1 Generating the PN Gravitational Waveform

1.1 The Post-Newtonian Expansion

In the post-Newtonian approach, we calculate the gravitational waveform from a binary source as follows:

$$h_{+,\times} = \frac{2M\eta v^2}{R}\left[h^{(0)}_{+,\times} + vh^{(1/2)}_{+,\times} + v^2h^{(2/2)}_{+,\times} + v^3h^{(3/2)}_{+,\times} + v^4h^{(4/2)}_{+,\times} + \cdots\right] \tag{1.1}$$

where $h_{+,\times}$ is the plus or cross polarization of the gravitational wave, $h^{(n/2)}_{+,\times}$ is the contribution to that polarization at the $n/2$ post-Newtonian order, R is the luminosity distance of the source, and (where m_1 and m_2 are the masses of the stars in the binary)

$$M \equiv m_1 + m_2 \qquad \text{(the system's total mass)} \tag{1.2a}$$

$$\eta \equiv \frac{m_1 m_2}{M^2} \qquad \text{(a dimensionless quantity that depends on the mass ratio)} \tag{1.2b}$$

$$\delta \equiv \frac{m_1 - m_2}{M} \qquad \text{(a dimensionless quantity that describes the mass difference)} \tag{1.2c}$$

$$v \equiv (M\omega)^{1/3} \qquad \text{(our dimensionless PN expansion parameter)} \tag{1.2d}$$

where ω is the angular rate of rotation evaluated in the binary's reference frame. The quantity δ does not actually appear in equation 1.1, but will be useful in what follows. Note that η and δ both express something about the difference between the stars' masses. Indeed,

$$\frac{m_1}{M} = \tfrac{1}{2}(1+\delta), \quad \frac{m_2}{M} = \tfrac{1}{2}(1+\delta), \quad \eta = \tfrac{1}{4}(1+\delta)(1-\delta) = \tfrac{1}{4}(1-\delta^2) \tag{1.2e}$$

showing that each quantity can be converted to the other. Strictly, one cannot determine the *sign* of δ from η, but we can (without loss of generality) define m_1 to be the larger of the stars' masses. With this convention, δ is always positive and the quantities really are inter-convertible.

Equation 1.1 assumes that we are using units where $G = c = 1$. In such a unit system, time, distance, and mass all have the same units, which we will take to be seconds (one solar mass = 4.927 µs). We will also sometimes express time and distance in years, where in each case, 1 y = $3.155\,76 \times 10^7$ s.

The quantity $M\omega$ is dimensionless in such units. For circular orbits in the Newtonian limit and the massive primary approximation $m_2 \ll m_1$, Kepler's third law $T^2 = (4\pi^2/GM)r^3$ implies (with $G = 1$) that

$$\frac{4\pi^2}{M}r^3 = T^2 = \frac{4\pi^2}{\omega^2} \quad \Rightarrow \quad v^3 \equiv M\omega = r^3\omega^3 \tag{1.3}$$

So in this case, v is actually the *orbital speed* of the satellite with mass m_2. In the circular orbit case with more general masses, v is technically $v = v_1 + v_2$ as measured in the binary system's CM frame, but still *approximately* characterizes the orbital speeds of the stars in the system.

This parameter's initial value v_0 is connected to the initial separation r_0 of the binary stars and the time to coalescence t_c. To 2PN order (and using spin quantities $\chi_{1\ell}, \chi_{2\ell}, \chi_s, \chi_a, \vec{\chi}_1$ and $\vec{\chi}_2$ to be defined later)

$$\frac{r_0}{M} \approx \left(\frac{1}{v_0}\right)^2 \Bigg\{1 - \left[1 - \frac{\eta}{3}\right]v_0^2 - \frac{1}{3}\Big[(1+\delta+\eta)\,\chi_{1\ell} + (1-\delta+\eta)\,\chi_{2\ell}\Big]v_0^3 + \left[\frac{19}{4}\eta + \frac{\eta^2}{9} + \frac{1}{2}\vec{\chi}_1\cdot\vec{\chi}_2 - \frac{3}{2}\chi_{1\ell}\chi_{2\ell}\right]v_0^4\Bigg\} \tag{1.4a}$$

$$\frac{t_c}{M} \approx \frac{5}{256\eta}\left(\frac{1}{v_0}\right)^8 \Bigg\{1 + \frac{32}{3}\left(\frac{743}{2688} + \frac{11}{32}\eta\right)v_0^2 + \frac{64}{3}\left(\frac{47}{40}\chi_s + \delta\frac{15}{32}\chi_a - \frac{3\pi}{10}\right)v_0^3 + \left[64\left(\frac{743}{2688} + \frac{11}{32}\eta\right)^2 + \frac{128}{9}\left(\frac{1\,855\,099}{14\,450\,688} + \frac{56\,975}{258\,048}\eta - \frac{371}{2048}\eta^2\right)\right]v_0^4\Bigg\} \tag{1.4b}$$

Because these expressions for r_0 and t_c involve η and spin quantities, in addition to v_0, the quantities $v_0, r_0/m$, and t_c are not strictly inter-convertible the way that η and δ are. (The first equation is from Kidder, *Phys Rev D* **52,** 2, 821ff (1995), equation 4.13, and the second is from Ali Wang's notes dated 2019-03-26.)

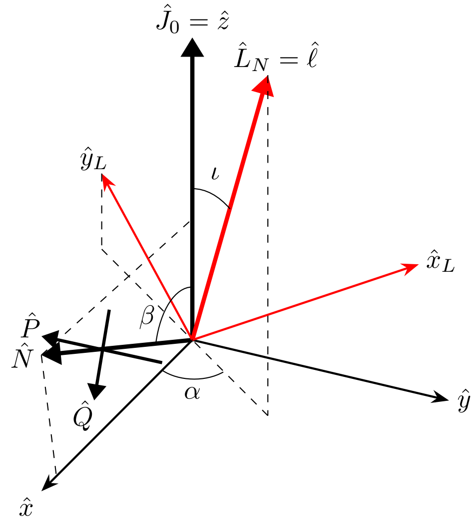

Figure 1: The coordinate systems we will use (adapted from Arun, et al., *Phys Rev D*, **79,** 104023 (2009), Figure 1). $\hat{N}$ points toward the source, and thus in the direction of gravitational wave propagation. The red vectors $(\hat{x}_L, \hat{y}_L, \hat{\ell})$ comprise an orthonormal triad such that $\hat{x}_L$ and $\hat{y}_L$ lie in the instantaneous plane of the precessing orbit and $\hat{\ell}$ is normal to that plane. (β here $= \theta$ in Arun, et al.)

