EECE423-01: 현대제어이론

Modern Control Theory

Chapter 2: Review of Linear Algebra

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◆ The main topics of this chapter are

1. Matrices, Vectors, Addition and Multiplication, Transpose
2. Linear Independence, Rank, Vector space, Image and Kernel space
3. Determinant, Inverse, Eigenvalues and Eigenvectors
4. Quadratic Form, Singular values, Cayley-Hamilton Theorem and Diagonalization
5. Jordan Canonical Form

1. Matrices, Vectors, Addition and Multiplication, Transpose

◆ What is a matrix?

- Matrix: A matrix is a rectangular array of numbers or functions enclosed in brackets

  - Examples: $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \end{bmatrix}$, $\begin{bmatrix} e^{3t} & \cos t \\ \sin t & -\sin t \\ t+1 & -t^2+t \end{bmatrix}$, $\begin{bmatrix} 1 & e^{-3t} \end{bmatrix}$, $\begin{bmatrix} 2t \\ -t \end{bmatrix}$

  - An $m \times n$ matrix: A matrix with m rows and n columns

  - $a_{11}, a_{12}, \ldots, a_{mn}$: Entries

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

  - Horizontal lines: Rows   Vertical lines: Columns

  - The matrix is called a square matrix if $m = n$

  - $a_{11}, a_{22}, \ldots, a_{nn}$: Diagonal entries

◆ What is a vector?

- Vector: A vector is a matrix with only one row or column
  - Examples: $\begin{bmatrix} 1 & e^{-3t} \end{bmatrix}$, $\begin{bmatrix} 2t \\ -t \end{bmatrix}$
  - A column vector with n components: $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$
    ($x_1, \ldots, x_n$: components)
  - A row vector with n components: $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$
    ($x_1, \ldots, x_n$: components)
  - Two matrices A and B are said to be equal, written $A = B$, if and only if they have same size and the corresponding entries are equal.
  - Matrices that are not equal are called different.

◆ Addition of matrices

- The sum of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size is written $A + B$ and has the entries $a_{ij} + b_{ij}$ obtained by adding the corresponding entries of A and B.

- Remark: Matrices of different sizes cannot be added.

- The product of any $m \times n$ matrix $A = [a_{ij}]$ and any scalar c is written cA and is the $m \times n$ matrix $cA = [ca_{ij}]$ obtained by multiplying each entry of A by c.

◆ Rules for matrix addition and scalar multiplication

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$ (written $A + B + C$)
- $A + 0 = A$
- $A + (-A) = 0$
- $c(A + B) = cA + cB$
- $(c + k)A = cA + kA$
- $c(kA) = (ck)A$ (written ckA)
- $1A = A$

◆ Examples

- $\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$

- $\left(\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\right) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} + \left(\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$

- $2\left(\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right) = 2\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 2 & 8 \end{bmatrix}$

- $(2+1)\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = 2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 6 & 9 \end{bmatrix}$

◆ Multiplication of matrices

- The product $C = AB$ (in this order) of an $m \times n$ matrix $A = [a_{ij}]$ times an $r \times p$ matrix $B = [b_{ij}]$ is defined if $n = r$ and the product $C = [c_{ij}]$ is an $m \times p$ matrix with entries

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

$$(i = 1, \ldots, m, \ j = 1, \ldots, p)$$

- Matrix multiplication is not commutative, i.e., $AB \neq BA$ in general

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- Example: $\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

◆ Properties of matrix multiplication

- $k(A)B = k(AB) = A(kB)$ (written kAB or AkB)
- $(AB)C = A(BC)$ (written ABC)
- $(A + B)C = AC + BC$
- $C(A + B) = CA + CB$

◆ Examples

- $\left(\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 0\end{bmatrix}\right)\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\left(\begin{bmatrix}2 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}2 & 3\\3 & 4\end{bmatrix}$

