EECE423-01: 현대제어이론

Modern Control Theory

Chapter 7: Observability

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◆ The main topics of this chapter are

1. Concept for Observability

2. Conditions Observability

3. Similarity Transform and Detectability

1. Concept for Observability

◆ Motivation of observability

For the LTI system

$$\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}, \quad x(t) \in \mathbb{R}^n, \; u(t) \in \mathbb{R}^k, \; y(t) \in \mathbb{R}^m$$

let us consider the following statment:

Can we uniformly determine the initial state $x(0) = x_0$ by using $y(t)$ and $u(t)$ in a finite time interval?

◆What is a observability?

When we can uniformly determine arbitrary inital state $x(0) = x_0$ by using $y(t), u(t)$ $(0 \leq t \leq s)$ for some finite $s \geq 0$, the LTI system

$$\begin{cases} \dfrac{dx(t)}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$$

is called an *observable*. Otherwise, the LTI system is called an *unobservable*.

2. Conditions for Observability

◆ Necessary and sufficient condition of observability

The LTI system $\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$ is observable

if and only if the following observability matrix has rank n.

$$U_\mathrm{o} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Based on this, we also say that the pair (C, A) is *observable*.

Proof: (1. necessary condition)

Suppose that $\text{rank}(U_\text{o}) \neq n$. Then, there exists a nonzero x_0 such that

$$U_\text{o} x_0 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_0 = 0.$$

This implies that

$$CA^i x_0 = 0 \ (0 \leq i \leq n-1).$$

We also obtain from the Cayley-Hamilton theorem that

$$Ce^{At} x_0 \equiv 0.$$

Thus, it readily follows that

$$\boxed{\begin{aligned} y(t) &= Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \\ &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \end{aligned}}$$

Because $y(t)$ is dependent only on $u(\cdot)$, x_0 cannot be determined.

(2. sufficient condition)

If $\mathrm{rank}(U_\mathrm{o}) = n$, the observability gramian

$$Y_s := \int_0^s e^{A^T t} C^T C e^{At} dt$$

is nonsingular for an arbitrary $s > 0$ (the proof will be provided later).

If we consider

$$e(t) := y(t) - \int_0^t Ce^{A(t-\tau)} Bu(\tau) d\tau - Du(t) = Ce^{At} x_0,$$

$e(t)$ can be determined since we know $y(t)$ and $u(t)$.

Then, integrating $e^{A^T t} C^T e(t)$ over $0 \leq t \leq s$ leads to

$$d(s) := \int_0^s e^{A^T t} C^T e(t) dt = \int_0^s e^{A^T t} C^T C e^{At} dt \cdot x_0 = Y_s x_0.$$

Since $d(s)$ is well-known and $|Y_s| \neq 0$,

we can uniformly determine x_0 by $x_0 = Y_s^{-1} d(s)$

(Y_s is a nonsingular matrix)

Assume that $|Y_s| = 0$ for some $s > 0$. Then, there exists a nonzero v

such that $Y_s v = 0$. Hence,

$$v^T Y_s v = \int_0^s v^T e^{A^T \tau} C^T C e^{A\tau} v d\tau = \int_0^s \|Ce^{A\tau} v\|_2^2 d\tau = 0$$

This implies that $Ce^{A\tau} v \equiv 0$ $(0 \leq \tau \leq s)$ and contradicts $\mathrm{rank}(U_\mathrm{o}) = n$.

◆ Example

$$\text{The LTI system} \begin{cases} \dot{x} &= \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & -1 \end{bmatrix} x \end{cases}$$

is observable, since the observability matrix

$$U_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$$

has rank 2.

◆ Properties of observability

For the LTI system $\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$,

the followings are equivalent.

(a) The pair (C, A) is observable.

(b) The observability matrix $U_{\mathrm{o}} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has rank n.

(c) The controllabilty gramian $Y_s := \int_0^s e^{A^T t} C^T C e^{At} dt$ is positive definite for every $s > 0$.

Proof: The equivalence between (a) and (b) together with the assertion (b)$\Rightarrow$(c) have already been shown. We show (c)$\Rightarrow$(b).

Suppose that the controllability matrix U_o is not of full rank.

