# Discrete Mathematics, Section 002, Spring 2016

## Lecture 9: Multisets, Inclusion-Exclusion formula

Zsolt Pajor-Gyulai
zsolt@cims.nyu.edu

Courant Institute of Mathematical Sciences

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# Outline

# Fundamental counting problems

So far we have studied three counting problems:

| | With repetition | Without repetition |
|---|---|---|
| **Ordered** | $n^k$ | $(n)_k$ |
| **Unordered** | ? | $\binom{n}{k}$ |

$$n: \text{ Size of universe,} \qquad k: \text{ Size of collection}$$

Now we fill in the gap.

# Multisets

## Multiset

An unordered collection of elements where an element can be included more than once.

For example:

$$\langle 1, 2, 3, 3 \rangle, \quad \langle 1, 5, 5, 7, 7, 9, 9 \rangle$$

These are still unordered collections and therefore

$$\langle 1, 2, 3, 3 \rangle = \langle 3, 2, 1, 3 \rangle = \langle 2, 3, 3, 1 \rangle$$

- **Multiplicity:** The multiplicity of an element is the number of time it is a member of the multiset. For example the multiplicity of 3 in $\langle 1, 2, 3, 3 \rangle$ is two.
- **Cardinality:** The cardinality of a multiset is the sum of multiplicities of its elements.

# Counting multisets

**'n multichoose k'**

Let $n, k \in \mathbb{N}$. The symbol $\left(\!\binom{n}{k}\!\right)$ denotes the number of multisets with cardinality equal to $k$ whose elements belong to an $n$-element set such as $\{1, 2, \ldots, n\}$.

| | With repetition | Without repetition |
|---|---|---|
| **Ordered** | $n^k$ | $(n)_k$ |
| **Unordered** | $\left(\!\binom{n}{k}\!\right)$ | $\binom{n}{k}$ |

$n$ : Size of universe, $k$ : Size of collection

# Examples

1. Do the two examples directly on your worksheet!
2. One element multisets are
$$\langle 1 \rangle, \ldots, \langle n \rangle$$
Therefore $\left(\!\binom{n}{1}\!\right) = n$.
3. There is only one $k$-element multiset with elements selected from $\{1\}$, namely
$$\langle \underbrace{1, \ldots, 1}_{k} \rangle,$$
and so $\left(\!\binom{1}{k}\!\right) = 1$.
4. If $n = 2$, then we have to select the multiplicity of 1, and then the remaining places get filled with 2-s. Therefore
$$\left(\!\binom{2}{k}\!\right) = k + 1$$
5. For special values, do Problem 2 on the Worksheet.

# 'Pascal's triangle' for multisets

| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $n = 0$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n = 1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $n = 2$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $n = 3$ | 1 | 3 | 6 | 10 | 15 | 21 | 28 |
| $n = 4$ | 1 | 4 | 10 | 20 | 35 | 56 | 84 |
| $n = 5$ | 1 | 5 | 15 | 35 | 70 | 126 | 210 |
| $n = 6$ | 1 | 6 | 21 | 56 | 126 | 252 | 462 |

## Proposition

$$\left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{n-1}{k}\!\!\right) + \left(\!\!\binom{n}{k-1}\!\!\right)$$

For the proof see the textbook (p103).

# Formula for $\left(\!\binom{n}{k}\!\right)$

## Proposition

$$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$$

Idea of the proof for $n \geq 1$:

- Order the elements in the multiset in increasing order.

$$\langle 1,1,1,2,3,3,5 \rangle \qquad \text{out of } \{1,2,3,4,5,6\}$$

- Put bars when a new element is used

$$\langle 1,1,1|,2|,3,3||,5|\rangle$$

- Replace the elements by stars

$$* * *|*|**||*|$$

# Formula for $\left(\!\binom{n}{k}\!\right)$

## Proposition

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Idea of the proof for $n \geq 1$:

- We can do this the other way around, take e.g.

$$* * * ||| * || * * * | * ||$$

- This translates into

$$\langle 1, 1, 1, 4, 6, 6, 6, 7 \rangle$$

- The original set can be also read off to be

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Do this yourself on Problem 3 on the worksheet!