1.2 Coordinate Systems

The coordinate system we are adopting follows the conventions established in Arun, Buonanno, Faye, and Ochsner (*Phys Rev D* **79,** 104023, 2009), since we will be lifting their "ready to use" gravitational wave equations from that article. (This will involve some renaming of our previously established conventions for parameters, but using their notation will likely reduce errors.)

In their coordinate system, the vector $\hat{N}$ defines the line of sight, pointing toward the viewer (and thus also in the direction of gravitational wave propagation). The polarization directions for the gravitational wave are $\hat{P}$ and $\hat{Q}$ such that

$$\hat{P} = \frac{\hat{N} \times \hat{J}_0}{|\hat{N} \times \hat{J}_0|}, \qquad \hat{Q} = \hat{N} \times \hat{P} \tag{1.5}$$

where $\hat{J}_0 = \hat{z}$ is the initial direction of the binary system's total angular momentum. We also define unit vectors $\hat{x}$ and $\hat{y}$ such that

$$\hat{y} = \frac{\hat{J}_0 \times \hat{N}}{|\hat{J}_0 \times \hat{N}|}, \qquad \hat{x} = \hat{y} \times \hat{N}$$

We can see from the diagram that $\hat{N}$ lies in the $\hat{x}\hat{z}$ plane, that $\hat{P}$ is opposite to the $\hat{y}$ direction.

We also define a "precessing" coordinate system such that

$$\hat{x}_L = \frac{\hat{J}_0 \times \hat{L}_N}{|\hat{J}_0 \times \hat{L}_N|} = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix}, \qquad \hat{y}_L = \hat{L}_N \times \hat{x}_L = \begin{bmatrix} -\cos\iota\cos\alpha \\ -\cos\iota\sin\alpha \\ \sin\iota \end{bmatrix} \tag{1.6}$$

where $\hat{L}_N \equiv \hat{\ell}$ is defined to be the direction of the system's Newtonian angular momentum $\vec{L}_N = \mu\vec{r}\times\vec{v}$, which is by definition normal to the binary's instantaneous orbital plane and the column vectors are components in the $\hat{x}\hat{y}\hat{z}$ basis. This means that the $\hat{x}_L$ and $\hat{y}_L$ unit vectors lie in the instantaneous orbital plane. Note also that since $\hat{x}_L$ is perpendicular to $\hat{J}_0$ unit vector, so it always lies in the $\hat{x}\hat{z}$ plane as well. Therefore the orbital plane intersects the $\hat{x}\hat{z}$ plane along the line parallel to $\hat{x}_L$.

The reason that we are interested in referring everything to $\hat{J}_0$ is that, under ordinary circumstances, the *direction* of the system's total angular momentum $\vec{J}$ is approximately conserved even as the system radiates both energy and angular momentum in the form of gravitational waves: $\hat{J} \approx \hat{J}_0$ (see Kidder, *Phys Rev D* **52,** 821–847, (1995): this is likely to be violated only very near to coalescence and even then only in the unlikely

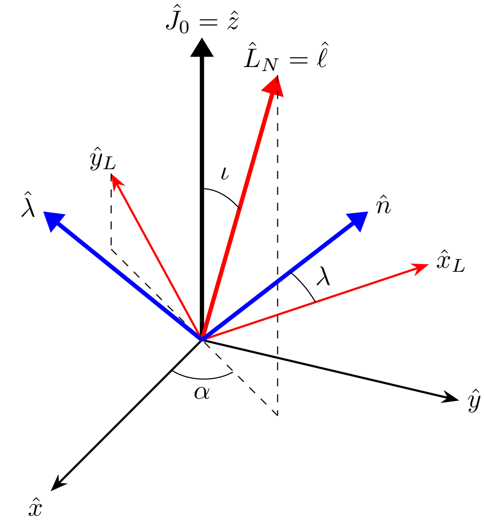

Figure 2: Definitions of quantities involved with the orbital phase. The unit vector $\hat{n}$ points from the more massive star to the lighter star at every instant, and the $\hat{\lambda} \equiv \hat{L}_N \times \hat{n}$ unit vector is perpendicular to $\hat{n}$ in the instantaneous plane of the orbit. The angle $\lambda(t)$ describes the cumulative angle of $\hat{n}$ relative to $\hat{x}_L$.

case that both stars' spins are opposite to the orbital angular momentum). Therefore, as the system evolves, $\vec{L}$ will approximately precess around the fixed direction of $\hat{J}_0$. However, because it is essential to have a *fixed* coordinate system as our basic reference and because $\hat{J}$ is not necessarily equal to $\hat{J}_0$, we emphasize that we choose our $\hat{z}$ axis to be aligned with the *initial* total angular momentum direction $\hat{J}_0$, which is not necessarily equal to $\hat{J}$ at later times.

1.3 Defining the Orbital Phase

Calculating phase angles in a precessing orbital plane is somewhat tricky. Figure 2 illustrates some of the necessary quantities. Define $\hat{n}$ to be direction of the lighter star in the binary relative to the more massive star (or, equivalently, from the system's center of mass). This vector will move in the instantaneous plane of the orbit. Define $\hat{\lambda} = \hat{L}_N \times \hat{n}$ to be a vector perpendicular to $\hat{n}$ but also in the instantaneous plane of the orbit: for a circular orbit, this would be the direction of $d\hat{n}/dt$. We define $\lambda(t)$ to be the cumulative (always increasing) angle between $\hat{n}$ and the basis vector $\hat{x}_L$ in the precessing $\hat{x}_L, \hat{y}_L\hat{L}_N$ frame, so that $\hat{n} = \cos\lambda\,\hat{x}_L + \sin\lambda\,\hat{y}_L$. (Note that $\hat{\lambda}$ indicates the direction that λ is increasing.)

