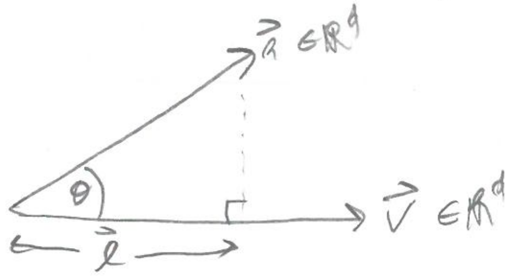
Lecture 9 Math 390.4 3/7/17

Let's do some linear algebra...



$\vec{\ell} = \text{proj}_{\vec{v}}(\vec{a})$ the "orthogonal projection of $\vec{a}$ onto $\vec{v}$"

perpendicular

By law of cosines, $\cos(\theta) = \dfrac{\vec{a}\cdot\vec{v}}{\|\vec{a}\|\|\vec{v}\|} = \dfrac{\|\vec{\ell}\|}{\|\vec{a}\|} \Rightarrow \|\vec{\ell}\| = \dfrac{\vec{a}\cdot\vec{v}}{\|\vec{v}\|}$

Rank[H] ?

$\frac{1}{\|\vec{v}\|^2}\vec{v}\vec{v}^T = \frac{1}{\|\vec{v}\|^2}\vec{v}\,[v_1 \ldots v_d]$

$= \left[\frac{v_1}{\|\vec{v}\|^2}\vec{v} \,\middle|\, \frac{v_2}{\|\vec{v}\|^2}\vec{v} \,\middle|\, \ldots \,\middle|\, \frac{v_d}{\|\vec{v}\|^2}\vec{v}\right]$

all lin dep

span$\{c, \ldots\}$ = span$\{\vec{v}\}$ = colsp(H)

dim(colsp(H)) = 1 = rank[H]

If $\vec{a} \perp \vec{v}$?

$H\vec{a} = \frac{1}{\|\vec{v}\|^2}\vec{v}\underbrace{\vec{v}^T\vec{a}}_{0} = \vec{0}_d$

Now we need direction... multiply by length 1, correct direction $\dfrac{\vec{v}}{\|\vec{v}\|}$

does length of $\vec{v}$ matter at all?

$$\text{proj}_{\vec{v}}(\vec{a}) = \underbrace{\frac{\vec{a}\cdot\vec{v}}{\|\vec{v}\|^2}}_{\|\vec{\ell}\|}\vec{v} = \frac{(\vec{a}\cdot\vec{v})\,\vec{v}}{\|\vec{v}\|^2} = \frac{\vec{v}\,\vec{v}^T\vec{a}}{\|\vec{v}\|^2} = \underbrace{\frac{1}{\|\vec{v}\|^2}\overbrace{\vec{v}\vec{v}^T}^{\text{outer product}}}_{H \in \mathbb{R}^{d\times d}}\vec{a} = H\vec{a}$$

H is called a projection matrix. It projects onto colsp($\vec{v}$)

Let's project $\vec{v}$ onto itself $\text{proj}_{\vec{v}}(\vec{v}) = H\vec{v} = \frac{1}{\|\vec{v}\|^2}\vec{v}(\vec{v}^T\vec{v}) = \vec{v}$ ✓

Proj $\vec{a} - \text{proj}_{\vec{v}}(\vec{a})$ onto $\vec{v}$?

$H(\vec{a} - H\vec{a}) = H\vec{a} - HH\vec{a} = H\vec{a} - H\vec{a} = \vec{0}$

What if I project twice in a row?

$$\text{proj}_{\vec{v}}(\text{proj}_{\vec{v}}(\vec{a})) = HH\vec{a} = \frac{1}{\|\vec{v}\|^2}\vec{v}\vec{v}^T\frac{1}{\|\vec{v}\|^2}\vec{v}\vec{v}^T\vec{a} = \frac{1}{(\|\vec{v}\|^2)^2}\vec{v}\overbrace{\vec{v}^T\vec{v}}^{\|\vec{v}\|^2}\vec{v}^T\vec{a}$$

$$= \frac{1}{\|\vec{v}\|^2}\vec{v}\vec{v}^T\vec{a} = H\vec{a} \Rightarrow HH = H \text{ "idempotency"}$$

Let's do $\vec{\ell} = \text{Proj}_V(\vec{a})$ where $V = [\vec{v}_1 | \vec{v}_2 | \ldots | \vec{v}_K] \in \mathbb{R}^{d\times K}$, full rank

We know $\vec{\ell} \in \text{Colsp}[V] \Rightarrow \vec{\ell} = V\vec{w}$ why?