- $\left(\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} + \begin{bmatrix}2 & 1\\1 & 0\end{bmatrix}\right)\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} + \begin{bmatrix}2 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}3 & 4\\2 & 3\end{bmatrix}$

- $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\left(\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} + \begin{bmatrix}2 & 1\\1 & 0\end{bmatrix}\right) = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} + \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}5 & 2\\2 & 1\end{bmatrix}$

◆ Transpose of a matrix, symmetric matrix

- The **transpose** of an $m \times n$ matrix $A = [a_{ij}]$ is the $n \times m$ matrix denoted by A^T and defined as

$$A^T = [a_{ji}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \ddots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

- If $A = A^T$, we call A is a **symmetric matrix**.

◆ Properties of transpose

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(cA)^T = cA^T$
- $(AB)^T = B^T A^T$
- If A is a square matrix, $|A| = |A^T|$
- If A is a nonsingular matrix, $(A^T)^{-1} = (A^{-1})^T$
- If A is a square matrix, the eigenvalues of A coincide with those of A^T

◆ Examples

- $\left(\begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix} + \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right)^T = \begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}^T + \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}^T = \begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}$
- $\left(2\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right)^T = 2\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}^T = \begin{bmatrix}2 & 0\\ 2 & 2\end{bmatrix}$
- $\left(\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}\right)^T = \begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}^T \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}^T = \begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}$
- $\left|\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}\right| = \left|\begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}^T\right| = 5$
- $\left(\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}^T\right)^{-1} = \left(\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}^{-1}\right)^T = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$

2. Linear Independence, Rank, Vector space, Image and Kernel space

◆ Linear independence

- **Linear combination** of a set of n vectors $v_1, v_2, \ldots, v_n$:

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

  where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars.

- A set of n vectors $v_1, v_2, \ldots, v_n$ is said to be **linearly independent** if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

  implies $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

- A set of n vectors $v_1, v_2, \ldots, v_n$ is said to be **linearly dependent** if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ that are not all zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

◆ Examples

- $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\ v_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$: linearly dependent
- $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$: linearly independent
- $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$: linearly independent

◆ Rank of a matrix

- The **rank** of a matrix A is the maximal number of linearly independent columns of A. It is denoted by $\operatorname{rank}(A)$.

- Examples

$$\operatorname{rank}\left(\begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}\right) = 2, \quad \operatorname{rank}\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}\right) = 1,$$

$$\operatorname{rank}\left(\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}\right) = 3, \quad \operatorname{rank}\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}\right) = 2$$

◆ Properties of rank

- Interchange of two rows (or columns) does not alter the value of the rank. Addition of a multiple of a row (or column) to another row (or column) does not alter the value of the rank.

- Multiplication of a row (or column) by a nonzero constant c does not alter the value of the rank.

- $\mathrm{rank}\ (A) = \mathrm{rank}\ (A^T)$.

- When A is an $n \times m$ matrix, $\mathrm{rank}\ (A) \leq \min(n, m)$.

- $\mathrm{rank}\ (AB) \leq \min(\mathrm{rank}\ (A), \mathrm{rank}\ (B))$.

◆ Vector space

- A nonempty set V of elements $a, b, \dots$ is called a real vector space (or real linear space), and these elements are called vectors, if, in V, there are defined two algebraic operations (called vector addition and scalar multiplication).

I. Vector addition ($a + b \in V$, $\forall a, b \in V$)

I.1. Commutativity: $a + b = b + a$, $\forall a, b \in V$

I.2. Associativity: $(a + b) + c = a + (b + c)$, $\forall a, b, c \in V$

I.3. $\exists$ unique $0 \in V$, such that $a + 0 = a$, $\forall a \in V$

I.4. For every a in V, there exists a unique vector in V, denoted by $-a$, and is such that $a + (-a) = 0$.

