EECE322-01: 자동제어공학개론

Introduction to Automatic Control

Chapter 4: Analysis of Feedback Systems

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◆ The main objectives of this chapter are

1. Basic Concepts of Control Systems

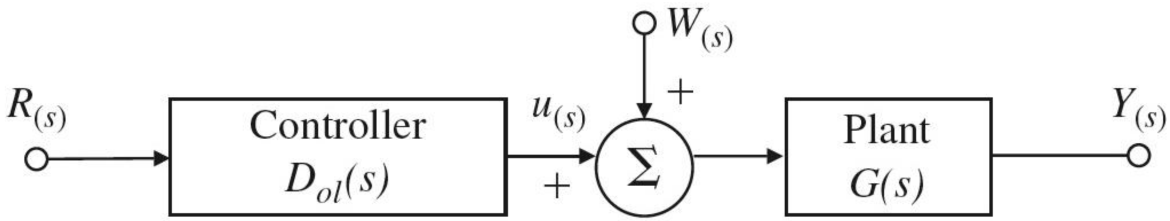
2. System Type

3. Introduction to PID Control

1. Basic Concepts of Control Systems

Equations of open-loop control systems

- Open-loop control system



- Output of open-loop control system

$$Y_{ol} = GD_{ol}R + GW$$

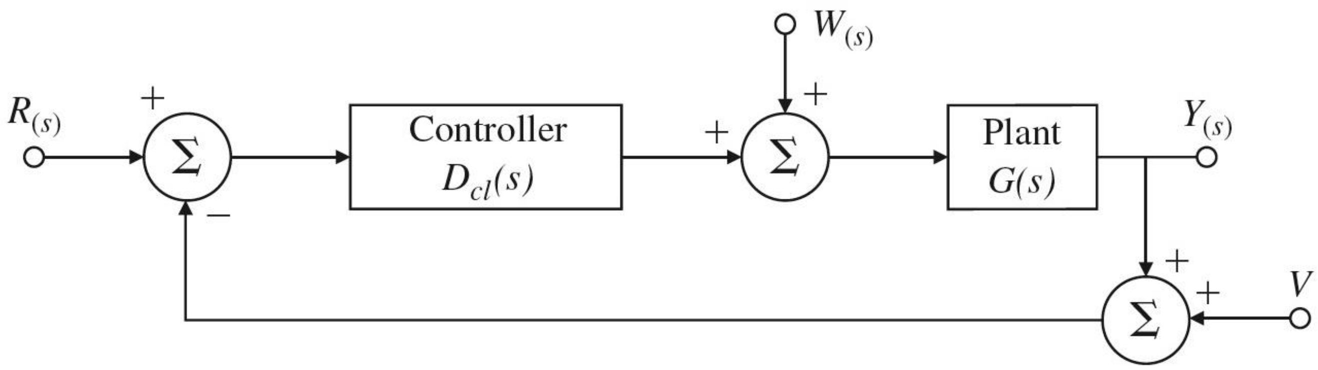
- Error (the difference between reference input and system output)

$$\begin{aligned} E_{ol} &= R - Y_{ol} \\ &= R - [GD_{ol}R + GW] \\ &= [1 - GD_{ol}]R - GW \\ &= [1 - T_{ol}]R - GW \end{aligned}$$

(open-loop tranfer function: $T_{ol} = GD_{ol}$)

Equations of closed-loop control systems

- Closed-loop control systems



- basic unitary feedback structure

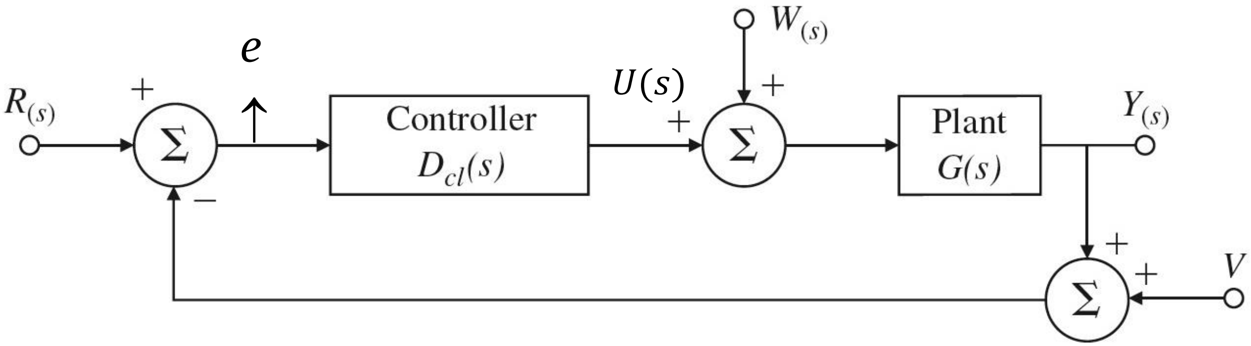
R: reference, the output is expected to track

W: disturbance, the control is expected to counteract so it does not disturb Y

V: sensor noise, the controller is supposed to ignore.

Transfer functions of closed-loop systems

- Closed-loop control system with disturbance and noise.



$$Y_{cl} = \frac{GD_{cl}}{1+GD_{cl}}R + \frac{G}{1+GD_{cl}}W - \frac{GD_{cl}}{1+GD_{cl}}V$$

$$U = \frac{D_{cl}}{1+GD_{cl}}R - \frac{GD_{cl}}{1+GD_{cl}}W - \frac{D_{cl}}{1+GD_{cl}}V$$

$$E(s) = R(s) - Y(s) - V(s)$$
$$Y(s) = G(s)(W(s) + D_{cl}(s)E(s))$$
$$Y(s) = G(s)\left(W(s) + D_{cl}(s)\left(R(s) - Y(s) - V(s)\right)\right)$$

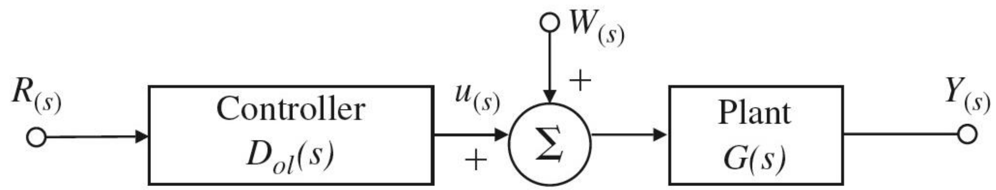
- Error: $E_{cl} = R - Y_{cl}$

$$E_{cl} = R - \left(\frac{GD_{cl}}{1+GD_{cl}}R + \frac{G}{1+GD_{cl}}w - \frac{GD_{cl}}{1+GD_{cl}}V\right)$$
$$= \frac{1}{1+GD_{cl}}R - \frac{G}{1+GD_{cl}}W + \frac{GD_{cl}}{1+GD_{cl}}V$$

- Closed loop transfer function: $T_{cl} = \dfrac{GD_{cl}}{1+GD_{cl}}$

Stability of open-loop systems

- Open loop systems



- Output of open-loop control system $Y_{ol} = GD_{ol}R + GW$

Let $G(s) = \dfrac{b(s)}{a(s)},\ D_{ol}(s) = \dfrac{c(s)}{d(s)}.$ $\quad Y_{ol}(s) = \dfrac{b(s)}{a(s)} \cdot \dfrac{c(s)}{d(s)} R(s) + \dfrac{b(s)}{a(s)} W(s)$

