Discrete Mathematics, 2016 Fall - Worksheet 23

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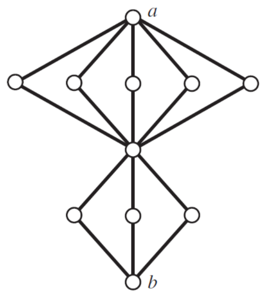
In all of the above problems explain your answer in full English sentences.

1. Let G be the graph in the figure.



   (a) How many different paths are there from a to b? $5 \cdot 3 = 15$

   (b) How many different walks are there from a to b? INFINITELY MANY

2. Prove that K_n is connected.

   Proof. Let $x, y \in V(K_n)$, then $xy \in E(K_n)$ and the path $P = x \sim y$ connects the two. Since the choice of x and y were arbitrary, this finishes the proof. □

3. Suppose G is a connected graph in which each vertex has even degree. Then, G has no cut edges.

   Proof. FTSC assume that there there is a cut edge $e = xy$. Then $G - e$ has two components G_1 and G_2 with $x \in G_1$ and $y \in G_2$. Then removing e decreases the degree of x by one which now becomes an odd number. However, the degree of other vertices in G_1 are unaffected and thus

$$\sum_{v \in V(G_1)} d_{G_1}(v) = \sum_{x \neq v \in V(G_1)} d_{G_1}(v) + d_{G_1}(x)$$

   which is the sum of an even and an odd number and therefore it is odd. However, this contradicts G_1 being a graph. □

4. List all the trees
   (a) with vertex set $\{1, 2, 3\}$. The only one is P_3
   (b) with vertex set $\{1, 2, 3, 4\}$. We have P_4 and the star.

5. a) Let T be a tree with $n \geq 1$ vertices. Prove that T has $n - 1$ edges.

   Proof. We prove this by induction. The base case checks out as $n = 1$ is just one vertex with $1 - 1 = 0$ edges.

   By the induction hypothesis, suppose the result is true for $n = k$ and let T be a tree on $n = k+1$ vertices. Let v be a leaf of T and let $T' = T - v$. Then T' is a tree with k vertices and thus it has $k - 1$ edges. Since v is a leaf, $d(v) = 1$ and therefore we only lost one edge by deleting it, that is T has $k + 1$ edges as required. □

   b) Prove the converse, i.e. that if G is a connected graph that has exactly $n - 1$ edges then G must be a tree.

   Proof. Suppose G is a connected graph with n vertices and $n - 1$ edges. We know that G has a spanning tree T. However,

$$|E(T)| = |V(T)| - 1 = |V(G)| - 1 = |E(G)|$$

   and it turns out that we did not throw out any edges. Therefore $G = T$ and thus it's a tree. □

6. Let T be a tree. Prove that the average degree of a vertex in T is less than 2.

Proof.

$$d_avg = \frac{1}{|V(G)|} \sum_{v \in V(G)} d_G(v) = 2\frac{|E(G)|}{|V(G)|} = 2\frac{n-1}{n} < 2$$

□