Failure Tolerance of Complex Networks

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1 Introduction

A complex network is broadly defined as a collection of interconnected and interacting systems [AB02], where the individual subsystems (or participating agents) themselves could be complex dynamical systems. The study of complex networks is a young and active area of scientific research (since 2000) inspired largely by the empirical study of real-world networks such as computer networks, technological networks, brain networks and social networks[LCOT13]. Since complex networks exist broadly in our real life, the study of robustness of complex networks gains more and more attention. Early works focused on the single, isolated networks without interaction with other networks[AJB00, SMA^{+}11]. Afterwards, effect of coupled networks are considered and analyzed using percolation theory[BPP^{+}10]. In this report, we analyze the failure tolerance of both single networks and coupled networks, and carry out a simulation to verify our conclusion.

2 Preliminaries

2.1 Percolation Theory

A complex network can be viewed as a graph where nodes are sites and edges are connections between them. A site is "occupied" with probability **p** or "empty" (in which case its edges are removed) with probability $1 - p$. For a given p, what is the probability that a path exists between top and bottom? Equivalently, one can ask, given a connected graph at what fraction $1 - p$ of failures the graph will become disconnected. This problem, known as site percolation, can be solved using Mean Field theory.

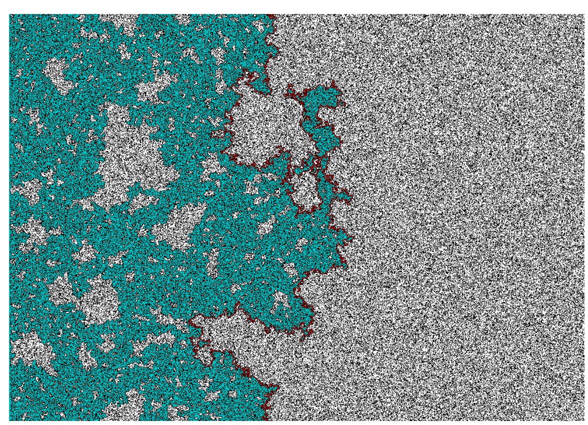


Figure 1: Percolation front on a square lattice at the percolation threshold (59.3%)

The Mean Field theory is applied to study phase transitions, which involves identification of a global parameter, called the ***order parameter***. It essentially quantifies the presence of order in the underlying system, which is zero (or negligible) in the disordered phase and takes on non-zero values in the ordered phase. In this case, order parameter, noted as p_∞, is the probability that there exists a path from the top to the bottom of the network. A phase transition is caused by a continuous or a discontinuous change of the order parameter from zero to a non-zero value when an intensive system parameter (e.g., site occupation parameter) is varied across the ***critical point***.

The nature of phase transition is often broadly classified into two types (i) **first-order** phase transition - when the order parameter changes discontinuously with the intensive parameter at the critical point and, (ii)**second-order** phase transition - when the order parameter varies continuously with the intensive parameter during the phase transition. A phase transition is marked by the presence of analytical singularities or discontinuities in the functions describing macroscopic physical parameters of the system. In the vicinity of the critical point marking the phase transition, the functional form of the order parameter is often modeled using a power law with a critical exponent as stated in Eqn. 2.1 below.

$$p_\infty \sim (p_c - p)^\beta, \tag{2.1}$$

where p_c is the critical point in this system.

2.2 Erdös-Rényi Network

The Erdös-Rényi(ER) network is a random graph obtained by randomly distributing M links between N nodes, being a statistical ensemble with equal probability for any generated configuration. For the ER network, since links are distributed in an uncorrelated way, the degree distribution is Poissonian, i.e., the frequency of nodes with k links is

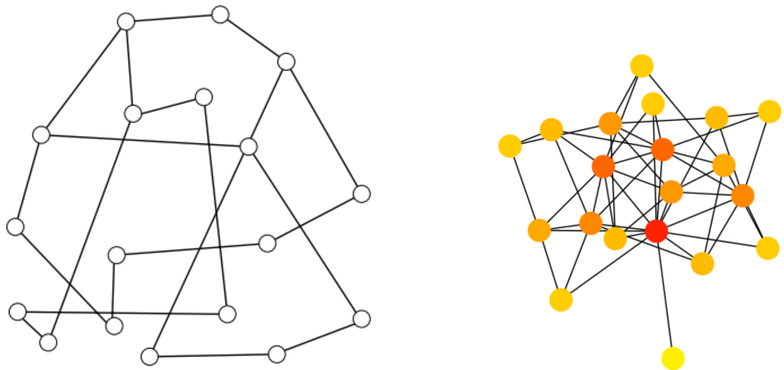
$$P(k) = exp(-\lambda)\frac{\lambda^k}{k!} \tag{2.2}$$

A typical ER network is shown in Fig. 3a.

