Math 390.4 / 650.3 Spring 2019
Midterm Examination Two

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Thursday, April 18, 2019

Full Name ______________________________

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Instructions

This exam is 110 minutes and closed-book. You are allowed **one** page (front and back) of a "cheat sheet." You may use a graphing calculator of your choice. Please read the questions carefully. If the question reads "compute," this means the solution will be a number otherwise you can leave the answer in *any* widely accepted mathematical notation which could be resolved to an exact or approximate number with the use of a computer. I advise you to skip problems marked "[Extra Credit]" until you have finished the other questions on the exam, then loop back and plug in all the holes. I also advise you to use pencil. The exam is 100 points total plus extra credit. Partial credit will be granted for incomplete answers on most of the questions. $\boxed{\text{Box}}$ in your final answers. Good luck!

Problem 1 Below are some theoretical questions related to OLS.

(a) [5 pt / 5 pts] Let $\boldsymbol{w} \in \mathbb{R}^{p+1}$ be a vector, let $\boldsymbol{y} \in \mathbb{R}^n$ be a vector constant with respect to $\boldsymbol{w}$ and let $\boldsymbol{X} \in \mathbb{R}^{n \times (p+1)}$ be a full-rank matrix constant with respect to $\boldsymbol{w}$. Find the $\boldsymbol{w}$ that solves the following equation by showing all steps:

$$\frac{\partial}{\partial \boldsymbol{w}} \left[\boldsymbol{y}^\top \boldsymbol{y} - 2\boldsymbol{w}\boldsymbol{X}^\top \boldsymbol{y} + \boldsymbol{w}^\top \boldsymbol{X}^\top \boldsymbol{X}\boldsymbol{w} \right] = 0$$

Problem 2 We continue now with questions related to OLS. Let $\boldsymbol{b} \in \mathbb{R}^{p+1}$ be the vector found in 1(a), let $\boldsymbol{y} \in \mathbb{R}^n$ be a constant vector, let $\boldsymbol{X} \in \mathbb{R}^{n \times (p+1)}$ be a constant full-rank matrix where the first column equals $\mathbf{1}_n$ and let $\boldsymbol{H}$ be the orthogonal projection matrix that we spoke about in class. Let $\hat{\boldsymbol{y}} \in \mathbb{R}^n$ be the orthogonal projection of $\boldsymbol{y}$ using $\boldsymbol{H}$ and let $\boldsymbol{e} \in \mathbb{R}^n$ be the difference of $\boldsymbol{y}$ and its orthogonal projection using $\boldsymbol{H}$. Further, let SST:$= ||\boldsymbol{y} - \bar{y}||^2$, SSR:$= ||\hat{\boldsymbol{y}} - \bar{y}||^2$ and SSE:$= ||\boldsymbol{e}||^2$.

(a) [3 pt / 8 pts] Prove $\boldsymbol{H}$ is symmetric.

(b) [3 pt / 11 pts] Compute $\left|\left|Proj_{\text{colsp}[\boldsymbol{X}]}(\mathbf{1}_n)\right|\right|^2$. Justify each non-trivial step.

(c) [2 pt / 13 pts] Let θ be the angle between $\boldsymbol{y}$ and $\hat{\boldsymbol{y}}$. As the number of columns grows larger and $\boldsymbol{X}$ remains full rank, what value does $\cos(\theta)$ converge to?

(d) [2 pt / 15 pts] Let $\boldsymbol{Q}$ denote $\boldsymbol{X}$ orthogonalized using the Gram-Schmidt algorithm. What is the dimension of $\boldsymbol{Q}$?

(e) [3 pt / 18 pts] Let $\boldsymbol{q}_{\cdot j}$ denote the jth column of $\boldsymbol{Q}$. Find $\boldsymbol{q}_{\cdot 1}$.

(f) [3 pt / 21 pts] Find $\left|\left|\boldsymbol{q}_{\cdot 3}^\top \boldsymbol{H}\right|\right|^2$.

