Discrete Mathematics, Section 001, Fall 2016

Lecture 5: Quantifiers, Operations on sets, combinatorial proofs

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Outline

Quantifiers

The following words appear frequently in proofs:

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Now we clarify and formalize them and introduce a brief notation using quantifiers.

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Mathematicians write this as

$\exists x \in \mathbb{N}$, such that x is prime and even.

where $\exists$ is called the **existential quantifier**.

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Statement

$\exists x \in \mathbb{Z}$, x is even and x is prime.

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Statement

$\exists x \in \mathbb{Z}$, x is even and x is prime.

Proof.

Consider the integer 2. Clearly, 2 is even and 2 is prime. Therefore 2 satisfies the required assertions. $\square$

Universal quantifier

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Or in a mathematician's way:

Statement

$\forall x \in \mathbb{Z}$, x is odd or even

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To prove a statement like that:
Let x be any element of $A \ldots$ (Show that x satisfies the assertions using only tha fact that it belongs to A.) Therefore x satisfies the required assertions.

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Proof.

Let $x \in A$; that is, x is an integer that is divisible by 6. This means there is an integer y such that $x = 6y$, which can be rewritten as

$$x = (2 \cdot 3)y = 2(3y).$$

Therefore x is divisible by 2 and therefore even. $\square$

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We can also write these as

- $\forall x \in \mathbb{Z}, \neg((x \text{ is even}) \wedge (x \text{ is odd}))$
- $\exists x \in \mathbb{Z}, \neg(x \text{ is prime})$

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When $\neg$ "moves" inside, it toggles between $\forall$ and $\exists$.

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- $\exists y, (\forall x + y = 0)$ False

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We can also write these as

- $\forall x, (\exists y, x + y = 0)$ True
- $\exists y, (\forall x + y = 0)$ False

Usually the parentheses are omitted, nevertheless the order matters!

Outline

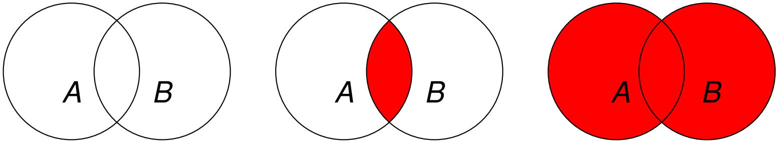
Unions and intersections

$A \cup B$

The **union** of two sets A and B is the set of all elements that are in A or B (or both).

$A \cap B$

The **intersection** of two sets A and B is the set of all elements that are both in A and B.



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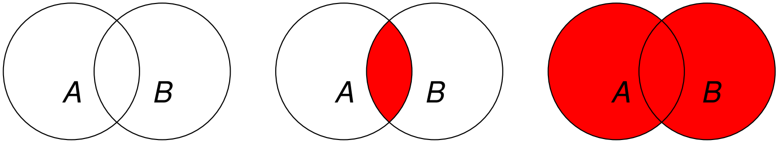


or

$$A = \{1, 2, 3, 4\} \qquad A \cup B = \{1, 2, 3, 4, 5, 6\} \quad A \cap B = \{3, 4\}$$
$$B = \{3, 4, 5, 6\}$$

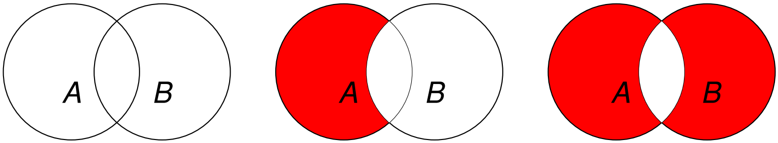
Difference and symmetric difference

$A - B$

The **set difference** $A - B$ of two sets A and B is the set of all elements of A that are not in B

$A \Delta B$

The **symmetric difference** of A and B is the set of all elements in A but not B or in B but not A.



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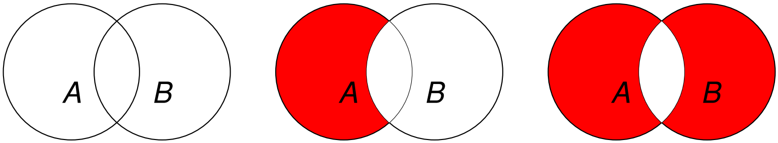

or

$$A = \{1, 2, 3, 4\} \qquad A - B = \{1, 2\} \qquad A\Delta B = \{1, 2, 5, 6\}$$
$$B = \{3, 4, 5, 6\} \qquad B - A = \{5, 6\}$$

More formal definitions

These operations can be formulated in terms of Boolean operators:

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- $A - B = \{x : (x \in A) \wedge (x \notin B)\}$
- $A \Delta B = (A - B) \cup (B - A)$

Cartesian products

$A \times B$

The **Cartesian product** of A and B is the set of all orered pairs (two-element lists) formed by taking an element of A together with an element of B in all possible ways. That is

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

$$A = \{1, 2, 3\}, \qquad B = \{3, 4, 5\}$$

$$A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5)\}$$

$$B \times A = \{(3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)\}$$

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In general $A \times B \neq B \times A$, but

Proposition

Let A and B be finite sets. Then $|A \times B| = |B \times A| = |A| \cdot |B|$.

Properties of set operations

Theorem

Let A, B, and C denote sets.

- *Commutative properties:*
$$A \cup B = B \cup A, \qquad A \cap B = B \cap A$$
- *Associative properties:*
$$A \cup (B \cup C) = (A \cup B) \cup C, \qquad A \cap (B \cap C) = (A \cap B) \cap C$$
- *Distributive properties:*
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
- $A \cup \emptyset = A$ *and* $A \cap \emptyset = \emptyset$.

