# Discrete Mathematics, 2016 Fall - Worksheet 12

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In all of the above problems explain your answer in full English sentences.

1. Calculate the first six terms of the sequence (that is, $a_0$ through $a_5$)

$$a_n = 2a_{n-1} + 2, \qquad a_0 = 1$$

**Solution.** *Simple computaton reveals* $a_1 = 4$*,* $a_2 = 10$*,* $a_3 = 22$*,* $a_4 = 46$*,* $a_5 = 94$*.*

2. Let $e_0 = 1$, $e_1 = 4$ and, for $n > 1$, let $e_n = 4(e_{n-1} - e_{n-2})$. What are the first five terms of the sequence $e_0, e_1, e_2, \dots$? Prove $e_n = (n+1)2^n$ for any natural $n$.

**Solution.** $e_0 = 1$*,* $e_1 = 4$*,* $e_2 = 12$*,* $e_3 = 32$*,* $e_4 = 80$*. We show the formula by strong induction. It clearly holds for* $n = 0, 1$*. Assume that it holds for any* $n < k$ *with* $k \geq 2$*. Then we have, in particular, that*

$$e_{k-1} = k2^{k-1}, \qquad e_{k-2} = (k-1)2^{k-2}.$$

*Then by the recursion,*

$$e_k = 4(e_{k-1} - e_{k-2}) = 4(k2^{k-1} - (k-1)2^{k-2}) = k2^{k+1} - (k-1)2^k = 2^k(2k - k + 1) = (k+1)2^k$$

3. Solve the following first order recurrence relations.

(a) $a_n = \frac{2}{3}a_{n-1}, a_0 = 4$.

**Solution.** *We know from class that the solution has the form* $a_n = c_1 \left(\frac{2}{3}\right)^n + c_2$*. Matching with the initial condition we get*

$$4 = a_0 = c_1 + c_2 = 4, \qquad \frac{8}{3} = a_1 = \frac{2}{3}c_1 + c_2.$$

*This can be solved to get* $c_1 = 4$*,* $c_2 = 0$ *and the solution to the recurrence relation is* $a_n = 4\left(\frac{2}{3}\right)^n$*.*

(b) $a_n = 2a_{n-1} + 2, a_0 = 2$.

**Solution.** *Again, by what we learned in class, we look for the solution in the form* $c_1 2^n + c_2$. *Matching with the initial condition gives*

$$2 = a_0 = c_1 + c_2, \qquad 6 = a_1 = 2c_1 + c_2$$

*which can be solved to get* $c_1 = 4$ *and* $c_2 = -2$ *and thus the solution is*

$$a_n = 4 \cdot 2^n - 2$$

4. Solve the following second order recurrence relations.

   (a) $a_n = 3a_{n-1} + 4a_{n-2}, a_0 = 3, a_1 = 2.$

   **Solution.** *The characteristic equation is* $r^2 = 3r + 4$ *which has roots* $r_1 = 4$ *and* $r_2 = -1$, *therefore we can look for the solutions in the form* $a_n = c_1 4^n + c_2(-1)^n$. *Matching the initial condition gives*

   $$3 = a_0 = c_2 + c_2, \qquad 2 = a_1 = c_1 4 - c_2$$

   *which can be solved to be* $c_1 = 5$ *and* $c_2 = -2$, *and the solution therefore is*

   $$a_n = 5 \cdot 4^n - 2(-1)^n$$

   (b) $a_n = -6a_{n-1} - 9a_{n-2}, a_0 = 3, a_1 = 6.$

   **Solution.** *The characteristic equation is* $r^2 = -6r - 9$ *which has a double root* $r = -3$, *therefore we can look for the solutions in the form* $a_n = c_1(-3)^n + c_2 n(-3)^n$. *Matching with inital conditions gives*

   $$3 = a_0 = c_1, \qquad 6 = a_1 = c_1(-3) + c_2(-3)$$

   *which has solution* $c_1 = 3$ *and* $c_2 = -5$ *and therefore the solution is*

   $$a_n = (3 - 5n) \cdot (-3)^n.$$

5. What can go wrong with the technique we used to solve the non-homogeneous equation on the slides? Try to solve

   - $a_n = 4a_{n-1} + 5a_{n-2} + 4, a_0 = 2, a_1 = 3.$

     **Solution.** *The characteristic equation of the homogeneous solution is* $r^2 = 4r + 5$ *which has roots* $r = 5, -1$ *and thus*

     $$a_n^h = c_1 5^n + c_2(-1)^n.$$

     *Now we can try a constant for the inhomogeneous solution:*

     $$c_3 = 4c_3 + 5c_3 + 4$$

     *which gives us* $c_3 = -1/2$ *and we have*

     $$a_n = c_1 5^n + c_2(-1)^n - \frac{1}{2}$$

*Matching with the initial condition gives*

$$2 = a_0 = c_1 + c_2 - \frac{1}{2}, \qquad 3 = a_1 = 5c_1 - c_2 - 1/2$$

*which has solution* $c_1 = 1$ *and* $c_2 = 3/2$ *and finally the solution is*

$$a_n = 5^n + \frac{3}{2}(-1)^n - \frac{1}{2}.$$

*In this case everything was alright.*

- $a_n = 3a_{n-1} - 2a_{n-2} + 5, a_0 = a_1 = 3.$

  **Solution.** *The characteristic equation of the homogeneous part is* $r^2 - 3r + 2 = 0$ *with roots* $r = 1, 2$*, which means*

  $$a_n^h = c_1 2^n + c_2.$$

  *If we try to find a constant solution to the inhomogeneous equation we get*

  $$c_3 = 3c_2 - 2c_3 + 5 \qquad \rightarrow 0 = 5$$

  *which is clearly impossible. No constant solves the inhomogeneous equation. Instead, we will try the next simplest* $a_n = c_3 n$*. Then plugging this back into the recurrence relation, we get*

  $$c_3 n = 3c_3(n-1) - 2c_3(n-2) + 5 \qquad \rightarrow c_3 = -5$$

  *and we can write the solution of the inhomogeneous equation in the form*

  $$a_n = c_1 2^n + c_2 - 5n$$

  *Matching with the initial condition is left to the reader.*

- $a_n = 2a_{n-1} - a_{n-2} + 2, a_0 = 4, a_1 = 2.$

  This is very similar to the previous example, except that now even $c_3 n$ will give no solution to the in homogeneous equation. Try $c_3 n^2$.