EECE423-01: 현대제어이론

Modern Control Theory

Chapter 1: Introduction

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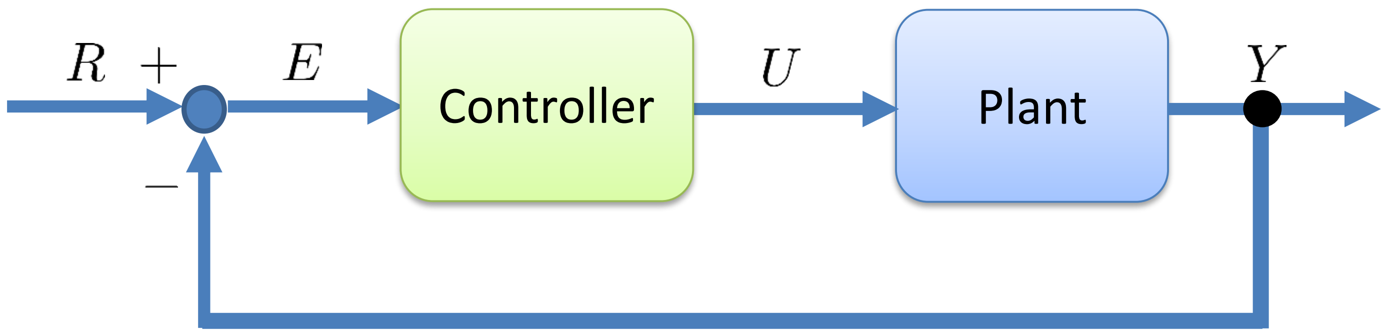
◆ The main objectives of this chapter are

1. Review on Classical Control Theory
2. Brief Introduction to Modern Control Theory
3. Comparison between Classical and Modern Control Theories
4. Summary of Modern Control Theory

1. Review on Classical Control Theory

◆ What is a control system?

1. Negative feedback of control systems



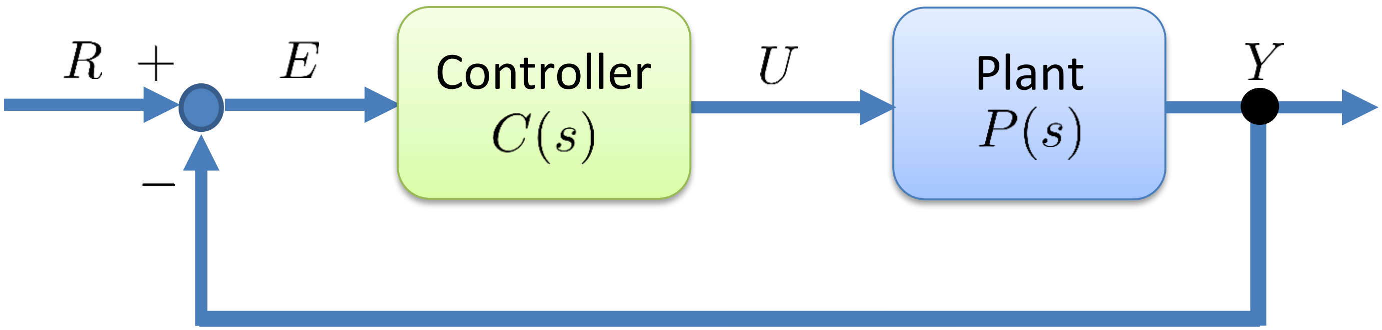
- R **:** Reference input U : Control input Y : Control output

2. Typical control objectives

- Analysis of input-output relation (ex. E/R, Y/R)
- Design of a controller to achieve a desired specification (ex. $E \to 0$)

