Discrete Mathematics, 2016 Spring - Worksheet 2

September 12, 2016

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In all of the above problems explain your answer in full English sentences.

1. Recast the following statements in the $if - then$ form.
   (a) The product of an odd integer and an even integer is even.
   If a is an odd integer and b is an even integer, then $a \cdot b$ is even.
   (b) The square of a prime number is not a prime.
   If p is a prime number, then p^2 is not a prime.
   (c) The product of two negative integers is negative.
   If a and b are negative integers, then $a \cdot b$ is negative.
   (d) The sum of three consecutive integers is divisible by three.
   If a, b, and c are consecutive integers, then $a + b + c$ is divisible by 3.

2. Consider the claim: 'If a guinea pig has a tail, its eyes are blue'. True or False? (Hint: Guinea pigs don't have tails.)

   True, because it is vacuously true.

3. Below you will find pairs of statements A and B. For each pair, please indicate which of the following three sentences are true and which are false:

   - If A, then B.
   - If B, then A.
   - A if and only if B.

   You may just write True or False.

   (a) A: $x > 0$, B: $x^2 > 0$. *TRUE, FALSE, FALSE*
   (b) A: $x < 0$, B: $x^3 < 0$. *TRUE, TRUE, TRUE*
   (c) A: $xy = 0$, B:$x = 0$ or $y = 0$. *TRUE, TRUE, TRUE*
   (d) A: $xy = 0$ B: $x = 0$ and $y = 0$ *FALSE, TRUE, FALSE*

4. Consider the two statements:

   (a) If A, then B.

   (b) If (not B), then not A.

   Under what circumstances are these statements true? When are they false? Explain why these statements are, in essence, identical.

   (a) is true if whenever A is true, then so is B. (b) is true if whenever B isn't true, A can't hold either. The only way either (a) or (b) would be false is there are circumstances under which A holds but B does not. This means that the two statements have the same truth value under all circumstances and they are, in essence, identical.

5. Write a proof of the following result:

   Proposition 1. *Let a, b, and c be integers. If $a|b$ and $b|c$, then $a|c$.*

   Proof. Since $a|b$, there is an integer k_1, such that $b = k_1 a$. Since $b|c$, there is an integer k_2 such that $c = k_2 b$. This means that

   $$c = k_2 b = k_2 k_1 a$$

   Since $k_1 k_2$ is an integer (by the virtue of k_1 and k_2 being integers, this means that $a|c$ and the claim is proved. ☐

6. Write a proof of the following result:

   Proposition 2. *Let x be an integer. Then x is even if and only if $x + 1$ is odd.*

   Proof. Let x be even. Then there is an integer k such that $x = 2k$. By adding 1 to both sides of this equation, we get $x + 1 = 2k + 1$, which means that $x + 1$ is odd.

   On the other hand, assume $x + 1$ is odd. This means that there is an integer k such that $x + 1 = 2k + 1$. Substracting 1 from both sides of this equation, we get $x = 2k$ which means that x is an even number.

   This proves the claim. ☐

7. Using Proposition 1, write a proof of the following result:

   Proposition 3. *Let a, b, c, and d be integers. If $a|b$, $b|c$, and $c|d$, then $a|d$.*

   Proof. Since $a|b$ and $b|c$, Proposition 1 implies that $a|c$. Using this, and $c|d$ another application of Proposition 1 yields that $a|d$ and the claim is proved. ☐

8. Write a proof for the following statements:

   (a) The sum of two odd integers is even.

Proof. (The statement can be rephrased as an 'if-then' statement as "If a, b are two odd integers, then $a + b$ is even." You don't need to say this but it might be helpful to do so initially.)
Let a and b be two odd integers. This means that there are integers k_1, k_2 such that $a = 2k_1 + 1$ and $b = 2k_2 + 1$, which implies

$$a + b = 2(k_1 + k_2) + 2 = 2(k_1 + k_2 + 1).$$

Since k_1 and k_2 are integers, so is $k_1 + k_2 + 1$ and therefore $a + b$ is even. $\square$

(b) If n is an odd integer, then $-n$ is also odd.

Proof. If n is an odd integer, there is an integer k such that $n = 2k + 1$. Then

$$-n = -(2k + 1) = -2k - 1 = -2k - 2 + 1 = 2(-k - 1) + 1.$$

Since k is an integer, so is $-k - 1$ and therefore $-n$ is odd. $\square$

(c) The product of an even integer and an odd integer is even.

Proof. Let a be an even integer and b be an odd integer. This means there are integers k_1 and k_2 such that $a = 2k_1$ and $b = 2k_2 + 1$. Thus

$$ab = 2k_1 \cdot (2k_2 + 1) = 2(2k_1k_2 + k_1).$$

Since k_1 and k_2 are integers, so is $2k_1k_2 + k_1$ and this shows that ab is even. $\square$

9. Suppose you are asked to prove a statement of the form 'A iff B'. The standard method is to prove both $A \Rightarrow B$ and $B \Rightarrow A$. Consider the following alternative proof strategy: Prove both $A \Rightarrow B$ and (not$A$) $\Rightarrow$ (not$B$). Explain why this would give a valid proof.

   A iff B is true provided A is true exactly when B is true. This is equivalent to saying that A is false exactly when B is false. Proving both $A \Rightarrow B$, and (not$A$) $\Rightarrow$ (not$B$) shows that when A is true, so is B and that when A is false so is B. Since there are no third option, this means exactly the above.