Discrete Mathematics, Section 002, Fall 2016

Lecture 6: Relations, Equivalence relations

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Outline

A different take on ordinary relations

Ordinary relations you are probably familiar with: $<, \leq, =, >, \geq$.

Let us look at them a little differently. For example, let

$$R_< = \{(a, b) : a \in \mathbb{Z}, b \in \mathbb{Z}, a < b\}.$$

Then

$$(a, b) \in R_< \qquad \Leftrightarrow \qquad a < b.$$

You can also think about this as:

'The pair (a, b) passes the test of $<$ if and only if $(a, b) \in R_<$'.

Note that $R_< \subseteq \mathbb{Z} \times \mathbb{Z}$.

Do Problem 1 on the Worksheet!

Abstract notion of a relation

- We want to have a notion of a general relation on a set A.
- We want to have a set R of all those ordered pairs

$$(a_1, a_2) \in A \times A$$

that 'pass the test' of the relation.

- In this case, we say a_1 is in relation R with a_2 or shorthand

$$a_1 R a_2 \qquad (\Leftrightarrow (a_1, a_2) \in R)$$

- Similarly, if $(a_1, a_2) \notin R$, we say a_1 is not in relation R with a_2 or shorthand

$$a_1 \not R a_2.$$

- This way, we can call the set R itself a relation.

$(a_1, a_2) \in R$	a_1 is in relation R with a_2	$a_1 R a_2$
$(a_1, a_2) \notin R$	a_1 is not in relation R with a_2	$a_1 \not R a_2$

Abstract notion of a relation

$(a_1, a_2) \in R$	a_1 is in relation R with a_2	$a_1 R a_2$
$(a_1, a_2) \notin R$	a_1 is not in relation R with a_2	$a_1 \not R a_2$

For example:

$$A = \{1, 2, 3, 4\} \qquad R = \{(1, 2), (1, 3), (2, 2), (3, 2)\}$$

- $1 R 2$
- $2 \not R 1$
- 4 is not in relation R with anything, not even with itself.
- Nothing is in relation R with 4, not even itself.
- $2 R 2$.

Abstract notion of a relation

Relation

A **relation** is a set of ordered pairs.

Relation on a set

We say R is a **relation on a set** A, provided $R \subseteq A \times A$.

Relation between sets

We say R is a **relation from set** A **to set** B provided $R \subseteq A \times B$.

Further example

Let $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6, 7\}$.

$$R = \{(1,1), (2,2), (3,3), (4,4)\}$$
$$S = \{(1,2), (3,2)\}$$
$$T = \{(1,4), (1,5), (4,7)\}$$
$$U = \{(4,4), (5,2), (6,2), (7,3)\}$$
$$V = \{(1,7), (7,1)\}$$

All of these are relations.

- R is the equality relation on A.
- S is a relation on A
- T is a relation from A to B.
- U is a relation from B to A.
- V is a relation but it is not from A to B nor from B to A.

Do Problem 2 on the worksheet!

Inverse relation

Inverse relation

Let R be a relation. The **inverse of** R, denoted R^{-1}, is the relation formed by reversing the order of all the ordered pairs in R. In symbols, $R^{-1} = \{(x, y) : (y, x) \in R\}$

$$R = \{(1, 5), (2, 6), (3, 7), (3, 8)\} \to R^{-1} = \{(5, 1), (6, 2), (7, 3), (8, 3)\}$$

Proposition

Let R be a relation. Then $(R^{-1})^{-1} = R$.

Proof.

Suppose $(x, y) \in R$. Then $(y, x) \in R^{-1}$ and thus $(x, y) \in (R^{-1})^{-1}$.

Now suppose $(x, y) \in (R^{-1})^{-1}$. Then $(y, x) \in R^{-1}$ and so $(x, y) \in R$. $\square$

Do Problems 3-4 on the Worksheet!

Properties of relations

Definition

Let R be a relation on a set A.

- If for all $x \in A$ we have xRx, we call R **reflexive.**
- If for all $x \in A$ we have $x \not R x$, we call R **irreflexive.**
- If for all $x, y \in A$ we have $xRy \Rightarrow yRx$, we call R **symmetric.**
- If for all $x, y \in A$, we have
$$(xRy \land yRx) \Rightarrow x = y,$$
we call R **antisymmetric.**
- If for all $x, y, z \in A$ we have
$$(xRy \land yRz) \Rightarrow xRz,$$
we call R **transitive.**

reflexive: $\forall x \in A, xRx$.
symmetric: $\forall x, y \in A, xRy \Rightarrow yRx$
transitive: $\forall x, y, z \in A, (xRy \wedge yRz) \Rightarrow xRz$
antisymmetric: $\forall x, y \in A, (xRy \wedge yRx) \Rightarrow x = y$.
irreflexive: $\forall x \in A, x \not R x$.

