Discrete Mathematics, Section 001, Fall 2016

Lecture 12: Linear recurrence relations

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Outline

How do we come up with complicated formulas?

Proposition

$$0^2 + 1^2 + \cdots + n^2 = \frac{(2n+1)(n+1)(n)}{6}$$

or

Proposition

The *n*th Fibonacci number is

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}$$

Recurrence relations

Definition

A recurrence relation is a formula that specifies how each term of a sequence is produced from earlier terms.

For example

$$a_n = 3a_{n-1} + 4a_{n-2}, \qquad a_0 = 3, \quad a_1 = 2.$$

Then

$$a_2 = 3a_1 + 4a_0 = 3 \times 2 + 4 \times 3 = 18,$$
$$a_3 = 3a_2 + 4a_1 = 3 \times 18 + 4 \times 2 = 62,$$
$$a_4 = 3a_3 + 4a_2 = 3 \times 62 + 4 \times 18 = 258.$$

We are going to learn how to derive

$$a_n = 4^n + 2(-1)^n$$

The order of a recurrence relation is the number of previous elements we need to compute the current one.

$$\begin{aligned}
\text{First order:} \quad & a_n = 3a_{n-1} + 5 \\
\text{Second order:} \quad & a_n = 3a_{n-1} + 2a_{n-2} + 6 \\
\vdots \quad & \quad \vdots \\
k\text{th order:} \quad & a_n = 2a_{n-1} + 4a_{n-2} - 8a_{n-3} + \cdots + 2a_{n-k} + 9
\end{aligned}$$

- For a kth order recurrence relation, we need to know the first k elements.
- When a recurrence relation has the above structure, it is linear, otherwise it is called non-linear.

Warm up by doing Problems 1-2 on the worksheet.

Outline

Simple cases

- Simplest example:

$$a_n = a_{n-1}, \qquad \to \qquad a_n = a_0.$$

- Slightly more difficult:

$$a_n = 2a_{n-1}, \qquad a_0 = 5.$$

Then the sequence is $5, 10, 20, 40, 80, \ldots$ and so

$$a_n = 5 \cdot 2^n.$$

More generally if $a_n = sa_{n-1}$,

$$a_n = a_0 \cdot s^n.$$

Consider

$$a_n = 2a_{n-1} + 3$$

and note

$$\begin{aligned}
a_1 &= 2a_0 + 3 \\
a_2 &= 2a_1 + 3 = 2^2 a_0 + 2 \cdot 3 + 3 \\
a_3 &= 2a_2 + 3 = 2^3 a_0 + 2^2 \cdot 3 + 2 \cdot 3 + 3 \\
a_4 &= 2a_3 + 3 = 2^4 a_0 + (2^3 + 2^2 + 2 + 1) \cdot 3 \\
\vdots &\quad \vdots \\
a_n &= 2a_{n-1} + 3 = 2^n a_0 + (2^{n-1} + 2^{n-2} + \cdots + 2 + 1) \cdot 3
\end{aligned}$$

Using $1 + 2 + \cdots + 2^{n-1} = 2^n - 1$, we can conjecture

$$a_n = 2^n(a_0 + 3) - 3.$$

This still needs to be proved (e.g. by induction)!

In general

Consider

$$a_n = s \cdot a_{n-1} + t$$

and note

$$\begin{aligned}
a_1 &= sa_0 + t \\
a_2 &= sa_1 + t = s^2 a_0 + s \cdot t + t \\
a_3 &= sa_2 + t = s^3 a_0 + s^2 \cdot t + s \cdot t + t \\
\vdots \quad & \quad \vdots \\
a_n &= sa_{n-1} + t = s^n a_0 + (s^{n-1} + s^{n-2} + \cdots + s + 1) \cdot t
\end{aligned}$$

Recall

$$1 + 2 + \cdots + s^{n-1} = \begin{cases} \frac{s^n - 1}{s - 1}, & s \neq 1, \\ n, & s = 1. \end{cases}$$