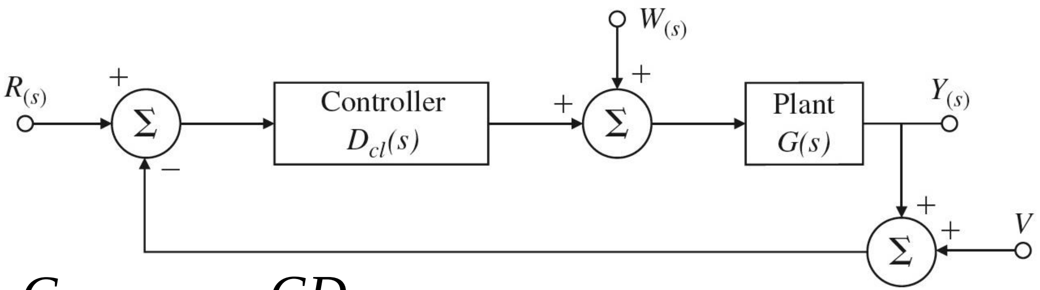
- Stability requirements: neither $a(s)$ nor $d(s)$ may have roots in the right half-plane.

- How about cancelling unstable pole by an RHP zero?
  → the slightest noise or disturbance will cause the output to grow until stopped by saturation or system failure.

- **An open-loop structure cannot be used to make an unstable plant to be stable.**

Stability of closed-loop systems

- Feedback systems



- Output: $Y_{cl} = \dfrac{GD_{cl}}{1+GD_{cl}}R + \dfrac{G}{1+GD_{cl}}W - \dfrac{GD_{cl}}{1+GD_{cl}}V$

- System poles: the roots of $1+GD_{cl} = 0.$

- Let $G(s) = \dfrac{b(s)}{a(s)},\ D_{cl}(s) = \dfrac{c(s)}{d(s)}.$

$$\frac{GD_{cl}}{1+GD_{cl}} = \frac{\frac{b(s)}{a(s)}\cdot\frac{c(s)}{d(s)}}{1+\frac{b(s)}{a(s)}\cdot\frac{c(s)}{d(s)}} = \frac{b(s)c(s)}{a(s)d(s)+b(s)c(s)}$$

- Characteristic equation: $1+GD_{cl} = 0,\ \ 1+\dfrac{b(s)c(s)}{a(s)d(s)} = 0,\ \ a(s)d(s)+b(s)c(s) = 0$

- One must still avoid unstable pole-zero cancellations. (unstable pole remains)

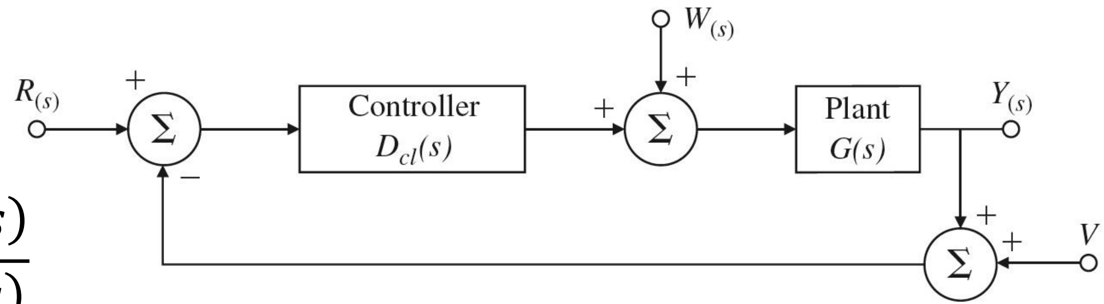
- **We can make an unstable plant to be stable.**

Example of stabilization

- Example (inverted pendulum)

  - $G(s) = \dfrac{1}{s^2 - 1}, \ = \dfrac{b(s)}{a(s)}$

  - let $D_{cl}(s) = \dfrac{K(s+\gamma)}{s+\delta}. \ = \dfrac{d(s)}{c(s)}$



- Characteristic equation:

$$a(s)d(s) + b(s)c(s) = (s^2 - 1)(s + \delta) + K(s + \gamma) = 0; \text{ when we take } \gamma = 1$$

$$\Rightarrow (s+1)\big((s-1)(s+\delta) + K\big) = (s+1)(s^2 + (\delta - 1\,) - \delta + K) = 0$$

- A simple solution:

  take $\gamma = 1$ and one can place the remaining two poles at any point desired.

$$\delta > 1, K > \delta$$

- Exercise: How to make the Characteristic equation.

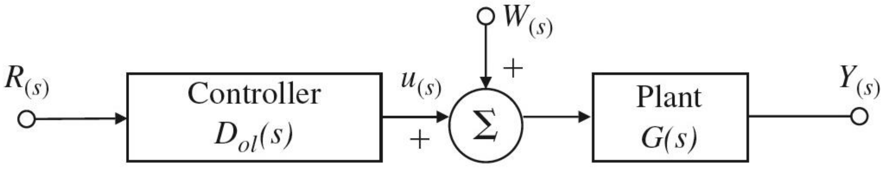
$$s^2 + 2\zeta\omega s + \omega^2 = 0 \ ?$$

Tracking

- The tracking problem is to cause the output to follow the reference input as closely as possible.

- Open-loop case:
  - consider stable plant with neither poles nor zeros in RHP
  - in principle the controller can be selected to cancel the transfer function of the plant and substitute whatever desired transfer function the engineer wishes.

  → problem:
  - the controller should be proper
  - the controller should not be too fast or produce large input
  - sensitivity problem (cancellation of almost neutrally stable poles)



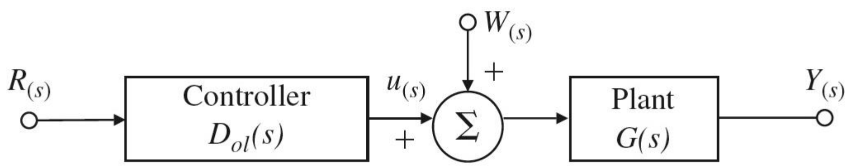
If, $G(s) = \dfrac{1}{s+3} \rightarrow D_{ol}(s) = s + 3\,??$

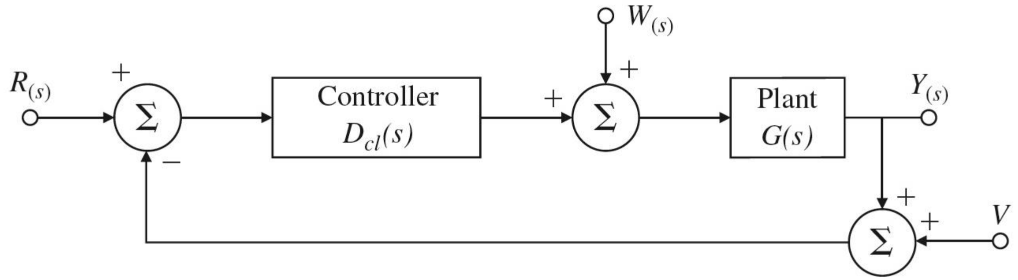
It is not particularly good method

Regulation

- The problem of regulation is to keep the error small when the reference is at most a constant set point and disturbances are present.