However, this is not quite the actual physical rotation rate, because the $\hat{x}_L, \hat{y}_L\hat{L}_N$ frame is precessing with respect to our fixed $\hat{x}\hat{y}\hat{z}$ coordinate system. So the rotation of $\hat{n}$ around $\hat{L}_N$ in an inertial frame will be a combination of the rotation in the precessing frame parameterized by Φ_L and the rotation of that precessing frame, which is parameterized by the angle $\alpha(t)$. The components of the $\hat{n}, \hat{\lambda}$, and $\hat{L}_N$ vectors in the fixed $\hat{x}\hat{y}\hat{z}$ coordinate system are

$$\hat{n} = \begin{bmatrix} -\sin\alpha\cos\lambda - \cos\iota\cos\alpha\sin\lambda \\ +\cos\alpha\cos\lambda - \cos\iota\sin\alpha\sin\lambda \\ \sin\iota\sin\lambda \end{bmatrix},\ \hat{\lambda} = \begin{bmatrix} +\sin\alpha\sin\lambda - \cos\iota\cos\alpha\cos\lambda \\ -\cos\alpha\sin\lambda - \cos\iota\sin\alpha\cos\lambda \\ \sin\iota\cos\lambda \end{bmatrix},\ \hat{L}_N = \begin{bmatrix} \sin\iota\cos\alpha \\ \sin\iota\sin\alpha \\ \cos\iota \end{bmatrix} \tag{1.7}$$

One can characterize the rotation rate of $\hat{n}$ in inertial space by the quantity $\omega \equiv (\vec{v}\cdot\hat{\lambda})/r$, where r is the distance between the stars, and $\vec{v} \equiv d\vec{r}/dt = d(r\hat{n})/dt$. This is the rotation rate that most simply expresses the physics of how the binary stars interact. A bit of mathematics shows that

$$\lambda(t) = \lambda(0) + \int_0^t [\,\omega - \cos\iota\dot{\alpha}\,]\,dt \tag{1.8}$$

where $\dot{\alpha} \equiv d\alpha/dt$ and $\lambda(0)$ is some initial phase constant. Note that ω, ι, and $\dot{\alpha}$ all depend on time t (as measured in the source frame).

1.4 Approximations and Assumptions

So far, we have not made any *obvious* approximations or assumptions, but there are implicit assumptions built into this whole approach. The most basic assumption is that **the gravitational fields are weak** enough that we can treat them as perturbations of flat spacetime, so that the pseudo-Euclidean coordinate systems previously described make sense. Not entirely unrelated is the assumption **that the binary system's stars are moving slowly** enough compared the speed of light that an expansion in terms of powers of the orbital speed makes sense. Both these assumptions break down in the case of black-hole binaries near to coalescence, so a different approach (usually numerical) must be used to calculate gravitational waves from such a source. (The "weak field" assumption also breaks down at points very near each star in the binary if they are neutron stars or black holes, but researchers have developed tricks for handling this in post-Newtonian calculations: see Blanchet, *Living Rev. Relativity,* **17,** (2014), 2, part B for a discussion.)

We also need to make certain assumptions to keep the expressions tractable. As gravitational wave emission tends to circularize the orbits of close binaries, we will assume **that the orbits are quasi-circular** instead of being elliptical. This is not an entirely satisfactory assumption, since not all binaries that might be observable by LISA will have had time to circularize, but while people are working on post-Newtonian expressions for elliptical orbits, they are more complicated and not nearly as well-developed at the present time as those for circular orbits. The reason that we describe the orbits "quasi-circular" is that as gravitational waves carry energy away from the system, the objects are spiralling in toward each other. However, for a quite a wide set of realistic systems, the time scale of this inspiral is so long compared to the orbital period that treating each individual orbit as being circular is an excellent approximation.

We will be assuming **that each binary star is spinning, and that each star's spin is not necessarily aligned with the system's orbital angular momentum.** This will cause the star's spin angular momenta to precess around the system's orbital angular momentum (a so-called spin-orbit interaction), and (because the direction $\hat{J}$ of the system's total angular momentum $\vec{J}$ is approximately conserved as discussed above), this will cause the orbital angular momentum $\vec{L}_N$ to wobble around $\hat{J}$ in a complicated manner. This is why we need the complicated coordinate systems described above. One should note that the time scale of this precession is typically very long compared to orbital frequency, so this does not materially affect the "quasi-circular" assumption above. The time scale of the effects of energy and angular momomentum loss due to gravitational waves is even longer than the precession time scale.

We will model the gravitational wave amplitudes up to 2PN order, which are the most accurate expressions available in the current literature. However, because LISA will be able to observe many wave cycles from typical binary sources, it will be able to measure the phase of the gravitational waves quite accurately. Therefore, to get good results, **we will model the orbital *phase* of the binary system to 3.5PN order** (again corresponding to the best published results currently available).

Finally, in this project **we will neglect spin-spin interactions.** These will be much weaker than the spin-orbit interactions for systems reasonably far from coalescence, which will apply to many systems LISA will be able to observe. This assumption also makes a number of equations much more tractable.

1.5 Parameters

For calculating the gravitational wave, we will take the basic parameters of the binary system to be

- M (the system's total mass $= m_1 + m_2$, where m_1 and m_2 are the binary stars' masses)
- δ (defined to be $(m_1 - m_2)/M$, with m_1 chosen to be the more massive star so that δ is always positive)
- f_0 (the initial value of the system's orbital frequency in the system's own reference frame)
- R (the system's luminosity distance from our solar system)
- β (the angle between the system's initial total angular momentum $\vec{J}_0$ and our line of sight)
- ψ (the angle of the system's initial total angular momentum $\vec{J}_0$ around our line of sight)
- λ_0 (the initial angle of the binary's $\hat{n}$ vector in the orbital plane relative to the $\hat{x}_L$ vector at $t = 0$).
- Θ (the altitude angle the source in the sky measured down from the ecliptic zenith)
- Φ (the azimuth angle of the source around the ecliptic relative to the detector's initial position)
- $\chi_{10x}, \chi_{10y}, \chi_{10z}$ (the components of star 1's spin in units of m_1^2. evaluated at $t = 0$)
- $\chi_{20x}, \chi_{20y}, \chi_{20z}$ (the components of star 2's spin in units of m_2^2. evaluated at $t = 0$)

Note that the components of the spin vectors $\vec{\chi}_{10}$ and $\vec{\chi}_{20}$ are evaluated in the fixed $\hat{x}\hat{y}\hat{z}$ coordinate system defined in section 1.2.

We can calculate the gravitational wave polarizations and phase without reference to the quantity ψ: that quantity makes a difference only in how we connect the gravitational wave polarizations to the detector. The parameters $M, \delta, R, \beta, \lambda_0, \Theta, \Phi$ appear explicitly in equations for the phase and/or wave polarization amplitudes, so we can calculate the derivatives of the gravitational wave signal with respect to these parameters mathematically (if somewhat tediously).

