Discrete Mathematics, Section 002, Spring 2016

Lecture 11: Induction

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Outline

The induction machine

What does it take for a line of Dominos to fall?

1. We need to be able to tip over the first domino.
2. We need to be sure that whenever a domino falls, it knocks over the next Domino in the line.

If these two things are satisfied, we can be sure that all the Domino's will fall.

The induction machine

Proposition

Let n be a positive integer. The sum of the first n odd natural numbers is n^2.

This is a statement about infinitely many equations:

$$\begin{aligned} 1 &= 1^2 \\ 1+3 &= 2^2 \\ 1+3+5 &= 3^2 \\ 1+3+5+7 &= 4^2 \\ &\vdots \end{aligned}$$

We could ask a computer to check each equation but it would last forever.

The induction machine

Proposition

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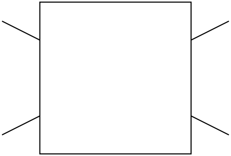
We are going to build a different kind of machine:



The induction machine

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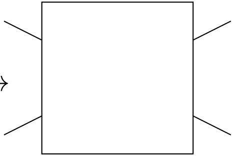
$1 = 1^2$ is true $\rightarrow$



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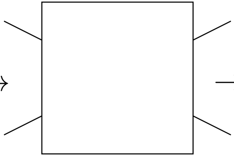

$\rightarrow$ $1 + 3 = 2^2$ is true

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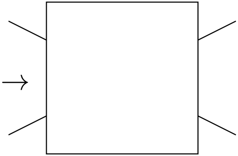

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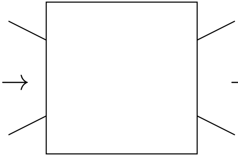

$\rightarrow$ $1 + 3 + 5 = 3^2$ is true

The induction machine

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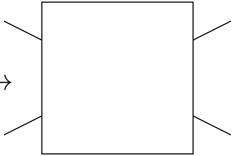

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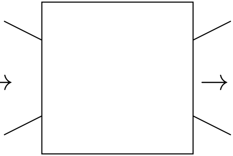


$\rightarrow$ $1 + 3 + 5 + 7 = 4^2$ is true

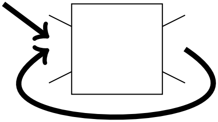
The induction machine

Proposition

Let n be a positive integer. The sum of the first n odd natural numbers is n^2.

If we can make sure that the machine works a 100% reliably, then we could just loop the output back on the input and 'ignite it with the first equation'.

$1 = 1^2$ is true



Then we can be sure, all the equations will be proved eventually, just as we could be sure that all the dominos will fall.

The induction machine

Proposition

Let n be a positive integer. The sum of the first n odd natural numbers is n^2.

1. Can we ignite?

$$\text{Obviously } 1 = 1^2.$$

2. Is the machine flawless?

$$\text{Feed: } 1 + 3 + 5 + \cdots + (2k-1) = k^2 \text{ is true}$$

Assuming this, add $2k+1$ to both sides and get

$$1 + 3 + \cdots + (2k-1) + (2k+1) = k^2 + (2k+1) =$$
$$= k^2 + 2k + 1 = (k+1)^2.$$

Therefore the output is

$$1 + 3 + 5 + \ldots (2k+1) = (k+1)^2 \text{ is true.}$$

The induction machine

Proposition

Let n be a positive integer. The sum of the first n odd natural numbers is n^2.

1. Can we ignite? $\rightarrow$ Yes.

$$\text{Obviously } 1 = 1^2.$$

2. Is the machine flawless? $\rightarrow$ Yes.

$$\text{Feed: } 1 + 3 + 5 + \cdots + (2k-1) = k^2 \text{ is true}$$

Assuming this, add $2k+1$ to both sides and get

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Therefore the output is

$$1 + 3 + 5 + \ldots (2k+1) = (k+1)^2 \text{ is true.}$$

Principle of mathematical induction

Theorem

Let A be a set of natural numbers. If

- $0 \in A$, and
- $\forall k \in \mathbb{N} : (k \in A \Rightarrow k + 1 \in A)$,

then $A = \mathbb{N}$.

Proof

Suppose for the sake of contradiction, that $A \neq \mathbb{N}$. Let $X = \mathbb{N} - A$, then $X \neq \emptyset$.

By the WOP, there is a smallest element in X. Note that $x \neq 0$ because $0 \in A$ is given. Therefore $x - 1 \in A$ by the definition of x.

By the second assumption, $x = (x - 1) + 1 \in A$, which is a contradiction.$\Rightarrow\Leftarrow$

Principle of mathematical induction

Proof by induction

To prove that every natural number has some property:

- Let A be the subset of $\mathbb{N}$ for which the result is true.
- (Basis step) Prove that $0 \in A$.
- (Inductive step) Prove that if $k \in A$, then $k + 1 \in A$. To do this:
  - Assume that the result is true for $n = k$. (Induction hypothesis)
  - Use the induction hypothersis to prove the result for $n = k + 1$.
- By the POMI, conclude $A = \mathbb{N}$.