◆ Classical control theory

1. Laplace transform of signals



- $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$

2. Transfer function

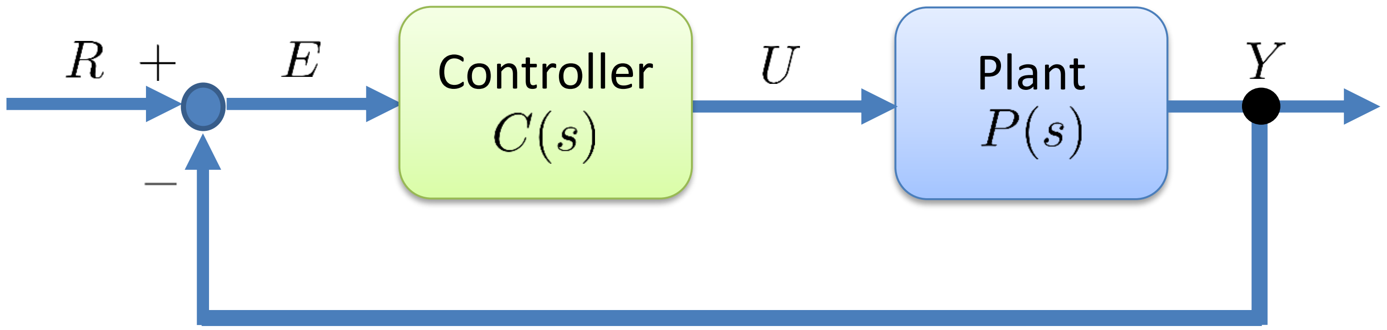
- $\frac{E(s)}{R(s)} = \frac{1}{1 + C(s)P(s)}, \quad \frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)}$

◆ Characteristics of classical control theory

1. One can easily obtain a frequency response by substituting $s = j\omega$ in transfer function such as

$$\frac{E(s)}{R(s)} = \frac{1}{1 + C(s)P(s)}, \qquad \frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)}$$

2. The signals are assumed to have scalar values ($R,\ E,\ U$ and Y are scalars)



◆ Disadvantages of classical control theory

1. One can easily obtain a frequency response by substituting

   $s = j\omega$ in transfer function such as

$$\frac{E(s)}{R(s)} = \frac{1}{1 + C(s)P(s)}, \qquad \frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)}$$

→ How can we derive the input/output relation in the time-domain?

2. The signals are assumed to have scalar values

   ($R,\ E,\ U$ and Y are scalars)

→ It would matter when the signals have vector values

2. Brief Introduction to Modern Control Theory

◆ Modern control theory

1. Modern control theory utilizes the **time-domain state-space representation**

2. The state space representation (also known as the "time-domain approach") provides a convenient and compact way to model and analyze systems with **multiple inputs and outputs**.

3. Modern control theory uses a mathematical model of a physical system related by **first-order differential equations**

- Explanation in Wikipedia-

3. Comparison between Classical and Modern Control Theories

◆ Example of mass-spring-damper system

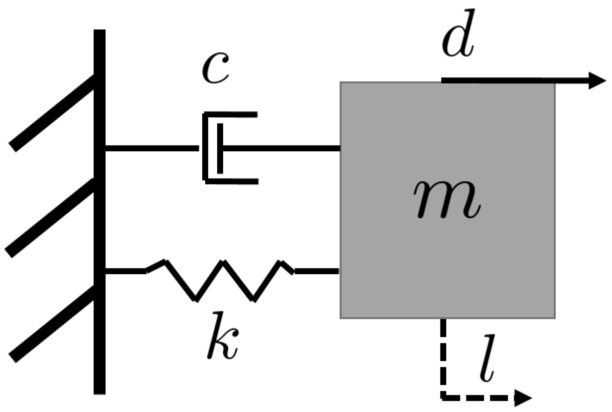


Figure 1.1 Mass-spring-damper system

- m: mass
- k: spring constant
- c: damper constant
- l: displacement of the mass
- d: external force

▪ **Dynamics of the system (with input d and output l)**

$$m\ddot{l} = d - c\dot{l} - kl$$

- **Transfer function approach**

$$\frac{L(s)}{D(s)} = \frac{1}{ms^2 + cs + k}$$

- **State-space approach**

$$\frac{d}{dt}\begin{bmatrix} l \\ \dot{l} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} l \\ \dot{l} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} d$$

$$l = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} l \\ \dot{l} \end{bmatrix}$$

◆ Transfer function approach vs State-space approach

- **Dynamics of the system (with input d and output l)**

$$m\ddot{l} = d - c\dot{l} - kl$$

• **Transfer function approach**	• **State-space approach**
$\frac{L(s)}{D(s)} = \frac{1}{ms^2 + cs + k}$	$\frac{d}{dt}\begin{bmatrix} l \\ \dot{l} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} l \\ \dot{l} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} d$ $l = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} l \\ \dot{l} \end{bmatrix}$
1. Input/output representation in the frequency-domain 2. Alegebraic equation 3. No state variable 4. Useful to consider steady state via final value theorem	1. Input/output representation in the time-domain 2. First-order differential equation 3. State variable $x := [l \ \dot{l}]^T$ 4. Useful to consider transient response

