zeros in LHP

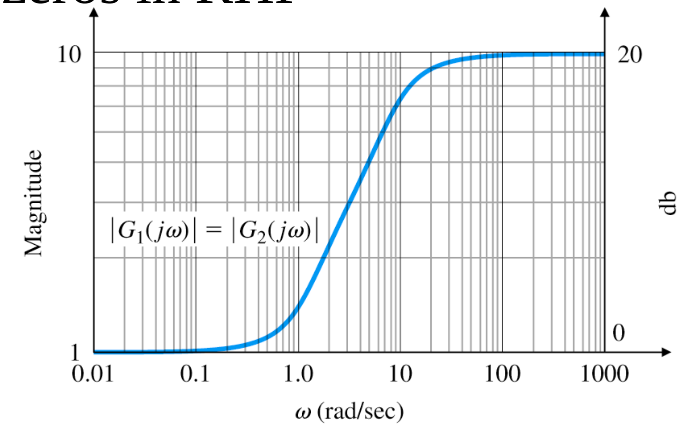
- Nonminimum phase systems: Systems with some zeros in RHP

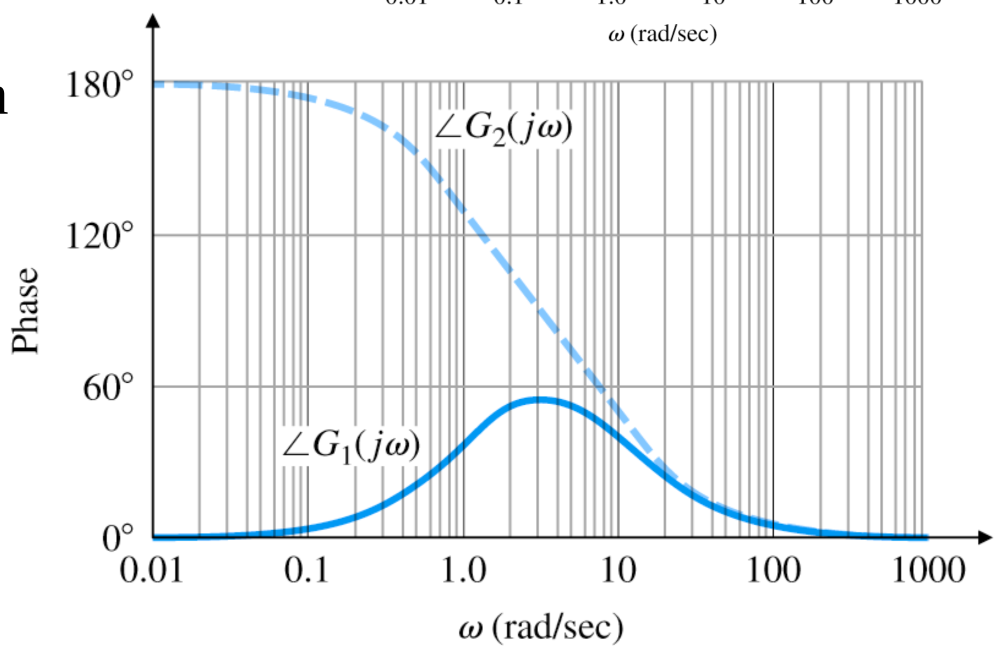
· minimum-phase: $G_1(s) = 10\dfrac{s+1}{s+10}$

· nonminimum-phase: $G_2(s) = 10\dfrac{s-1}{s+10}$

$$|G_1(j\omega)| = |G_2(j\omega)|$$

- net change in phase of nonminimum phase systems is greater than that of minimum phase systems



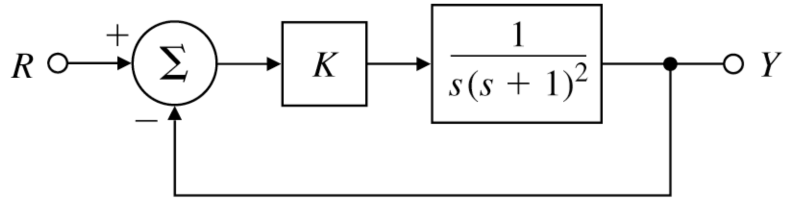


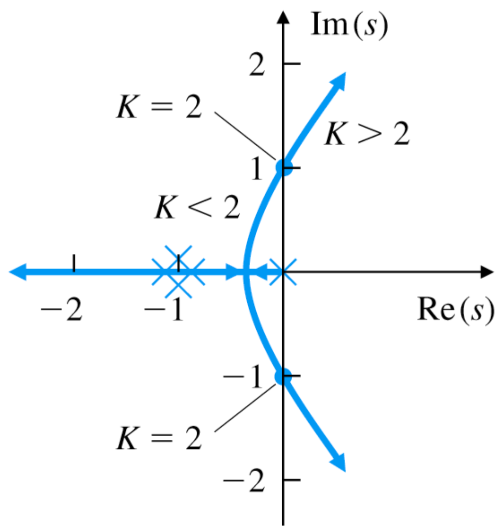
3. Nyquist plot and Nyquist stability criterion

Neutral stability

- Can we determine the stability of the closed-loop system from the Bode plot of the open-loop transfer function?

- Example:





- For all points s on the root locus,

$$1+KG(s)=0 \rightarrow \left|KG(s)\right|=1 \quad \text{and} \quad \angle G(s)=180^\circ \ \left(G(s)=-1/K\right)$$

- At the point of neutral stability($K=2,\ s=j1=j\omega_0$),

$$\left|KG(j\omega_0)\right|=1 \quad \text{and} \quad \angle G(j\omega_0)=180^\circ$$ **Neutral stability condition**

There is some relation between the stability of closed loop system and the frequency response of the plant.

Stability using Bode plots – specific case

- Close-loop stability from Bode plot

$$\frac{K}{s(s+1)^2} \rightarrow \frac{K}{(j\omega)(j\omega+1)^2}$$

- For $K = 2$ (neutrally stable),

$$|KG(j\omega)| = 1 \text{ at } \omega \text{ where } \angle G(j\omega) = -180°$$

- $|KG(j\omega)| < 1$ at ω where $\angle G(j\omega) = -180°$

$\Leftrightarrow K < 2 \rightarrow$ stable

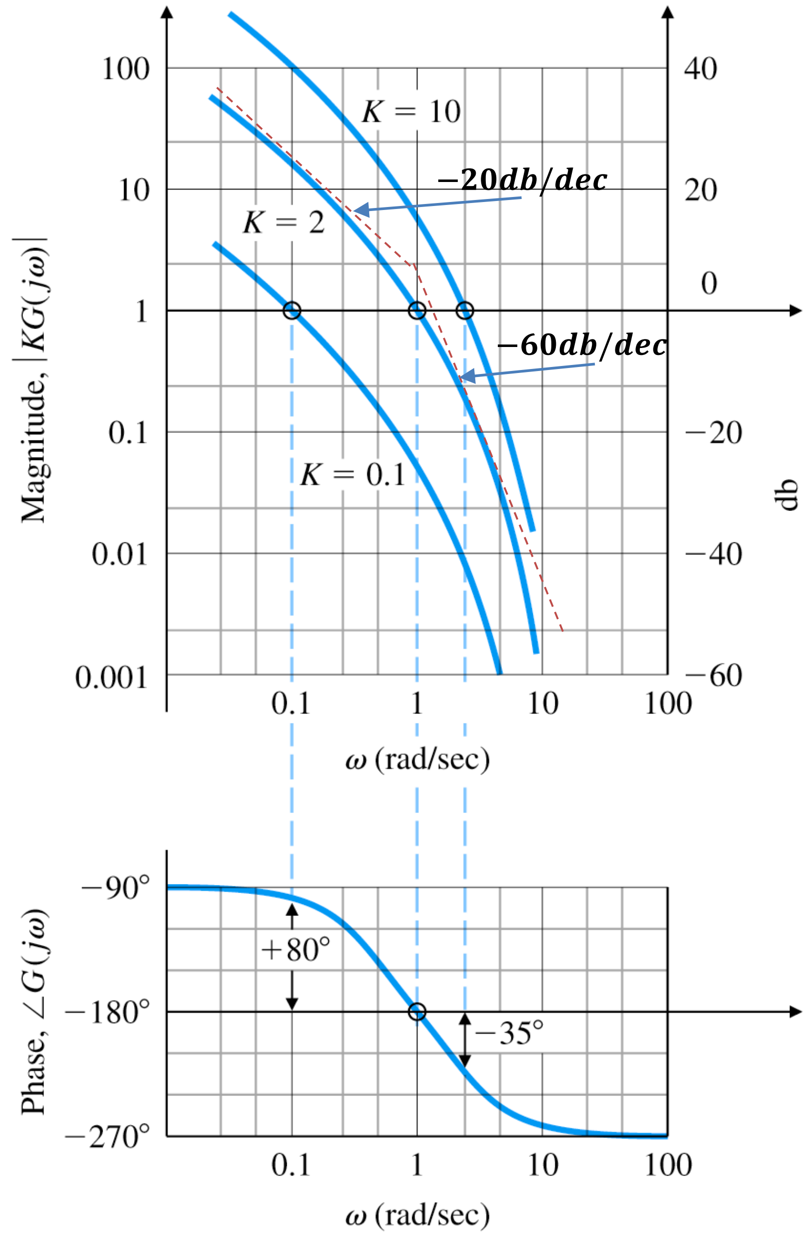
- $|KG(j\omega)| > 1$ at ω where $\angle G(j\omega) = -180°$

$\Leftrightarrow K > 2 \rightarrow$ unstable

$\mapsto$ Stability criterion:

$$|KG(j\omega)| < 1 \text{ at } \angle G(j\omega) = -180°$$

- Note: In this example, increasing the gain makes the system unstable.



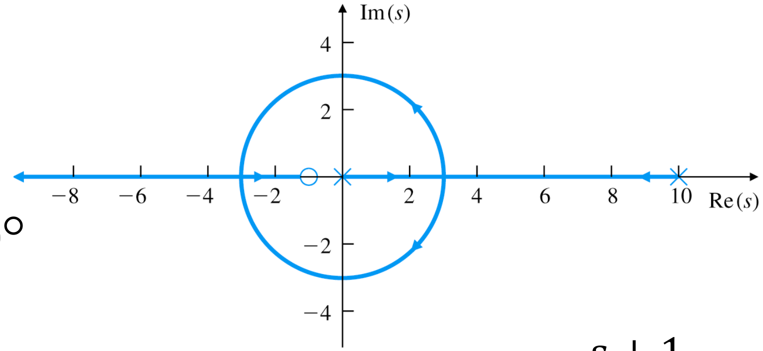
Limitations of Bode plots in stability analysis

- Does increasing the gain (from the gain of neutral stability) increases or decreases the system's stability?
  - Usually increasing gain makes the system less stable.

- There are systems where increasing gain leads from instability to stability.

  $\rightarrow$ Stability criterion:

  $$\left|KG(j\omega)\right| > 1 \text{ at } \omega \text{ where } \angle G(j\omega) = -180^\circ$$



$$KG(s) = K\frac{s+1}{s(s-10)}$$

- There are systems where $\left|KG(j\omega)\right|$ crosses magnitude=1 more than once.

  $\rightarrow$ Perform a rough sketch of the root locus.

  $\rightarrow$ Use Nyquist stability criterion.

Motivations of the Nyquist stability criterion

- In most cases, increasing gain results in instability.

- There are some cases when the system is unstable with decreased gains.

- Nyquist stability criterion provides the complete answer.
  - It relates the **open-loop frequency response** to **the number of RHP poles of the closed-loop system**.
  - You should be able to draw a complex valued function on the complex plane.

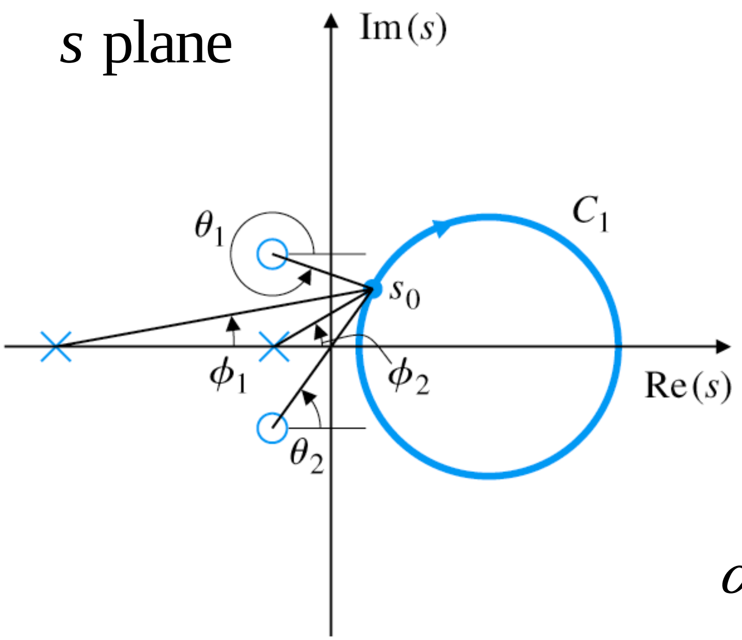
- Nyquist stability criterion is based on the argument principle.

The Argument Principle

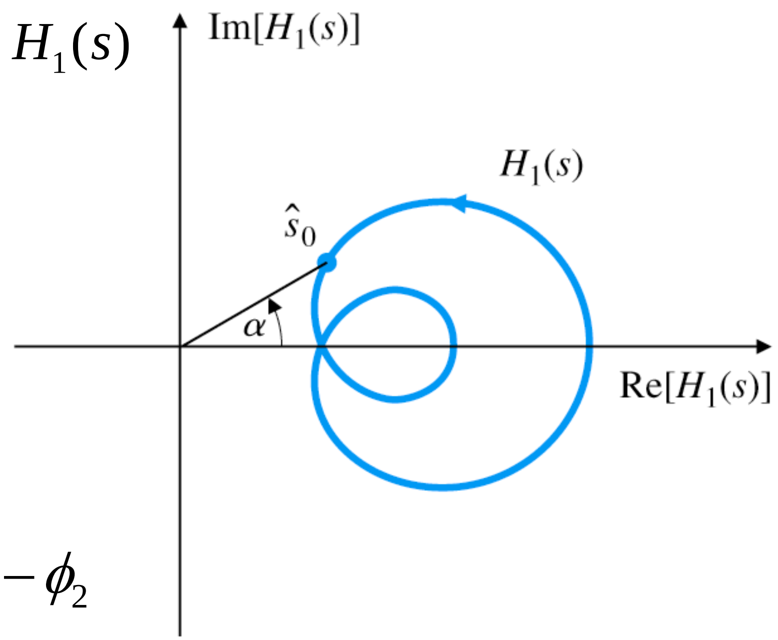
- Key concept to understand the Nyquist Stability Criterion.

