Discrete Mathematics, Section 001, Fall 2016

Lecture 13: Functions

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Outline

What is a function?

Intuitively: a function is a 'rule' or 'mechanism' that transforms one quantity into another.

- $f(x) = x^2 + 4$ takes an integer x and transforms it into the integer $x^2 + 4$.
- $g(x) = |x|$ takes the integer x and returns x if $x \geq 0$ and $-x$ if $x < 0$.

Abstract sense: Functions are special types of relations.

Definition

A relation f is called a **function** provided $(a, b) \in f$ and $(a, c) \in f$ imply $b = c$.

In other words, f is not a function if there are a, b, c such that $(a, b), (a, c) \in f$ but $b \neq c$.

What is a function?

Definition

A relation f is called a **function** provided $(a, b) \in f$ and $(a, c) \in f$ imply $b = c$.

For example, consider

$$f = \{(1, 2), (2, 3), (3, 1), (4, 7)\}, \qquad g = \{(1, 2), (1, 3), (4, 7)\}.$$

Here f is a function, but g is not because $(1, 2), (1, 3) \in g$ but $2 \neq 3$.

Get used to this by doing Problem 1 on the worksheet.

Functional notation

Definition

A relation f is called a **function** provided $(a, b) \in f$ and $(a, c) \in f$ imply $b = c$.

Definition

Let f be a function. If $(a, b) \in f$, we write

$$f(a) = b.$$

By the definition of a function this is unambiguous, as for any object a, there is at most one b such that (a, b).

Functional notation

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$$f = \{(1, 2), (2, 3), (3, 1), (4, 7)\}, \qquad g = \{(1, 2), (1, 3), (4, 7)\}.$$

- $f(1) = 2$, $f(2) = 3$, $f(3) = 1$, $f(4) = 7$ but , e.g. $f(10)$ is undefined.
- $g(1)$ is ambiguous.

Example

Express the integer function $f(x) = x^2$ as a relation.

- Option 1: List elements

$$f = \{\dots, (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), \dots\}$$

- Option 2: Set-builder notation

$$f = \{(x, y) : x, y \in \mathbb{Z}, y = x^2\}$$

Outline

Definitions

Definition

Let f be a function. The set of all possible first elements of the ordered pairs in f is called the **domain** of f and is denoted $\mathrm{dom} f$.

In other words,

$$\mathrm{dom} f = \{a : \exists b, (a, b) \in f\} = \{a : f(a) \text{ is defined}\}$$

Definition

Let f be a function. The set of all possible second elements of the ordered pairs in f is called the **image** of f and is denoted $\mathrm{im} f$.

In other words,

$$\mathrm{im} f = \{b : \exists a, (a, b) \in f\} = \{b : b = f(a) \text{ for some } a\}$$

Example

- Let $f = \{(1,2),(2,3),(3,1),(4,7)\}$. Then

$$\mathrm{dom} f = \{1,2,3,4\}, \qquad \mathrm{im} f = \{1,2,3,7\}.$$

- Let $f = \{(x,y) : x, y \in \mathbb{Z}, y = x^2\}$. Then

$$\mathrm{dom} f = \mathbb{Z}, \qquad \mathrm{im} f = \{y \in \mathbb{Z} : y \text{ is a perfect square}\}.$$

Practice this by doing Problem 2 on the Worksheet.

Yet another notation

Definition

Let f be a function and let A and B sets. We say that f is a function from A to B provided $\mathrm{dom} f$=A and $\mathrm{im} f \subseteq B$. In this case we write $f : A \to B$ and also say that f is a mapping from A to B.

Saying $f : A \to B$ means three things:

- f is a function.
- $\mathrm{dom} f = A$.
- $\mathrm{im} f \subseteq B$.

For example,

$$\sin : \mathbb{R} \to \mathbb{R}, \qquad \sin : \mathbb{R} \to [-1, 1]$$

To show that $f : A \to B$, the above three conditions need to be checked.

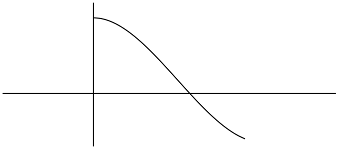
Outline

The usual function graph

Graphs provide excellent visualization of functions whose inputs and outputs ar real numbers.

