# Math 390.4 / 650.3 Spring 2019
# Midterm Examination Two

### Professor Adam Kapelner

### Thursday, April 18, 2019

Full Name _____

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signature                                                    date

## Instructions

This exam is 110 minutes and closed-book. You are allowed **one** page (front and back) of a "cheat sheet." You may use a graphing calculator of your choice. Please read the questions carefully. If the question reads "compute," this means the solution will be a number otherwise you can leave the answer in *any* widely accepted mathematical notation which could be resolved to an exact or approximate number with the use of a computer. I advise you to skip problems marked "[Extra Credit]" until you have finished the other questions on the exam, then loop back and plug in all the holes. I also advise you to use pencil. The exam is 100 points total plus extra credit. Partial credit will be granted for incomplete answers on most of the questions. Box in your final answers. Good luck!

**Problem 1** Below are some theoretical questions related to OLS.

(a) [5 pt / 5 pts]   Let $\boldsymbol{w} \in \mathbb{R}^{p+1}$ be a vector, let $\boldsymbol{y} \in \mathbb{R}^n$ be a vector constant with respect to $\boldsymbol{w}$ and let $\boldsymbol{X} \in \mathbb{R}^{n \times (p+1)}$ be a full-rank matrix constant with respect to $\boldsymbol{w}$. Find the $\boldsymbol{w}$ that solves the following equation by showing all steps:

$$\frac{\partial}{\partial \boldsymbol{w}} \left[ \boldsymbol{y}^\top \boldsymbol{y} - 2\boldsymbol{w} \boldsymbol{X}^\top \boldsymbol{y} + \boldsymbol{w}^\top \boldsymbol{X}^\top \boldsymbol{X} \boldsymbol{w} \right] = 0$$

**Problem 2** We continue now with questions related to OLS. Let $\boldsymbol{b} \in \mathbb{R}^{p+1}$ be the vector found in 1(a), let $\boldsymbol{y} \in \mathbb{R}^n$ be a constant vector, let $\boldsymbol{X} \in \mathbb{R}^{n \times (p+1)}$ be a constant full-rank matrix where the first column equals $\boldsymbol{1}_n$ and let $\boldsymbol{H}$ be the orthogonal projection matrix that we spoke about in class. Let $\hat{\boldsymbol{y}} \in \mathbb{R}^n$ be the orthogonal projection of $\boldsymbol{y}$ using $\boldsymbol{H}$ and let $\boldsymbol{e} \in \mathbb{R}^n$ be the difference of $\boldsymbol{y}$ and its orthogonal projection using $\boldsymbol{H}$. Further, let SST:= $||\boldsymbol{y} - \bar{y}||^2$, SSR:= $||\hat{\boldsymbol{y}} - \bar{y}||^2$ and SSE:= $||\boldsymbol{e}||^2$.

(a) [3 pt / 8 pts]   Prove $\boldsymbol{H}$ is symmetric.

(b) [3 pt / 11 pts]   Compute $\left\| Proj_{\text{colsp}[\boldsymbol{X}]}(\mathbf{1}_n) \right\|^2$. Justify each non-trivial step.

(c) [2 pt / 13 pts]   Let $\theta$ be the angle between $\boldsymbol{y}$ and $\hat{\boldsymbol{y}}$. As the number of columns grows larger and $\boldsymbol{X}$ remains full rank, what value does $\cos(\theta)$ converge to?

(d) [2 pt / 15 pts]   Let $\boldsymbol{Q}$ denote $\boldsymbol{X}$ orthogonalized using the Gram-Schmidt algorithm. What is the dimension of $\boldsymbol{Q}$?

(e) [3 pt / 18 pts]   Let $\boldsymbol{q}_{.j}$ denote the $j$th column of $\boldsymbol{Q}$. Find $\boldsymbol{q}_{.1}$.

(f) [3 pt / 21 pts]   Find $\left\| \boldsymbol{q}_{.3}^{\top} \boldsymbol{H} \right\|^2$.