- **Contour evaluation**
  Given a transfer function $H(s)$ (with poles and zeros), evaluate $H(s)$ for values of s on the clockwise contour C.
  - For some test point s_0, compute $H(s_0)$ and draw it on the complex plane and do this for various s_0 on the contour C.

s plane



Plot of $H_1(s)$



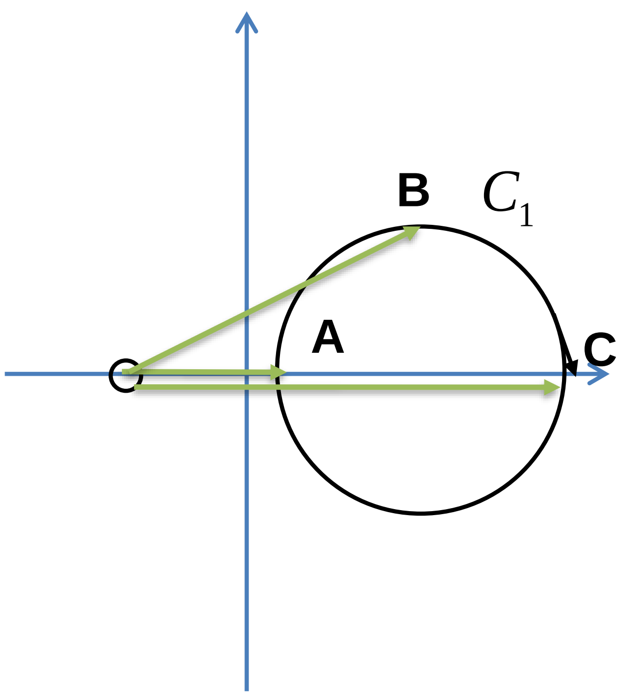
$$\alpha = \theta_1 + \theta_2 - \phi_1 - \phi_2$$

Example

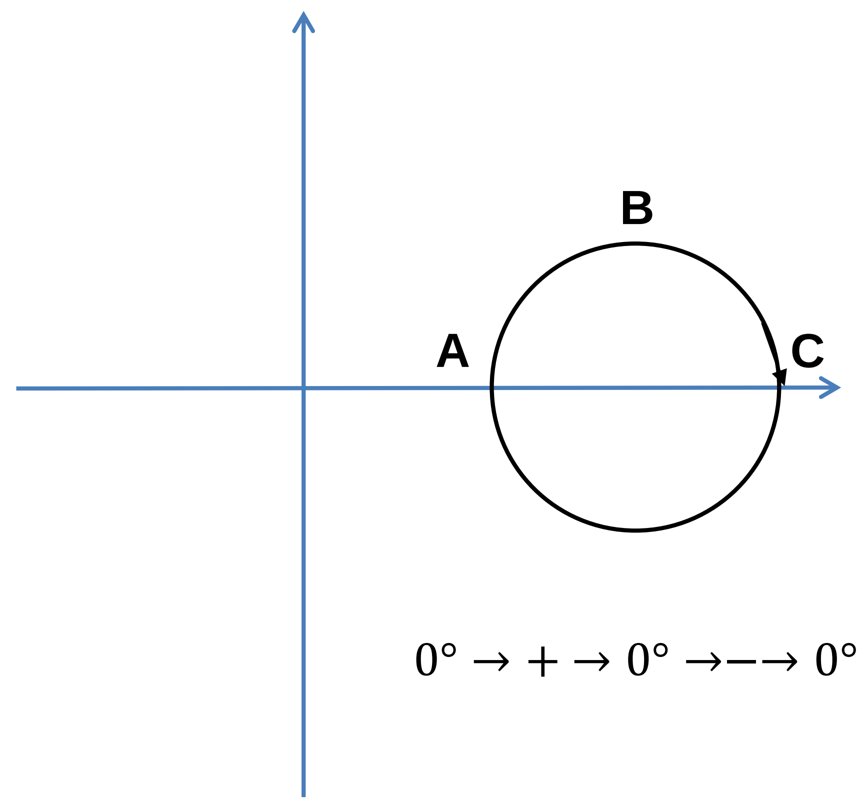
- Encirclement of the origin when C contains no pole/zero

- Case 1: One zero at -1, $H(s) = s + 1$

s plane



Plot of $H(s)$

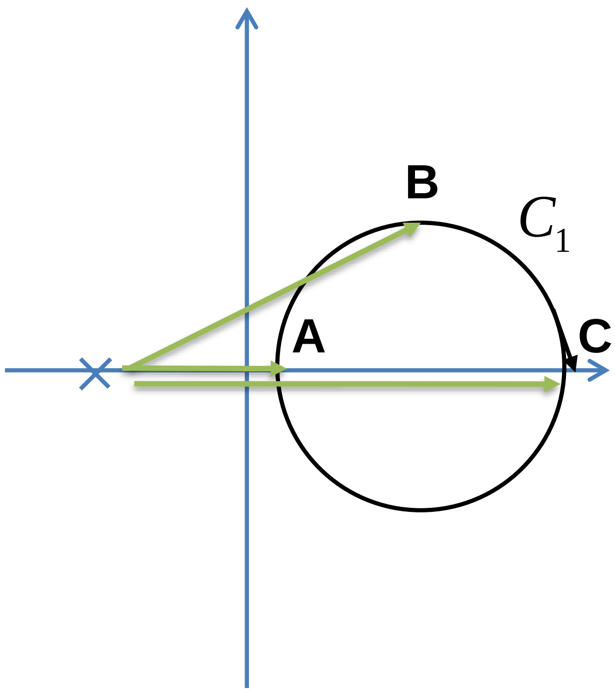


Consider Magnitude & Phase of $H(s)$

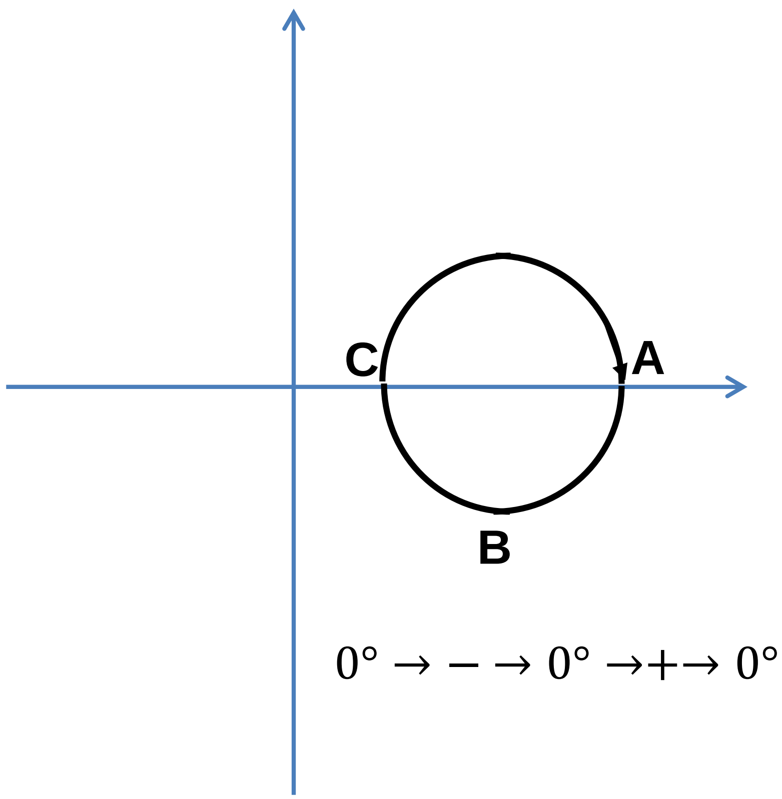
Example

- Encirclement of the origin when C contains no pole/zero
- Case 2: One pole at -1, $H(s) = \dfrac{1}{s+1}$

s plane



Plot of $H(s)$

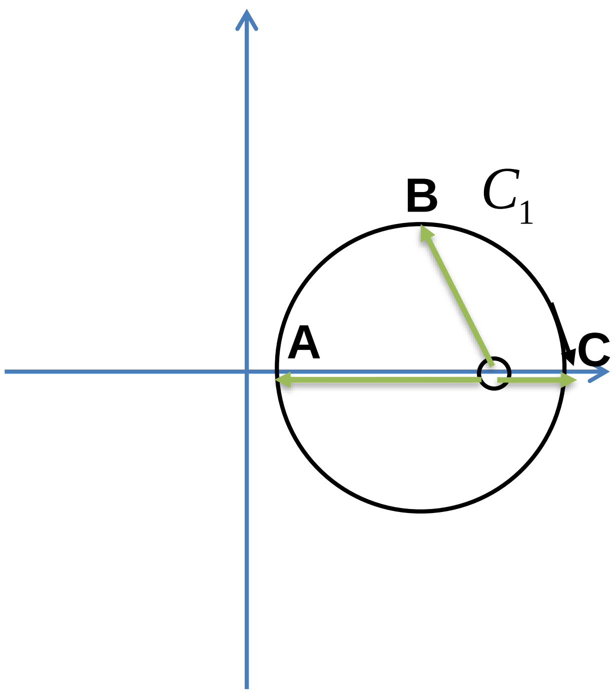


Consider Magnitude & Phase of $H(s)$

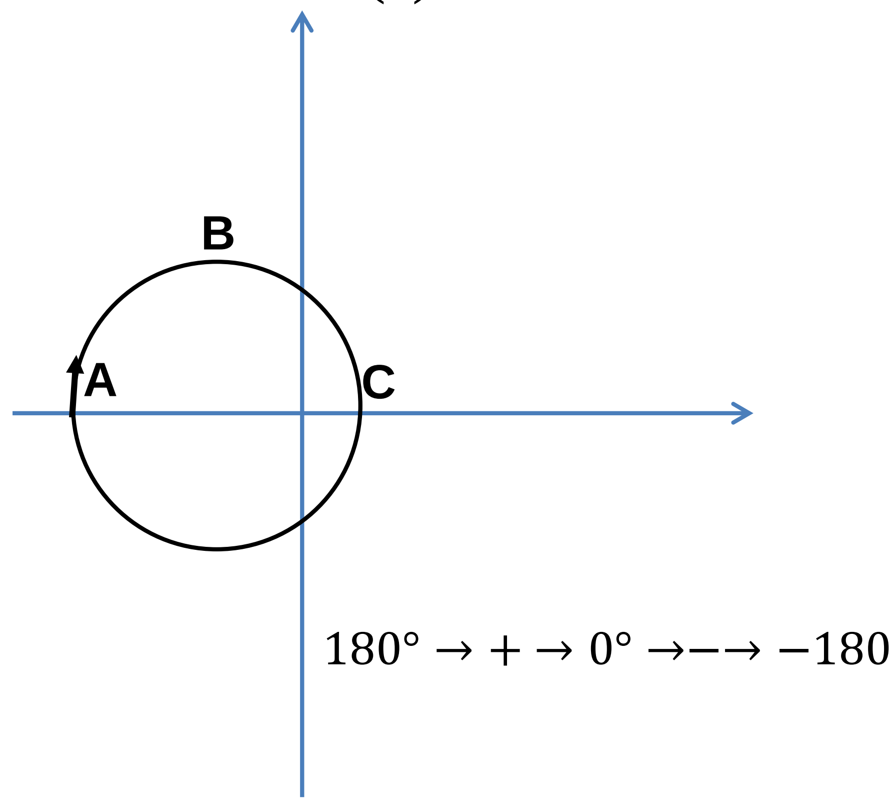
Example

- Encirclement of the origin when C contains pole or zero
- Case 1: One zero at -1, $H(s) = s - 1$

s plane



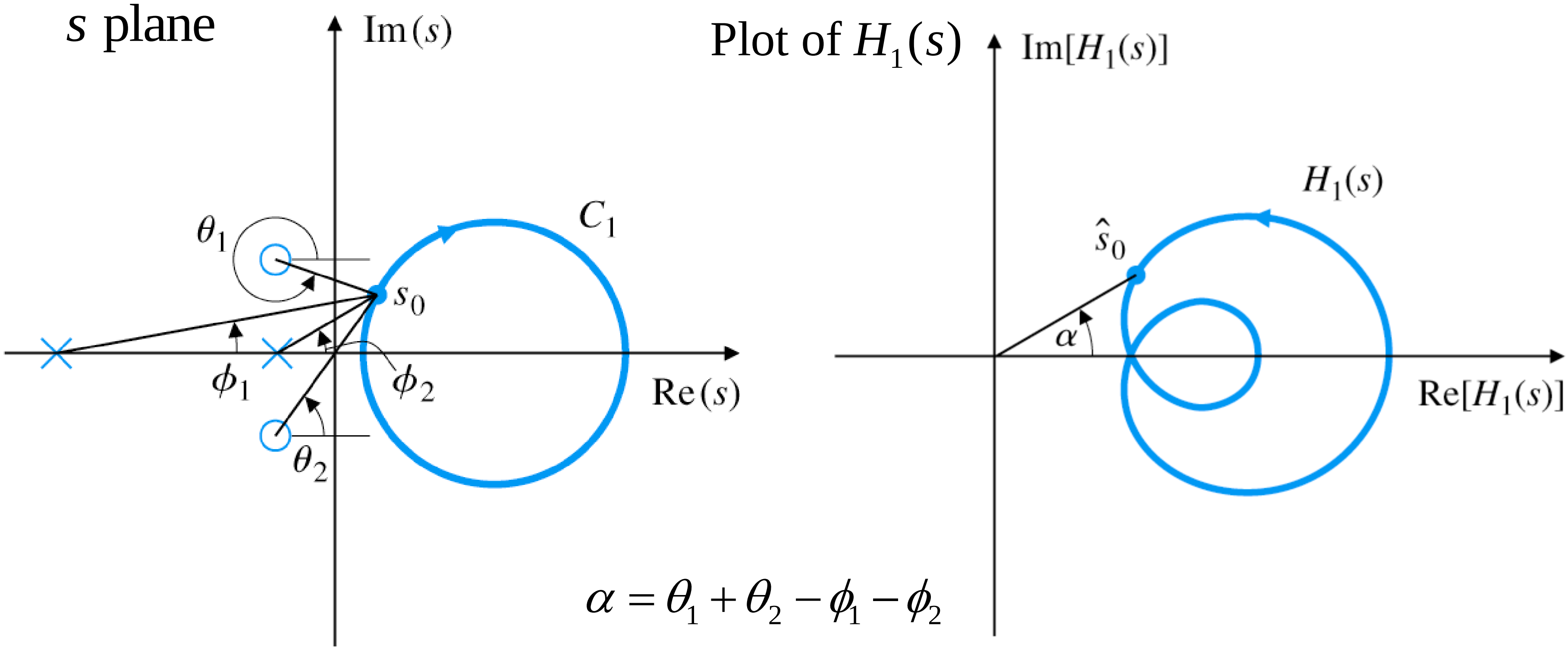
Plot of $H(s)$



Consider Magnitude & Phase of $H(s)$

The Argument Principle - C contains no pole/zero

- Encirclement of the origin when C contains no pole/zero.



$$\alpha = \theta_1 + \theta_2 - \phi_1 - \phi_2$$

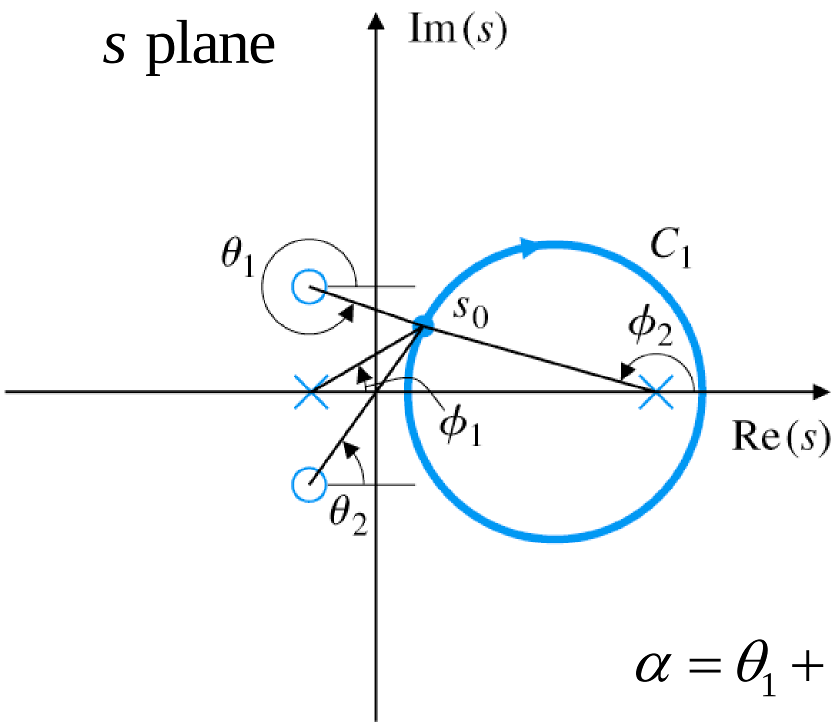
- α increases and decreases and return to the original value.

