Discrete Mathematics, Section 002, Fall 2016

Lecture 8: Binomial and Multinomial coefficients

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October 2, 2016

Outline

Let $A = 2^{\{1,2\}}$ and R be the has-the-same-size relation.

Equivalence class	Size of class
$[\emptyset]$	1
$[\{1\}]$	2
$[\{1,2\}]$	1

Also

$$(x+y)^2 = 1 \cdot x^2 + 2 \cdot xy + 1 \cdot y^2$$

Let $A = 2^{\{1,2,3\}}$ and R be the has-the-same-size relation.

Equivalence class	Size of class
$[\emptyset]$	1
$[\{1\}]$	3
$[\{1,2\}]$	3
$[\{1,2,3\}$	1

Also

$$(x+y)^3 = 1 \cdot x^2 + 3 \cdot x^2 y + 3 \cdot xy^2 + 1 \cdot y^3$$

Let $A = 2^{\{1,2,3,4\}}$ and R be the has-the-same-size relation.

Equivalence class	Size of class
$[\emptyset]$	1
$[\{1\}]$	4
$[\{1,2\}]$	6
$[\{1,2,3\}$	4
$[\{1,2,3,4\}]$	1

Also

$$(x+y)^4 = 1 \cdot x^4 + 4 \cdot x^3 y + 6 \cdot x^2 y^2 + 4 \cdot x y^3 + 1 \cdot y^4$$

The connection

Note that

$$(x+y)^2 = (x+y)(x+y) = xx + xy + yx + yy$$

also

$$(x+y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$

In general

$$(x+y)^n = \underbrace{(x+y)}_{1}\underbrace{(x+y)}_{2}\dots\underbrace{(x+y)}_{n}$$

- To form a term, we make a lists of length n of x-s and y-s.
- Then we need to identify the ones that only differ in the order of the x,y-s.

This defintes an equivalence relation R on the set of such lists.

The connection

In general

$$(x+y)^n = \underbrace{(x+y)}_{1}\underbrace{(x+y)}_{2}\dots\underbrace{(x+y)}_{n}$$

- To form a term, we make a lists of length n of x-s and y-s.
- Then we need to identify the ones that only differ in the order of the x,y-s.

This defintes an equivalence relation R on the set of such lists.

$$|[\underbrace{x\dots x}_{n-k}\underbrace{y\dots y}_{k}]| = ?$$

The connection

For example

$$|[xxxyyy]| = ?$$

We only have to keep track of where the y-s are.

List in class	Pos. of y-s
$xxxyyy$	$\{4,5,6\}$
$\dots$	$\dots$
$xyxyxy$	$\{2,4,6\}$
$yxyxyx$	$\{1,3,5\}$
$\dots$	$\dots$
$yyyxxx$	$\{1,2,3\}$

On the right hand side we list all three element subsets of $\{1,2,3,4,5,6\}$.

Similarly,

$$|[\underbrace{x\ldots x}_{n-k}\underbrace{y\ldots y}_{k}]|$$

is given by the number of all k element subsets of $\{1, 2, \ldots, n\}$.

Definition

Let $n, k \in \mathbb{N}$. The symbol $\binom{n}{k}$ denotes the number of k-element subsets of an n-element set. It is called the **binomial coefficient** and we say it as n choose k.

Binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Do Problems 1-3 on the Worksheet!

Examples

- Evaluate $\binom{n}{0}$.
  The only zero element subset is $\emptyset$,
  $$\binom{n}{0} = 1$$
- Evaluate $\binom{n}{1}$.
  There are exactly n ways to choose 1 element out of n to form a 1 element subset.
  $$\binom{n}{1} = n$$

Examples

- Relate $\binom{n}{n-k}$ and $\binom{n}{k}$.
  Choosing a k element subset specifies exactly an $n-k$ element subset which is the complement. This works both ways and therefore

Proposition

Let $n, k \in \mathbb{N}$ with $0 \leq k \leq n$. Then

$$\binom{n}{k} = \binom{n}{n-k}$$

- This allows us to compute e.g.

$$\binom{5}{0} = 1, \quad \binom{5}{1} = 5 \quad \binom{5}{2} = \binom{5}{3} = ?,$$
$$\binom{5}{4} = \binom{5}{1} = 5, \quad \binom{5}{5} = \binom{5}{0} = 1.$$

A combinatorial proof

Pascal's identity

Let n and k be integers with $0 < k < n$. Then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof.

Question: How many k element subsets does the set $\{1, 2, \ldots, n\}$ have?