- Note that when $\mathrm{dom} f, \mathrm{im} f \subseteq \mathbb{R}$, then $f \in \mathbb{R} \times \mathbb{R}$.
- If f is nice enough, the points in $\mathbb{R} \times \mathbb{R}$ that belong to f give a nice curve.

E.g., when $f : [0,2] \to \mathbb{R}$ is defined by $f(x) = 1 - x^2 + 0.3x^3$,



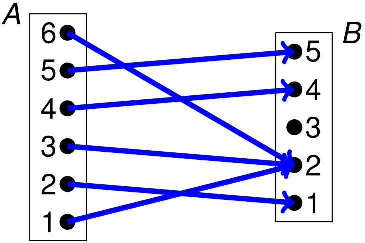
This, however, doesn't work when $\mathrm{dom} f$ is more complicated, e.g.

$$f : 2^A \to \mathbb{N}, \qquad f(B) = |B|$$

Alternative for finite sets

Consider $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 2, 3, 4, 5\}$ and

$$f : A \to B, \qquad f = \{(1, 2), (2, 1), (3, 2), (4, 4), (5, 5), (6, 2)\}.$$

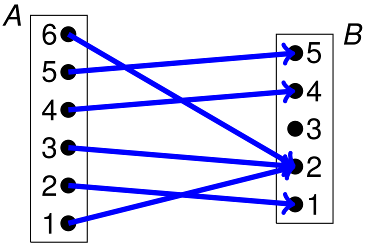


From this, it is easy to read off $\mathrm{im} f = \{1, 2, 4, 5\}$.

Of course, we can draw arrows for any relations. $f : A \to B$ means that

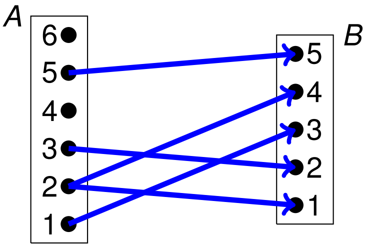
- There is an arrow from all ‘dots’ in A.
- There is at most one arrow from all ‘dots in A.

Consider the same sets as on the previous slide.



$$f = \{(1,2),(2,1),(3,2),(4,4),(5,5),(6,2)\}$$

$$f : A \to B$$



$$g = \{(1,3),(2,1),(2,4),(3,2),(4,4),(5,5)\}$$

Not $g : A \to B$.

Outline

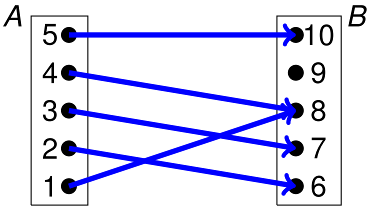
Inverse as a relation

Recall the inverse of a relation:

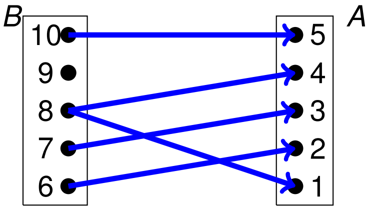
$$R^{-1} = \{(x, y) : (y, x) \in R\}$$

Is the inverse of a function also a function? $\rightarrow$ Nope

$$A = \{1, 2, 3, 4, 5\}, \qquad B = \{6, 7, 8, 9, 10\}$$



$f = \{(1, 8), (2, 6), (3, 7), (4, 8), (5, 10)\}$



$f^{-1} = \{(8, 1), (6, 2), (7, 3), (8, 4), (10, 5)\}$

One to one functions

We would like to characterize those functions whose inverse is also a function.

Definition

A function f is called **one to one** provided that, whenever $(x, b), (y, b) \in f$, we must have $x = y$.

In other words, $x \neq y$, then $f(x) \neq f(y)$.

Do the first half of Problem 4 on the worksheet!

Proposition

Let f be a function. The inverse relation f^{-1} is a function if and only if f is one to one.

Proof.

First suppose f^{-1} is a function. Assume $(x, b), (y, b) \in f$, then we have $(b, x), (b, y) \in f^{-1}$ and by the definition of a function, we have that $x = y$. This proves that f is one to one.

Now suppose f is one to one. Assume $(b, x), (b, y) \in f^{-1}$, then $(x, b), (y, b) \in f$ and by f being one-to-one, we see $x = y$. This proves that f^{-1} is a function. □

Now do the second part of Problem 4 on the worksheet.