2.3 Barabási-Albert Network

The Barabási-Albert network is a network which was grown under the preferential attachment rule, i.e., at each iteration a new node is added to the network and connected to m already existing nodes with a probability of linking to a certain node proportional to the actual degree (number of links) of that node. BA network belongs to scale-free networks, whose degree distribution follows a power law.

$$P(k) = \begin{cases} ck^{-\gamma} & m \leq k \leq K \\ 0 & otherwise \end{cases}, \tag{2.3}$$



(a) A typical ER network (b) A typical BA network

Figure 2: Two types of single networks

where γ is the degree exponent and K an upper limit due to the system finite size. As shown in Fig. 3b, the power-law degree distribution favors the existence of highly connected nodes when compared with the Poisson distribution.

3 Single Network Robustness

3.1 Molloy-Reed criterion

For a giant component to exist each node that belongs to it must be connected to at least two other nodes on average. Therefore, the average degree k_i of a randomly chosen node i that is part of the giant component should be at least 2. Denote with $P(k_i|i \leftrightarrow j)$ the joint probability that a node in a network with degree k_i is connected to a node j that is part of the giant component. This conditional probability allows us to determine the expected degree of node i as

$$\langle k_i|i \leftrightarrow j\rangle = \sum_{k_i} k_i P(k_i|i \leftrightarrow j) \tag{3.1}$$

We can write the probability use Bayes' theorem in the last term.

$$P(k_i|i \leftrightarrow j) = \frac{P(k_i, i \leftrightarrow j)}{P(i \leftrightarrow j)} = \frac{P(i \leftrightarrow j|k_i)p(k_i)}{P(i \leftrightarrow j)} \tag{3.2}$$

For a network with degree distribution p_k, in the absence of degree correlations, we can write

$$P(i \leftrightarrow j|k_i) = \frac{k_i}{N-1}, \tag{3.3}$$

which expresses the fact that we can choose between $N-1$ nodes to link to, each with probability $1/(N-1)$ and that we can try this k_i times.

$$P(i \leftrightarrow j) = \frac{L}{N(N-1)/2} = \frac{\langle k\rangle N/2}{N(N-1)/2} = \frac{\langle k\rangle}{N-1}, \tag{3.4}$$

where L is the number of total links in the network. We can replace Eqn. 3.3, Eqn. 3.4 into Eqn. 3.1 and get

$$\sum_{k_i} k_i P(k_i|i \leftrightarrow j) = \sum_{k_i} k_i \frac{k_i p(k_i)}{\langle k \rangle} = \frac{\sum_{k_i} {k_i}^2 p(k_i)}{\langle k \rangle} = \frac{\langle k^2 \rangle}{\langle k \rangle} \tag{3.5}$$

With that we arrive at the **Molloy-Reed criterion**, providing the condition to have a giant component as

$$\kappa = \frac{\langle k^2 \rangle}{\langle k \rangle} > 2 \tag{3.6}$$

Networks with $\kappa < 2$ lack a giant component, being fragmented into many disconnected components. The Molloy-Reed criterion links the networks integrity, as expressed by the presence or the absence of a giant component, to $\langle k \rangle$ and $\langle k^2 \rangle$. It is valid for any degree distribution p_k.

3.2 Critical Threshold

The random removal of an f fraction of nodes has two consequences:

- It alters the degree of some nodes, as nodes that were previously connected to the removed nodes will lose some links $[k \rightarrow k' \leq k]$.
- It changes the degree distribution, as the neighbors of the missing nodes will have an altered degree $[p_k \rightarrow p'_k]$.