II. Scalar multiplication ($ca \in V$, $\forall c \in R, \forall a \in V$)

II.1. Distributivity: $c(a + b) = ca + cb$, $\forall c \in R, \forall a, b \in V$

II.2. Distributivity: $(c + k)a = ca + ka$, $\forall c, k \in R, \forall a \in V$

II.3. Associativity: $c(ka) = (ck)a$, $\forall c, k \in R, \forall a \in V$

II.4. $\forall a \in V$, $1a = a$

◆ Dimension, span and basis

- The maximum number of linearly independent vectors in V is called the **dimension** of V and is denoted by $\mathrm{dim}V$.

- The set of all linear combinations of given vectors $v_1, \ldots, v_n$ is called the **span** of these vectors.

- A set of vectors is a **basis** for a vector space V if

  (1) the vectors in the set are linearly independent

  (2) all the elements in V can be described by a linear combination of the vectors (i.e., the vectors span V)

◆ Image (range)

- Given an $m \times n$ matrix M, the **image** or range of M is the span (set of all possible linear combinations) of its column vectors.

$$\mathrm{Im}\ M := \{y \in R^m \mid \exists x \in R^n,\ y = Mx\}$$

- The **image** of M is a linear subspace of R^m, and its dimension coincides with the rank of the matrix M.

◆ Kernel space (null space)

- Given an $m \times n$ matrix M, the **kernel** or null space of M is the set

$$\mathrm{Ker}\ M := \{x \in R^n \mid Mx = 0\}$$

- The **kernel** of M is a linear subspace of R^n, and its dimension is called the nullity of the matrix M.

◆ Fundamental theorem of linear equations

- For every $m \times n$ matrix M, the following relation holds:

$$\dim \text{Ker } M + \dim \text{Im } M = n$$

- Example $M = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & -1 & 2 \end{bmatrix}$

$$\text{Ker } M = \left\{ c_1 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \;\middle|\; \forall c_1 \in R \right\}$$

$$\text{Im } M = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \;\middle|\; \forall c_1, c_2 \in R \right\}$$

$$\rightarrow \dim \text{Ker } M + \dim \text{Im } M = 1 + 2 = 3$$

3. Determinant, Inverse, Eigenvalues and Eigenvectors

◆ Determinant

- A **determinant** is a scalar and is defined for a square matrix.

- The **determinant** of an $n \times n$ matrix A is denoted by $|A|$.

- $|A|$ is defined as follows:

$$|A| := \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

or

$$|A| := \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

where M_{ij} is the determinat of the submatrix of A obtained from omitting the ith row and jth column.

◆ Properties of determinant

- Interchange of two rows (or columns) multiplies the value of the determinant by -1.

- Addition of a multiple of a row (or column) to another row (or column) does not alter the value of the determinant.

- Multiplication of a row (or column) by a nonzero constant c multiplies the value of the determinant by c.

- When A and B are $n \times n$ matrices, $|AB| = |A||B|$

- When A is an $n \times n$ matrix, $\mathrm{rank}\ (A) = n$ if and only if $|A| \neq 0$

◆ Examples

- $\left|\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}\right| = 1\left|\begin{bmatrix} 6 & 4 \\ 0 & 2 \end{bmatrix}\right| - 3\left|\begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix}\right| + 0\left|\begin{bmatrix} 2 & 6 \\ -1 & 0 \end{bmatrix}\right| = -12$
- $\left|\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}\right| = -1\left|\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\right| = -1$
- $\left|\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}\right| = \left|\begin{bmatrix} 2 & 3 \\ 7 & 10 \end{bmatrix}\right| = -1$
- $\left|\begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}\right| = 2\left|\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}\right| = -2$
- $\left|\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right| = \left|\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}\right| = \left|\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\right|\left|\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right| = -1$

◆ Inverse matrix

- The **inverse** of an $n \times n$ matrix A is denoted by A^{-1} and is an $n \times n$ matrix such that $AA^{-1} = A^{-1}A = I$.