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(g) [3 pt / 24 pts]    If you add one column to $\boldsymbol{X}$ and it remains full rank and recompute $\hat{\boldsymbol{y}}$, circle all quantities below that change:

    i) $n$

    ii) $p$

    iii) $\boldsymbol{b}$

    iv) SST

    v) SSR

    vi) SSE

    vii) $\dim[\boldsymbol{H}]$

    viii) $\operatorname{rank}[\boldsymbol{H}]$

    ix) $Proj_{\operatorname{colsp}[\boldsymbol{X}]}(\boldsymbol{y})$

**Problem 3**  This question is about modeling price of cars in the `cars` dataset:

```
1  > dim(cars)
2  [1] 93 27
3  > summary(cars)
4      Manufacturer       Model          Type         Min.Price         Price
5   Chevrolet: 8    100    : 1    Compact:16    Min.   : 6.70   Min.   : 7.40
6   Ford     : 8    190E   : 1    Large  :11    1st Qu.:10.80   1st Qu.:12.20
7   Dodge    : 6    240    : 1    Midsize:22    Median :14.70   Median :17.70
8   Mazda    : 5    300E   : 1    Small  :21    Mean   :17.13   Mean   :19.51
9   Pontiac  : 5    323    : 1    Sporty :14    3rd Qu.:20.30   3rd Qu.:23.30
10  Buick    : 4    535i   : 1    Van    : 9    Max.   :45.40   Max.   :61.90
11  (Other)  :57    (Other):87
12     Max.Price          MPG.city        MPG.highway              AirBags
13  Min.   : 7.9    Min.   :15.00   Min.   :20.00   Driver & Passenger:16
14  1st Qu.:14.7    1st Qu.:18.00   1st Qu.:26.00   Driver only       :43
15  Median :19.6    Median :21.00   Median :28.00   None              :34
16  Mean   :21.9    Mean   :22.37   Mean   :29.09
17  3rd Qu.:25.3    3rd Qu.:25.00   3rd Qu.:31.00
18  Max.   :80.0    Max.   :46.00   Max.   :50.00
19
20  DriveTrain   Cylinders       EngineSize        Horsepower          RPM
21  4WD  :10    3     : 3    Min.   :1.000   Min.   : 55.0   Min.   :3800
22  Front:67    4     :49    1st Qu.:1.800   1st Qu.:103.0   1st Qu.:4800
23  Rear :16    5     : 2    Median :2.400   Median :140.0   Median :5200
24              6     :31    Mean   :2.668   Mean   :143.8   Mean   :5281
25              8     : 7    3rd Qu.:3.300   3rd Qu.:170.0   3rd Qu.:5750
26              rotary: 1    Max.   :5.700   Max.   :300.0   Max.   :6500
27
28   Rev.per.mile   Man.trans.avail  Fuel.tank.capacity   Passengers
29  Min.   :1320    No :32          Min.   : 9.20    Min.   :2.000
30  1st Qu.:1985    Yes:61          1st Qu.:14.50    1st Qu.:4.000
31  Median :2340                    Median :16.40    Median :5.000
32  Mean   :2332                    Mean   :16.66    Mean   :5.086
33  3rd Qu.:2565                    3rd Qu.:18.80    3rd Qu.:6.000
34  Max.   :3755                    Max.   :27.00    Max.   :8.000
35
36      Length          Wheelbase        Width        Turn.circle      Rear.seat.room
37  Min.   :141.0   Min.   : 90.0   Min.   :60.00   Min.   :32.00   Min.   :19.00
38  1st Qu.:174.0   1st Qu.: 98.0   1st Qu.:67.00   1st Qu.:37.00   1st Qu.:26.00
39  Median :183.0   Median :103.0   Median :69.00   Median :39.00   Median :27.50
40  Mean   :183.2   Mean   :103.9   Mean   :69.38   Mean   :38.96   Mean   :27.83
41  3rd Qu.:192.0   3rd Qu.:110.0   3rd Qu.:72.00   3rd Qu.:41.00   3rd Qu.:30.00
42  Max.   :219.0   Max.   :119.0   Max.   :78.00   Max.   :45.00   Max.   :36.00
```

4

```
43                                                          NAs    :2
44    Luggage.room          Weight          Origin               Make
45   Min.   : 6.00    Min.   :1695    USA    :48    Acura Integra: 1
46   1st Qu.:12.00    1st Qu.:2620    non-USA:45    Acura Legend : 1
47   Median :14.00    Median :3040                  Audi 100     : 1
48   Mean   :13.89    Mean   :3073                  Audi 90      : 1
49   3rd Qu.:15.00    3rd Qu.:3525                  BMW 535i     : 1
50   Max.   :22.00    Max.   :4105                  Buick Century: 1
51   NAs    :11                                     (Other)      :87
```

Below are the outputs for a few different OLS models for variable `price`:

| Model 1 | (Intercept) | TypeLarge | TypeMidsize | TypeSmall | TypeSporty | TypeVan |
|---|---|---|---|---|---|---|
| | 18.212500 | 6.087500 | 9.005682 | -8.045833 | 1.180357 | 0.887500 |

| Model 2 | TypeCompact | TypeLarge | TypeMidsize | TypeSmall | TypeSporty | TypeVan |
|---|---|---|---|---|---|---|
| | 18.21250 | 24.30000 | 27.21818 | 10.16667 | 19.39286 | 19.10000 |

(a) [2 pt / 26 pts]   What is the R code used to fit Model 1?

