Discrete Mathematics, 2016 Fall - Worksheet 17

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In all of the above problems explain your answer in full English sentences.

1. Please express the following permutations in disjoint cycle form.

   (a) $\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{bmatrix}.$

   Solution.
   $$(123456)$$

   (b) $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \end{bmatrix}$

   Solution.
   $$(124)(365)$$

2. Prove that in the cycle decomposition produced by the algorithm discussed on the slides, the resulting cycles are pairwise disjoint.

   Proof. For the sake of contradiction, assume that some $s \in A$ that is not an element of some cycle $(t, \pi(t), \pi^{(2)}(t), \dots)$ but some $\pi^{(k)}(s)$ is. Let (k) the smallest such iterate. Then $\pi^{(k-1)}(s)$ is not in the cycle. However, $\pi^{(k)}(s)$ being in the cycle implies that $\pi^{(m)}(t) = \pi^{(k)}(s)$ for some m. However. This means that if $a = \pi^{(k-1)}(s)$ and $b = \pi^{(m-1)}(t)$ then $a \neq b$ (since one of them is in the cycle, the other one isn't), but
   $$\pi(a) = \pi(b),$$
   contradicting the one-to-one-ness of π.

   □

3. How many permutations in S_n have exactly one cycle?

   Solution. *Note that without loss of generality, we can assume that our cycle looks like*
   $$(1a_2a_3 \dots a_n),$$
   because we can always cyclically 'rotate' a cycle. Now for $a_2 \dots a_n$, we can choose any arrangements of the remaining $n-1$ numbers, and thus the answer is $(n-1)!$.

4. Let π, σ be given by

$$\pi = (1)(2,3,4,5)(6,7,8,9), \qquad \sigma = (1,3,5,7,9,2,4,6,8)$$

Calculate the following

(a) $\pi \circ \sigma$

$$(1,4,7,6,9,3,2,5,8)$$

(b) $\sigma \circ \pi$

$$(1,3,6,9,8,2,5,4,7)$$

(c) π^{-1}

$$(1)(2,5,4,3)(6,9,8,7)$$

(d) $\pi^{-1} \circ \pi$.

$$id = (1)(2)(3)(4)(5)(6)(7)(8)(9)$$

5. Write the following permutations as the composition of transpositions and determine whether the permutation is even or odd.

(a) $(1,3)(2,4,5) = (1,3) \circ (2,5) \circ (2,4)$

(b) $(1,2,4,3)(5) = (1,3) \circ (1,4) \circ (1,2)$

(c) $[(1,3)(2,4,5)]^{-1} = (1,3)(2,5,4) = (1,3) \circ (2,4) \circ (2,5)$.

6. Prove the following group facts:

(a) If $(G, *)$ is a group and $g \in G$, then $(g^{-1})^{-1} = g$.

Proof. This follows from

$$g^{-1} * g = g * g^{-1} = e$$

□

(b) If $(G, *)$ is a group with identity element e, then $e^{-1} = e$.

Proof. This is an immediate consequence of

$$e * e = e$$

□

(c) If $(G, *)$ is a group and $g, h \in G$, then $(g * h)^{-1} = h^{-1} * g^{-1}$.

Proof. Note that on one hand,

$$h^{-1} * g^{-1} * g * h = h^{-1} * e * h = h^{-1} * h = e.$$

On the other hand,

$$g * h * h^{-1} * g^{-1} = g * e * g^{-1} = gg^{-1} = e,$$

which proves the claim. □

7. Show that the alternating group $(A_n, \circ)$ is indeed a group.

Proof. First we have to show that A_n is closed under $\circ$. This indeed holds as if $\tau, \sigma \in A_n$, then

$$\tau = \tau_1 \circ \cdots \circ \tau_a, \qquad \sigma = \sigma_1 \circ \cdots \circ \sigma_b$$

where τ_i and σ_i are transpositions and a, and b are even numbers. Then

$$\tau \circ \sigma = \tau_1 \circ \cdots \circ \tau_a \circ \sigma_1 \circ \cdots \circ \sigma_b$$

which means that $\tau \circ \sigma \in A_n$ as $a + b$ is even.

Associativity and the inverse element $e = id_{\{1,\ldots,n\}}$ are inherited from S_n. To show that there are inverses, note that the inverse of $\tau \in A_n$ in S_n is

$$\tau^{-1} = \tau_a^{-1} \circ \cdots \circ \tau_1^{-1}$$

which is a decomposition into an even number of transpositions. Thus $\tau^{-1} \in A_n$. $\square$