Discrete Mathematics, 2016 Spring - HW 6 Solutions

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General comments:

Section 20

1) Please state the contrapositive of each of the following statements:
(a) If p is prime, the 2p 2 is divisible by p.
(b) If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.
(c) If the battery is fully charged, the car will start.
(d) If A or B, then C.

Solution:
(a) If 2p 2 is not divisible by p, then p is not prime, (or equivaletly, p is composite).
(b) If the parallelogram is not a rhombus, then the diagonals of a parallelogram are not perpendicular.
(c) If the car will not start, then the battery is not fully charged.
(d) If not C, then not (A or B), or equivalently, not A and not B.

10) Prove by contradiction: Let a be a number with a $>$ 1. Prove that $\sqrt{a}$ is strictly between 1 and a.

Proof by contradiction:

Suppose $\sqrt{a}$ is not strictly between 1 and a, or equivalently, $\sqrt{a} \geqslant$ a. Then, square both sides of the inequality, then: a$\geqslant$a^2.

Rearrange the inequality, a - a^2 = a(1-a)$\geqslant$ 0, a(a-1)$\leqslant$ 0, so 0$\leqslant$a$\leqslant$1, which contradicts the given condition that a$>$1. Therefore, $\sqrt{a}$ is strictly between 1 and a. $\square$

(12) Prove by contradiction: A positive integer is divisible by 10 if and only if its last digit (when written in base ten) is a zero. You may assume that every positive integer N can be expressed as:
$N = d_k 10^k + d_{k1} 10^{k1} + \quad + d_1 10 + d_0$
where the numbers d_0 through d_k are in the set $\{0, 1, . . . , 9\}$ and $d_k \neq 0$. In this notation, d_0 is the ones digit of N's base ten representation.

Proof by contradiction:
Suppose the last digit of a positive integer x is not zero, then we can rewrite the number x in the following form:

$x = d_k 10^k + d_{k1} 10^{k1} + \quad + d_1 10 + d_0$, where the last digit of x, d_0, is not zero. Then $x = d_k 10^k + d_{k1} 10^{k1} + \quad + d_1 10 + d_0 \equiv d_0 \pmod{10}$. Since $d_0$is a postive integer in the set $\{0, 1, . . . , 9, d_0 \pmod{10} \neq 0 \pmod{10}$. Thus, x is not divisidble by 10, which contradicts the given condition that this positive integer is divisible by 10.

Similarly, by assumng that a positive integer is not divisible by 10, we can show that the last digit of the number written in base 10 is not a zero and reach a contradiction with the given condition.
□

(15) Prove the converse of the Addition Principle. The converse of a statement If A, then B is the statement If B, then A. In other words, your job is to prove the following: Let A and B be finite sets. If $|A \cup B| = |A| + |B|$, then $A \cap B = \emptyset$.

Proof by contradiction:
Suppose $A \cap B \neq \emptyset$, meaning that the finite set A and B have at least one element in common. So the total number of the set $|A \cup B| = |A| + |B| - | A \cap B |$, where $| A \cap B | \geqslant 1$. Thus, $| A \cap B | \leqslant |A| + |B| - 1$, which contradicts our given assumption that $|A \cup B| = |A| + |B|$.
□

Section 21

(4-5) Prove the follwoing statements by smallest counterexample:
(a) n! $\leq$ n^n for all positive integers n.
(b)$\begin{pmatrix} \mathbf{2n} \\ \mathbf{n} \end{pmatrix} \leq 4^n$ for all natural numbers n

Proof by smallest counterexample:
(a)
For the sake of contradiction, suppose there were positive integers n such that n! $>$ n^n, then there is a smallest positive integer x such that x! $>$ x^x. Since 1! $>$ 1^1, x$\geq$2. Hence, 1$\leq$ n = (x-1) is a smaller integer than x. So (x-1)! $\leq$ (x-1)$^{x-1}$. It follows that (x-1)! $\leq$ (x-1)$^{x-1}$¡ (x)$^{x-1}$. Multiply both sides of the inequality with positive integer x, we get (x-1)! $\cdot$x ¡ (x)^x, x! ¡ (x)^x. We reach a contradiction.
□

