Discrete Mathematics, 2016 Spring - Worksheet 4

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Instructor: Zsolt Pajor-Gyulai, CIMS

In all of the above problems explain your answer in full English sentences.

1. Solve the equation $n! = 720$ for n.

   Obviously $n = 6$ *solves this equation. Since* $n_1 < n_2$ *implies* $n_1! < n_2!$*, this is the only possible solution.*

2. There are six different French books, eight different Russian books and five different Spanish books.

   (a) In how many different ways can these books be arranged on a bookshelf? *This is just the same as ordering 19 books and so the answer is* $19!$.

   (b) In how many different ways can these books be arranged if all books in the same language are grouped together? *In this case we can first choose the order of the books* $3!$ *ways and then within each group order the books on the same language (so* $6!$, $!8!$ *and* $5!$ *ways respectively. Thus, the total number of ways is* $3!6!8!5!$.

3. Calculate the following products:

   (a) $\prod_{k=1}^{4}(2k+1) = (2+1)(2\cdot 2+1)(2\cdot 3+1)(2\cdot 4+1) = 3\cdot 5\cdot 9 = 135$

   (b) $\prod_{k=1}^{n}\frac{1}{k} = 1\cdot\frac{1}{2}\cdot\ldots\cdot\frac{1}{n} = \frac{1}{1\cdot\ldots\cdot n} = \frac{1}{n!}$

4. Can factorial be extended to negative integers? Think about the formula $n! = n(n-1)!$. What would be the value of $(-1)!$?

   No, note that $1 = 0! = 0(-1)!$ *and therefore* $(-1)! = 1/0$ *which is not defined.*

5. Write out the following sets by listing their elements between curly braces and find their cardinality.

   (a) $A = \{x \in \mathbb{N} : x \leq 10 \text{ and } 3|x\} = \{0, 3, 6, 9\}$ and $|A| = 4$.

   (b) $B = \{x \in \mathbb{Z} : x \text{ is a prime and } 2|x\} = \{2\}$ and $|B| = 1$

(c) $C = \{x \in \mathbb{Z} : 10|x \text{ and } x|100\} = \{-100, -50, -20, -10, 10, 20, 50, 100\}$ and $|C| = 8$

(d) $D = \{x \in \mathbb{Z} : 1 \leq x^2 \leq 2\} = \{-1, 1\}$ and $|D| = 2$.

6. For each of the following sets, find a way to rewrite the set using set builder notation.

   (a) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = \{x \in \mathbb{N} : 1 \leq x \leq 10\}$

   (b) $\{-8, -6, -4, -2, 0, 2, 4, 6, 8\} = \{x \in \mathbb{Z} : x = 2k \in \mathbb{Z}, -4 \leq k \leq 4\}$

   (c) $\{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\} = \{x \in \mathbb{Z} : x = k^2, k \in \mathbb{Z}, k = 1, ..., 10\}$

7. (a) Let $A = \{x \in \mathbb{Z} : 4|x\}$ and let $B = \{x \in \mathbb{Z} : 2|x\}$. Prove that $A \subseteq B$.

   We are going to prove that if $x \in A$ then $x \in B$. Let $x \in A$, then $4|x$ and therefore there is an integer k such that $x = 4k = 2(2k)$. Since $2k$ is an integer, this means that $2|x$ and thus $x \in B$ and the claim is proved.

   (b) Generalize the previous problem. Let $a, b \in \mathbb{Z}$ and let

   $$A = \{x \in \mathbb{Z} : a|x\}, \qquad B = \{x \in \mathbb{Z} : b|x\}.$$

   Find and prove a necessary and sufficient conditions for $A \subseteq B$. In other words, find and prove a theorem of the form

   "$A \subseteq B$ if and only if *some condition involving a and b*."

   Proposition 1. *$A \subseteq B$, if and only if $b|a$.*

   Proof. We first show that $A \subseteq B$ implies $b|a$, by showing that $b \nmid a$ implies $A \not\subseteq B$, which in turn we will show by exhibiting a counterexample. Note that $a \in A$. However $a \notin B$ in this case which means exactly that $A \not\subseteq B$.

   To show the other direction, assume $b|a$. This means that there is an integer k such that $a = kb$. Then if $x \in A$, i.e. there is an integer k_1 such that $x = k_1 a = k_1(kb) = k_1 kb$ and thus $b|x$ and therefore $x \in B$. This proves $A \subseteq B$. $\square$

8. Compute each of the following by writing either $\in$ or $\subseteq$ in place of $\bigcirc$.

   - $2 \in \{1, 2, 3\}$
   - $\{2\} \subseteq \{1, 2, 3\}$
   - $\{2\} \in \{\{1\}, \{2\}, \{3\}\}$
   - $\emptyset \subseteq \{1, 2, 3\}$
   - $\mathbb{N} \subseteq \mathbb{Z}$
   - $\{2\} \subseteq \mathbb{Z}$

9. This problem is about power sets.

   (a) Write out the elements and give the cardinality of the set $2^{\emptyset}$. (Hint: Start with the cardinality.)

   $2^{\emptyset} = \{\emptyset\}$ *and* $|2^{\emptyset}| = 1$

   (b) Find the cardinality of the following sets.

      i. $|2^{2^{\{1,2,3\}}}| = 2^{|2^{\{1,2,3\}}|} = 2^{2^{|\{1,2,3\}|}} = 2^{2^3} = 2^8$

      ii. $|\{x \in 2^{\{1,2,3,4\}} : |x| = 1\}| = 4$ because there are four one element subsets of $\{1, 2, 3, 4\}$.

   (c) $\{2\} \in 2^{\mathbb{Z}}$

10. (Russel's paradox) Consider the set of all sets R that are not elements of themselves, i.e. $x \in R$ if x is a set but $x \notin x$. Does R contain itself as an element? The answer to this question signifies the breakdown of naive set theory which led to the development of axiomatic set theory. (You need to take a course in mathematical logic to learn more about this.)

    On one hand, if $R \in R$ then R is a set that is an element of itself and then $R \notin R$, a contradiction. However if $R \notin R$, then R is not an element of itself and thus $R \in R$ another contradiction. The only way this does not break the universe is if the question itself is non-sensical, i.e. there is no such set R in the first place.