- As s traverses C_1, the angle α will not undergo a net change of 360° as long as there are no poles or zeros within C_1.

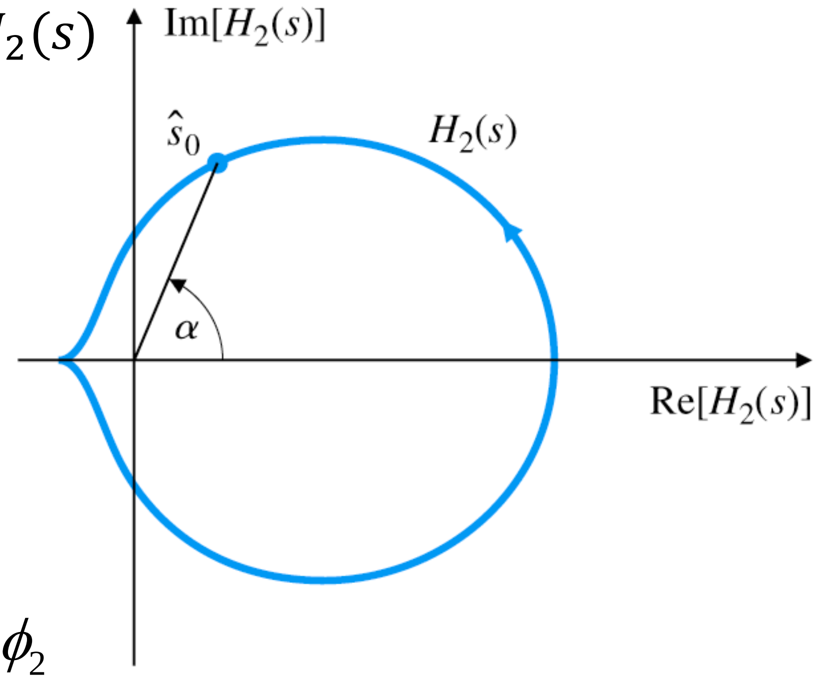
$\rightarrow$ The plot of $H_1(s)$ will not encircle the origin.

The Argument Principle - C contains pole or zero

- Encirclement of the origin when C contains pole or zero.



Plot of $H_2(s)$



$$\alpha = \theta_1 + \theta_2 - \phi_1 - \phi_2$$

- As s traverses C_1, the angle ϕ_2 will undergo a net change of $-360°$.

$\rightarrow$ As s traverses C_1, the angle α will undergo a net change of $+360°$.

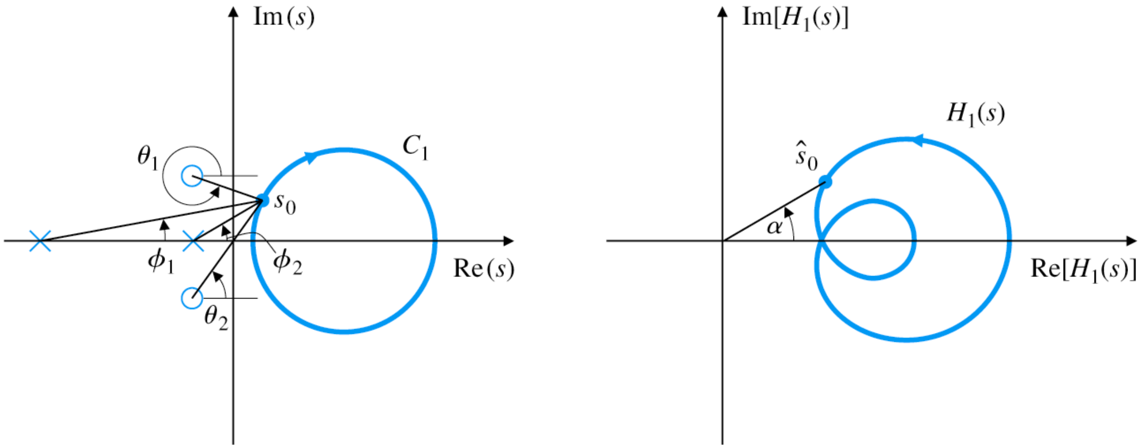
$\rightarrow$ The plot of $H_2(s)$ will encircle the origin in counterclockwise direction.

- $Z > P \rightarrow (Z - P)$ clockwise encirclements around the origin
- $Z < P \rightarrow (P - Z)$ counterclockwise encirclements around the origin

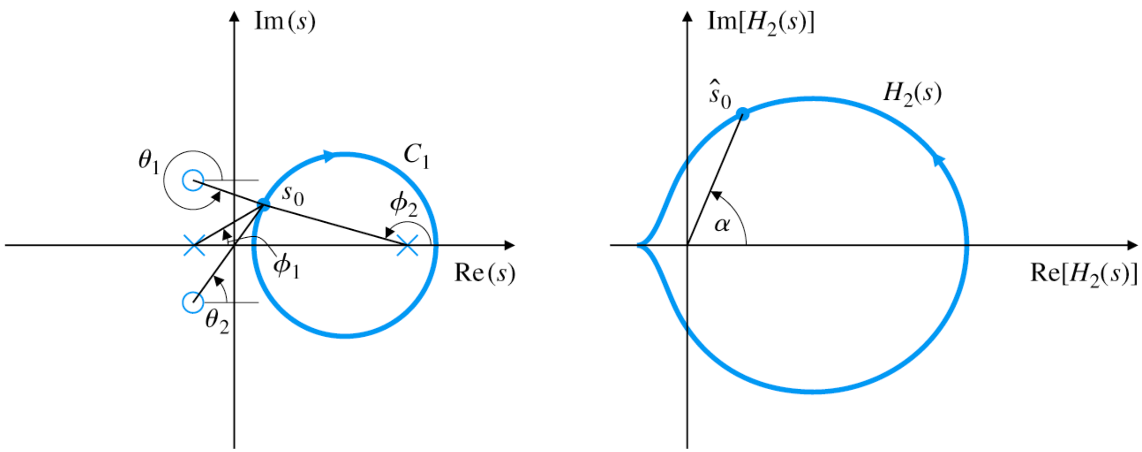
Summary of The Argument Principle

A contour map of a complex function will encircle the origin ***Z*-*P* times clockwise**.
Z is the number of zeros and *P* is the number of poles of the function inside the contour.

No poles in C_1



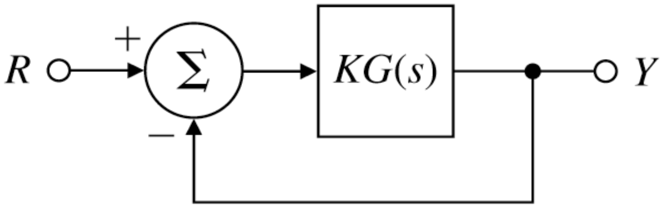
A pole in C_1



From the Argument Principle to stability analysis

- Application of the argument principle to the basic closed-loop system.

$$\frac{Y(s)}{R(s)} = T(s) = \frac{KG(s)}{1+KG(s)}.$$

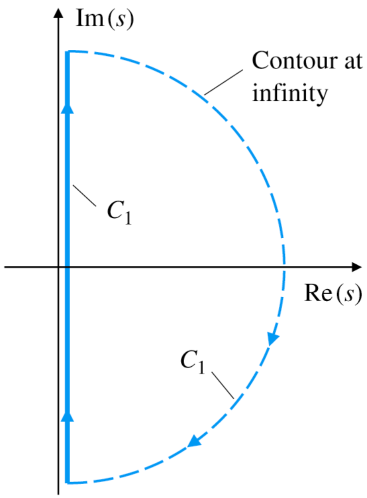


- A contour map of a complex function $H(s)$ will encircle the origin **Z-P times,** where Z is the number of zeros and P is the number of poles of $H(s)$ inside the contour.

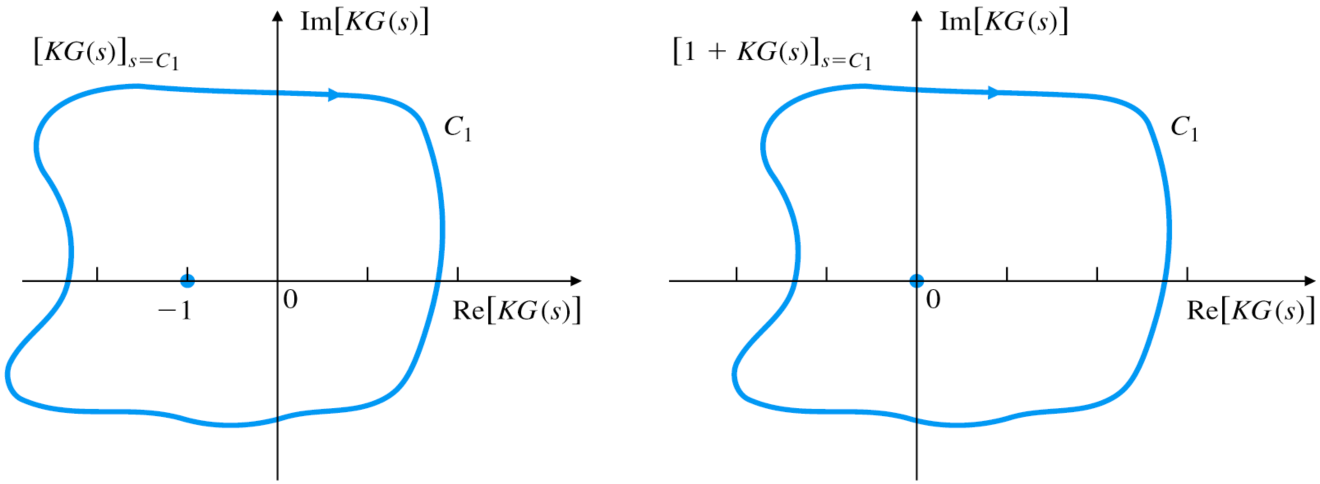
→ A contour map of $1+KG(s)$ will encircle the origin **Z-P times,** where Z is the number of zeros and P is the number of poles of $1+KG(s)$ inside the contour.

Contour selecting and the Nyquist plot

- Contour representing RHP and the Nyquist plot



Contour



Nyquist plot

- Closed-loop poles are the solutions of

$$1 + KG(s) = 0 \rightarrow KG(s) = -1.$$

- Encirclements of $0 + j0$ by C_1 contour evaluation of $\left(1 + KG(s)\right)$

$\leftrightarrow$ Encirclements of $-1 + j0$ by C_1 contour evaluation of $KG(s)$

Closed-loop poles in RHP

- Consider a contour representing RHP:

→ A contour map of $1+KG(s)$ will encircle the **origin** N=Z-P times,
where Z: the number of RHP zeros of $1+KG(s)$,
P: the number of RHP poles of $1+KG(s)$

→ A contour map of $KG(s)$ will encircle **-1** N=Z-P times,
where Z: the number of RHP zeros of $1+KG(s)$,
P: the number of RHP poles of $1+KG(s)$

→ A contour map of $KG(s)$ will encircle **-1** N=Z-P times,
where Z: the number of RHP zeros of $1+KG(s)$,
P: the number of RHP poles of $KG(s)$

Complement

$$1+KG\left(s\right)=1+K\frac{b\left(s\right)}{a\left(s\right)}=\frac{a\left(s\right)+Kb\left(s\right)}{a\left(s\right)}$$

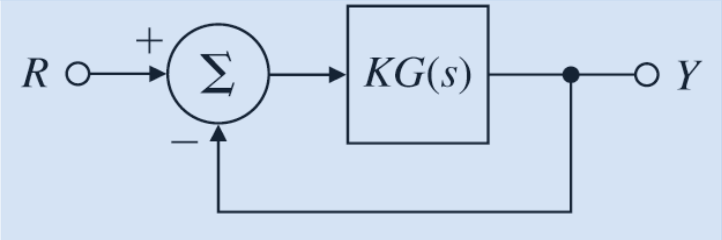
$$\rightarrow\begin{cases}\text{poles of } G(s)\left(=b(s)/a(s)\right)=\text{poles of }\left(1+KG(s)\right)\\ \text{closed-loop poles} = \text{zeros of }\left(1+KG(s)\right)\end{cases}$$

$$\rightarrow\begin{cases}\text{poles of } G(s) \text{ in RHP} = \text{poles of }\left(1+KG(s)\right)\text{ in RHP}\\ \text{closed-loop poles in RHP} = \text{zeros of }\left(1+KG(s)\right)\text{ in RHP}\end{cases}$$

Nyquist Stability Criterion

For the basic feedback system

with $\dfrac{Y(s)}{R(s)} = T(s) = \dfrac{KG(s)}{1+KG(s)}$, we have



$$Z=N+P$$

Z: the number of RHP poles of closed-loop system.

N: the number of **clockwise encirclement** of $KG(s)$ about -1.

P: the number of RHP poles of open-loop system.

- Stability of the closed-loop system can be determined in terms of the number of RHP poles of the open-loop system, and the Nyquist plot.

- If $G(s)$ has P unstable poles, then Nquist plot should encircle the point -1 N times counterclockwise so that the closed loop system is stable (Z=0).

- Usually, we draw the plot with K=1.

Nyquist plot

- Procedure of plotting the Nyquist plot

1. Plot $KG(s)$ for the contour C_1.

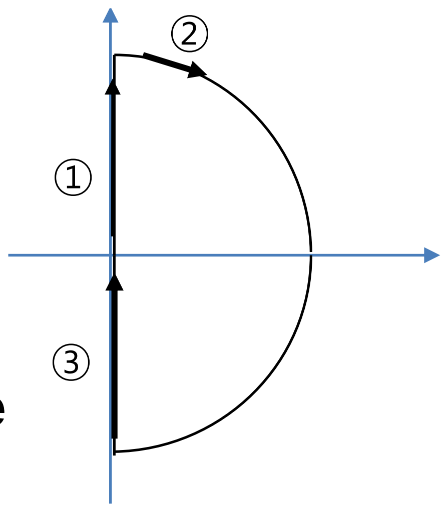
- Plot $KG(s)$ for $-j\infty \leq s \leq +j\infty$.

- The magnitude of $KG(j\omega)$ will be small at high frequencies.

- The Nyquist plot will always be symmetric with respect to real axis.