◆ Another example (multi-input multi-output case)

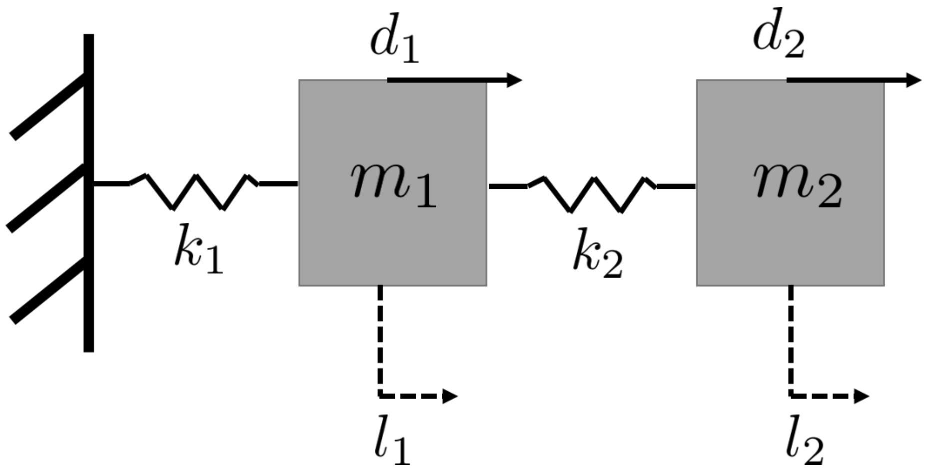


Figure 1.2 2-Mass-spring system

- **Dynamics of the system**

$$m_1\ddot{l}_1 = d_1 - k_1 l_1 - k_2(l_1 - l_2), \quad m_2\ddot{l}_2 = d_2 + k_2(l_1 - l_2)$$

→ **Input/output relation with inputs** d_1, d_2 **and** l_1, l_2 **outputs**

◆ Approach 1: Transfer function approach

- **Dynamics of the system (with inputs d_1, d_2 and outputs l_1, l_2)**

$$m_1\ddot{l}_1 = d_1 - k_1 l_1 - k_2(l_1 - l_2), \quad m_2\ddot{l}_2 = d_2 + k_2(l_1 - l_2)$$

$$\frac{L_1(s)}{D_1(s)} = \frac{1}{m_1 m_2 s^4 + (m_1 k_2 + m_2 k_1 + m_2 k_2)s^2 + k_1 k_2}$$

$$\frac{L_1(s)}{D_2(s)} = \frac{k_2}{m_1 m_2 s^4 + (m_1 k_2 + m_2 k_1 + m_2 k_2)s^2 + k_1 k_2}$$

$$\frac{L_2(s)}{D_1(s)} = \frac{k_2}{m_1 m_2 s^4 + (m_1 k_2 + m_2 k_1 + m_2 k_2)s^2 + k_1 k_2}$$

$$\frac{L_2(s)}{D_2(s)} = \frac{m_1 s^2 + k_1 + k_2}{m_1 m_2 s^4 + (m_1 k_2 + m_2 k_1 + m_2 k_2)s^2 + k_1 k_2}$$

→ **Independent treatment of the Input/output relation**

◆ Approach 2: State-space approach

- **Dynamics of the system (with inputs d_1, d_2 and outputs l_1, l_2)**

$$m_1\ddot{l}_1 = d_1 - k_1 l_1 - k_2(l_1 - l_2), \quad m_2\ddot{l}_2 = d_2 + k_2(l_1 - l_2)$$

$$\frac{d}{dt}\begin{bmatrix} l_1 \\ \dot{l}_1 \\ l_2 \\ \dot{l}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix}\begin{bmatrix} l_1 \\ \dot{l}_1 \\ l_2 \\ \dot{l}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

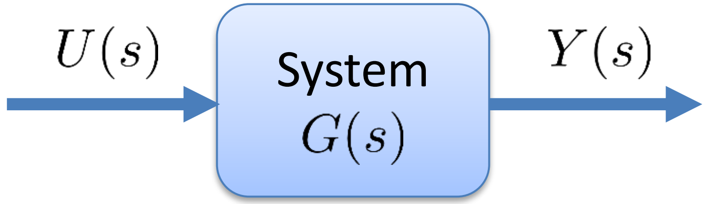
$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} l_1 \\ \dot{l}_1 \\ l_2 \\ \dot{l}_2 \end{bmatrix}$$