- Open-loop case: the controller has no influence at all on the response to disturbances



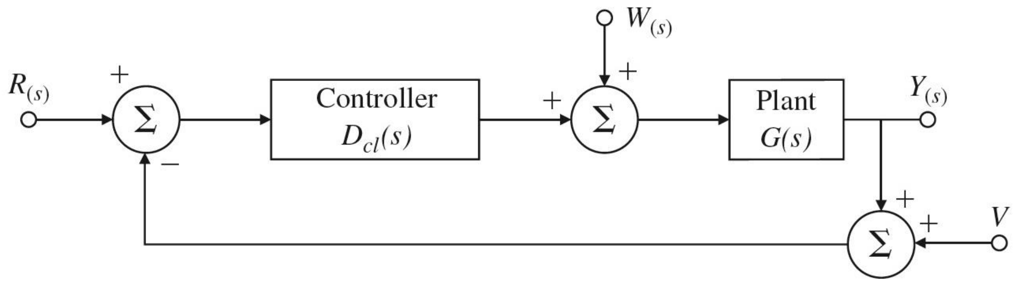


- Feedback case:

$$E_{cl} = \frac{1}{1+GD_{cl}}R - \frac{G}{1+GD_{cl}}W + \frac{GD_{cl}}{1+GD_{cl}}V$$

  - We can do something.
  - What happens if we increase D_{cl}? $\rightarrow$ Transfer function $\approx$ unity.
  - We should compromise between disturbance rejection and noise attenuation.

Remarks on regulation



$$E_{cl} = \frac{1}{1+GD_{cl}} R - \frac{G}{1+GD_{cl}} W + \frac{GD_{cl}}{1+GD_{cl}} V$$

- Notes:

  - The frequency content of most plant disturbances occurs at very low frequencies (most common case is a bias).

  - Good sensor will have no bias and can be constructed to have very little noise over the entire range of low frequencies of interest.

  → We design the controller transfer function to be large at low frequencies and we make it small at higher frequencies.

Sensitivity – open-loop case

- Sensitivity of steady-state gain (transfer function at s=0) to parameter changes

- Sensitivity of steady-state gain (transfer func. at $s = 0$) to parameter changes

- Plant gain change: G $\to G + \delta G$

- Open-loop: $\left(\text{Steady-state gain: } T_{ol} = GD_{ol}\right)$

$$T_{ol} + \delta T_{ol} = D_{ol}\left(G + \delta G\right) = D_{ol}G + D_{ol}\delta G = T_{ol} + D_{ol}\delta G$$

Normalized error in gain: $\dfrac{\delta T_{ol}}{T_{ol}} = \dfrac{D_{ol}\delta G}{D_{ol}G} = \dfrac{\delta G}{G}$

$$\left(\begin{array}{ll} \text{Sensitivity } S \text{ of the gain } T_{ol} \text{ with repect to the change } G & \\ \qquad := \dfrac{\text{fractional change in } T_{ol}}{\text{fractional change in } G} & \to S_G^T := \dfrac{\dfrac{\delta T_{ol}}{T_{ol}}}{\dfrac{\delta G}{G}} \end{array}\right)$$

$$\to S_G^{T_{ol}} := \text{sensitivity of } T_{ol} \text{ with respect to } G = 1$$

Sensitivity – closed-loop case

- Closed-loop

- Steady-state gain: $T_{cl} + \delta T_{cl} = \dfrac{(G+\delta G)D_{cl}}{1+(G+\delta G)D_{cl}}$, $\quad T_{cl} = \dfrac{GD_{cl}}{1+GD_{cl}}$ $\left(T_{cl}(G) := \dfrac{GD_{cl}}{1+GD_{cl}}\right)$

1st-order variation: $\delta T_{cl} = \dfrac{dT_{cl}}{dG}\delta G$ $\left(T_{cl}(G+\delta G) = T_{cl}(G) + \dfrac{dT_{cl}}{dG}\delta G + \dfrac{1}{2}\dfrac{d^2T_{cl}}{dG^2}(\delta G)^2 + \cdots\right)$

$$\frac{\delta T_{cl}}{T_{cl}} = \frac{1}{T_{cl}}\frac{dT_{cl}}{dG}\delta G = \left(\frac{G}{T_{cl}}\frac{dT_{cl}}{dG}\right)\frac{\delta G}{G} = (\text{sensitivity})\frac{\delta G}{G}$$

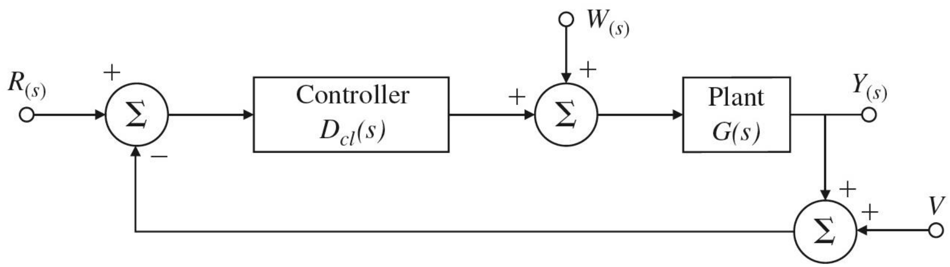
$S_G^{T_{cl}} :=$ sensitivity of T_{cl} with respect to G

$$:= \frac{G}{T_{cl}}\frac{dT_{cl}}{dG}$$

$$S_G^{T_{cl}} = \frac{G}{GD_{cl}/(1+GD_{cl})}\frac{(1+GD_{cl})D_{cl} - D_{cl}(GD_{cl})}{(1+GD_{cl})^2} = \frac{1}{1+GD_{cl}}$$

- Sensitivity of $T_{ol}(T_{cl})$ with respect to $D_{ol}(D_{cl})$

Advantages of feedback



$$E_{cl} = \frac{1}{1+GD_{cl}} R - \frac{G}{1+GD_{cl}} W + \frac{GD_{cl}}{1+GD_{cl}} V$$

Major advantage of feedback: System **errors to constant disturbances can be made smaller** with feedback than they are in open-loop systems by a factor of $S = \dfrac{1}{1+DG(0)}$,where $DG(0)$ is the loop gain at s=0.

Advantage of feedback: In feedback control, the error in the overall transfer function gain is **less sensitive to variations in the plant gain** by a factor of $S = \dfrac{1}{1+DG}$ compared with errors in open-loop control.

Sensitivity and complementary sensitivity functions

Sensitivity function: $\mathrm{S} = \dfrac{1}{1+DG}$ **Loop Gain**: $L = DG$

Complementary sensitivity function: $\mathrm{T} = 1 - \mathrm{S} = \dfrac{DG}{1+DG}$

- Error in the closed-loop

$E_{cl} = SR - SGW + TV$ (TV= error term due to sensor noise)

- Control objective: Minimize E_{cl}

- $\mathrm{T} = 1 - \mathrm{S} \rightarrow$ Impossible to minimize both T and S at the same time.

