Discrete Mathematics, Section 001, Fall 2016

Lecture 23: Connections, Trees

Zsolt Pajor-Gyulai
zsolt@cims.nyu.edu

Courant Institute of Mathematical Sciences

December 7, 2016

Outline

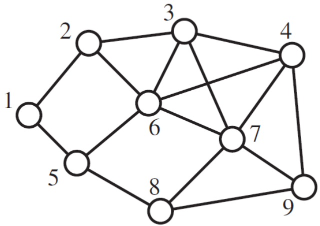
Definition

Let $G = (V, E)$ be a graphs. A **walk** in G is a sequence (or lists) if vertices, with each vertex adjacent to the next; that is,

$$W = (v_0, v_1, \ldots, v_l) \qquad v_0 \sim v_1 \sim v_2 \sim \cdots \sim v_l$$

The **length** of this walk is l.

Note that there are $l + 1$ vertices on the walk.



- $1 \sim 2 \sim 3 \sim 4$.

This is a $(1, 4)$ walk.

- $W = 1 \sim 2 \sim 3 \sim 6 \sim 2 \sim 1 \sim 5$.
- $W^{-1} = 5 \sim 1 \sim 2 \sim 6 \sim 3 \sim 2 \sim 1$.

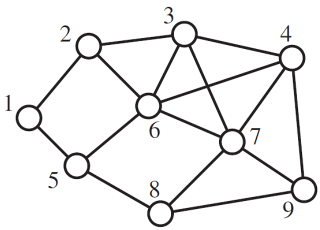
If $W = v_0 \sim v_1 \sim \cdots \sim v_{l-1} \sim v_l$, then its **reversal** $W^{-1} = v_l \sim v_{l-1} \sim \cdots \sim v_1 \sim v_0$ is also a walk.

Definition

Let $G = (V, E)$ be a graphs. A **walk** in G is a sequence (or lists) if vertices, with each vertex adjacent to the next; that is,

$$W = (v_0, v_1, \ldots, v_l) \qquad v_0 \sim v_1 \sim v_2 \sim \cdots \sim v_l$$

The **length** of this walk is l.



- 9

This is a walk of length zero.

- $1 \sim 5 \sim 1 \sim 5 \sim 1$

This is a **closed** walk.

- The sequence $(1, 6, 7, 9)$ is not a walk.

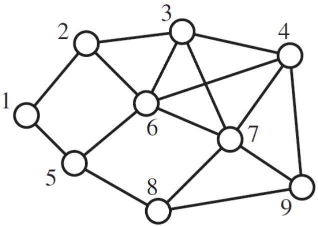
Concatenation

Let G be a graph. Suppose W_1 and W_2 are the following walks:

$$W_1 = v_0 \sim v_1 \sim \cdots \sim v_l, \qquad W_2 = w_0 \sim w_1 \sim \dots w_k$$

and suppose $v_l = w_0$. Their **concatenation**, denoted $W_1 + W_2$, is the walk

$$v_0 \sim v_1 \sim \cdots \sim (v_l = w_0) \sim w_1 \sim \cdots \sim w_k.$$



The concatenation of

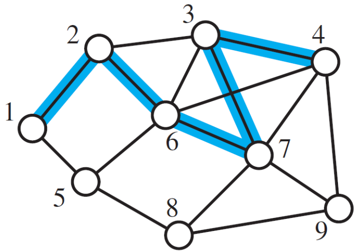
$$1 \sim 2 \sim 3 \sim 4 \qquad 4 \sim 7 \sim 3 \sim 2$$

and is

$$1 \sim 2 \sim 3 \sim 4 \sim 7 \sim 3 \sim 2$$

Definition

A **path** in a graph is a walk in which no vertex is repeated.



- The walk $1 \sim 2 \sim 6 \sim 7 \sim 3 \sim 4$ is a path.
- It is also called a $(1, 4)$ path.
- Note that the definition implies that no edge will be repeated either.

Do Problem 1!

Proposition

Let P be a path in a graph G. Then P does not traverse any edge of G more than once.

Proposition

Let P be a path in a graph G. Then P does not traverse any edge of G more than once.