(g) [3 pt / 24 pts] If you add one column to $\boldsymbol{X}$ and it remains full rank and recompute $\hat{\boldsymbol{y}}$, circle all quantities below that change:

i) n

ii) p

iii) $\boldsymbol{b}$

iv) SST

v) SSR

vi) SSE

vii) $\dim[\boldsymbol{H}]$

viii) $\text{rank}[\boldsymbol{H}]$

ix) $Proj_{\text{colsp}[\boldsymbol{X}]}(\boldsymbol{y})$

Problem 3

This question is about modeling price of cars in the `cars` dataset:

```
> dim(cars)
[1] 93 27
> summary(cars)
    Manufacturer      Model          Type      Min.Price         Price      
 Chevrolet: 8    100     : 1   Compact:16   Min.   : 6.70   Min.   : 7.40  
 Ford     : 8    190E    : 1   Large  :11   1st Qu.:10.80   1st Qu.:12.20  
 Dodge    : 6    240     : 1   Midsize:22   Median :14.70   Median :17.70  
 Mazda    : 5    300E    : 1   Small  :21   Mean   :17.13   Mean   :19.51  
 Pontiac  : 5    323     : 1   Sporty :14   3rd Qu.:20.30   3rd Qu.:23.30  
 Buick    : 4    535i    : 1   Van    : 9   Max.   :45.40   Max.   :61.90  
 (Other)  :57    (Other):87                                                
   Max.Price        MPG.city      MPG.highway                   AirBags  
 Min.   : 7.9   Min.   :15.00   Min.   :20.00   Driver & Passenger:16  
 1st Qu.:14.7   1st Qu.:18.00   1st Qu.:26.00   Driver only       :43  
 Median :19.6   Median :21.00   Median :28.00   None              :34  
 Mean   :21.9   Mean   :22.37   Mean   :29.09                          
 3rd Qu.:25.3   3rd Qu.:25.00   3rd Qu.:31.00                          
 Max.   :80.0   Max.   :46.00   Max.   :50.00                          

 DriveTrain  Cylinders    EngineSize      Horsepower         RPM      
 4WD  :10   3      : 3   Min.   :1.000   Min.   : 55.0   Min.   :3800  
 Front:67   4      :49   1st Qu.:1.800   1st Qu.:103.0   1st Qu.:4800  
 Rear :16   5      : 2   Median :2.400   Median :140.0   Median :5200  
            6      :31   Mean   :2.668   Mean   :143.8   Mean   :5281  
            8      : 7   3rd Qu.:3.300   3rd Qu.:170.0   3rd Qu.:5750  
            rotary: 1    Max.   :5.700   Max.   :300.0   Max.   :6500  

  Rev.per.mile  Man.trans.avail Fuel.tank.capacity   Passengers   
 Min.   :1320   No :32          Min.   : 9.20      Min.   :2.000  
 1st Qu.:1985   Yes:61          1st Qu.:14.50      1st Qu.:4.000  
 Median :2340                   Median :16.40      Median :5.000  
 Mean   :2332                   Mean   :16.66      Mean   :5.086  
 3rd Qu.:2565                   3rd Qu.:18.80      3rd Qu.:6.000  
 Max.   :3755                   Max.   :27.00      Max.   :8.000  

     Length        Wheelbase         Width        Turn.circle    Rear.seat.room
 Min.   :141.0   Min.   : 90.0   Min.   :60.00   Min.   :32.00   Min.   :19.00  
 1st Qu.:174.0   1st Qu.: 98.0   1st Qu.:67.00   1st Qu.:37.00   1st Qu.:26.00  
 Median :183.0   Median :103.0   Median :69.00   Median :39.00   Median :27.50  
 Mean   :183.2   Mean   :103.9   Mean   :69.38   Mean   :38.96   Mean   :27.83  
 3rd Qu.:192.0   3rd Qu.:110.0   3rd Qu.:72.00   3rd Qu.:41.00   3rd Qu.:30.00  
 Max.   :219.0   Max.   :119.0   Max.   :78.00   Max.   :45.00   Max.   :36.00  
```

```
                                                                                   NAs    :2
  Luggage.room        Weight         Origin                  Make
 Min.    : 6.00   Min.    :1695   USA      :48    Acura Integra: 1
 1st Qu.:12.00   1st Qu.:2620   non-USA:45    Acura Legend : 1
 Median :14.00   Median :3040                 Audi 100     : 1
 Mean   :13.89   Mean   :3073                 Audi 90      : 1
 3rd Qu.:15.00   3rd Qu.:3525                 BMW 535i     : 1
 Max.   :22.00   Max.   :4105                 Buick Century: 1
 NAs    :11                                  (Other)      :87
```