In general

Consider

$$a_n = s \cdot a_{n-1} + t$$

and note

$$a_n = sa_{n-1} + t = s^n a_0 + (s^{n-1} + s^{n-2} + \cdots + s + 1) \cdot t$$

Recall

$$1 + 2 + \cdots + s^{n-1} = \begin{cases} \frac{s^n - 1}{s-1}, & s \neq 1, \\ n, & s = 1. \end{cases}$$

Proposition

When $s \neq 1$,

$$a_n = s^n \left(a_0 + \frac{t}{s-1} \right) - \frac{t}{s-1}$$

and $a_n = a_0 + nt$ when $s = 1$.

In general

Proposition

When $s \neq 1$,

$$a_n = s^n \left(a_0 + \frac{t}{s-1} \right) - \frac{t}{s-1}$$

and $a_n = a_0 + nt$ when $s = 1$.

Proof.

Straightforward by induction, left to the reader. $\square$

A more convenient form to remember when $s \neq 1$:

Proposition

When $s \neq 1$, there are constants c_1, c_2 such that

$$a_n = c_1 s^n + c_2.$$

For example, let us solve $a_n = 5a_{n-1} + 3$ where $a_0 = 1$.

- By the proposition, $a_n = c_1 5^n + c_2$.
- Note that

$$\begin{aligned} a_0 &= 1 = c_1 + c_2 \\ a_1 &= 8 = 5c_1 + c_2. \end{aligned}$$

- Solving these, we find $c_1 = \frac{7}{4}$ and $c_2 = -\frac{3}{4}$ and thus

$$a_n = \frac{7}{4} \cdot 5^n - \frac{3}{4}$$

Practice this by doing Problem 3 on the WS.

Outline

An example

Consider

$$a_n = 5a_{n-1} - 6a_{n-2}$$

- Based on the previous case, our first guess is that the solution should be some kind of power.
- Should it be 5^n or 6^n?
- Let's stay safe and try r^n with r left to be figured out later:

$$r^n = 5r^{n-1} - 6r^{n-2}.$$

- Dividing both sides by r^{n-2} gives

$$r^2 = 5r - 6.$$

- This has roots $r = 2, 3$, and we get two solutions:

$$a_n = 2^n, \qquad a_n = 3^n$$

In general

Proposition

If r is a root of the equation $x^2 = s_1 x + s_2$, then $a_n = r^n$ solves

$$a_n = s_1 a_{n-1} + s_2 a_{n-2}$$

Proof.

We directly verify

$$s_1 r^{n-1} + s_2 r^{n-2} = r^{n-2}(s_1 r + s_2) = r^{n-2} r^2 = r^n$$

which proves the claim. □

But note that $r^0 = 1$, so this only gives solutions with $a_0 = 1$!

Ways to build other solutions

Proposition

If $a_n^{(1)}$ and $a_n^{(2)}$ are both solutions of the recurrence relation

$$a_n = s_1 a_{n-1} + s_2 a_{n-2}$$

then for any real numbers c_1, c_2, the linear combination

$$c_1 a_n^{(1)} + c_2 a_n^{(2)}$$

is also a solution.

Proof.

We directly verify

$$s_1(c_1 a_{n-1}^{(1)} + c_2 a_{n-1}^{(2)}) + s_2(c_1 a_{n-2}^{(1)} + c_2 a_{n-2}^{(2)}) =$$
$$= c_1(s_1 a_{n-1}^{(1)} + s_2 a_{n-2}^{(1)}) + c_2(s_1 a_{n-1}^{(2)} + s_2 a_{n-2}^{(2)}) = c_1 a_n^{(1)} + c_2 a_n^{(2)}$$

which proves the Proposition. $\square$

When $r_1 \neq r_2$

When the equation $x^2 = s_1 x + s_2$ has two distinct roots, then r_1^n and r_2^n are both solutions and thus

$$a_n = c_1 r_1^n + c_2 r_2^n$$

is also a solution for any numbers c_1 and c_2. This we can adjust to any initial conditions:

$$a_0 = c_1 + c_2, \qquad a_1 = c_1 r_1 + c_2 r_2.$$

We can always solve these equations to be

$$c_1 = \frac{a_1 - a_0 r_2}{r_1 - r_2}, \qquad c_2 = \frac{-a_1 + a_0 r_1}{r_1 - r_2}$$