Then, there exists a nonzero vector $v \in \mathbb{R}^n$ such that

$$CA^k v = 0 \text{ for } k = 0, 1, \ldots, n-1.$$

It readily follows from Cayley-Hamilton theorem that

$$CA^k v = 0 \text{ for all } k \in \mathbb{N}.$$

Hence, $Ce^{At}v = 0, \quad \forall t \geq 0.$

Therefore, $Y_s v = 0$ and this contradicts (c).

3. Similarity Transform and Detactability

◆ Motivation of similarity transform

$$\text{The LTI system} \begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases} \text{is unobservable}$$

$$\Updownarrow$$

$$\operatorname{rank}(U_\mathrm{o}) = r < n, \;\; \text{where } U_\mathrm{o} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

What if the system is unobservable?

$\rightarrow$ Similarity transform may play an important role.

◆ Review of similarity transform

Two continuous-time LTI systems

$$\begin{cases} \dfrac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} \dfrac{d\tilde{x}}{dt} &= \tilde{A}\tilde{x} + \tilde{B}u \\ y &= \tilde{C}\tilde{x} + \tilde{D}u \end{cases}$$

Algebraically equivalent

are called *algebraically equivalent* if there exists a nonsingular matrix T such that the followings hold:

$$\tilde{A} := TAT^{-1},\ \tilde{B} := TB,\ \tilde{C} := CT^{-1},\ \tilde{D} := D$$

The corresponding map $\tilde{x} = Tx$ is called a similarity transform.

◆ Observable decomposition

Suppose that the LTI system $\begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$

is unobservable, i.e.,

$$\text{rank}(U_{\text{o}}) = r < n, \quad \text{where } U_{\text{o}} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Then, there exists a similarity transform matrix T such that the following assertions are true.

(a) The transformed pair has the form

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{C} = CT^{-1} = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix},$$

where $\tilde{A}_{11} \in \mathbb{R}^{r \times r}$ and $\tilde{C}_1 \in \mathbb{R}^{m \times r}$.

(b) The pair $(\tilde{C}_1, \tilde{A}_{11})$ is observable.

This is called a *observable decomposition*.

proof: (a) Let $w_1,\ w_2,\dots,w_r$ be linearly independent rows of the observability matrix U_o. We complete them by $n-r$ linearly independent vectors $w_{r+1}, w_{r+2},\dots,w_n$ such that the matrix

$$T = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

is nonsingular and show that T has the desired property.

Because of the Cayley-Hamilton theorem, each vector $w_1 A, \ldots, w_r A$ can be written as a linear combination of $w_1, \ldots, w_r$. Hence, there is an $r \times r$ matrix $\tilde{A}_{11}$ such that $\begin{bmatrix} w_1 A \\ \vdots \\ w_r A \end{bmatrix} = \tilde{A}_{11} \begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix}$

and we can write with certain matrices $\tilde{A}_{21}, \tilde{A}_{22}$

$$TA = \begin{bmatrix} w_1 A \\ \vdots \\ w_r A \\ \vdots \\ w_n A \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_r \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} T$$

$$\Rightarrow TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}.$$

Similarly, it is possible to represent each row of C as a linear combination of $w_1, \ldots, w_r$. Thus, there is matrix $\tilde{C}_1$ such that

$$C = \tilde{C}_1 \begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix} = \tilde{C}_1 T \;\Rightarrow \tilde{C}_1 = CT^{-1}.$$

This completes the proof of (a).

(b) It readily follows that

$$U_{\mathrm{o}}T^{-1} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} T^{-1} = \begin{bmatrix} CT^{-1} \\ CT^{-1}TAT^{-1} \\ \vdots \\ CT^{-1}(TAT^{-1})^{n-1} \end{bmatrix} = \begin{bmatrix} \tilde{C}_1 & 0 \\ \tilde{C}\tilde{A}_{11} & 0 \\ \vdots & \vdots \\ \tilde{C}\tilde{A}_{11}^{n-1} & 0 \end{bmatrix}$$

Because of the Cayley-Hamilton theorem, for every $l \geq r$,

$\tilde{A}_{11}^{l}$ is a linear combination of $I,\ \tilde{A}_{11},\ \ldots,\ \tilde{A}_{11}^{r-1}$. Thus,

$$\operatorname{rank}(U_{\mathrm{o}}T^{-1}) = \operatorname{rank}\left(\begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_1\tilde{A}_{11} \\ \vdots \\ \tilde{C}_1\tilde{A}_{11}^{r-1} \end{bmatrix}\right) = \operatorname{rank}(U_{\mathrm{o}}) = r.$$

This completes the proof of (b).

