Discrete Mathematics, Section 001, Fall 2016

Lecture 15: Counting Functions, Pigeonhole principle

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Outline

1. Counting functions
2. Applications of the pigeonhole principle
3. Cantor's theorem

Number of functions between finite sets

Q: Let A and B be finite sets. How many functions from A to B are there?

$$A = \{1, 2, \ldots, a\}, \qquad B = \{1, 2, \ldots, b\}$$

- We can write every function as

$$f = \{(1, ?), (2, ?), (3, ?), \ldots, (a, ?)\}$$

  where the ?s are entries from B.
- There are b choices for each $?$.

Proposition

Let A and B be finite sets with $|A| = a$ and $|B| = b$. The number of functions from A to B is b^a.

If $A = \{1, 2, 3\}$ and $B = \{4, 5\}$ then all the functions $f : A \to B$:

$$\begin{array}{ll}
\{(1,4),(2,4),(3,4)\} & \{(1,5),(2,4),(3,4)\} \\
\{(1,4),(2,4),(3,5)\} & \{(1,5),(2,4),(3,5)\} \\
\{(1,4),(2,5),(3,4)\} & \{(1,5),(2,5),(3,4)\} \\
\{(1,4),(2,5),(3,5)\} & \{(1,5),(2,5),(3,5)\}
\end{array}$$

As predicted, there are $2^3 = 8$ functions.

Notation

The set of all functions from A to B is denoted by B^A.

Just as it was the case with 2^A (all subsets of A), this is just a notation which is convenient because then

$$|B^A| = |B|^{|A|}$$

Do Problem 1 on the worksheet.

Q: Let A and B be finite sets with $|A| = a$ and $|B| = b$.

- How many functions $f : A \to B$ are one-to-one?
- How many are onto?

- Let $|A| > |B|$, and assume f is one-to-one. Without loss of generality assume

$$A = \{1, 2, \ldots, |A|\}, \qquad B = \{1, 2, \ldots, |B|\}$$

  - Assume that we find what gets mapped to $\{1, 2, \ldots, |B|\}$.
  - We still have $|A| - |B|$ elements in A to map somewhere, but we ran out of distinct elements in B!

  Therefore f cannot be one-to-one.
- Let $|A| < |B|$, and assume f is onto.
  - Assume that we map all elements in A to an element in B.
  - We still have $|B| - |A|$ elements in B that we haven't covered yet!

  Therefore f cannot be onto.

The pigeonhole principle

Proposition (The pigeonhole principle)

Let A and B be finite sets and let $f : A \to B$. If $|A| > |B|$, then f is not one-to-one. If $|A| < |B|$, then f is not onto.

The contrapositive of this statement is

Proposition

Let A and B be finite sets and let $f : A \to B$. If f is a bijection, then $|A| = |B|$.

Fun fact: For infinite sets, this becomes the definition of the cardinalities being equal:

- $|\mathbb{N}| = |\mathbb{Z}|$ (later today)
- $|\mathbb{R}| \neq |\mathbb{Z}|$.

Q: Let A and B be finite sets with $|A| = a$ and $|B| = b$.

- How many functions $f : A \to B$ are one-to-one?
- How many are onto?

$$(1, ?), (2, ?), \ldots, (a, ?)$$

- One-to-one when $|A| \leq |B|$:
  The ?-s have to be filled with elements from B without repetition. Therefore

$$\#\text{One-to-one functions} = (b)_a = \frac{b!}{(b-a)!}$$

- Onto when $|A| \geq |B|$: The ?-s have to be filled with elements from B with every element used at least once.

$$\#\text{Onto functions} = \sum_{j=0}^{b} (-1)^j \binom{b}{j} (b-j)^a. \qquad (WS2)$$

Outline

Proposition

Let $n \in \mathbb{N}$. Then there exist positive integers a and b, with $a \neq b$, $a, b \leq 11$, such that $n^a - n^b$ is divisible by 10.

E.g. $n = 17$, then

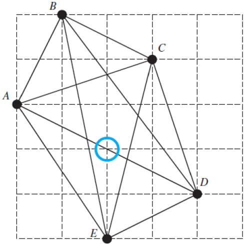
$$17^6 - 17^2 = 24,137,569 - 289 = 24,137,280$$

Proof.

