Discrete Mathematics, 2016 Spring - HW 6

October 14, 2016

Instructor: Zsolt Pajor-Gyulai

Courant Institute of Mathematical Sciences, NYU

To get full credit in all of the problems, use rigorous justification and unless otherwise indicated, make sure that your solution reads as a perfect English sentence. You should only assume integers, operations and order relations as given. If you use a statement or a definition from the textbook, make sure to indicate it.

Section 20

1) Please state the contrapositive of each of the following statements:
   (a) If p is prime, the $2^p - 2$ is divisible by p.
   (b) If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.
   (c) If the battery is fully charged, the car will start.
   (d) If A or B, then C.

10) Prove by contradiction: Let a be a number with $a > 1$. Prove that $\sqrt{a}$ is strictly between 1 and a.

Section 21

4-5) Prove the following statements by smallest counterexample
   (a) $n! \leq n^n$ for all positive integers n.
   (b) $\binom{2n}{n} \leq 4^n$ for all natural numbers n.

7) The **Fibonacci numbers** are the list of integers $(1, 1, 2, 3, 5, 8, \ldots) = (F_0, F_1, F_2, \ldots)$ where

$$F_0 = 1, \qquad F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \text{ for } n \geq 2.$$

   (a) Read the proof of the fact that for all $n \in \mathbb{N}$, we have $F_n \leq 1.7^n$ on p133 in the textbook.
   (b) Now prove using the smallest counterexample method that $F_n > 1.6^n$ whenever $n \geq 29$.

Section 22

4-5,16) Prove the following equations and inequalities by induction.

(a) $1^3 + 2^3 + \ldots n^3 = \frac{n^2(n+1)^2}{4}$.

(b) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$

10) Prove, by induction, that the sum of the angles of a convex n-gon (with $n \geq 3$) is $180(n-2)$ degrees.

12) The *Towers of Hanoi* is a puzzle consisting of a board with three dowels and a collection of n disks of n different sizes (radii). The disks have holes drilled through their centers so that they can fit on the dowels on the board. Initially, all the disks are on the first dowel and are arranged in size order (from the largest on the bottom to the smallest on the top).

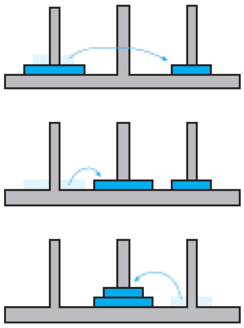

Figure 1: Solution of the Towers of Hanoi puzzle for $n = 2$.

The object is to move all the disks to another dowel in as few moves as possible. Each moves consists of taking the top disk off one of the stacks and placing it on another stack, with the added condition that you may not place a larger disk atop a smaller one. The figure shows how to solve the puzzle in three moves when $n = 2$. Prove that for every positive integer n, the puzzle can be solved in $2^n - 1$ moves.