Discrete Mathematics, Section 001, Fall 2016

Lecture 21: Public key cryptography

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Outline

The setting

In the schoolyard, Alice secretly wants to tell Bob that she likes him. Unfortunately, gossip girl Eve is listening. Can Alice tell Bob the secret?

- They could create a secret code, and talk in that code but Eve would hear the code then!
- They could make up the code privately, but that is impractical, it's hard to get privacy on the schoolyard.
- Fortunately, Bob is a math prodigy knows another way.

Key: Find a procedure that is easy to do but it is extremely hard to undo.

Factoring primes

- If you have primes p and q computing $n = pq$ is extremely easy.
- However, if you are given n and the knowledge that it is a product of primes, finding these prime factors is extremely hard.

Conjecture

There is no computationally efficient procedure for factoring positive integers.

Think

$$p, q \text{ are 500 digit numbers} \quad \rightarrow \quad n \text{ is a 1000 digit number}$$

In this case our best algorithms and computers would take about a century.

Procedure

1. Alice wants to say 'I like you', but she can only tell Bob a big number. So first, she encodes this using ASCII:

I		I	i	k	e		y	o	u
073	032	108	105	107	101	032	121	111	117

So Alice's number is:

$$M = 73032108105107101032121111117$$

2. Bob, in his mind creates a pair of functions E and D that are inverses of one another:

$$D(E(M)) = M$$

Procedure

1. Alice wants to say 'I like you', but she can only tell Bob a big number. So first, she encodes this using ASCII:

$$M = 73032108105107101032121111117$$

2. Bob, in his mind create a pair of functions E and D that are inverses of one another:

$$D(E(M)) = M$$

3. Bob tells Alice the function E. Eve overhears this but she's bad at math so she has no idea how to invert E to get D.
4. Using Bob's encryption function E, Alice computes $N = E(M)$ and tells N to Bob. Eve overhears this, but since she doesn't know what D is, she can't extract M.
5. Bob now uses his decryption function D in his mind to get

$$M = D(N) = D(E(M)).$$

1. In private, Bob creates a public encryption function E and a secret decryption function D.
2. Bob sends his public encryption function E to Alice. (E ←)
3. In private, Alice writes her message in ASCII, M. She uses Bob's function E to calculate $N = E(M)$.
4. Alice sends N to Bob. (N →)
5. In private, Bob uses his decryption function D to calculate $M = D(N)$. He now has Alice's message.

Eve sees E and N, but cannot calculate M from these.

Outline

The encryption function

Let n be a large integer. Set the encryption function to be

$$E(M) = M^2 \bmod n$$

What should n be?

- If Alice wants to send a message with c characters, then M will have $3c$ digits (up to the initial zeroes) and M^2 will have $6c$ digits.
- If the digits of n is more than $6c$ then $M^2 \bmod n = M^2$ and Eve can get to the message by computing the ordinary square root.
- You don't want to loose information so the number of digits of n should be at least $3c$.

$$\#\text{digits}(M) < \#\text{digits}(n) < 2 \cdot \#\text{digits}(M)$$

The encryption function

In our example,

$$M = 730321081051071010321211111117$$

which has 29 digits. So if Bob expects such a large message, he can look up two big primes

$$p = 977555333111111, \qquad q = 988666444111111$$

and multiply them to get

$$n = 966476155599814608692148154321$$

which will do. This is what Bob sends Alice who then encrypts

$$N = E(M) = M^2 \bmod n = 412976518048000543454453602839$$

which she sends back to Bob.

The encryption function

In our example,

$$M = 730321081051071010321211111117$$

$$p = 977555333111111, \qquad q = 988666444111111$$

$$n = 966476155599814608692148154321$$

$$N = E(M) = M^2 \bmod n = 412976518048000543454453602839$$

- Eve now has both n and N but she doesn't have p and q. Therefore she cannot invert E to find D and can't extract M.
- Bob knows p and q and N and as we will see, he can invert easily.

Square roots in $\mathbb{Z}_n$

$$E(M) = M^2 \bmod n$$

If Bob gets $N \in \mathbb{Z}_n$, then to find $M = D(N)$, he needs to take a square root in $\mathbb{Z}_n$.

In other words, we need to solve $x \otimes x = N$.

For example, by simple brute force

- In $\mathbb{Z}_{59}$, we have $\sqrt{17} = 28, 31$.
- In $\mathbb{Z}_{59}$, we have no $\sqrt{18}$.
- In $\mathbb{Z}_{1121}$, we have $\sqrt{17} = 146, 500, 621, 975$.
- Of course, for huge n, this is ridiculous.

Quadratic residues

$$E(M) = M^2 \bmod n$$

The encrypted message will be a 'perfect square' in $\mathbb{Z}_n$.

Definition

Let n be a positive integer and let $a \in \mathbb{Z}_n$. If there is an element $b \in \mathbb{Z}_n$ such that $a = b \otimes b$, we call a a **quadratic residue modulo** n. Otherwise we call a a quadratic nonresidue.

Motivation behind the name:

$$b \otimes b = b^2 \bmod n$$

So N is a quadratic residue modulo n.

