Discrete Mathematics, 2016 Fall - Worksheet 16

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Instructor: Zsolt Pajor-Gyulai, CIMS

In all of the above problems explain your answer in full English sentences.

1. For each of the pair of functions below, determine which of $g \circ f$ and $f \circ g$ is defined. If one or both are defined, find the resulting functions. If both are defined, determine wheteher $g \circ f = f \circ g$.

   Clearly, $f \circ g \neq g \circ f$.

   (a) $f = \{(1,2),(2,3),(3,4)\}$ and $g = \{(1,3),(2,4),(3,1)\}$
   Note that Im(f) $= \{2,3,4\} \not\subseteq$ Dom(g) $= \{1,2,3\}$ and therefore $g \circ f$ is not defined. Similarly, Im(g) $= \{1,3,4\} \not\subseteq \{1,2,3\} =$ Dom(g) and therefore $f \circ g$ is not defined either.

   (b) $f = \{(1,2),(2,3),(3,4)\}$ and $g = \{(2,1),(3,1),(4,1)\}$
   Note that Im(g) $= \{1\} \subseteq \{1,2,3\} =$ Dom(f) and $f \circ g$ is defined and

   $$f \circ g = \{(2,2),(3,2),(4,2)\}.$$

   Also note that Im(f) $= \{2,3,4\} =$ Dom(g) and so $g \circ f$ is defined and

   $$g \circ f = \{(1,1),(2,1),(3,1)\}.$$

   Clearly $g \circ f \neq f \circ g$.

   (c) $f = \{(1,4),(2,4),(3,3),(4,4)\}$ and $g = \{(1,1),(2,1),(3,4),(4,4)\}$.
   Note that Im(g) $= \{1,4\} \subseteq \{1,2,3,4\} =$ Dom(f) and $f \circ g$ is defined with

   $$f \circ g = \{(1,4),(2,4),(3,4),(4,4)\}.$$

   Note also that Im(f) $= \{3,4\} \subseteq \{1,2,3,4\} =$ Dom(g) and thus $g \circ f$ is defined with

   $$g \circ f = \{(1,4),(2,4),(3,4),(4,4)\}.$$

   Clearly, $f \circ g = g \circ f$.

   (d) $f(x) = 1 - x$ and $g(x) = 2 - x$ for $x \in \mathbb{R}$.
   Note that Dom(f) = Im(f) = Dom(g) = Im(g) $= \mathbb{R}$ and therefore both $f \circ g$ and $g \circ f$ are defined. Moreover,

   $$f \circ g(x) = 1 - g(x) = 1 - (2 - x) = x - 1, \qquad g \circ f(x) = 2 - f(x) = 2 - (1 - x) = x + 1.$$

   In particular, $f \circ g \neq g \circ f$.

2. Suppose A, B, and C are sets and $f : A \to B$ and $g : B \to C$. Prove the following:

   (a) If f and g are one-to-one, so is $g \circ f$.

   Proof. We use the direct method, assume $g \circ f(x) = g \circ f(y)$. Then

   $$g(f(x)) = g(f(y)).$$

   Since g is one to one, this implies

   $$f(x) = f(y).$$

   Since f is one to one, this implies $x = y$ and therefore $g \circ f$ is one to one. $\square$

   (b) If f and g are onto, so is $g \circ f$.

   Proof. Let $c \in C$. Then since g is onto, there is a $b \in B$ such that $g(b) = c$. Since f is onto, there is an $a \in A$ such that $f(a) = b$. Then

   $$g \circ f(a) = g(f(a)) = g(b) = c.$$

   Since c was arbitrary, we conclude that $g \circ f$ is onto. $\square$

   (c) If f and g are bijections, so is $g \circ f$.

   Proof. Since f and g are bijections they are both one to one and onto. By (a) and (b), this implies that $g \circ f$ is also one to one and onto. Thus it is a bijection. $\square$

3. Define the operation $*$ on the integers defined by $x * y = |x - y|$.

   (a) Is $*$ closed on the integers?

   Solution. *If $x, y \in \mathbb{Z}$, then so is $|x - y|$ and therefore $*$ is closed on the integers.*

   (b) Is $*$ commutative?

   Solution. *For any $x, y \in \mathbb{Z}$, $|x - y| = |-(y - x)| = |y - x|$ and thus $*$ is commutative.*

   (c) Is $*$ associative?

   Solution. *$*$ is not associative, e.g.*

   $$(6 * 3) * 2 = |6 - 3| * 2 = 3 * 2 = |3 - 2| = 1 \neq 5 = |6 - 1| = 6 * |3 - 2| = 6 * (3 * 2)$$

   (d) Does $*$ have an identity element? If so, does every integer have an inverse?

   Solution. *$*$ does not have an identity element. To see this, assume FTSC that $e \in \mathbb{Z}$ is an identity element, i.e*

   $$|x - e| = x * e = x.$$

   But clearly $|x - e| \geq 0$ so this indentity cannot hold for any $x < 0$ $\Rightarrow\Leftarrow$.