- If A has an **inverse**, the inverse is unique.

- If A **has an inverse**, then A is called a **nonsingular** matrix.
  If A **has no inverse**, then A is called a **singular** matrix.

◆ Computation method for a inverse matrix

- The inverse of a nonsingular matrix A $(n \times n)$ can be given by

$$A^{-1} = \frac{1}{|A|}\mathrm{adj}(A)$$

where $\mathrm{adj}(A)$ is adjoint matrix with its (i, j)th element A_{ij} given by

$$A_{ij} = (-1)^{i+j} M_{ji}$$

(M_{ij} is the determinat of the submatrix of A obtained from omitting the ith row and jth column)

- If you want to compute a inverse matrix by hand, a numerical method such as Gauss Elimination can be helpful, but this course omits the details for a limited time.

◆ Properties of a inverse matrix

- The inverse A^{-1} of an $n \times n$ matrix A exists if and only if $|A| \neq 0$

- If $A\ (n \times n)$ and $B\ (n \times n)$ are nonsingular, $(AB)^{-1} = B^{-1}A^{-1}$

- If $A\ (n \times n)$ is a nonsingular matrix and α is a scalar, $(\alpha A)^{-1} = A^{-1}/\alpha$

- If $A\ (n \times n)$ is a nonsingular matrix, $|A^{-1}| = 1/|A|$

Examples

- For $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, there is no inverse of A

- $\left(\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$

- $\left(2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

- $\left| \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \right| = 1 / \left| \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right| = \frac{1}{5}$

◆ Eigenvalues and Eigenvectors

- For a given $n \times n$ matrix A, if there exists a scalar λ such and a nonzero vector v such that

$$Av = \lambda v$$

Then, v is called an **eigenvector** of A corresponding to this **eigenvalue** λ.

◆ Properties of eigenvalues

- The **eigenvalues** of an $n \times n$ matrix A are the roots of the characteristic equation

$$\det(sI - A) = 0$$

- When A is an $n \times n$ matrix and has the eigenvalues $\lambda_1, \ldots, \lambda_n$ (allowed for multiplicity), $|A|$ is equal to $\lambda_1 \lambda_2 \cdots \lambda_n$

◆ Example

- $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ has its characteristic equation given by

$$\det(sI - A) = s^3 + s^2 - 21s - 45 = (s-5)(s+3)^2 = 0$$

For $\lambda_1 = 5$, $(\lambda_1 I - A)v_1 = \begin{bmatrix} 7 & -2 & 3 \\ -2 & 4 & 6 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = 0 \rightarrow v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

For $\lambda_2 = -3$, $(\lambda_2 I - A)v_2 = \begin{bmatrix} -1 & -2 & 3 \\ -2 & -4 & 6 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0$

$$\rightarrow v_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

◆ Summary of determinant, inverse matrix and eigenvalue

- For an $n \times n$ matrix A, the followings are equivalent:

(a) A is a nonsingular (i.e., there exists a A^{-1}).

(b) $|A| \neq 0$.

(c) rank $(A) = n$.

(d) 0 is not an eigenvalue of A.

◆ Advanced issue 1 - Characteristics of Determinant

- When A is an $n \times n$ matrix and D is an $m \times m$ matrix, we obtain the following relations:

$$\left|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right| = |A||D - CA^{-1}B| \quad (\text{when } |A| \neq 0)$$
$$= |D||A - BD^{-1}C| \quad (\text{when } |D| \neq 0)$$

