# 1 Calculating $\frac{\partial \dot{v}}{\partial \delta}$,$\frac{\partial \dot{v}}{\partial \chi_{1\ell}}$, and $\frac{\partial \dot{v}}{\partial \chi_{2\ell}}$

Equation (1.9) in the Theory1 document provides an expression for $\dot{v}$, if we let $\alpha = \frac{5\dot{v}}{32\eta v^9}$ Then:

## 1.1 $\frac{\partial \dot{v}}{\partial \delta}$

$$\frac{\partial \dot{v}}{\partial \delta} = \frac{32}{5}\left[\left(\frac{\partial \eta}{\partial \delta}v^9 = 9\eta v^8\frac{\partial v}{\partial \delta}\right)\alpha + \eta v^9\frac{\partial \alpha}{\partial \delta}\right] \tag{1}$$

Where:

$$\begin{aligned}
\frac{\partial \alpha}{\partial \delta} =& v\left[-2\frac{\partial v}{\partial \delta}\left(\frac{743}{336}+\frac{11}{4}\eta\right)\right] \\
&+ v^2\left[-\frac{11}{4}\frac{\partial \eta}{\partial \delta}+3\frac{\partial v}{\partial \delta}\left(4\pi-\frac{47}{3}\chi_s-\frac{25}{4}\delta\chi_a\right)\right] \\
&+ v^3\left[-\frac{47}{3}\frac{\partial \chi_s}{\partial \delta}-\frac{25}{4}\left(\chi_a+\delta\frac{\partial \chi_a}{\partial \delta}\right)+4\frac{\partial v}{\partial \delta}\left(\frac{34103}{18144}+\frac{13661}{2016}\eta+\frac{59}{18}\eta^2\right)\right] \\
&+ v^4\left[\frac{\partial \eta}{\partial \delta}\left(\frac{13661}{2016}+\frac{59}{9}\eta\right)+5\frac{\partial v}{\partial \delta}\left(\chi_s\left[-\frac{31811}{1008}+\frac{5034}{84}\eta\right]+\delta\chi_a\left[-\frac{473}{84}+\frac{1231}{56}\eta\right]+\frac{4159}{672}\pi+\frac{189}{8}\pi\eta\right)\right] \\
&+ v^5\left[\frac{\partial \chi_s}{\partial \delta}\left(-\frac{31811}{1008}+\frac{5039}{84}\eta\right)+\frac{\partial \eta}{\partial \delta}\left(\frac{5039}{84}\chi_s+\frac{1231}{56}\delta\chi_a+\frac{189}{8}\pi\right)+\left(\chi_a+\delta\frac{\partial \chi_a}{\partial \delta}\right)\left(-\frac{473}{84}+\frac{1231}{56}\right)\right] \\
&+ v^5\left[6\frac{\partial v}{\partial \delta}\left(\frac{16447322263}{139208800}-\frac{1712}{105}\gamma_E+\frac{16}{3}\pi^2+\left[-\frac{56198689}{217728}+\frac{451}{48}\pi^2\right]\eta+\frac{541}{896}\eta^2-\frac{5605}{2592}\eta^3-\frac{856}{315}-\frac{856}{105}\ln\left(16v^2\right)\right)\right] \\
&+ v^6\left[\frac{\partial \eta}{\partial \delta}\left(\left[-\frac{56198689}{217728}+\frac{451}{48}\pi^2\right]+\frac{541}{448}\eta-\frac{5605}{864}\eta^2\right)+7\pi\frac{\partial v}{\partial \delta}\left(-\frac{4415}{4032}+\frac{358675}{6048}\eta+\frac{91495}{1512}\eta^2\right)\right] \\
&+ v^7\left[\pi\frac{\partial \eta}{\partial \delta}\left(\frac{358675}{6048}+\frac{91495}{756}\eta\right)\right]
\end{aligned}$$

and:

$$\frac{\partial \eta}{\partial \delta} = \frac{\partial}{\partial \delta}\left(\frac{1}{4}\left(1-\delta^2\right)\right) = -\frac{1}{2}\delta \tag{2}$$

$$\frac{\partial \chi_s}{\partial \delta} = \frac{\partial}{\partial \delta}\left(\chi_{1\ell}\left(1+\delta\right)^2+\chi_{2\ell}\left(1-\delta\right)^2\right) = 2\left(\chi_{1\ell}\left(1+\delta\right)-\chi_{2\ell}\left(1-\delta\right)\right) \tag{3}$$

$$\frac{\partial \chi_a}{\partial \delta} = \frac{\partial}{\partial \delta}\left(\chi_{2\ell}\left(1-\delta\right)-\chi_{1\ell}\left(1+\delta\right)\right) = -\left(\chi_{2\ell}+\chi_{1\ell}\right) \tag{4}$$

### 1.2 $\frac{\partial \dot{v}}{\partial \chi_{1\ell}}$

$$\frac{\partial \dot{v}}{\partial \chi_{1\ell}} = \frac{32}{5}\eta \left[9v^8 \frac{\partial v}{\partial \chi_{1\ell}}\alpha + v^9 \frac{\partial \alpha}{\partial \chi_{1\ell}}\right] \tag{5}$$

Where:

$$\begin{aligned}
\frac{\partial \alpha}{\partial \chi_{1\ell}} =& v\left[-2\frac{\partial v}{\partial \chi_{1\ell}}\left(\frac{743}{336}+\frac{11}{4}\eta\right)\right] \\
&+ v^2\left[3\frac{\partial v}{\partial \chi_{1\ell}}\left(4\pi - \frac{47}{3}\chi_s - \delta\frac{25}{4}\chi_a\right)\right] \\
&+ v^3\left[-\frac{47}{3}\frac{\partial \chi_s}{\partial \chi_{1\ell}} - \delta\frac{25}{4}\frac{\partial \chi_a}{\partial \chi_{1\ell}} + 4\frac{\partial v}{\partial \chi_{1\ell}}\left(\frac{34103}{18144}+\frac{13661}{2016}\eta+\frac{59}{18}\eta^2\right)\right] \\
&+ v^4\left[5\frac{\partial v}{\partial \chi_{1\ell}}\left(\left[-\frac{31811}{1008}+\frac{5039}{84}\right]\chi_s + \delta\left[-\frac{473}{84}+\frac{1231}{56}\eta\right]\chi_a + \frac{4159}{672}\pi + \frac{189}{8}\pi\eta\right)\right] \\
&+ v^5\left[\left(-\frac{31811}{1008}+\frac{5039}{85}\eta\right)\frac{\partial \chi_s}{\partial \chi_{1\ell}} + \delta\left(-\frac{473}{84}+\frac{1231}{56}\eta\right)\frac{\partial \chi_a}{\partial \chi_{1\ell}}\right] \\
&+ v^5\left[6\frac{\partial v}{\partial \chi_{1\ell}}\left(\frac{6447322263}{139208800} - \frac{1712}{105}\gamma_E + \frac{16}{3}\pi^2 + \left[-\frac{5619869}{217728}+\frac{451}{48}\pi^2\right]\eta + \frac{541}{896}\eta^2 - \frac{5605}{592}\eta^3 - \frac{856}{105}\ln\left(16v^2\right) - \frac{856}{315}\right)\right] \\
&+ v^6\left[7\pi\frac{\partial v}{\partial \chi_{1\ell}}\left(-\frac{4415}{4032}+\frac{358675}{6048}\eta+\frac{91495}{1512}\right)\right]
\end{aligned}$$

and:

$$\frac{\partial \chi_s}{\partial \chi_{1\ell}} = \frac{\partial}{\partial \chi_{1\ell}}\left(\chi_{1\ell}\left(1+\delta\right)^2 + \chi_{2\ell}\left(1-\delta\right)^2\right) = \left(1+\delta\right)^2 \tag{6}$$

$$\frac{\partial \chi_a}{\partial \chi_{1\ell}} = \frac{\partial}{\partial \chi_{1\ell}}\left(\chi_{2\ell}\left(1-\delta\right) - \chi_{1\ell}\left(1+\delta\right)\right) = -\left(1+\delta\right) \tag{7}$$

