analytic_orbits

July 23, 2025

Our goal is to find the analytical expression for a circular orbit in the spacetime of a rotating black hole, which is described by the Kerr metric. In Boyer-Lindquist coordinates, the non-zero components of the metric tensor $g_{\mu\nu}$ are:

$$g_{tt} = -\left(1 - \frac{2Mr}{\rho^2}\right)$$
$$g_{t\phi} = -\frac{2Mar\sin^2(\theta)}{\rho^2}$$
$$g_{rr} = \frac{\rho^2}{\Delta}$$
$$g_{\theta\theta} = \rho^2$$
$$g_{\phi\phi} = \left(r^2 + a^2 + \frac{2Ma^2r\sin^2(\theta)}{\rho^2}\right)\sin^2(\theta)$$

Where the auxiliary functions ρ^2 and Δ are defined as:

$$\rho^2 = r^2 + a^2\cos^2(\theta) \quad \text{and} \quad \Delta = r^2 - 2Mr + a^2$$

The Kerr metric possesses an axial symmetry around the polar axis. The simplest and most physically relevant orbits to study are those that lie within the plane of this symmetry: the **equatorial plane**.

Physical Assumption: We restrict our analysis to orbits confined to the equatorial plane.

Mathematical Consequence: This physical assumption corresponds to setting the polar angle to a constant value of $\theta = \frac{\pi}{2}$. This greatly simplifies the metric components, as $\sin(\theta) = 1$ and $\cos(\theta) = 0$, which in turn means $\rho^2 = r^2$. The metric components become:

$$g_{tt} = -\left(1 - \frac{2M}{r}\right)$$
$$g_{t\phi} = -\frac{2Ma}{r}$$
$$g_{rr} = \frac{r^2}{\Delta}$$
$$g_{\theta\theta} = r^2$$

$$g_{\phi\phi} = r^2 + a^2 + \frac{2Ma^2}{r}$$

These simplified components form the mathematical foundation for our specific problem.

In General Relativity, the path of a free-falling test particle is a **geodesic**. The geodesic equation is the relativistic analogue of Newton's second law, describing how a particle moves through curved spacetime. It is given by:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

Here, τ is the proper time (the time measured by a clock on the particle), and $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols, which encode the gravitational field's effect on the particle's acceleration.

To find the specific conditions for a circular orbit, we translate the physical definition of "circular" into mathematical constraints on the particle's motion.

Physical Assumption: The particle is in a stable, circular orbit.

Mathematical Consequence: This statement has two powerful implications: 1. The orbital radius is constant, so there is no radial velocity: $\frac{dr}{d\tau} = 0$. 2. The orbit is not spiraling in or out, so there is no radial acceleration: $\frac{d^2r}{d\tau^2} = 0$.

We apply these conditions to the radial component ($\alpha = r$) of the geodesic equation. The equation becomes:

$$0 + \Gamma^r_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

Because the radial and polar velocities ($dr/d\tau$ and $d\theta/d\tau$) are zero, the only terms in the sum over μ and ν that survive are those where the particle has velocity: the time (t) and azimuthal (ϕ) components. This reduces the sum to four terms, yielding our master equation for circular orbits:

$$\Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2 + 2\Gamma^r_{t\phi}\frac{dt}{d\tau}\frac{d\phi}{d\tau} + \Gamma^r_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 = 0$$

This equation provides the fundamental relationship between the orbital radius and the angular velocity that must be satisfied for a circular orbit to exist.

To solve our master equation, we must first calculate the required Christoffel symbols. The general formula is:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\kappa}\left(g_{\mu\kappa,\nu} + g_{\nu\kappa,\mu} - g_{\mu\nu,\kappa}\right)$$

where the comma denotes a partial derivative (e.g., $g_{\mu\nu,\kappa} = \frac{\partial g_{\mu\nu}}{\partial x^\kappa}$).

