REALITY GAME

Consider the following multiplayer *pari mutuel betting* game. There are N players, each independently deciding what portion of their wealth to bet on a coin toss coming up HEADS, with the remainder being automatically bet on TAILS. In particular, let $w_i(k)$ denote the wealth of player i at time k, and let $s_i(k) \in [0,1]$ denote the portion of their wealth that player i decides to bet on HEADS. Thus, for the n^{th} time step, player i bets $\pi_i(n) = s_i(n)w_i(n)$ on HEADS and $(1 - s_i(n))\, w_i(n)$ on TAILS. As a result, the total amount of money bet on HEADS by all the players together is $\pi(n) = \sum_{i=1}^{N} \pi_i(n)$.

Let $W(n) = \sum_{i=1}^{N} w_i(n)$ denote the total wealth among all the players. If the n^{th} toss comes up HEADS, player i receives a portion of $\pi(n)$ proportional to their bet on HEADS, i.e.

$$w_i(n+1) = W(n)\frac{\pi_i(n)}{\pi(n)}.$$

If, on the other hand, the n^{th} toss comes up TAILS, player i receives a portion of $\sum_{i=1}^{N} w_i(n) - \pi(n)$ proportional to their bet on TAILS, i.e.

$$w_i(n+1) = W(n)\frac{w_i(n) - \pi_i(n)}{\sum_{i=1}^{N} w_i(n) - \pi(n)}.$$

Can you see why $W(n)$ remains invariant?

The special nature of this game lies in the so-called *reality map*, i.e. the way bets influence the probabilities of the outcomes. In particular, if, in a particular round, a total amount p is bet on an event (e.g. HEADS), then that event will occur with probability

$$q_\alpha(p) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{\pi\alpha\left(p - \frac{1}{2}\right)}{1 - (2p-1)^2}\right) \tag{1}$$

where α is a parameter.

To begin with, we consider all stationary strategies, which are in one-to-one correspondence with all numbers in the interval $[0,1]$. Your first task is to create a computational environment for simulating this game between an arbitrary number of players, playing any of these stationary strategies, for an arbitrary number of rounds. Once your simulation is running, gather data and assess the relative success of the different stationary strategies played against one another for different values of the parameter α. Do the wealth levels settle down after a sufficient number of rounds?

Now consider a single new *rational* player who enters the game and is not restricted to stationary strategies. To begin with, assume that they know the stationary strategies of all other players. Design an optimal strategy for this single rational player.

Finally, relax the assumption that the single rational player knows the stationary strategies of the other players. If they only know that all other players are restricted to play stationary strategies, can they infer their opponents' strategies from the way they play over time? If so, can they still exploit their optimal strategy?