2. Evaluate the number of clockwise encirclement of -1 and call that N.

-If encirclements are in the counterclockwise direction, then N is negative



3. Determine P.

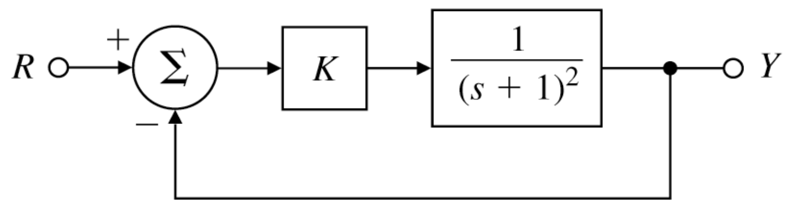
4. $Z = N + P$

$Z = 0 \rightarrow$ The closed loop system is stable

Otherwise, the closed loop system is unstable

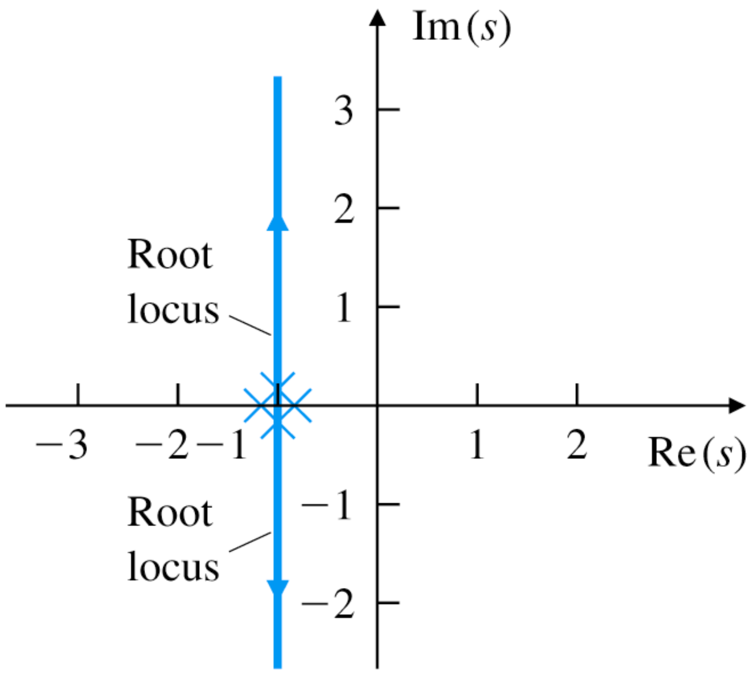
Example

- Example 6.8: Nyquist plot for a second order system



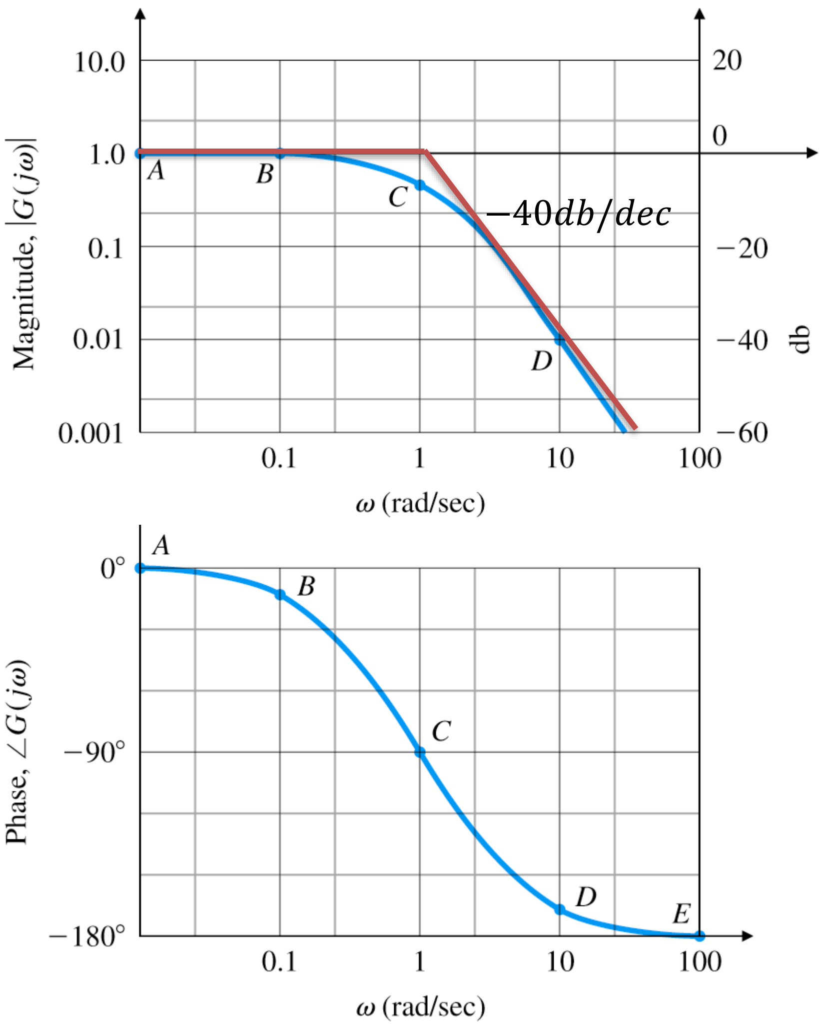
- Root locus

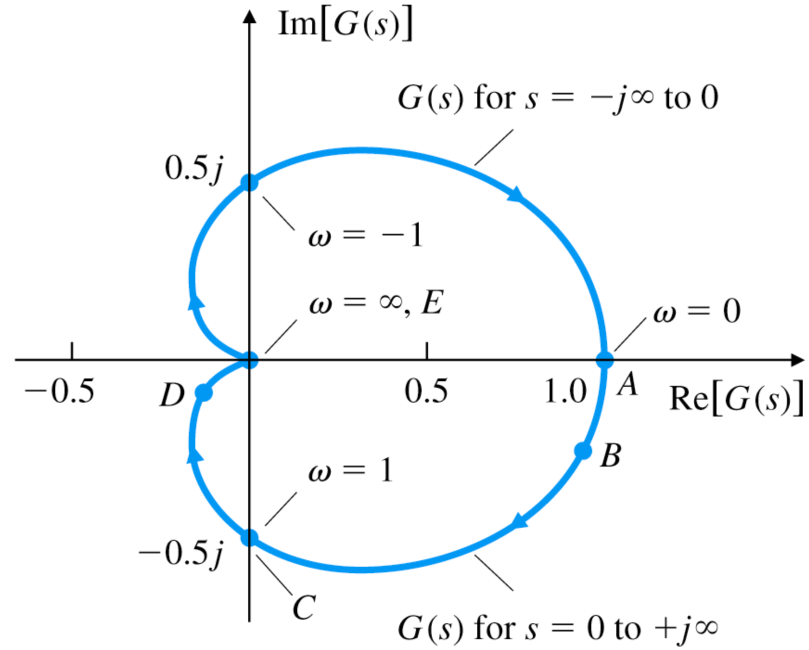
- Stable for all positive K



- Nyquist plot $KG(s) = \dfrac{1}{(s+1)^2}$ for $K = 1$

$\boxed{K = 1}$





$P = 0, \quad N = 0$

$Z = N + P = 0$

$\rightarrow$ stable

- No positive value of K causes the polar plot to encircle -1.

$\rightarrow$ Stable for all $K > 0$.

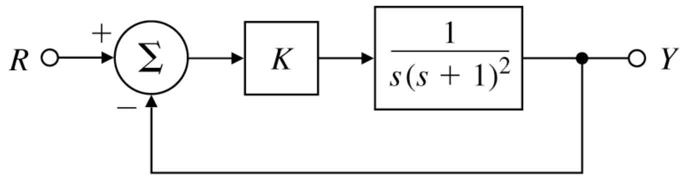
Remark

- Remark:
  - Encirclement of -1 by $KG(s)$ = Encirclement of $-1/K$ by $G(s)$

  $\rightarrow$ Count the number of encirclement of $-1/K$ by $G(s)$.

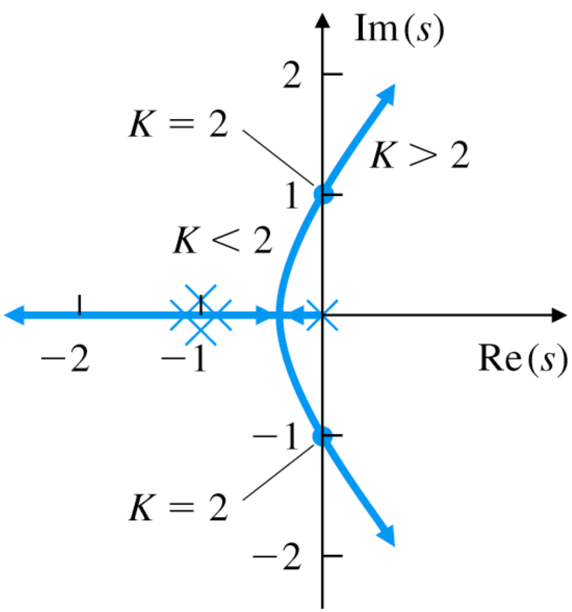
Example

- Example 6.9: Nyquist plot for a third order system



$$G(s) = \frac{1}{s(s+1)^2}$$

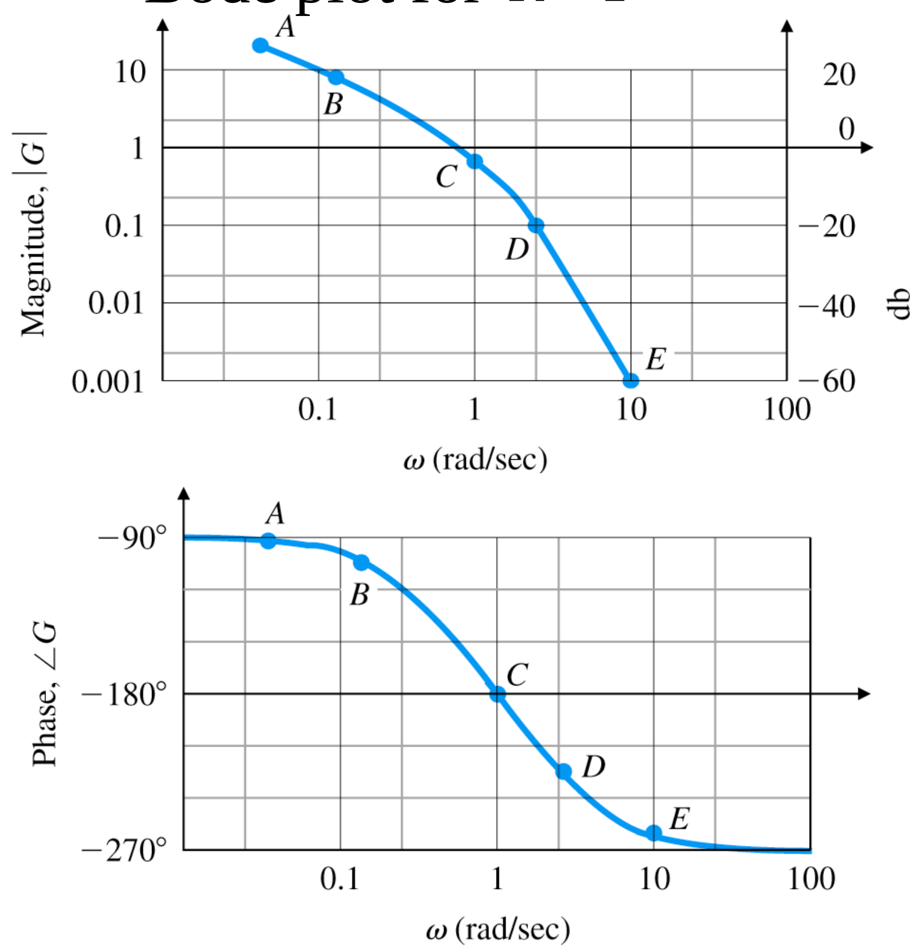
- Root locus



Stable for small K,

unstable for large K

- Bode plot for $K = 1$

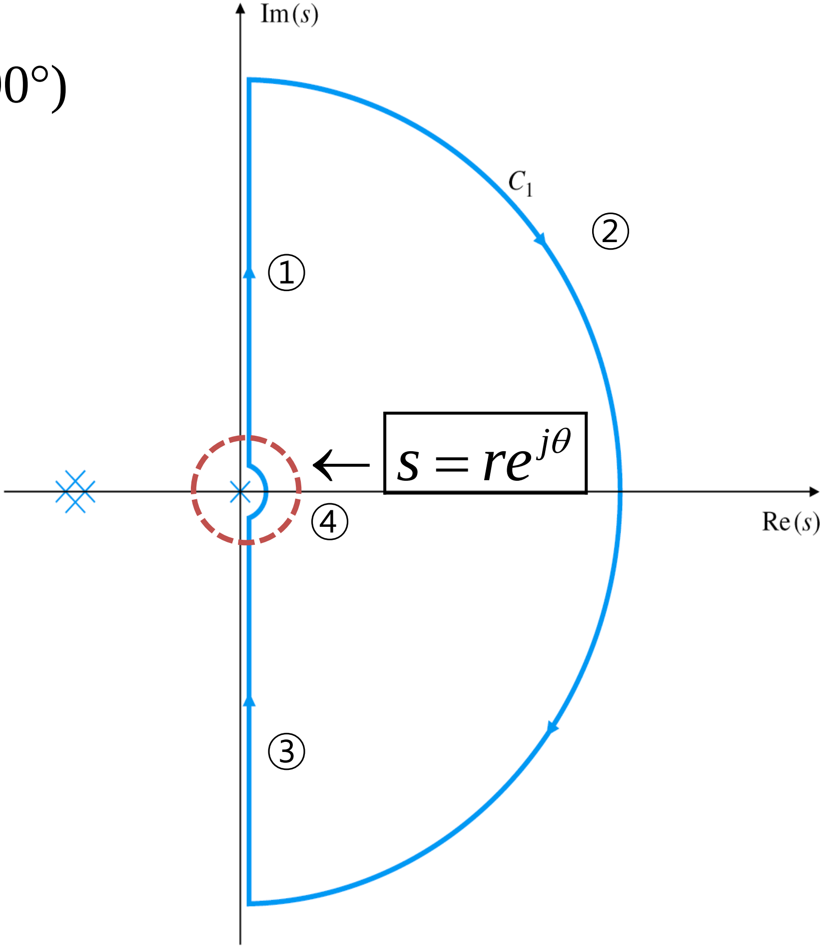


- Poles of $KG(s)$ at $s=0 \to$ Modify the contour C_1.

For s on small arc, $s = re^{j\theta}$, $r << 1$, $-90° \le \theta \le 90°$ $(\theta = -90° \to 0° \to 90°)$

$$\Rightarrow G(s) = \frac{1}{s(s+1)^2} \cong \frac{1}{s} = \frac{1}{re^{j\theta}} = \frac{1}{r} e^{j(-\theta)},$$

$\frac{1}{r} >> 1$, $-90° \le -\theta \le 90°$ $(-\theta = 90° \to 0° \to -90°)$



- The Nyquist plot of $G(j\omega)$ crosses the real axis at $\omega = 1$ with $|G(j1)| = 0.5$.