Therefore the result is true for all natural numbers.

An example

Proposition

Let $n \in \mathbb{N}$. Then

$$0^2 + 1^2 + \dots n^2 = \frac{(2n+1)(n+1)n}{6}, \qquad (*)$$

Proof

We prove this result by induction on n. Let $A \subseteq \mathbb{N}$ be the subset for which the claim is true.

- **Basis step:** Note that when $n = 0$, both sides of $(*)$ is zero and therefore $n \in A$.
- **Induction hypothesis:** Suppose the result is true for $n = k$, i.e. $k \in A$ and

$$0^2 + 1^2 + \dots + k^2 = \frac{(2k+1)(k+1)k}{6}, \qquad (**).$$

[...]

Proposition

Let $n \in \mathbb{N}$. Then

$$0^2 + 1^2 + \dots n^2 = \frac{(2n+1)(n+1)n}{6}, \qquad (*)$$

Proof

[...]

- **Induction step:** Now we show that under the induction hypothesis, the result holds for $n = k + 1$. Indeed, add $(k+1)^2$ to both sides of $(**)$,

  $$0^2 + 1^2 + \dots (k+1)^2 = \frac{(2k+1)(k+1)k}{6} + (k+1)^2 =$$

  $$(\text{Probl 1 on WS}) = \frac{[(2(k+1)+1][(k+1)+1][k+1]}{6}.$$

  and therefore $k + 1 \in A$.

[...]

An example

Proposition

Let $n \in \mathbb{N}$. Then

$$0^2 + 1^2 + \ldots n^2 = \frac{(2n+1)(n+1)n}{6}, \qquad (*)$$

Proof.

[...]

- We have shown that $0 \in A$ and $(k \in A) \Rightarrow [(k+1) \in A]$. Therefore, by induction, $A = \mathbb{N}$ and the claim is true for all natural numbers.

$\square$

The basis step doesn't need to be about 0.

Proposition

Let n be a positive integer. Then

$$2^0 + 2^1 + \cdots + 2^{n-1} = 2^n - 1$$

Proof

We prove this by induction on n.

Basis step: The case $n = 1$ is true because $2^0 = 1$ and $2^1 - 1$.

Induction hypothesis: Suppose the result is true for $n = k$, i.e.

$$2^0 + 2^1 + \cdots + 2^{k-1} = 2^k - 1, \qquad (***)$$

[...]

The basis step doesn't need to be about 0.

Proposition

Let n be a positive integer. Then

$$2^0 + 2^1 + \cdots + 2^{n-1} = 2^n - 1.$$

Proof

[...]

Induction step: Now we show that under the ind. hyp., the result is true for $n = k + 1$. Indeed by adding 2^k to both sides of $(***)$,

$$2^0 + 2^1 + \cdots + 2^k = 2^k + 1 + 2^k = 2 \cdot 2^k + 1 = 2^{k+1} + 1.$$

which is exatly what we needed to show.

Practice this on Problem 2, (a)-(c) on the worksheet!

Example with an inequality

Proposition

Let n be a natural number. Then

$$10^0 + 10^1 + \cdots + 10^n < 10^{n+1}$$

Proof.

The proof is by induction on n. The basis case, when $n = 0$, is clear because $10^0 < 10^1$.

Assume that the result holds for $n = k$; that is, we have

$$10^0 + 10^1 + \cdots + 10^k < 10^{k+1}.$$

To show that the Proposition is true when $n = k + 1$, we add 10^{k+1} to both sides and find

$$10^0 + 10^1 + \ldots 10^{k+1} < 10^{k+1} + 10^{k+1} =$$

$$= 2 \cdot 10^{k+1} < 10 \cdot 10^{k+1} = 10^{k+2}.$$

Therefore the result holds when $n = k + 1$. $\square$

Now do Problem 2 (d)-(e).

Further example

Proposition

Let $n \in \mathbb{N}$. Then $4^n - 1$ is divisible by 3.

Proof.

The proof is by induction on n. The basis case, $n = 0$, is clear since $4^0 - 1 = 1 - 1 = 0$ is divisible by 3.

Suppose that the result is true for $n = k$; that is, $3 | 4^k - 1$. To show that then the result is also true for $n = k + 1$, note

$$4^{k+1} - 1 = 4 \cdot 4^k - 1 = 4(4^k - 1) + 4 - 1 = 4(4^k - 1) + 3.$$

Since $4^k - 1$ and 3 are both divisible by 3, it follows that $4(4^k - 1) + 3$ is also divisible by 3, which means that $4^{k+1} - 1$ is divisible by 3 too. $\square$

Outline

Principle of Mathematical Induction - strong version

We like induction, because

- The basis case is usually easy.
- When proving the induction step we can work with the case $n = k$ assumed true by the induction hypothesis.
- This means that we have a lot more to work with than just the assumptions of the theorem.