The problem with the remaining $\chi_{10x}, \chi_{10y}, \chi_{10z}, \chi_{20x}, \chi_{20y}, \chi_{20z}$ and f_0 parameters is that they do *not* appear in the equations explicitly, but rather shape the future evolution of the binary system in a manner that we can only calculate numerically. We will describe below how to calculate derivatives with respect to these quantities.

1.6 Calculating the PN Factor v

The equation of evolution for the unitless post-Newtonian expansion parameter $v \equiv (M\omega)^{1/3}$ is

$$\begin{aligned}\dot{v} =& \frac{32\eta v^9}{5}\Bigg[1 - \left(\frac{743}{336} + \frac{11}{4}\eta\right)v^2 + \left(4\pi - \frac{47}{3}\chi_s - \delta\frac{25}{4}\chi_a\right)v^3 + \left(\frac{34\,103}{18\,144} + \frac{13\,661}{2016}\eta + \frac{59}{18}\eta^2\right)v^4 \\ &+ \left(\left[-\frac{31\,811}{1008} + \frac{5039}{84}\eta\right]\chi_s + \delta\left[-\frac{473}{84} + \frac{1231}{56}\eta\right]\chi_a + \frac{4159\pi}{672} + \frac{189\pi}{8}\eta\right)v^5 \\ &+ \left(\frac{16\,447\,322\,263}{139\,708\,800} - \frac{1712}{105}\gamma_E + \frac{16}{3}\pi^2 + \left[-\frac{56\,198\,689}{217\,728} + \frac{451}{48}\pi^2\right]\eta + \frac{541}{896}\eta^2 - \frac{5605}{2592}\eta^3 - \frac{856}{105}\ln[16v^2]\right)v^6 \\ &+ \pi\left(-\frac{4415}{4032} + \frac{358\,675}{6048}\eta + \frac{91\,495}{1512}\eta^2\right)v^7\Bigg] \end{aligned} \tag{1.9}$$

where the dot indicates a derivative with respect to dimensionless time $\tau_s \equiv t_s/M$ in the source frame,

$$\chi_s \equiv \frac{\chi_1 m_1^2 \hat{S}_1 + \chi_2 m_2^2 \hat{S}_2}{M^2} \cdot \hat{L}_N, \qquad \chi_a \equiv \frac{\chi_2 m_2 \hat{S}_2 - \chi_1 m_1 \hat{S}_1}{M} \cdot \hat{L}_N \tag{1.10a}$$

$$\gamma_E \equiv \text{Euler constant} \equiv 0.5772156649015328606 \tag{1.10b}$$

and δ and η are as defined in equations 1.2e. (This equation comes from equation 32 in Buonanno, Cook, and Pretorius, *Phys Rev D* **75,** 124018, 2007.) In the approximation we are using (where we are ignoring the spin-spin interaction as being negligible), the projections of $\hat{S}_1$ and $\hat{S}_2$ on the Newtonian orbital angular momentum vector $\hat{L}_N$ are constant in time, so χ_s and χ_a are also constant in time. Since $m_1/M = \frac{1}{2}(1+\delta$ and $m_1/M = \frac{1}{2}(1-\delta$, if we define $\chi_{1\ell} = \chi_1 \hat{S}_1 \cdot \hat{L}_N$ and $\chi_{2\ell} = \chi_2 \hat{S}_2 \cdot \hat{L}_N$ then

$$\chi_s = \tfrac{1}{4}\chi_{1\ell}(1+\delta)^2 + \tfrac{1}{4}\chi_{2\ell}(1-\delta)^2 \quad \text{and} \quad \chi_a = \tfrac{1}{2}\chi_{2\ell}(1-\delta) - \tfrac{1}{2}\chi_{1\ell}(1+\delta) \tag{1.11}$$

We see, therefore, that the PN expansion parameter v will depend only on the fixed parameters $\delta, \chi_{1\ell}$, and $\chi_{2\ell}$ in addition to time.

(Technically, $\chi_1, \chi_{1\ell}, \chi_2$, and $\chi_{2\ell}$ are fixed only if use the so-called "secularly constant" forms of the spins $\vec{S}^c_{1,2}$, which are related to stars' actual spin angular momenta $\vec{S}_{1,2}$ as discussed in section VII of Blanchet, Buonanno, and Faye, *Phys Rev D* **74,** 104034, 2006. The difference in the magnitudes and directions of these vectors is of order v_0^2 at time $t = 0$. However, how exactly we characterize the stars' spins is not particularly important, and the "secularly constant" expressions for the spins are what actually impact the wave and are therefore what we can measure, so it makes sense to use these forms as our parameters. If one desires, one can then connect the parameters and their calculated uncertainties to the actual spin angular momenta using the equations in the referenced paper. In practice, however, this will rarely be truly relevant.)

We evolve equation 1.9 numerically as follows. Let $\tau_n \equiv t_n/M$ be the (unitless) time at the present moment ("now"), as measured in the source frame, $\Delta\tau$ be the current unitless time step in that frame, and let $v_p \equiv v(\tau_n - \Delta\tau), v_n \equiv v(\tau_n)$, and $v_f \equiv v(\tau_n + \Delta\tau)$. We can then write equation 1.9 in the difference form

$$\dot{v} \equiv \frac{dv}{d\tau} \approx \frac{v_f - v_p}{2\Delta\tau} = f(v_n, \delta, \chi_{1\ell}, \chi_{2\ell}) \quad \Rightarrow \quad v_f = v_p + 2\Delta\tau f + \mathcal{O}(\Delta\tau^3) \tag{1.12}$$

where $f = f(v_n, \delta, \chi_{1\ell}, \chi_{2\ell})$ is the entire right side of equation 1.9. If we know v_p, v_n, and the fixed values of $\delta, \chi_{1\ell}$, and $\chi_{2\ell}$, we can calculate v_f at the next time step. Iterating this equation allows us to calculate v_f at successive future time steps as far as we need to go.

Because the difference approximation to the derivative appearing on the left is centered about the time where we are evaluating the right side of the equation, the calculation will be accurate "to second order in $\Delta\tau$," meaning that the first error term in the calculation of v_f will be of order of magnitude not of $\Delta\tau_s^2$ but rather $\Delta\tau^3$, a factor of $\Delta\tau^2$ smaller than the term involving $f()$.