Below are the outputs for a few different OLS models for variable `price`:

Model 1
```
(Intercept)   TypeLarge TypeMidsize   TypeSmall TypeSporty     TypeVan
  18.212500    6.087500    9.005682   -8.045833   1.180357    0.887500
```

Model 2
```
TypeCompact   TypeLarge TypeMidsize   TypeSmall TypeSporty     TypeVan
   18.21250    24.30000    27.21818    10.16667   19.39286    19.10000
```

(a) [2 pt / 26 pts] What is the `R` code used to fit Model 1?

(b) [2 pt / 28 pts] Which model *most likely* has higher R^2?

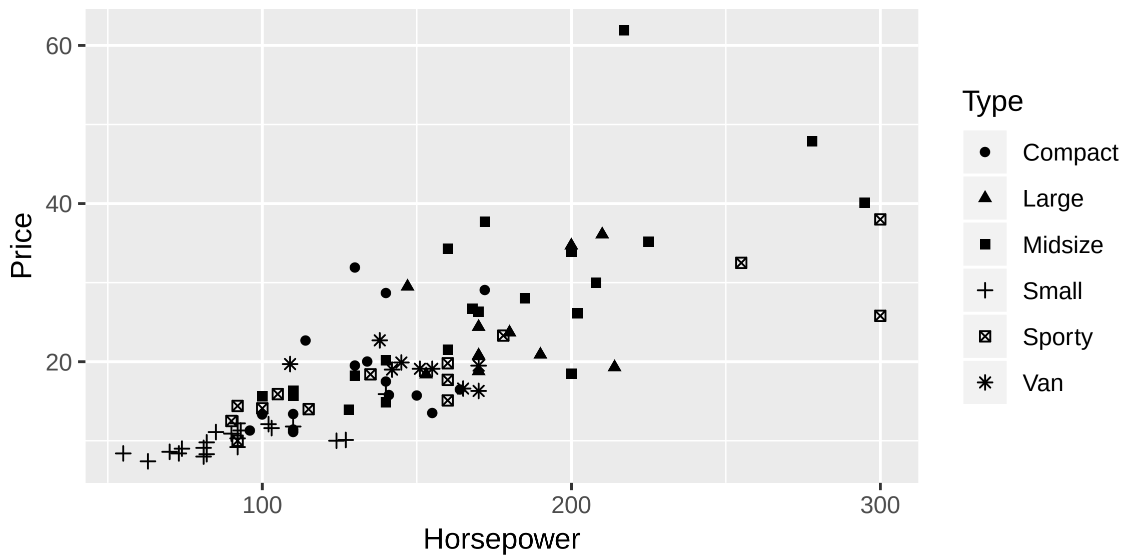
i) Model 1
ii) Model 2
iii) They have equal R^2
iv) Not enough information to tell

(c) [2 pt / 30 pts] Which model *most likely* has higher oosR^2?

i) Model 1
ii) Model 2
iii) They have equal oosR^2
iv) Not enough information to tell

(d) [4 pt / 34 pts] Assume the dataframe `cars` is sorted by variable `Type` in ascending alphabetical order of the factor level name. Find $\boldsymbol{X}^\top \boldsymbol{X}$ explicitly for Model 2.

Consider the following plot:



(e) [4 pt / 38 pts] Write ggplot code (as best as you can) to generate this figure.

Consider the following OLS linear model for target variable `price`:

Model 3

```
            (Intercept)             Horsepower
             1.45938866             0.12788635
              TypeLarge           TypeMidsize
             5.13487179            -4.98652796
              TypeSmall             TypeSporty
             2.42815602             2.23460382
                TypeVan  Horsepower:TypeLarge
            25.53605395            -0.02922214
 Horsepower:TypeMidsize  Horsepower:TypeSmall
             0.04973893            -0.05888501
  Horsepower:TypeSporty     Horsepower:TypeVan
            -0.02985597            -0.1807183
```

(f) [2 pt / 40 pts] What is the `R` code used to fit Model 3?

