1A Rotational Mechanics

No. 1

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- The centre of mass of N discrete point masses is

$$\underset{\sim}{R}_0 = (x_0, y_0, z_0) = \frac{1}{M} \sum_i^N m_i \underset{\sim}{r}_i$$

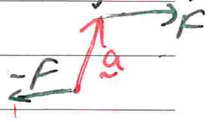
- For a continuous body:

$$\underset{\sim}{R}_0 = \frac{1}{M} \iiint \underset{\sim}{r}\, dm$$

- COM is important because in many cases we can model an extended body as a point mass located at the COM.
- The principle of superposition allows us to find the COM of a composite system
  - ↳ objects with holes ~~in~~ can be treated as the outer object with the hole as a negative mass object.

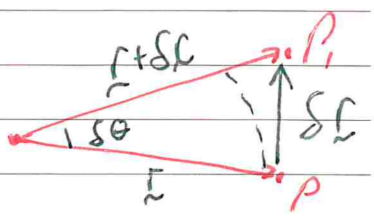
Moments

- The moment of a vector field about an axis is the scalar value of the field × perpendicular distance.
- Torque is the moment of force: $\underset{\sim}{\tau} = \underset{\sim}{r} \times \underset{\sim}{F}$
- For an object to be in equilibrium, $\sum_i \underset{\sim}{F}_i = 0$ and $\sum_i \underset{\sim}{\tau}_i = 0$ (about any axis).
  - ↳ law of the lever: $F_1 l_1 = F_2 l_2$
- A couple is a pair of equal opposite but displaced forces
  - ↳ torque in this case is $\underset{\sim}{a} \times \underset{\sim}{F}$, independent of origin



Circular motion

- Motion with respect to an origin can be split into an angular and radial component.



- For a tiny segment, $|\underset{\sim}{r}| \approx |\underset{\sim}{r} + \delta \underset{\sim}{r}|$

$$\therefore |\delta \underset{\sim}{r}| = l\, \delta\theta$$

- The linear velocity is $v = \frac{d|\underset{\sim}{r}|}{dt} = l \frac{d\theta}{dt}$
- Angular velocity is defined as $\dot{\underset{\sim}{\theta}} = \underset{\sim}{\omega} = \frac{d\underset{\sim}{\theta}}{dt}$ ← direction of $\underset{\sim}{\theta}$ determined using RH grip rule.

- Likewise, linear acceleration: $a = r\frac{d^2\theta}{dt^2}$
  angular acceleration: $\boldsymbol{\alpha} = \ddot{\boldsymbol{\theta}}$

$$\Rightarrow \quad \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \qquad \mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r}$$

- Suppose an object in circular motion has linear acceleration: $F_m = \frac{d(mv)}{dt} = mr\ddot{\theta}$

$$\Rightarrow \quad \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = mr^2\ddot{\boldsymbol{\theta}}$$

- This is the rotational equivalent of Newton's 2^{nd} law, where mr^2 is the 'mass' ⟵ moment of inertia I.

$$I = \sum_i^N m_i r_i^2 \ (\text{discrete}) \qquad I = \iiint r^2 \, dm \ (\text{cont}).$$

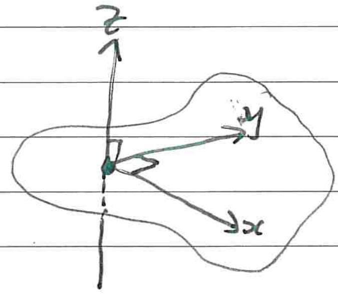
  ↳ r is the perpendicular distance to the axis.

- The parallel axis theorem lets us calculate I about any axis parallel to an axis through the COM, given I_0 and the separation

$$I = Ma^2 + I_0$$

- The perpendicular axis theorem (for laminas) relates I for 3 perpendicular axes:

$$I_z = I_x + I_y$$



Angular momentum

- Defined by $\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,\mathbf{r} \times \mathbf{v}$
- For a system of N particles:

true for any axis.

$$\mathbf{L} = \sum_i^N m_i(\mathbf{r}_i \times \mathbf{v}_i) \Rightarrow \frac{d\mathbf{L}}{dt} = \sum_i^N \boldsymbol{\tau}_i \Rightarrow \mathbf{L} = I\boldsymbol{\omega}$$

- The sum of all internal torques is zero by Newton's 3^{rd}
  ↳ total angular momentum of a system is constant if no external torque is applied.