Square roots when n is a prime

Things are relatively simple when n is a prime.

Proposition

Let p be a prime and let $a \in \mathbb{Z}_p$. Then a has at most two square roots in $\mathbb{Z}_p$.

FTSC suppose that a has three (or more) square roots in $\mathbb{Z}_p$.

Note that if x is a square root of a,

$$(-x) \otimes (-x) = (-x)^2 \bmod p = x^2 \bmod p = x \otimes x = a.$$

and therefore $-x = 0 \ominus x$ (in $\mathbb{Z}_p$) is also a square root.

Since a has three or more roots, we can choose two of them $x, y \in \mathbb{Z}_p$, such that $x \neq \pm y$. Then

$$(x - y)(x + y) \bmod p = x^2 - y^2 \bmod p = x^2 \ominus y^2 = a - a = 0$$

[...]

Proposition

Let p be a prime and let $a \in \mathbb{Z}_p$. Then a has at most two square roots in $\mathbb{Z}_p$.

[...]

Since a has three or more roots, we can choose two of them $x, y \in \mathbb{Z}_p$, such that $x \neq \pm y$. Then

$$(x - y)(x + y) \bmod p = x^2 - y^2 \bmod p = x^2 \ominus y^2 = a - a = 0$$

But this means that $p|(x - y)(x + y)$, which implies (as p is prime) that

$$p|(x - y) \qquad \text{or} \qquad p|(x + y)$$

However, since $x \neq \pm y$ and $0 \leq x, y < p$, this is not possible. $\Rightarrow\Leftarrow$. □

An auxiliary result

Fermat's little theorem

Let p be a prime and let a be an integer. Then

$$a^p \equiv a \pmod{p}$$

For example, $5^{23} \equiv 5 \pmod{23}$.

This has a generalization:

Euler's theorem

Let n be a positive integer and let a be an integer relatively prime to n. Then

$$a^{\varphi(n)} \equiv 1 \pmod{n}$$

For proofs, see the textbook.

Identification of quadratic residues in $\mathbb{Z}_p$ for special p.

Proposition

Let p be a prime of the form $p = 4k + 3$ for some $k \in \mathbb{N}$. Let $a \in \mathbb{Z}_p$ be a quadratic residue. Then the square roots of a in $\mathbb{Z}_p$ are

$$\left[\pm a^{\frac{p+1}{4}}\right] \mod p.$$

By the restriction on p, $\frac{p+1}{4}$ is an integer and everything makes sense. Since a is a quadratic residue, there is an $x \in \mathbb{Z}_p$ such that $x^2 \equiv a \pmod{p}$. Then

$$\left[\pm a^{\frac{p+1}{4}}\right]^2 \equiv \left[x^{\frac{p+1}{2}}\right]^2 = x^{p+1} = x^p x \equiv x^2 \equiv a \pmod{p}.$$

where Fermat's little theorem was used. By the previous Proposition, there are no more roots. □

$n = pq$ where p and q are primes of the form $4k + 3$.

We still want to solve $x \otimes x = N$ in $\mathbb{Z}_n$. This is the same as

$$x^2 \bmod n = N$$

which means that for some $k \in \mathbb{Z}$,

$$x^2 = kn + N = k(pq) + N$$

But this means both

$$x^2 = (kp)q + N, \qquad x^2 = (kq)p + N$$

or

$$x^2 \bmod p = N \qquad x^2 \bmod q = N.$$

But if x_1 is a square root of N in $\mathbb{Z}_p$ and x_2 is a square root of N in $\mathbb{Z}_q$ (which we now know how to compute for our special primes), then an x satisfying both

$$x = x_1 + k_1 q \qquad x = x_2 + k_2 p$$

for some $k_1, k_2 \in \mathbb{Z}$ does the job.

In our example

For Bob's public key

$$n = 966476155599814608692148154321$$

he has that

$$p = 977555333311111, \qquad q = 988666444411111$$

are both of the form $4k + 3$ and therefore

- $\sqrt{N}$ in $\mathbb{Z}_p$ is

  $$\pm N^{\frac{p+1}{4}} \bmod p = \pm N^{244388833327778} \bmod p = \pm 869000357225117$$

- $\sqrt{N}$ in $\mathbb{Z}_q$ is

  $$\pm N^{\frac{q+1}{4}} \bmod q = \pm N^{988666444411111} \bmod q = \pm 154623490704086$$

In other words

$$x_1 = 869000357225117, 108554976085994$$

$$x_2 = 154623490704086, 834042953707025$$

Chinese remainder theorem

$$x_1 = 869000357225117, 108554976085994$$

$$x_2 = 154623490704086, 834042953707025$$

This gives four equation of the form

$$x = x_1 + k_1 p \qquad x = x_2 + k_2 q$$

or in other words

$$x \equiv x_1 \pmod{p}, \qquad x \equiv x_2 \pmod{q}$$

Chinese remainder theorem

The pair of congruences

$$x \equiv x_1 \pmod{p}, \qquad x \equiv x_2 \pmod{q}$$

has a unique solution x_0 with $0 \leq x_0 < pq$. Furthermore, every mutual solution to these congruences differs from x_0 by a multiple of mn.