This method is called a "leapfrog" method because it uses the past value of v and the present value of f to leap over the present moment to the future value of v. In addition to being simple and second-order accurate, this method also has the advantage of being a "simplectic integrator" that happens to preserve the energy of the system being modeled. Research done by Thummim Mekuria in the summer of 2021 showed that this characteristic was especially important for accurately modeling the evolution of the spins in particular: leapfrog modeling equations remained stable much longer than non-simplectic methods (such as Runge-Kutta methods) as the binary system approached coalescence.

However, it is tricky to start a leapfrog method at $\tau = 0$, because equation 1.12 requires knowing v a step before $\tau = 0$, which we do not know. We therefore calculate the first step using the Euler method:

$$\frac{dv}{d\tau} \approx \frac{v_1 - v_0}{\Delta\tau} = f(v_0, \delta, \chi_{1\ell}, \chi_{2\ell}) \quad \Rightarrow \quad v_1 = v_0 + \Delta\tau f + \mathcal{O}(\Delta\tau^2) \tag{1.13}$$

where $v_0 \equiv v(0)$ and $v_1 \equiv v(0+\Delta\tau_s)$. Because the difference approximation for the derivative is *not* centered in this case, this calculation is accurate only to first order in $\Delta\tau_s$ and is not simplectic. But since we only do this calculation once at the beginning, the larger error will not hurt us badly.

1.7 Calculating Derivatives of v

The gravitational wave from the source depends on v both directly in equation 1.1 and indirectly through the phase, which depends on the integral of ω given in equation 1.8, where $\omega = v^3/M$. Therefore, evaluate the partial derivatives of the gravitational wave signal h with respect to the parameters that we need to calculate the parameter uncertainties, we will need to calculate the partial derivatives of v with respect to whatever parameters it depends on.

Consider the parameter δ as an example (calculating derivatives with respect to the parameters $\chi_{1\ell}$ and $\chi_{2\ell}$ will be completely analogous). The first step is to take the derivative of both sides of equation 1.9:

$$\frac{\partial \dot{v}}{\partial \delta} = \left[\frac{\partial f}{\partial \delta} + \frac{\partial f}{\partial \eta}\frac{d\eta}{d\delta} + \frac{\partial f}{\partial \chi_s}\frac{\partial \chi_s}{\partial \delta} + \frac{\partial f}{\partial \chi_a}\frac{\partial \chi_a}{\partial \delta}\right] + \frac{\partial f}{\partial v}\frac{\partial v}{\partial \delta} \equiv g + \frac{\partial f}{\partial v}\frac{\partial v}{\partial \delta} \tag{1.14}$$

where g stands for the quantity in brackets. To calculate $\partial v/\partial\delta$ as a function of time, we can integrate the equation above using the same kind of leap-frog calculation that we are using to calculate v:

$$\left(\frac{\partial v}{\partial \delta}\right)_f = \left(\frac{\partial v}{\partial \delta}\right)_p + 2\Delta\tau \left[g_n + \left(\frac{\partial f}{\partial v}\right)_n \left(\frac{\partial v}{\partial \delta}\right)_n\right] \tag{1.15}$$

where (as before) the p, n and f subscripts refer to the past, current, and future values of the subscripted quantity. But initializing this calculation is tricky, as we do not know the initial value of $\partial v/\partial\delta$. The solution is to use a centered difference based on equation 1.9 to calculate the initial value of $\partial\dot{v}/\partial\delta$:

$$\left(\frac{\partial \dot{v}}{\partial \delta}\right)_0 \approx \left[\frac{\dot{v}(\delta + \Delta\delta) - \dot{v}(\delta - \Delta\delta)}{2\Delta\delta}\right]_0 = \frac{f(v_0, \delta + \Delta\delta, \chi_{1\ell}, \chi_{2\ell}) - f(v_0, \delta - \Delta\delta, \chi_{1\ell}, \chi_{2\ell})}{2\Delta\delta} \tag{1.16}$$

where v_0 is the initial value of v, $f()$ is again the entire right side of equation 1.9, and $\Delta\delta$ is some suitably small increment in the value of δ (for example, $\Delta\delta = 10^{-6}$). We then substitute the calculated result for $(\partial\dot{v}/\partial\delta)_0$ into equation 1.14, and solve for $(\partial v/\partial\delta)_0$:

$$\left(\frac{\partial v}{\partial \delta}\right)_0 = \frac{(\partial\dot{v}/\partial\delta)_0 - g_0}{(\partial f/\partial v)_0} \tag{1.17}$$

where (except for the $(\partial\dot{v}/\partial\delta)_0$ term), evaluating the quantities on the right at $\tau = 0$ simply means substituting $v = v_0$ when doing the calculation. With the value of $(\partial v/\partial\delta)_0$ in hand, we can then do an Euler integration step analogous to equation 1.13 to calculate $(\partial v/\partial\delta)_1$ and then use equation 1.15 to calculate future values of this quantity.

As mentioned before, calculating $\partial v/\partial\chi_{1\ell}$ and $\partial v/\partial\chi_{2\ell}$ is completely analogous.

Calculating the derivative of v with respect to f_0 is different. Because we do not have an analytic expression that gives v in terms of $v_0 = (2\pi M f_0)^{1/3}$, we are going to have to calculate this numerically. We do this by running two parallel versions of the iterative calculation described in equation 1.12, one initialized so that the initial value of v is $v_0 + \Delta v$, and one initialized so that the initial value of v is $v_0 - \Delta v$, where Δv is some suitably small number (perhaps $\Delta v = 10^{-6}$. Let the result of the first and second calculations at the nth (current) time step be v_{n+} and v_{n-}, respectively. Then at that same step,

$$\left(\frac{\partial v}{\partial v_0}\right)_n \approx \frac{v_{n+} - v_{n-}}{2\Delta v} \tag{1.18}$$

Note then that the definition of v_0 in terms of f_0 and M implies that

$$\left(\frac{\partial v}{\partial M}\right)_n = \left(\frac{\partial v}{\partial v_0}\right)_n \left(\frac{\partial v_0}{\partial M}\right)_n = \left(\frac{\partial v}{\partial v_0}\right)_n \frac{1}{3}(2\pi M f_0)^{-2/3}(2\pi f_0) = \left(\frac{\partial v}{\partial v_0}\right)_n \frac{v_0}{3M} \tag{1.19a}$$

$$\left(\frac{\partial v}{\partial f_0}\right)_n = \left(\frac{\partial v}{\partial v_0}\right)_n \left(\frac{\partial v_0}{\partial f_0}\right)_n = \left(\frac{\partial v}{\partial v_0}\right)_n \frac{1}{3}(2\pi M f_0)^{-2/3}(2\pi M) = \left(\frac{\partial v}{\partial v_0}\right)_n \frac{v_0}{3f_0} \tag{1.19b}$$

This allows us to find the partial derivatives with respect to M and f_0 of any quantities that depend on v.

