# Discrete Mathematics, Section 002, Spring 2016

## Lecture 3: Counterexamples,Boolean Algebra, Lists

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# Refresher question

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**Answer**

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# Outline

1. Counterexamples
2. Boolean Algebra
3. Lists

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$$a = 5, b = -5 \qquad \rightarrow \qquad 5 \cdot (-1) = -5, \qquad (-5) \cdot (-1) = 5$$

This works!

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**Refuting a false if-then statement via a counterexample.**

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Only shows that $xy = 1$. $\rightarrow$ $x = y = -1$ works too!

This would yield $a = -b$ and now it is easy to pick the counterexample.

# Outline

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Useful for reasoning about numbers:

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- Value of an expression like $3x - 4$ depends on $x$.
- When e.g. $x = 1$, the value is $3(-1) - 4 = -1$.

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- Values of expressions can be summarized in **Truth tables.**

Truth table of $x \wedge y$ (AND):

| x | y | x$\wedge$ y |
|---|---|---|
| True | True | True |
| True | False | False |
| False | True | False |
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Truth table of $x \vee y$ (OR):

| x | y | x∨ y |
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| True | True | True |
| True | False | True |
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Truth table of $\neg x$ (NOT):

| x | ¬ x |
|---|---|
| True | False |
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# Identities

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$$(x + y)^2 = x^2 + 2xy + y^2$$

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| x | y | x∧ y | $\neg(x \wedge y)$ | ¬ x | ¬ y | (¬ x)∨ (¬ y) |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F |
| T | F | F | T | F | T | T |
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| F | T | F | T | T | F | T |
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We say $\neg(x \wedge y)$ and $(\neg x) \vee (\neg y)$ are **logically equivalent.**

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Basic algebraic properties of the operations $\wedge, \vee, \neg$:

- **(Commutative properties)**

$$x \wedge y = y \wedge x \qquad , x \vee y = y \vee x.$$

- (**Associative properties**)

$$(x \wedge y) \wedge z = x \wedge (y \wedge z), \qquad (x \vee y) \vee z = x \vee (y \vee z).$$

- (**Identity elements**)

$$x \wedge True = x, \qquad x \vee False = x.$$

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To show that two Boolean expressions are logically equivalent: Construct a truth table showing the values of the two expressions for all possible values of the variables.

- **(Distributive properties)**

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

- (**DeMorgan's Laws**)

$$\neg(x \wedge y) = (\neg x) \vee (\neg y), \qquad \neg(x \vee y) = (\neg x) \wedge (\neg y).$$

- $\neg(\neg x) = x$.
- $x \wedge x = x$ and $x \vee x$.
- $x \wedge (\neg x) = \mathit{False}$ and $x \vee (\neg x) = \mathit{True}$.

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- We also introduce *implies* ($\rightarrow$) and *if-and-only-if* ($\leftrightarrow$).

Models $A \Rightarrow B$, only false when $A$ is true but $B$ is not.

| x | y | x$\rightarrow$ y |
|---|---|---|
| True | True | True |
| True | False | False |
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Models $A \Leftrightarrow B$, false when one is true but the other one is false.

| x | y | x$\leftrightarrow$ y |
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| True | True | True |
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# Outline

# Collections

In this course we consider two types of collections:

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- **Sets**: Unordered collections.

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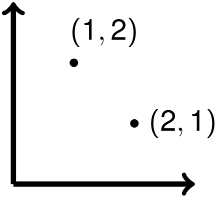
- **Ordered pair:** A list of length two.

Example of lists:

- A point on the $x - y$ plane.



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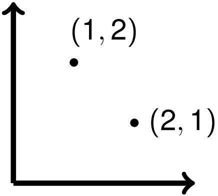


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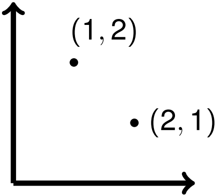


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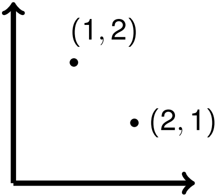


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**Equality of lists**

Two lists are **equal** provided they have the same length, and elements in the corresponding positions on the two lists are equal.