- Just as linear impulse is $\Delta p = \int F dt$, we can define angular impulse: $\Delta \underset{\sim}{L} = \int \underset{\sim}{\tau}\, dt$

Rotational energy

- For a single particle at a distance r_i from the axis:

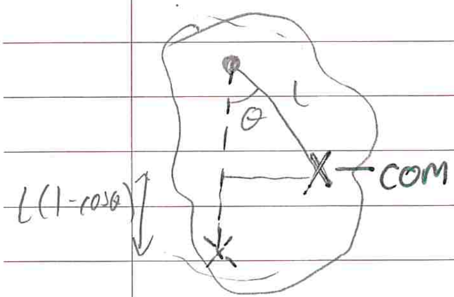
$$KE_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 \omega^2$$

$$\Rightarrow RKE_{total} = \frac{1}{2}\left(\sum_i^N m_i r_i^2\right)\omega^2 = \frac{1}{2} I\omega^2$$

- For a general solid in motion, its total KE is the sum of linear KE of COM and rotational KE about COM.

$$\therefore KE_{total} = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2$$

- This makes it very easy to analyse a physical pendulum



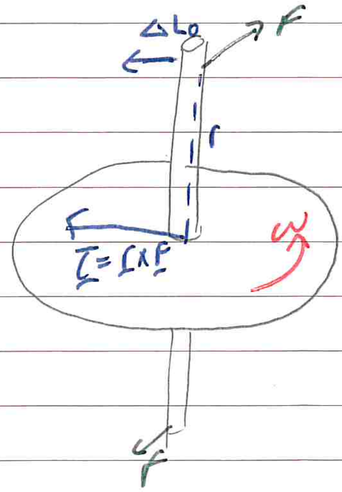
- no linear motion $\therefore E_{tot} = GPE + RKE$

  i.e $E_{tot} = mgl(1-\cos\theta) + \frac{1}{2} I\dot{\theta}^2$

- then set $\dot{E}_{tot} = 0$, ~~with~~ use small angle approx.

  $\dot{E}_{tot} = 0 \Rightarrow \ddot{\theta} + \frac{mgl}{I}\theta = 0$ i.e SHM.

Gyroscopes

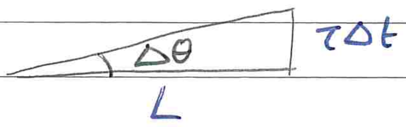


- If a force F is applied to a rotating flywheel, a torque is produced.
- This cause a change in $\underset{\sim}{L}$: $\Delta \underset{\sim}{L} = \underset{\sim}{\tau}\,\Delta t$
- But $L_0 = I\omega$ and ω is constant

  $\therefore |\underset{\sim}{L_0} + \Delta \underset{\sim}{L}| = |\underset{\sim}{L_0}| \Rightarrow$ only direction changing

  i.e circular motion.

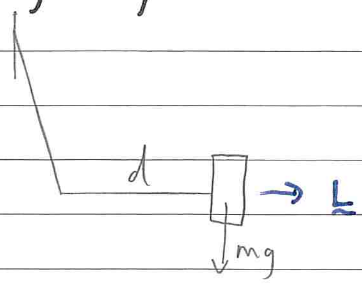
- Thus there is precession about an axis $\perp$ to both $\underset{\sim}{L}$ and τ.



$$\Omega = \frac{\Delta\theta}{\Delta t} = \frac{\tau}{L}$$ (rate of precession)

$$\therefore \underset{\sim}{\tau} = I \underset{\sim}{\Omega} \times \underset{\sim}{\omega}$$

e.g bicycle wheel on string



$$\tau = Mgd$$

$$\therefore \Omega = \frac{Mgd}{I\omega}$$