(b) [2 pt / 28 pts]   Which model *most likely* has higher $R^2$?

   i) Model 1

   ii) Model 2

   iii) They have equal $R^2$

   iv) Not enough information to tell

(c) [2 pt / 30 pts]   Which model *most likely* has higher $oosR^2$?

   i) Model 1

   ii) Model 2

   iii) They have equal $oosR^2$

   iv) Not enough information to tell

(d) [4 pt / 34 pts]   Assume the dataframe `cars` is sorted by variable `Type` in ascending alphabetical order of the factor level name. Find $\boldsymbol{X}^\top \boldsymbol{X}$ explicitly for Model 2.

Consider the following plot:
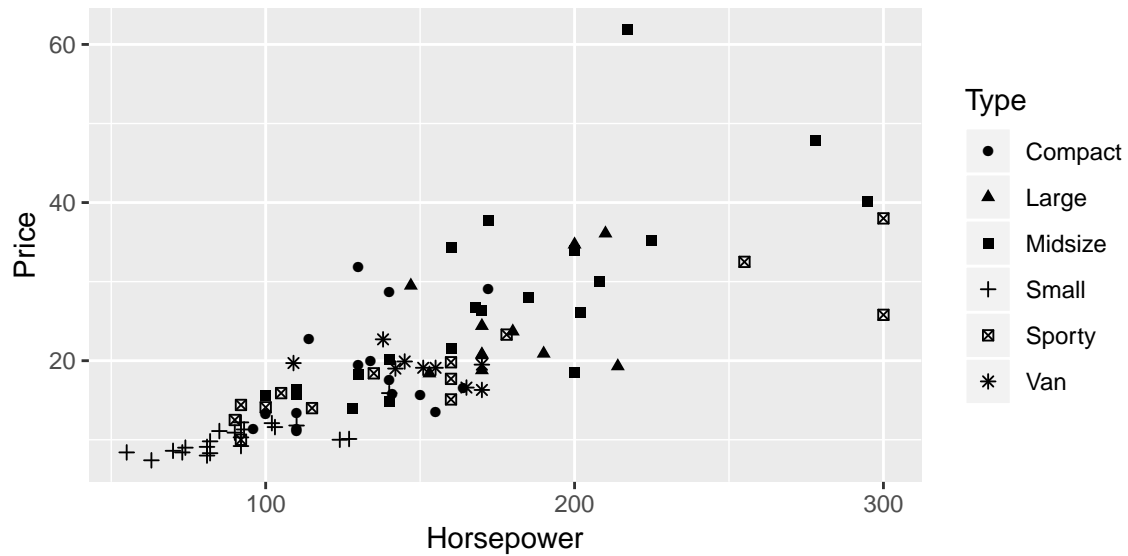


(e) [4 pt / 38 pts]   Write ggplot code (as best as you can) to generate this figure.

Consider the following OLS linear model for target variable `price`:

Model 3

| | (Intercept) | | Horsepower |
|---|---|---|---|
| | 1.45938866 | | 0.12788635 |
| | TypeLarge | | TypeMidsize |
| | 5.13487179 | | -4.98652796 |
| | TypeSmall | | TypeSporty |
| | 2.42815602 | | 2.23460382 |
| | TypeVan | | Horsepower:TypeLarge |
| | 25.53605395 | | -0.02922214 |
| Horsepower:TypeMidsize | | Horsepower:TypeSmall | |
| 0.04973893 | | -0.05888501 | |
| Horsepower:TypeSporty | | Horsepower:TypeVan | |
| -0.02985597 | | -0.1807183 | |

(f) [2 pt / 40 pts]   What is the R code used to fit Model 3?

