EECE423-01: 현대제어이론

Modern Control Theory

Chapter 4: Response of LTI Systems

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◆ The main topics of this chapter are

1. Basic Concepts of Linear Systems

2. Equivalent State-Space Systems

3. Matrix Exponential

4. Solutions to LTI Systems

Appendix: Solutions to LTV Systems

1. Basic Concepts of Linear Systems

◆ Causality

- Intuitive interpretaion: If a system has the property that the output *before* some time t does not depend on the input *after* time t.

  Such a system is called *causal*.

- Mathematical interpretaion: An operator $\mathbf{T}$ is said to be *causal* if

$$(\mathbf{T}f)_\tau = (\mathbf{T}f_\tau)_\tau, \quad \forall \tau \geq 0$$

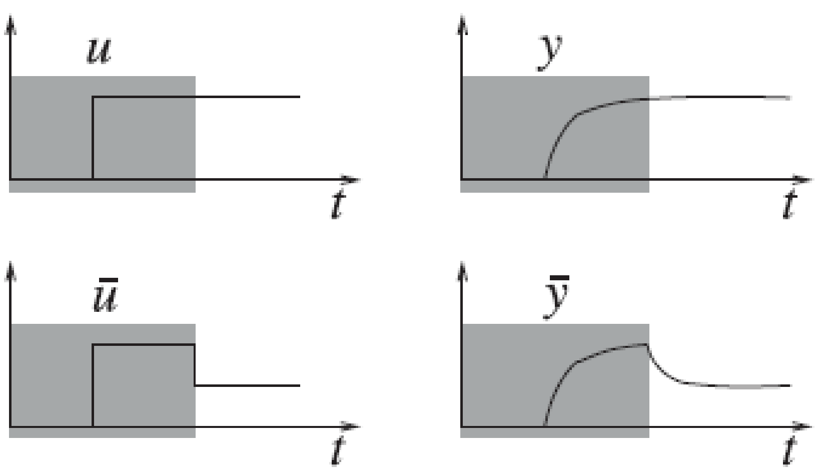
for an arbitray f, where

$$f_\tau(t) = \begin{cases} f(t) & 0 \leq t \leq \tau \\ 0 & \tau < t \end{cases}$$

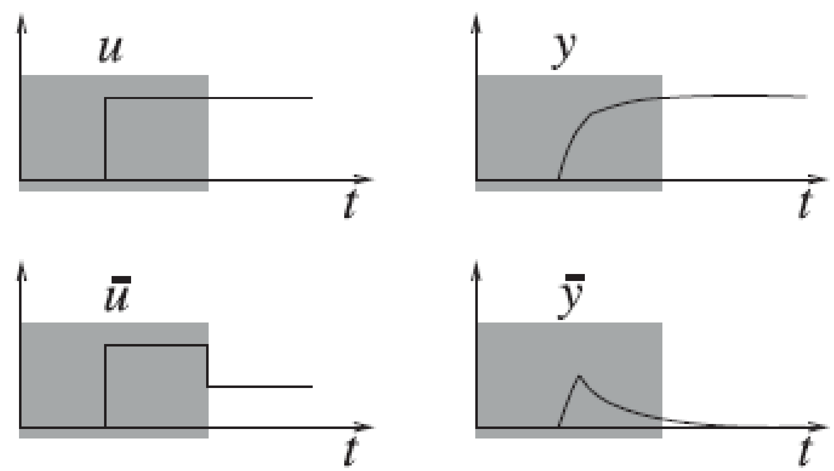
and it is called the truncation of f to the interval $[0, \tau]$.

- Remark: When we consider the *causality* of the oprator describing the input/output behavior for LTI systems, the effect of the initial state (i.e., $x(0)$) is implicitly ignored. In general, there are many different outputs according to different initial condtions, even for the same inputs.

◆ Example



(a) Causal system



(b) Noncausal system

◆ Time invariance

- Intuitive interpretaion: If a system has the property that time-shifting of its inputs results in time-shifting of the outputs, the systems is called *time-invariant*.