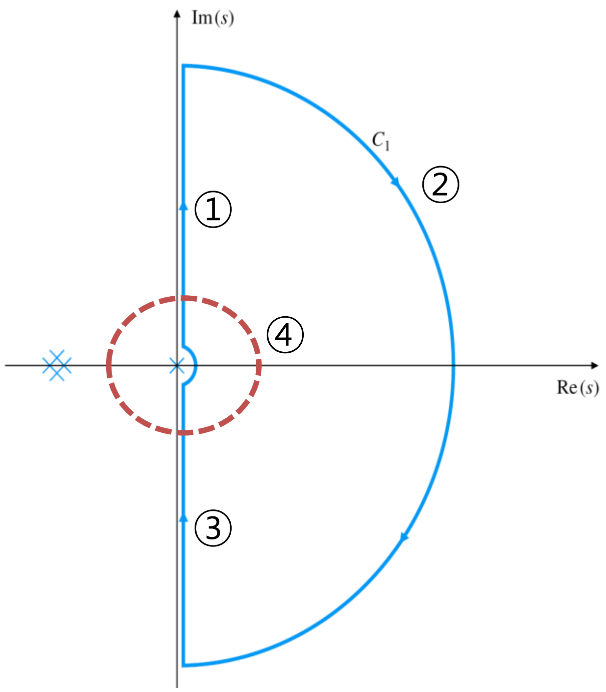
$$1 + KG(s) = 0, \quad G(s) = -1/K$$

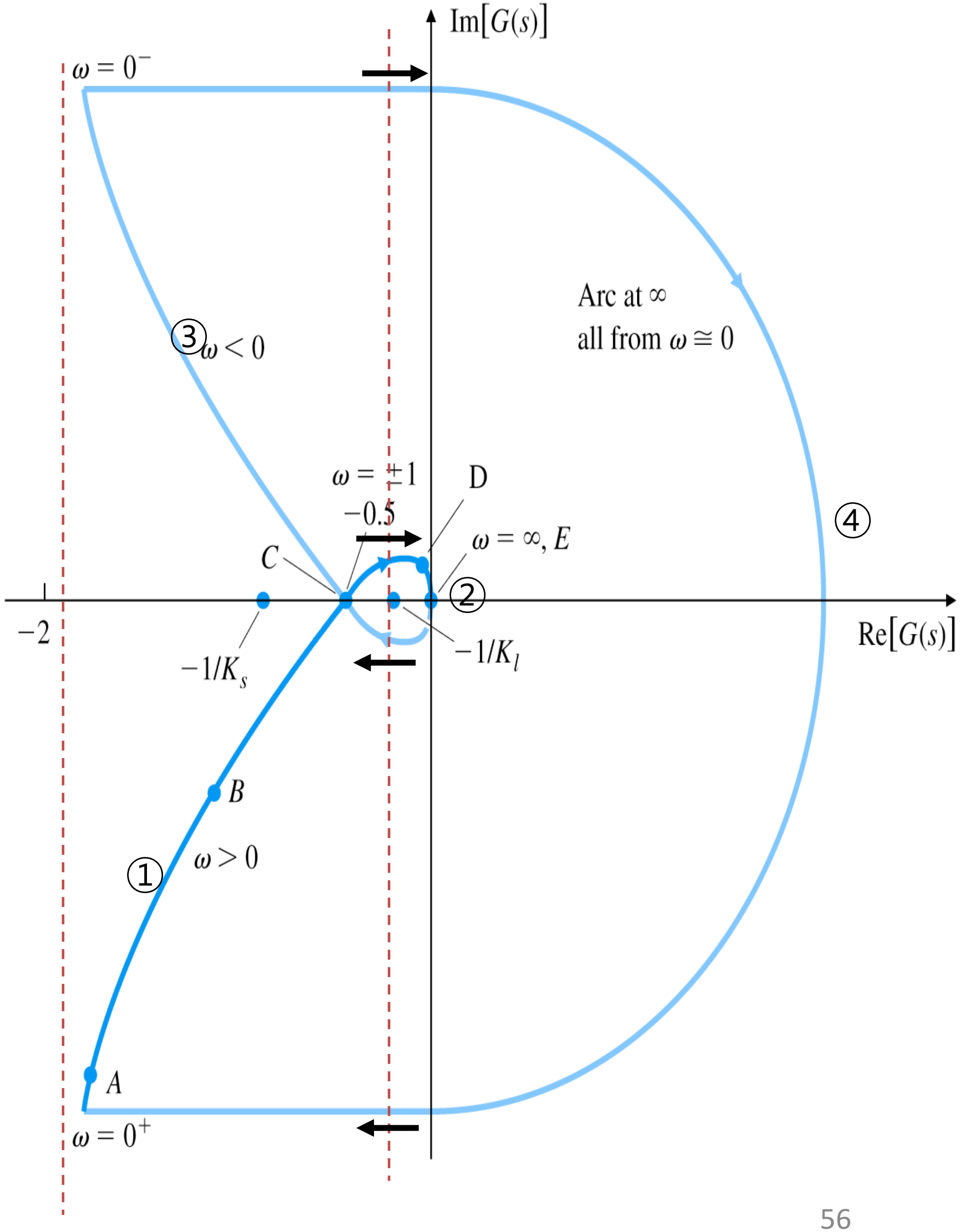
$$-0.5 < -1/K < 0 \;\rightarrow N = 2,$$

$$\rightarrow Z = N + P = 2 + 0 = 2 \rightarrow \text{unstable}$$

$$-1/K < -0.5 \;\rightarrow N = 0,$$

$$\rightarrow Z = N + P = 0 + 0 = 0 \rightarrow \text{ stable}$$





Example

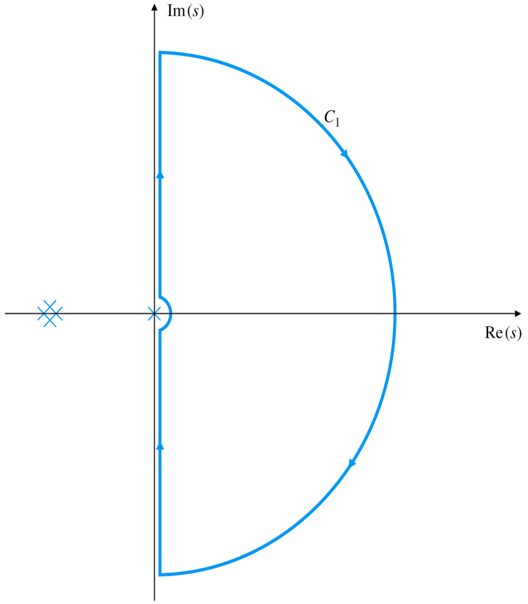
<Extension> Case where $G(s)=\dfrac{1}{s^k(s+1)^2}$, $k\geq 1$

For s on small arc, $s=re^{j\theta}$, $r<<1$, $-90^\circ\leq\theta\leq 90^\circ$ $(\theta=-90^\circ\rightarrow 0^\circ\rightarrow 90^\circ)$

$$\Rightarrow G(s)=\frac{1}{s^k(s+1)^2}\cong\frac{1}{s^k}=\frac{1}{r^k e^{jk\theta}}=\frac{1}{r^k}e^{j(-k\theta)},$$

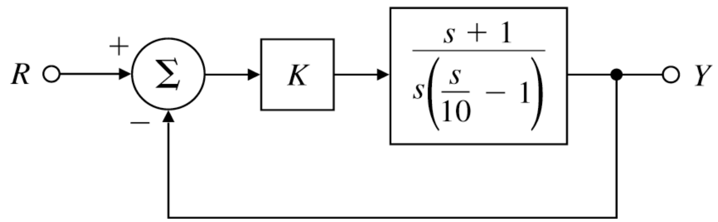
$\dfrac{1}{r^k}>>1$, $-k90^\circ\leq -k\theta\leq k90^\circ$ $(-k\theta=k90^\circ\rightarrow 0^\circ\rightarrow -k90^\circ)$

$\rightarrow$ Nyquist plot at infinity executes $k/2$ clockwise rotations.

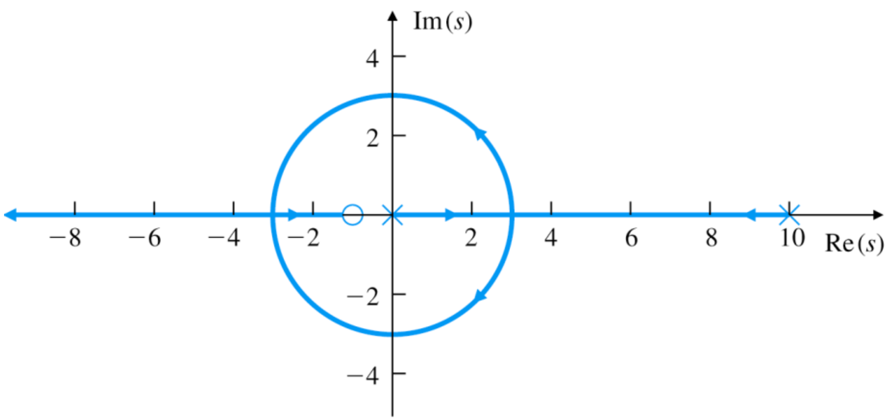


Example

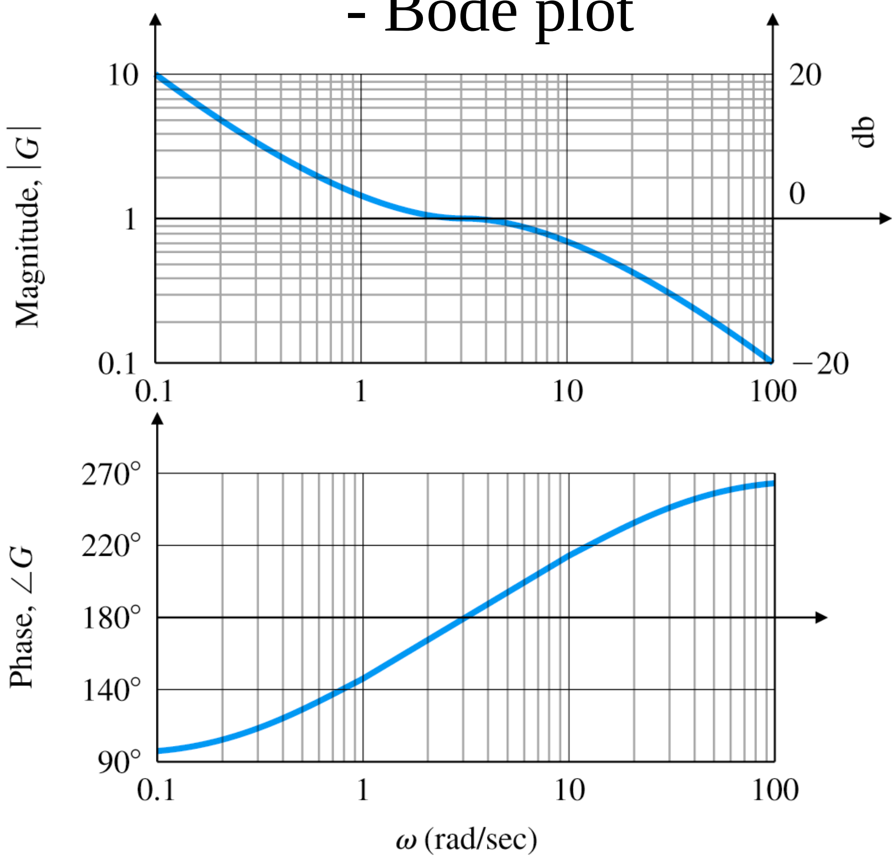
- Example 6.10: Nyquist plot for an open-loop unstable system



- Root locus

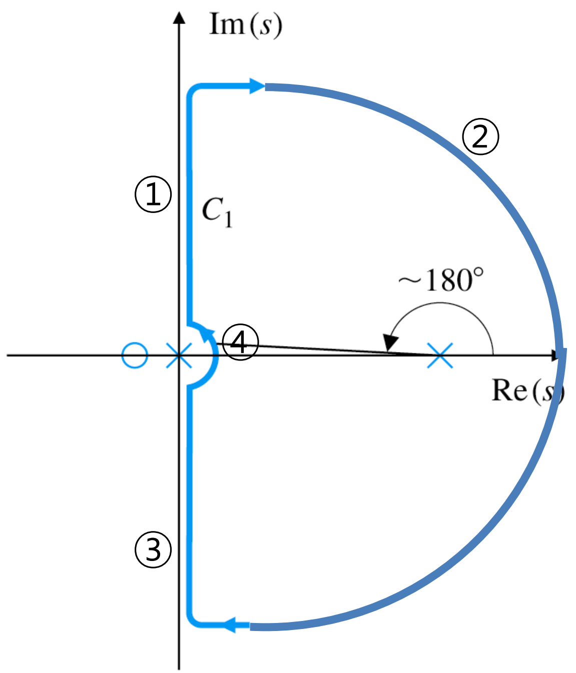


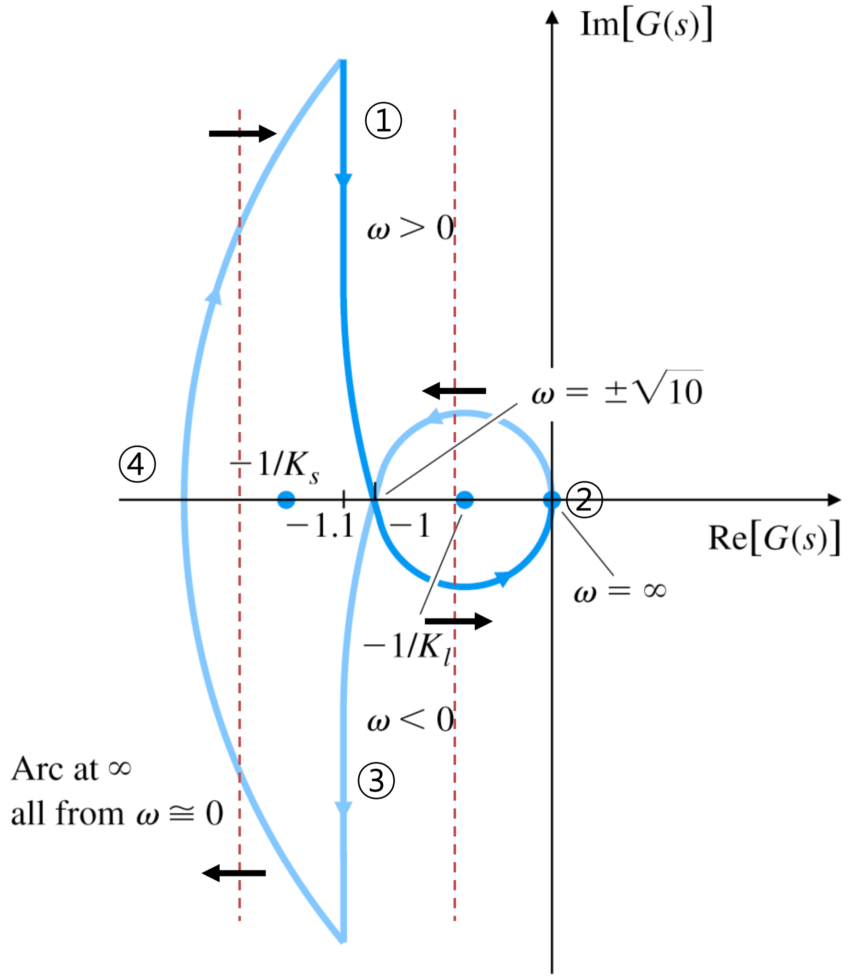
- Bode plot



- Unstable → Impossible to determine its frequency response experimentally.

$$G(s)=\frac{s+1}{s\left(\frac{s}{10}-1\right)} \rightarrow P=1$$





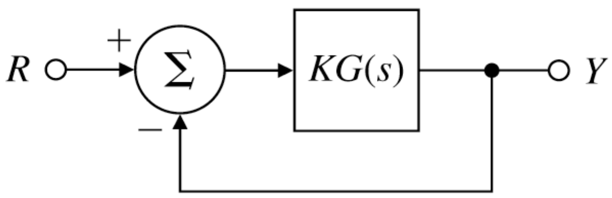
- The Nyquist plot crosses the real axis at $\omega \cong 3$ with $|G(j\omega)|=1$.

