Discrete Mathematics, 2016 Spring - Worksheet 5

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Instructor: Zsolt Pajor-Gyulai, CIMS

In all of the above problems explain your answer in full English sentences.

1. Rewrite the following sentences using the quantifier notation.

   (a) Every integer is a prime.

   $$\forall x \in \mathbb{Z}, x \text{ is a prime.}$$

   (b) There is an integer whose square is 2.

   $$\exists x \in \mathbb{Z} \text{ such that } x^2 = 2.$$

   (c) All integers are divisible by 5.

   $$\forall x \in \mathbb{Z}, 5|x.$$

   (d) Some integer is divisible by 7.

   $$\exists x \in \mathbb{Z} \text{ such that } 7|x.$$

   (e) For every integer x, there is an integer y such that $xy = 1$.

   $$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \text{ such that } xy = 1.$$

   (f) There is an integer x and an integer y such that $x/y = 10$.

   $$\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z} \text{ such that } x/y = 10.$$

   (g) There is an integer that, when multiplied by any integer, always gives the result 0.

   $$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, xy = 0.$$

   (h) No matter what integer you choose, there is always another integer that is larger.

   $$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \text{ such that } y > x.$$

2. Write the negation of each of the sentences in the previous problem, first with quantifiers and then in plain English. In the firs way, move the $\neg$ symbol as far to the right as possible.

(a) Every integer is a prime.
$$\exists x \in \mathbb{Z}, \neg(x \text{ is a prime}).$$

(b) There is an integer whose square is 2.
$$\forall x \in \mathbb{Z}, \neg(x^2 = 2).$$

(c) All integers are divisible by 5.
$$\exists x \in \mathbb{Z}, \neg(5|x).$$

(d) Some integer is divisible by 7.
$$\forall x \in \mathbb{Z}, \neg(7|x).$$

(e) For every integer x, there is an integer y such that $xy = 1$.
$$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg(xy = 1).$$

(f) There is an integer x and an integer y such that $x/y = 10$.
$$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg(x/y = 10).$$

(g) There is an integer that, when multiplied by any integer, always gives the result 0.
$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \neg(xy = 0).$$

(h) No matter what integer you choose, there is always another integer that is larger.
$$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg(y > x).$$

3. Label each of the following sentences about integers as either true or false. (No need to prove them)

   (a) $\forall x, \forall y, xy = 0$ FALSE
   (b) $\forall x, \exists y, xy = 0$ TRUE, y=0
   (c) $\exists x, \forall y, xy = 0$ TRUE, x=0
   (d) $\exists x, \exists y, xy = 0$ TRUE

4. Let $A = \{1, 2, 3, 4, 5\}$ and let $B = \{4, 5, 6, 7\}$. Compute

   (a) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$
   (b) $A \cap B = \{4, 5\}$
   (c) $A - B = \{1, 2, 3\}$
   (d) $B - A = \{6, 7\}$
   (e) $A \Delta B = \{1, 2, 3, 6, 7\}$
   (f) $A \times B = \{(1, 4), (1, 5), (1, 6), (1, 7), (2, 4), (2, 5), (2, 6), (2, 7), (3, 4), (3, 5), (3, 6), (3, 7), (4, 4), (4, 5), (4, 6), (4, 7), (5, 4), (5, 5), (5, 6), (5, 7)\}$

(g) $B \times A = \{(4,1),(4,2),(4,3),(4,4),(4,5),(5,1),(5,2),(5,3),$
$(5,4),(5,5),(6,1),(6,2),(6,3),(6,4),(6,5),(7,1),(7,2),(7,3),(7,4),(7,5)\}$

5. Prove the following theorems and illustrate them with a Venn-diagram (you can look at p57 for what this means).(First DeMorgan's Law)

Theorem 1. *Let A, B, and C sets. Then*

$$A - (B \cup C) = (A - B) \cap (A - C)$$

Proof.

$$\begin{aligned} A - (B \cup C) &= \{x \in A : x \notin B \cup C\} = \{x \in A : \neg((x \in B) \vee (x \in C))\} = \\ &= \{x \in A : \neg(x \in B) \wedge \neg(x \in A)\} = \\ &= \{x \in A : \neg(x \in B)\} \cap \{x \in A : \neg(x \in C)\} = (A - B) \cap (A - C) \end{aligned}$$

where the DeMorgan law for the Boolean operators was used to obtain the second equality. □

6. Let A and B be sets with $|A| = 10$ and $|B| = 7$.

(a) Calculate $|A \cap B| + |A \cup B|$.

Since $|A \cup B| = |A| + |B| - |A \cup B|$, the answer is $10 + 7 = 17$.

(b) Find an upper bound y and a lower bound x for $|A \cup B|$, that are sharp. That is

$$x \le |A \cup B| \le y.$$

To show that your answer is sharp, find sets such that $|A \cup B| = x$ and $|A \cup B| = y$ exactly.

$$10 = |A| \le |A \cup B| \le |A| + |B| = 17$$

The lower bound is sharp as shown by the example $B \subseteq A$, while the upper bound is sharp as shown by the example $A \cap B = \emptyset$.

Remark. *In general, one has the bounds*

$$\max(|A|, |B|) \le |A \cup B| \le |A| + |B|$$

7. Prove the following proposition:

Proposition 1. *Let n be an integer. Then*

$$2^0 + 2^1 + ... + 2^{n-1} = 2^n - 1$$

Solution is on the last slide.