Discrete Mathematics, 2016 Spring - HW 8

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To get full credit in all of the problems, use rigorous justification and unless otherwise indicated, make sure that your solution reads as a perfect English sentence. You should only assume integers, operations and order relations as given. If you use a statement or a definition from the textbook, make sure to indicate it.

Section 24

10-11) Let a, b, c real numbers.

(a) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$. For which values of a and b is f one-to-one? ... onto $\mathbb{R}$?

(b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax^2 + bx + c$. For which values of a,b,c is f one-to-one?...onto $\mathbb{R}$?

23) Let $f : A \to B$ be a function. This induces a function (denoted again by f with a big abuse of notation) $f : 2^A \to 2^B$ the following way. For $x \subseteq A$, define

$$f(X) = \{f(x) : x \in X\},$$

the set of all values f takes when applied to elements of X. For example if $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(x) = x^2$ and $X = \{1, 3, 5\}$ then $f(X) = \{1^2, 3^2, 5^2\} = \{1, 9, 25\}$. This is called the **image of X under f**. In all the following examples, find $f(X)$.

(a) $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = |x|$, and $X = \{-1, 0, 1, 2\}$.

(b) $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2^x$, and $X = [-1, 1]$.

(c) $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x$, and $X = [0, \pi]$.

24) In the same spirit as the previous problem we can define the **inverse image of a set $Y \subseteq X$ under f**. Formally, if $f : A \to B$ is a function and $Y \subseteq B$, then define

$$f^{-1}(Y) = \{x \in A : f(x) \in Y\},$$

or in other words the set of all elements of A that are mapped into a value in Y. For example suppose $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$ and $Y = \{4, 9\}$, then $f^{-1}(Y) = \{-3, -2, 2, 3\}$. In the following examples, find $f^{-1}(Y)$.

(a) $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = |x|$, and $Y = \{1, 2, 3\}$.

(b) $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$, and $Y = \{-2, 3, 4\}$.

(c) $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1/(1 + x^2)$, and $Y = \{-1/2\}$.

19) Let n be a positive integer. Let A_n be the set of positive divisors of n that are less than $\sqrt{n}$ and let B_n be the set of positive divisors of n that are greater than $\sqrt{n}$. That is:

$$A_n = \{d \in \mathbb{N} : d|n, d < \sqrt{n}\}, \qquad B_n = \{d \in \mathbb{N} : d|n, d > \sqrt{n}\}.$$

For example when $n = 24$, then $\sqrt{24} \approx 4.899$ and so $A_{24} = \{1, 2, 3, 4\}$ and $B_{24} = \{6, 8, 12, 24\}$.

(a) Find a bijection $f : A_n \to B_n$. This implies that $|A_n| = |B_n|$.

(b) Prove that a positive integer has an odd number of positive divisors if and only if n is a perfect square. (Hint: Perfect squares have a divisor that is not in $A_n \cup B_n$).

21) Let f be a function. We say that f is two-to-one provided for each $b \in \mathrm{im}f$ there are exactly two elements $a_1, a_2 \in \mathrm{dom}f$ such that $f(a_1) = f(a_2) = b$. For a positive integer n, let A be a $2n$ element set and B be an n-element set. How many functions $f : A \to B$ are two-to-one?

Section 25

3) How large a group of people do we need to consider to be certain that three members of the group have the same initials (first, middle, last)?

12) (a) Read the discussion before and proof of Theorem 25.3 on p179-180.

(b) Find a sequence of nine distinct integers that does not contain a monotone subsequence of length four. Generalize your construction by showing how to construct (for every positive integer n) a sequence of n^2 distinct integers that does not contain a monotone subsequence of length $n + 1$. (Use the hint at the back of the textbook)