Theorem

If the equation $x^2 = s_1 x + s_2$ has two distinct roots $r_1 \neq r_2$, then all the solutions of the recurrence relation $a_n = s_1 a_{n-1} + s_2 a_{n-2}$ is of the form

$$c_1 r_1^n + c_2 r_2^n.$$

Example

Consider

$$a_n = 5a_{n-1} - 6a_{n-2}$$

We have found that $r = 2, 3$ and therefore

$$a_n = c_1 2^n + c_2 3^n.$$

Let's assume $a_0 = -1$ and $a_1 = 1$, which means

$$-1 = c_1 + c_2, \qquad 1 = 2c_1 + 3c_2.$$

This has solution

$$c_1 = -4, \qquad c_2 = 3$$

and therefore

$$a_n = -4 \cdot 2^n + 3 \cdot 3^n.$$

Do this yourself on Problem 4(a).

Complex roots

Consider

$$a_n = 2a_{n-1} - 2a_{n-2}, \qquad a_0 = 1, a_1 = 3.$$

The associated quadratic equation is $x^2 = 2x - 2$ or

$$0 = x^2 - 2x + 2$$

The quadratic formula yields that this has two complex roots $1 \pm i$ and

$$a_n = c_1(1+i)^n + c_2(1-i)^n.$$

The equations to solve are

$$1 = c_1 + c_2, \qquad 3 = c_1(1+i) + c_2(1-i).$$

which has solutions $c_1 = \frac{1}{2} - i$ and $c_2 = \frac{1}{2} + i$. Therefore

$$a_n = \left(\frac{1}{2} - i\right)(1+i)^n + \left(\frac{1}{2} + i\right)(1-i)^n.$$

Repeated roots

What happens when $r_1 = r_2$, e.g.

$$a_n = 4a_{n-1} - 4a_{n-2}, \qquad a_0 = 1, a_1 = 3.$$

The equation $x^2 = 4x - 4$ can be written

$$0 = x^2 - 4x + 4 = (x-2)(x-2)$$

has only one repeated root $r = 2$, which only gives

$$a_n = c2^n.$$

But this is not enough to handle the initial condition:

$$1 = c, \qquad 3 = c2$$

which cannot be satisfied by a single c.

Repeated roots

$$a_n = 4a_{n-1} - 4a_{n-2}, \qquad a_0 = 1, a_1 = 3.$$

The first five values are $1, 3, 8, 20, 48$. Let's see what c would need to be used in each step if $a_n = c2^n$.

$$\begin{array}{llcl}
n = 0 & 1 = c & \to & c = 1 \\
n = 1 & 3 = c2 & \to & c = \frac{3}{2} \\
n = 2 & 8 = c2^2 & \to & c = 2 \\
n = 3 & 20 = c2^3 & \to & c = \frac{5}{2} \\
n = 4 & 48 = c2^4 & \to & c = 3.
\end{array}$$

We can read of the pattern $c(n) = (1 + \frac{n}{2})$ and so

$$a_n = \left(1 + \frac{n}{2}\right) 2^n = 2^n + \frac{1}{2} n2^n$$

Repeated roots

Theorem

Let the quadratic equation $x^2 - s_1 x - s_2 = 0$ have exactly one solution $r \neq 0$. Then every solution of the recurrence relation

$$a_n = s_1 a_{n-1} + s_2 a_{n-2}$$

is of the form

$$c_1 r^n + c_2 n r^n.$$

Proof

Since the quadratic equation has only one root, factorization gives

$$x^2 - s_1 x - s_2 = (x - r)^2,$$

and therefore $s_1 = 2r$ and $s_2 = -r^2$ and the recurrence relation becomes

$$a_n = 2r a_{n-1} - r^2 a_{n-2}$$

[...]