- Mathematical interpretaion: An operator $\mathbf{T}$ is said to be *time-invariant* if

$$\mathbf{T}S_\tau = S_\tau\mathbf{T}, \quad \forall\tau \in \mathbb{R}$$

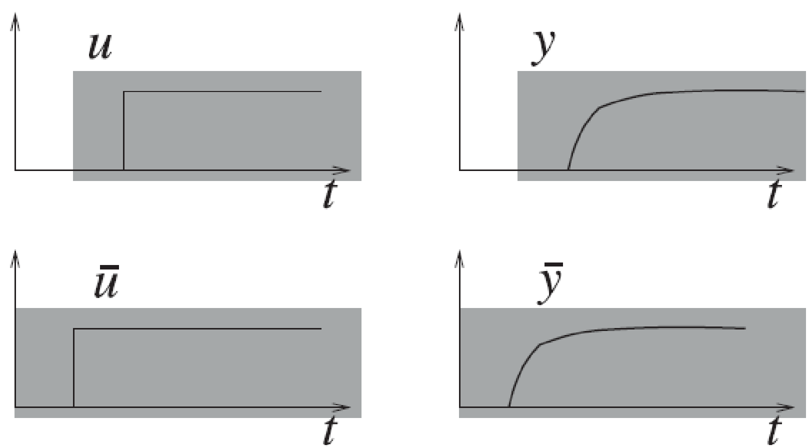
where S_τ is a shift operator defined as

$$S_\tau(f(t)) := f(t-\tau)$$

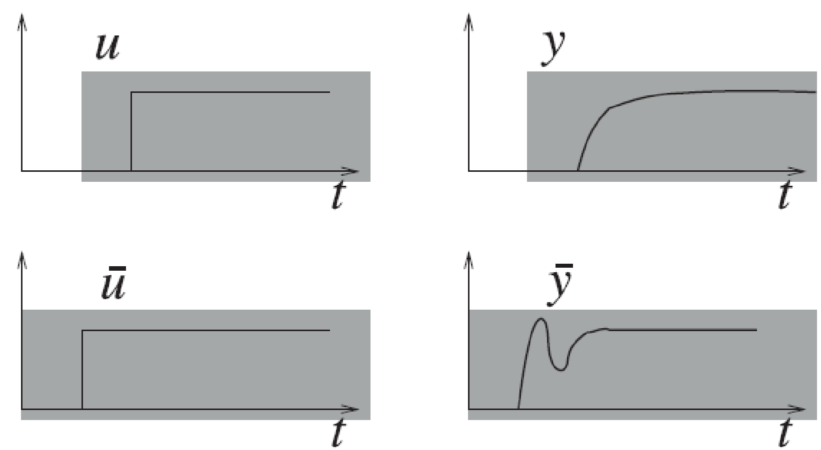
with a constant τ.

- Remark: Similarly for the *causality*, the effect of the initial state is implicitly igonored when we consider the *time-invariance* of the operator describing the input/output behavior for LTI systems.

◆ Example



(a) Time-invariant system



(b) Not time-invariant system

◆ Linearity

- Intuitive interpretaion: A system is regarded as a *linaer* system when it can be viewed as a linear map from its inputs to corresponding outputs.

- Mathematical interpretaion: Let y_1 and y_2 be the outputs of a state-space system corresponding to the inputs u_1 and u_2, respectively. Then, the system is *linear* in the sense that for every $\alpha, \beta \in \mathbb{R}$, $\alpha y_1 + \beta y_2$ is the output corresponding to the input $\alpha u_1 + \beta u_2$.

- Remark: $\alpha y_1 + \beta y_2$ is one of the outputs corresponding to the input $\alpha u_1 + \beta u_2$. In general, there may be many other outputs (obtained from different initial conditions) that will not be of this form.