→ **Systematic treatment of the Input/output relation**

◆ Classical control theory vs Modern control theory

Classical control theory	Modern control theory
• **Transfer function approach**	• **State-space approach**
1. Input/output representation in the frequency-domain	1. Input/output representation in the time-domain
2. Alegebraic equation	2. First-order differential equation
3. No state variable	3. State variable
4. Useful to consider steady state via final value theorem	4. Useful to consider transient response
5. Independent treatment of the input/output relation	**5. Systematic treatment of the input/output relation**

◆ Transform of transfer function to state-space representation



Let $G(s) = \dfrac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$

If we define the state variable as

$$X(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} U(s)$$

Then, we obtain

$$Y(s) = (b_{n-1}s^{n-1} + \cdots + b_1 s + b_0) X(s)$$

This means

$$\boxed{\begin{aligned} y(t) &= b_0 x(t) + b_1 \frac{dx(t)}{dt} + \cdots + b_{n-1} \frac{d^{n-1}x(t)}{dt^{n-1}} \\ \frac{d^n x(t)}{dt^n} &= -a_0 x(t) - a_1 \frac{dx(t)}{dt} - \cdots - a_{n-1} \frac{d^{n-1}x(t)}{dt^{n-1}} + u(t) \end{aligned}}$$

$$y(t) = b_0 x(t) + b_1 \frac{dx(t)}{dt} + \cdots + b_{n-1} \frac{d^{n-1}x(t)}{dt^{n-1}}$$

$$\frac{d^n x(t)}{dt^n} = -a_0 x(t) - a_1 \frac{dx(t)}{dt} - \cdots - a_{n-1} \frac{d^{n-1}x(t)}{dt^{n-1}} + u(t)$$

Here, let us further define

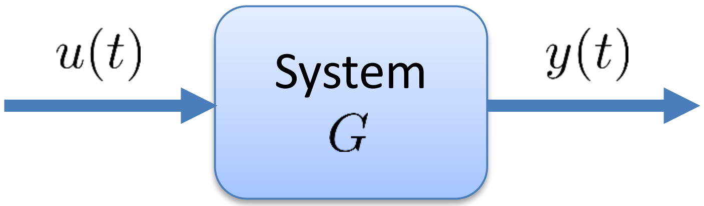
$$x_1(t) := x(t),\ \ x_2(t) := \frac{dx(t)}{dt},\ \ \cdots,\ \ x_n(t) := \frac{d^{n-1}x(t)}{dt^{n-1}}$$

Then, we have the following state-space representation:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & \cdots & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & \cdots & b_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

◆Transform of state-space representation to transfer function



$$G : \begin{cases} \dfrac{dt}{dx} = Ax + Bu \\ y = Cx \end{cases}$$

Let us assume that $x(0) = 0$.

Then, applying Laplace transform to G leads to

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s)$$

Thus, the transfer function is described by

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B}$$

◆ Example 1 : Cruise control model

Equations of motion

$$u - b\dot{x} = m\ddot{x} \rightarrow \quad \ddot{x} + \frac{b}{m}\dot{x} = \frac{u}{m}$$

$$(\dot{x} := v) \quad \dot{v} + \frac{b}{m}v = \frac{u}{m}$$

Transfer function: $\dfrac{V(s)}{U(s)} = \dfrac{1/m}{s + b/m}$

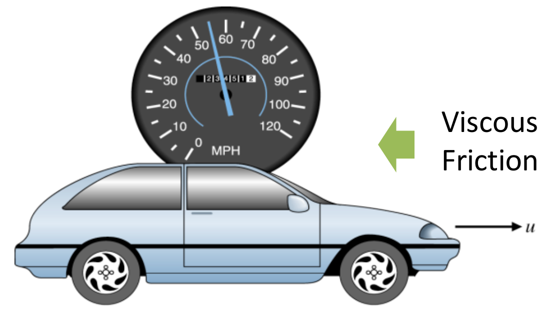
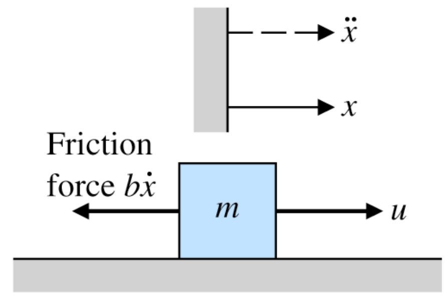


