Discrete Mathematics, 2016 Spring - Worksheet 3

September 14, 2016

Instructor: Zsolt Pajor-Gyulai, CIMS

In all of the above problems explain your answer in full English sentences.

1. Disprove the following statements:

   (a) If a and b are integers with $a|b$, then $a \leq b$.

   For example, $2|-4$ *but* $2 > -4$.

   (b) If a, b, and c are positive integers with $a|(bc)$, then $a|b$ or $a|c$.

   For example, $6|12 = 3 \cdot 4$ *but neither* $6|3$ *nor* $6|4$. *Thus the statement is false.*

   (c) If p and q are prime, then $p + q$ is composite.

   For example, 2 *and* 3 *are primes, but* $5 = 2 + 3$ *is not composite. Thus the statement is false.*

   (d) Two right triangles have the same area if and only if the lengths of their hypotenuses are the same.

   Consider two right triangles, one with side length $3, 4, 5$ *and one with* $\sqrt{12.5}$, $\sqrt{12.5}$, 5. *They both have hypotenuse* 5 *but the area of the first one is* $3 \cdot 4/2 = 6$, *while the area other one is* $\sqrt{12.5} \cdot \sqrt{12.5}/2 = 6.25$. *Thus the statement is false.*

2. What does it mean for an if and only if statement to be false? What properties should a couterexample for an if-and-only-if statement have?

   An if and only if statement, $A \Leftrightarrow B$ *is false, if there are circumstances under which* A *is true but* B *is not, or the other way around. Therefore a counterexample must exhibit such a circumstance.*

3. Evaluate the following Boolean expressions:

   (a) $True \wedge True \wedge True \wedge True \wedge False$. *FALSE*

   (b) $(\neg True) \vee True$. *TRUE*

   (c) $\neg(True \vee True)$. *FALSE*

   (d) $(True \vee True) \wedge False$. *FALSE*

   (e) $True \vee (True \wedge False)$. *TRUE*

4. Prove the following Boolean identities by truth tables:
   (a) $\neg(x \wedge y) = (\neg x) \vee (\neg y)$ and $\neg(x \vee y) = (\neg x) \wedge (\neg y)$ (DeMorgan's laws).
   (b) $x \to y = (\neg x) \vee y$.
   (c) $x \leftrightarrow y = (\neg x) \leftrightarrow (\neg y)$.

   where $=$ stands for logically equivalent.

   For each part, write down the two truth tables and check that they are identical.

5. Find a logically equivalent Boolean expression to $x \leftrightarrow y$ only in terms of the basic Boolean operations $\wedge, \vee, \neg$.

$$x \leftrightarrow y = (x \to y) \wedge (y \to x) = ((\neg x) \vee y) \wedge ((\neg y) \vee x)$$

6. How many different binary operation can there be?

   Since there are 4 different different inputs, each of which we can assign two possible values, this number is $2^4 = 16$.

7. A person's inititals are the two-element lists consisting of the initial letters of their first and last names. For example, mines are ZP.
   (a) How many possible initials are there?

   Since we have 26 *possibilities both for the first and the second letters, by the multiplication principle, we have* $26^2 = 676$ *possible initials.*

   (b) How many initials are there where the two letters are different?

   Since we have 26 *possibilites for the first letter and once that is chosen we have* 25 *possibilities for the second one, the multiplication principle implies that there are* $25 \cdot 26 = 650$ *such initials.*

8. A club has 10 members.
   (a) A club has 10 members who wish to elect a president and a vice-president. How many ways can these positions be filled (assuming the club is not a cult-of personality dictatorship and one person can only have one title)?

   We have 10 *ways to elect the president and once that is done we have* 9 *ways to elect a VP. The multiplication principle implies that there are* $10 \cdot 9 = 90$ *possible outcomes of the election.*

   (b) Now suppose the club also wants to elect a secretary and a treasurer. How many outcomes are there for the election then?

   Once we picked the president (10 ways) and the VP (9 ways) then we have 8 people to choose the treasurer from which leaves us with 7 choices for the treasurer. Thus we have $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ *possible outcomes.*

9. In how many ways can a black rook and a white rook be placed on different squares of a chess board such that neither is attacking the other?

   We can put the black rook anywhere (64 choices) which blocks out exactly one row and a column, so exactly 15 places (don't double count the position of the black rook!). This means that we have to put the white rook somewhere on the remaining $64 - 15 = 49$ *places. Therefore we have* $64 \cdot 49 = 3136$ *total possibilities.*

   Further problems to practice:

10. License plates in a cetrtain state consist of six characters: The first three characters ar uppercase letters (A-Z), and the last three characters are digits (0-9).
    (a) How many license plates are possible?
    (b) How many license plates are possible if no character maybe repeated on the same plate?
11. A telephone number (in the US and Canada) is a ten digin number whose first digit cannot be a 0 or a 1. How many telephone numbers are possible?
12. A US Social Security number is a nine-digit number. The first digit may be zero.
    (a) How many of these are even?
    (b) How many have all of their digits even?
    (c) How many read the same backwards? (e.g. 122979221)