Theorem

Then every solution of the recurrence relation

$$a_n = s_1 a_{n-1} + s_2 a_{n-2}$$

is of the form

$$c_1 r^n + c_2 n r^n.$$

Proof

[...]

$$a_n = 2r a_{n-1} - r^2 a_{n-2}$$

Then the given form satisfies the recurrence relation.

$$\begin{aligned} &2r(c_1 r^{n-1} + c_2(n-1)r^{n-1}) - r^2(c_1 r^{n-2} + c_2(n-2)r^{n-2}) = \\ &= (2c_1 r^n - c_1 r^n) + (2c_2(n-1)r^n - c_2(n-2)r^n) = \\ &= c_1 r^n + c_2 n r^n. \end{aligned}$$

[...]

Theorem

Then every solution of the recurrence relation

$$a_n = s_1 a_{n-1} + s_2 a_{n-2}$$

is of the form

$$c_1 r^n + c_2 n r^n.$$

Proof

[...] To show that every solution can be written in this form, note that matching the initial conditions gives

$$a_0 = c_1, \qquad a_1 = r(c_1 + c_2).$$

Since by assumption $r \neq 0$,

$$c_1 = a_0 \qquad c_2 = \frac{a_0 r - a_1}{r}.$$

which means that for every initial condition the solution can be given in the desired form and the theorem is proved. $\square$

Example

Consider

$$a_n = 8a_{n-1} - 16a_{n-2}, a_0 = 2, a_1 = 5$$

The corresponding quadratic equation is

$$0 = x^2 - 8x + 16 = (x-4)^2$$

and therefore by the Theorem,

$$a_n = c_1 4^n + c_2 n 4^n.$$

To match the initial conditions, write

$$2 = a_0 = c_1, \qquad 5 = a_1 = 4(c_1 + c_2),$$

which implies $c_1 = 2$ and $c_2 = -\frac{3}{4}$ and therefore the solution is

$$a_n = 2 \cdot 4^n - \frac{3}{4} n 4^n.$$

Do Problem 4 (b) on the worksheet!

Non-homogeneous equation

Consider

$$a_n = 5a_{n-1} - 6a_{n-2} + 2, a_0 = 1, a_1 = 2.$$

- We already know how to solve the homogeneous equation

  $$a_n^h = 5a_{n-1}^h - 6a_{n-2}^h$$

  to get

  $$a_n^h = c_1 2^n + c_2 3^n$$

- We also know that the sum of solutions is a solution.
- Therefore we can try to find the inhomogeneous solution as

  $$a_n = c_1 2^n + c_2 3^n + a_n^i$$

  where a_n^i is any solution of the inhomogeneous equation.
- This is already adjustable to the inital conditions by finding the right c_1 and c_2.

Non-homogeneous equation

Consider

$$a_n = 5a_{n-1} - 6a_{n-2} + 2, a_0 = 1, a_1 = 2.$$

- Try to find the inhomogeneous solution as

$$a_n = c_1 2^n + c_2 3^n + a_n^i$$

- This is already adjustable to the inital conditions by finding the right c_1 and c_2.
- Easiest thing to try is $a_n^i = c_3$. Then

$$c_3 = 5c_3 - 6c_3 + 2$$

which works if $c_3 = 1$. Therefore

$$a_n = c_1 2^n + c_2 3^n + 1,$$

and after matching the initial conditions gives $c_1 = -1, c_2 = 1$ and thus

$$a_n = -2^n + 3^n + 1.$$

Do Problem 5 on the worksheet!