Discrete Mathematics, Section 001, Fall 2016

Lecture 10: Contrapositive, Contradiction, Smallest counterexample

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Outline

Proof by contrapositive

So far we learned to prove $A \Rightarrow B$ statements directly. Here is an alternative way:

$$a \to b = \neg a \vee b = \neg a \vee \neg(\neg b) = \neg(\neg b) \vee \neg a = \neg b \to \neg a$$

where $=$ now means logical equivalence. Therefore,

$$A \Rightarrow B \qquad \text{is the same as} \qquad (\text{not } B) \Rightarrow (\text{not } A)$$

Proof by contrapositive

To prove 'If A, then B': Assume $(\neg B)$ and work to prove $(\neg A)$.

Proof by contrapositive

Proposition

Let R be an equivalence relation on a set A and let $a, b \in A$. If $a \not R b$, then $[a] \cap [b] = \emptyset$.

Proof.

Let R be an equivalece relation on a set A and let $a, b \in A$. We prove the contrapositive of the statement.

Suppose $[a] \cap [b] \neq \emptyset$.

we have aRb. □

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Suppose $[a] \cap [b] \neq \emptyset$. Thus there is an $x \in [a] \cap [b]$; that is, $x \in [a]$ and $x \in [b]$.

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Suppose $[a] \cap [b] \neq \emptyset$. Thus there is an $x \in [a] \cap [b]$; that is, $x \in [a]$ and $x \in [b]$. Hence xRa and xRb. By symmety aRx, and since xRb, by transitivity we have aRb. $\square$

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To see the advantage, try thinking of a direct proof!

Formulate some theorems like this in Problem 1 and prove one in Problem 2 on the Worksheet!

Reductio ad absurdum (Proof by contradiction)

Note that

$$a \to b = \neg a \vee b = \neg a \vee b \vee False = \\ = \neg(a \wedge \neg b) \vee False = (a \wedge \neg b) \to False$$

where $=$ again means logical equivalence.

$A \Rightarrow B$ is the same as $(A \text{ and } not(B)) \Rightarrow Impossible$

What on earth does this mean?

- If A and $not(B)$ are simultaneously true that implies the impossible.
- In other words $A \Rightarrow B$ is the same as A and $not(B)$ being impossible to be simultaneously true.

Reductio ad absurdum (Proof by contradiction)

$$A \Rightarrow B \qquad \text{is the same as} \qquad (A \text{ and } (\text{not } B)) \Rightarrow \mathit{Impossible}$$

If A and (not B) are both true that implies the impossible.

Proof by contradiction

To prove "If A then B":

- We assume the conditions in A.
- Suppose for the sake of contradiction, (not B).
- Argue until we reach a contradiction (something impossible).

Thus we have reached a contradiction. Therefore the supposition (not B) must be false. Hence B is true.

The last two sentece is often abbreviated by $\Rightarrow\Leftarrow$.

Reductio ad absurdum (Proof by contradiction)

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$\Rightarrow\Leftarrow$.

Reductio ad absurdum (Proof by contradiction)

Proposition

No integer is both even and odd

Proof.

Let x be an integer.

Suppose, for the sake of contradiction, that x is both even and odd.

$$\Rightarrow\Leftarrow$$

Therefore x is not both even and odd, and the propostition is proved. $\square$

Reductio ad absurdum (Proof by contradiction)

Proposition

No integer is both even and odd

Proof.

Let x be an integer.

Suppose, for the sake of contradiction, that x is both even and odd.

Since x is even, we know $2|x$; that is, there is an integer a such that $x = 2a$. Since x is odd, we know that there is an integer b such that $x = 2b + 1$.

$\Rightarrow\Leftarrow$

Therefore x is not both even and odd, and the propostition is proved. $\square$

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Therefore $2a = 2b + 1$. Dividing both sides by 2 gives $a - b = \frac{1}{2}$. Note that $a - b$ is an integer since a and b are integers. However, $\frac{1}{2}$ is not an integer.$\Rightarrow\Leftarrow$

Therefore x is not both even and odd, and the propostition is proved. $\square$

Now practice how to set up such proofs by doing Problem 3 on the worksheet!

Proving that a set is empty

Proposition

No integer is both even and odd

Proving that a set is empty

Proposition

Let

$$X = \{x \in \mathbb{Z} : x \text{ is even}\}, \qquad Y = \{x \in \mathbb{Z} : x \text{ is odd}\}.$$

Then $X \cap Y = \emptyset$.

Proving that a set is empty

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Proving that a set is empty

To prove a set is empty:
Assume the set is nonempty and argue to a contradiction.

Proving uniqueness

Proving uniqueness

To prove there is at most one object that satisfies certain conditions:

Suppose there are two different objects, x and y, that satisfy the conditions and argue to a contradiction.