This formula requires the **inverse metric tensor**, $g^{\alpha\kappa}$. Finding the inverse of a 4x4 matrix can be algebraically intensive. However, we can gain significant insight by representing our simplified equatorial metric as a matrix.

Mathematical Simplification: Let's assign variables to our equatorial metric components: $a = g_{tt}$, $b = g_{rr}$, $c = g_{\theta\theta}$, $d = g_{\phi\phi}$, and $e = g_{t\phi}$. The metric tensor $g_{\mu\nu}$ takes the form:

$$g_{\mu\nu} = \begin{bmatrix} a & 0 & 0 & e \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ e & 0 & 0 & d \end{bmatrix}$$

The inverse of this matrix, $g^{\mu\nu}$, can be calculated to be:

$$g^{\mu\nu} = \begin{bmatrix} \frac{d}{ad-e^2} & 0 & 0 & -\frac{e}{ad-e^2} \\ 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & \frac{1}{c} & 0 \\ -\frac{e}{ad-e^2} & 0 & 0 & \frac{a}{ad-e^2} \end{bmatrix}$$

Physical Insight from the Math: The structure of this inverse matrix provides a crucial simplification. We are calculating $\Gamma^r_{\mu\nu}$, so we only need the $\alpha = r$ row of the inverse metric. From the matrix above, we can see that the only non-zero component in this row is $g^{rr} = 1/b = 1/g_{rr}$. This means that when we sum over κ in the Christoffel symbol formula, the only term that survives is the one where $\kappa = r$.

The simplification from the inverse metric allows us to write the Christoffel symbols in a much more manageable form:

$$\Gamma^r_{tt} = \frac{1}{2}g^{rr}\left(g_{tr,t} + g_{tr,t} - g_{tt,r}\right) = \frac{1}{2}g^{rr}(-g_{tt,r})$$

$$\Gamma^r_{t\phi} = \frac{1}{2}g^{rr}\left(g_{tr,\phi} + g_{\phi r,t} - g_{t\phi,r}\right) = \frac{1}{2}g^{rr}(-g_{t\phi,r})$$

$$\Gamma^r_{\phi\phi} = \frac{1}{2}g^{rr}\left(g_{\phi r,\phi} + g_{\phi r,\phi} - g_{\phi\phi,r}\right) = \frac{1}{2}g^{rr}(-g_{\phi\phi,r})$$

Substituting these into our master equation from Cell 2 gives:

$$-\frac{1}{2}g^{rr}\left(g_{tt,r}\left(\frac{dt}{d\tau}\right)^2 + 2g_{t\phi,r}\frac{dt}{d\tau}\frac{d\phi}{d\tau} + g_{\phi\phi,r}\left(\frac{d\phi}{d\tau}\right)^2\right) = 0$$

Since g^{rr} is non-zero, we can divide it out, leaving a quadratic equation for the ratio $\frac{dt}{d\phi} = \frac{dt/d\tau}{d\phi/d\tau}$. This ratio represents the amount of coordinate time that passes for each radian of angular travel, as measured by a distant observer. Solving this quadratic equation yields:

$$\frac{dt}{d\phi} = \frac{1}{g_{tt,r}}\left(-g_{t\phi,r} \pm \sqrt{g^2_{t\phi,r} - g_{tt,r}g_{\phi\phi,r}}\right)$$

Using our calculated derivatives for the equatorial plane:

$$g_{tt,r} = \frac{-2M}{r^2} \quad , \quad g_{t\phi,r} = \frac{2Ma}{r^2} \quad , \quad g_{\phi\phi,r} = 2r - \frac{2Ma^2}{r^2}$$

Substituting these values and performing the algebraic simplification leads to the final, elegant result:

$$\frac{dt}{d\phi} = a \pm \sqrt{\frac{r^3}{M}}$$

Physical Interpretation: The two solutions correspond to the two possible orbital directions. The `+` sign is for **prograde** orbits (co-rotating with the black hole), and the `-` sign is for **retrograde** orbits (moving against the spin). The term `a` represents the effect of **frame-dragging**—the twisting of spacetime by the black hole's rotation, which "helps" prograde orbits and "hinders" retrograde orbits relative to a non-rotating (Schwarzschild) black hole.