1.8 Calculating the Received Wave Phase $\Psi_r(t)$

Equation 1.8 expressed in unitless form implies that

$$\lambda(\tau) = \lambda_0 + \int_0^\tau \left[v - \cos\iota \frac{d\alpha}{d\tau_s}\right] d\tau = \lambda_0 + \Lambda(\tau) \tag{1.20}$$

where Λ is the value of the integral.

The phase that appears in the gravitational waves also depends on the way that the wave gets reflected by the curvature of spacetime near the source. To at least 1.5PN order, we can model this effect as a kind of phase shift (see Blanchet, *Class Quantum Gravity* **25,** 575 (1996) for a discussion. [**Note:** We need to verify that this is true to 3.5PN order.] Define

$$\Psi(\tau) = \lambda(\tau) - 2v^3 \ln(v/w) \tag{1.21}$$

where w is an arbitrary constant. Choosing the value of w simply shifts the gravitational wave phase around by a bit, which is equivalent to choosing a slightly different definition of $\tau = 0$. For the sake of simplicity, we will choose $w = v_0$ in this case, which will imply that the phase shift is equal to zero at the time we are defining to be $\tau = 0$.

Note for what follows that

$$\frac{d\Psi}{d\tau} = v - \cos\iota \frac{d\alpha}{d\tau} - \left[2v^2 \ln\frac{v}{v_0} + 2\frac{v^3}{v/v_0}\left(\frac{1}{v_0}\right)\right]\dot{v} = v - \cos\iota\frac{d\alpha}{d\tau} - 2v^2\left[\ln\frac{v}{v_0} + 1\right] f \tag{1.22}$$

where $\dot{v} = f$ as defined above.

Now, $\Psi(\tau)$ gives us the phase in the frame of the source. But the actual wave phase $\Psi_r(\tau_r)$ as received by LISA is different because of three additional effects: (1) retardation due to the wave-travel-time between the source and LISA, (2) a cosmological redshift $z(R)$ that depends on the cosmological distance R the source is from the earth, and (3) a Doppler shift due to the orbital motion of LISA around the sun. Since the retardation simply involves a phase shift, we can throw this away: we will pretend that wave transmission is instantaneous even though we know that we are actually observing waves that the source has produced some time in the past. The cosmological redshift has the effect of introducing a dilation of the unitless time τ_r between events as measured in the solar system frame compared to the unitless time τ between those events measured at the source:

$$\tau_r = (z+1)\tau \quad \text{or} \quad \tau = \frac{\tau_r}{z+1} \quad \Rightarrow \quad \frac{d}{d\tau_r} = \frac{d\tau_r}{d\tau}\frac{d}{d\tau} = \frac{1}{z+1}\frac{d}{d\tau} \tag{1.23}$$

Therefore, when we define, for example, a time step $\Delta\tau_r$ in the solar system frame, we will have to translate that to a step $\Delta\tau = \Delta\tau_r/(z+1)$ in the source frame. In what follows, we will assume that we do this automatically. Fortunately, since z is a constant factor, this will not present much of a problem.

The Doppler shift equation tells us that the wave frequency will be shifted by a factor of approximately $1+dR/dt$, as long as dR/dt (the rate at which the distance between the detector and the source is increasing) is much smaller than 1. In our particular case, where we will assume that the LISA detector is orbiting the sun in a circle at a constant speed of $V_0 = 9.936 \times 10^{-5}$ equal to the earth's mean orbital speed and the source is located at altitude and azimuthal angles of Θ and Φ relative to the ecliptic zenith and around the ecliptic, respectively, we have

$$\frac{d\Psi_r}{d\tau_r} = \Big[1 + V_0 \sin\Theta \sin(\Omega M\tau_r + \Phi)\Big]\frac{d\Psi}{d\tau_r} = \frac{1}{1+z}\Big[1 + V_0 \sin\Theta \sin(\Omega t_r + \Phi)\Big]\frac{d\Psi}{d\tau} \tag{1.24}$$

where Ω is the angular frequency of the earth's orbit ($= 2\pi/\text{y}$) and remember that time in the solar system frame is $t_r = M\tau_r$. (Note that this equation implicitly defines $\Phi = 0$ to be the direction between the sun and LISA at time $\tau_r = \tau = 0$.)

We will have to calculate the actual phase $\Psi_r(\tau)$ by integrating this expression. Assume that we have calculated everything at the present time step n. Then we can calculate

$$\left(\frac{d\Psi}{d\tau}\right)_n = v_n - \cos\iota_n \left(\frac{d\alpha}{d\tau}\right)_n - 2v_n^2\left[\ln\frac{v_n}{v_0} + 1\right] f_n \quad \text{and} \tag{1.25a}$$

$$\Psi_{rf} = \Psi_{rp} + \frac{2\Delta\tau_r}{1+z}\Big[1 + V_0 \sin\Theta \sin(\Omega t_{rn} + \Phi)\Big]\left(\frac{d\Psi}{d\tau}\right)_n \tag{1.25b}$$

But note that $\Delta\tau_r/(1+z) = \Delta\tau$, so we could write the latter more simply as

$$\Psi_{rf} = \Psi_{rp} + 2\Delta\tau\Big[1 + V_0 \sin\Theta \sin(\Omega t_{rn} + \Phi)\Big]\left(\frac{d\Psi}{d\tau}\right)_n \tag{1.25c}$$

This makes complete sense, as the phase *should* be a frame-independent scalar.

As usual, we will need to start this calculation with an Euler-approximation initial step (note that $(\Psi_r) = \lambda_0$ by definition). This will give us the wave phase $\Psi_r(\tau_r)$ to put into the expressions for the gravitational wave polarizations h_+ and $h_\times$.