◆ Characteristics of LTI systems

Let us consider the following state-space system:

$$\begin{cases} \dfrac{dx}{dt} & = Ax + Bu \\ y & = Cx + Du \end{cases}$$

- This system is causal, linear, and time-invariant.
- This system is called a continuous-time linear time-invariant (LTI) system

2. Equivalent State-Space Systems

◆ Mathematical representations of control systems

$$
\begin{cases}
\dfrac{dx}{dt} &= Ax + Bu \\
y &= Cx + Du
\end{cases}
$$

Not uniformly determined

(e.g., controllable canonical form,
observable canonical form)

Uniformly determined

(e.g., $C(sI - A)^{-1}B + D$)

$$
G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} + d
$$

→ There are various equivalent state-space equations for a given transfer function

◆ Similarity transformation

Let us consider the follwoing state-space system:

$$\begin{cases} \dfrac{dx}{dt} & = Ax + Bu \\ y & = Cx + Du \end{cases}$$

For a given nonsingular matrix T, define a new state vector as

$$\tilde{x} := Tx$$

Here, the relation between x and $\tilde{x}$ is one-to-one and onto.

For $\tilde{x}$, we can obtain the following:

$$\begin{cases} \dfrac{d\tilde{x}}{dt} & = T\dfrac{dx}{dt} = TAx + TBu = TAT^{-1}\tilde{x} + TBu \\ y & = Cx + Du = CT^{-1}\tilde{x} + Du \end{cases}$$

This can be written as

$$\begin{cases} \dfrac{d\tilde{x}}{dt} & = \tilde{A}\tilde{x} + \tilde{B}u \\ y & = \tilde{C}\tilde{x} + \tilde{D}u \end{cases}$$

where

$$\tilde{A} := TAT^{-1},\ \tilde{B} := TB,\ \tilde{C} := CT^{-1},\ \tilde{D} := D$$

This procedure is called a *similarity transformation* or an *equivalent transformation*.

◆ Algebraically equivalent

Two continuous-time LTI systems

$$\begin{cases} \dfrac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du \end{cases} \qquad \text{or} \qquad \begin{cases} \dfrac{d\tilde{x}}{dt} &= \tilde{A}\tilde{x} + \tilde{B}u \\ y &= \tilde{C}\tilde{x} + \tilde{D}u \end{cases}$$

respectively, are called *algebraically equivalent* if there exists a nonsingular matrix T such that the followings hold:

$$\tilde{A} := TAT^{-1},\ \tilde{B} := TB,\ \tilde{C} := CT^{-1},\ \tilde{D} := D$$

The corresponding map $\tilde{x} = Tx$ is called

a *similarity transformation* or an *equivalent transformation.*

◆ Properties of algebraically equivalent

Suppose that two state-space LTI systems are algebraically equivalent.

(1) With every input u, both systems assocaite the same set of outputs y. However, the output is generally not the same for the same initial conditions, except the case when the initial condtions are zero.

(2) The systems are zero-state equivalent, i.e., both systems have the same transfer function. However, zero-state equivalece does not imply algebraic equivalence.

$$
\begin{aligned}
\tilde{C}(sI-\tilde{A})^{-1}\tilde{B}+\tilde{D} &= CT^{-1}(sI-TAT^{-1})^{-1}TB+D \\
&= C(T^{-1}(sI-TAT^{-1})T)^{-1}B+D \\
&= C(sT^{-1}IT-T^{-1}TAT^{-1}T)^{-1}B+D \\
&= C(sI-A)^{-1}B+D
\end{aligned}
$$

◆ Example

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 2 \end{bmatrix},\ D = 1$$

Let $T = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ together with $T^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

Then, $\tilde{A} = TAT^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},\ \tilde{B} = TB = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\tilde{C} = CT^{-1} = \begin{bmatrix} -1 & -2 \end{bmatrix},\ \tilde{D} = D = 1$$

$$\to C(sI - A)^{-1}B + D = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} = \frac{s^2 + 4s + 1}{s^2 + 3s + 2}$$

3. Matrix Exponential

◆ Characteristics of solutions to LTI state-space systems

Consider the solutions $x(t)$ and $y(t)$ for the following state-space system:

$$\begin{cases} \dfrac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

(1) If we can explicitly compute $x(t)$, $y(t)$ is readily obtained through $y = Cx + Du$.