On the other hand note that we have something even more.

- When the machine proves $k \to k+1$, it has already proved

$$0 \to 1 \to 2 \to \cdots \to k-1 \to k.$$

- This means that we can proceed cumulatively, i.e.

$$0 \to 1, \qquad 0, 1 \to 2 \qquad 0, 1, 2 \to 3 \qquad \ldots$$

$$\ldots \qquad 0, 1, \ldots, k \to k+1.$$

Principle of Mathematical Induction - strong version

Theorem

Let A be a set of natural numbers. If

- $0 \in A$ and
- for all $k \in \mathbb{N}$, $0, 1, \ldots, k \in A$ implies $k + 1 \in A$,

then $A = \mathbb{N}$.

To prove that every natural number has some property:

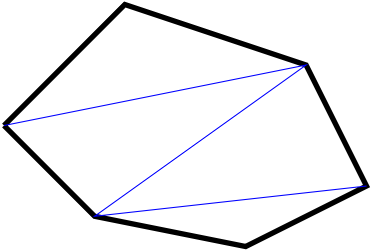
- Let A be the set of natural numbers for which the result is true.
- Prove $0 \in A$. (Basis step)
- Prove that $0, 1, \ldots, k \in A$ implies $k + 1 \in A$. (inductive step)
- By the SPOMI, $A = \mathbb{N}$.

An example

Definition

A **triangulation** of a polygon is to draw diagonals through the interior of the poligons so that

- The diagonals do not cross each other.
- Every region created is a triangle.



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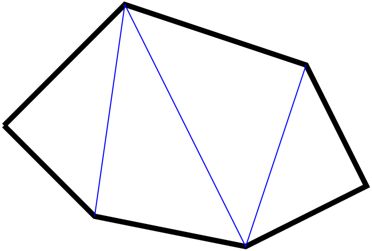
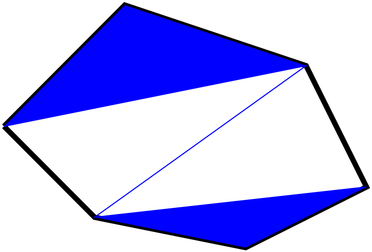


An example

Definition

A triangle in a triangulation is **exterior** if two of their three sides are on the exterior of the original polygon.



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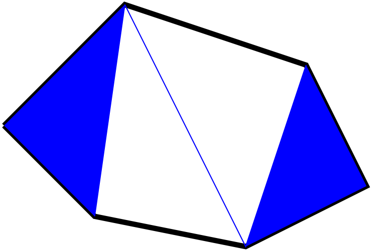


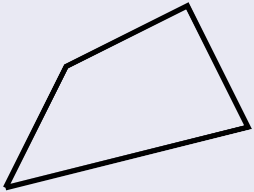
Proposition

If a polygon with four or more sides is triangulated, then at least two of the triangles formed are exterior.

Proof

Let n be the number of sides of the polygon. We prove the Proposition by strong induction on n.

Base case: For $n = 4$, the only way to triangulate the polygon is to draw in one of the two possible diagonals. Each way, the resulting two triangles are exterior.



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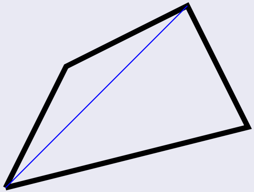

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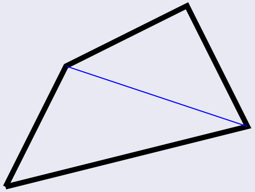


[...]

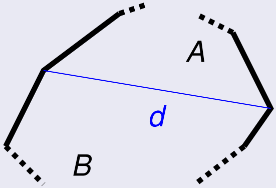
Proposition

If a polygon with four or more sides is triangulated, then at least two of the triangles formed are exterior.

Proof

[...]

Strong induction hypothesis: Suppose that the result has been proved for all polygons with $n = 4, 5, ..., k$ sides and let P be any triangulated polygon with $k + 1$ sides. We will show that at least two of the sides are exterior.



Let d be one of the diagonals. Then d separates P into two triangulated polygons A and B with fewer sides than P.

[...]

Proposition

If a polygon with four or more sides is triangulated, then at least two of the triangles formed are exterior.

Proof

[...]

- If A is not a triangle, then since A has at least four but at most k sides, the strong induction hypothesis implies that two or more A-triangles are exterior to A. The only way for one of these triangles not to be exterior to P as well is if one of the sides is d. Since that can happen to at most one of the triangles exterior to A, the other one also has to be exterior to P.
- If A is a triangle then A is an exterior triangle.
- Similarly, there is a triangle in B exterior to P.

Therefore, we have at least two exterior triangles to P. $\square$

Do problem 3-4 on the worksheet!