Associativity for unions

Theorem

Let A, B, and C denote sets. Then

$$A \cup (B \cup C) = (A \cup B) \cup C.$$

We use Boolean identities

Proof.

$$\begin{aligned} A \cup (B \cup C) &= \{x : (x \in A) \vee (x \in B \cup C)\} = \\ &= \{x : (x \in A) \vee ((x \in B) \vee (x \in C))\} = \\ &= \{x : ((x \in A) \vee (x \in B)) \vee (x \in C)\} = \\ &= \{x : (x \in A \cup B) \vee (x \in C)\} = \\ &= (A \cup B) \cup C \end{aligned}$$

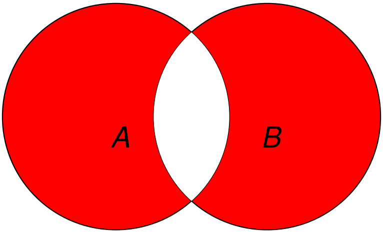
□

Alternative representation for the symmetric difference

Theorem

Let A and B sets. Then

$$A \Delta B = (A \cup B) - (A \cap B).$$



Proof.

Let A and B be sets.

(1) Suppose $x \in A \Delta B$.

Therefore $x \in (A \cup B) - (A \cap B)$.

(2) Suppose $x \in (A \cup B) - (A \cap B)$.

Therefore $A \Delta B = (A \cup B) - (A \cap B)$. $\square$

Proof.

Let A and B be sets.

(1) Suppose $x \in A \Delta B$. Thus $x \in (A - B) \cup (B - A)$. This means either $x \in A - B$ or $B - A$.

- Suppose $x \in A - B$.

- Suppose $x \in B - A$.

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- Suppose $x \in B - A$. Then $x \in (A \cup B) - (A \cap B)$ similarly.

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Outline

1. Quantifiers
2. Operations on sets
3. Combinatorial proofs

The size of a union

Proposition (Inclusion-Exclusion formula)

Let A and B be finite sets. Then $|A| + |B| = |A \cup B| + |A \cap B|$.

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Proof.

Assign label $'A'$ to the objects in A and label $'B'$ to objects in B.

On one hand we clearly handed out $|A| + |B|$ labels.

On the other hand, there are $|A \cup B|$ objects that got at least a label and exactly $|A \cap B|$ elements got doubly labeled. Therefore $|A \cup B| + |A \cap B|$ counts all elements that receive a label, double counting the ones that have two labels. Therefore the total number of labels also equals this number.

Since $|A + B|$ and $|A \cup B| + |A \cap B|$ give the answer to the same question, they must be equal. $\square$

Inclusion-exclusion formula for two sets

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Then $|A| = 500$ and $|B| = 200$. Also note

$$A \cap B = \{x \in \mathbb{Z} : 1 \leq x \leq 1000 \text{ and } 10|x\}$$

and therefore $|A \cap B| = 100$.

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$$|A \cup B| = |A| + |B| - |A \cap B| = 500 + 200 - 100 = 600.$$

Therefore there are 600 integers in the range 1 to 1000 that are divisible by either 2 or 5.

A combinatorial proof

To prove an equation of the form LHS=RHS:

- Pose a question of the form, 'In how many ways...?'
  - On one hand argue why LHS is the correct answer.
  - On the other hand argue why RHS is the correct answer.
- Therefore LHS=RHS

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Definition

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What about two or more sets?

Disjoint, pairwise disjoint sets

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For example, $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$ and $C = \{7, 8, 9\}$ are pairwise disjoint.

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The sets $A_1, A_2, \ldots A_n$ are **pairwise disjoint** provided $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

For example, $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$ and $C = \{7, 8, 9\}$ are pairwise disjoint.

Extended addition principle

If the sets $A_1, A_2, \ldots, A_n$ are pairwise disjoint sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$$

Disjoint, pairwise disjoint sets

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In a more compact notation:

$$|\cup_{k=1}^{n} A_k| = \sum_{k=1}^{n} |A_k|$$

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Note that this is stronger than requiring

$$A_1 \cup ... \cup A_n = \emptyset.$$

Consider for example

$$A_1 = \{1, 2\}, \qquad A_2 = \{2, 3\}, \qquad A_3 = \{3, 4\}.$$

Then

$$A_1 \cap A_2 = \{2\} \neq \emptyset, \qquad A_1 \cup A_2 \cup A_3 = \emptyset.$$

Problem 7 from Worksheet 5

Proposition

$$2^0 + 2^1 + \dots 2^{n-1} = 2^n - 1$$

Let $A = \{x \in 2^{\{1,\dots,n\}} : x \neq \emptyset\}$. What is $|A|$?

- Easy answer: $|A| = 2^n - 1$.
- On the other hand, let

$$A_j = \{x \in 2^{\{1,\dots,n\}} : \text{largest element in } x \text{ is } j\}$$

  - A subset cannot have two largest elements $\rightarrow A_j \cap A_i = \emptyset$.
  - Every nonempty set has a largest element $\rightarrow A = \cup_{j=1}^{n} A_j$.
  - $|A_j| = 2^{j-1}$ because j is in every $x \in A_j$ and x can be completed by any subset of $\{1, \dots, j-1\}$.

By the extended addition principle:

$$|A| = |A_1| + \dots + |A_n| = 2^0 + \dots 2^{n-1}$$

Since both the left and the right hand side answer the same question, they must be equal.