Discrete Mathematics, Section 002, Spring 2016

Lecture 24: Simple graphs and their subgraphs

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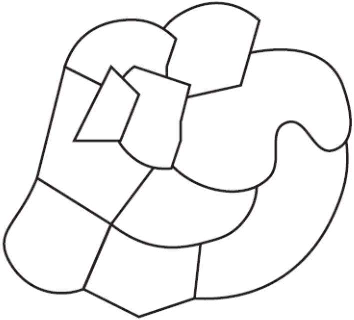
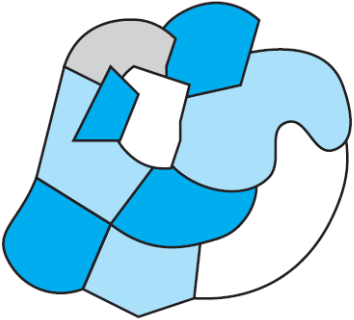
Outline

Map coloring



How many colors do you need to color a map such that bordering countries have different colors?

Map coloring



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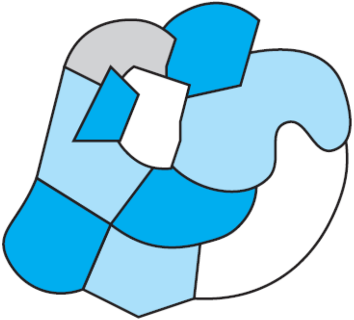
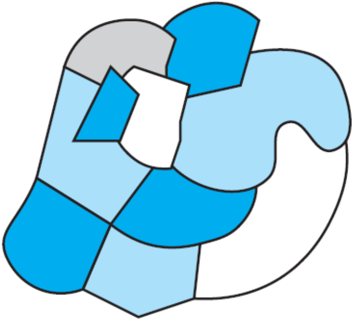
Map coloring



How many colors do you need to color a map such that bordering countries have different colors?

- Can this map be colored with fewer than four colors?
- Is there another map that can be colored with fewer than four colors?
- Is there a map that requires more than four colors?

Map coloring



How many colors do you need to color a map such that bordering countries have different colors?

- Can this map be colored with fewer than four colors? $\rightarrow$ No
- Is there another map that can be colored with fewer than four colors? $\rightarrow$ Yes
- Is there a map that requires more than four colors? $\rightarrow$ No

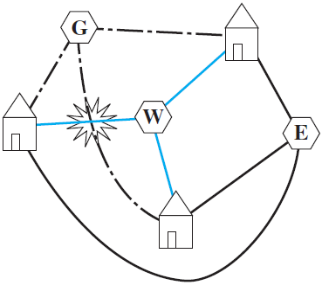
Map coloring $\to$ scheduling

If you think map coloring is useless:

Assume a university has a lot of students and a lot of courses. How many exams does a university need to hold so that nobody has a conflict.

Problem	Map Coloring	Exam scheduling
Assign	colors	time slots
to	countries	courses
condition	common border	common student
	$\Rightarrow$ different colors	$\Rightarrow$ different slots
objective	fewest color	fewest time slots

Three utilities



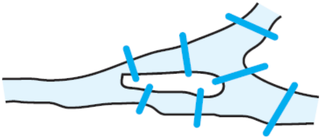
Gas-Electric-Water needs to be connected to all houses and pipes cannot cross.

This is not possible!

If you think this is silly:

On a printed circuit board, resistors, capacitors, etc. are printed on a flat piece of plastic. Connections between these are by printing metal wires on the surface. If two crossed, that would be short circuit and one has to use multi-layers which are expensive.

Seven bridges



Is there a tour of the city we can take through so that we cross every bridge exactly once?

This is not possible!

If you think this is silly:

Think about organizing garbage collection in the city and optimizing the path of the garbage truck.