Proposition

Let a and b be numbers with $a \neq 0$. There is at most one number x with $ax + b = 0$.

Proving uniqueness

Proposition

Let a and b be numbers with $a \neq 0$. There is at most one number x with $ax + b = 0$.

Proof.

Suppose there are two different numbers x and y such that $ax + b = 0$ and $ay + b = 0$. This gives

$$ax + b = ay + b.$$

Substracting b from both sides gives $ax = ay$. Since $a \neq 0$, we can divide both sides by a to give $x = y$. $\Rightarrow\Leftarrow$. □

Now practice writing some proofs like this by doing Problem 4 on the worksheet!

Caveats

- Proof by contradiction is often easier than the direct proof because there are more things to work with.
- Sometimes this is not required though.
- Here is how to tell when you can simplify a proof of “If A then B” by contradiction.
    - You assumed A and (not B), but you only used A and reached B and (not B). Then you really have a direct proof and you should remove the extraneous proof by contradiction apparatus.
    - You assumed A and (not B), but you only used (not B), and the contradiction you reached was A and (not A). Then you really have a proof by contrapositive and you should rewrite the proof in that form.

Sudoku

With Sudoku puzzles, countradiction reasoning is what you do.

			4		9			8
					3		1	
5	6	7		2	8			

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Claim

The 1 for the middle box must go to the left on the 2 in the bottom row.

Proof.

□

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Claim

The 1 for the middle box must go to the left on the 2 in the bottom row.

Proof.

Suppose 1 goes in the top row. Then the 1 for the left box cannot be in the top row and cannot be in the middle row and cannot be in the bottom row. $\Rightarrow\Leftarrow$. $\square$

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Claim

The 1 for the middle box must go to the left on the 2 in the bottom row.

Proof.

Suppose 1 goes in the middle row. But then we would have two 1s in the middle row. $\Rightarrow\Leftarrow$. $\square$

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Outline

Motivation

- Proving something by contradiction is essentially assuming there is a counterexample and showing that it would break the universe.
- You can view this really as a proof by the lack of counterexample.
- Sometimes there is a natural ordering associated with certain statements. For example, when we prove something for all natural numbers, then these natural numbers are ordered by the usual ordering.
- In these cases if there is a counterexample, e.g. a natural number for which the statement is not true, then there has to be a smallest such counterexample.
- This gives us an extra tool to work with as we can use that for anything smaller than the smaller counterexample, the result holds.

Example

Proposition

Every natural number is either even or odd

Proof.

Suppose, for the sake of contradiction, that there were an integer x that is neither even nor odd.

$\ldots$

$\Rightarrow\Leftarrow$. Therefore every natural is either even or odd. $\square$

Example

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Suppose, for the sake of contradiction, that there were an integer x that is neither even nor odd. So there is no integer b with $x = 2b$, and there is no integer b such that $x = 2b + 1$. . . . $\Rightarrow\Leftarrow$. Therefore every natural is either even or odd. $\square$

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$\Rightarrow\Leftarrow$. Therefore every natural is either even or odd. $\square$

We are stuck here, we have to come up with something more!

Proposition

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Suppose, for the sake of contradiction, that there were an natural x that is neither even nor odd.

we have a contradiction, so the supposition is false and the proposition is proved. $\square$

Proposition

Every natural number is either even or odd

Proof.

Suppose, for the sake of contradiction, that there were an natural x that is neither even nor odd. Then there is a SMALLEST natural number, x, that is neither even nor odd.

we have a contradiction, so the supposition is false and the proposition is proved. □

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Proof.

Suppose, for the sake of contradiction, that there were an natural x that is neither even nor odd. Then there is a SMALLEST natural number, x, that is neither even nor odd.

Since $x - 1$ is a smaller natural than x, we know that $x - 1$ is either even or odd.

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Every natural number is either even or odd

Proof.

Suppose, for the sake of contradiction, that there were an natural x that is neither even nor odd. Then there is a SMALLEST natural number, x, that is neither even nor odd.

We know $x \neq 0$, because 0 is even. Therefore $x \geq 1$.

Since $x - 1$ is a smaller natural than x, we know that $x - 1$ is either even or odd.

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Since $0 \leq x - 1$ is a smaller natural than x, we know that $x - 1$ is either even or odd.

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Since $0 \leq x - 1$ is a smaller natural than x, we know that $x - 1$ is either even or odd.

- Suppose $x - 1$ is odd, then $x - 1 = 2a + 1$ for some $a \in \mathbb{Z}$. Thus $x = 2a + 2 = 2(a + 1)$ and so x is even.$\Rightarrow\Leftarrow$.

we have a contradiction, so the supposition is false and the proposition is proved. $\square$

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Since $0 \leq x - 1$ is a smaller natural than x, we know that $x - 1$ is either even or odd.