(2) $x(t)$ is uniformly determined according to $x(0)$ together with $u(t)$ $(t \geq 0)$.

(3) $x(0)$ is called an *initial vector* or an *initial value*.

◆ Homogenous LTI systems

As a preliminary step to obtain $x(t)$ (as well as $y(t)$),

$$\text{we assume } u(t) \equiv 0 \text{ in } \begin{cases} \dfrac{dx}{dt} & = Ax + Bu \\ y & = Cx + Du \end{cases}.$$

Then, we consider the following homogenous LTI system:

$$\frac{dx}{dt} = Ax \quad \text{with} \qquad x(0) = x_0 \in \mathbb{R}^n$$

◆ Solutions to homogenous LTI systems

$$\frac{dx}{dt} = Ax \quad \text{with} \quad x(0) = x_0 \in \mathbb{R}^n$$

In the case of a scalar system (i.e., $n = 1$ and $A = a$),

$$x(t) = e^{at} x_0$$

Similarly, in the case of a multi-variable system (i.e., $n \geq 2$),

$$x(t) = e^{At} x_0$$

$\rightarrow$ How can we define e^{At}?

◆ Definition of matrix exponential

$$\frac{dx}{dt} = Ax \quad \text{with} \quad x(0) = x_0 \in \mathbb{R}^n$$

$$\rightarrow x(t) = e^{At} x_0$$

Here, e^{At} is called a *state transition matrix.*

Motivated by Taylor series of the scalar exponential,

$$\boxed{e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{n!}A^nt^n + \cdots}$$

◆ Properties of matrix exponential

(1) The ith column of e^{At} is the unique solution to

$$\frac{dx}{dt} = Ax, \quad x(0) = e_i, \quad t \geq 0,$$

where e_i is the ith standard basis of $\mathbb{R}^n$.

(2) For every $t, \tau \in \mathbb{R}$,

$$e^{At} e^{A\tau} = e^{A(t+\tau)}.$$

(3) For every $t \in \mathbb{R}$, e^{At} is nonsingular and

$$(e^{At})^{-1} = e^{-At}.$$

(4) Regarding to derivative, the followig holds:

$$\frac{de^{At}}{dt} = A + A^2t + \frac{1}{2!}A^3t^2 + \cdots = Ae^{At} = e^{At}A.$$

(5) Regarding to integral, the following holds:

$$\int e^{At}dt = C + It + \frac{1}{2!}At^2 + \frac{1}{3!}A^2t^3 + \cdots.$$

Indeed, when $|A| \neq 0$,

$$\int e^{At}dt = A^{-1}e^{At} + C' = e^{At}A^{-1} + C' \quad (C' := C - A^{-1}).$$

(6) When $AB = BA$, the following holds:

$$e^{(A+B)t} = e^{At}e^{Bt}.$$

In general, $e^{(A+B)t} \neq e^{At}e^{Bt}$ when $AB \neq BA$.

◆ Advanced Property of matrix exponential

(7) For every $n \times n$ matrix A, there exists n scalar functions

$\alpha_0(t),\ \alpha_1(t), \ldots, \alpha_{n-1}(t)$ for which

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i(t) A^i, \quad \forall t \in \mathbb{R}.$$

Proof: Recall the Cayley-Hamilton theorem that

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_n I = 0,$$

where the a_i $(i = 1, \ldots, n)$ are the coefficients of the characteristic

equation of A, i.e., $|sI - A| = 0$. Therefore,

$$A^n = -a_1 A^{n-1} - a_2 A^{n-2} - \cdots - a_{n-1} A - a_n I.$$

This implies that

$$
\begin{aligned}
A^{n+1} &= -a_1 A^n - a_2 A^{n-1} - \cdots - a_{n-1} A - a_n A \\
&= a_1(a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_n I) - a_2 A^{n-1} - \cdots - a_n A \\
&= (a_1^2 - a_2) A^{n-1} + (a_1 a_2 - a_3) A^{n-2} + \cdots + (a_1 a_{n-1} - a_n) A + a_1 a_n I.
\end{aligned}
$$

It turns out that A^{n+1} can be written as a linear combination of $A^{n-1}, \ldots, A, I$.

