Discrete Mathematics, Section 001, Fall 2016

Lecture 13: Sequences generated by polynomials

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Outline

Consider the following identities:

$$0^2 + 1^2 + \cdots + n^2 = \frac{(2n+1)(n+1)n}{6}$$

$$0^3 + 1^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

- These are all polynomial expressions.
- Proving them is simple by induction.
- How do we come up with them in the first place?

We are going to learn:

- How to decide whether a sequence of numbers is generated by a polynomial expression.
- How to determine the polynomial in question.

The difference operator

Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers. We can form the new sequence

$$a_1 - a_0, \qquad a_2 - a_1, \qquad a_3 - a_2, \qquad \ldots$$

Definition

If a is a sequence, then Δa is a sequence defined by

$$\Delta a_n = a_{n+1} - a_n.$$

Δ is called the **difference operator**.

For example

$$\begin{array}{rccccccccccccc} a: & 0 & & 2 & & 7 & & 15 & & 26 & & 40 & & 57 \\ \Delta a: & & 2 & & 5 & & 8 & & 11 & & 14 & & 17 & \end{array}$$

The difference operator

What does Δ do to sequences given by polynomials. For example if $a_n = n^3 - 5n + 1$,

$$\begin{aligned}\Delta a_n &= a_{n+1} - a_n = \\ &= [(n+1)^3 - 5(n+1) + 1] - [n^3 - 5n + 1] = \\ &= n^3 + 3n^2 + 3n + 1 - 5n - 5 + 1 - n^3 + 5n - 1 = \\ &= 3n^2 + 3n - 4\end{aligned}$$

Δ took a degree-3 polynomial and turned it into a degree-2 polynomial.

The difference operator

Proposition

Let a be a sequence of numbers in which a_n is given by a degree-d polynomial in n where $d \geq 1$. Then Δa is a sequence given by a polynomial of degree $d-1$.

Proof

Suppose

$$a_n = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0, \qquad c_d \neq 0.$$

Then

$$\begin{aligned}\Delta a_n = a_{n+1} - a_n = \\ &= \left[c_d(n+1)^d + c_{d-1}(n+1)^{d-1} + \ldots c_1(n+1) + c_0\right] - \\ &- \left[c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0\right].\end{aligned}$$

[...]

Proposition

Let a be a sequence of numbers in which a_n is given by a degree-d polynomial in n where $d \geq 1$. Then Δa is a sequence given by a polynomial of degree $d - 1$.

Proof

[...]

$$\begin{aligned}
\Delta a_n = & \left[c_d(n+1)^d + c_{d-1}(n+1)^{d-1} + \ldots c_1(n+1) + c_0\right] - \\
& - \left[c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0\right] = \\
& = c_d\left[(n+1)^d - n^d\right] + c_{d-1}\left[(n+1)^{d-1} - n^{d-1}\right] + \ldots \\
& + c_1[(n+1) - n] + c_0[1-1]
\end{aligned}$$

[...]

Proposition

Let a be a sequence of numbers in which a_n is given by a degree-d polynomial in n where $d \geq 1$. Then Δa is a sequence given by a polynomial of degree $d-1$.

Proof

[...]

$$\Delta a_n = c_d\left[(n+1)^d - n^d\right] + c_{d-1}\left[(n+1)^{d-1} - n^{d-1}\right] + \dots + c_1[(n+1)-n] + c_0[1-1]$$

Note that by the Binomial theorem,

$$(n+1)^j = n^j + \sum_{k=0}^{j-1}\binom{j}{k}n^k \qquad j = 0,\dots d$$

and therefore $(n+1)^j - n^j$ is a polynomial of degree $j-1$. This proves the claim. $\square$

Multiple applications of Δ

a :	0		2		7		15		26		40		57
Δa :		2		5		8		11		14		17	
$\Delta^2 a$:			3		3		3		3		3		
$\Delta^3 a$:				0		0		0		0			

Corollary

If a sequence a is generated by a polynomial of degree d, then $\Delta^{d+1} a$ is the all-zeros sequence.

Now we seek to prove the converse of this!

Properties of Δ

Proposition

Let a, b, be sequences of numbes and let s be a number.

1. If, for all n, then $\Delta(a_n + b_n) = \Delta a_n + \Delta b_n$.
2. If, for all n, then $\Delta(s a_n) = s \Delta a_n$.

Proof.

(1),(2) are simple (Worksheet Problem 1). $\square$

Binomial coefficients as polynomials

Let

$$a_n = \binom{n}{3} = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} = \frac{1}{6} n(n-1)(n-2)$$

Strictly speaking we only proved this for $n \geq 3$. Note, however that it applies perfectly to $n = 0, 1, 2$ as well.