(g) [1 pt / 41 pts]   Which model *most likely* has higher $R^2$?

    i) Model 2

    ii) Model 3

    iii) They have equal $R^2$

    iv) Not enough information to tell

(h) [2 pt / 43 pts]   Which model *most likely* has higher $oosR^2$?

    i) Model 2

    ii) Model 3

    iii) They have equal $oosR^2$

    iv) Not enough information to tell

(i) [3 pt / 46 pts]   Interpret the number -0.1807183 for term `Horsepower:TypeVan` in Model 3.

**Problem 4**  This question is about OLS again. For the questions concerned with out of sample, consider running the code using split-sample or gathering future data under stationarity. Consider the following code:

```
n = 100
x = runif(n, 0, 1)
X = cbind(1, x)
beta = c(1, 1)
delta = rnorm(n, mean = 0, sd = 0.1)
y = X %*% beta + delta

mod1 = lm(y ~ 0 + X)
```

(a) [2 pt / 48 pts]　　What is $f(x)$ in this case? $f$ is defined as we did in class.

(b) [3 pt / 51 pts]　　Circle all the following that are true for `mod1`.

　　　i) $b$ will be very close to $\beta$

　　　ii) $b$ will not be very close to $\beta$

　　　iii) $s_e$ will be very small

　　　iv) $s_e$ will not be very small

　　　v) $\text{ooss}_e$ will be very small

　　　vi) $\text{ooss}_e$ will not be very small

Now consider running the following code after running the first chunk of code:

```
1 x_prime = x + rnorm(n, mean = 0, sd = 1e-6)
2 X = cbind(X, x_prime)
3 mod2 = lm(y ~ 0 + X)
```

(c) [1 pt / 52 pts]　　In the case of model 2, what is $p$?

(d) [4 pt / 56 pts]　　Circle all the following that are true for `mod2`.

　　　i) $b$ will be very close to $\beta$

　　　ii) $b$ will not be very close to $\beta$

　　　iii) $s_e$ will be very small

　　　iv) $s_e$ will not be very small

　　　v) $\text{ooss}_e$ will be very small

　　　vi) $\text{ooss}_e$ will not be very small

Now consider running the following code after running the two previous chunks of code:

```
1 mod3 = lm(y ~ poly(x, 6))
```

(e) [4 pt / 60 pts]    Circle all the following that are true for `mod3`.

    i) $\boldsymbol{b}$ will be very close to $\beta$

    ii) $\boldsymbol{b}$ will not be very close to $\beta$

    iii) $s_e$ will be very small

    iv) $s_e$ will not be very small

    v) $\text{ooss}_e$ will be very small

    vi) $\text{ooss}_e$ will not be very small

**Problem 5** This question is about the concept of model validation and the strategy we discussed in class. Let's say we divide scramble the rows of $\mathbb{D}$ then create a partition

$$
\mathbb{D} = \left[ \begin{array}{c} \mathbb{D}_{\text{train}} \\ \hline \mathbb{D}_{\text{select}} \\ \hline \mathbb{D}_{\text{test}} \end{array} \right]
$$

in a 3:1:1 ratio train : select : test (in number of rows).

We then fit $g_1 = \mathcal{A}(\mathcal{H}, \mathbb{D}_{\text{train}})$, $g_2 = \mathcal{A}(\mathcal{H}, \mathbb{D}_{\text{test}})$ and $g_{\text{final}} = \mathcal{A}(\mathcal{H}, \mathbb{D})$. Which of the following statement(s) can be employed as a means of *honest* model validation?

(a) [3 pt / 63 pts]    We wish to select a model out of $M$ candidate models $g_1, g_2, \ldots, g_M$. Which of the following are recommended strategies of doing so?

    i) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}_{\text{train}}$ and then choosing the model with lowest error on $\mathbb{D}_{\text{train}}$.

    ii) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}$ and then choosing the model with lowest error on $\mathbb{D}$.

    iii) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}_{\text{select}}$ and then choosing the model with lowest error on $\mathbb{D}_{\text{select}}$.

    iv) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}_{\text{select}}$ and then $\mathbb{D}_{\text{test}}$ and then choosing the model with lowest error on $\mathbb{D}_{\text{test}}$.