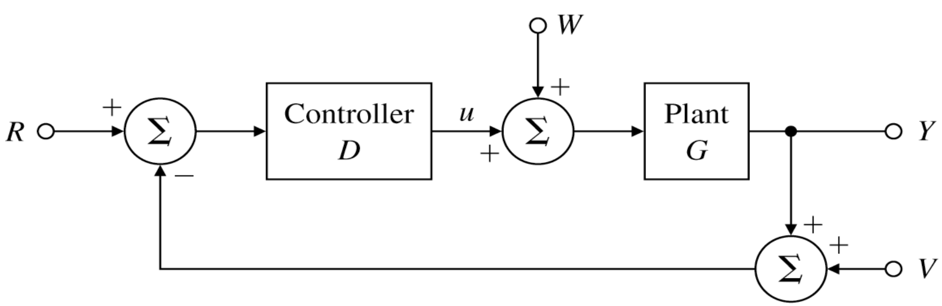
- The reference and disturbance energies are typically concentrated in a band of frequencies below some limit(ω_c).

- The sensor can be designed so that tha sensor noise V is small in the low frequency band below ω_c.

Frequency Separation

At low frequency, $\mathrm{S} \rightarrow 0$.

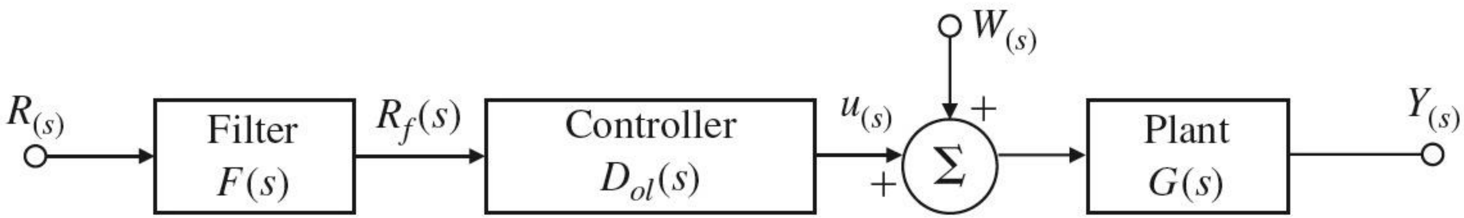
At high frequency, $\mathrm{T} \rightarrow 0$.



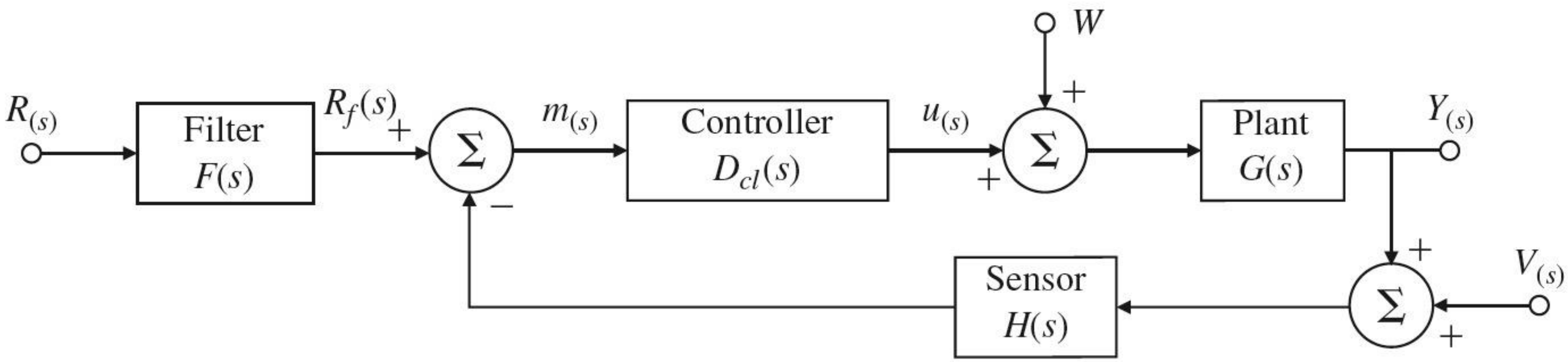
The filtered case

- General case including a dynamic filter on the input and dynamics in the sensor.

- Filtered open loop system:



- Filtered closed loop system:



- sensitivity functions: $S_F^{T_{cl}} = 1.0,\ S_G^{T_{cl}} = \dfrac{1}{1+GD_{cl}H},\ S_H^{T_{cl}} = \dfrac{GD_{cl}H}{1+GD_{cl}H}$

2. System Type

Motivation of system type

- Pole at the Origin

-In a regulator design problem, it is often assumed that both reference and disturbances are constant functions and also take $D(0)$ and $G(0)$ to be finite constants.

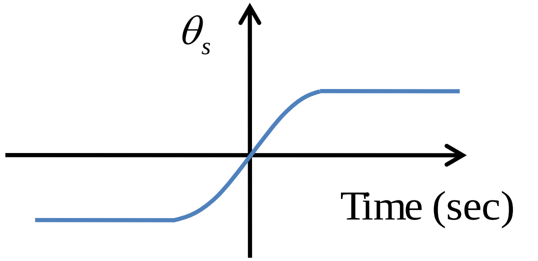
-We will consider the possibility that either or both $D(s)$ and $G(s)$ have poles at s=0.

-For example, the proportional-integral-derivative (PID) control:

$$D(s) = k_p + \frac{k_i}{s} + k_d s$$

System type

- Reference input is usually not constant
  → But it can be approximated as a polynomial in time



- E.g., reference input for elevation angle.
  S-shape → approximated as a linear function.

- General method: the input is represented as a polynomial in time
  → We will discuss the steady state tracking errors for polynomials of different degrees.

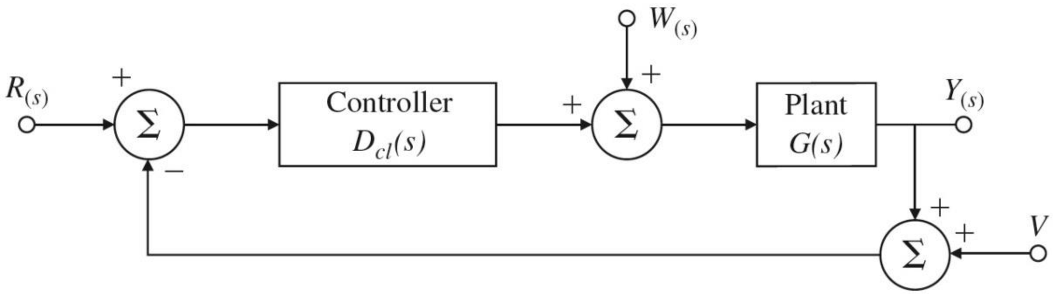
- Some control systems can follow the reference input (step, ramp, …) without error in steady state.

- **System Type**: Degree of the polynomial reference input (or disturbance input) for which the steady-state system error is nonzero finite constant.