Outline

1. Motivating examples
2. Simple graphs
3. Subgraphs

Definition of a simple graph

Definition

A **simple graph** is a pair $G = (V, E)$ where V is a nonempty finite set and E is a set of two element subsets of V

For example,

$$G = (\{1, 2, 3, 4, 5, 6, 7\}, \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{5, 6\}\})$$

Here

$$V = \{1, 2, 3, 4, 5, 6, 7\} \qquad E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{5, 6\}\}$$

Terminology

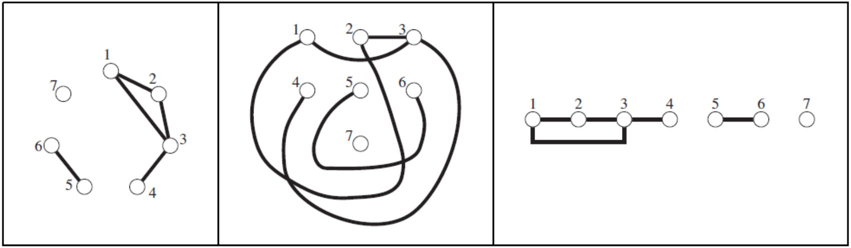
The elements of V are called **vertices** (singular: vertex), and the elements of E are called **edges**.

Visualization of a simple graph

$$G = (\{1, 2, 3, 4, 5, 6, 7\}, \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{5, 6\}\})$$

- For every $v \in V$, draw a node.
- For every $e \in E$, draw a line connecting

$$v_i, v_j \in V \qquad \text{where} \qquad e = \{v_i, v_j\}.$$



All three are all good visualizations of G however the left and rightmost are obviously clearer. (Do Problems 1-2 on WS).

Warnings

- $e \in E$ is a two element set.
- Therefore an edge cannot be $\{u, u\}$ (No loops.)
- No parallel edges either.

This is why we are talking about simple graphs. Since we are only talking about simple graphs in this class, we will simply say graph.

Adjacency

Definition

Let $G = (V, E)$ be a graph and let $u, v \in V$. We say that u is **adjacent** to v provided $\{u, v\} \in E$.

- The notation $u \sim v$ means that u is adjacent to v.
- If $\{u, v\} \in E$, then u and v are called it's **endpoints.**
- When there is no risk of confusion, we will simply write

$$uv \qquad \text{for} \qquad \{u, v\}.$$

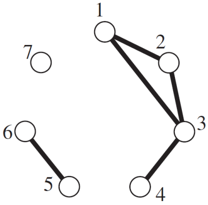
- To say that u is an endpoint of an edge e, we will write $u \in e$ and say that u is **incident** on e.
- We DO NOT say that u and v are 'connected'.

Degree of vertices

Definition

Let $G = (V, E)$ be a graph. The **neighborhood** of a vertex v is defined to be

$$N(v) = \{u \in V : u \sim v\}$$

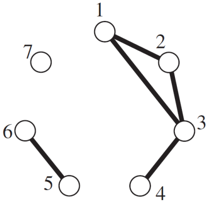


$N(1) = \{2, 3\}$ $N(2) = \{1, 3\}$
$N(3) = \{1, 2, 4\}$ $N(4) = \{3\}$
$N(5) = \{6\}$ $N(6) = \{5\}$
$N(7) = \emptyset$

Degree of vertices

Definition

Let $G = (V, E)$ be a graph and let $v \in V$. The degree of v is the number of edges with which v is incident. The degree of v is denoted $d_G(v)$ or, if there is no risk of confusion, simply $d(v)$.



$$d(1) = 2 \quad d(2) = 2 \quad d(3) = 3 \quad d(4) = 1$$
$$d(5) = 1 \quad d(6) = 1 \quad d(7) = 0$$

Note that

- $d(v) = |N(v)|$
- In the example, $\sum_{v \in V} d(v) = 10$ which is also $2|E|$.