Suppose FTSC that some path P in a graph G traverses the edge $e = uv$ more than once. Then

$$P = \cdots \sim u \sim v \sim \cdots \sim u \sim v \sim \ldots \qquad \text{or}$$
$$P = \cdots \sim u \sim v \sim \cdots \sim v \sim u \sim \ldots$$

Either case, we repeated u and thus P is not a path. $\Rightarrow\Leftarrow$. $\square$

Proposition

Let P be a path in a graph G. Then P does not traverse any edge of G more than once.

Suppose FTSC that some path P in a graph G traverses the edge $e = uv$ more than once. Then

$$P = \cdots \sim u \sim v \sim \cdots \sim u \sim v \sim \ldots \qquad \text{or}$$
$$P = \cdots \sim u \sim v \sim \cdots \sim v \sim u \sim \ldots$$

Either case, we repeated u and thus P is not a path. $\Rightarrow\Leftarrow$. $\square$

We have the following alternative definition:

Definition

A **path** is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set

$$E = \{v_i v_{i+1} : 1 \leq i < n\}$$

A path on n vertices is denoted by P_n.

Connections

- We want to capture the notion when vertices are connected.
- Two vertices are connected if there is a path connecting them.

Definition

Let G be a graph and let $u, v \in V(G)$. We say that u is **connected to** v provided there is a (u, v)-path in G.

We will show that this is an equivalence relation on $V(G)$.

Theorem

Let G be a graph. The is-connected-to relation is an equivalence relation on $V(G)$.

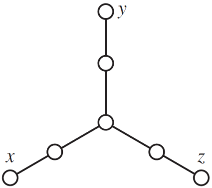
- Reflexive:
  If v is a vertex, then the path (v) is a (v, v)-path and so v is connected to v.
- Symmetric:
  Suppose, in a graph G, vertex u is connected to vertex v. This means there is a (u, v)-path P in G. Then it's reversal P^{-1} is a (v, u) path, and so v is connected to u.
- Transitive:
  Suppose x is connected to y, which is connected to z. This means that there is a (x, y) path P and a (y, z) path Q in G. We want to show that there is also an (x, z)-path. The obvious candidate is the concatenation $P + Q$.

This, however, doesn't work.

Theorem

Let G be a graph. The is-connected-to relation is an equivalence relation on $V(G)$.

- Transitive:
  Suppose x is connected to y, which is connected to z. This means that there is a (x, y) path P and a (y, z) path Q in G. We want to show that there is also an (x, z)-path. The obvious candidate is the concatenation $P + Q$.



  $P + Q$ might repeat vertices and therefore it might not be a path only a walk. We need to show that we can construct a path out of it.

Lemma

Let G be a graph and let $x, y \in V(G)$. If there is an (x, y)-walk in G, then there is an (x, y)-path in G.

Suppose there is an (x, y)-walk in the graph G. By the well ordering principle, there is a shortest (x, y)-walk P. We claim that this walk is actually a path.

Suppose FTSC, that P is not an (x, y)-path. Then there must be a repeated vertex u on the path:

$$P = x \sim \ldots ? \sim u \sim \cdots \sim u \sim ?? \sim \cdots \sim y.$$

Form a new walk P' by removing the blue part. Since $u \sim ??$, P' will be a walk shorter than P. $\Rightarrow\Leftarrow$ $\square$

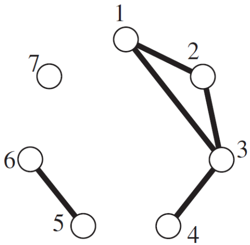
Theorem

Let G be a graph. The is-connected-to relation is an equivalence relation on $V(G)$.

- Transitive:
  Suppose x is connected to y, which is connected to z. This means that there is a (x, y) path P and a (y, z) path Q in G. We want to show that there is also an (x, z)-path. The concatenation $P + Q$ is an (x, z)-walk. Therefore by the Lemma, there is an (x, z)-path as well and thus x is connected to y. $\square$

Q: What are the equivalence classes for this equivalence relation?

- If u and v are in the same equivalence class then there is a path joining them.
- If u and v are in different equivalence classes then there is no path joining them.



The equivalence classes are

$$\{1,2,3,4\} \qquad \{5,6\} \qquad \{7\}$$

Consider the induced subgraphs

$$G[\{1,2,3,4\}] \qquad G[\{5,6\}] \qquad G[\{7\}]$$

Definition

A **component** of G is a subgraph of G induced on an equivalence class of the is-connected-to relation on $V(G)$.

