Discrete Mathematics, 2016 Fall - Worksheet 19

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In all of the above problems explain your answer in full English sentences.

1. In the context of $\mathbb{Z}_{10}$, calculate
   (a) $3 \oplus 3 = (3+3) \bmod 10 = 6$
   (b) $7 \otimes 3 = (7 \cdot 3) \bmod 10 = 21 \bmod 10 = 1$
2. In the context of $\mathbb{Z}_{12}$, calculate
   (a) $9 \oplus 8 = 17 \bmod 12 = 5$
   (b) $11 \otimes 5 = 55 \bmod 12 = 7$
3. In the context of $\mathbb{Z}_9$, calculate
   (a) $5 \ominus 8 = (-3) \bmod 9 = 6$
   (b) $8 \ominus 5 = 3 \bmod 9 = 3$
4. In the context of $\mathbb{Z}_{10}$, calculate
   (a) $8 \oslash 7 = 8 \otimes 7^{-1} = 8 \otimes 3 = 4$
   (b) $5 \oslash 9 = 5 \otimes 9^{-1} = 5 \otimes 1 = 5$
5. In $\mathbb{Z}_{431}$, find 29^{-1}. Use the Euclidean algorithm to find the solution of the Diophantine equation

$$29x + 431y = 1$$

$$\begin{aligned} 431 &= 14 \cdot 29 + 25 \\ 29 &= 1 \cdot 25 + 4 \\ 25 &= 6 \cdot 4 + 1 \end{aligned}$$

Now working backwards,

$$1 = 25 - 6 \cdot 4 = 25 - 6(29 - 25) = 7 \cdot 25 - 6 \cdot 29 = 7 \cdot (431 - 14 \cdot 29) - 6 \cdot 29 = 7 \cdot 431 - 104 \cdot 29$$

and therefore $x = -104$ and $y = 7$ is a solution and therefore $29^{-1} = (-104) \bmod 431 = 327$.

6. Solve

   (a) $4 \otimes (x \ominus 8) = 9$ in $\mathbb{Z}_{11}$. Multiplying by $4^{-1} = 3$ from the left we get

   $$x \ominus 8 = 3 \otimes 9 = 27 \text{ mod } 11 = 5$$

   $\oplus 8$ both sides gives

   $$x = 5 \oplus 8 = 13 \text{ mod } 11 = 2.$$

   (b) $2 \otimes x = 3$ in $\mathbb{Z}_{10}$. 2 does not have a reciprocal in $\mathbb{Z}_1 0$ because $gcd(2, 10) = 2 \neq 1$. Trying all the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, we can see that none of them gives a solution and therefore the equation has no solutions.

7. Find all solutions of

   (a) $3x \equiv 17 \pmod{20}$
   We first solve $3 \otimes x_0 = 17$ in $\mathbb{Z}_{20}$. Since $3^{-1} = 7$, we have

   $$x_0 = 3^{-1} \otimes 17 = 7 \otimes 17 = 119 \text{ mod } 20 = 19.$$

   Therefore we can get all the solutions of the congruence in as

   $$x = 19 + 20k, \qquad k \in \mathbb{Z}.$$

   (b) $2x \equiv 12 \pmod{15}$ Since $2^{-1} = 8$ in $\mathbb{Z}_{15}$, we have $x_0 = 8 \cdot 12 \text{ mod } 15 = 6$ and the set of solutions of the congruence is given by

   $$x = 6 + 15k, \qquad k \in \mathbb{Z}.$$