Theorem

Let $G = (V, E)$. The sum of the degrees of the vertices in G is twice the number of edges; that is,

$$\sum_{v \in V} d(v) = 2|E|$$

Suppose the vertex set is $\{v_1, v_2, \ldots, v_n\}$. Consider the rectangular array where the i, j entry is 1 if $v_i \sim v_j$ and 0 otherwise. In our example,

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the **adjacency matrix** of the graph.

Theorem

Let $G = (V, E)$. The sum of the degrees of the vertices in G is twice the number of edges; that is,

$$\sum_{v \in V} d(v) = 2|E|$$

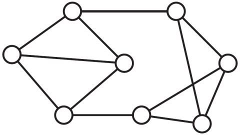
Consider the rectangular array where the i, j entry is 1 if $v_i \sim v_j$ and 0 otherwise.

Q: How many ones are in the adjacency matrix?

- For each $e \in E$, there are exactly two 1-s in the matrix. Indeed, if $v_i v_j \in E$ then there is a 1 at ij and at ji. So one answer is $2|E|$.
- In row i, there is a 1 for every vertex adjacent to v_i. That is, the total number of 1-s in row i is $d(i)$. Therefore the second answer is $\sum_{v \in B} d(v)$. □

Further notation

- **Maximum and minimum degree:** The maximum and minimum degree of a vertex in G is denoted by $\Delta(G)$ and $\delta(G)$ respectively.
- If all vertices in G have the same degree r, we call G r-regular.



- The **order** of G is $|V|$. The size of G is $|E|$.
- When the vertex and edge sets are not named, we will refer to them as $V(G)$ and $E(G)$.
- We call G a **complete graph** if all pairs of distinct vertices are adjacent. A complete graph of n vertices: K_n.

Outline

1. Motivating examples
2. Simple graphs
3. Subgraphs

Definition

Let G and H be graphs. We call G a **subgraph** of H provided $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$.

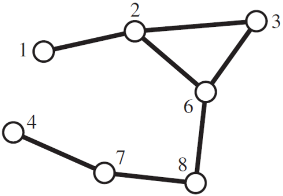
For example,

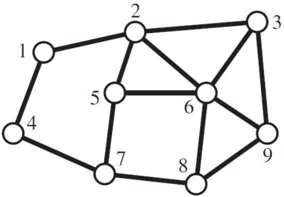
$$V(G) = \{1, 2, 3, 4, 6, 7, 8\}$$

$$E(G) = \{\{1, 2\}, \{2, 3\}, \{2, 6\}, \{3, 6\}, \{4, 7\}, \{6, 8\}, \{7, 8\}\}$$

$$V(G) = \{1, 2, 3, 4, 6, 7, 8, 9\}$$

$$E(G) = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 5\}, \{2, 6\}, \{3, 6\}, \{3, 9\}, \{4, 7\} \{5, 6\}, \{5, 7\}, \{6, 8\}, \{6, 9\} \{7, 8\}, \{8, 9\}\}$$





Spanning subgraphs

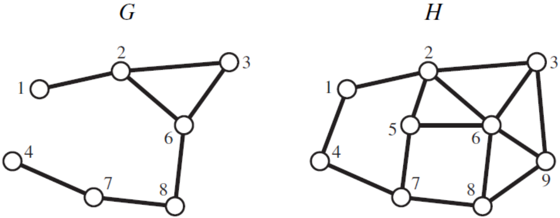
One way to get subgraphs is to delete edges. Let H be a graph, then deleting the edge $e \in E(H)$, we get the graph $H - e$ which formally is the graph

$$V(H - e) = V(H), \qquad E(H - e) = E(H) - \{e\}$$

If we only remove edges, we get a spanning subgraph.

Definition

Let G and H be graphs. We call G a **spanning subgraph** of H provided G is a subgraph of H and $V(G) = V(H)$.