Definition

A **component** of G is a subgraph of G induced on an equivalence class of the is-connected-to relation on $V(G)$.

Extreme cases:

- In an edgeless graph, each of it's vertices forms a component unto itself.
- It is possible that there is only one component.

Definition

A graph is called connected provided each pair of vertices in the graph is connected by a path; that is, for all $x, y \in V(G)$, there is an (x, y)-path.

Do Problem 2!

Cut edges and cut vertices

Definition

Let G be a graph. A vertex $v \in V(G)$ is a **cut vertex** of G provided $G - v$ has more components than G. Similarly, an edge $e \in E(G)$ is called a **cut edge** of G provided $G - e$ has more components than G.

In the special case when G is connected:

- A cut vertex v is such that $G - v$ is disconnected.
- An edge e is a cut edge if $G - e$ is disconnected.

However the cutting does not result in big disconnectedness.

Theorem

Let G be a connected graph and suppose $e \in E(G)$ is a cut edge of G. Then $G - e$ has exactly two components.

Theorem

Let G be a connected graph and suppose $e \in E(G)$ is a cut edge of G. Then $G - e$ has exactly two components.

By the definition of a cutting edge, $G - e$ has at least two components. We will show that it has exactly two.

FTSC assume that it is not so and let a, b, c be three vertices of $G - e$, all in different components. This means that there is no path connecting any two of them.

Let P be an (a, b)-path in G. Since this does not survive for $G - e$, P must traverse e. Suppose that $e = xy$ and

$$P = a \sim \cdots \sim x \sim y \sim \cdots \sim b.$$

Similarly, since G is connected, there is a (c, a)-path Q from c to a that must use the edge $e = xy$.

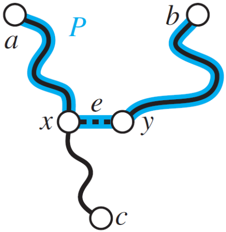
Q: Which vertex, x or y appears first on Q on the way from c to a?

Theorem

Let G be a connected graph and suppose $e \in E(G)$ is a cut edge of G. Then $G - e$ has exactly two components.

Q: Which vertex, x or y appears first on Q on the way from c to a?

- If x appears first, then the (c, x) portion of Q concatenated to the (x, a)-portion of P^{-1} produces a (c, a) walk in $G - e$. Therefore we also have a (c, a) path in $G - e$, which contradicts the assumption. $\Rightarrow\Leftarrow$.

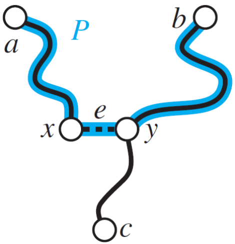


Theorem

Let G be a connected graph and suppose $e \in E(G)$ is a cut edge of G. Then $G - e$ has exactly two components.

Q: Which vertex, x or y appears first on Q on the way from c to a?

- If y appears first, then the (c, y) portion of Q concatenated to the (x, b)-portion of P produces a (c, b) walk in $G - e$. Therefore we also have a (c, b) path in $G - e$, which contradicts the assumption. $\Rightarrow\Leftarrow$.



Do Problem 3!

Outline

Cycles

Definition

A **cycle** is a walk of length at least three in which the first and last vertex are the same, but no other vertices are repeated.

The term can also refer to the (sub)graph consisting of the vertices and edges of such a walk. In other words, a cycle is a graph of the form $G = (V, E)$ where

$$V = \{v_1, v_2, \ldots, v_n\}, \qquad E = \{v_1 v_2, v_2 v_3, \ldots, v_{n-1} v_n, v_n v_1\}.$$

A cycle on n vertices is denoted C_n.

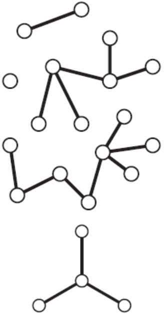
Forests and trees

Definition

Let G be a graph. If G contains no cycles, then we call G **acyclic** or alternatively a **forest**.

Definition

A **tree** is a connected, acyclic graph.



- A tree is a connected forest.
- On the left is a forest with five components, all are trees.
- Only tree on two vertices is K_2.
- Trees on three and four vertices:

Do Problem 4!