- $\binom{n}{k}$, by definition.
- When forming a k-element subset of $\{1, 2, \ldots, n\}$ we either put the element n into it or not.
  - If it does then there are $\binom{n-1}{k-1}$ choices complete the subset.
  - If it does not then there are $\binom{n-1}{k}$ choices for the subset.

Therefore $\binom{n-1}{k-1} + \binom{n-1}{k}$ also gives an answer. $\square$

Pascal's triangle

Pascal's identity

Let n and k be integers with $0 < k < n$. Then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\begin{array}{lccccccccc}
n = 0: & & & & & 1 & & & & \\
n = 1: & & & & 1 & & 1 & & & \\
n = 2: & & & 1 & & 2 & & 1 & & \\
n = 3: & & 1 & & 3 & & 3 & & 1 & \\
n = 4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

The kth element (starting from 0) in the n-th row gives $\binom{n}{k}$. Do Problems 4-5 on the Worksheet!

Formula for $\binom{n}{k}$

Theorem

Let n and k be integers with $0 \leq k \leq n$. Then

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof

Consider A to be all the rearrangements of the numbers $\{1, 2, \ldots, n\}$ and for each rearrangement consider the subset given by the first k elements. We define an equivalence relation under which two rearrangements are equivalent if they give the same subset. Indeed, define the equivalence relation R such that two rearrangements are equivalent provided the first k elements (and therefore the last $n-k$ automatically) are the same but possibly rearranged amongst each other. [...]

Formula for $\binom{n}{k}$

Theorem

Let n and k be integers with $0 \leq k \leq n$. Then

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof.

[...] The cardinality of each equivalence class is the number of arrangements of the first k elements times the number of arrangements of the last $n-k$ elements which is clearly $k! \cdot (n-k)!$. There is a one to one correspondence between subsets of k elements and the equivalence classes and therefore

$$\binom{n}{k} = \frac{|A|}{|[.]|} = \frac{n!}{k!(n-k)!},$$

and the theorem is proved. □

Now use this to do Problem 6-8 on the Worksheet!

Outline

Multinomial coefficients

Alternative way to think of $\binom{n}{k}$

Let A be an n-element set and we want to hand out k labels saying ‘good’ and $n-k$ labels saying ‘bad’. How many ways can we do this? $\quad\to\quad \binom{n}{k}$.

This can be generalized:

Multinomial coefficient

Let $\binom{n}{a\ b\ c}$ be the number of ways to label the elements of an n-element set with three types of labels, where we hand out a label of Type I, b label of Type II and c label of Type III.

Compute this for a few simple cases by doing Problem 9 from the worksheet!

Formula for the multinomial coefficient

Propositon

$$\binom{n}{a\ b\ c} = \begin{cases} \frac{n!}{a!b!c!} & \text{if } a+b+c=n \\ 0 & \text{otherwise} \end{cases}$$

Proof.

If $a+b+c \neq n$, then there is no way to hand out exactly one label to all elements. Therefore in this case $\binom{n}{a\ b\ c} = 0$.
[...]

□

Formula for the multinomial coefficient

Propositon

$$\binom{n}{a\ b\ c} = \begin{cases} \frac{n!}{a!b!c!} & \text{if } a+b+c=n \\ 0 & \text{otherwise} \end{cases}$$

Proof.

[...] Assume $a+b+c=n$. On the set D of all rearrangements of the set of n elements N, define the equivalence relation R where two arrangements are equivalent if their first a elements form the same subset of N and similarly the following b elements and the remaining c elements after.

$$\underbrace{*\ *\ *\ *\ *\ *\ *\ **}_{a} \mid \underbrace{*\ *\ *\ *\ *}_{b} \mid \underbrace{*\ *\ *\ *\ *\ *\ *\ **}_{c}$$

[...] $\square$

Formula for the multinomial coefficient

Propositon

$$\binom{n}{a\ b\ c} = \begin{cases} \frac{n!}{a!b!c!} & \text{if } a+b+c=n \\ 0 & \text{otherwise} \end{cases}$$

Proof.

[...]

$$\underbrace{* * * * * * * *}_{a} \mid \underbrace{* * * * *}_{b} \mid \underbrace{* * * * * * * *}_{c}$$

In any equivalence class, there are those rearrangements where these subsets are rearranged individually. A labelling can be identified with an equivalence class. This can be done $a!b!c!$ ways. Therefore

$$\binom{n}{a\ b\ c} = \frac{|D|}{|[d]|} = \frac{n!}{a!b!c!}, \qquad d \in D.$$

□

Multinomial theorem

Theorem

$$(x+y+z)^n = \sum_{a+b+c=n} \binom{n}{a\ b\ c} x^a y^b z^c$$

where the sum is over all natural numbers a, b, c, with $a+b+c=n$.