Consider the 11 natural number

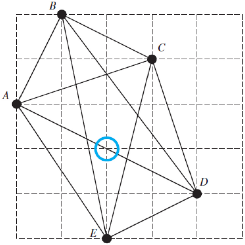
$$n^1 \qquad n^2 \qquad n^3 \qquad \dots n^{11}$$

The last digits of these numbers takes values in $\{0, 1, 2, \dots, 9\}$. Since there are 10 possible digits and 11 different numbers, two of these numbers, let's say, n^a, n^b, will share the same last digit. Therefore $10 | n^b - n^a$. □



Proposition

Given five distinct lattice points in the plane (points with integer coordinates), at least one of the line segments determined by these points has a lattice point as its midpoint

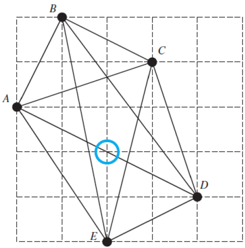


Proposition

Given five distinct lattice points in the plane (points with integer coordinates), at least one of the line segments determined by these points has a lattice point as its midpoint

Recall:

If (a, b) and (c, d) are two points in the plane then the midpoint of the line segment connecting them is $(\frac{a+c}{2}, \frac{b+d}{2})$.



Proposition

Given five distinct lattice points in the plane (points with integer coordinates), at least one of the line segments determined by these points has a lattice point as its midpoint

Proof.

Each lattice point is one of the following type:

$$(even, even) \qquad (even, odd) \qquad (odd, even) \qquad (odd, odd)$$

Since we are given five lattice points, the pigeonhole principle implies that two of these points have to be of the same type, let's say (a, b) and (c, d). Then both $a + c$ and $b + d$ are even and therefore the midpoint $(\frac{a+c}{2}, \frac{b+d}{2})$ of the connecting segment has integer coordinates. $\square$

Do Problem 3 on the worksheet.

Outline

Cardinality of infinite sets

Definition

Two infinite sets A and B have the same cardinality if there is a bijection $f : A \to B$.

For example $f : \mathbb{N} \to \mathbb{Z}$, defined by

$$f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even,} \\ (n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

n	0	1	2	3	4	5	6	7	8	9	...
$f(n)$	0	1	-1	2	-2	3	-3	4	-4	5	...

- Every natural appears exactly once in the first row. (One-to-one)
- Every integer appears exactly once in the second row. (Onto)

Therefore f is a bijection and $\mathbb{N}$ and $\mathbb{Z}$ has the same cardinality.

Cantor's theorem

Q: Is this true for all infinite sets? -> Nope

Theorem(Cantor's theorem)

Let A be a set. If $f : A \to 2^A$, then f is not onto.

Proof.

Let A be a set and $f : A \to 2^A$. To show that f is not onto, we have to find a $B \in 2^A$ for which $\nexists a \in A$ with $f(a) = B$. Let

$$B = \{x \in A : x \notin f(x)\}.$$

FTSC, assume $\exists a \in A$ such that $f(a) = B$. Then either a is in B or not.

- If $a \in B$, then $a \notin f(a)$. But $f(a) = B$ implies $a \in f(a)$. $\Rightarrow\Leftarrow$.
- If $a \notin B = f(a)$, then by definition, $a \in B$. $\Rightarrow\Leftarrow$.

Therefore there is no such a and f is not onto. $\square$

A more intuitive reading

A is the set of people in a company (not excluding companies with inifnitely many employees, actually that is the interesting case)

- The company names each committee after one of the employees.
- Let "Joe" be the name of the committee of the people who are not a member of the committee named after them.

Q:Is Joe a member of "Joe"?

- If yes then no by definition.
- If no then yes by definition.

Each case is a contradiction!

Proposition

$''|\mathbb{R}| > |\mathbb{N}|''$.

Proof.

Define the function $f : 2^{\mathbb{N}} \to \mathbb{R}$ by

$$f(A) = \sum_{a \in A} 10^{-a}.$$

In decimals, $f(A)$ is a number with ones exactly at every position corresponding to all $a \in A$. E.g.

$$f(\{1,2,4\}) = 10^{-1} + 10^{-2} + 10^{-4} = 0.1101$$

Therefore if $A_1 \neq A_2$, then $f(A_1) \neq f(A_2)$ and f is one-to one. Therefore

$$''|\mathbb{N}| < |2^{\mathbb{N}}| \leq |\mathbb{R}|''.$$

$\square$