EECE423-01: 현대제어이론

Modern Control Theory

Chapter 6: Controllability

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◆ The main topics of this chapter are

1. Concept for Controllability

2. Conditions for Controllability

3. Similarity Transform and Stabilizability

1. Concepts for Controllability

◆ Motivation of controllability

For the LTI system

$$\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}, \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^k,\ y(t) \in \mathbb{R}^m$$

let us consider the following statment:

> Is there a control input $u(t)$ that move $x(t)$ from any initial state x_0 to any other final state x_1 in a finite time interval?

◆What is a controllability?

If there exist $s \geq 0$ and $u(t)$ $(0 \leq t \leq s)$ that move $x(t)$ from

$x(0) = x_0$ to $x(s) = x_1$ for arbitrary $x_0, x_1 \in \mathbb{R}^n$, the LTI system

$$\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$$

is called a *controllable*. Otherwise, the LTI system is called an *uncontrollable*.

2. Conditions for Controllability

◆Necessary and sufficient condition of controllability

$$\text{The LTI system} \begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases} \text{is controllable}$$

if and only if the following controllability matrix has rank n.

$$U_{\mathrm{c}} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

Based on this, we also say that the pair (A, B) is *controllable*.

Proof: (1. necessary condition)

The solution $x(t)$ at $t = s$ is given by

$$x(s) = e^{As}\left(x(0) + \int_0^s e^{-A\tau}Bu(\tau)d\tau\right)$$

and we can obtain

$$e^{-As}x(s) - x(0) = \int_0^s e^{-A\tau}Bu(\tau)d\tau$$

Here, it readily follows from Cayley-Hamilton Theorem that

$$e^{At} = q_1(t)I + q_2(t)A + \cdots + q_n(t)A^{n-1}$$

where $q_i(t)$ is an adequately defined scalar function.

If we define $h_i = \int_0^s q_i(-\tau)u(\tau)d\tau$,

$$e^{-As}x(s) - x(0) = \int_0^s e^{-A\tau}Bu(\tau)d\tau$$

$$= Bh_1 + \cdots + A^{n-1}Bh_n = U_c \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

To put it another way,

$$\boxed{e^{-As}x(s) - x(0) = U_c \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}}$$

To exist $h_1, \dots, h_n$ (and thus $u(t)$ $(0 \le t \le s)$ that satisfy

$$\boxed{e^{-As}x(s) - x(0) = U_{\mathrm{c}} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}}$$

for arbitrary $x(0) = x_0$ and $x(s) = x_1$ in $\mathbb{R}^n$,

$$\boxed{U_{\mathrm{c}} \text{ should have rank } n.}$$

(2. sufficient condition)

If $\mathrm{rank}(U_\mathrm{c}) = n$, the controllability gramian

$$W_s := \int_0^s e^{At} B B^T e^{A^T t} dt$$

is nonsingular for an arbitrary $s > 0$ (the proof will be provided later).

If we consider for some $s > 0$ the control input defined as

$$u(t) = B^T e^{A^T(s-t)} W_s^{-1}(-e^{As}x_0 + x_1), \quad 0 \leq t \leq s$$

then we obtain the following:

$$x(s) = e^{As}x_0 + \int_0^s e^{A(s-\tau)} B B^T e^{A^T(s-\tau)} d\tau \cdot W_s^{-1}(-e^{As}x_0 + x_1)$$

$$= e^{As}x_0 - e^{As}x_0 + x_1 = x_1.$$

(W_s is a nonsingular matrix)

Assume that $|W_s| = 0$ for some $s > 0$. Then, there exists a nonzero v such that $W_s v = 0$. Hence,

$$v^T W_s v = \int_0^s v^T e^{A\tau} B B^T e^{A^T\tau} v d\tau = \int_0^s \|B^T e^{A^T\tau} v\|_2^2 d\tau = 0$$

This implies that $B^T e^{A^T\tau} v \equiv 0 \quad (0 \le \tau \le s)$. For $f(\tau) := B^T e^{A^T\tau}$,

we obtain $f(0)v = 0, \dfrac{df(\tau)}{d\tau} v|_{\tau=0} = 0, \cdots, \dfrac{d^{n-1}f(\tau)}{d\tau^{n-1}} v|_{\tau=0} = 0$, and thus

$$\begin{bmatrix} B^T \\ \vdots \\ B^T(A^T)^{n-1} \end{bmatrix} v = U_{\mathrm{c}}^T v = 0 \ (v \neq 0).$$ This contradicts $\mathrm{rank}(U_{\mathrm{c}}) = n$.

◆ Example

The LTI system $\dot{x} = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -1 & -1 \\ 2 & -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} u$

is controllable, since the controllability matrix

$$U_\mathrm{c} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 & 1 \\ 0 & 2 & 3 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 3 & 3 \end{bmatrix}$$

has rank 3.

◆ Properties of controllability

For the LTI system $\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$,

the followings are equivalent.