(b) [3 pt / 66 pts]    We wish to select a model out of $M$ candidate models $g_1, g_2, \ldots, g_M$ and then provide an estimate of model generalization error. Which of the following are recommended strategies of doing so?

    i) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}_{\text{select}}$ and then choosing the model with lowest error on $\mathbb{D}_{\text{select}}$ and providing the estimate of that error.

    ii) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}_{\text{select}}$ and then choosing the model with lowest error on $\mathbb{D}_{\text{select}}$ and then testing on $\mathbb{D}$ and providing the estimate using that error.

iii) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}_{\text{select}}$ and then choosing the model with lowest error on $\mathbb{D}_{\text{select}}$ and then testing on $\mathbb{D}_{\text{test}}$ and providing the estimate using that error.

iv) Fitting $g_1, g_2, \ldots, g_M$ to $\mathbb{D}_{\text{train}}$ and then testing on $\mathbb{D}_{\text{select}}$ and then $\mathbb{D}_{\text{test}}$ and then choosing the model with lowest error on $\mathbb{D}_{\text{test}}$ and providing the estimate using that error.

(c) [2 pt / 68 pts]    Would your answer in (b) be able to provide an estimate of the variability in the generalization error? Yes / No.

Consider the selection of the model $g_1, g_2, \ldots, g_M$ to be termed "tuning". Imagine we used the protocol pictured below.
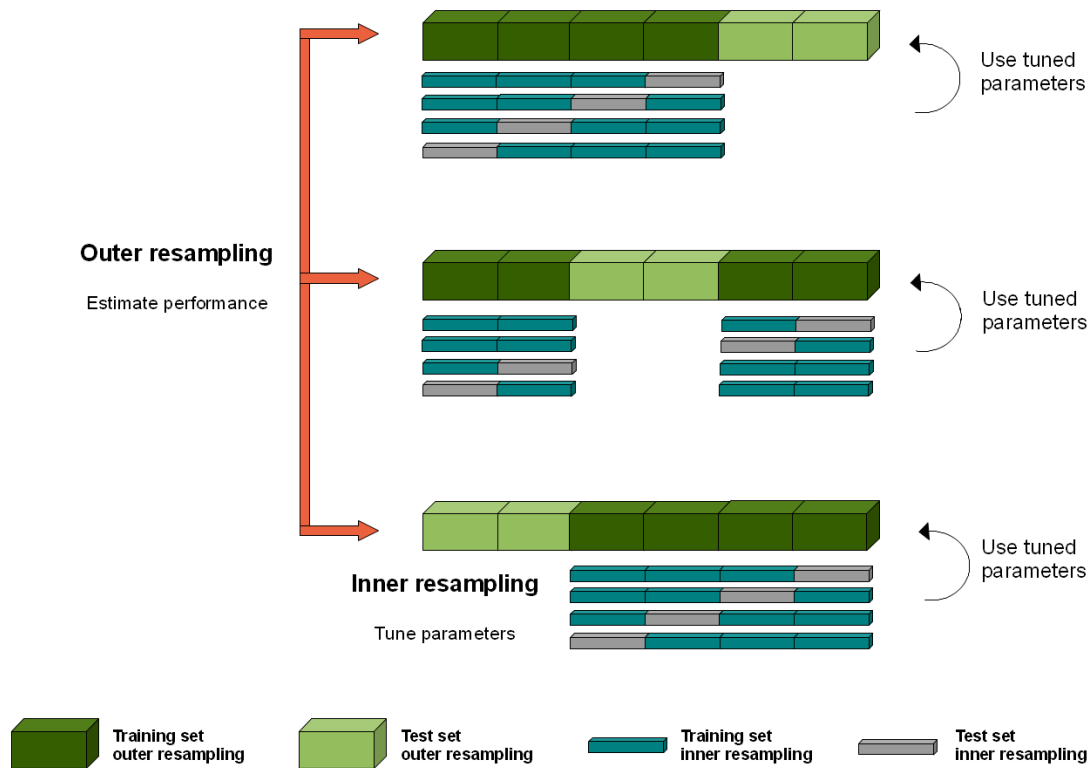


(d) [3 pt / 71 pts]    What are the number of folds in the inner loop and the outer loop *in our problem* respecting the ratio given in the problem description?

(e) [4 pt / 75 pts]    What are the two main advantages of the protocol above over the answer you gave in (b)?