E.g. lists $(a, b, c)$ and $(x, y, z)$ are equal if $a = x$, $b = y$, and $c = z$.

# Counting two element lists

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Write out all possibilities:

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We wrote this in a manner ensuring we have neither repeated nor omitted any such lists.

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- First row contains all lists beginning with 1, second row contains the ones beginning with 2, etc.
- There are 4 such rows.
- Each row contains four lists.

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There are $n \cdot n = n^2$ such lists.

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- There are 5 rows, each containing 4 elements.

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- There are 5 rows, each containing 4 elements.
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- The choices are not independent, but that doesn't matter.

# Counting two element lists

**Question**

How many ordered pairs are there where the entries in the list may be any of the digits $1, 2, 3, 4, 5$, but where the two numbers in the list are different?

$$\begin{array}{ccccc} - & (1,2) & (1,3) & (1,4) & (1,5) \\ (2,1) & - & (2,3) & (2,4) & (2,5) \\ (3,1) & (3,2) & - & (3,4) & (3,5) \\ (4,1) & (4,2) & (4,3) & - & (4,5) \\ (5,1) & (5,2) & (5,3) & (5,4) & - \end{array}$$

- There are 5 rows, each containing 4 elements.
- 5 choices for the first element and 4 for the second.
- The choices are not independent, but that doesn't matter.

There are $5 \cdot 4 = 20$ such lists.

**Theorem (Multiplication Principle)**

Consider two-element lists for which there are $n$ choices for the first element, and for each choice of the first element, there are $m$ choices for the second element. Then the number of such lists is $nm$.

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## Proof.

Construct a chart of all possible lists. Each row of this chart contains all the two-element lists that begin with a particular element. Since there are $n$ choices for the first element, there are $n$ rows in the chart. Since, for each choice of the first element, there are $m$ choices for the second element, we know that every row of the chart has $m$ entries. Therefore the number of lists is

$$\underbrace{m + m + \cdots + m}_{n \text{ times}} = n \times m$$

$\square$

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(1,1,1) & (1,1,2) & (1,1,3) \\
(1,2,1) & (1,2,2) & (1,2,3) \\
(1,3,1) & (1,3,2) & (1,3,3) \\
(2,1,1) & (2,1,2) & (2,1,3) \\
(2,2,1) & (2,2,2) & (2,2,3) \\
(2,3,1) & (2,3,2) & (2,3,3) \\
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- Every row of the chart corresponds to a fixed combination of the first two elements.
- Every row consists of 3 elements.
- How many rows are there in this chart?

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- How many rows are there in this chart?

We have solved this problem! $\rightarrow$ $3 \times 3 = 9$.

# Longer lists

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- How many rows are there in this chart? $\rightarrow 3 \times 3 = 9$

There are $9 \times 3 = 3 \times 3 \times 3 = 27$ such lists.

## Question

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- For the first element, we have 5 choices.

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- For the third place, we have to choose from the 3 remaining.

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- For the first element, we have 5 choices.
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- For the third place, we have to choose from the 3 remaining.

There are $5 \times 4 \times 3 = 60$ such lists.

# In general

How many lists are there of length $k$, in which each element of the list is selected from among $n$ possibilities.

1. Repetitions allowed:

$$\underbrace{n \times n \times \cdots \times n}_{k \ \ times} = n^k$$

2. Repetitions not allowed:

$$n \times (n-1) \times (n-2) \times \cdots \times n-(k-1)$$

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when $k \leq n$, and 0 if $k > n$.

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**Notation**

The number

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## Theorem

*The number of lists of length $k$ whose elements are chosen from a pool of $n$ possible elements is*

$$= \begin{cases} n^k & \textit{if repetitions are permitted} \\ (n)_k & \textit{if repetitions are forbidden} \end{cases}$$