System type for reference tracking

- Unity feedback, reference tracking (no disturbance, no sensor noise)

- Steady state error analysis for polynomial inputs



$$\boxed{W = 0, V = 0}$$

$$E_{cl} = \frac{1}{1+DG}R - \frac{G}{1+DG}W + \frac{DG}{1+DG}V$$

- Negative unity feedback system with $W = V = 0$.

$$E = \frac{1}{1+L}R = \mathbb{S}R \qquad \left(L(s) = G(s)D_{cl}(s)\right)$$

- Polynomial input $r(t) = t^k / k!1(t)$ $(R(s) = 1/s^{k+1})$:

$$\lim_{t\to\infty} e(t) = e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+G(s)D_{cl}(s)}\frac{1}{s^{k+1}} = \lim_{s\to 0}\frac{1}{1+GD_{cl}}\frac{1}{s^k}$$

System type for reference tracking

· Type 0 system (GD_{cl} has no pole at the origin): $G(0)D_{cl}(0) = K_p$, $R(s) = 1/s$

$$\rightarrow e_{ss} = \lim_{s\to 0} s \frac{1}{1+GD_{cl}} \frac{1}{s} = \lim_{s\to 0} \frac{1}{1+GD_{cl}} = \frac{1}{1+GD_{cl}(0)}$$

$\rightarrow$ Type 0 system, $GD_{cl}(0) := K_p$ (position error constant)

· Type n system (GD_{cl} has n poles at the origin):

$$GD_{cl}(s) = \frac{GD_{clo}(s)}{s^n}, \qquad GD_{clo}(0) = K_n$$

$$e_{ss} = \lim_{s\to 0} s \frac{1}{1+\dfrac{GD_{clo}(s)}{s^n}} \frac{1}{s^{k+1}} =$$

$$= \lim_{s\to 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$$

(Note that the transfer function from r to e has the zero s^n.)

System type for reference tracking

· Type n system (GD_{cl} has n poles at the origin): $e_{ss} = \lim_{s\to 0} \dfrac{s^n}{s^n + K_n} \dfrac{1}{s^k}$

$$n > k \Rightarrow e_{ss} = 0 \quad \left(e_{ss} = \lim_{s\to 0} \frac{s^{n-k}}{s^n + K_n} = 0 \right)$$

$$n < k \Rightarrow e_{ss} \to \infty \quad \left(e_{ss} = \lim_{s\to 0} \frac{1}{s^n + K_n} \frac{1}{s^{k-n}} = 0 \right)$$

$$n = k = 0 \Rightarrow e_{ss} = 1/(1 + K_0) \quad \left(e_{ss} = \lim_{s\to 0} \frac{1}{1 + K_0} \frac{1}{1} = \frac{1}{1 + K_0} \right)$$

$$n = k \neq 0 \Rightarrow e_{ss} = 1/K_n \quad \left(e_{ss} = \lim_{s\to 0} \frac{1}{s^n + K_n} = \frac{1}{K_n} \right)$$

$K_0 = K_p = \lim_{s\to 0} L(s)$, $\quad n = 0 \quad$ (K_p: position constant)

$K_1 = K_v = \lim_{s\to 0} sL(s)$, $\quad n = 1 \quad$ (K_v: velocity constant)

$K_2 = K_a = \lim_{s\to 0} s^2 L(s)$, $\quad n = 2 \quad$ (K_a: acceleration constant)

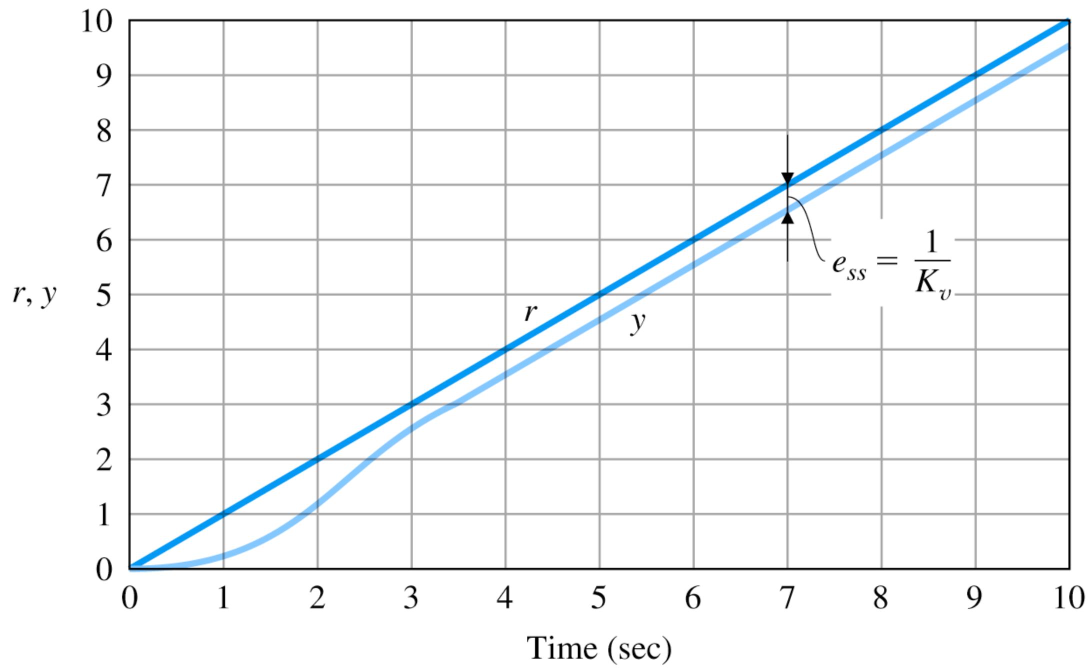
- Stability: The closed-loop system must be stable!!!

Summary of system type for reference tracking

- Errors as a function of system type

	Input		
Type	Step (Position)	Ramp (Velocity)	Parabola (Acceleration)
Type 0	$1/(1+K_p)$	∞	∞
Type 1	0	$1/K_v$	∞
Type 2	0	0	$1/K_a$

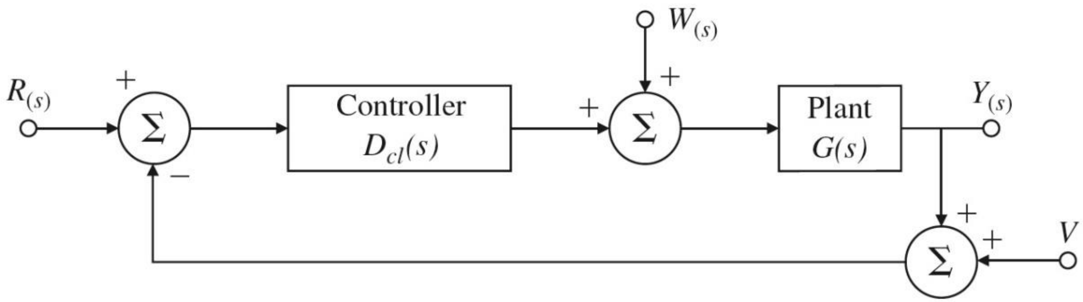
- Type 1 system
  Relationship between
  ramp response and K_v



Example of system type for reference tracking

- System Type for Speed Control

Speed control with P control: $D_{cl}(s) = k_p, G(s) = \dfrac{A}{\tau s + 1}$



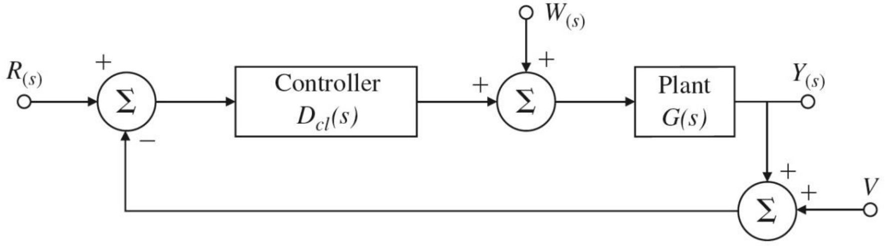
$$\Rightarrow L(s) = G(s)D_{cl}(s) = \frac{k_p A}{\tau s + 1}$$

$$\rightarrow n = 0 \text{ (Type 0)}, \quad K_p = k_p A, \;\; e_{ss} = \frac{1}{1 + k_p A}$$

