Discrete Mathematics, 2016 Fall - Worksheet 6

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In all of the above problems explain your answer in full English sentences.

1. Similar to what we did on the slide for $<$, define the corresponding relation set to two of the integer relations $\leq, =, >, \geq$.

$$R_{\leq} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \leq y\}$$
$$R_{=} = \{(x, x) \in \mathbb{Z} \times \mathbb{Z} : x \in \mathbb{Z}\}$$
$$R_{>} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x > y\}$$
$$R_{\geq} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \geq y\}$$

2. Write the following relations on the set $\{1, 2, 3, 4, 5\}$ as sets of ordered pairs.

   (a) The $\leq$ relation.

   $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 4), (4, 5), (5, 5)\}$

   (b) The 'divides' relation.

   $$R_{|} = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 4), (3, 3), (4, 4), (5, 5)\}$$

   (c) The $=$ relation.

   $$R_{=} = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$$

3. Each of the following is a relation on the set $\{1, 2, 3, 4, 5\}$. Express these relations in words and then find their inverses.

   (a) $R = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$. Being consecutive integers.

   $$R^{-1} = \{(2, 1), (3, 2), (4, 3), (5, 4)\}$$

   (b) $R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3), (4, 4), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)\}$ Greater than or equal.

   $R^{-1} = \{(1, 1), (1, 2), (2, 2), (1, 3), (2, 3), (3, 3), (1, 4), (2, 4), (3, 4), (4, 4), (1, 5), (2, 5), (3, 5), (4, 5), (5, 5)\}$

(c) $R = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$. Sum to 6.

$$R^{-1} = \{(5,1),(4,2),(3,3),(2,4),(1,5)\}$$

4. What is the inverse of the following relations?

   (a) $\leq$.

   $$\geq$$

   (b) $\{(x,y) : x,y \in \mathbb{Z}, x-y=1\}$.

   $$\{(x,y) : x,y \in \mathbb{Z}, y-x=1\}$$

   (c) $\{(x,y) : x,y \in \mathbb{Z}, xy > 0\}$.

   $$\{(x,y) : x,y \in \mathbb{Z}, xy > 0\}.$$

5. For each of the following relations on $\{1,2,3,4,5\}$ determine whether the relation is reflexive, irreflexive, symmetric, antisymmetric, and/or transitive:

   - $R = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$. Reflexive, Symmetric, Transitive, Antisymmetric
   - $R = \{(1,2),(2,3),(3,4),(4,5)\}$. Irreflexive, Antisymmetric (Vacuously)
   - $R = \{(1,1),(1,2),(1,3),(1,4),(1,5)\}$. Antisymmetric, Transitive
   - $R = \{(1,1),(1,2),(2,1),(3,4),(4,3)\}$. Symmetric
   - $R = \{1,2,3,4,5\} \times \{1,2,3,4,5\}$. Reflexive, Symmetric, Transitive

6. For the following relations on the set of humans beings, please determine whether the relation if reflexive, irreflexive, symmetric, antisymetric, and/or transitive.

   (a) has the last name as REFLEXIVE, SYMMETRIC, ANTISYMMETRIC, TRANSITIVE

   (b) is the child of IRREFLEXIVE, ANTISYMMETRIC (Vacuously)

   (c) is married to SYMMETRIC, IRREFLEXIVE

   (d) has a common parent as REFLEXIVE, SYMMETRIC

7. Consider the relation $|$ (divisible) on

   (a) On the naturals. REFLEXIVE, ANTISYMMETRIC, TRANSITIVE

   (b) On the integers. REFLEXIVE, TRANSITIVE

   Decide what properties do they have.

8. Show that the following relation is an equivalence relation:

   $$A = \{B \in 2^{\mathbb{Z}} : |B| < \infty\}, \qquad R = \{(B,C) : B,C \in A, |B| = |C|\}$$

Proof. We have to show that R is reflexive, symmetric and transitive.

Obviously $|B| = |B|$ for every $B \in A$ and therefore $(B, B) \in R$ and R is reflexive. Also if $(B, C) \in R$, i.e. $|B| = |C|$, then $|C| = |B|$, i.e. $(C, B) \in R$ and therefore R is symmetric. Finally, if $(B, C) \in R$, i.e. $|B| = |C|$ and $(C, D) \in R$ i.e., $|C| = |D|$ for some $B, C, D \in A$, then $|B| = |D|$ and thus $(B, D) \in R$ and therefore R is transitive. $\square$

9. Which of the following are equivalence relations?

   (a) $R = \{(1,1),(1,2),(2,1),(2,2),(3,3)\}$ on the set $\{1,2,3\}$. THIS ONE.

   (b) $|$ on $\mathbb{Z}$. NOT THIS ONE

   (c) $\leq$ on $\mathbb{Z}$. NOT THIS ONE

   (d) Is-an-anagram-of on the set of English words. THIS ONE

10. For each of the following congruences, find all integers N, with $N > 1$, that make the congruence true

   (a) $23 \equiv 13 \pmod{N}$ $N = 1, 2, 5, 10$

   (b) $10 \equiv 5 \pmod{N}$ $N = 1, 5$

   (c) $6 \equiv 60 \pmod{N}$ $N = 1, 2, 3, 6, 9, 18, 27, 54$