- Suppose $x - 1$ is odd, then $x - 1 = 2a + 1$ for some $a \in \mathbb{Z}$. Thus $x = 2a + 2 = 2(a + 1)$ and so x is even. $\Rightarrow\Leftarrow$.
- Suppose $x - 1$ is even, then $x - 1 = 2a$ for some $a \in \mathbb{Z}$. Thus $x = 2a + 1$ and so x is odd. $\Rightarrow\Leftarrow$.

we have a contradiction, so the supposition is false and the proposition is proved. □

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Suppose, for the sake of contradiction, that there were an natural x that is neither even nor odd. Then there is a SMALLEST natural number, x, that is neither even nor odd.

We know $x \neq 0$, because 0 is even. Therefore $x \geq 1$.

Since $0 \leq x - 1$ is a smaller natural than x, we know that $x - 1$ is either even or odd.

- Suppose $x - 1$ is odd, then $x - 1 = 2a + 1$ for some $a \in \mathbb{Z}$. Thus $x = 2a + 2 = 2(a + 1)$ and so x is even.$\Rightarrow\Leftarrow$.
- Suppose $x - 1$ is even, then $x - 1 = 2a$ for some $a \in \mathbb{Z}$. Thus $x = 2a + 1$ and so x is odd.$\Rightarrow\Leftarrow$.

In every case, we have a contradiction, so the supposition is false and the proposition is proved. $\square$

Extension to all integers

Corollary

Every integer is either even or odd.

Proof.

Let $x \in \mathbb{Z}$. If $x \geq 0$ then the previous proposition applies and x is either even or odd. Otherwise $x < 0$ in which case $-x > 0$ and so $-x$ is either even or odd.

- If $-x$ is even, then $-x = 2a$ for some $a \in \mathbb{Z}$. But then $x = -2a = 2(-a)$ and x is even as $-a \in \mathbb{Z}$.
- If $-x$ is odd then $-x = 2a + 1$ for some $a \in \mathbb{Z}$. But then $x = -2a - 1 = 2(-a - 1) + 1$ and x is odd as $-a - 1 \in \mathbb{Z}$.

In every case, x is either even or odd. □

Proof template for smallest counterexample

Proof by smallest counterexample

1. First, let x be a smallest counterexample to the result we are trying to prove. It must be clear that there can be such an x.
2. Second, rule out x being the very smallest possibility. This is usually easy and is called the basis step.
3. Third consider an instant x' of the result that is just smaller than x. Use the fact that according to the assumption, the result is true for x' but false for x, to reach $\Rightarrow\Leftarrow$.

Try to write a proof like this yourself by doing Problem 5 on the worksheet!

Another example

Proposition

Let n be a positive integer. Then $\sum_{k=1}^{n}(2k-1) = n^2$.

Proof

Suppose the claim is false. Then there is a smallest positive integer x for which

$$\sum_{k=1}^{x}(2k-1) \neq x^2.$$

Clearly $x \neq 1$, because $1 = 1^2$ and therefore $x > 1$. Since x was the smallest counterexample,

$$\sum_{k=1}^{x-1}(2k-1) = (x-1)^2.$$

[...]

Another example

Proposition

Let n be a positive integer. Then $\sum_{k=1}^{n}(2k-1)=n^2$.

Proof.

[...]

$$\sum_{k=1}^{x-1}(2k-1)=(x-1)^2.$$

Add $2x-1$ to both sides to get

$$\sum_{k=1}^{x}(2k-1)=(x-1)^2+(2x-1)=$$
$$=x^2-2x+1+2x-1=x^2.$$

contradicting that x is a counterexample. $\Rightarrow\Leftarrow$. $\square$

The well ordering principle

The following statement is crucial in the above proofs.

Well ordering principle

Every nonempty set of natural numbers contains a least element.

For example,

- $P = \{x \in \mathbb{N} : x \text{ is prime}\}$. Least element is 2.
- $X = \{x \in \mathbb{N} : x \text{ is neither even nor odd}\}$. This is of course the emptyset, but if it was not, there would be a smallest element.

However,

- $Y = \{y \in \mathbb{Q} : y \geq 0, y \notin \mathbb{Z}\}$. This has no smallest element, if there was, you could prove that every rational is an integer, which is obviously false.

The WOP is a property of the natural numbers but not a property of the integers or the rationals.

The well ordering principle

To prove a statement about the natural numbers

1. Suppose for the sake of contradiction, that the statement is false, let $X \subseteq \mathbb{N}$ to be the set of counterexamples.
2. By assumption $X \neq \emptyset$ and thus by the well ordering principle it has a smallest element.
3. We know $x \neq 0$ because (show result holds for 0).
4. Therefore $x - 1 \in \mathbb{N}$ and the statement is true for $x - 1$.
5. (Argue for contradiction) $\Rightarrow\Leftarrow$.

In the remaining time, practice this by doing Problems 6-8 on the worksheet.