(g) [1 pt / 41 pts] Which model *most likely* has higher R^2?

i) Model 2

ii) Model 3

iii) They have equal R^2

iv) Not enough information to tell

(h) [2 pt / 43 pts] Which model *most likely* has higher oosR^2?

i) Model 2

ii) Model 3

iii) They have equal oosR^2

iv) Not enough information to tell

(i) [3 pt / 46 pts] Interpret the number -0.1807183 for term `Horsepower:TypeVan` in Model 3.

Problem 4 This question is about OLS again. For the questions concerned with out of sample, consider running the code using split-sample or gathering future data under stationarity. Consider the following code:

```
n = 100
x = runif(n, 0, 1)
X = cbind(1, x)
beta = c(1, 1)
delta = rnorm(n, mean = 0, sd = 0.1)
y = X %*% beta + delta

mod1 = lm(y ~ 0 + X)
```

(a) [2 pt / 48 pts] What is $f(x)$ in this case? f is defined as we did in class.

(b) [3 pt / 51 pts] Circle all the following that are true for `mod1`.

i) $\boldsymbol{b}$ will be very close to β

ii) $\boldsymbol{b}$ will not be very close to β

iii) s_e will be very small

iv) s_e will not be very small

v) ooss_e will be very small

vi) ooss_e will not be very small

Now consider running the following code after running the first chunk of code:

```
x_prime = x + rnorm(n, mean = 0, sd = 1e-6)
X = cbind(X, x_prime)
mod2 = lm(y ~ 0 + X)
```

(c) [1 pt / 52 pts] In the case of model 2, what is p?

(d) [4 pt / 56 pts] Circle all the following that are true for `mod2`.

i) $\boldsymbol{b}$ will be very close to β

ii) $\boldsymbol{b}$ will not be very close to β

iii) s_e will be very small

iv) s_e will not be very small

v) ooss_e will be very small

vi) ooss_e will not be very small

Now consider running the following code after running the two previous chunks of code:

```
mod3 = lm(y ~ poly(x, 6))
```

(e) [4 pt / 60 pts] Circle all the following that are true for `mod3`.

i) $\boldsymbol{b}$ will be very close to β

ii) $\boldsymbol{b}$ will not be very close to β

iii) s_e will be very small

iv) s_e will not be very small

v) ooss_e will be very small

vi) ooss_e will not be very small

Problem 5 This question is about the concept of model validation and the strategy we discussed in class. Let's say we divide scramble the rows of $\mathbb{D}$ then create a partition

$$\mathbb{D} = \begin{bmatrix} \mathbb{D}_{\text{train}} \\ \hline \mathbb{D}_{\text{select}} \\ \hline \mathbb{D}_{\text{test}} \end{bmatrix}$$

in a 3:1:1 ratio train : select : test (in number of rows).

We then fit $g_1 = \mathcal{A}(\mathcal{H}, \mathbb{D}_{\text{train}})$, $g_2 = \mathcal{A}(\mathcal{H}, \mathbb{D}_{\text{test}})$ and $g_{\text{final}} = \mathcal{A}(\mathcal{H}, \mathbb{D})$. Which of the following statement(s) can be employed as a means of *honest* model validation?

(a) [3 pt / 63 pts] We wish to select a model out of M candidate models $g_1, g_2, \ldots, g_M$. Which of the following are recommended strategies of doing so?

i) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}_{\text{train}}$ and then choosing the model with lowest error on $\mathbb{D}_{\text{train}}$.

ii) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}$ and then choosing the model with lowest error on $\mathbb{D}$.

iii) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}_{\text{select}}$ and then choosing the model with lowest error on $\mathbb{D}_{\text{select}}$.

iv) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}_{\text{select}}$ and then $\mathbb{D}_{\text{test}}$ and then choosing the model with lowest error on $\mathbb{D}_{\text{test}}$.

(b) [3 pt / 66 pts] We wish to select a model out of M candidate models $g_1, g_2, \ldots, g_M$ and then provide an estimate of model generalization error. Which of the following are recommended strategies of doing so?

i) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}_{\text{select}}$ and then choosing the model with lowest error on $\mathbb{D}_{\text{select}}$ and providing the estimate of that error.

