1. **Problem 1**. The behavior of polynomials fit to data tends to be erratic near the boundaries. The polynomials fit beyond the boundary knots behave even more wildly than the corresponding global polynomials in that region. Assuming the function is linear near the boundaries (where we have less information anyway) is often considered reasonable. A *natural quadratic spline* adds additional constraints, namely that the function is linear beyond the boundary knots. Let

$$\boldsymbol{y(x)} = \begin{cases} \sum_{j=1}^{3} \beta_j x^{j-1} + \sum_{k=1}^{K} \theta_k (x-\xi_k)_+^2 & for\ x \in [\xi_1, \xi_K] \\ \sum_{j=1}^{2} \beta_j x^{j-1} + \sum_{k=1}^{K} \theta_k (x-\xi_k)_+^2 & \text{for } x \text{ outside interval } (\xi_1, \xi_K) \end{cases}$$

Determine a set of bases for y(x) such that $\frac{d^2}{\partial x} y(x)$= 0 for x outside interval (ξ_1, ξ_K)

Answer:

Since we are concerned for a set of bases outside the interval(ξ_1, ξ_K), the second equation above is all that we are concerned with.

$$y(x) = \sum_{j=1}^{2} \beta_j X^{j-1} + \sum_{k=1}^{K} \theta_k (x-\xi_k)_+^2 \qquad (1)$$

Expanding (1)

$$y(x) = \beta_1 + x\beta_2 + \sum_{k=1}^{K} \theta_k x^2 - 2\theta_k x \xi_k + \theta_k \xi_k^2 \qquad (2)$$

The constraints require that f''(x) = 0. Calculating f'(x):

$$\frac{\partial}{\partial x} y(x) = \beta_2 + \sum_{k=1}^{k} 2\theta_k X - 2\theta_k \xi_k \qquad (3)$$

Taking the derivative of the first derivative (3) and setting equal to 0 yields:

$$\frac{\partial}{\partial x} y'(x) = 2 \sum_{k=1}^{k} \theta_k$$

$$\sum_{k=1}^{K} \theta_k = 0 \qquad (4)$$

Expanding (4) and solving for θ_K

$$\theta_K + \sum_{k=1}^{K-1} \theta_k = 0$$

$$\theta_K = -\sum_{k=1}^{K-1} \theta_k \qquad (5)$$

Expanding the right summation in (1) as we just did above and plugging in the value of θ_K

$$y(x) = \beta_1 + x\beta_2 + \sum_{k=1}^{K-1} \theta_k (x - \xi_k)_+^2 + \theta_K (x - \xi_K)_+^2$$

$$y(x) = \beta_1 + x\beta_2 + \sum_{k=1}^{K-1} \theta_k (x - \xi_k)_+^2 - \sum_{k=1}^{K-1} \theta_k (x - \xi_K)_+^2 \qquad (6)$$

Joining like terms in (6) yields:

$$y(x) = \beta_1 + x\beta_2 + \sum_{k=1}^{K-1} \theta_k [(x - \xi_k)_+^2 - (x - \xi_K)_+^2] \qquad (7)$$

Finally, from (7) we arrive at our basis functions:

$H_1(X) = 1$

$H_2(X) = X$

$H_{k+2}(X) = \{(x - \xi_k)_+^2 - (x - \xi_K)_+^2\}_{k=1,\ldots,K-1}$