- When B is an $n \times m$ matrix and C is an $m \times n$ matrix, we obtain the following relation:

$$|I_n + BC| = |I_m + CB|$$

◆ Advanced issue 2 - Matrix Inversion Lemma

- If A is an $n \times n$ matrix and D is an $m \times m$ matrix,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}$$

$$\text{where} \quad S = D - CA^{-1}B$$

$$(\text{when} \quad |A| \neq 0, \ |S| \neq 0)$$

$$= \begin{bmatrix} K^{-1} & -K^{-1}BD^{-1} \\ -D^{-1}CK^{-1} & D^{-1} + D^{-1}CK^{-1}BD^{-1} \end{bmatrix}$$

$$\text{where} \quad K = A - BD^{-1}C$$

$$(\text{when} \quad |D| \neq 0, \ |K| \neq 0)$$

- If A is an $n \times n$ nonsingular matrix and B is an $n \times m$ matrix and C is an $m \times n$ matrix,

$$(A + BC)^{-1} = A^{-1} - A^{-1}B(I_m + CA^{-1}B)^{-1}CA^{-1}$$

4. Quadratic Form, Singular values, Cayley-Hamilton Theorem and Diagonalization

◆ Quadratic form

- A quadratic form in the components $x_1, \ldots, x_n$ of a vector $x := [x_1 \ \cdots \ x_n]^T$ is a sum of n^2 terms, namely,

$$x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

- Example

$$\begin{aligned}
&2x_1^2 - 2x_1x_2 + 4x_1x_3 + x_2^2 + 6x_3 \\
=&2x_1^2 - x_1x_2 + 2x_1x_3 + x_2^2 - x_2x_1 + 6x_3^2 + 2x_3x_1 \\
=&\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{aligned}$$

◆ Positive definite matrix

- A symmetric matrix $A(= A^T)$ is said to be positive definite, if

$$x^T Ax > 0 \quad (\forall x \neq 0)$$

- A symmetric matrix $A(= A^T)$ is said to be semi-positive definite, if

$$x^T Ax \geq 0 \quad (\forall x \neq 0)$$

◆Basic property of a (semi-)positive definite matrix

- λ_i $(i = 1, \ldots, n)$: Eigenvalues of an $n \times n$ symmetric matrix A

(1) A is a positive definite matrix $\Leftrightarrow \lambda_i > 0$ $(\forall i)$

(2) A is a semi-positive definite matrix $\Leftrightarrow \lambda_i \geq 0$ $(\forall i)$

*All eigenvalues of a real symmetric matrix are real

◆ Advanced properties of a (semi-)positive definite matrix

- The followings are equivalent for a symmetric $n \times n$ matrix Q.

(1) Q is a positive definite matrix.

(2) All eigenvalues of Q are strictly positive.

(3) There exists an $n \times n$ nonsingular matrix H such that

$$Q = H^T H.$$

◆ Example

$$\text{For } A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 6 \end{bmatrix}$$

- $x^T A x = (x_1 - x_2)^2 + (x_1 + 2x_3)^2 + 2x_3^2$

  $\rightarrow x^T A x = 0$ only if $x_1 = x_2 = x_3 = 0$

  $\rightarrow A$ is a positive definite matrix

- $\det(sI - A) = (s-2)(s^2 - 7s + 1) = 0$

  $\rightarrow$ Eigenvalues $\lambda = 2, (7 \pm \sqrt{5})/2$ are larger than 0

  $\rightarrow A$ is a positive definite matrix

◆ Singular values

Let A be an $m \times n$ matrix, and consider the matrix $A^T A$.

Because $A^T A$ is an $n \times n$ (semi-)positive definite matrix,

- its eigenvalues are real
- its eigenvalues are equal or larger than 0
- let $\lambda_1, \ldots, \lambda_n$ denote the eigenvaules of $A^T A$ with repetiions.

The numbers $\sigma_1, \ldots, \sigma_n$ ($\sigma_i := \sqrt{\lambda_i}$) are called

the **singular values** of A.

