Q2

June 17, 2020

1 Question 2. Multilinear Algebra

Given

$$A * B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, A * C = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}, C * D + D = \begin{pmatrix} 6 & 4 \\ 16 & 10 \end{pmatrix}, y = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T \tag{1}$$

find the vector of coefficients $\hat{\beta}$ by solving the following optimization problem:

$$\hat{\beta} = argmin_{\beta} \| y - \{ [(A \otimes C)^T * (B^T \otimes A^T)] \ [(B \odot C) * (A \odot D) + A * B \odot D] \} \beta \|_2^2 \tag{2}$$

Simplify the above expression to an appropriate form before solving the optimization problem.

Hint: $(A \otimes B) * (C \otimes D) = (A * C) \otimes (B * D); (A \odot B) * (C \odot D) = (A * C) \odot (B * D)$

[1]:
```
import numpy as np
from tensorly.tenalg import khatri_rao
from IPython.core.display import display, HTML
display(HTML("<style>.container { width:100% !important; }</style>"))
```

```
<IPython.core.display.HTML object>
```

2 *Note: Following section is for notes and references. The answer is below this section*

3 Hadamard Product

Element wise matrix multiplication E.g.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \tag{3}$$

$$A * B = \begin{pmatrix} 1 * 5 & 2 * 6 \\ 3 * 7 & 4 * 8 \end{pmatrix} \qquad = \begin{pmatrix} 5 & 12 \\ 21 & 32 \end{pmatrix} \tag{4}$$

Code

```
import numpy as np

A = np.array([[1,2],[3,4]])
B = np.array([[5,6],[7,8]])

ham = A*B
```

3.1 # Kronecker Product

Denoted $A \otimes B$ where A $\in \mathbb{R}^{IxJ}$ and B $\in \mathbb{R}^{KxL}$. The result is a matrix of size (IK) x (JL) and defined by:

$$\begin{pmatrix} a_{11}B & a_{12}B & ... & a_{1J}B \\ a_{21}B & a_{22}B & ... & a_{2J}B \\ ... & ... & ... & ... \\ a_{I1}B & a_{I2}B & ... & a_{IJ}B \end{pmatrix} \tag{5}$$

$$A \otimes B = \begin{pmatrix} 1*B & 2*B \\ 3*B & 4*B \end{pmatrix} = \begin{pmatrix} 5 & 6 & 10 & 12 \\ 7 & 8 & 14 & 16 \\ 15 & 18 & 20 & 24 \\ 21 & 24 & 28 & 32 \end{pmatrix} \tag{6}$$

Code

```
import numpy as np

A = np.array([[1,2],[3,4]])
B = np.array([[5,6],[7,8]])

kron = np.kron(A,B)
```

3.2 # Khati-Rao Product

"Matching columnwise" Kronecker product. Denoted $A \odot B$ where A $\in \mathbb{R}^{IxK}$ and B $\in \mathbb{R}^{JxK}$. A $\odot$ B is a matrix of size (IJ) x (K) and computed by

$$A \odot B = [a_1 \otimes b_1 \qquad a_2 \otimes b_2 \quad .... \quad a_k \otimes b_k] \tag{7}$$

$$A \odot B = \begin{pmatrix} 1*5 & 2*6 \\ 1*7 & 2*8 \\ 3*5 & 4*6 \\ 3*7 & 4*8 \end{pmatrix} \qquad = \begin{pmatrix} 5 & 12 \\ 7 & 16 \\ 15 & 24 \\ 21 & 32 \end{pmatrix} \tag{8}$$

Code

```
from tensorly.tenalg import khatri_rao

A = np.array([[1,2],[3,4]])
B = np.array([[5,6],[7,8]])

kr = khatri_rao([A,B])
```

3.3 # Answer:

A few things to note first

$$I. \quad **A*B** \in \mathbb{R}^{2x2}, \quad **A*C** \in \mathbb{R}^{2x2} \quad \therefore A \in \mathbb{R}^{2x2},\ B \in \mathbb{R}^{2x2}\ , and \ \ C \in \mathbb{R}^{2x2} \tag{9}$$

$$II. \quad (A \otimes B)^T = (A^T \otimes B^T) \tag{10}$$

$$III. \quad (A * C) = (C * A) \ - \ commutative\ property\ holds\ for\ Hadamard\ product \tag{11}$$

$$IV. \quad (A \odot B) + (A \odot C) = A \odot (B + C) - distributive\ property\ holds\ for\ Khatri\ Rao\ product \tag{12}$$

$$\hat{\beta} = argmin_{\beta} \|y - \{[(A \otimes C)^T * (B^T \otimes A^T)]\ [(B \odot C) * (A \odot D) + A * B \odot D]\}\beta\|_2^2 \qquad (1) \tag{13}$$

Rearraning the elements in the 1st square bracket of (1) according to item (II.) above and applying the hint $(A \otimes B) * (C \otimes D) = (A * C) \otimes (B * D)$

$$\hat{\beta} = argmin_{\beta} \|y - \{[(A^T \otimes C^T) * (B^T \otimes A^T)]\ [(B \odot C) * (A \odot D) + A * B \odot D]\}\beta\|_2^2 \qquad = argmin_{\beta} \|y - \{[(A^T * B^T) \otimes (C^T \tag{14}$$

Following the commutative property for Hadamard product as noted above in (III.) and reapplying item (II.) equation (2) can be written as

$$= argmin_{\beta} \|y - \{[(A * B)^T \otimes (A * C)^T]\ [(B \odot C) * (A \odot D) + A * B \odot D]\}\beta\|_2^2 \qquad (3) \tag{15}$$

Rearanging elements in the second square bracket of (3) according to the hint $(A \odot B) * (C \odot D) = (A * C) \odot (B * D)$

$$= argmin_{\beta} \|y - \{[(A * B)^T \otimes (A * C)^T]\ [(B * A) \odot (C * D) + A * B \odot D]\}\beta\|_2^2 \qquad (4) \tag{16}$$

Notating $(A * B) = (B * A) = \gamma$ (4) becomes

$$= argmin_{\beta} \| y - \{[(A * B)^T \otimes (A * C)^T] \, [\gamma \odot (C * D) + \gamma \odot D]\} \beta \|_2^2 \qquad (5) \qquad (17)$$

Notating item (IV.) above and factoring out γ

$$= argmin_{\beta} \| y - \{[(A * B)^T \otimes (A * C)^T] \, [\gamma \odot ((C * D) + D) \,]\} \beta \|_2^2 \qquad (6) \qquad (18)$$

Replacing γ with $\gamma = (A * B)$ in (6) we arrive at

$$\hat{\beta} = argmin_{\beta} \| y - \{[(A * B)^T \otimes (A * C)^T] \, [(A * B) \odot ((C * D) + D) \,]\} \beta \|_2^2 \qquad (7) \qquad (19)$$

$$\boxed{\hat{\beta} = argmin_{\beta} \| y - X\beta \|_2^2 \qquad \therefore X = \{[(A * B)^T \otimes (A * C)^T] \, [(A * B) \odot ((C * D) + D) \,]\}} \qquad (8)$$

(20)

[2]:
```
ab = np.array([[1,2],[3,4]])
ac = np.array([[5,6],[7,8]])
cdd = np.array([[6,4],[16,10]])
y = np.array([1,2,3,4])
```

3.3.1 Using eqn. (8) above, create X matrix

[3]:
```
#left square bracket - Kronecker product
M1 = np.kron(ab.T, ac.T)
```

[4]:
```
#right square bracket - khatri rao prodcut
M2 = khatri_rao([ab, cdd])
```

[5]:
```
# X - matrix
X = M1.dot(M2)
```

3.3.2 Using numpy to solve OLS for beta

[6]:
```
betas_np = np.linalg.lstsq(X,y, rcond=None)[0]
betas_np
```

[6]:
```
array([-0.0309884 ,  0.03603101])
```

3.3.3 Manual calculation of OLS for sanity check $\hat{\beta} = (X^T X)^{-1} X^T y$

[7]:
```
betas_ols = np.linalg.pinv(X.T.dot(X)).dot(X.T.dot(y))
betas_ols
```

[7]:
```
array([-0.0309884 ,  0.03603101])
```

$$\boxed{\hat{\beta} = \begin{pmatrix} -0.0309884 \\ 0.03603101 \end{pmatrix}} \tag{21}$$

[ ]:
```

```