Example of system type for reference tracking

- System Type Using Integral Control

Speed control with PI control:



$$u(t) = k_p e(t) + k_I \int^t e(\tau) d\tau$$

$$D_{cl}(s) = k_p + \frac{k_I}{s}, \ \ G(s) = \frac{A}{\tau s + 1}$$

$$\Rightarrow L(s) = G(s) D_{cl}(s) = \frac{A\left(k_p s + k_I\right)}{s\left(\tau s + 1\right)}$$

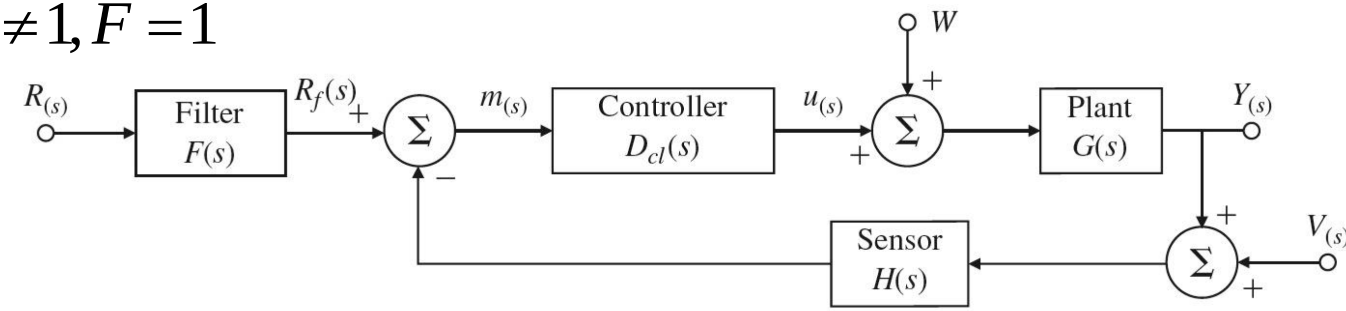
$$\rightarrow n = 1 \text{ (Type 1)}, \quad K_v = \lim_{s \to 0} sL(s) = Ak_I, \ \ e_{ss} = \frac{1}{Ak_I}$$

- Robustness of system type:

In the unity feedback structure, the system type is a robust property with respect to parameter changes which do not remove the poles at the origin.

System type for reference tracking: general case

- General Case: $H \neq 1, F = 1$



$$\frac{Y(s)}{R(s)} = \mathrm{T}(s) = \frac{GD}{1+GDH}$$

$$E(s) = R(s) - Y(s) = R(s) - \mathrm{T}(s)R(s) = \left[1-\mathrm{T}(s)\right]R(s)$$

$$E(s) = \left[1-\mathrm{T}(s)\right]R(s) = \mathrm{S}R \qquad \left(\mathrm{S} = \frac{1+(H-1)DG}{1+HDG}\right)$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[1-\mathrm{T}(s)\right]R(s)$$

- For $r(t) = t^k / k!1(t)$ $(R(s) = 1/s^{k+1})$:

$$E(s) = \frac{1}{s^{k+1}}\left[1-\mathrm{T}(s)\right] \to e_{ss} = \lim_{s\to 0} s\frac{1-\mathrm{T}(s)}{s^{k+1}} = \lim_{s\to 0}\frac{1-\mathrm{T}(s)}{s^k}$$

- If e_{ss} is a nonzero constant for some k, the system type is k.

$\leftarrow \left[1-\mathrm{T}(s)\right]$ has k zeros at the origin (i.e., at $s=0$).

Example of general case

- System Type for a Servo with Tachometer Feedback

  - Motor position control with non-unity feedback:

$$G(s) = \frac{1}{s(\tau s+1)}, \;\; D(s) = k_p, \;\; H(s) = 1 + k_t s$$

Sol. $E(s) = R(s) - Y(s) = R(s) - \mathrm{T}(s)R(s)$

$$= R(s) - \frac{DG(s)}{1+HDG(s)}R(s) = \frac{1+\left(H(s)-1\right)DG(s)}{1+HDG(s)}R(s)$$

$$e_{ss} = \lim_{s\to 0} sR(s)\left[1-\mathrm{T}(s)\right], \quad R(s) = 1/s^{k+1}$$

$$e_{ss} = \lim_{s\to 0} s\frac{\left[1-\mathrm{T}(s)\right]}{s^{k+1}} = \begin{cases} 0, & k=0 \\ \dfrac{1+k_t k_p}{k_p}, & k=1 \end{cases} \rightarrow \text{Type 1}, \;\; K_v = \frac{k_p}{1+k_t k_p}$$

System type for regulation and disturbance rejection

- System type is defined similarly as in the case of reference input.

- System type w.r.t. disturbance input: Degree of polynomial disturbance input that results in a nonzero constant steady-state error.

$$E_{cl} = \frac{1}{1+DG}R - \frac{G}{1+DG}W + \frac{DG}{1+DG}V$$

- System's ability to reject disturbance inputs:

$$\frac{E(s)}{W(s)} = \frac{-Y(s)}{W(s)} = T_w(s), \; T_w(s) = s^n T_{0,w}(s), \; T_{0,w}(0) = 1/K_{n,w}$$

$$\left(= -\frac{G}{1+DG}\right)$$

(Note that the transfer function from w to e has the zero s^n.)

$$\left(W(s) = \frac{1}{s^{k+1}}\right) \rightarrow e_{ss} = -y_{ss} = \lim_{s\to 0}\left[sT_w(s)\frac{1}{s^{k+1}}\right] = \lim_{s\to 0}\left[T_{0,w}(s)\frac{s^n}{s^k}\right]$$

$$\Rightarrow \begin{cases} n > k \rightarrow e_{ss} = -y_{ss} = 0 \\ n < k \rightarrow |e_{ss}| = |y_{ss}| = \infty \\ n = k \rightarrow e_{ss} = -y_{ss} = 1/K_{n,w} \end{cases}$$

Example of system type

- System Type for a DC Motor Position Control

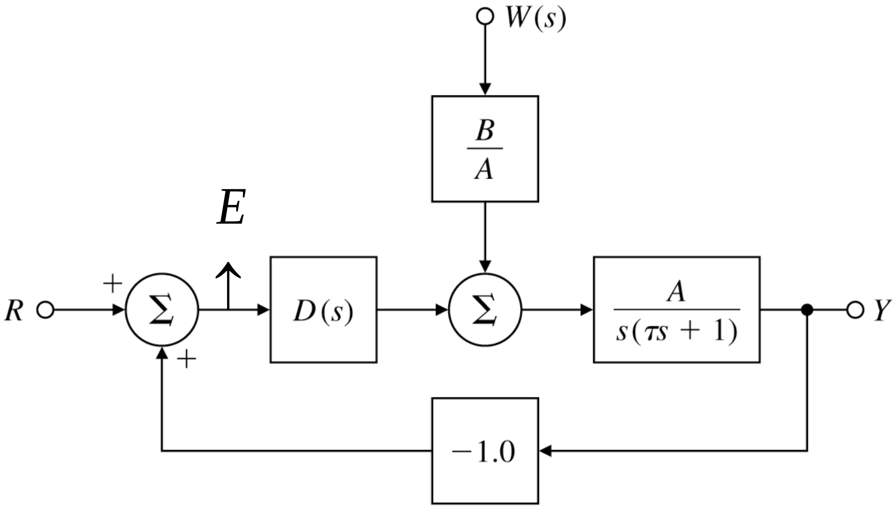
- Determine system types and steady state errors.