◆ Example

For $A = \begin{bmatrix} 5 & 2 \\ -3 & 0 \end{bmatrix}$

- $|\lambda I - A| = \lambda^2 - 5\lambda + 6 = 0$

  $\rightarrow$ Eigenvalues: $\lambda = 2, 3$

- $|\lambda I - A^T A| = \lambda^2 - 38\lambda + 36 = 0$

  $\rightarrow$ Singular values: $\sigma = \sqrt{19 + 5\sqrt{13}}, \sqrt{19 - 5\sqrt{13}}$

◆ Cayley-Hamilton Theorem

- For an $n \times n$ matrix A whose characteristic equation given by

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n = 0$$

the following relation holds:

$$a(A) = A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1}A + a_n I = 0$$

- Example $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

$$a(s) = s^2 - s - 3 = 0$$

$$a(A) = A^2 - A - 3I = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

◆ Diagonalizable matrix

- A square $n \times n$ matrix A is called diagonalizable if there exist matrices P and P^{-1} such that $P^{-1}AP$ is a diagonal matrix.

- An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

- If $v_1, \ldots, v_n$ are linearly independent eigenvectors of A corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively,

$$P^{-1}AP = \Lambda := \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \text{ where } P := [v_1 \ v_2 \ \cdots \ v_n]$$

$$\rightarrow A^n = P\Lambda^n P^{-1}$$

◆ Sufficient condition for diagonalizability

- If a square $n \times n$ matrix A has different eigenvalues $\lambda_1, \ldots, \lambda_n$ (i.e., $\lambda_i \neq \lambda_j, \forall i, j$), A is a diagonalizable matrix.

◆ Example

For $A_1 = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}$, $\lambda_1 = 1$, $\lambda_2 = -1$ and $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\rightarrow$ diagonalizable matrix

◆ Issues on diagonalizablity of a matrix

- If the characteristic equation $\det(sI - A) = 0$ for an $n \times n$ matrix A has multiple roots, there could exist a case such that A is non-diagonalizable.

- It is not true that a matrix whose characteristic equation has multiple roots is always non-diagonalizable.

- In contrast to the case of a nonsingular matrix, $|A| = 0$ does not mean A is non-diagonalizable and $|A| \neq 0$ does not mean A is diagonalizable.

◆ Example1

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \text{ has the characteristic equation } (s-1)^2 = 0$$

$$\text{and only one corresponding eigenvector } v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

→ non-diagonalizable matrix

◆ Example2

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \text{ has the characteristic equation } (s-3)s = 0$$

$$\text{and the corresponding eigenvectors } v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

→ diagonalizable matrix

◆ Example3

$$A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ has the characteristic equation } (s-2)(s+1)^2 = 0$$

$$\text{and the corresponding eigenvectors } v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_{-11} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v_{-12} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\rightarrow$ diagonalizable matrix

5. Jordan Canonical Form

◆ Jordan canonical form (for a non-diagonalizable matrix)

- For every $n \times n$ matrix A, there exists a nonsingular matrix P that transforms A into

$$J = P^{-1}AP = \begin{bmatrix} J_{p_1}(\lambda_1) & & & \\ & J_{p_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{p_N}(\lambda_N) \end{bmatrix}$$

where $J_{p_i}(\lambda_i)$ is a Jordan block defined as

$$J_{p_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & \lambda_i \end{bmatrix} \qquad (p_i \times p_i)$$

and each λ_i is an eigenvalue of A.

◆ Geometric and Algebraic Multiplicities

- For an eigenvalue λ_i of an $n \times n$ matrix A, assume that
  1. There are κ_i Jordan blocks which have λ_i as diagonal elements
  2. Each Jordan block has size of $n_{ip} \times n_{ip}$ $(p = 1, \ldots, \kappa_i)$

Then, the geometric and algebraic multiplicities are defined as follows:

- κ_i: Geometric Multiplicity
- $\mathrm{rank}(A - \lambda_i I) = n - \kappa_i$

- $n_i := n_{i1} + \cdots + n_{i\kappa_i}$: Algebraic Multiplicity
- $\det(sI - A) = \prod_{i=1}^{l} (s - \lambda_i)^{n_i}$

◆ Non-derogatory matrix and derogatory matrix

Case I: For all eigenvalues λ_i, $\kappa_i = n_i$ $(i = 1, \ldots, l)$.