Of course, we will also need derivative versions of this equation. Again taking the parameter δ as an example, we will calculate

$$\left(\frac{\partial\Psi_r}{\partial\delta}\right)_f = \left(\frac{\partial\Psi_r}{\partial\delta}\right)_p + 2\Delta\tau\Big[1 + V_0 \sin\Theta \sin(\Omega t_{rn} + \Phi)\Big]\left[\frac{\partial}{\partial\delta}\left(\frac{d\Psi}{d\tau}\right)\right]_n, \tag{1.26a}$$

$$\begin{aligned}\text{where}\quad \left[\frac{\partial}{\partial\delta}\left(\frac{d\Psi}{d\tau}\right)\right]_n &= \left(\frac{\partial v}{\partial\delta}\right)_n + \sin\iota_n\left(\frac{\partial\iota}{\partial\delta}\right)_n\left(\frac{d\alpha}{d\tau}\right)_n - \cos\iota_n\left[\frac{\partial}{\partial\delta}\left(\frac{d\alpha}{d\tau}\right)\right]_n \\ &\quad - 4v_n\left(\frac{\partial v}{\partial\delta}\right)_n\left[\ln\frac{v_n}{v_0} + 2\right] f_n - 2v_n^2\left[\ln\frac{v_n}{v_0} + 1\right]\left(\frac{\partial f}{\partial\delta}\right)_n\end{aligned} \tag{1.26b}$$

(This equation illustrates the quantities that we will need to evaluate when we evolve ι and α.) We can initialize with an Euler step, and note that since $(\Psi_r)_0 = \lambda_0$ is a basic parameter, $(\partial\Psi_r/\partial q)_0 = 0$ for any parameter q other than λ_0. Calculating derivatives with respect to $\chi_{1\ell}, \chi_{2\ell}$ and v_0 will be similar.

Calculating derivatives with respect to Θ and Φ will be simpler. For example,

$$\left(\frac{\partial\Psi_r}{\partial\Theta}\right)_f = \left(\frac{\partial\Psi_r}{\partial\Theta}\right)_p + 2\Delta\tau\Big[1 + V_0 \cos\Theta \sin(\Omega t_{rn} + \Phi)\Big]\left[\frac{\partial}{\partial\delta}\left(\frac{d\Psi}{d\tau}\right)\right]_n \tag{1.27}$$

Start (as usual) with an Euler integration step using a zero initial value.

Even though equation 1.25c does not seem to depend on z, it actually does. Consider a given system with all parameters remaining the same except for z (which is a function of R), which is different for the two systems. Our simulation will also keep $\Delta\tau_r$ the same, which means that we should go back to equation 1.25b to see how the phase values depend on z. Taking the derivative of that equation yields

$$\begin{aligned}\left(\frac{\partial\Psi_r}{\partial z}\right)_f &= \left(\frac{\partial\Psi_r}{\partial z}\right)_p - 2\Delta\tau_r\frac{z}{(1+z)^2}\Big[1 + V_0 \sin\Theta \sin(\Omega t_{rn} + \Phi)\Big]\left(\frac{d\Psi}{d\tau}\right)_n \\ &= \left(\frac{\partial\Psi_r}{\partial z}\right)_p - 2\Delta\tau\frac{z}{1+z}\Big[1 + V_0 \sin\Theta \sin(\Omega t_{rn} + \Phi)\Big]\left(\frac{d\Psi}{d\tau}\right)_n\end{aligned} \tag{1.28}$$

As usual, we start with an Euler step using a zero initial value.

Equations 1.20 and 1.21 imply that $\partial\Psi_r/\partial\lambda_0 = 1$.

1.9 Evolving the Spin Variables

For circular orbits, the evolution equations for the spin directions are (in the $\hat{x}\hat{y}\hat{z}$ basis) are (to 2PN order)

$$\frac{d\hat{\chi}_1}{d\tau} = \vec{\Omega}_1 \times \hat{\chi}_1, \qquad \frac{d\hat{\chi}_2}{d\tau} = \vec{\Omega}_2 \times \hat{\chi}_2, \quad \text{where} \tag{1.29a}$$

$$\begin{aligned}\vec{\Omega}_1 = v^5 \Bigg\{ & \frac{3}{4} + \frac{\eta}{2} - \frac{3}{4}\delta + v^2 \left[\frac{9}{16} + \frac{5}{4}\eta - \frac{\eta^2}{24} + \delta\left(-\frac{9}{16} + \frac{5}{8}\eta\right)\right] \\ & + v^4 \left[\frac{27}{32} + \frac{3}{16}\eta - \frac{105}{32}\eta^2 - \frac{\eta^3}{48} + \delta\left(\frac{-27}{32} + \frac{39}{8}\eta - \frac{5}{32}\eta^2\right)\right]\Bigg\}\hat{L}_N\end{aligned} \tag{1.29b}$$

and $\vec{\Omega}_2$ is the same with $\delta \to -\delta$ (see Blanchet, *Living Reviews in Relativity* **14,** 2 (2014), equations 388 and 394). Using the Vector class, we should be able to calculate these quantities easily in code and use the leapfrog method to integrate:

$$\hat{\chi}_{1f} = \hat{\chi}_{1p} + 2\Delta\tau_s \vec{\Omega}_{1n} \times \hat{\chi}_{1n} \tag{1.30}$$

and similarly for $\hat{\chi}_2$. This will give us the components of these unit vectors in the $\hat{x}\hat{y}\hat{z}$ basis as a function of time, which is precisely what we need for calculating the gravitational wave amplitude (see below). The initial values for this equation can be easily calculated from the spin parameters (for example, $(\hat{\chi}_1)_x$ at time $t = 0$ is $= \chi_{10x}/\chi_1$, where the conserved value of $\chi_1 = (\chi_{10x}^2 + \chi_{10y}^2 + \chi_{10z}^2)^{1/2}$).