The relation $=$ (equality) on the integers:

- Reflexive: Any integer is equal to itself.
- Symmetric: If $x = y$ then $y = x$.
- Transitive: If $x = y$ and $y = z$ then $x = z$.
- Trivially antisymmetric.
- Not irreflexive.

Practice this by Problems 6-7 on the worksheet!

Outline

1. Relations
2. Equivalence relations

Some relations we come across in math, have strong resemblence to equality.

Definition

Le R be a relation on a set A. We say R is an **equivalence relation** provided it is reflexive, symmetric and transitive.

For example:

$$A = \{B \in 2^{\mathbb{Z}} : |B| < \infty\}, \qquad R = \{(B, C) : B, C \in A, |B| = |C|\}$$

Prove this on Problem 8 on the Worksheet!

Another example:

Congruence modulo n

Let n be a positive integer. We say that x and y are congruent modulo n, and write

$$x \equiv y \pmod{n}$$

provided $n|(x-y)$.

For example:

- $3 \equiv 13 \pmod 5$ because $5|-10 = 3 - 13$.
- $4 \equiv 4 \pmod 5$ because $5|0 = 4 - 4$.
- $16 \not\equiv 3 \pmod 5$ because $5 \nmid 13 = 16 - 3$.

Get familiar by doing Problem 10 on the worksheet!

Theorem

Let n be a positive integer. The 'is congruent to mod n' relation is an equivalence relation on the set of integers.

Proof.

Let n be a positive integer and let $\equiv$ denote congruence mod n. We need to show that $\equiv$ is reflexive, symmetric, and transitive.

- Claim: $\equiv$ is reflexive.
- Claim: $\equiv$ is symmetric.
- Claim: $\equiv$ is transitive. $\cdots$ $\cdots$ Thus $\equiv$ is transitive.

Therefore $\equiv$ is an equivalence relation. $\square$

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- Claim: $\equiv$ is reflexive. Let x be an arbitrary integer.

  Therefore $x \equiv x$. Thus $\equiv$ is reflexive.
- Claim: $\equiv$ is symmetric.
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- Claim: $\equiv$ is reflexive. Let x be an arbitrary integer. Since $0 \cdot n = 0$, we have $n|0$, which we can rewrite as $n|(x - x)$. Therefore $x \equiv x$. Thus $\equiv$ is reflexive.
- Claim: $\equiv$ is symmetric.
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- Claim: $\equiv$ is symmetric. Let x and y be integers and suppose $x \equiv y$.

  Therefore $y \equiv x$. Thus $\equiv$ is symmetric.
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- Claim: $\equiv$ is reflexive. Let x be an arbitrary integer. Since $0 \cdot n = 0$, we have $n|0$, which we can rewrite as $n|(x - x)$. Therefore $x \equiv x$. Thus $\equiv$ is reflexive.
- Claim: $\equiv$ is symmetric. Let x and y be integers and suppose $x \equiv y$. This means that $n|(x - y)$.

  and so $n|(y - x)$. Therefore $y \equiv x$. Thus $\equiv$ is symmetric.
- Claim: $\equiv$ is transitive.$\cdots$ $\cdots$ Thus $\equiv$ is transitive.

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- Claim: $\equiv$ is reflexive. Let x be an arbitrary integer. Since $0 \cdot n = 0$, we have $n|0$, which we can rewrite as $n|(x - x)$. Therefore $x \equiv x$. Thus $\equiv$ is reflexive.
- Claim: $\equiv$ is symmetric. Let x and y be integers and suppose $x \equiv y$. This means that $n|(x - y)$. So there is an integer k such that $(x - y) = kn$. But then $(y - x) = (-k)n$ and so $n|(y - x)$. Therefore $y \equiv x$. Thus $\equiv$ is symmetric.
- Claim: $\equiv$ is transitive. $\cdots$ (HW) $\cdots$ Thus $\equiv$ is transitive.

Therefore $\equiv$ is an equivalence relation. □