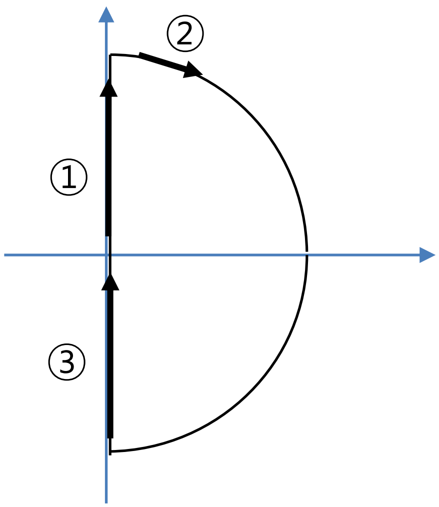
$$-1<-1/K<0 \rightarrow K>1 \rightarrow \; N=-1, \;\; Z=N+P=-1+1=0 \rightarrow \text{ stable}$$

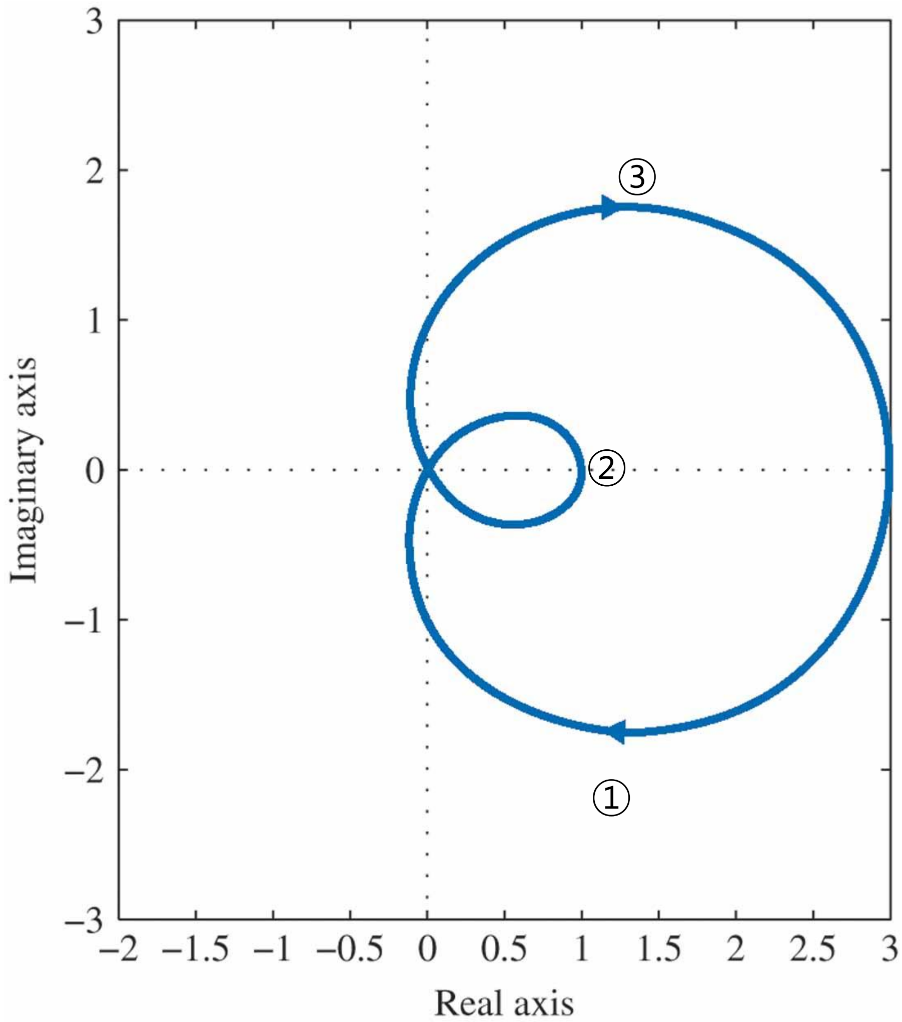
$$-1/K<-1 \rightarrow 0<K<1 \rightarrow N=1, \;\; Z=1+1=2 \rightarrow \text{ unstable}$$

Example

- Example 6.11: Draw the Nyquist plot for $G(s) = \dfrac{s^2+3}{(s+1)^2}$. Determine the stability for positive K.



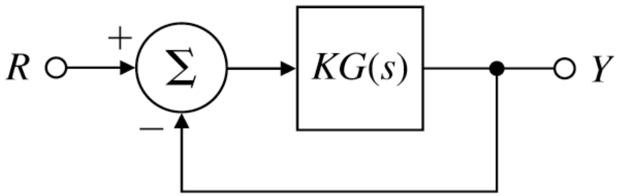




4. Stability margins

Gain margin and phase margin

- In many cases, **the system is stable for all small gain values** and becomes unstable if the gain increases past a certain critical point.



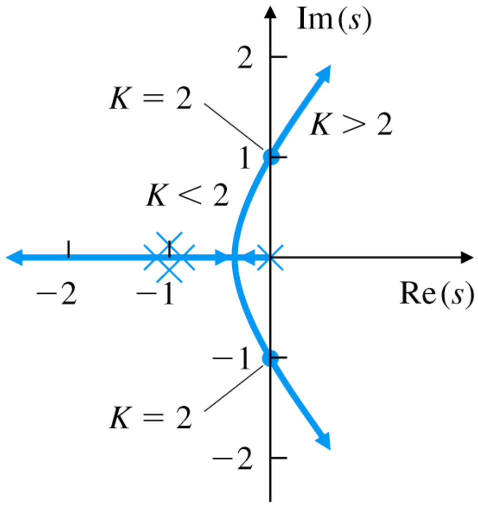
- Gain Margin (GM): **the factor** by which the gain can be raised before instability results.
- Phase Margin (PM): the amount by which the phase of $G(j\omega)$ exceeds -180 deg when $|KG(j\omega)|=1$.

Note:

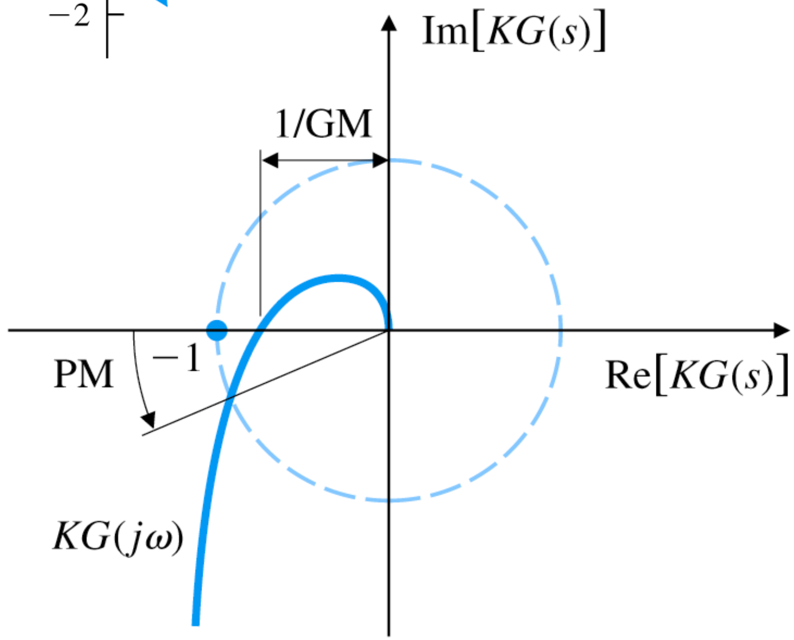
- only applicable for systems for which '**the system is stable for small gains**'.
- the stability is for the feedback systems.

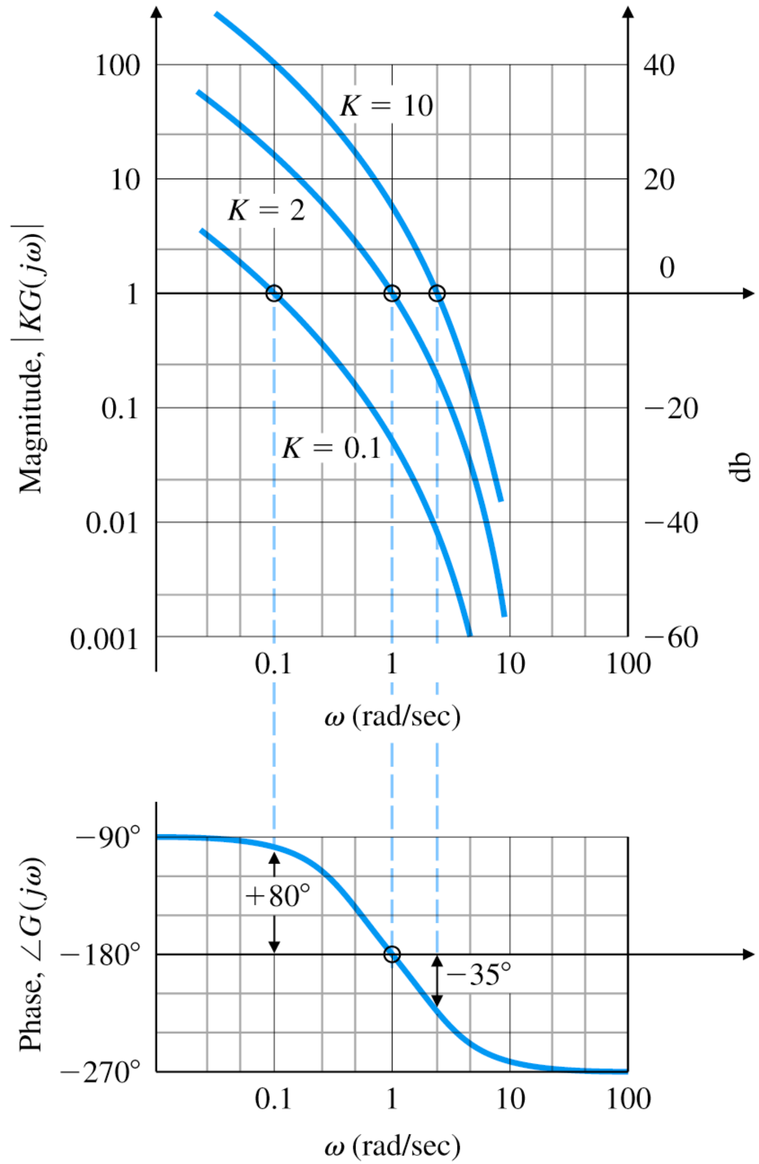
GM and PM from Nyquist plot and Bode plot

- Gain/phase margin from Nyquist plot and Bode plot



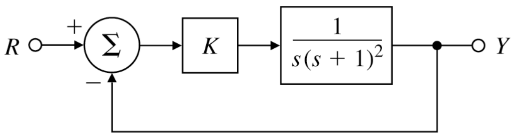
$$G(s) = \frac{1}{s(s+1)^2}$$

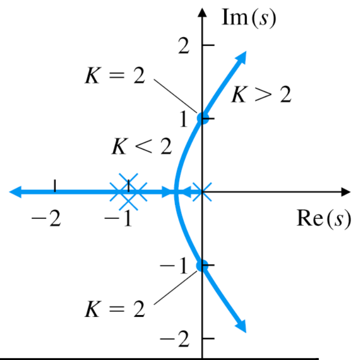




GM and PM from Nyquist plot and Bode plot

$$KG(s) = K\frac{1}{s(s+1)^2}$$





$$\boxed{K = 0.1 \rightarrow \text{GM} = 20 \text{ (26 db)}}$$

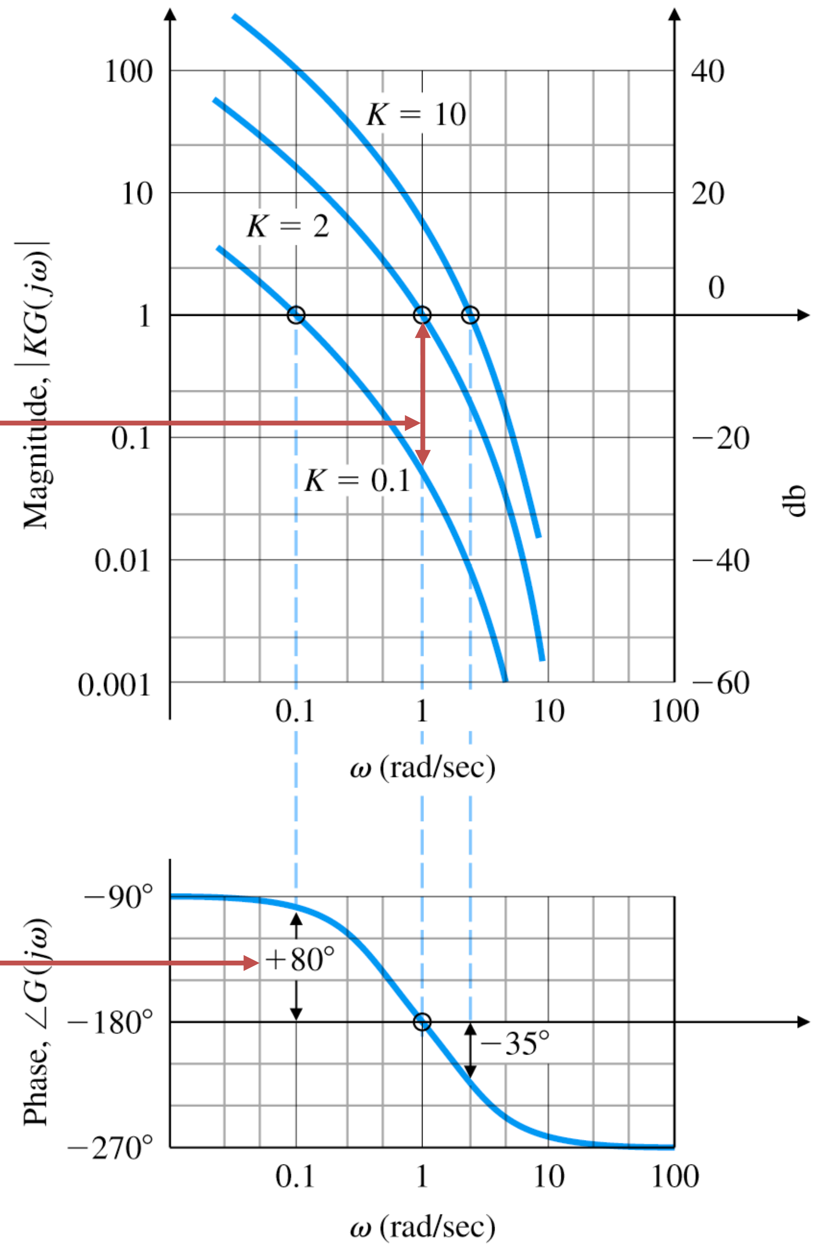
$$K = 2 \rightarrow \text{GM} = 1 \text{ (0 db)}$$