$\rightarrow$ All the Jordan blocks have size of 1×1.

$\rightarrow$ A is diagonalizable.

Case II: For all eigenvalues λ_i, $\kappa_i = 1$ $(i = 1, \ldots, l)$.

$\rightarrow$ There exists only one Jordan block for each eigenvalue λ_i.

$\rightarrow$ In this case, we call A a **non-derogatory** matrix.

Case III: There exist an eigenvalue λ_i such that $\kappa_i \geq 2$.

$\rightarrow$ There exist two or more Jordan blocks for an eigenvalue λ_i.

$\rightarrow$ In this case, we call A a **derogatory** matrix.

◆ Non-derogatory case (one Jordan block for each eigenvalue λ_i)

Let us assume for an $n \times n$ matrix A that

$$\det(sI - A) = (s - \lambda_1)^{n_1}(s - \lambda_2)^{n_2} \cdots (s - \lambda_l)^{n_l}$$

$$(\lambda_i \neq \lambda_j,\ \forall i, j; \quad n_1 + n_2 + \cdots + n_l = n)$$

For an eigenvalue λ_i with the algebraic multiplicity n_i,

let us define the vectors $v_{i,1}, \ldots, v_{i,n_i}$ as follows:

$$\boxed{\begin{aligned} (\lambda_i I - A)v_{i,1} &= 0 \\ (\lambda_i I - A)v_{i,2} &= -v_{i,1} \\ &\vdots \\ (\lambda_i I - A)v_{i,n_i} &= -v_{i,n_i-1} \end{aligned}}$$

Then, we define the $n \times n_i$ matrix $P(\lambda_i)$ as

$$P(\lambda_i) := \begin{bmatrix} v_{i,1} & v_{i,2} & \cdots & v_{i,n_i} \end{bmatrix}$$

Applying this procedure to all the eigenvalues $\lambda_1, \ldots, \lambda_l$ leads to

$$P(\lambda_1) := \begin{bmatrix} v_{1,1} & \cdots & v_{1,n_1} \end{bmatrix}, \ \ldots, \ P(\lambda_l) := \begin{bmatrix} v_{l,1} & \cdots & v_{l,n_l} \end{bmatrix}$$

Here, if we define the $n \times n$ matrix P as

$$P := \begin{bmatrix} P(\lambda_1) & P(\lambda_2) & \cdots & P(\lambda_l) \end{bmatrix}$$

we can obtain the following Jordan canonical form:

$$J = P^{-1}AP = \begin{bmatrix} J_{n_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{n_l}(\lambda_l) \end{bmatrix}$$

◆ Example

For $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix}$, we have $\det(sI - A) = (s+1)^2(s+2) = 0$

For $\lambda_1 = -1$, $\operatorname{rank}(\lambda_1 I - A) = \operatorname{rank}\left(\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 2 & 5 & 3 \end{bmatrix}\right) = 2 = 3 - 1$

$\rightarrow A$ is a non-derogatory matrix.

From $(\lambda_1 I - A)v_{1,1} = 0, \quad (\lambda_1 I - A)v_{1,2} = -v_{1,1},$

$$v_{1,1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad v_{1,2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

On the other hand, for $\lambda_2 = -2$,

$$(\lambda_2 I - A)v_{2,1} = 0 \text{ leads to } v_{2,1} = \begin{bmatrix} -1/2 \\ 1 \\ -2 \end{bmatrix}$$

By defining $P := \begin{bmatrix} v_{1,1} & v_{1,2} & v_{2,1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1/2 \\ -1 & 0 & 1 \\ 1 & -1 & -2 \end{bmatrix}$,

$$\rightarrow P^{-1}AP = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

◆ Derogatory case (p_i Jordan block for each eigenvalue λ_i)