a) Proportional control : $D(s) = k_p$

T_w: transfer function from W to E

$$T_w(s) = \frac{(-1.0)\dfrac{A}{s(\tau s+1)}}{1-(-1.0)\dfrac{A}{s(\tau s+1)}D(s)}\frac{B}{A}$$

$$= \frac{-B}{s(\tau s+1)+Ak_p} = s^0 T_{0,w}$$



DC motor with unity feedback

$\rightarrow$ System type $n = 0$, $K_{0,w} = \dfrac{-Ak_p}{B}$, $e_{ss} = T_{0,w}(0) = \dfrac{-B}{Ak_p}$

Note: Type 1 system with respect to reference input

b) PI control: $D(s) = k_p + k_I / s$

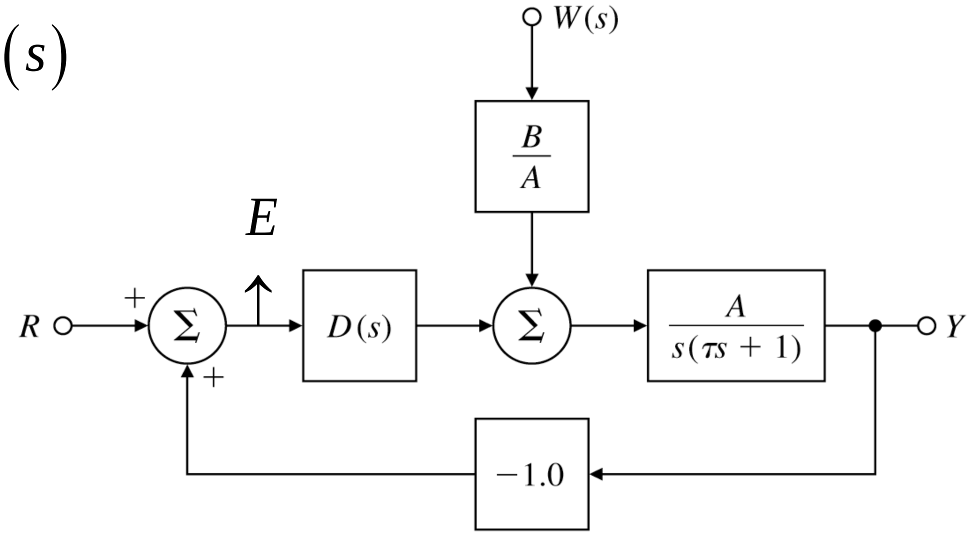
$$T_w(s) = \frac{-Bs}{s^2(\tau s+1) + (k_p s + k_I)A} = sT_{0,w}(s)$$

$\rightarrow \qquad n = 1$

$$K_{n,w} = \frac{Ak_I}{-B}$$

$\rightarrow$ System type $n = 1$

$$e_{ss} = T_{0,w}(0) = \frac{-B}{Ak_I}$$



DC motor with unity feedback

Note: Type 2 system with respect to reference input

3. Introduction to PID Control

Three degrees of freedom controller: PID control

- Properties of a PID control

  - Proportional feedback control reduces the error response to disturbances, but results in nonzero steady-state error to constant inputs.
  - A term proportional to the integral of the error eliminates the steady-state error to constant inputs.
  - A term proportional to the derivative of the error improve the dynamic response (anticipatory term).

Structure of PID control: $D(s) = k_P + \frac{k_I}{s} + k_D s$

k_p: proportional term, $\frac{k_I}{s}$: integral term, k_D: derivative term

Proportional control (P control)

- Proportional Control (P)

$$u = k_p e, \quad \frac{U(s)}{E(s)} = D_{cl}(s) = k_p$$

- For 2nd order plant $G(s) = \dfrac{A}{s^2 + a_1 s + a_2}$,

  Closed-loop characteristic equation:

$$1 + k_p G(s) = 0$$

$$s^2 + a_1 s + a_2 + k_p A = 0$$

$$\left(\text{Note: } \omega_n = \sqrt{a_2 + k_p A} \uparrow, \ \zeta = \frac{a_1}{2\omega_n} \downarrow \right)$$

- The system with proportional control usually has a steady-state error in respose to a constant reference input or to a constant disturbance input.

Proportional-integral control (PI control)

- Proportional plus Integral Control (PI)

$$u(t) = k_p e(t) + k_I \int_{t_0}^{t} e(\tau) d\tau$$

$$\frac{U(s)}{E(s)} = D_{cl}(s) = k_p + \frac{k_I}{s}$$

- Integral term raises the type to Type 1.
- The system can reject completely constant bias disturbances.

Ex. PI control in a speed control

Effects of PI control on the steady-state error to a step disturbance:

$$Y = \frac{A}{\tau s + 1}(U + W)$$

PI control: $U = k_p (R - Y) + k_I \dfrac{R - Y}{s}$

Example of PI control – first order systems

- Speed control

Effects of PI control on the steady-state error to a step disturbance:

$$Y = \frac{A}{\tau s + 1}(U + W)$$

PI control: $U = k_p\left(R - Y\right) + k_I \frac{R - Y}{s}$

$$\left(\tau s + 1\right)Y = A\left(k_p + \frac{k_I}{s}\right)\left(R - Y\right) + AW$$

$$\left(\tau s^2 + \left(Ak_p + 1\right)s + Ak_I\right)Y = A\left(k_p s + k_I\right)R + sAW$$

- Characteristic equation:

$$\tau s^2 + \left(Ak_p + 1\right)s + Ak_I = 0$$

$\rightarrow \omega_n = \sqrt{Ak_I / \tau},\ \zeta = (Ak_p + 1) / 2\tau\omega_n$

$\rightarrow$ May result in an unsatisfactory lightly damped response.

Example of PI control – second order system

- For 2nd order plant $G(s) = \dfrac{A}{s^2 + a_1 s + a_2}$,

  Characteristic equation: $1 + \dfrac{k_p s + k_I}{s} \dfrac{A}{s^2 + a_1 s + a_2} = 0$

$$s^3 + a_1 s^2 + a_2 s + A k_p s + A k_I = s^3 + a_1 s^2 + (a_2 + A k_p) s + A k_I = 0$$

$\rightarrow$ Controller parameters can be used to set only two of the coefficients.

