Discrete Mathematics, 2016 Spring - Worksheet 4

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In all of the above problems explain your answer in full English sentences.

1. Solve the equation $n! = 720$ for n.
2. There are six different French books, eight different Russian books and five different Spanish books.
   (a) In how many different ways can these books be arranged on a bookshelf?
   (b) In how many different ways can these books be arranged if all books in the same language are grouped together?
3. Calculate the following products:
   (a) $\prod_{k=1}^{4}(2k+1)$
   (b) $\prod_{k=1}^{n}\frac{1}{k}$
4. Can factorial be extended to negative integers? Think about the formula $n! = n(n-1)!$. What would be the value of $(-1)!$?
5. Write out the following sets by listing their elements between curly braces and find their cardinality.
   (a) $\{x \in \mathbb{N} : x \leq 10 \text{ and } 3|x\}$
   (b) $\{x \in \mathbb{Z} : x \text{ is a prime and } 2|x\}$
   (c) $\{x \in \mathbb{Z} : 10|x \text{ and } x|100\}$
   (d) $\{x \in \mathbb{Z} : 1 \leq x^2 \leq 2\}$
6. For each of the following sets, find a way to rewrite the set using set builder notation.
   (a) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
   (b) $\{-8, -6, -4, -2, 0, 2, 4, 6, 8\}$
   (c) $\{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$
7. (a) Let $A = \{x \in \mathbb{Z} : 4|x\}$ and let $B = \{x \in \mathbb{Z} : 2|x\}$. Prove that $A \subseteq B$.

(b) Generalize the previous problem. Let $a, b \in \mathbb{Z}$ and let

$$A = \{x \in \mathbb{Z} : a|x\}, \qquad B = \{x \in \mathbb{Z} : b|x\}.$$

Find and prove a necessary and sufficient conditions for $A \subseteq B$. In other words, find and prove a theorem of the form

"$A \subseteq B$ if and only if *some condition involving a and b.*"

8. Compute each of the following by writing either $\in$ or $\subseteq$ in place of $\bigcirc$.

- $2 \bigcirc \{1, 2, 3\}$
- $\{2\} \bigcirc \{1, 2, 3\}$
- $\{2\} \bigcirc \{\{1\}, \{2\}, \{3\}\}$
- $\emptyset \bigcirc \{1, 2, 3\}$
- $\mathbb{N} \bigcirc \mathbb{Z}$
- $\{2\} \bigcirc \mathbb{Z}$

9. This problem is about power sets.

(a) Write out the elements and give the cardinality of the set $2^{\emptyset}$. (Hint: Start with the cardinality.)

(b) Find the cardinality of the following sets.

i. $2^{2^{\{1,2,3\}}}$

ii. $\{x \in 2^{\{1,2,3,4\}} : |x| = 1\}$

(c) Complete $\{2\} \bigcirc 2^{\mathbb{Z}}$ by a $\subseteq$ or $\in$.

10. (Russel's paradox) Consider the set of all sets R that are not elements of themselves, i.e. $x \in R$ if x is a set but $x \notin x$. Does R contain itself as an element? The answer to this question signifies the breakdown of naive set theory which led to the development of axiomatic set theory. (You need to take a course in mathematical logic to learn more about this.)

Optional programming exercises (no credit)

PE20) Note that $10! = 10 \cdot 9 \cdot 8 \cdot ... \cdot 2 \cdot 1 = 3628800$. Find the sum of the digits in the number $100!$.

PE34) 145 is a curious number as $1! + 4! + 5! = 1 + 24 + 120 = 145$. Find the sum of all numbers which are equal to the sum of the factorial of their digits.