Figure 2.1 Cruise-control model



State variable

$x := [x \ \ v]^T$

Input u

Output

$y := v$

$$\begin{cases} \dfrac{dx}{dt} &= Ax + Bu \\ y &= Cx \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

◆ Example 2 : Bridged tee circuit

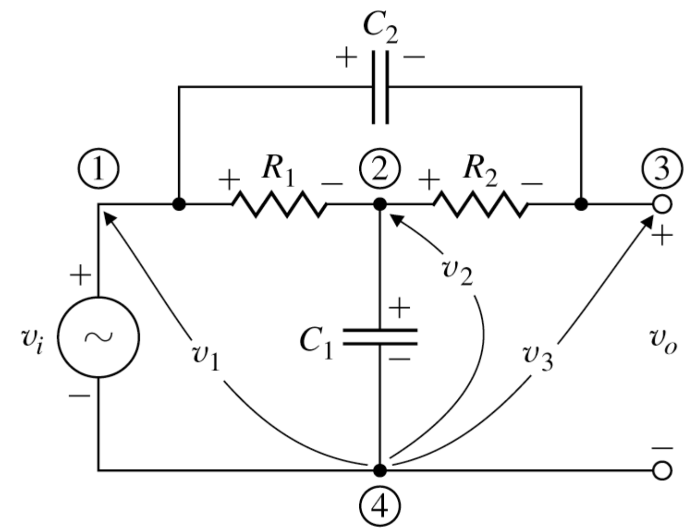
node 1: $v_1 = v_i$

node 2 (KCL):

$$-\frac{v_1 - v_2}{R_1} + \frac{v_2 - v_3}{R_2} + C_1 \frac{dv_2}{dt} = 0$$

node 3 (KCL):

$$\frac{v_3 - v_2}{R_2} + C_2 \frac{d\left(v_3 - v_1\right)}{dt} = 0$$



State variable

$x := [v_{c1}\ \ v_{c2}]^T$

$(v_{c1} = v_2,\ \ v_{c2} = v_1 - v_3)$

Input

$u := v_i$

Output

$y := v_3$

$$\begin{cases} \dfrac{dx}{dt} & = Ax + Bu \\ y & = Cx \end{cases}$$

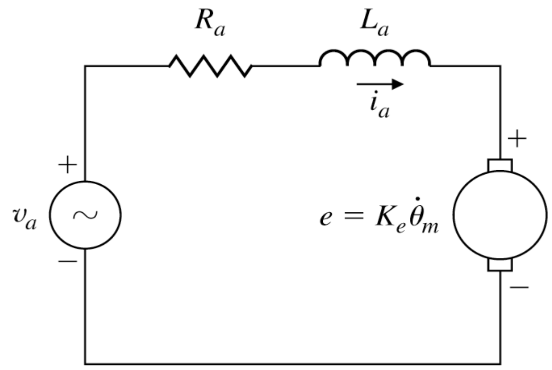
$$A = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & -\frac{1}{C_1 R_2} \\ -\frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix}$$

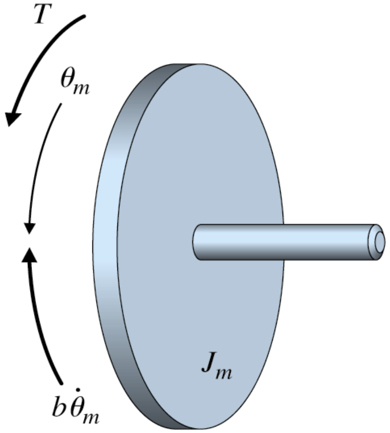
$$B = \begin{bmatrix} \frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \\ \frac{1}{C_2 R_2} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 \end{bmatrix}$$

◆ Example 3 : DC motor in state-space representation

$$J_m \ddot{\theta}_m + b\dot{\theta}_m = K_t i_a$$

$$L_a \frac{di_a}{dt} + R_a i_a = v_a - K_e \dot{\theta}_m$$





State variable

$$x := [\theta_m \ \dot{\theta}_m \ i_a]^T$$

Input

$$u := v_a$$

Output

$$y := \theta_m$$

$$\begin{cases} \dfrac{dx}{dt} &= Ax + Bu \\ y &= Cx \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{b}{J_m} & \frac{K_t}{J_m} \\ 0 & -\frac{K_e}{L_a} & -\frac{R_a}{L_a} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L_a} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