$$K = 10 \rightarrow \text{GM} = 0.2 \text{ (−14 db)}$$

$$\boxed{K = 0.1 \rightarrow \text{PM} \cong 80°}$$

$$K = 2 \rightarrow \text{PM} \cong 0°$$

$$K = 10 \rightarrow \text{PM} \cong -35°$$



Crossover frequency

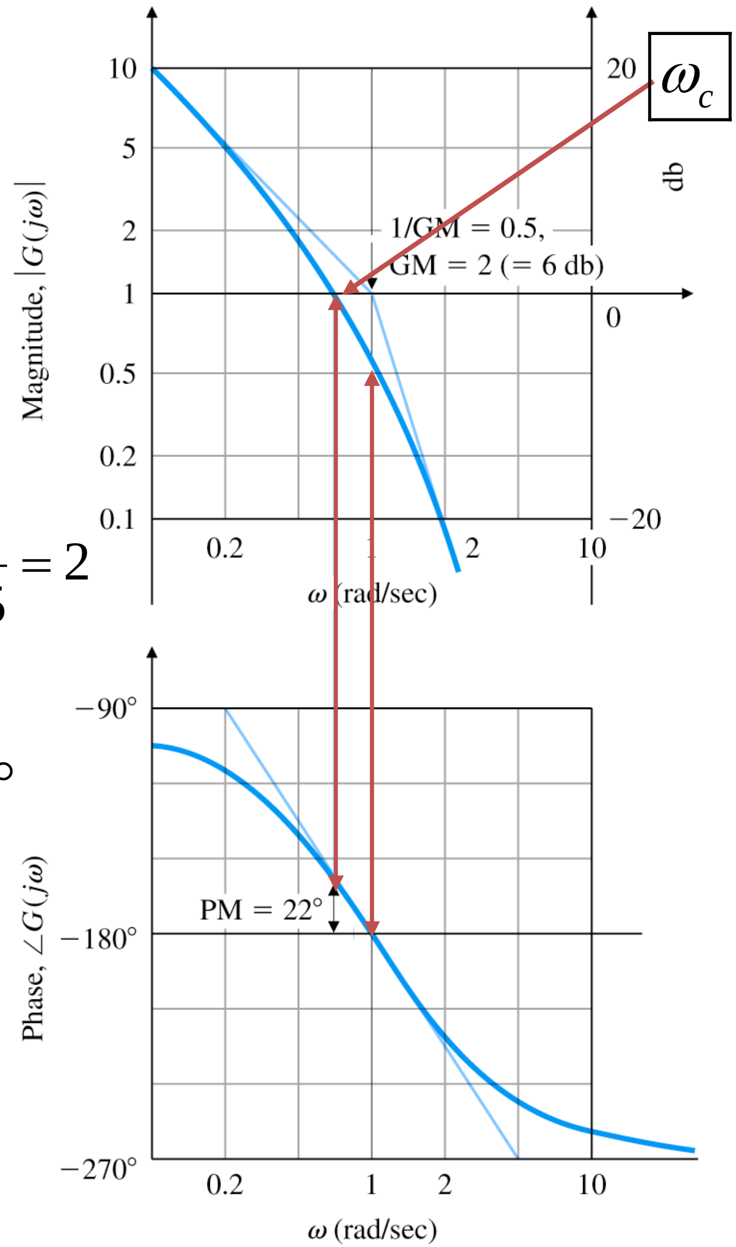
- Crossover frequency (ω_c):
  Frequency at which the gain is unity (0 db)

- Bode plots for $KG(s) = K\dfrac{1}{s(s+1)^2}$

  for $K = 1$

$$\mathrm{GM} = \frac{1}{0.5} = 2$$

$$\mathrm{PM} \cong 22°$$



Computing phase margin

- Find PM for different values of K.

$K = 5 \rightarrow \text{PM} = -22°$

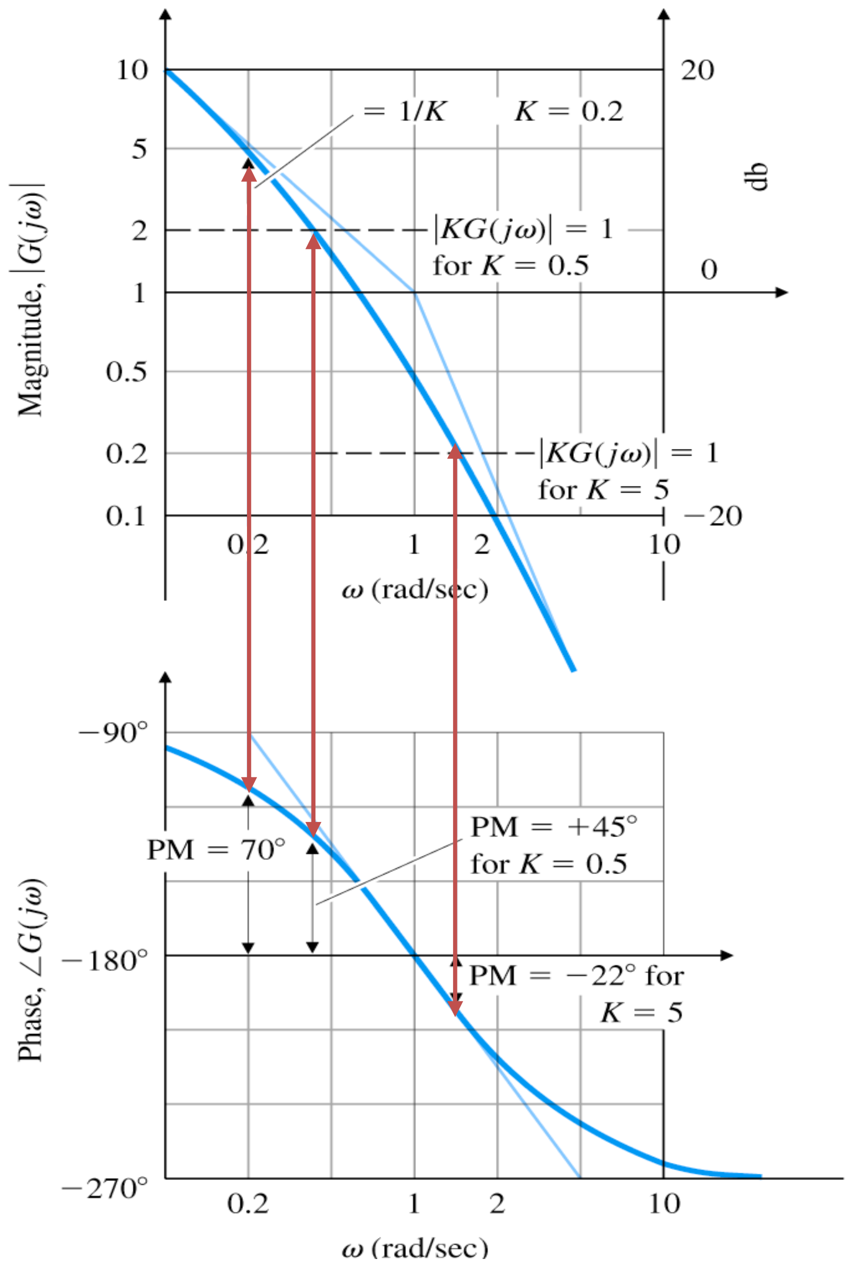
$K = 0.5 \rightarrow \text{PM} = +45°$

- Find K for a desired PM

$\text{PM} = +70° \rightarrow \omega = 0.2$

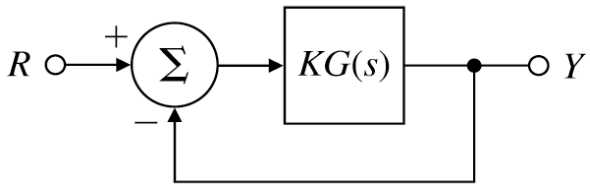
$\rightarrow 1/K = |G(j0.2)| = 5$

$\rightarrow K = 0.2$



Phase margin and damping ratio

- Relation between PM and damping ratio.



Open-loop 2nd-order system: $G(s) = \dfrac{\omega_n^2}{s(s+2\zeta\omega_n)}$

Closed-loop system (with unity feedback): $T(s) = \dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

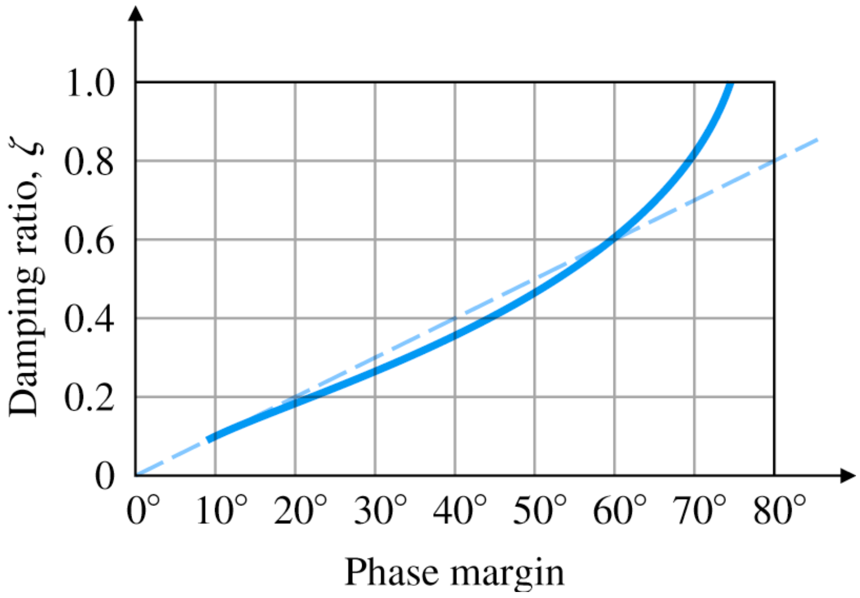
$$PM = \tan^{-1}\left[\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4}-2\zeta^2}}\right]$$

(Approximately a straight line up to about PM = 60°)

- A straight line approximation:

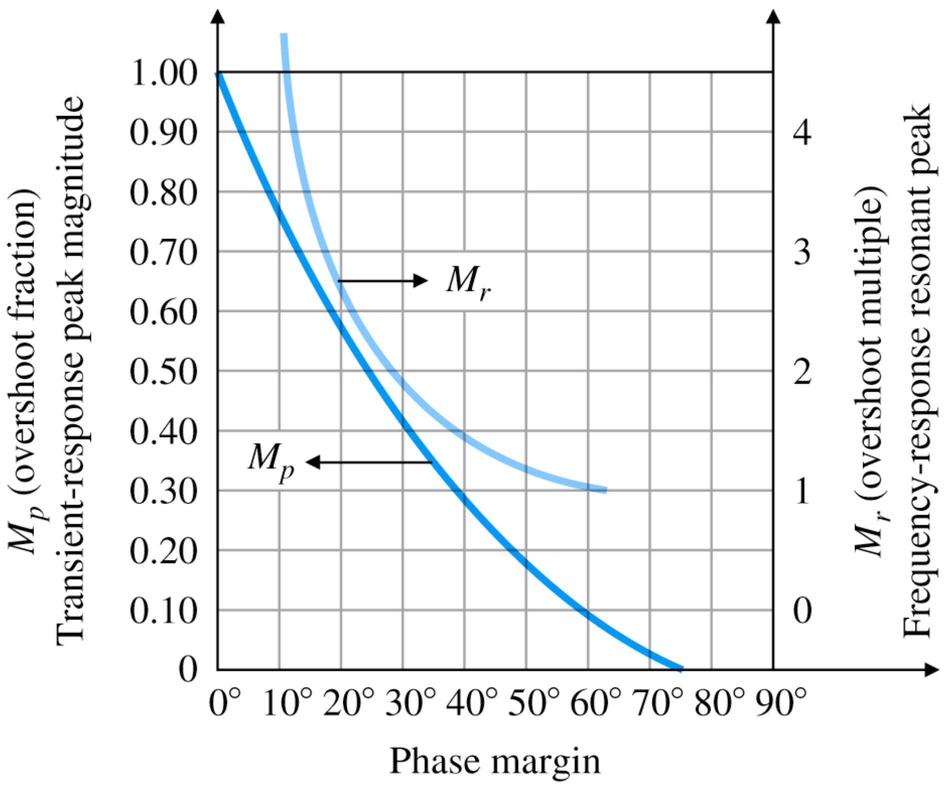
$$\zeta \cong \frac{PM}{100} \quad (\text{below PM} = 70°)$$

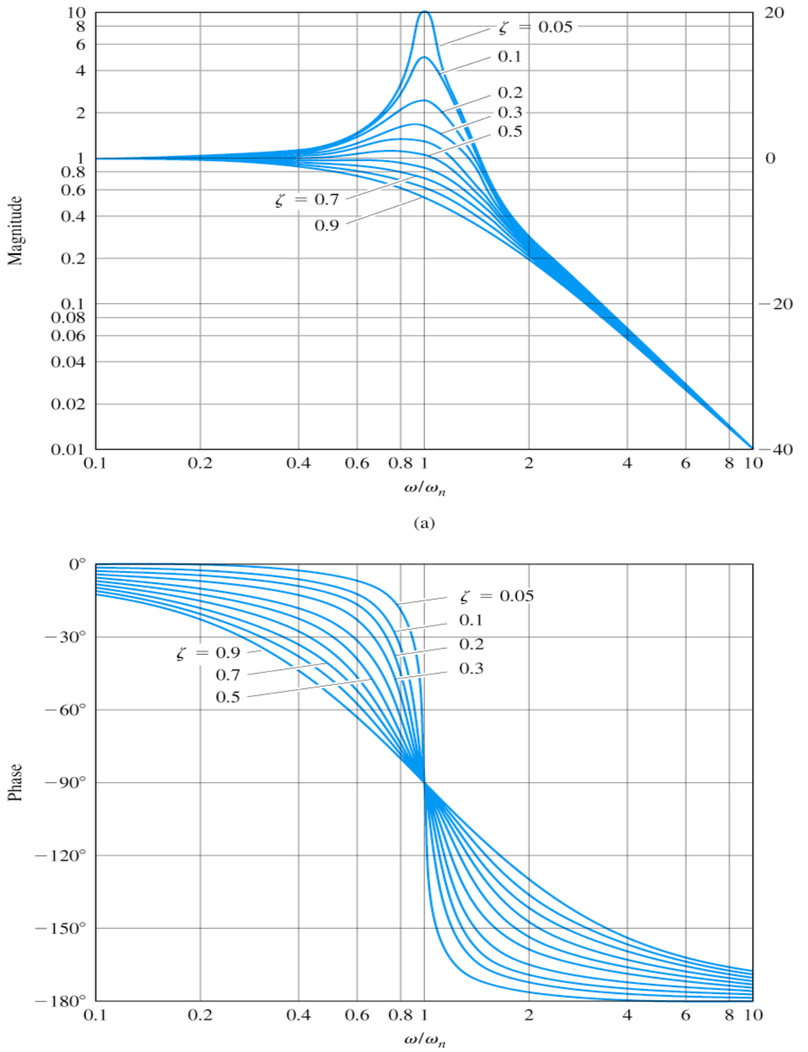
- The gain margin for the 2nd-order system is infinite.