In our example

We earlier got

$$x_1 = 869000357225117, 108554976085994$$

$$x_2 = 154623490704086, 834042953707025$$

If p^{-1} is the reciprocal of p in $\mathbb{Z}_q$,

$$M = x_0 = x_1 + \left[((x_2 - x_1) \bmod q) \otimes p^{-1}\right] p \bmod n$$

Two independent possibilites for x_1 and x_2 gives us four x_0-s. The Euclidean algorithm yields

$$-174529541694500p + 172568094394491q = 1$$

and thus in $\mathbb{Z}_q$,

$$p^{-1} = -174529541694500 \bmod q = 814136902716611$$

In our example

We earlier got

$$x_1 = 869000357225117, 108554976085994$$

$$x_2 = 154623490704086, 834042953707025$$

For each pair x_1, x_2,

$$M = x_1 + \left[((x_2 - x_1) \bmod q) \otimes p^{-1}\right] p \bmod n$$

$$p^{-1} = 814136902716611$$

This gives the following four M values:

$$M_1 = 730321081051071010321211111117$$

$$M_2 = 811685135902651145620250169817$$

$$M_3 = 893444047494707507660027043204$$

$$M_4 = 154791019697163463071897984504$$

In our example

This gives the following four M values:

$$M_1 = 730321081051071010321211111117$$

$$M_2 = 811685135902651145620250169817$$

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$$M_4 = 154791019697163463071897984504$$

To see if any of this is the actual message, do inverse ASCII:

$M_1 \rightarrow$ I like you $\quad M_2 \rightarrow Junk \quad M_3 \rightarrow Junk \quad M_4 \rightarrow Junk$

So the message is clearly discernable for Bob.

Summary

After Bob gets N,

- Compute $\sqrt{N}$ in $\mathbb{Z}_p$ and $\mathbb{Z}_q$.
- This gives four pairs for (x_1, x_2).
- For each pair solve the Chinese remainder problem and get an M.
- Out of the four M-s, three will be gibberish, the remaining one is the original M.

Outline

Euler's totient

Definition

For every positive integer n, let $\varphi(n)$ denote the number of integers from 1 to n inclusive that are relatively prime to n.

For example,

- For $n = 14$, $\{1, 3, 5, 9, 11, 13\}$ are the relative primes to n and therefore $\varphi(14) = 6$.
- For $n = 15$, $\{1, 2, 4, 7, 8, 11, 13, 14\}$ are the relative primes to n and therefore $\varphi(15) = 8$.

Theorem

If the prime factorization of n is $p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$, then

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_t}\right)$$

Euler's totient

Theorem

If the prime factorization of n is $p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$, then

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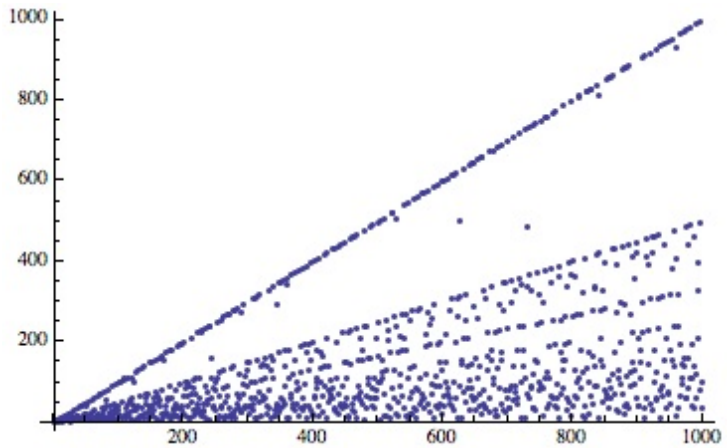

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The encryption function

$$E(M) = M^e \bmod n, \qquad D(N) = N^d \bmod n$$

for appropriately chosen e and d and $n = pq$ as before.

$$D(E(M)) = D(M^e \bmod n) = M^{ed} \bmod n$$

By Euler's theorem, we know that

$$M^{\varphi(n)} = 1 \qquad \text{in } \mathbb{Z}_n$$

and therefore

$$M^{k\varphi(n)+1} = (M^{\varphi(n)})^k M = 1^k M = M \qquad \text{in } \mathbb{Z}_n$$

and so we want $ed = k\varphi(n) + 1$.

The encryption function

$$E(M) = M^e \bmod n, \qquad D(N) = N^d \bmod n$$

We want $ed = k\varphi(n) + 1$.

- Since Bob knows p and q, he can compute $\varphi(n)$ easily.
- Bob selects his favorite $e \in \mathbb{Z}_{\varphi(n)}$.
- Then he computes $d = e^{-1} \in \mathbb{Z}_{\varphi(n)}$.
- He tells Alice n and e publicly (so that Eve can hear), but keeps p, q and d a secret.
- Alice computes $N = E(M)$ using Bob's key and sends N to Bob.
- Eve knows n, e, but he cannot find $D(N)$ as he can't find $\varphi(n)$.
- Bob decodes $M = D(N)$ and they live happily ever after.