After we randomly remove an f fraction of nodes, a node with degree k becomes a node with degree k' with probability

$$P(k \rightarrow k') = \begin{pmatrix} k \\ k' \end{pmatrix} f^{k-k'} (1-f)^{k'} \qquad k' \leq k \tag{3.7}$$

Suppose the degree distribution in the original network is p_k, i.e. there is p_k probability to have a node of degree k, the probability that we have a new node with degree k' in the new network is

$$p'_{k'} = \sum_{k=k'}^{\infty} p_k P(k \rightarrow k') \tag{3.8}$$

With $p'_{k'}$ defined, we can write the expectation of degree in the new network,

$$\begin{aligned} \langle k' \rangle_f &= \sum_{k'=0}^{\infty} k' p'_{k'} \\ &= \sum_{k'=0}^{\infty} \sum_{k=k'}^{\infty} p_k \frac{k(k-1)!}{(k'-1)!(k-k')!} f^{k-k'} (1-f)^{k'-1} (1-f) \end{aligned} \tag{3.9}$$

Note that we can change the sum order from $\sum_{k'=0}^{\infty}\sum_{k=k'}^{\infty}$ to $\sum_{k=0}^{\infty}\sum_{k'=0}^{k}$, and get

$$\begin{aligned}\langle k'\rangle_f =&\sum_{k=0}^{\infty}\sum_{k'=0}^{k} p_k \frac{k(k-1)!}{(k'-1)!(k-k')!} f^{k-k'}(1-f)^{k'-1}(1-f)\\ =&\sum_{k=0}^{\infty}(1-f)kp_k\sum_{k'=0}^{k}\frac{(k-1)!}{(k-k')!(k'-1)!}f^{k-k'}(1-f)^{k'-1}\\ =&\sum_{k=0}^{\infty}(1-f)kp_k\sum_{k'=0}^{k}P(k-1\to k'-1)\\ =&\sum_{k=0}^{\infty}(1-f)kp_k\\ =&(1-f)\langle k\rangle\end{aligned} \tag{3.10}$$

Similarly, we can get

$$\left\langle k'^2\right\rangle_f = (1-f)^2\left\langle k^2\right\rangle + f(1-f)\langle k\rangle \tag{3.11}$$

To ensure the integrity of the new network, its Molloy-Reed criterion $\kappa' = \frac{\langle k'^2\rangle_f}{\langle k'\rangle_f}$ should be above 2, thus we can get the critical threshold

$$f_c = 1 - \frac{1}{\frac{\langle k^2\rangle}{\langle k\rangle} - 1} \tag{3.12}$$

No matter what degree distribution p_k is, as long as more than f_c nodes are **randomly** removed, the network will break down. The critical threshold f_c depends only on $\langle k\rangle$ and $\langle k^2\rangle$, quantities that are uniquely determined by the degree distribution p_k.

3.3 Case Study

3.3.1 Erdös-Rényi Network

ER network's degree observes to Poisson distribution as defined in Eqn. 2.2, hence we get

$$\begin{aligned}\langle k\rangle =&\sum_{k=0}^{\infty} ke^{-\lambda}\frac{\lambda^k}{k!}\\ =&\lambda e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\end{aligned} \tag{3.13}$$

Notice that $\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}$ is the taylor series expansion of e^{λ}, leads to

$$\langle k\rangle = \lambda e^{-\lambda}e^{\lambda} = \lambda \tag{3.14}$$

Similarly, we can solve $\left\langle k^2\right\rangle$

$$\begin{aligned}\left\langle k^2\right\rangle =&\sum_{k=0}^{\infty} k^2 e^{-\lambda}\frac{\lambda^k}{k!}\\ =&\lambda^2-\lambda\end{aligned} \tag{3.15}$$

Then the critical threshold for an ER network is

$$f_c = 1 - \frac{1}{\lambda} \tag{3.16}$$

Hence, the denser is a random network, the higher is its f_c, i.e. the more nodes we need to remove to break it apart. Furthermore Eqn. 3.16 predicts that f_c is always finite, hence a random network must break apart after the removal of a finite fraction of nodes.