Proving a function is one to one

To show that f is one-to-one:

- **Direct Method:** Suppose $f(x) = f(y)$.... Therefore $x = y$ and f is one-to-one.
- **Contrapositive method:** Suppose $x \neq y$. ... Therefore $f(x) \neq f(y)$ and f is one-to-one.
- **Contradiction method:** Suppose $f(x) = f(y)$ but $x \neq y$. ... $\Rightarrow\Leftarrow$ and f is one-to-one.

Example

Let $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 3x + 4$. Then f is one-to-one.

Proof.

Suppose $f(x) = f(y)$. Then $3x + 4 = 3y + 4$. Substracting 4 from both sides gives $3x = 3y$. Dividing both sides by 3 gives $x = y$. Therefore f is one-to-one. $\square$

Example

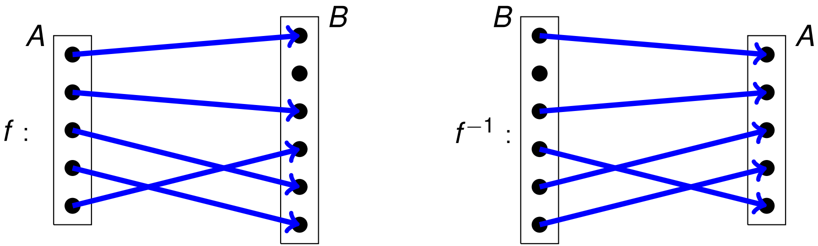
Let $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$. Then f is not one-to-one.

Proof.

Notice that $f(3) = f(-3) = 9$, but $3 \neq -3$ and thus f is not one to one. $\square$

A more focused question

Q: If $f : A \to B$, when is it true that $f^{-1} : B \to A$?



So this doesn't work. In order to rule this out, we introduce the following notion.

Definition

Let $f : A \to B$. We say that f is **onto** B provided that for every $b \in B$, there is an $a \in A$ so that $f(a) = b$. In other words, $\mathrm{im} f = B$.

Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{7, 8, 9, 10\}$ and

$$f = \{(1, 7), (2, 7), (3, 8), (4, 9), (5, 9), (6, 10)\},$$

$$g = \{(1, 7), (2, 7), (3, 7), (4, 9), (5, 9), (6, 10)\}.$$

Then $f : A \to B$ is onto, but $g : A \to B$ is not onto!

Proving a function is onto

To show $f : A \to B$ is onto:

- **Direct method:** Let b be an arbitrary element of B. Explain how to find/construct and element $a \in A$ such that $f(a) = b$. Therefore f is onto.
- **Set method:** Show that the sets B and $\mathrm{im} f$ are equal.

Example

Let $f : \mathbb{Q} \to \mathbb{Q}$ be defined by $f(x) = 3x + 4$. Prove that f is onto $\mathbb{Q}$.

Proof.

Let $b \in \mathbb{Q}$ be arbitrary. We seek an $a \in \mathbb{Q}$ such that $f(a) = b$. Let $a = \frac{1}{3}(b - 4)$. Since b is a rational number, so is a. Notice that

$$f(a) = 3\left[\frac{1}{3}(b-4)\right] + 4 = (b-4) + 4 = b.$$

Therefore $f : \mathbb{Q} \to \mathbb{Q}$ is onto. $\square$

Now we answer the question:

Theorem

Let A and B be sets and let $f : A \to B$. The inverse relation f^{-1} is a function from B to A if and only if f is one to one and onto.

Definition

Let $f : A \to B$. We call f a **bijection** provided it is both one-to-one and onto.

Example

Let A be the set of even integers and let B be the set of odd integers. The function $f : A \to B$ defined by $f(x) = x + 1$ is a bijection.

Proof

To show that f is one to one, suppose $f(x) = f(y)$ where x and y are even integers. Thus

$$f(x) = f(y) \quad \Rightarrow \quad x + 1 = y + 1 \quad \Rightarrow \quad x = y.$$

Hence f is one-to-one. [...]

Definition

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Proof

To see that f is onto B, let $b \in B$. By definition $b = 2k + 1$ for some $k \in \mathbb{Z}$. Let $a = 2k$; clearly a is even. Then

$$f(a) = a + 1 = 2k + 1 = b$$

so f is onto.

Since f is both one-to-one and onto, f is a bijection. □

Do Problems 5-6 on the worksheet.