Applying the same procedure for increasing powers of A, every A^k $(k \geq 0)$ can be written as

$$
A^k = \overline{a}_{n-1}(k) A^{n-1} + \cdots + \overline{a}_1(k) A + \overline{a}_0(k) I \quad \forall k \geq 0.
$$

with appropriate coefficients $\overline{a}_i(k)$.

By substituting this into the definition of e^{At}, we obtain

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{i=0}^{n-1} \overline{a}_i(k) A^i.$$

Exchanging the order of summation means

$$e^{At} = \sum_{i=0}^{n-1} \left(\sum_{k=0}^{\infty} \frac{t^k \overline{a}_i(k)}{k!} \right) A^i.$$

Defining $\alpha_i(t) := \sum_{k=0}^{\infty} \frac{t^k \overline{a}_i(k)}{k!}$ completes the proof.

◆ Methods for computing matrix exponential

Question: How can we compute e^{At}?

(1) Direct computation of $e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{n!}A^nt^n + \cdots$

$\rightarrow$ In general, only an approximate computation is possible.

(2) Laplace transform of

$$\frac{d}{dt}e^{At} = Ae^{At}, \quad e^{A\cdot 0} = I, \quad t \geq 0.$$

$\rightarrow$ Let $f(t) := e^{At}$, and use $\mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0)$.

In other words, we have

$$sF(s) - I = AF(s)$$

where $F(s) := \mathcal{L}\{e^{At}\}$. Hence,

$$F(s) = (sI - A)^{-1} \Leftrightarrow e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

Thus,

$$\boxed{e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}}$$

(3) When A is a diagonalizable matrix,

$$T^{-1}AT = \Lambda := \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

where $\lambda_1, \ldots, \lambda_n$ is the eigenvalues of A.

Here, if we note that

$$A = T\Lambda T^{-1} = T \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} T^{-1}$$

together with

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{n!}A^nt^n + \cdots$$

then we obtain

$$
\begin{aligned}
e^{At} &= TIT^{-1} + T(\Lambda t)T^{-1} + T\left(\frac{1}{2!}\Lambda^2 t^2\right)T^{-1} + \cdots \\
&= Te^{\Lambda t}T^{-1} \\
&= T\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} T^{-1}
\end{aligned}
$$

In other words,

$$
\boxed{e^{At} = T\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} T^{-1}}
$$

(4) When A is a non-diagonalizable matrix,

$$J = P^{-1}AP = \begin{bmatrix} J_{p_1}(\lambda_1) & & & \\ & J_{p_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{p_N}(\lambda_N) \end{bmatrix}$$

where $J_{p_i}(\lambda_i)$ is a Jordan block defined as

$$J_{p_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & \lambda_i \end{bmatrix} \qquad (p_i \times p_i)$$

and each λ_i is an eigenvalue of A.

Similarly for the case of a diagonalizable matrix,

$$A = PJP^{-1} = P \begin{bmatrix} J_{p_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{p_N}(\lambda_N) \end{bmatrix} P^{-1}$$

and thus

$$e^{At} = Pe^{Jt}P^{-1} = P \begin{bmatrix} e^{J_{p_1}(\lambda_1)t} & & \\ & \ddots & \\ & & e^{J_{p_N}(\lambda_N)t} \end{bmatrix} P^{-1}$$

where $e^{J_{p_i}(\lambda_i)t}$ is given by

$$e^{J_{p_i}(\lambda_i)t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \dfrac{t^{p_i-1}}{(p_i-1)!}e^{\lambda_i t} \\ & e^{\lambda_i t} & \ddots & \vdots \\ & & \ddots & te^{\lambda_i t} \\ & & & e^{\lambda_i t} \end{bmatrix}$$