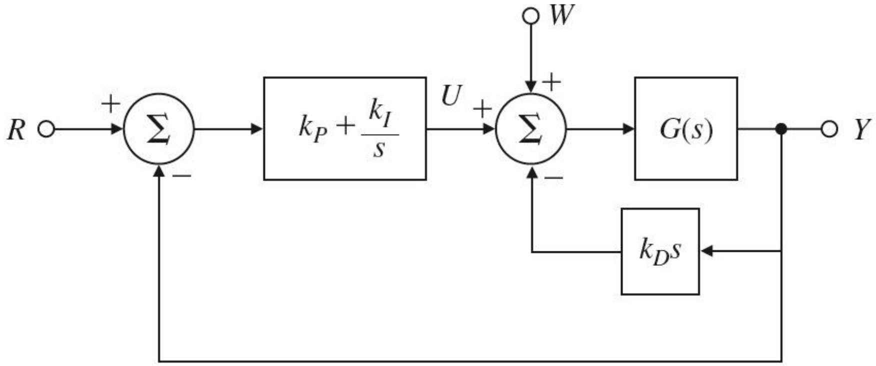
Proportional-integral-derivative control (PID control)

- Proportional-Integral-Derivative Control (PID control)

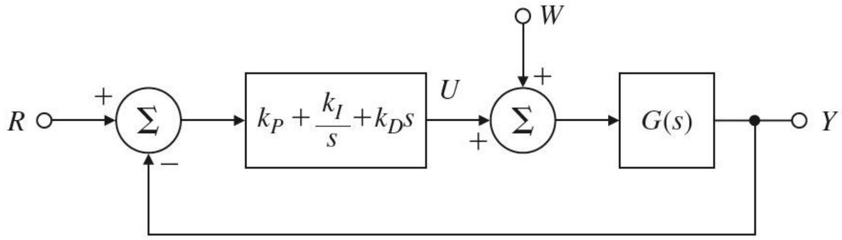
$$u(t) = k_P e(t) + k_I \int^t e(\tau)d\tau + k_D \frac{de(t)}{dt}$$

$$\text{transfer function: } D_{cl}(s) = \frac{U(s)}{E(s)} = k_P + \frac{k_I}{s} + k_D s = \frac{k_P s + k_I + k_D s^2}{s}$$

D term in feedback



D term in forward path



- With derivative in the feedback, the reference is not differentiated: it can avoid undesirable response to sudden change

Example of PID control – second order systems

- Speed control with the second order plant

  - For 2nd order plant $G(s) = \dfrac{A}{s^2 + a_1 s + a_2}$,

  Characteristic equation: $1 + (k_p + \dfrac{k_I}{s} + k_D s)\dfrac{A}{s^2 + a_1 s + a_2} = 0$

$$s^3 + a_1 s^2 + a_2 s + A(k_p s + k_I + k_D s^2) = 0$$

$$\rightarrow s^3 + (a_1 + Ak_D)s^2 + (a_2 + Ak_p)s + Ak_I = 0$$

$\rightarrow$ Controller parameters can be used to set all coefficients.

$\rightarrow$ The roots can be uniquely, and in theory, arbitrarily determined.

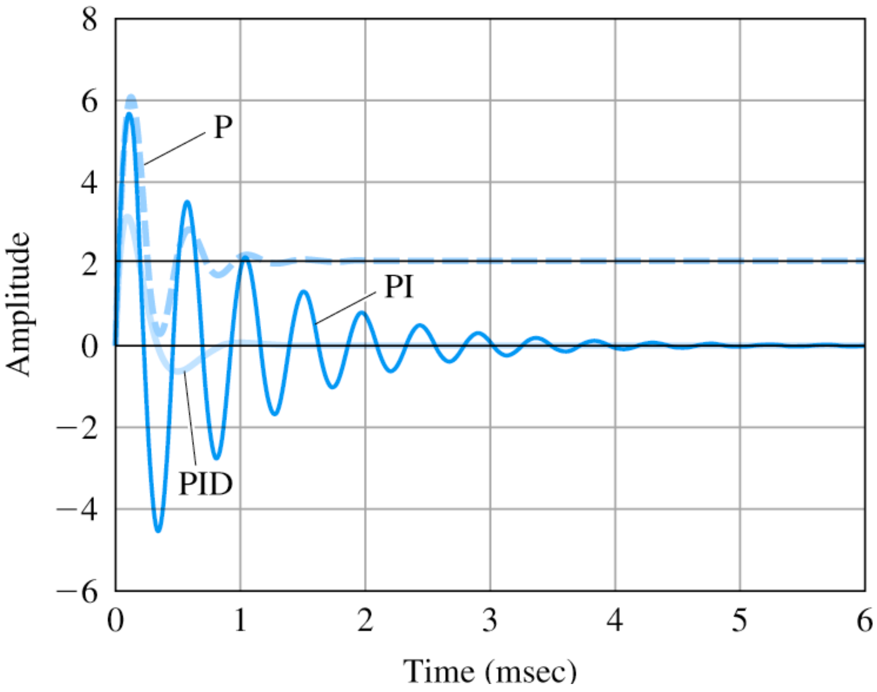
Comparison by simulation

- PID Control of Motor Speed

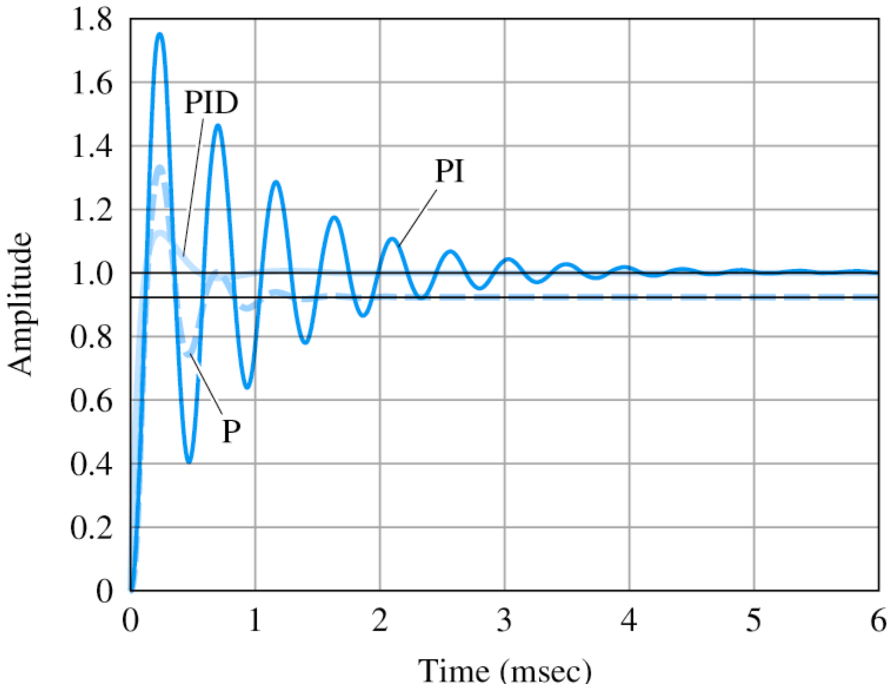
$$\left(\left[(L_a s + R_a)(J_m s + b) + K_t K_e\right]\Omega_m = K_t V_a + K_w W\right)$$

Plant Parameters: see textbook.

Controller parameters: $k_p = 3, \quad k_I = 15\,\mathrm{sec}^{-1}, \quad k_D = 0.3\,\mathrm{sec}$



Step disturbance



Step reference