(a) The pair (A, B) is controllable.

(b) The controllabilty matrix $U_{\mathrm{c}} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ has rank n.

(c) The controllabilty gramian $W_s := \displaystyle\int_0^s e^{At} B B^T e^{A^T t} dt$ is positive definite for every $s > 0$.

Proof: The equivalence between (a) and (b) together with the assertion (b)$\Rightarrow$(c) have already been shown. We show (c)$\Rightarrow$(b).

Suppose that the controllability matrix U_c is not of full rank.

Then, there exists a nonzero vector $v \in \mathbb{R}^n$ such that

$$v^T A^k B = 0 \text{ for } k = 0, 1, \ldots, n-1.$$

It readily follows from Cayley-Hamilton theorem that

$$v^T A^k B = 0 \text{ for all } k \in \mathbb{N}.$$

Hence, $v^T e^{At} B = 0, \quad \forall t \geq 0.$

Therefore, $v^T W_s = 0$ and this contradicts (c).

3. Similarity Transform and Stabilizability

◆ Motivation of similarity transform

$$\text{The LTI system } \begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases} \text{ is uncontrollable}$$

$$\Updownarrow$$

$$\operatorname{rank}(U_{\mathrm{c}}) = r < n, \ \text{ where } U_{\mathrm{c}} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

What if the system is uncotrollable?

$\rightarrow$ Similarity transform may play an important role.

◆ Review of similarity transform

Two continuous-time LTI systems

$$\begin{cases} \dfrac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} \dfrac{d\tilde{x}}{dt} &= \tilde{A}\tilde{x} + \tilde{B}u \\ y &= \tilde{C}\tilde{x} + \tilde{D}u \end{cases}$$

Algebraically equivalent

are called *algebraically equivalent* if there exists a nonsingular matrix T such that the followings hold:

$$\tilde{A} := TAT^{-1}, \ \tilde{B} := TB, \ \tilde{C} := CT^{-1}, \ \tilde{D} := D$$

The corresponding map $\tilde{x} = Tx$ is called a similarity transform.

◆ Controllable decomposition

Suppose that the LTI system $\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$

is uncotrollable, i.e.,

$$\operatorname{rank}(U_{\mathrm{c}}) = r < n, \text{ where } U_{\mathrm{c}} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

Then, there exists a similarity transform matrix T such that the following assertions are true.

(a) The transformed pair has the form

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{B} = TB = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix},$$

where $\tilde{A}_{11} \in \mathbb{R}^{r \times r}$ and $\tilde{B}_1 \in \mathbb{R}^{r \times k}$.

(b) The pair $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable.

This is called a *controllable decomposition*.

proof: (a) Let $v_1,\ v_2, \dots, v_r$ be linearly independent columns of the controllability matrix U_c. We complete them by $n-r$ linearly independent vectors $v_{r+1}, v_{r+2}, \dots, v_n$ such that the matrix

$$Q = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

is nonsingular and show that $T = Q^{-1}$ has the desired property.

Because of the Cayley-Hamilton theorem, each vector $Av_1, \ldots, Av_r$ can be written as a linear combination of $v_1, \ldots, v_r$. Hence, there is an $r \times r$ matrix $\tilde{A}_{11}$ such that

$$\begin{bmatrix} Av_1 & \cdots & Av_r \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix} \tilde{A}_{11}$$

and we can write with certain matrices $\tilde{A}_{12}, \tilde{A}_{22}$

$$AQ = AT^{-1} = \begin{bmatrix} Av_1 & \cdots & Av_r & Av_{r+1} & \cdots & Av_n \end{bmatrix}$$

$$= \begin{bmatrix} v_1 & \cdots & v_r & v_{r+1} & \cdots & v_n \end{bmatrix} \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}$$

$$= T^{-1} \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \quad \Rightarrow TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}.$$

Similarly, it is possible to represent each column of B as a linear combination of $v_1, \ldots, v_r$. Thus, there is matrix $\tilde{B}_1$ such that

$$B = \begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix} \tilde{B}_1 = \begin{bmatrix} v_1 & \cdots & v_r & v_{r+1} & \cdots & v_n \end{bmatrix} \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}$$

$$= T^{-1} \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} \Rightarrow TB = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}.$$

This completes the proof of (a).

(b) It readily follows that

$$\begin{aligned} TU_\mathrm{c} &= T\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \\ &= \begin{bmatrix} TB & TAT^{-1}TB & \cdots & (TAT^{-1})^{n-1}TB \end{bmatrix} \\ &= \begin{bmatrix} \tilde{B}_1 & \tilde{A}_{11}\tilde{B}_1 & \cdots & \tilde{A}_{11}^{r-1}\tilde{B}_1 & \cdots & \tilde{A}_{11}^{n-1}\tilde{B}_1 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \end{aligned}$$

Because of the Cayley-Hamilton theorem, for every $l \geq r$,

$\tilde{A}_{11}^{l}$ is a linear combination of $I,\ \tilde{A}_{11},\ \ldots,\ \tilde{A}_{11}^{r-1}$. Thus,

$$\mathrm{rank}(TU_\mathrm{c}) = \mathrm{rank}\left(\begin{bmatrix} \tilde{B}_1 & \tilde{A}_{11}\tilde{B}_1 & \cdots & \tilde{A}_{11}^{r-1}\tilde{B}_1 \end{bmatrix}\right) = \mathrm{rank}(U_\mathrm{c}) = r.$$

This completes the proof of (b).