To put it another way,

$$e^{At} = P \begin{bmatrix} e^{J_{p_1}(\lambda_1)t} & & \\ & \ddots & \\ & & e^{J_{p_N}(\lambda_N)t} \end{bmatrix} P^{-1}$$

with

$$e^{J_{p_i}(\lambda_i)t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \frac{t^{p_i-1}}{(p_i-1)!}e^{\lambda_i t} \\ & e^{\lambda_i t} & \ddots & \vdots \\ & & \ddots & te^{\lambda_i t} \\ & & & e^{\lambda_i t} \end{bmatrix}$$

◆ Example

For $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, let us compute e^{At}.

$$\text{(a) } (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2}{s+1} - \dfrac{1}{s+2} & \dfrac{1}{s+1} - \dfrac{1}{s+2} \\ \dfrac{-2}{s+1} + \dfrac{2}{s+2} & \dfrac{-1}{s+1} + \dfrac{2}{s+2} \end{bmatrix}$$

Thus,

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

(b) A has eigenvalues $\lambda_1 = -1,\ \lambda_2 = -2$ together with

corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Then, by $T = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $T^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$,

$$
\begin{aligned}
e^{At} = T \begin{bmatrix} e^{-t} & \\ & e^{-2t} \end{bmatrix} T^{-1} &= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & \\ & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}
\end{aligned}
$$

◆ Case study for state transition matrix

- $\Phi(t) := e^{At}$
- $\Phi_{ij}(t)$: (i, j)th element of $\Phi(t)$
- λ_k $(k = 1, \ldots, n)$: kth eigenvalue of A

(1) $\mathrm{Re}(\lambda_k) < 0$ $(\forall k = 1, \ldots, n)$

$$\Rightarrow \Phi_{ij}(t) \to 0 \ (t \to \infty), \quad \forall i = 1, \ldots, n; \ \forall j = 1, \ldots, n$$

(2) There exists a λ_k such that $\mathrm{Re}(\lambda_k) > 0$

$$\Rightarrow \text{There exists a } \Phi_{ij}(t) \text{ such that } \Phi_{ij}(t) \to \infty \ (t \to \infty)$$

(3) $\mathrm{Re}(\lambda_k) \leq 0$ $(\forall k = 1, \ldots, n)$

(i) All the Jordan blocks corresponding to $\mathrm{Re}(\lambda_k) = 0$ have size of 1

$$\Rightarrow \Phi_{ij}(t) \text{ is bounded for } t \geq 0, \quad \forall i = 1, \ldots, n;\ \forall j = 1, \ldots, n$$

(ii) There exists a Jordan block corresponding to $\mathrm{Re}(\lambda_k) = 0$ with size larger than 1

$$\Rightarrow \text{There exists a } \Phi_{ij}(t) \text{ such that } \Phi_{ij}(t) \to \infty \ (t \to \infty)$$

→ This will be also considered in ‘Chapter 5. Stability.’

◆ Another interpretation for state transition matrix

(a) There exists a complex eigenvalue of A

$\Rightarrow e^{At}$ is vibrational as t becomes larger

(b) All the eigenvalues of A are real

$\Rightarrow e^{At}$ is not vibrational as t becomes larger

(c) 0 is an eigenvalue of A

$\Rightarrow e^{At}$ plays as an integrator

4. Solutions to LTI Systems

◆ Solutions to general case

Consider the general case $u(t) \not\equiv 0$ for the following system:

$$\frac{dx}{dt} = Ax + Bu$$

Assume that

$$x(t) = e^{At}(x(0) + z(t)), \quad z(0) = 0$$

Then, $x(t)$ satisfies $x(0) = x_0$.