Induced subgraphs

When removing a vertex $v \in V(H)$, we need to be more careful as we have to remove all the edges that v is incident on.

$$V(H - v) = V(H) - \{v\} \qquad E(H - v) = \{e \in E(H) : v \notin e\}$$

If we only remove vertices, we get an induced subgraph.

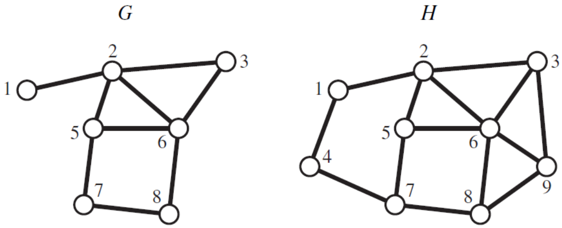
Definition

Let H be a graph and let A be a subset of the vertices of H; that is, $A \subseteq V(H)$. The **subgraph of H induced on A** is the graph $H[A]$ defined by

$$V(H[A]) = A, \qquad E(H[A]) = \{xy \in E(H) : x \in A, y \in A\}.$$

Induced subgraphs

In our earlier example, if $A = \{1, 2, 3, 5, 6, 7, 8\}$,



where $G = H[A]$.

Do Problem 3 on the worksheet.

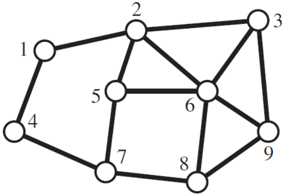
Cliques

Subgraphs that are complete graphs have a special name.

Definition

Let G be a graph. A subset of vertices $S \subseteq V(G)$ is called a **clique** provided any two distinct vertices in S are adjacent. The **clique number** of G is the size of a largest clique; it is denoted by $\omega(G)$.

In other words a set $S \subseteq V(G)$ is a clique provided $G[S]$ is a complete graph.



Some of the cliques:

$\{1, 4\}$ $\{2, 5, 6\}$ $\{9\}$ $\{2, 3, 6\}$

$\{6, 8, 9\}$ $\{4\}$ $\emptyset$

We also have $\omega(H) = 3$.

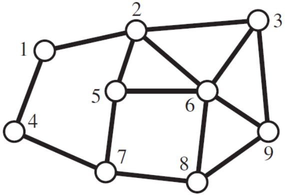
Independent sets

Edgeless induced graphs also have a special name.

Definition

Let G be a graph. A subset of vertices $S \subseteq V(G)$ is called an **independent set** provided no two vertices in S are adjacent. The **independence number of** G is the size of a largest independent set; it is denoted $\alpha(G)$.

In other words, a set $S \subseteq V(G)$ is independent if $G[S]$ is an edgeless graph.



Some of the independent sets:

$\{1, 3, 5\}$ $\{1, 7, 9\}$ $\{4\}$ $\{4, 6\}$

$\{1, 3, 5, 8\}$ $\{1, 3, 7\}$ $\emptyset$

We also have $\alpha(H) = 4$.

Complements

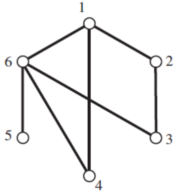
Cliques and Independets sets are flipsides of the same coin.

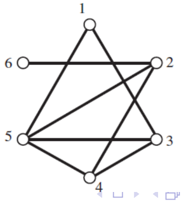
Definition

Let G be a graph. The complement of G is the graph denoted $\bar{G}$ defined by

$$V(\bar{G}) = V(G), \qquad E(\bar{G}) = \{xy : x, y \in V(G), x \neq y, xy \notin E(G)\}$$

In other words the complement is formed by removing all the edges from G and replacing them by all possible edges that are not in G.





Cliques, independent sets in complementing graphs

Proposition

Let G be a graph. A subset of $V(G)$ is a clique of G if and only if it is an independent set of $\bar{G}$. Furthermore,

$$\omega(G) = \alpha(\bar{G}), \qquad \alpha(G) = \omega(\bar{G})$$

Do Problem 4-5!