The equation of evolution for $\hat{L}_N$ is to leading order (see equation 3.20 in Arun, et al. *Phys Rev D* **79,** 104023 (2009) with a necessary correction to fix the units) is

$$\begin{aligned}\frac{d\hat{L}_N}{d\tau} &= -\frac{v}{M^2\eta}\left[\frac{d\vec{S}_1}{d\tau} + \frac{d\vec{S}_2}{d\tau}\right] = -\frac{v}{\eta}\left[\chi_1 \frac{m_1^2}{M^2}\frac{d\hat{\chi}_1}{d\tau} + \chi_2 \frac{m_2^2}{M^2}\frac{d\hat{\chi}_2}{d\tau}\right] \\ &= -\frac{v}{4\eta}\left[\chi_1(1+\delta)^2 \frac{d\hat{\chi}_1}{d\tau} + \chi_2(1-\delta)^2\frac{d\hat{\chi}_2}{d\tau}\right]\end{aligned} \tag{1.31}$$

where we can calculate $d\hat{\chi}_1/d\tau_s$ and $d\hat{\chi}_2/d\tau_s$ using equations 1.29. We can readily integrate this using an analogous leapfrog scheme, and from the components of $\hat{L}_N$, we can calculate the angles ι and α at any given time step:

$$\iota = \cos^{-1}(\hat{L}_N)_z, \quad \alpha = \tan^{-1}\frac{(\hat{L}_N)_y}{(\hat{L}_N)_x} \tag{1.32}$$

(To get unambiguous results, one should use Xojo's `Atan2(,)` function to return the value of α. Also, since I believe that α should be accumulating with time, we should keep track when the value of α jumps suddenly and add the appropriate multiple of 2π to keep α increasing or decreasing monotonically.)

[**Note:** I am troubled by equation 1.31. It *looks* like it is coming from $\vec{J} = \vec{L} + \vec{S}_1 + \vec{S}_2 =$ constant (where $\vec{L}$ is the "secularly constant" version of the orbital angular momentum), the assumption that $\vec{L} = \vec{L}_N$, that $|\vec{L}_N| \equiv \mu r v = M\eta r v$, and that $r/M = 1/v^2$. But $\vec{L}$ is *not* equal to $\vec{L}_N$, $r/M = 1/v^2$ only to lowest order and (when radiation is considered) $\vec{J}$ is not constant. But the change in $\vec{J}$ is on the radiation reaction time scale, which is long compared to the precession time scale, so it may be that we can ignore $d\vec{J}/d\tau$ at the precession time scale and (since apparently the change in $\vec{J}$ is parallel to $\hat{L}_N$ for circular orbits when we ignore spin-spin effects) the long-term change in $\vec{J}$ by the increase in the magnitude of v in the expression $|\vec{L}_N| \approx M^2\eta/v$. Also, perhaps the the distinction between $\vec{L}$ and $\vec{L}_N$ might not be relevant in the circular motion case or maybe cancels some of the terms in the expansion of r/M. We will go with this equation for now. But in the long run, I'd like to see some justification of equation, particularly for the "secularly constant" versions of the spin and angular momenta that we are assuming here. (But this $d\vec{L}_N/dt = -d\vec{S}_1/dt - d\vec{S}_2/dt$ also appears in Blanchet et al. *Phys Rev D* **74,** 104034 (2006), which does explicitly use the "secularly constant" spin variables in the same section, though they also say that this equation is "at leading order." The question is probably mainly how accurate the angular momentum magnitude really is.)]

To calculate derivatives with respect to the spin parameters, note that the phase depends on ι, α, χ_s and χ_a. Since the last two are constant in time, they can be readily calculated at time $t = 0$ using the spin parameters (which are the spin vector components at time $t = 0$, and from those equations, one can also readily calculate $\partial\chi_a/\partial\chi_{10x}, \partial\chi_a/\partial\chi_{10y}$, etc. as analytical expressions. Then if we need to calculate a derivative of the received phase with respect to, say, χ_{20y}, the process would be

$$\frac{\partial\Psi_r}{\partial\chi_{20y}} = \frac{\partial\Psi_r}{\partial v}\frac{\partial v}{\partial\chi_s}\frac{\partial\chi_s}{\partial\chi_{20y}} + \frac{\partial\Psi_r}{\partial v}\frac{\partial v}{\partial\chi_a}\frac{\partial\chi_a}{\partial\chi_{20y}} \tag{1.33}$$

where all of these derivatives are analytical expressions. On the other hand, the phase also depends on ι and $d\alpha/d\tau$, which cannot be calculated analytically at a given time from the initial spin parameters. To handle these calculations, we will need to run 13 parallel cases of just equations 1.30 and the equivalent for equation 1.31, one "base case" with the central initial spin parameter values, and a pair for each of the six parameters, one with the parameter bumped up by a small amount (say, $\Delta\chi = 10^{-6}$), and one bumped down. Then, to calculate the derivative of, say, ι_n with respect to, say χ_{10y}, we calculate the centered difference

$$\left(\frac{\partial\iota}{\partial\chi_{10y}}\right)_n \approx \frac{\iota_n(\chi_{10y} + \Delta\chi) - \iota_n(\chi_{10y} - \Delta\chi)}{2\Delta\chi} \tag{1.34}$$

where the value $\iota(\chi_{10y} + \Delta\chi)$ is the value of ι at step n as calculated by the case where the parameter χ_{10y} has been bumped up by $\Delta\chi$ and so on. Calculating the derivatives of α will be similar.

The gravitational wave polarization equations refer to the spin components directly, but to their values at the current instant, not at at time $t = 0$, so we will have to do something similar with these:

$$\begin{aligned}\left(\frac{\partial h}{\partial\chi_{10y}}\right)_n &= \frac{\partial h}{\partial\chi_{1x}}\left(\frac{\partial\chi_{1x}}{\partial\chi_{10y}}\right)_n + \frac{\partial h}{\partial\chi_{1y}}\left(\frac{\partial\chi_{1y}}{\partial\chi_{10y}}\right)_n + \frac{\partial h}{\partial\chi_{1z}}\left(\frac{\partial\chi_{1z}}{\partial\chi_{10y}}\right)_n \\ &\quad + \frac{\partial h}{\partial\chi_{2x}}\left(\frac{\partial\chi_{2x}}{\partial\chi_{10y}}\right)_n + \frac{\partial h}{\partial\chi_{2y}}\left(\frac{\partial\chi_{2y}}{\partial\chi_{10y}}\right)_n + \frac{\partial h}{\partial\chi_{2z}}\left(\frac{\partial\chi_{2z}}{\partial\chi_{10y}}\right)_n \end{aligned} \tag{1.35a}$$

$$\text{where} \quad \left(\frac{\partial\chi_{1x}}{\partial\chi_{10y}}\right)_n = \frac{\chi_{1x}(\chi_{10y} + \Delta\chi) - \chi_{1x}(\chi_{10y} - \Delta\chi)}{2\Delta\chi}, \text{ etc.} \tag{1.35b}$$

The first partial in each term on the right side of equation 1.35a (the partials of h) can be calculated analytically from the equations for the gravitational wave polarization. We must calculate all six of these terms because *any* of the spin components at step n might change even if only one of the initial spin parameters is bumped.