Let us assume for an $n \times n$ matrix A that

$$\det(sI - A) = (s - \lambda_1)^{n_1}(s - \lambda_2)^{n_2} \cdots (s - \lambda_l)^{n_l}$$
$$(\lambda_i \neq \lambda_j, \ \forall i, j; \quad n_1 + n_2 + \cdots + n_l = n)$$

For an eigenvalue λ_i with the geometric multiplicity p_i, i.e.,

$$\mathrm{rank}(\lambda_i I - A) = n - p_i \ (p_i \geq 2)$$

we can compute the following r vectors through trial and error:

$$\begin{aligned} (\lambda_i I - A)v_{i,1,1} &= 0 \\ (\lambda_i I - A)v_{i,1,2} &= -v_{i,1,1} \\ &\vdots \\ (\lambda_i I - A)v_{i,1,m_{i1}} &= -v_{i,1,m_{i1}-1} \end{aligned} \qquad \cdots \qquad \begin{aligned} (\lambda_i I - A)v_{i,p_i,1} &= 0 \\ (\lambda_i I - A)v_{i,p_i,2} &= -v_{i,p_i,1} \\ &\vdots \\ (\lambda_i I - A)v_{i,p_i,m_{ip_i}} &= -v_{i,p_i,m_{ip_i}-1} \end{aligned}$$

$$m_{i1} + m_{i2} + \cdots + m_{ip_i} = n_i$$

Then, we define the $n \times n_i$ matrix $P(\lambda_i)$ as

$$P(\lambda_i) := \begin{bmatrix} v_{i,1,1} & \cdots & v_{i,1,m_{i1}} & \cdots & v_{i,p_i,1} & \cdots & v_{i,p_i,m_{ip_i}} \end{bmatrix}$$

Applying this procedure to all the eigenvalues $\lambda_1, \ldots, \lambda_l$ leads to

$$P := \begin{bmatrix} P(\lambda_1) & P(\lambda_2) & \cdots & P(\lambda_l) \end{bmatrix}$$

Then, we can obtain the following Jordan canonical form:

$$J = P^{-1}AP = \begin{bmatrix} J_{m_{11}}(\lambda_1) & & & & & & \\ & \ddots & & & & & \\ & & J_{m_{1p_1}}(\lambda_1) & & & & \\ & & & \ddots & & & \\ & & & & J_{m_{l1}}(\lambda_l) & & \\ & & & & & \ddots & \\ & & & & & & J_{m_{lp_l}}(\lambda_l) \end{bmatrix}$$

◆ Example

$$\text{For } A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix}, \quad \det(sI - A) = (s+1)^3 s = 0$$

For $\lambda_1 = -1$,

$$\text{rank}(\lambda_1 I - A) = \text{rank}\left(\begin{bmatrix} -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \right) = 2 = 4 - 2$$

$\rightarrow A$ is a derogatory matrix.

For $\lambda_1 = -1$, there exsit 2 (=geometric multiplicity) Jordan blocks

and the sum of their sizes is 3 (=algebraic multiplicity)

$\rightarrow$ Jordan blocks for λ_1 have sizes of 1 and 2.

From $(\lambda_1 I - A)v_{1,k,1} = 0 \quad (k = 1, 2)$,

$$v_{1,1,1} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad v_{1,2,1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Becuase $v_{1,1,1} \notin \mathrm{Im}\,(\lambda_1 I - A)$, there is no $v_{1,1,2}$ but $v_{1,2,2}$ such that

$$(\lambda_1 I - A)v_{1,2,2} = -v_{1,2,1} \text{ and thus } v_{1,2,2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

On the other hand, for $\lambda_2 = 0$,

$$(\lambda_2 I - A)v_{2,1,1} = 0 \text{ leads to } v_{2,1,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By defining $P := \begin{bmatrix} v_{1,1,1} & v_{1,2,1} & v_{1,2,2} & v_{2,1,1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$,

$$\rightarrow P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$