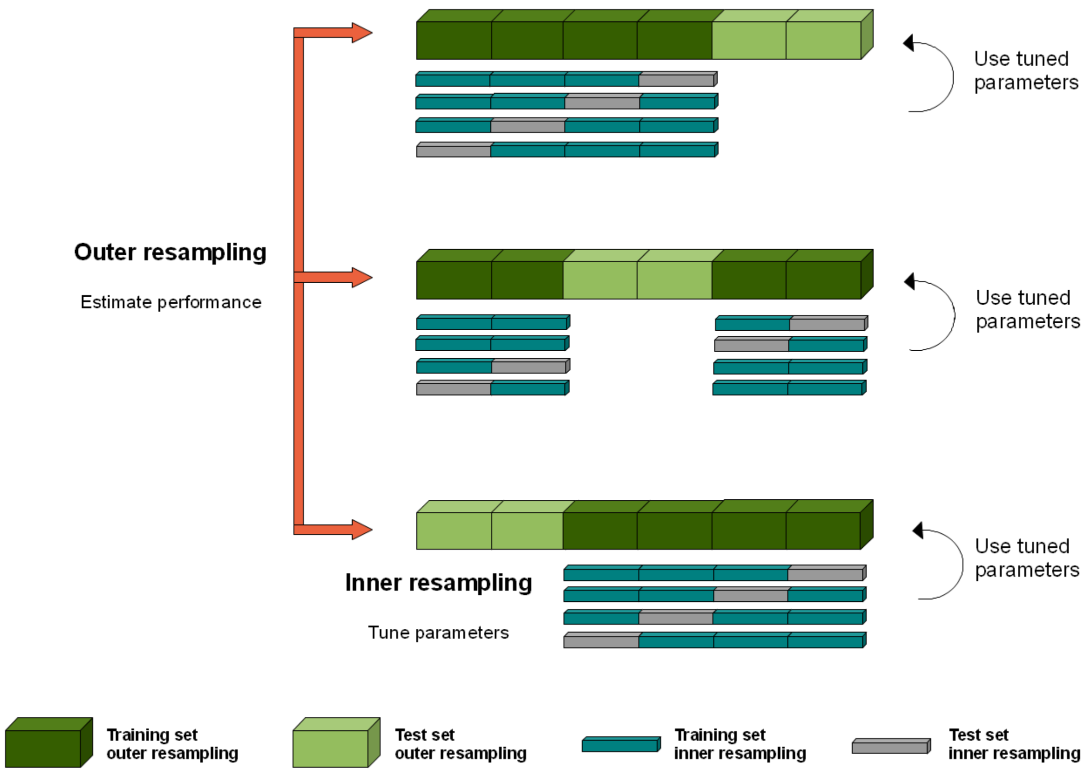
ii) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}_{\text{select}}$ and then choosing the model with lowest error on $\mathbb{D}_{\text{select}}$ and then testing on $\mathbb{D}$ and providing the estimate using that error.

iii) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}_{\text{select}}$ and then choosing the model with lowest error on $\mathbb{D}_{\text{select}}$ and then testing on $\mathbb{D}_{\text{test}}$ and providing the estimate using that error.

iv) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}_{\text{select}}$ and then $\mathbb{D}_{\text{test}}$ and then choosing the model with lowest error on $\mathbb{D}_{\text{test}}$ and providing the estimate using that error.

(c) [2 pt / 68 pts] Would your answer in (b) be able to provide an estimate of the variability in the generalization error? Yes / No.

Consider the selection of the model $g_1, g_2, \ldots, g_M$ to be termed "tuning". Imagine we used the protocol pictured below.



(d) [3 pt / 71 pts] What are the number of folds in the inner loop and the outer loop *in our problem* respecting the ratio given in the problem description?

(e) [4 pt / 75 pts] What are the two main advantages of the protocol above over the answer you gave in (b)?

