Discrete Mathematics, 2016 Spring - HW 9

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To get full credit in all of the problems, use rigorous justification and unless otherwise indicated, make sure that your solution reads as a perfect English sentence. You should only assume integers, operations and order relations as given. If you use a statement or a definition from the textbook, make sure to indicate it.

Section 26

6-7) (a) Let A and B be sets and suppose $f : A \to B$ is one-to-one and onto. Prove that then

$$f \circ f^{-1} = \mathrm{id}_B, \qquad \text{and} \qquad f^{-1} \circ f = \mathrm{id}_A$$

(b) Suppose A and B are sets, and f and g are functions with $f : A \to B$ and $g : B \to A$. Prove that if $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$ then f is one to one and $g = f^{-1}$.

13-14) Let A be a set and $f : A \to A$. Then $f \circ f$ is also a function from A to itself. Let us write $f^{(n)}$ for the n-fold composition

$$f^{(n)} = f \circ f \circ \cdots \circ f.$$

Of course $f^{(1)} = f$.

(a) What is a sensible meaning for $f^{(0)}$?

(b) If f is invertible, prove that $(f^{-1})^{(n)} = (f^{(n)})^{-1}$.

(c) Find a formula for the n-th iteration of the function $f :: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$ given the starting value $x_0 = 1$. That is, find the nth term of

$$f(1), f^{(1)}(1), f^{(2)}(1), \ldots$$

Section 27

1) Consider the permutation $\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 1 & 6 & 5 & 3 & 8 & 9 & 7 \end{bmatrix}$. Express π as

(a) As a set of ordered pairs.

(b) In cycle notation.

4) How many permutations in S_n do not have a cycle of length one in their disjoint cycle notation.

13) Let $\pi = (1,2)(3,4,5,6,7)(8,9,10,11)(12) \in S_{12}$.

   (a) Find the smallest positive integer k for which

   $$\pi^{(k)} = \pi \circ \pi \circ \cdots \circ \pi = \iota.$$

   (b) Generalize the previous argument. If π's are disjoint cycles have lenghts $n_1, n_2, \ldots, n_t$, what is the smallest integer k so that $\pi^{(k)} = \iota$?

11) Let $\tau_1, \tau_2, \ldots, \tau_a$ be transpositions and suppose

$$\pi = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_a.$$

Prove that

$$\pi^{-1} = \tau_a \circ \tau_{a-1} \circ \cdots \circ \tau_1$$

Reading) Read Definition 27.14, the following discussion, and the proof of Lemma 27.13 on pages 193-195.

Section 40

2) Let $*$ be a binary operation defined on the real numbers $\mathbb{R}$ by $x * y = x + y - xy$.

   (a) Is $*$ closed on the real numbers?

   (b) Is $*$ commutative?

   (c) Is $*$ associative?

   (d) Does $*$ have an identity element? If so, does every real number have an inverse?

13) Let A be a set. Prove that $(2^A, \Delta)$ is a group.