(b)
For the sake of contradiction, suppose there were positive integers n such that $\begin{pmatrix} 2n \\ n \end{pmatrix} > 4^n$. Then there is a smallest positive integer x such that $\begin{pmatrix} 2x \\ x \end{pmatrix} > 4^x$. When n = 1, we have $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 < 4^1$, so x $\geq$2. Since 1$\leq$x-1 is a smaller positive integer than x, we know that $\begin{pmatrix} 2(x\text{-}1) \\ x\text{-}1 \end{pmatrix} \leq 4^{x-1}$. Since $\begin{pmatrix} 2(x\text{-}1) \\ x\text{-}1 \end{pmatrix} = \begin{pmatrix} 2x\text{-}2 \\ x\text{-}1 \end{pmatrix} = \frac{(2x-2)!}{(x-1)!(2x-2-x+1)!} = \frac{(2x-2)!}{(x-1)!(x-1)!}$, then $\frac{(2x-2)!(2x-1)(2x)}{(x-1)!(x-1)!\cdot x\cdot x} = \begin{pmatrix} 2x \\ x \end{pmatrix} = \frac{(2x-2)!(2x-1)\cdot 2}{(x-1)!(x-1)!\cdot x} \leq 4^{x-1}\cdot \frac{(2x-1)\cdot 2}{x} = 4^x\text{-}4^{x-1}\frac{2}{x} < 4^x$. So $\begin{pmatrix} 2x \\ x \end{pmatrix} < 4^x$. We reach a contradiction.
□

(7) The Fibonacci numbers are the list of integers (1, 1, 2, 3, 5, 8, . . .) = (F$_0$, F$_1$, F$_2$, . . .), where F$_0$ = 1, F$_1$ = 1, F$_n$ = F$_{n-1}$ + F$_{n-2}$, for n $\geq$ 2.
(a) Read the proof of the fact that for all n $\in$ N, we have F$_n \leq$ 1.7 n on p133 in the textbook.
(b) Now prove using the smallest counterexample method that F$_n$ $>$ 1.6 ^{n} whenever n $\geq$ 29.
Proof:

(b)
Here I will show how to prove the proposition by contradiction. For the sake of contradiction, suppose that the statement that $F_n > 1.6^n$, where n $\geq$ 29, were false. Let X be the set of counterexamples, X = { n$\in$N, n$\geqslant$ 29, $F_n \leq 1.6^n$ }. Thus, by the Well-Ordering Principle, we know that the set X contains at least one element x. Since when n = 29, $F_{29} > 1.6^{29}$ and when x = 30, $F_{30} > 1.6^{30}$. So x$\geq$31. F_{x-1} and F_{x-2} are the two numbers before x and $F_{x-1} > 1.6^{x-1}$ and $F_{x-2} > 1.6^{x-2}$. So $F_x = F_{x-1} + F_{x-2}$, $F_x > 1.6^{x-1}+1.6^{x-2} = 1.6^x(1.6^{-1}+1.6^{-2})$. Since $1.6^x(1.6^{-1}+1.6^{-2}) > 1.6^x$. So $F_x > 1.6^x$, and here we reach a contradiction.
□

Section 22

(4-5,16) Prove the following equations and inequalities by induction.
(a) $\mathbf{1^3 + 2^3 + \ldots n^3 = \frac{n^2(n+1)^2}{4}}$.
(b) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$.
(c) $\mathbf{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^n} \geq 1 + \frac{n}{2}}$.
Proof by induction:
These statesments can be easily proved by following the steps below:
(i) First check the case when n=1 (initial case).
(ii)Then assume n=k holds, and show n=k+1 holds (inductive case).
(iii) Conclude that the hypothesis holds for all n $\in$ N.

(10) Prove, by induction, that the sum of the angles of a convex n-gon (with n 3) is 180(n2) degrees. Proof by induction:
Similar to Problem (4-5,16), follow the same scheme as shown before.

(12) The Towers of Hanoi. Prove that for every positive integer n, the puzzle can be solved in $\mathbf{2^n}$ -1 moves.

Proof by induction:

This statement can be proved using many different methods, mostly by arguments. However, the most logical and efficient method is to show by induction.

(i) Check the case when n = 1: When n = 1, we only need to move once. So we need only 1 = 2^1-1 move. So the hypothesis is proved for the case n = 1.

(ii) Now suppose the hypothesis is true when n = k, then we know we need x^k-1 moves. THen when n = k+1, we can move the k plactes to the second dowel with 2^k-1 moves, leaving (k+1)th place to its original position. Then we move this plate to the third dowel. So there are 2^k-1+1 moves. We can then move all the plates on the second dowel onto the third dowel with another 2^k-1 moves. So in total we need (2^k -1 +1) + (2^k -1) = $2^k \cdot 2$ -1 = 2^{k+1} -1 moves. Hence, we show that to complete a towel with n = k+1 plates, we need $2^{k+1} - 1$ moves. Therefore, we prove the case for n = k+1.

(iii) Therefore, it follows that the hypotheis is true for all positive integer n.
$\square$