◆ Interpretation of observable decomposition

$$\begin{cases} \dfrac{d\tilde{x}(t)}{dt} & = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \\ y(t) & = \tilde{C}\tilde{x}(t) + \tilde{D}u(t) \end{cases}, \qquad \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix}$$

(a) The system is divided into the observable part $(\tilde{C}_1, \tilde{A}_{11})$ and the unobservable part $(0, \tilde{A}_{22})$.

(b) The state of the observable part can be reconstructed from the output.

(c) The state of the unobservable part cannot be reconstructed from the output.

(d) En estimate of the whole state is only possible when the state of the unobservable part tends to 0 as $t \to \infty$ (if $u = 0$).

◆ Detectability

When the LTI system $\begin{cases} \dfrac{dx(t)}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$ is unobservable,

this system is called a *detectable*, if the matrix $\tilde{A}_{22}$ in the following

normal form is stable.

$$\begin{cases} \dfrac{d\tilde{x}(t)}{dt} &= \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \\ y(t) &= \tilde{C}\tilde{x}(t) + \tilde{D}u(t) \end{cases}, \quad \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix}$$

◆ Popov-Belevitch-Hautus (PBH) test

$$\text{The LTI system} \begin{cases} \dfrac{dx(t)}{dt} & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}$$

(a) is observable if and only if

$$\operatorname{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = n \quad \text{for every} \quad \lambda \in \mathbb{C}.$$

(b) is detectable if and only if

$$\operatorname{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = n \quad \text{for every} \quad \lambda \in \mathbb{C} \quad \text{with} \quad \operatorname{Re}(\lambda) \geq 0.$$

proof: (a)–necessary condition

Let (C, A) is observable and suppose that

$$\operatorname{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) < n.$$

Then, there exists a nonzero vector v such that

$$Av = \lambda v \text{ and } Cv = 0.$$

This implies that

$$A^l v = \lambda^l v \quad \text{for every} \quad l \geq 1.$$

Thus,

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} v = 0.$$

This is a contradiction to the assumed observability.

(a)–sufficient condition

Suppose that (C, A) is unobservable and consider the decomposition

$$\begin{cases} \dfrac{d\tilde{x}(t)}{dt} & = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \\ y(t) & = \tilde{C}\tilde{x}(t) + \tilde{D}u(t) \end{cases}, \qquad \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix}$$

Then, for every eigenvalue λ of $\tilde{A}_{22}$, we see that

$$\operatorname{rank}\left(\begin{bmatrix} \tilde{A} - \lambda I \\ \tilde{C} \end{bmatrix}\right) < n.$$

This means that

$$\operatorname{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A} - \lambda I \\ \tilde{C} \end{bmatrix} T\right) < n.$$

(b)–necessary condition

Let (C, A) is detectable and suppose that

$$\operatorname{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) < n \quad \text{for a } \lambda \quad \text{ with } \operatorname{Re}(\lambda) \geq 0.$$

Because

$$\operatorname{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} \tilde{A} - \lambda I \\ \tilde{C} \end{bmatrix}\right) < n,$$

there exists a nonzero vector $v = [v_1^T \ v_2^T]^T$ such that

$$\tilde{A}_{11} v_1 = \lambda v_1, \ \tilde{C}_1 v_1 = 0 \text{ and } \tilde{A}_{21} v_1 + \tilde{A}_{22} v_2 = \lambda v_2$$

Since $(\tilde{C}_1, \tilde{A}_{11})$ is observable, $v_1 = 0$ and thus λ is an eigenvalue of $\tilde{A}_{22}$.

This is a contradiction to the assumed detectability.

(b)–sufficient condition

Suppose that (C, A) is not detectable.

Then, there exists an eigenvalue λ of $\tilde{A}_{22}$ with $\mathrm{Re}(\lambda) \geq 0$.

By using this λ, we see that

$$\mathrm{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = \mathrm{rank}\left(\begin{bmatrix} \tilde{A} - \lambda I \\ \tilde{C} \end{bmatrix}\right) < n.$$