Problem 6 Consider the following code:

```
compute_distance_matrix = function(X){
  n = nrow(X)
  D = matrix(NA, n, n)
  for (i_1 in 1 : (n - 1)){
    for (i_2 in (i_1 + 1) : n){
      D[i_1, i_2] = sqrt(sum((X[i_1, ] - X[i_2, ])^2))
    }
  }
  D
}

pacman::p_load(Rcpp)
cppFunction('
  NumericMatrix compute_distance_matrix_cpp(NumericMatrix X) {
    int n = X.nrow();
    int p = X.ncol();
    NumericMatrix D(n, n);
    std::fill(D.begin(), D.end(), NA_REAL);

    for (int i_1 = 0; i_1 < (n - 1); i_1++){
      for (int i_2 = i_1 + 1; i_2 < n; i_2++){
        int sqd_diff = 0;
        for (int j = 0; j < p; j++){
          sqd_diff += pow(X(i_1, j) - X(i_2, j), 2);
        }
        D(i_1, i_2) = sqrt(sqd_diff);
      }
    }
    return D;
  }
')
```

We now profile both functions using a matrix $\boldsymbol{X}$ that has n in the 100's via the code:

```
system.time({
  D = compute_distance_matrix(X)
})
system.time({
  D = compute_distance_matrix_cpp(X)
})
```

(a) [2 pt / 77 pts] Which function registers a faster profiling time and by how much? Provide a multiple.

(b) [2 pt / 79 pts] Explain why this should be.

(c) [2 pt / 81 pts] You wish to recode the `R` function `sort` using `Rcpp`. Assume your C++ code is bug-free. Is this endeavor fruitful? Why or why not?

Problem 7 Consider the following dataset:

```
> pacman::p_load(ggplot2, dplyr, magrittr)
> D = ggplot2::txhousing
> dim(D)
[1] 8602    9
> summary(D)
     city                 year          month
 Length:8602         Min.   :2000   Min.   : 1.000
 Class :character    1st Qu.:2003   1st Qu.: 3.000
 Mode  :character    Median :2007   Median : 6.000
```

```
                          Mean    :2007    Mean    : 6.406
                          3rd Qu.:2011    3rd Qu.: 9.000
                          Max.    :2015    Max.    :12.000

     sales              volume              median
 Min.   :   6.0   Min.   :8.350e+05   Min.   : 50000
 1st Qu.:  86.0   1st Qu.:1.084e+07   1st Qu.:100000
 Median : 169.0   Median :2.299e+07   Median :123800
 Mean   : 549.6   Mean   :1.069e+08   Mean   :128131
 3rd Qu.: 467.0   3rd Qu.:7.512e+07   3rd Qu.:150000
 Max.   :8945.0   Max.   :2.568e+09   Max.   :304200
 NAs    :568      NAs    :568         NAs    :616
    listings        inventory          date
 Min.   :    0   Min.   : 0.000   Min.   :2000
 1st Qu.:  682   1st Qu.: 4.900   1st Qu.:2004
 Median : 1283   Median : 6.200   Median :2008
 Mean   : 3217   Mean   : 7.175   Mean   :2008
 3rd Qu.: 2954   3rd Qu.: 8.150   3rd Qu.:2012
 Max.   :43107   Max.   :55.900   Max.   :2016
 NAs    :1424    NAs    :1467
```

(a) [2 pt / 83 pts] Write `dplyr` code below to update `D` to convert the city variable into a nominal factor variable.

(b) [5 pt / 88 pts] Write `dplyr` code below to update `D` to create a new character variable called `month_date` which has a string timestamp with format MM/YYYY or M/YYYY, then sort by date (earliest first) and then drop columns `month`, `year` and `date`.

(c) [2 pt / 90 pts] Write `dplyr` code below to "windsorize" `D` on the `volume` variable. This means it will only contain rows that are between the 5%ile and 95%ile of volumes.

(d) [3 pt / 93 pts] Write `dplyr` code below to summarize the data in `D` by providing the average volume in each month.

We now wish to predict the target `volume` based on the other variables as features. Consider the following code after the first chunk has been executed:

```
> D %<>% na.omit
> pacman::p_load(mlr)
> modeling_task = makeRegrTask(data = D, target = "volume")
> algorithm = makeLearner("regr.lm")
> validation = makeResampleDesc("CV", iters = 5)
> resample(algorithm, modeling_task, validation, measures = list(rmse))$aggr
34120325
```

(e) [3 pt / 96 pts] Interpret the output, 34120325, as best as you can.

(f) [2 pt / 98 pts] What simple transformation can be done to one of the variables in the dataset that would likely increase predictive performance?

Consider the following code:

```
> X = model.matrix(volume ~ . * . * ., D)
```

(g) [2 pt / 100 pts] In one sentence (or less) answer the following: which procedure could you use to build a model predicting `volume` based on the features now found in the design matrix `X`?