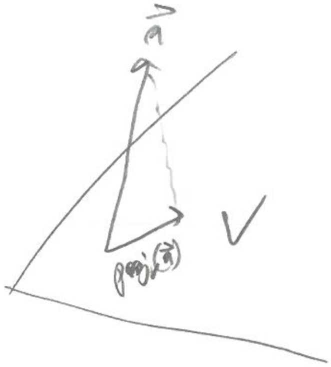
where $\vec{w} = [w_1 \ldots w_K]$ vector of

$H\vec{v} = \|\vec{e}\|\,\vec{v}$ when $\|\vec{e}\| = 1$ $\Rightarrow$ $\vec{v}$ is an eigenvector of H
and its eigenvalue $\lambda = 1$.

Other eigenvectors?

$H\vec{w} = \lambda\vec{w}$ No... since all $H\vec{w} = c\vec{v}$ if $\vec{w} \neq \vec{v} \Rightarrow c = 0$.

$\mathrm{rank}(H) = \#\{\text{non-zero eigenvalues}\} = 1$



$\forall \vec{v}_j \quad \vec{a} - Proj_V(\vec{a}) \perp \vec{v}_j$ why?

$$\Rightarrow \left(\vec{a} - V\vec{w}\right) \cdot \vec{v}_j = 0$$

$$\vec{v}_1^T(\vec{a} - V\vec{w}) = 0,$$
$$\vec{v}_2^T(\vec{a} - V\vec{w}) = 0$$
$$\vdots$$
$$\vec{v}_K^T(\vec{a} - V\vec{w}) = 0$$

$$\Rightarrow \begin{bmatrix} \leftarrow \vec{v}_1^T \rightarrow \\ \leftarrow \vec{v}_2^T \rightarrow \\ \vdots \\ \leftarrow \vec{v}_K^T \rightarrow \end{bmatrix} (\vec{a} - V\vec{w}) = \vec{0}_K$$

$$\Rightarrow V^T(\vec{a} - V\vec{w}) = \vec{0}_K$$

$$\Rightarrow V^T\vec{a} - V^TV\vec{w} = \vec{0}_K$$

$$\Rightarrow V^T\vec{a} = V^TV\vec{w}$$

$$\Rightarrow \vec{w} = (V^TV)^{-1}V^T\vec{a}$$

$$Proj_V(\vec{a}) = V\vec{w} = \underbrace{V(V^TV)^{-1}}_{H} V^T\vec{a} = H\vec{a}$$

Note: if $V = \begin{bmatrix} \uparrow \\ \vec{v} \\ \downarrow \end{bmatrix} \Rightarrow H = \vec{v}(\vec{v}^T\vec{v})^{-1}\vec{v}^T = \dfrac{\vec{v}\vec{v}^T}{\|\vec{v}\|^2}$ same as before

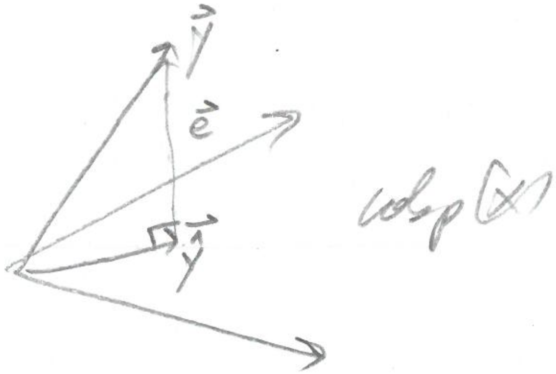
<u>Properties of Projection Matrices</u>