4. Summary of Modern Control Theory

Modern control theory is the base in the following areas:

1. Robust control theory

2. Optimal control theory

3. Estimation theory

4. Nonlinear control theory

5. And so on..

→ They have been developed considerably by using advanced arguments on **Linear Algebra**

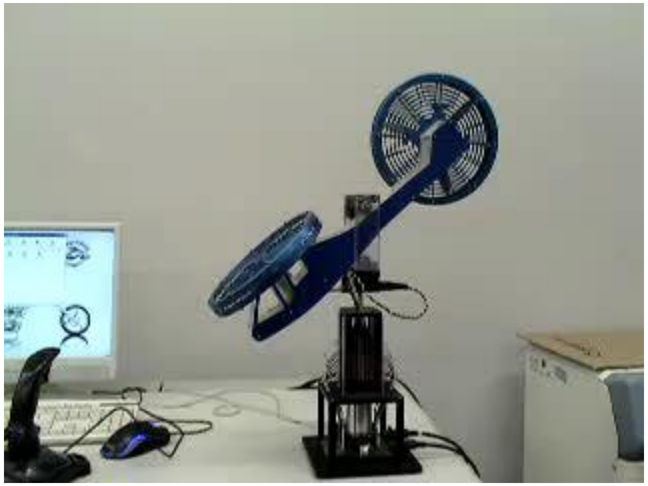
◆ Example for robust control theory

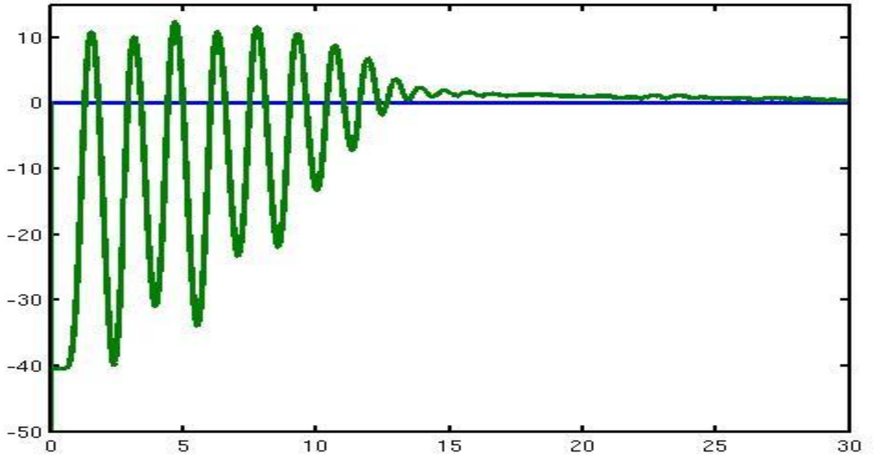
- Balance control of cart-driven inverted pendulum

→ The inverted pendulum maintains the balance for different lengths of stick by using **only one controller**

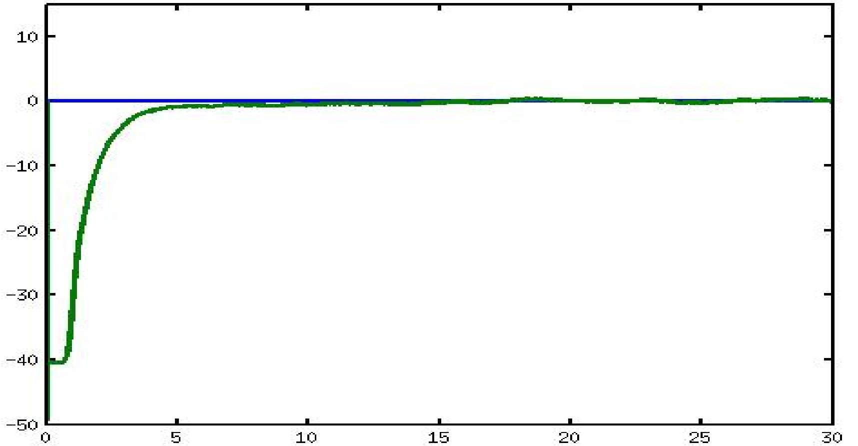
◆ Example for estimation theory

- Attitude control of 2 degree of freedom helicopter





Results for a conventional control method



Results for a newly developed control method