Discrete Mathematics, 2016 Fall - Worksheet 18

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In all of the above problems explain your answer in full English sentences.

1. For the given integers a, b, find the integers q and r such that $a = qb + r$ and $0 \leq r < b$.
   (a) $a = 100$, $b = 3$
   $$100 = 33 \cdot 3 + 1$$
   (b) $a = -100$, $b = 3$
   $$-100 = (-34) \cdot 3 + 2$$

2. For the given integers a, b, compute a div b and a mod b.
   (a) $a = 99$, $b = 3$.
   $$a \text{ div } b = 33 \qquad a \text{ mod } b = 0$$
   (b) $a = -99$,$b = 3$.
   $$a \text{ div } b = -33 \qquad a \text{ mod } b = 0$$
   (c) $a = 10$, $b = 3$.
   $$a \text{ div } b = 3 \qquad a \text{ mod } b = 1$$

3. Please calculate:
   (a) $gcd(20, 25)$ We write
   $$\begin{aligned} 20 &= 0 \cdot 25 + 20 \\ 25 &= 1 \cdot 20 + 5 \\ 20 &= 4 \cdot 5 + 0 \end{aligned}$$
   and thus $gcd(20, 25) = 5$.
   (b) $gcd(-89, -98)$. We write
   $$\begin{aligned} -89 &= 1 \cdot (-98) + 9 \\ -98 &= -11 \cdot 9 + 1 \\ 9 &= 9 \cdot 1 + 0 \end{aligned}$$
   and thus $gcd(-89, -98) = 1$

4. For each pair of integers a, b in the previous problem, find integers x and y such that $ax + by = gcd(a, b)$. Working our way backwards from the bottom in the first case, we can write

$$gcd(20, 25) = 5 = 25 - 1 \cdot 20 = (-1) \cdot 20 + 1 \cdot 25$$

and hence $x = -1$ and $y = 1$. In the second one,

$$gcd(-89, -98) = 1 = 11 \cdot 9 + (-1) \cdot 98 = 11 \cdot (98 - 89) - 98 = 11 \cdot (-89) + (-10) \cdot (-98)$$

and hence $x = 11$ and $y = -10$.

5. Let a and b be positive integers. Prove that 2^a and $2^b - 1$ are relatively prime.

Note that by the theorem in class, we can conclude that 2^a and 2^b are relative primes if we can find integers $x, y \in \mathbb{Z}$ such that

$$x2^a + y(2^b - 1) = 1$$

If $a \leq b$ then $y = -1$ and $x = 2^{b-a}$ works. On the other hand if $a > b$, then the situation is more complicated. Let N be a number such that $2^N b > a$. Then we write

$$(2^b - 1)(2^b + 1) = 2^{2b} - 1$$

$$(2^{2b} - 1)(2^{2b} + 1) = 2^{4b} - 1$$

$$\vdots$$

$$(2^{2^{N-1}b} - 1)(2^{2^{N-1}b} + 1) = 2^{2^N b} - 1$$

Then take $x = 2^{2^N b - a}$ and

$$y = -(2^b + 1)(2^{2b} + 1) \ldots (2^{2^{N-1}b} + 1)$$

6. Decide if the following diophantine equations have a solution or not and if yes find a solution:

- $3x + 4y = 2$
  Since $gcd(3, 4) = 1$, the solution exists and one possible solution is given by $x = -2$, $y = 2$. (taking $t = 2$, $u = -1$, $v = 1$ on the slides)
- $6x - 2y = 4$ Since $gcd(6, 2) = 2|4$, the solution exists. We can write $gcd(6, 2) = 6 - 2 \cdot 2$ so $u = 1$ and $v = -2$. Since $t = 2$, we have $x = 2$, $y = -4$.