This is a polynomial!

$$\begin{array}{rcccccccccccccccc}
a: & 0 & & 0 & & 0 & & 1 & & 4 & & 10 & & 20 & & 35 \\
\Delta a: & & 0 & & 0 & & 1 & & 3 & & 6 & & 10 & & 15 & \\
\Delta^2 a: & & & 0 & & 1 & & 2 & & 3 & & 4 & & 5 & & 6 \\
\Delta^3 a: & & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\Delta^4 a: & & & & & 0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

You can find Pascal's triangle in this in the $\nearrow$ direction

Binomial coefficients as polynomials

$a:$	0		0		0		1		4		10		20		35	
$\Delta a:$		0		0		1		3		6		10		15		
$\Delta^2 a:$			0		1		2		3		4		5		6	
$\Delta^3 a:$				1		1		1		1		1		1		
$\Delta^4 a:$					0		0		0		0		0		0	

$$\begin{aligned}
\Delta\binom{n}{3} &= \binom{n+1}{3} - \binom{n}{3} \\
&= \frac{1}{6}(n+1)n(n-1) - \frac{1}{6}n(n-1)(n-2) \\
&= \frac{(n^3-n)-(n^3-3n^2+2n)}{6} = \frac{3n^2-3n}{6} \\
&= \frac{1}{2}n(n-1) = \binom{n}{2}
\end{aligned}$$

Binomial coefficients as polynomials

Proposition

If $k > 0$ and $a_n = \binom{n}{k}$, then

1) $\Delta a_n = \binom{n}{k-1}$.
2) $a_0 = \Delta a_0 = \Delta^2 a_0 = \cdots = \Delta^{k-1} a_0 = 0$ but $\Delta^k a_0 = 1$.

Proof.

To see 1), note that.

$$\Delta \binom{n}{k} = \binom{n+1}{k} - \binom{n}{k} = \binom{n}{k-1} \qquad k \geq n.$$

by Pascal's identity. For $k > n$, all three binomial coefficients are zero and therefore the identity holds again.

2) is homework. $\square$

Characterizing properties of polynomial sequences

For the sequence $a_n = \binom{n}{k}$, we know

- $\Delta^{k+1} a_0 = 0$ for all n.
- The value of a_0.
- The value of $\Delta^j a_0$ for all $1 \leq j < k$.

This is enough to determine a polynomial sequence a_n!

Proposition

Let a and b be sequences of numbers and let k be a positive integer. Suppose that

- $\Delta^k a_n$ and $\Delta^k b_n$ are zero for all n,
- $a_0 = b_0$,
- $\Delta^j a_0 = \Delta^j b_0$ for all $1 \leq j < k$.

Then $a_n = b_n$.

Characterizing properties of polynomial sequences

Proposition

Let a and b be sequences of numbers and let k be a positive integer. Suppose that

- $\Delta^k a_n$ and $\Delta^k b_n$ are zero for all n,
- $a_0 = b_0$,
- $\Delta^j a_0 = \Delta^j b_0$ for all $1 \leq j < k$.

Then $a_n = b_n$.

Proof

We prove this by induction on k. The basis case is $k = 1$, in which case we already have

$$\Delta a_n = \Delta b_n = 0$$

which means that both sequences are constants. Since we have $a_0 = b_0$, this means that the sequences are identical.[...]

Characterizing properties of polynomial sequences

Hypotheses:

- $\Delta^k a_n$ and $\Delta^k b_n$ are zero for all n,
- $a_0 = b_0$,
- $\Delta^j a_0 = \Delta^j b_0$ for all $1 \leq j < k$.

Proof.

[...]Suppose now that the result is true for $k = l$ and let a and b be sequences satisfying the hypotheses of the theorem with $k = l + 1$. Form the new sequences

$$a'_n = \Delta a_n, \qquad b'_n = \Delta b_n.$$

Clearly, a'_n and b'_n satisfies the hypotheses with $k = l$ and therefore $a' = b'$.
[...]

$\square$

Characterizing properties of polynomial sequences

Proof.

[...]

$$a'_n = \Delta a_n, \qquad b'_n = \Delta b_n.$$

Then $a'_n = b'_n$ for every n.

Now we show that $a_n = b_n$ for every n. Suppose FTSC that a and b were different. Then there is a smallest m such that

$$a_m \neq b_m$$

Note that $m \neq 0$ because $a_0 = b_0$ and that by the minimality of m, we have $a_{m-1} = b_{m-1}$. Then

$$a_m - a_{m-1} = a'_{m-1} = b'_{m-1} = b_m - b_{m-1}$$

which implies

$$a_m - b_m = a_{m-1} - b_{m-1} = 0.$$

This means $a_m = b_m \Rightarrow\Leftarrow$. $\square$

Theorem

Let a be a sequence of numbers. The terms a_n can be expressed as a polynomial expression in n if and only if there is a non-negative integer k such that for all $n \geq 0$, we have $\Delta^{k+1} a_n = 0$. In this case,

$$a_n = a_0 \binom{n}{0} + (\Delta a_0) \binom{n}{1} + (\Delta^2 a_0) \binom{n}{2} + \cdots + (\Delta^k a_0) \binom{n}{k}.$$

Proof.

We have already proved that a_n is a polynomial of degree d, then $\Delta^{d+1} a_n = 0$, we only have to prove the other direction and the formula. This is going to be a reading exercise on your homework. $\square$

An example

$$\begin{array}{rccccccccccccc}
a: & 0 & & 2 & & 7 & & 15 & & 26 & & 40 & & 57 \\
\Delta a: & & 2 & & 5 & & 8 & & 11 & & 14 & & 17 & \\
\Delta^2 a: & & & 3 & & 3 & & 3 & & 3 & & 3 & & \\
\Delta^3 a: & & & & 0 & & 0 & & 0 & & 0 & & &
\end{array}$$

Then by the theorem with $k = 3$,

$$a_n = 0\binom{n}{0} + 2\binom{n}{1} + 3\binom{n}{2} = 0 + 2n + 3\frac{n(n-1)}{2} = \frac{n(3n+1)}{2}.$$

Do problems 2-3 from the worksheet!