### 1.3 $\frac{\partial \dot{v}}{\partial \chi_{2\ell}}$

$$\frac{\partial \dot{v}}{\partial \chi_{2\ell}} = \frac{32}{5}\eta \left[9v^8 \frac{\partial v}{\partial \chi_{2\ell}}\alpha + v^9 \frac{\partial \alpha}{\partial \chi_{2\ell}}\right] \tag{8}$$

Where:

$$\begin{aligned}
\frac{\partial \alpha}{\partial \chi_{2\ell}} =& v\left[-2\frac{\partial v}{\partial \chi_{2\ell}}\left(\frac{743}{336}+\frac{11}{4}\eta\right)\right] \\
&+ v^2\left[3\frac{\partial v}{\partial \chi_{2\ell}}\left(4\pi - \frac{47}{3}\chi_s - \delta\frac{25}{4}\chi_a\right)\right] \\
&+ v^3\left[-\frac{47}{3}\frac{\partial \chi_s}{\partial \chi_{2\ell}} - \delta\frac{25}{4}\frac{\partial \chi_a}{\partial \chi_{2\ell}} + 4\frac{\partial v}{\partial \chi_{2\ell}}\left(\frac{34103}{18144}+\frac{13661}{2016}\eta+\frac{59}{18}\eta^2\right)\right] \\
&+ v^4\left[5\frac{\partial v}{\partial \chi_{2\ell}}\left(\left[-\frac{31811}{1008}+\frac{5039}{84}\right]\chi_s + \delta\left[-\frac{473}{84}+\frac{1231}{56}\eta\right]\chi_a + \frac{4159}{672}\pi + \frac{189}{8}\pi\eta\right)\right] \\
&+ v^5\left[\left(-\frac{31811}{1008}+\frac{5039}{85}\eta\right)\frac{\partial \chi_s}{\partial \chi_{2\ell}} + \delta\left(-\frac{473}{84}+\frac{1231}{56}\eta\right)\frac{\partial \chi_a}{\partial \chi_{2\ell}}\right] \\
&+ v^5\left[6\frac{\partial v}{\partial \chi_{2\ell}}\left(\frac{6447322263}{139208800} - \frac{1712}{105}\gamma_E + \frac{16}{3}\pi^2 + \left[-\frac{5619869}{217728}+\frac{451}{48}\pi^2\right]\eta + \frac{541}{896}\eta^2 - \frac{5605}{592}\eta^3 - \frac{856}{105}\ln\left(16v^2\right) - \frac{856}{315}\right)\right] \\
&+ v^6\left[7\pi\frac{\partial v}{\partial \chi_{2\ell}}\left(-\frac{4415}{4032}+\frac{358675}{6048}\eta+\frac{91495}{1512}\right)\right]
\end{aligned}$$

and:

$$\frac{\partial \chi_s}{\partial \chi_{2\ell}} = \frac{\partial}{\partial \chi_{2\ell}}\left(\chi_{2\ell}\left(1+\delta\right)^2 + \chi_{2\ell}\left(1-\delta\right)^2\right) = \left(1-\delta\right)^2 \tag{9}$$

$$\frac{\partial \chi_a}{\partial \chi_{2\ell}} = \frac{\partial}{\partial \chi_{2\ell}}\left(\chi_{2\ell}\left(1-\delta\right) - \chi_{2\ell}\left(1+\delta\right)\right) = \left(1-\delta\right) \tag{10}$$