① H is symmetric $\left(V(V^TV)^{-1}V^T\right)^T = (V^T)^T\left((V^TV)^{-1}\right)^T V^T = V(V^TV)^{-1}V^T$ ✓

A symmetric, Is inverse symmetric? $A^{-1}A = I \Rightarrow A^T(A^{-1})^T = I^T = I \Rightarrow A(A^{-1})^T = I \Rightarrow (A^{-1})^T = A^{-1}$

② Idempotent

$$\left(V(V^TV)^{-1}V^T\right)\left(V(V^TV)^{-1}V^T\right) = H$$



$\text{rank}(H) = p+1$

What are the residuals?

$$\vec{e} = \vec{y} - \hat{\vec{y}} = \vec{y} - H\vec{y} = (I-H)\vec{y}$$

In OLS $\vec{e} \perp \hat{\vec{y}}$ Proof? $\left((I-H)\vec{y}\right)\cdot(H\vec{y})$

$$= (\vec{y} - H\vec{y})^T H\vec{y}$$

$$= (\vec{y}^T - \vec{y}^T H^T) H\vec{y}$$

$$= \vec{y}^T H\vec{y} - \vec{y}^T H\vec{y} = 0$$

$H^T = H$
$HH = H$

Is $I-H$ a projection matrix?

① $(I-H)^T = I^T - H^T = I - H$ symmetric

② $(I-H)(I-H) = (I - H - H + \overset{H}{\overset{\|}{HH}}) = (I-H)$ idempotent

$\Rightarrow$ Yes

What is it projecting onto?

$$(I-H)\vec{e} = \vec{e}$$

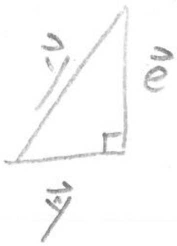
span of all vectors $\perp X$.

$\uparrow$ projection onto residual space i.e. $\text{Colsp}[X^\perp]$

Since $\text{span}[X] + \text{span}[X^\perp] = \mathbb{R}^n$

$\Rightarrow \text{rank}[X^\perp] = n-(p+1)$ since $\text{rank}(X) = p+1$ (the OLS assumption)

"d.o.f. of the residuals"

Since 

$\Rightarrow \|\vec{y}\|^2 = \|\hat{\vec{y}}\|^2 + \|\vec{e}\|^2$ by Pythagorean thm

Is this true? 

$\vec{y} - \bar{y} = \vec{y} - \vec{1}_n \bar{y}$

is $\vec{1}_n \in \text{Colsp}[X]$? Yes It gets projected onto itself!

$H(\vec{y} - \vec{1}_n \bar{y}) = H\vec{y} - \bar{y} H \vec{1}_n = \hat{\vec{y}} - \vec{1}_n \bar{y}$

Does $\vec{e} = \vec{y} - \hat{\vec{y}} = \vec{y} - \bar{y} - \hat{\vec{y}} + \bar{y} = (\vec{y} - \bar{y}) - (\hat{\vec{y}} - \bar{y})$ ✓

$$\Rightarrow \underbrace{\|\vec{y} - \bar{y}\|^2}_{SST} = \underbrace{\|\hat{\vec{y}} - \bar{y}\|^2}_{SSR} + \underbrace{\|\vec{e}\|^2}_{SSE} = \sum e_i^2$$

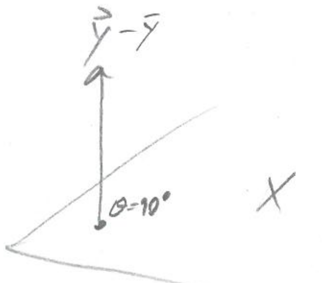
$// \quad \sum (y_i - \bar{y})^2 \qquad // \quad \sum (\hat{y}_i - \bar{y})^2$

$SST = SSR + SSE$

"adj"

$R^2 = \frac{SSR}{SST} = \cos^2(\theta(\vec{y}, \hat{\vec{y}})) \in [0,1]$

"hyp"



$R^2 = 0$

R^2 high