◆ Interpretation of controllable decomposition

$$\begin{cases} \dfrac{d\tilde{x}(t)}{dt} & = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \\ y(t) & = \tilde{C}\tilde{x}(t) + \tilde{D}u(t) \end{cases}, \qquad \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}$$

(a) The system is divided into the controllable part $(\tilde{A}_{11}, \tilde{B}_1)$ and the uncontrollable part $(\tilde{A}_{22}, 0)$.

(b) The controllable part can always be stabilized by an adequate controller.

(c) The uncontrollable part cannot be affected by a control at all.

(d) The system can only be stabilized when the uncontrollable part is stable.

◆ Stabilizability

When the LTI system $\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$ is uncontrollable,

this system is called a *stabilizable*, if the matrix $\tilde{A}_{22}$ in the following

normal form is stable.

$$\begin{cases} \dfrac{d\tilde{x}(t)}{dt} & = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \\ y(t) & = \tilde{C}\tilde{x}(t) + \tilde{D}u(t) \end{cases}, \quad \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}$$

◆ Popov-Belevitch-Hautus (PBH) test

The LTI system $\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$

(a) is controllable if and only if

$$\operatorname{rank}(\begin{bmatrix} A - \lambda I & B \end{bmatrix}) = n \quad \text{for every} \quad \lambda \in \mathbb{C}.$$

(b) is stabilizable if and only if

$$\operatorname{rank}(\begin{bmatrix} A - \lambda I & B \end{bmatrix}) = n \quad \text{for every} \quad \lambda \in \mathbb{C} \quad \text{with} \quad \operatorname{Re}(\lambda) \geq 0.$$

proof: (a)–necessary condition

Let (A, B) is controllable and suppose that

$$\mathrm{rank}(\begin{bmatrix} A - \lambda I & B \end{bmatrix}) < n.$$

Then there exists a nonzero vector v such that

$$v^T A = \lambda v^T \text{ and } v^T B = 0.$$

This implies that

$$v^T A^l = \lambda^l v^T \quad \text{for every} \quad l \geq 1.$$

Thus,

$$v^T \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0.$$

This is a contradiction to the assumed controllability.

(a)–sufficient condition

Suppose that (A, B) is uncontrollable and consider the decomposition

$$\begin{cases} \dfrac{d\tilde{x}(t)}{dt} &= \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \\ y(t) &= \tilde{C}\tilde{x}(t) + \tilde{D}u(t) \end{cases}, \qquad \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}$$

Then, for every eigenvalue λ of $\tilde{A}_{22}$, we see that

$$\operatorname{rank}\left(\begin{bmatrix} \tilde{A} - \lambda I & \tilde{B} \end{bmatrix}\right) < n.$$

This means that

$$\operatorname{rank}\left(\begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) = \operatorname{rank}\left(T^{-1}\begin{bmatrix} \tilde{A} - \lambda I & \tilde{B} \end{bmatrix}\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}\right) < n.$$

(b)–necessary condition

Let (A, B) is stabilizable and suppose that

$$\mathrm{rank}(\begin{bmatrix} A - \lambda I & B \end{bmatrix}) < n \quad \text{for a } \lambda \quad \text{with } \mathrm{Re}(\lambda) \geq 0.$$

Because

$$\mathrm{rank}\left(\begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) = \mathrm{rank}\left(\begin{bmatrix} \tilde{A} - \lambda I & \tilde{B} \end{bmatrix}\right) < n,$$

there exists a nonzero vector $v = [v_1^T \ v_2^T]^T$ such that

$$v_1^T \tilde{A}_{11} = \lambda v_1^T, \ v_1^T \tilde{B}_1 = 0 \text{ and } v_1^T \tilde{A}_{12} + v_2^T \tilde{A}_{22} = \lambda v_2^T$$

Since $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable, $v_1 = 0$ and thus λ is an eigenvalue of $\tilde{A}_{22}$.

This is a contradiction to the assumed stabilizability.

(b)–sufficient condition

Suppose that (A, B) is not stabilizable.

Then, there exists an eigenvalue λ of $\tilde{A}_{22}$ with $\mathrm{Re}(\lambda) \geq 0$.

By using this λ, we see that

$$\operatorname{rank}\left(\begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} \tilde{A} - \lambda I & \tilde{B} \end{bmatrix}\right) < n.$$