3.3.2 BA Network(scale-free)

BA network's degree observes to heavy-tail distribution as defined in Eqn. 2.3, hence we get

$$\langle k \rangle = \sum_{k=m}^{\infty} ck^{1-\gamma} = \frac{c}{2-\gamma} x^{2-\gamma} \Big|_{x=m}^{x=\infty} \tag{3.17}$$

$$\langle k^2 \rangle = \sum_{k=m}^{\infty} ck^{2-\gamma} = \frac{c}{3-\gamma} x^{3-\gamma} \Big|_{x=m}^{x=\infty} \tag{3.18}$$

Then we have

$$f_c = \begin{cases} 1 & 0 < \gamma < 3 \\ 1 - \frac{1}{\frac{\gamma-2}{\gamma-3}m - 1} & \gamma > 3 \end{cases} \tag{3.19}$$

For $\gamma > 3$ the critical threshold f_c depends only on γ and m, hence f_c is independent of the network size N. In this regime a scale-free network behaves like a random network: it falls apart once a finite fraction of its nodes are removed. For $\gamma < 3$, in order to fragment an infinite scale-free network we must remove all of its nodes. Scale-free networks can withstand an arbitrary level of random failures without breaking apart. The hubs are responsible for this remarkable robustness. Indeed, random node failures by definition are blind to degree, affecting with the same probability a small or a large degree node. Yet, in a scale-free network we have far more small degree nodes than hubs. Therefore, random node removal will predominantly remove one of the numerous small nodes as the chances of selecting randomly one of the few large hubs is negligible. These small nodes contribute little to a networks integrity, hence their removal does little damage.

4 Coupled Network Robustness

Let's consider two networks, namely A and B. When a node A_i in A is removed, its connected nodes in A and B are removed as well. The same procedure takes place recurrently and can cause a cascaded failure between these two networks. In Sec. 3, we show that in single networks, when a fraction of nodes f is removed, a percolation transition occurs at a certain threshold f_c. Below this threshold, a giant mutually connected connected cluster exists, otherwise, the entire system turns completely fragmented.

4.1 Coupled Erdös-Rényi Network

For two coupled ER networks, the problem can be solved using generating functions [SBC+08]. When the two networks have the same degree, $\langle k \rangle_A = \langle k \rangle_B = \lambda$, the value of f_c is

$$f_c = 1 - \frac{1}{2\lambda z(1-z)}, \tag{4.1}$$

where $z \approx 0.28467$ is the solution of equation $z = exp((z-1)/2z)$. Approximately, we have

$$f_c = 1 - \frac{2.4554}{\lambda} \tag{4.2}$$

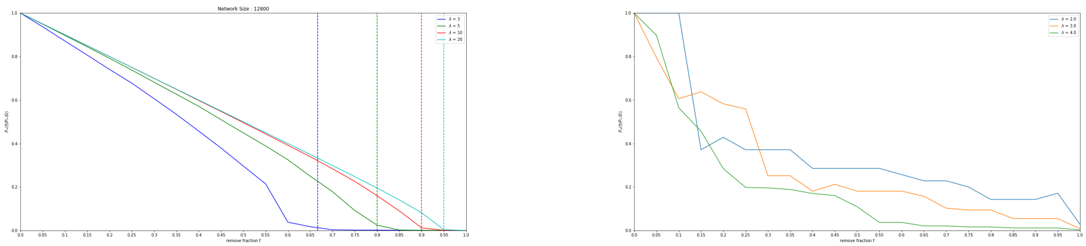
Compared to Eqn. 3.16, this reveals that coupled ER networks are more vulnerable than stand-alone ER networks. As the ratio between the average degrees of two networks decreases, the threshold f_c increases, i.e., the coupled network becomes more resilient to failures. In the limit where this ratio becomes zero, the single-network feature is recovered, obviously.

4.2 Scale-free Networks

Analogously to ER networks, the coupling between scale-free networks significantly increases the vulnerability of the system, with a lower f_c compared to the case of a single network. Since hubs can have a low-degree counterpart node, their vulnerability evinces with the coupling. In contrast to single networks, the broader the degree distribution the lower is f_c, i.e., the smaller the fraction of nodes that needs to be removed to fully fragment the system.

5 Simulations

In single networks, we both visualize the case of ER network and scale-free network. We use P_∞ , the probability of one node belonging to the largest connected center, to measure the connectivity of networks. As plotted below, a second-order phase transition take place in both types of networks when more nodes are being removed. We use Configure Model to generate a scale-free network of arbitrary γ. Specifically, it first generates a random degree sequence and wires nodes together afterwards.



(a) ER network

(b) Scale-free network

Figure 3: Single Network Simulation

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