-2.0.1 Deriving the 4-Velocity Components `dt/dτ` and `dφ/dτ`

Our final goal is to find the individual components of the 4-velocity, which describe the rate of change of coordinates with respect to the particle's own proper time. This requires combining all our previous results.

Step 1: The Normalization Condition

The foundation of our derivation is the fact that the 4-velocity ($U^\mu = dx^\mu/d\tau$) of any massive particle has a constant magnitude of -1 (in units where c=1). This is expressed as:

$$-1 = g_{\mu\nu}U^\mu U^\nu = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

For our circular orbit in the equatorial plane, this expands to:

$$-1 = g_{tt}\left(\frac{dt}{d\tau}\right)^2 + 2g_{t\phi}\frac{dt}{d\tau}\frac{d\phi}{d\tau} + g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2$$

Step 2: The Strategy for `dt/dτ`

Our strategy is to relate `dt/dτ` to the quantity we already know, `dt/dφ`, using the chain rule:

$$\frac{dt}{d\tau} = \frac{dt}{d\phi} \cdot \frac{d\phi}{d\tau}$$

To find the unknown term `dφ/dτ`, we can rearrange the normalization equation. By factoring out $(\frac{d\phi}{d\tau})^2$, we get:

$$-1 = \left(\frac{d\phi}{d\tau}\right)^2 \left[g_{tt}\left(\frac{dt}{d\phi}\right)^2 + 2g_{t\phi}\left(\frac{dt}{d\phi}\right) + g_{\phi\phi}\right]$$

Solving for `dφ/dτ` and substituting it back into the chain rule gives us our desired expression:

$$\frac{dt}{d\tau} = \frac{dt/d\phi}{\sqrt{-\left[g_{tt}(dt/d\phi)^2 + 2g_{t\phi}(dt/d\phi) + g_{\phi\phi}\right]}}$$

This powerful formula gives the total time dilation factor using only the metric components and the coordinate velocity.

Step 3: Substitution and Final Result for `dt/dτ`

We now substitute our known quantities for the equatorial plane into this formula: * $g_{tt} = -\left(1 - \frac{2M}{r}\right)$ * $g_{t\phi} = -\frac{2Ma}{r}$ * $g_{\phi\phi} = r^2 + a^2 + \frac{2Ma^2}{r}$ * $\frac{dt}{d\phi} = a \pm \sqrt{\frac{r^3}{M}}$

Plugging these into the equation above results in a very large and complex expression. However, after a significant amount of algebraic simplification, this expression reduces to the final result:

$$\frac{dt}{d\tau} = \frac{r^{3/2} \pm a\sqrt{M}}{r^{3/2} - 3Mr^{1/2} \pm 2a\sqrt{M}} \cdot \sqrt{\frac{r}{r - 2M \pm 2a\sqrt{M/r}}}$$

Step 4: Finding the Proper Angular Velocity `dφ/dτ`

With the total time dilation factor known, we can now easily find the final component of the 4-velocity, `dφ/dτ`, which represents the angular velocity as measured by the orbiting particle's own clock. The simplest method is to rearrange the chain rule:

$$\frac{d\phi}{d\tau} = \frac{dt/d\tau}{dt/d\phi}$$

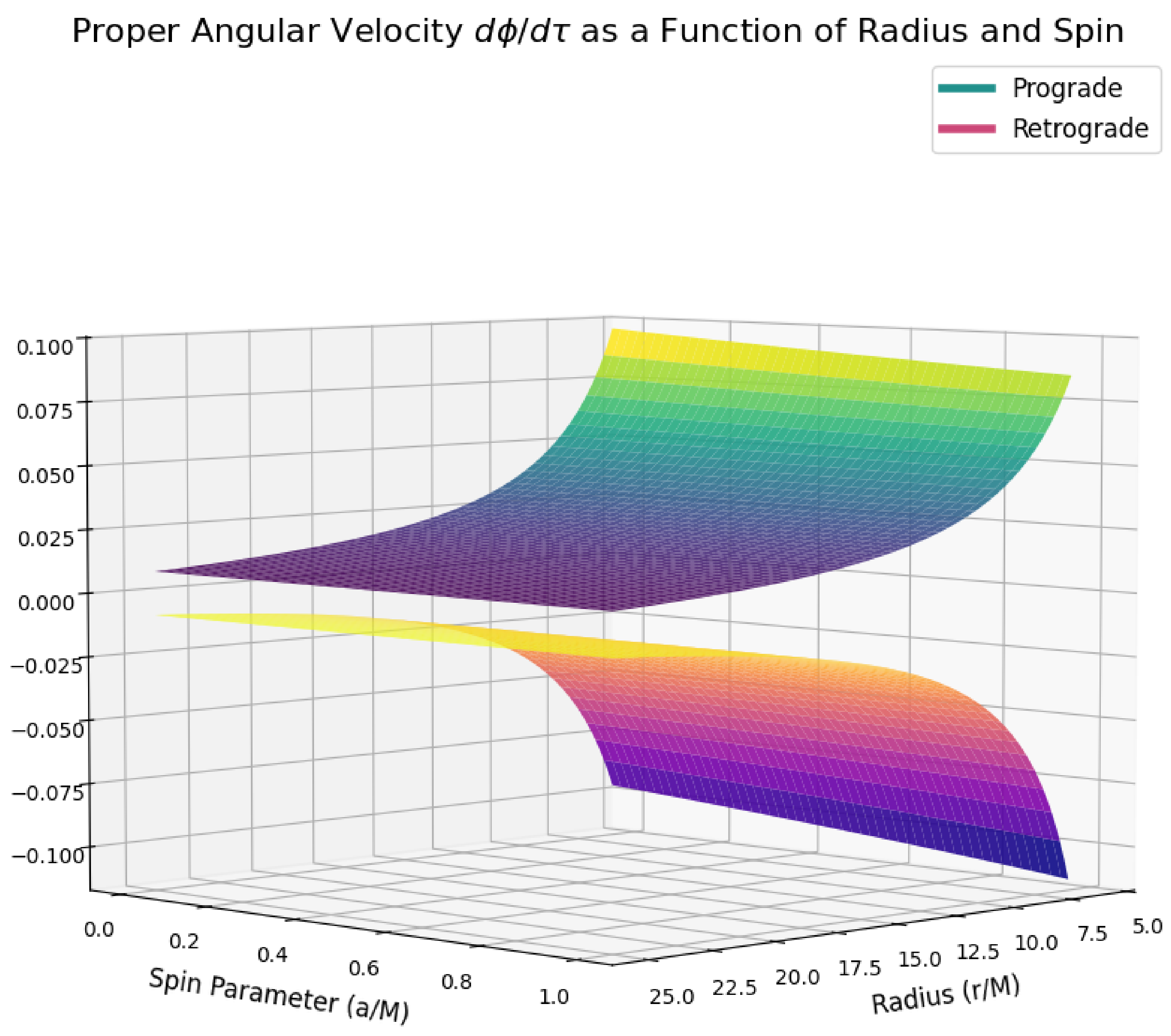
We can now substitute our final expression for `dt/dτ` from Step 3 and our known expression for `dt/dφ`. After simplification, this yields the compact result:

$$\frac{d\phi}{d\tau} = \frac{\pm\sqrt{M}}{r^{3/4}\sqrt{r^{3/2} - 3Mr^{1/2} \pm 2a\sqrt{M}}}$$

As always, the top signs correspond to prograde orbits and the bottom signs to retrograde orbits. With this, we have fully solved for all non-zero components of the 4-velocity for a circular equatorial orbit in Kerr spacetime.

