Discrete Mathematics, Section 001, Fall 2016

Lecture 24: Eulerian graphs, Coloring

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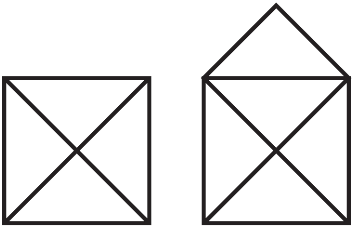
Outline

Can we draw the graph without lifting our pen?

Definition

Let G be a graph. A walk in G that traverses every edge exactly once is called an **Eulerian trail**. If, in addition, the trail begins and ends at the same vertex, we call the walk an **Eulerian tour**. If G has an Eulerian tour, we call G Eulerian.



Questions:

- Which graphs have Eulerian trails?
- Which graphs have Eulerian tours?

Now we give a complete answer.

Basic observations

If G has an Eulerian trail, then G has at most one nontrivial component.

We call a component *trivial* if it contains only one vertex. If there are two or more components that are not trivial, then two edges in different components cannot be included in the same walk. Therefore we can restrict ourself to connected graphs WLOG.

If G has an Eulerian trail, then it has at most two vertices of odd degree

Let v be neither the first or the last vertex on an Eulerian trail. Then the trail goes in and out v the same number of times. As every edge is traversed exactly once, this implies that $d(v)$ must be even. Therefore only the first and the last vertices on the trail can have an odd degree.

Basic observations

If G has an Eulerian trail that begins at a vertex a and ends at a vertex b with $a \neq b$, then vertices a and b have odd degree.

The trail chooses an edge to leave a as the first step of the walk. Then everytime the trail comes back to a it must leave on an edge that it has not previously traversed.

If G has an Eulerian tour, then all vertices have an even degree.

In this case, every exit from a vertex v we have an associated entrance. Since these all must happen on different edges, this implies $d(v)$ being even for every vertex of G.

Basic observations

If G is a connected Eulerian graph, then G has an Euler tour that begins/ends at any vertex.

Let's assume that an Eulerian tour starts at $a \in V(G)$ and assume that the first visited vertex after a is b:

$$W = a \sim b \sim \cdots \sim a.$$

Then we can start the tour at b:

$$W' = b \sim \cdots \sim a \sim b.$$

This way we can shift the tour to start at any vertex.

Theorem

Let G be a connected graph. Then G is Eulerian if and only if all vertices have even degree

We prove this by induction on $|G|$. We have seen ($\Rightarrow$) already. To prove ($\Leftarrow$), we will use strong induction on $|G|$.

Basis case: When $|G| = 1$, let v be the only point of G. Then it has no edges and therefore (v) is trivially an Eulerian tour.

Induction hypothesis: Assume that the result is true for all $|G| = k$ with $k \leq n$.

Let G be a graph with $|G| = n + 1$. Pick an arbitrary vertex v and start forming a walk that does not traverse the same edge twice. If we reach a vertex where there are no more free edges to leave on, it must be the initial point because the degree of all vertices is even. Let H be the longest such walk. FTSC assume it is not an Eulerian tour.

[...]

Theorem

Let G be a connected graph. Then G is Eulerian if and only if all vertices have even degree

[...]

Consider the graph $G - E(P)$:

- It does not contain any edges incident on v.
- All components have less vertices than n and thus by the induction hypothesis, they containan Eulerian tours.

Since G is connected, there must be a component that contains a point w that is on the walk P. Let us call the Eulerian tour of this component Q.

[...]

Theorem

Let G be a connected graph. Then G is Eulerian if and only if all vertices have even degree

[...]

Consider the following walk. Follow P from v to w, then traverse Q from w to w, then traverse the rest of P from w to v. This walk is longer than P and no edges are repeated along it which contradicts the maximality of P. $\Rightarrow\Leftarrow$. $\square$

Theorem

Let G be a connected graph. Then G has an Eulerian trail if and only if the number of vertices with an odd degree is 0 or 2.

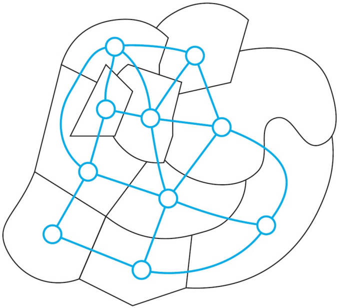
Once again, we have already seen ($\Rightarrow$). We use the previous theorem to prove ($\Leftarrow$).

- If there are no odd vertices then there is an Eulerian tour which serves as an Eulerian trail too.
- If there are two odd vertices then
  - Add an extra edge connecting them (temporarily allow for parallel edges and note that it does not ruin the proof of the previous theorem).
  - In the resulting graph every vertex has even degree and therefore there is an Eulerian tour.
  - Now remove the extra edge and the tour breaks up to a trail.

$\square$

Outline

Statement of the problem



Statement of the problem

Let G be a graph. To each vertex of G, we wish to assign a color such that adjacent vertices receive different colors. We want to achieve this using as few colors as possible.

We have seen applications:

- Map coloring
  - Vertices: Countries
  - Colors: Colors
- Exam scheduling
  - Vertices: Courses
  - Colors: Exams

Definition

Let G be a graph and let k be a positive integer. A k-**coloring** of G is a function

$$f : V(G) \to \{1, 2, \ldots, k\}$$

We call this coloring **proper** provided

$$\forall xy \in E(G), \qquad f(x) \neq f(y).$$

If a graph has a proper k-coloring, we call it k-colorable.