# Formula for $\left(\!\binom{n}{k}\!\right)$

**Proposition**

$$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$$

Idea of the proof for $n \geq 1$:

- For a general $n$ and $k$ there are $k$ stars and $n-1$ bars.
- We need to choose the $k$ places where to put the stars, then the rest of the places will be occupied by the bars.

For a formalized proof, see the textbook.

# Outline

# Inclusion-exclusion formula for two sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

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Then $|A| = 500$ and $|B| = 200$. Also note

$$A \cap B = \{x \in \mathbb{Z} : 1 \leq x \leq 1000 \text{ and } 10|x\}$$

and therefore $|A \cap B| = 100$.

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$$|A \cup B| = |A| + |B| - |A \cap B| = 500 + 200 - 100 = 600.$$

Therefore there are 600 integers in the range 1 to 1000 that are divisible by either 2 or 5.

# Inclusion-exclusion formula for three sets

$$|A \cup B \cup C| = |A| + |B| + |C|$$

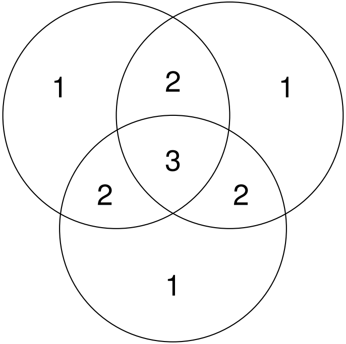


Figure: Number of times elements in each 'compartment' are counted.

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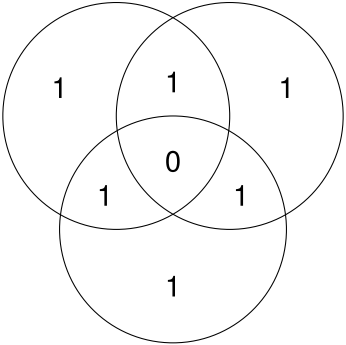


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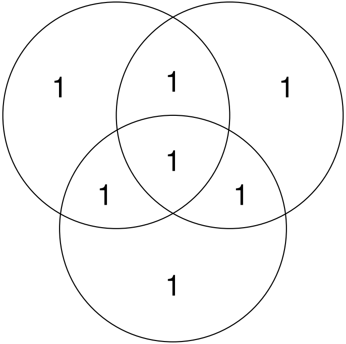
# Inclusion-exclusion formula for three sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

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# Inclusion-exclusion formula for three sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

At an art academy, there are 43 students taking ceramics, 57 students taking painting, and 29 students taking sculpture. There are 10 students in both ceramics and painting, 5 in both painting and sculpture, 5 in both ceramics and sculpture, and 2 taking all three courses. How many students are taking at least one course at the art academy?

$$|C \cup P \cup S| = |C| + |P| + |S| - |C \cap P| - |C \cap S| - |P \cap S| + |C \cap P \cap S| =$$

$$= 43 + 57 + 29 - 10 - 5 - 5 + 2 = 111$$

# Inclusion-exclusion formula for more sets

## Four sets

$$
\begin{aligned}
|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - \\
- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + \\
+ |A \cap B \cap D| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - \\
- |A \cap B \cap C \cap D|.
\end{aligned}
$$

Try this with Problem 4 on the worksheet!

# Inclusion-exclusion formula for more sets

**Theorem**

$$\begin{aligned}|A_1 \cdots \cup A_n| = |A| + \cdots + |A_n| - \\ - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n| + \\ + |A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| - \\ - \cdots + \ldots\ldots \\ \pm |A_1 \cap \cdots \cap A_n|.\end{aligned}$$

For a proof, see the textbook.