By substituting $x(t)$ into $\dfrac{dx}{dt} = Ax + Bu$, we obtain

$$Ae^{At}(x(0) + z(t)) + e^{At}\dot{z}(t) = Ae^{At}(x(0) + z(t)) + Bu$$

This implies that

$$\dot{z}(t) = e^{-At}Bu(t)$$

This together with $z(0) = 0$ leads to

$$z(t) = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

Thus, we can obtain the following solution:

$$\boxed{x(t) = e^{At}\left(x(0) + \int_0^t e^{-A\tau}Bu(\tau)d\tau\right)}$$

We return to the following case with the output equation:

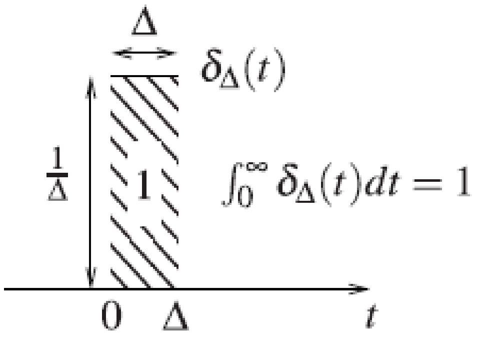
$$\begin{cases} \frac{dx}{dt} & = Ax + Bu \\ y & = Cx + Du \end{cases}$$

Then, we have the following solution:

$$\boxed{y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)}$$

◆ Impulse signal

$$\delta(t) := \lim_{\Delta \to 0} \delta_\Delta(t)$$



◆ Impulse response

$$\begin{cases} \dfrac{dx}{dt} & = Ax + Bu \\ y & = Cx \end{cases}$$

Compute $y(t)$ when $u(t) = \delta(t)e_i$ with $x(0) = 0$.

By substituting $u(t) = \delta(t)e_i,\ x(0) = 0$ toether with $D = 0$ into

$$\boxed{y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)}$$

we obtain

$$\begin{aligned} y(t) = C\int_0^t e^{A(t-\tau)}B\delta(\tau)e_i d\tau &= Ce^{At}\lim_{\Delta\to 0}\int_0^{\Delta} e^{-A\tau}B\frac{1}{\Delta}e_i d\tau \\ &= Ce^{At}Be_i \end{aligned}$$

Thus, we have

$$\boxed{y(t) = Ce^{At}Be_i}$$

Appendix: Solutions to LTV Systems

◆ Causality

- Intuitive interpretaion: If a system has the property that the output *before* some time t does not depend on the input *after* time t.

  Such a system is called *causal*.

- Mathematical interpretaion: The state-space system is *causal* in the sense that if y is one of the outputs that corresponds to the input u, then for every other input $\overline{u}$ for which

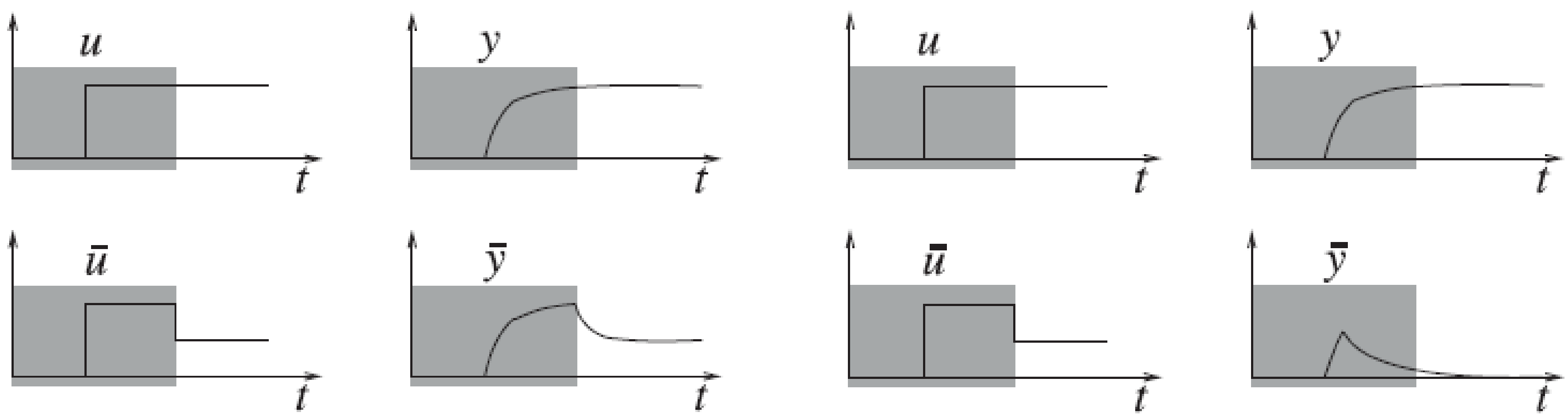
$$\overline{u}(t) = u(t), \quad 0 \leq \forall t < T$$

