Discrete Mathematics, 2016 Spring - Worksheet 14

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In all of the above problems explain your answer in full English sentences.

1. Which of the following relations are functions?

   (a) $\{(1,2),(3,4)\}$ This one.

   (b) $\{(x,y) : x,y \in \mathbb{Z}, y = 2x\}$ This one.

   (c) $\{(x,y) : x,y \in \mathbb{Z}, x + y = 0\}$ This one.

   (d) $\{(x,y) : x,y \in \mathbb{Z}, xy = 0\}$ Not this one, $(0,y)$ is in it for every $y \in \mathbb{Z}$.

   (e) $\{(x,y) : x,y \in \mathbb{Z}, y = x^2\}$ This one.

   (f) $\emptyset$ This one vacuously.

   (g) $\{(x,y) : x,y \in \mathbb{Q}, x^2 + y^2 = 1\}$ Not this one, e.g. $(0,1)$ and $(0,-1)$ are both in it.

   (h) $\{(x,y) : x,y \in \mathbb{Z}, x|y\}$ Not this one, e.g. $(2,4)$ and $(2,8)$ are both in it.

   (i) $\{(x,y) : x,y \in \mathbb{N}, x|y, \text{ and } y|x\}$ Not this one, eg. $(1,1)$ and $(1,-1)$ are both in it.

   (j) $\{(x,y) : x,y \in \mathbb{N}, \binom{x}{y} = 1\}$ Not this one, e.g $(2,0)$ and $(2,2)$ are both in it.

2. For those relations that are functions in Problem 1, find their domain and image.

   - For the function in (a), the domain is $(1,3)$, while the image is $(2,4)$.
   - For the function in (b), the domain is $\mathbb{Z}$ while the image is the even numbers.
   - For the function in (c), both the domain and the image are $\mathbb{Z}$.
   - For the function in (e), the domain is $\mathbb{Z}$ while the image are those integers that are themselves squares of an integer.
   - For the function in (f), both the domain and the image are empty.

3. For each of the following functions f, find the image of the function, im.

   (a) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x + 1$.

   Solution. *The image of the function is all odd integers.*

   (b) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{1+x^2}$.

Solution. *To find the image of the function, look at the equation* $b = \frac{1}{1+x^2}$*. Rearranging this gives*

$$x^2 = \frac{1}{b} - 1.$$

Clearly, this equation only has solution(s) when $b \in (0,1]$*, otherwise the right hand side is negative. Therefore Im*$(f) = (0,1]$*.*

(c) $f : [-1,1] \to \mathbb{R}$ defined by $f(x) = \sqrt{1-x^2}$.

Solution. *By the definition of the square root,* $\sqrt{1-x^2}$ *is always non-negative and therefore we only have to check, when is there a solution to* $b = \sqrt{1-x^2}$ *with* $b \geq 0$*. Squaring this gives* $b^2 = 1 - x^2$ *and thus* $x^2 = 1 - b^2$*. This equation has a solution if and only if* $|b| \leq 1$ *and in this case the solution is in Dom*$(f) = [-1,1]$*. Combining this with* $b \geq 0$*, we get Im*$(f) = [0,1]$*.*

4. Which of the functions in Problem 1 are one-to-one? What are the inverses of these functions?

- The function in (a) is one-to-one and its inverse is given by $\{(2,1),(4,3)\}$.
- The function (b) is one-to-one and its inverse is given by $f^{-1} : \{\text{even numbers}\} \to \mathbb{Z}$, given by

$$\{(x,y) : x, y \in \mathbb{Z}, x \text{ is even}, \text{y} = \text{x}/2\}$$

- The function in (c) is one-to-one and it is its own inverse.
- The function in (f) is one-to-one vacuously and is its own inverse.

5. For each of the functions, determine whether the function is one-to-one, onto, or both. Prove your assertions.

(a) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x^2$.

Solution. f *is not one to one as e.g.* $2(-2)^2 = 8 = 2 \cdot 2^2$*. Neither is it onto as* $f(x) \geq 0$ *for every* $x \in \mathbb{Z}$ *and therefore e.g.* -2 *is not attained.*

(b) $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(x) = (-1)^x\left(\lfloor x/2 \rfloor + 1\right)$, where $\lfloor . \rfloor$ is the integer part function.

Solution.
- *To see that this function is one to one, let us assume that there are* $x, y \in \mathbb{N}$ *such that* $f(x) = f(y)$*. Note that* $f(x)$ *is positive if and only if* x *is even and therefore* x *and* y *are both simultaneously even or odd.*
  If they are both even, $\lfloor x/2 \rfloor = x/2$ *and* $\lfloor y/2 \rfloor = y/2$ *and therefore*

  $$x/2 + 1 = f(x) = f(y) = y/2 + 1$$

  which yields $x = y$*.*
  If they are both odd, $\lfloor x/2 \rfloor = (x-1)/2$ *and* $\lfloor y/2 \rfloor = (y-1)/2$ *and therefore*

  $$\frac{x-1}{2} + 1 = f(x) = f(y) = \frac{y-1}{2} + 1$$

  from which $x = y$*.*

- *To show that f is not onto, note that $|f(x)| = \lfloor x/2 \rfloor + 1 > 1$ which implies that 0 is not in the image.*

6. Give an example of a set A and a function $f : A \to A$ where f is onto but not one to one. Also give one where f is one-to-one but not onto.

   Solution. *This example was mentioned in class. $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} x + 3 & x \leq 0 \\ x - 3 & x > 0 \end{cases}$$

   is not one-to-one as e.g. $f(-3) = f(3) = 1$. I leave the verification that it is onto to you.

   The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x}{1+|x|}$ does the job as you can verify.

   Note that in both examples, the set A was infinite. Indeed, as we will discuss it next time, this cannot happen for finite sets.