[10]:
```python
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
from matplotlib.lines import Line2D

# --- 1. Define the function for d(phi)/d(tau) ---
def calculate_u_phi(r, a, M=1.0, prograde=True):
    """
    Calculates the proper angular velocity (u^phi) for a circular orbit.

    Args:
        r (np.ndarray): Array of radii.
        a (np.ndarray): Array of spin parameters.
        M (float): Mass of the black hole (default is 1).
        prograde (bool): True for prograde (co-rotating), False for retrograde.

    Returns:
        np.ndarray: The calculated u^phi values.
    """
    sign = 1.0 if prograde else -1.0

    # The term inside the main square root in the denominator
    # This can be negative for r < ISCO, causing a domain error.
    denominator_sqrt_term = r**1.5 - 3 * M * r**0.5 + sign * 2 * a * np.sqrt(M)

    # Set any values inside the ISCO to NaN so they are not plotted
    denominator_sqrt_term[denominator_sqrt_term < 0] = np.nan

    # Calculate the full denominator
    denominator = r**0.75 * np.sqrt(denominator_sqrt_term)

    # Calculate u^phi
    u_phi = (sign * np.sqrt(M)) / denominator

    return u_phi

# --- 2. Set up the data grid for the plot ---
# MODIFICATION: Changed the bounds for the radius 'r'
r_vals = np.linspace(6, 25, 200)  # r/M from 2 to 6
a_vals = np.linspace(0, 1, 200)   # a/M from 0 to 1 (physical range)

# Create a 2D grid of (r, a) points using meshgrid
R, A = np.meshgrid(r_vals, a_vals)
```

```python
# --- 3. Calculate the Z values (u_phi) for both cases ---
U_phi_prograde = calculate_u_phi(R, A, prograde=True)
U_phi_retrograde = calculate_u_phi(R, A, prograde=False)

# --- 4. Create the 3D Plot ---
fig = plt.figure(figsize=(14, 10))
ax = fig.add_subplot(111, projection='3d')

# Plot the surfaces
# Using a colormap helps visualize the height (z-value)
# Using transparency (alpha) helps see both surfaces
surf_pro = ax.plot_surface(R, A, U_phi_prograde, cmap='viridis', alpha=0.9)
surf_retro = ax.plot_surface(R, A, U_phi_retrograde, cmap='plasma', alpha=0.9)

# Set labels and title with LaTeX for nice formatting
ax.set_title(r'Proper Angular Velocity $d\phi/d\tau$ as a Function of Radius and Spin', fontsize=16)
ax.set_xlabel(r'Radius (r/M)', fontsize=12, labelpad=10)
ax.set_ylabel(r'Spin Parameter (a/M)', fontsize=12, labelpad=10)
ax.set_zlabel(r'Proper Angular Velocity $d\phi/d\tau$', fontsize=12, labelpad=10)

# MODIFICATION: Changed the viewing angle for a different perspective
ax.view_init(elev=5, azim=45)

# Create a custom legend since plot_surface doesn't have a 'label' argument
legend_elements = [Line2D([0], [0], color=plt.cm.viridis(0.5), lw=4, label='Prograde'),
                   Line2D([0], [0], color=plt.cm.plasma(0.5), lw=4, label='Retrograde')]
ax.legend(handles=legend_elements, loc='upper right', fontsize=12)

plt.show()
```



Proper Angular Velocity $d\phi/d\tau$ as a Function of Radius and Spin
Prograde
Retrograde
0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.075
−0.100
0.0
0.2
0.4
0.6
0.8
1.0
Spin Parameter (a/M)
25.0
22.5
20.0
17.5
15.0
12.5
10.0
7.5
5.0
Radius (r/M)