  for some $T > 0$, the system exhibits (at least) one output $\overline{y}$ that satisfies

$$\overline{y}(t) = y(t), \quad 0 \leq \forall t < T$$

- Remark: The *causality* property does not mean "for every input $\overline{u}$ that matches u on $[0, T)$, every output $\overline{y}$ matches y on $[0, T)$." In general, only one output $\overline{y}$ (obtained with the same initial condition) will match y.

◆ Example



(a) Causal system (b) Noncausal system

◆ Homogenous LTV systems

As a preliminary step to obtain $x(t)$ (as well as $y(t)$),

$$\text{we assume } u(t) \equiv 0 \text{ in } \begin{cases} \dfrac{dx(t)}{dt} & = A(t)x(t) + B(t)u(t) \\ y(t) & = C(t)x(t) + D(t)u(t) \end{cases}.$$

Then, we consider the following homogenous LTI system:

$$\frac{dx(t)}{dt} = A(t)x(t) \quad \text{with} \quad x(0) = x_0 \in \mathbb{R}^n$$

◆ Peano-Baker series

$$\frac{dx(t)}{dt} = A(t)x(t) \quad \text{with} \quad x(0) = x_0 \in \mathbb{R}^n$$

The unique solution is given by

$$x(t) = \Phi(t,0)x_0, \quad x_0 \in \mathbb{R}^n, \quad t \geq 0$$

where

$$\begin{aligned}\Phi(t,0) := I &+ \int_0^t A(s_1)ds_1 + \int_0^t A(s_1)\int_0^{s_1} A(s_2)ds_2ds_1 \\ &+ \int_0^t A(s_1)\int_0^{s_1} A(s_2)\int_0^{s_2} A(s_3)ds_3ds_2ds_1 + \cdots .\end{aligned}$$

The $n \times n$ matrix $\Phi(t,0)$ is called the state transition matrix.

◆ Properties of state transition matrix

1. $\Phi(t,0)$ is the unique solution to
$$\frac{d}{dt}\Phi(t,0) = A(t)\Phi(t,0), \quad \Phi(0,0) = I, \quad t \geq 0$$

2. $\Phi(0,0) = I$

3. For every $t_1, t_2 \geq 0$,
$$\Phi(t_2,t_1)\Phi(t_1,0) = \Phi(t_2,0)$$

4. $\Phi(t,0) = \Phi(0,t)^{-1}$

The proofs are readily followed by the definition of $\Phi(t,0)$

◆ Solutions to general LTV systems

We return to the following case with the output equation:

$$\begin{cases} \dfrac{dx(t)}{dt} & = A(t)x(t) + B(t)u(t) \\ y(t) & = C(t)x(t) + D(t)u(t) \end{cases}$$

Then, we have the following solutions:

$$\boxed{\begin{aligned} x(t) &= \Phi(t,0)x(0) + \int_0^t \Phi(t,\tau)B(\tau)u(\tau)d\tau \\ y(t) &= C(t)\Phi(t,0)x(0) + \int_0^t C(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau + Du(t) \end{aligned}}$$