Resonant peak, overshoot

- Resonant peak vs PM, Overshoot vs PM (or damping ratio)



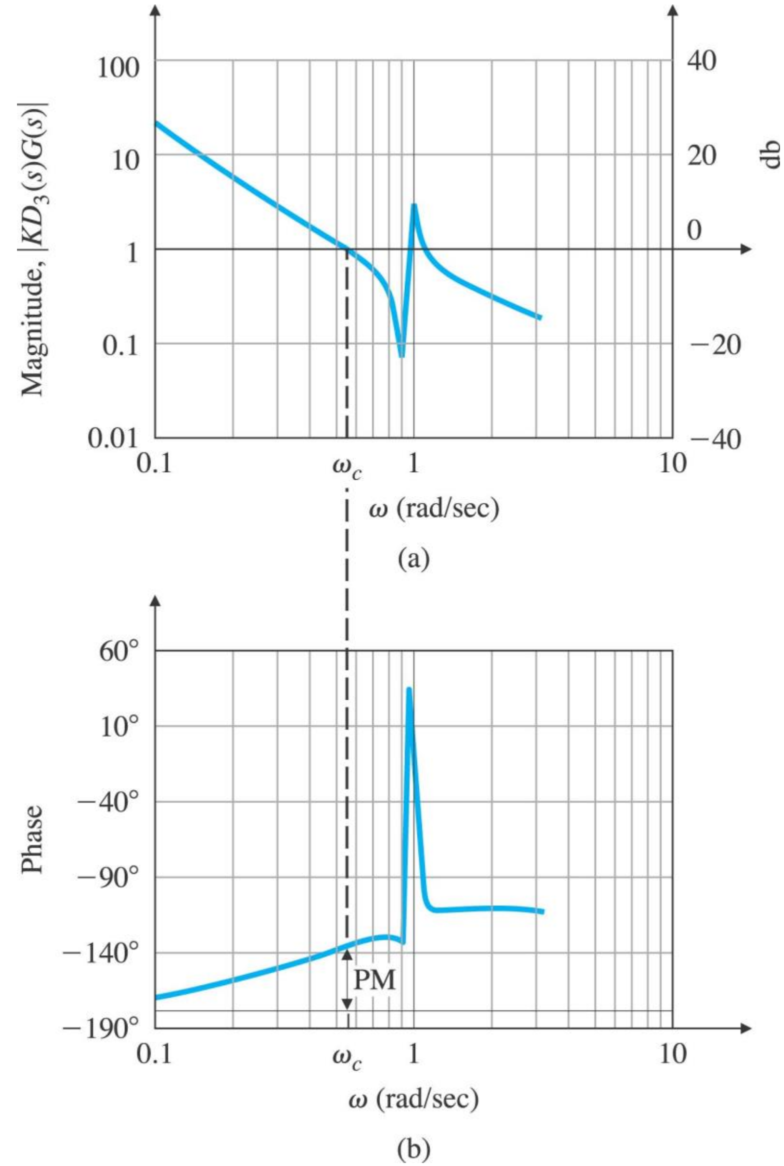


Summary of stability margins

- Summary

  - Design guideline for stability: PM $\geq 30°$
  - The crossover frequency describes the system's speed of response.
  - For 1st and 2nd-order systems, the phase never crosses the 180° line.

  $\rightarrow$ GM is always ∞
  - For higher-order systems, it is possible to have more than one frequency where $|KG(j\omega)| = 1$ or where $\angle KG(j\omega) = 180°$.
  - Conservative assessment: use the crossover freq. with mim. value of PM.



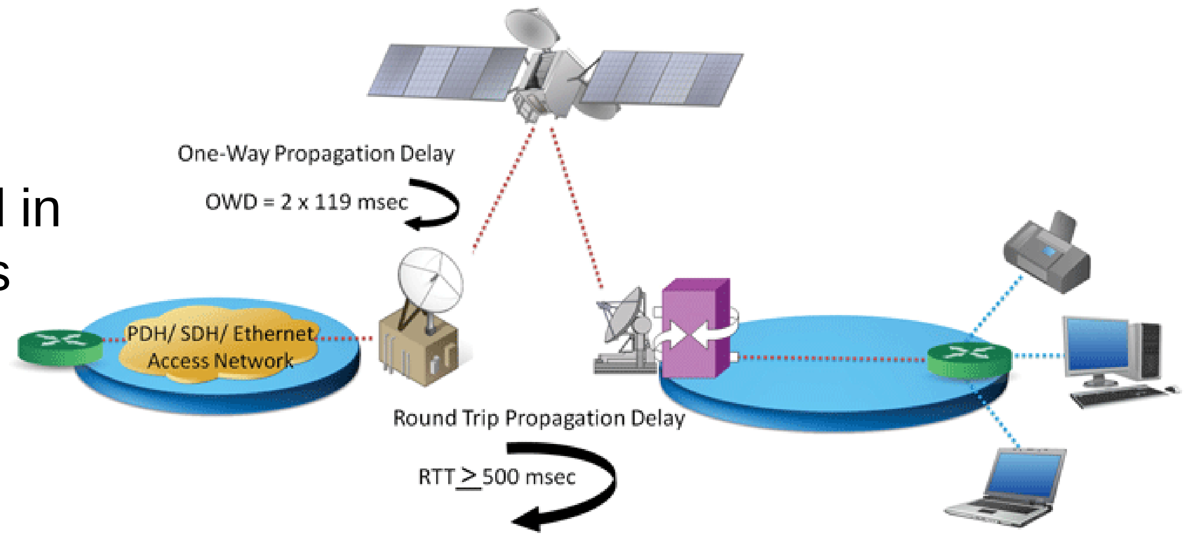
$\boxed{\text{PM} \cong 43°}$

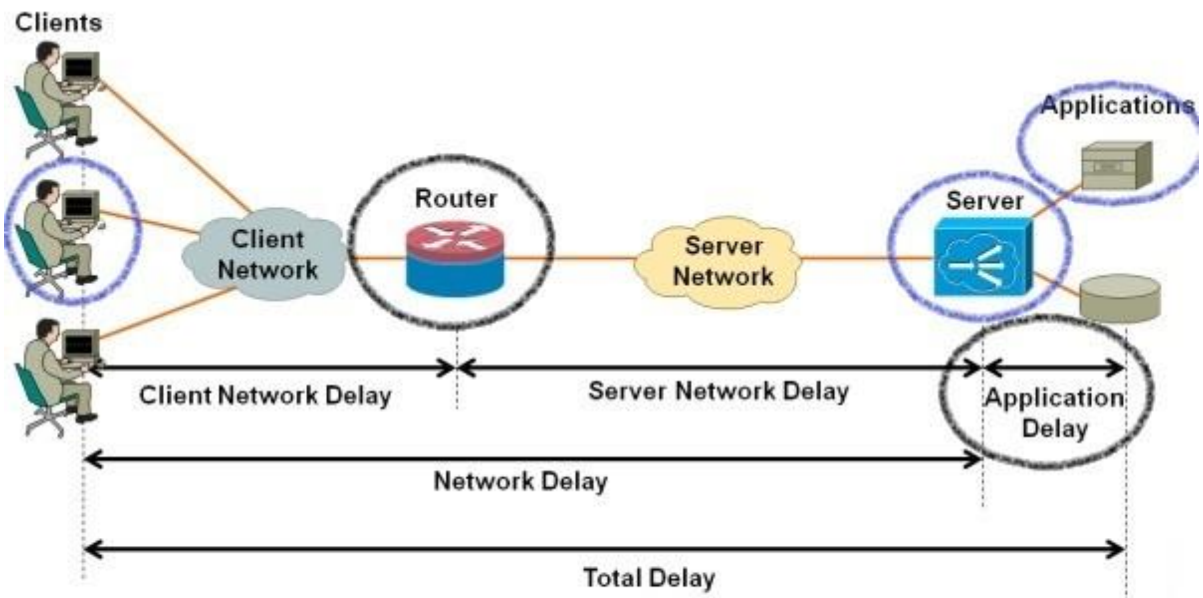
5. Effect of time-delay

Time-delay, network-delay

- Examples of time-delay

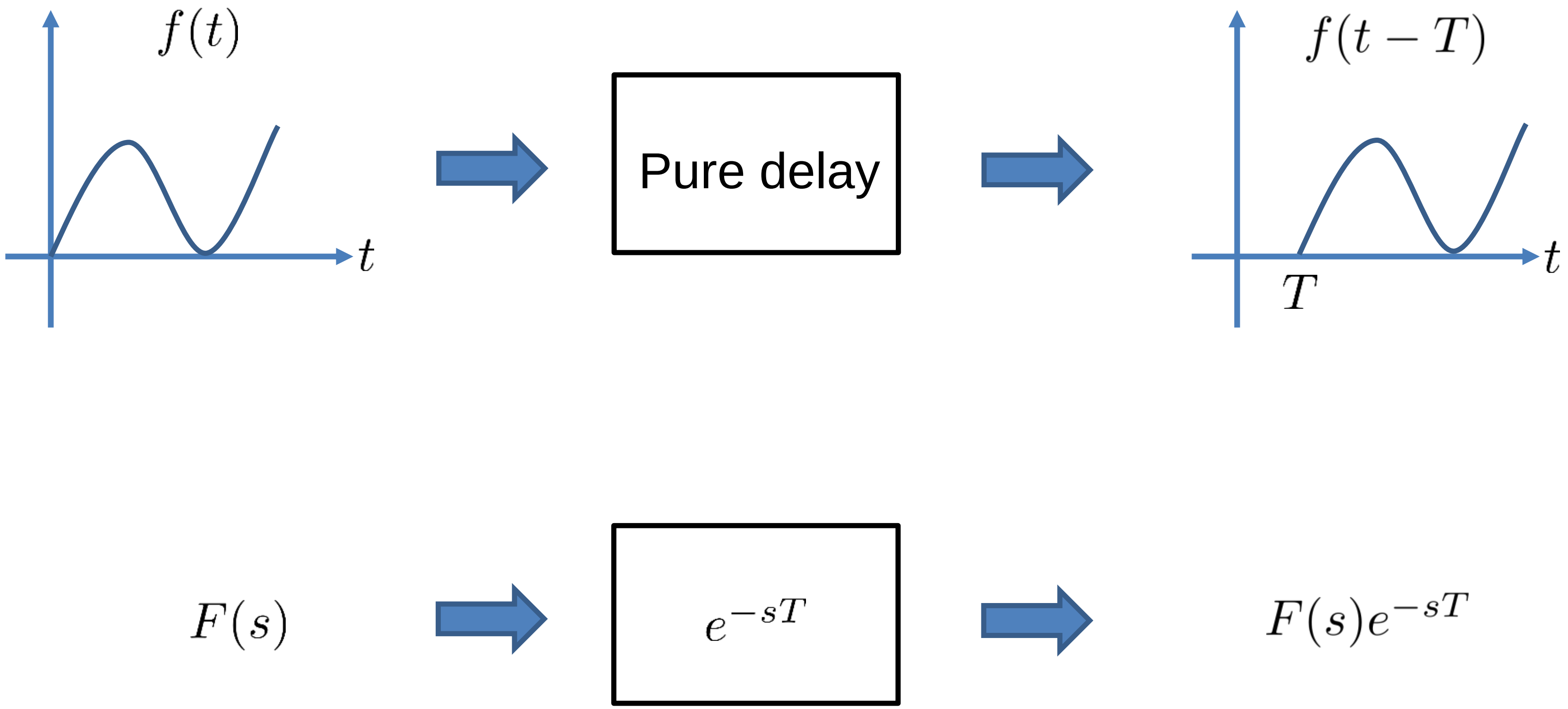
Time-delay is often occurred in feedback loops with wireless network control systems





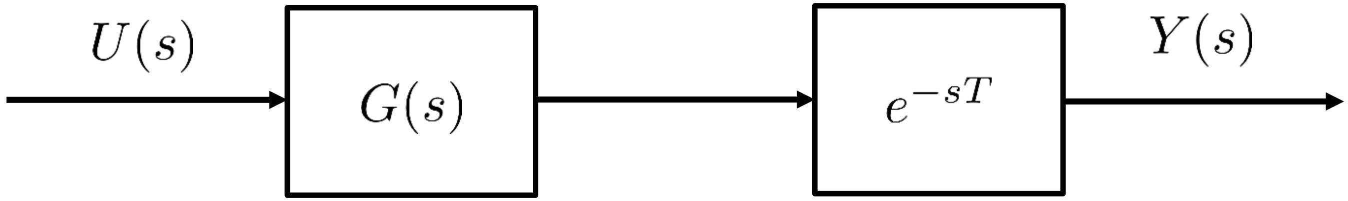
Mathematical description of time-delay

- Time-delay



Time-delay in open-loop systems

- Open-loop control system with time-delay



- Poles of system with time-delay

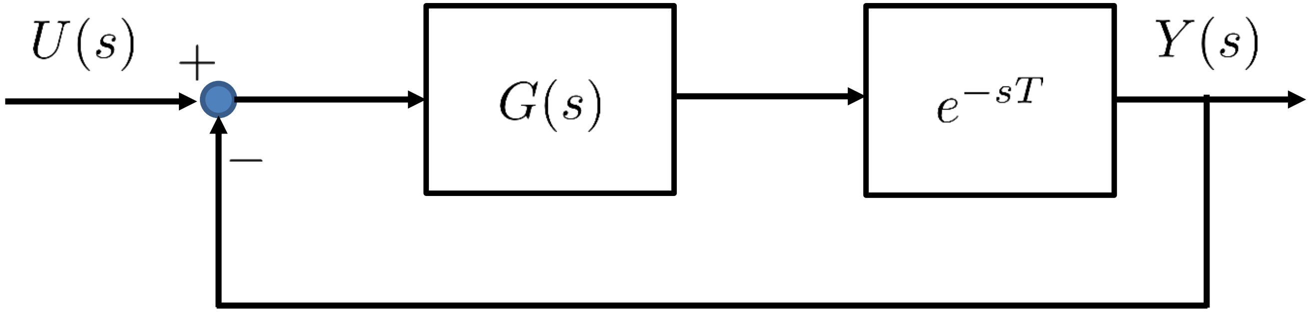
$$G(s) = \frac{b(s)}{a(s)} \Rightarrow \frac{b(s)e^{-sT}}{a(s)}$$

$\rightarrow$ Poles are independent of time-delay in open-loop systems

$\rightarrow$ Time-delay does not affect the stability of open-loop systems

Time-delay in closed-loop systems

- Closed-loop system with time-delay



- Poles of system with time-delay

$$G(s) = \frac{b(s)}{a(s)} \Rightarrow 1 + G(s)e^{-sT} = 1 + \frac{b(s)}{a(s)}e^{-sT} = 0$$

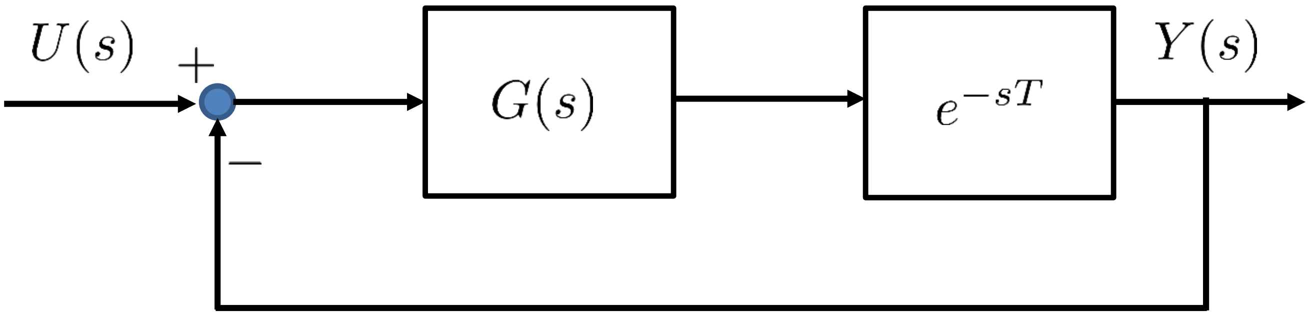
$$\Rightarrow a(s) + b(s)e^{-sT} = 0$$

$\rightarrow$ Poles are dependent on time-delay in closed-loop systems

$\rightarrow$ Time-delay do affect the stability of closed-loop systems

Stability of time-delay feedback systems

- Necessary and sufficient condition for stability



All the roots of $a(s) + b(s)e^{-sT} = 0$ are in OLHP

$\Updownarrow$

The closed-loop system is stable

- There are generally an infinite number of roots