- For a vertex v, the value $f(v)$ is its color.
- $\{1, \ldots, k\}$ is our palette.
- The proper coloring means that for any edge, the two endpoints are colored differently.
- The definition does not require that we use all colors (or in other words that f is onto).

Definition

Let G be a graph. The smallest positive integer k for which G is k-colorable is called the **chromatic number** of G. The chromatic number of G is denoted $\chi(G)$.

For example, consider K_n.

- We can properly color K_n using n colors.
- We cannot do better than that because every vertex is adjacent to every other!

Therefore $\chi(K_n) = n$.

Proposition

If G is a subgraph of a graph H, then $\chi(G) \leq \chi(H)$.

Given a proper coloring of H, let every vertex $v \in V(G)$ have the same color as it does in H. This results in a proper coloring of G. □

Easy bounds

Proposition

Let G be a graph with maximum degree Δ. Then $\chi(G) \leq \Delta + 1$.

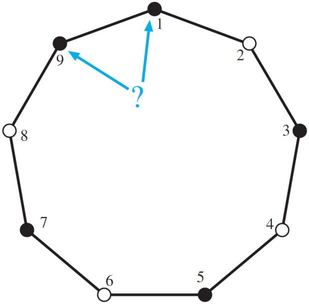
Note that one can color naively one vertex after another. Since every point has at most Δ neighbors, there is always a color on the palette of size $\Delta + 1$ that is different from all the neighbor's colors. $\square$

Proposition

Let G be a graph with at least one edge. Then $\chi(G) \geq 2$.

Let $xy \in E(G)$ be an edge. Then x and y have to have different colors. $\square$

Chromatic number of cycles



- If n is even, then the black-white-black pattern works.
- If n is odd, then this does not work and we need one more color.

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

Note that when n is odd, $\chi(C_n) = 3$, even though $d(v) = 2$ for every $v \in V(G)$.

One and two colorable graphs

Proposition

A graph G is one colorable if and only if it is edgeless

Follows from our easy lower bound. □

Definition

A graph G is called **bipartite** provided it is 2-colorable

Let G be a bipartite graph and select a proper two coloring f. Let

- Let $X = \{v \in V(G) : f(v) = 1\}$
- Let $Y = \{v \in V(G) : f(v) = 2\}$

Then $\{X, Y\}$ is a partition of $V(G)$. If $e \in E(G)$, then one of its endpoints is in X, and the other one is in Y.

$\{X, Y\}$ is called a bipartition of G.

Examples of bipartite graphs

- C_n is bipartite for every even n.
- Trees

Proposition

Trees are bipartite

We prove this by induction on the number of vertices.

Basis case: Clearly a tree with one vertex is bipartite as $\chi(K_1) = 1 \leq 2$.

Induction hypothesis: Every tree with n vertices is bipartite.

Let T be a tree with $n + 1$ vertices and let v be a leaf. Let $T' = T - v$. Since T' is a tree with n vertices, it is bipartite by the induction hypothesis. Let f be a proper 2-coloring of T'.

Now consider v-s only neighbor w. Whatever $f(w)$ is, assign v the other color. This gives a proper two coloring of T. $\square$

Complete bipartite graphs

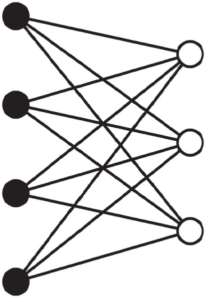


Figure : This is $K_{4,3}$.

Definition

Let n, m be positive integers. The **complete bipartite graph** $K_{n,m}$ is a graph whose vertices can be partitioned $V = X \cup Y$ such that

- $|X| = n$,
- $|Y| = m$,
- $xy \in E(K_{n,m})$ for all $x \in X$ and $y \in Y$.
- No edge has both its endpoints in either X or Y.

Criterion for bipartite-ness

How can one check whether a graph is bipartite?

- Showing that a graph is bipartite is easy. Just start coloring points alternating the colors and if it works the graph is bipartite.
- If it is not bipartite than choosing two adjecent points, assinging them different colors and then coloring with alternating colors will break down.

Alternatively, this can be seen to be captured by the following theorem.

Theorem

A graph is bipartite if and only if it does not contain an odd cycle.

However no such criterion is known for more than two colors.

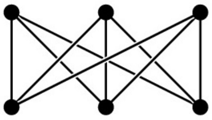
Outline

Planar graphs

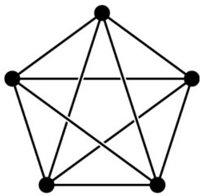
Theorem

A **planar graph** is a graph that can be drawn in the plane without any two edges crossing each other.

Examples of non-planar graphs:



$K_{3,3}$



K_5

Theorem (Kuratowski)

A graph G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.

(A subdivision of a graph G is formed from G by replacing edges with paths.)

Coloring of planar graphs

To prove that for any planar graphs G, $\chi(G) \leq 6$ is not hard. Neither is to show that $\chi(G) \leq 5$ just a little lengthier. However the proof of the following theorem is hundreds of pages long.

The 4 coloring theorem

If G is a planar graph, then $\chi(G) \leq 4$.

For example, graphs coming from maps are planar and therefore 4-colorable.