## Problem 6   Consider the following code:

```
compute_distance_matrix = function(X){
    n = nrow(X)
    D = matrix(NA, n, n)
    for (i_1 in 1 : (n - 1)){
        for (i_2 in (i_1 + 1) : n){
            D[i_1, i_2] = sqrt(sum((X[i_1, ] - X[i_2, ])^2))
        }
    }
    D
}

pacman::p_load(Rcpp)
cppFunction('
    NumericMatrix compute_distance_matrix_cpp(NumericMatrix X) {
        int n = X.nrow();
        int p = X.ncol();
        NumericMatrix D(n, n);
        std::fill(D.begin(), D.end(), NA_REAL);

        for (int i_1 = 0; i_1 < (n - 1); i_1++){
            for (int i_2 = i_1 + 1; i_2 < n; i_2++){
                int sqd_diff = 0;
                for (int j = 0; j < p; j++){
                    sqd_diff += pow(X(i_1, j) - X(i_2, j), 2);
                }
                D(i_1, i_2) = sqrt(sqd_diff);
            }
        }
        return D;
    }
')
```

We now profile both functions using a matrix $X$ that has $n$ in the 100's via the code:

```
system.time({
  D = compute_distance_matrix(X)
})
system.time({
  D = compute_distance_matrix_cpp(X)
})
```

(a) [2 pt / 77 pts]  Which function registers a faster profiling time and by how much? Provide a multiple.

(b) [2 pt / 79 pts]  Explain why this should be.

(c) [2 pt / 81 pts]  You wish to recode the R function sort using Rcpp. Assume your C++ code is bug-free. Is this endeavor fruitful? Why or why not?

## Problem 7  Consider the following dataset:

```
> pacman::p_load(ggplot2, dplyr, magrittr)
> D = ggplot2::txhousing
> dim(D)
[1] 8602     9
> summary(D)
    city                 year          month
 Length:8602        Min.   :2000   Min.   : 1.000
 Class :character   1st Qu.:2003   1st Qu.: 3.000
 Mode  :character   Median :2007   Median : 6.000
```

```
10                    Mean    :2007    Mean    : 6.406
11                    3rd Qu.:2011    3rd Qu.: 9.000
12                    Max.    :2015    Max.    :12.000
13
14     sales              volume                median
15   Min.   :    6.0   Min.   :8.350e+05   Min.   :  50000
16   1st Qu.:   86.0   1st Qu.:1.084e+07   1st Qu.:100000
17   Median :  169.0   Median :2.299e+07   Median :123800
18   Mean   :  549.6   Mean   :1.069e+08   Mean   :128131
19   3rd Qu.:  467.0   3rd Qu.:7.512e+07   3rd Qu.:150000
20   Max.   : 8945.0   Max.   :2.568e+09   Max.   :304200
21   NAs    :568       NAs    :568         NAs    :616
22     listings          inventory              date
23   Min.   :     0   Min.   : 0.000   Min.   :2000
24   1st Qu.:   682   1st Qu.: 4.900   1st Qu.:2004
25   Median :  1283   Median : 6.200   Median :2008
26   Mean   :  3217   Mean   : 7.175   Mean   :2008
27   3rd Qu.:  2954   3rd Qu.: 8.150   3rd Qu.:2012
28   Max.   : 43107   Max.   :55.900   Max.   :2016
29   NAs    :1424     NAs    :1467
```

(a) [2 pt / 83 pts]    Write `dplyr` code below to update `D` to convert the city variable into a nominal factor variable.

(b) [5 pt / 88 pts]    Write `dplyr` code below to update `D` to create a new character variable called `month_date` which has a string timestamp with format MM/YYYY, then sort by date (earliest first) and then drop columns `month`, `year` and `date`.

(c) [2 pt / 90 pts]    Write `dplyr` code below to "windsorize" `D` on the `volume` variable. This means it will only contain rows that are between the 5%ile and 95%ile of volumes.

(d) [3 pt / 93 pts]    Write **dplyr** code below to summarize the data in **D** by providing the average volume in each month.

We now wish to predict the target **volume** based on the other variables as features. Consider the following code after the first chunk has been executed:

```
> D % <> % na.omit
> pacman::p_load(mlr)
> modeling_task = makeRegrTask(data = D, target = "volume")
> algorithm = makeLearner("regr.lm")
> validation = makeResampleDesc("CV", iters = 5)
> resample(algorithm, modeling_task, validation, measures = list(rmse))$
    aggr
34120325
```

(e) [3 pt / 96 pts]    Interpret the output, 34120325, as best as you can.

(f) [2 pt / 98 pts]    What simple transformation can be done to one of the variables in the dataset that would likely increase predictive performance?

Consider the following code:

```
> X = model.matrix(volume ~ . * . * ., D)
```

(g) [2 pt / 100 pts]    In one sentence (or less) answer the following: which procedure could you use to build a model predicting **volume** based on the features now found in the design matrix **X**?

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