Theorem

Let T be a tree. For any two vertices a and b in $V(T)$, there is a unique (a, b) path. Conversely, any graph G with this property must be a tree.

$(\Rightarrow)$ Suppose T is a tree and let $a, b \in V(T)$. We need to prove that there is a unique (a, b)-path in T.

Since T is connected, there is at least one (a, b) path. FTSC assume that there were two (or more) different (a, b)-paths P or Q in T; Let x be the first place where they differ, i.e.

$$P : a \sim \cdots \sim x \sim y \sim \cdots \sim b, \qquad Q : a \sim \cdots \sim x \sim z \sim \cdots \sim b$$

This also means that the edge xy cannot be in Q. Consider the graph $T - xy$.

Theorem

Let T be a tree. For any two vertices a and b in $V(T)$, there is a unique (a, b) path. Conversely, any graph with this property must be a tree.

Consider the graph $T - xy$.

- Consider the following (x, y)-walk in T. Start at x, follow P^{-1} from x to a, then follow Q from a to b and then follow P^{-1} from b to y. This does not traverse xy and therefore there is a walk and thus a path from x to y in $T - xy$ Call it R.
- If we now concatenate R and xy, we get a cycle in a tree which is impossible. $\Rightarrow\Leftarrow$

$(\Leftarrow)$ FTSC assume that G has a cycle. Then take two points x, y on the cycle. Then clearly we have at least two ways to go from x to y. (Two direction along the cycle.)$\Rightarrow\Leftarrow$. $\square$

Leafs

Definition

A **leaf** of a graph is a vertex of degree one.

Theorem

Every tree with at least two vertices has at least two leafs.

Let T be a tree with at least two vertices and let P be a longest path in T. Then P has two or more vertices, say

$$P = v_0 \sim v_1 \sim \cdots \sim v_l, \qquad l \geq 1.$$

We claim that v_0 and v_l must be leaves. If this was not so, then v_0 would have another neighbor x that cannot be on P because that would give a cylce. But then appending x to the beginning of P would give a longer path contradicting the maximality of P. Similar argument proves that v_l is a leaf.

Leafs

Proposition

Let T be a tree and let v be a leaf of T. Then $T - v$ is a tree.

Clearly, $T - v$ cannot contain a cylce, because then the same cycle would be there in T as well, which is impossible as T is a tree.

To show that $T - v$ is connected, let $a, b \in V(T - v)$. Since T is connected, there is an (a, b) path P in T. FTSC assume that

$$P = a \sim \cdots \sim v \sim \cdots \sim b$$

Since v is not any of the endpoints, this would mean that $d(v) = 2$, which is not the case as v is a leaf. $\Rightarrow\Leftarrow$. This means that v cannot be on the path P and therefore P is an (a, b)-path in $T - v$ as well.

Since the choice of a, b were arbitrary, $T - v$ is connected. This finishes the proof.

Further characterization of trees

Theorem

Let G be a connected graph. Then G is a tree if and only if every edge is a cutting edge.

In this sense, trees are minimally connected graphs. For the proof, see the textbook.

Theorem

Let G be a connected graph on $n \geq 1$ vertices. Then G is a tree if and only if G has exactly $n - 1$ edges.

Do Problem 5(a)

Spanning subgraphs

Definition

Let G be a graph. A **spanning tree** of G is a spanning subgraph of G that is a tree.

Theorem

A graph has a spanning tree if and only if it is connected.

($\Rightarrow$) Suppose G has a spanning tree T. Let $u, v \in V(G)$. Since T is spanning, we have $V(T) = V(G)$, and so $u, v \in V(T)$. Since T is connected, there is a (u, v)-path P in T. Since T is a subgraph, P is also a (u, v)-path of G.

Spanning subgraphs

Definition

Let G be a graph. A **spanning tree** of G is a spanning subgraph of G that is a tree.

Theorem

A graph has a spanning tree if and only if it is connected.

($\Leftarrow$) Suppose G is connected. Let T be a spanning connected subgraph of G with the least number of edges. We claim that T is a tree.

By construction, T is connected. We show that every edge of T is a cut edge. Otherwise, if $E(T)$ were not a cut edge of T, then $T - e$ would be a smaller spanning connected subgraph of G. $\Rightarrow\Leftarrow$. $\square$

Do Problem 5(b) and Problem 6