# Discrete Mathematics, Section 002, Spring 2016

## Lecture 4: Factorials, Sets

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# Outline

1. Factorials and products
2. Introduction to sets, Subsets

# Factorials

A special case of what we did last time:

> How many lists of length $n$ can we make using $n$ elements without repetition?

or alternatively

> How many ways can we order $n$ elements?

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- Note that

$$n! = n \cdot (n-1)!$$

- Also

$$\begin{aligned}(n)_k &= n(n-1)\dots(n-k+1) = \\ &= \frac{n(n-1)\dots 2\cdot 1}{(n-k)(n-k-1)\dots 2\cdot 1} = \frac{n!}{(n-k)!}\end{aligned}$$

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     One, the empty list.

   - When $n = 1$, $n! = n(n-1)!$ becomes $1 = 1 \cdot 0!$
   - At the end of the day we make the definition like this, because it's convenient.

# Product notation

Another way to write factorials:

$$n! = \prod_{k=1}^{n} k$$

- $\prod$ stands for product.
- $k$ is a dummy variable, a place holder that ranges from the lower value to the upper value.

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Another example,

$$\prod_{k=1}^{5}(2k+3) = (2\cdot 1+3)(2\cdot 2+3)(2\cdot 3+3)(2\cdot 4+3)(2\cdot 5+3)$$

# Product notation

Interpretation as a for loop to compute

$$\prod_{k=1}^{n} f_k$$

```
def evaluate():
    prod = 1
    for k in range(1,n+1):
        prod *= f_k

    return prod
```

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- Empty product:

$$0! = \prod_{k=1}^{0} k = 1$$

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- Simplest way to specify: list elements.

$$\left\{2, 3, \frac{1}{2}\right\} \qquad \left\{3, \frac{1}{2}, 2\right\} \qquad \left\{2, 2, 3, \frac{1}{2}\right\}$$

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These are all the same set!

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**Membership in a set**

An object $x$ that belongs to a set $A$ is called an **element** of it.

$$\text{Notation:} \qquad x \in A$$

For example, $x \in \mathbb{Z}$ reads:

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- "$x$ is in $\mathbb{Z}$"
- "$x$ is an integer"

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A set $A$ with $|A| \in \mathbb{N}$ is **finite**, otherwise it's **infinite**.

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- $|\emptyset| = 0$

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  For example,

$$\mathbb{N} = \{x : x \in \mathbb{Z}, x \geq 0\}$$

  This is a set of objects satisfying
  - $x \in \mathbb{Z}$
  - $x \geq 0$

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Now practice these on the worksheet!

# Equality of sets

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To show $A = B$, we have the following template:

Let $A$ and $B$ be the sets.

- Suppose $x \in A$. . . . Therefore $x \in B$.
- Suppose $x \in B$. . . . Therefore $x \in A$.

Therefore $A = B$.

## Proposition

The following two sets are equal:

$$E = \{x \in \mathbb{Z} : x \text{ is even}\},$$

$$F = \{x \in \mathbb{Z} : x = a + b \text{ where } a \text{ and } b \text{ are both odd}\}.$$

## Proof.

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Let $E$ and $F$ as in the statement of the proposition. We seek to prove that $E = F$.

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**Proof.**

Let $E$ and $F$ as in the statement of the proposition. We seek to prove that $E = F$.

Suppose $x \in E$. Therefore $x$ is even, hence $2|x$ and so $x = 2y$ for some $y \in \mathbb{Z}$.

Therefore $x$ is the sum of two odd numbers and so $x \in F$.

Suppose $x \in F$.

Therefore $x \in E$. $\square$

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Suppose $x \in E$. Therefore $x$ is even, hence $2|x$ and so $x = 2y$ for some $y \in \mathbb{Z}$. Note that $2y + 1$ and $-1$ are both odd and that $x = 2y = (2y + 1) + (-1)$. Therefore $x$ is the sum of two odd numbers and so $x \in F$.

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Suppose $x \in F$. Therefore $x$ is the sum of two odd integers. It was shown in Exercise 5.1 that $x$ is then even. Therefore $x \in E$. $\square$

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- $\emptyset \subseteq A$.
- If $B \subseteq A$ and $B \neq A, \emptyset$, we call it a **proper subset.**

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To show that $A \subseteq B$:
Let $x \in A, \ldots$ Therefore $x \in B$.

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## Proposition

Let $P$ be the set of Pythagorean triples.

$$P = \{(a, b, c) : a, b, c \in \mathbb{Z} \text{ and } a^2 + b^2 = c^2\}$$

and

$$T = \{(p, q, r) : p = x^2 - y^2, q = 2xy, \text{ and } r = x^2 + y^2 \text{ where } x, y \in \mathbb{Z}\}$$

Then $T \subseteq P$.

$$T = \{(p,q,r) : p = x^2 - y^2, q = 2xy,$$
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For example $x = 3$, $y = 2$,

$p = x^2 - y^2 = 9 - 4 = 5, \qquad q = 2xy = 12, \qquad r = x^2 + y^2 = 13$

and $(p,q,r) \in P$.

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**Proof.**

Let $P$ and $T$ as in the statement of the proposition.

Let $(p, q, r) \in T$.

Therefore $(p, q, r) \in P$. $\square$

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**Proof.**

Let $P$ and $T$ as in the statement of the proposition.

Let $(p,q,r) \in T$. Therefore there are integers $x$ and $y$ such that $p = x^2 - y^2$, $q = 2xy$, and $r = x^2 + y^2$.

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$$p^2 + q^2 = (x^2 - y^2)^2 + (2xy)^2 = (x^4 - 2x^2y^2 + y^4) + 4x^2y^2 =$$
$$= x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2 = r^2$$

Therefore $(p, q, r) \in P$. $\square$

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For example,

- $x \in \{x\}$
- But $x \subseteq \{x\}$ or $x = \{x\}$ are incorrect.

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For example, what about $A = \{1, 2, 3\}$?

| Number of elements | Subsets | Number |
|---|---|---|
| 0 | $\emptyset$ | 1 |
| 1 | {1},{2},{3} | 3 |
| 2 | {1,2},{1,3},{2,3} | 3 |
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Total: 8.

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Total: 8.

## Alternatively

For every element, we have two choices independently of each other: include/not include.

Therefore $|A| = 2^3 = 8$.

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Let $A$ be a finite set and let $n = |A|$. Let the $n$ elements of $A$ be named $a_1, a_2, \ldots, a_n$. To each subset $B$ of $A$, we can associate a list of length $n$; each element of the list is one of the words "yes" or "no". The $k$th element of the list is "yes" precisely when $a_k \in B$. This establishes a one to one correspondence with $yes/no$ lists of length $n$. The number of such lists is $2^n$, so the number of subsets of $A$ is $2^n$ where $n = |A|$. $\square$

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This is a so-called **bijective** proof.

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The notation is created so that

$$|2^A| = 2^{|A|}$$