Discrete Mathematics, Section 002, Fall 2016

Lecture 7: Equivalence classes, Partitions

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September 28, 2016

Outline

Equivalence Classes

Do Problem 1 on your worksheet!

Equivalence Classes

We have seen on Problem 1 on the worksheet that two numbers are congruent mod 2 if and only if they are either both odd or both even.

- Any two odd numbers are congruent mod 2.
- Any two even numbers are congruent mod 2.

$$\textit{even} + \textit{odd} \to \text{all } \mathbb{Z}$$

Definition

Let R be an equivalence relation on a set A and let $a \in A$. The **equivalence class** of a, denoted $[a]$, is the set of all elements of A related to a, that is

$$[a] = \{x \in A : xRa\}.$$

Example: Do Problem 2 on Worksheet!

Definition

Let R be an equivalence relation on a set A and let $a \in A$. The **equivalence class** of a, denoted $[a]$, is the set of all elements of A related to a, that is

$$[a] = \{x \in A : xRa\}.$$

For example, let $\equiv \pmod 2$. Then

$$[1] = \{x \in \mathbb{Z} : x \equiv 1 \pmod 2\}$$

This is the set of all integers x such that

$$2|(x-1), \qquad \text{i.e.} \qquad x - 1 = 2k$$

for some $k \in \mathbb{Z}$. Therefore $x = 2k + 1$ and thus x is odd.

Definition

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$$[a] = \{x \in A : xRa\}.$$

For example, let $\equiv \pmod 2$. Then

$$[1] = \{x \in \mathbb{Z} : x \equiv 1 \pmod 2\} = \text{odd numbers}$$

$$[0] = \{x \in \mathbb{Z} : x \equiv 0 \pmod 2\} =$$

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What about $[3]$? $\rightarrow$ Problem 3 Worksheet!

Equivalence class fun facts.

1. Every element is the member of its own equivalence class.

Proposition

Let R be an equivalence relation on a set A and let $a \in A$. Then

$$a \in [a].$$

Proof.

Note that $[a] = \{x \in A : xRa\}$. To show that $a \in [a]$, we just need to show that aRa, and that is true by definition since R is reflexive. $\square$

2. The union of all equivalence classes is A.

Proposition

$$\cup_{a \in A}[a] = A, \qquad \text{(Problem 4 on Worksheet!)}$$

3. Equivalent elements have identical equivalence classes.

Proposition

Let R be an equivalence relation on a set A and let $a, b \in A$. Then aRb if and only if $[a] = [b]$.

Proof.

($\Rightarrow$) Suppose aRb, we will show that $[a]$ and $[b]$ are the same. Suppose $x \in [a]$. This means that xRa. Since aRb, we have (by transitivity) xRb. Therefore $x \in [b]$.

On the other hand suppose $y \in [b]$, i.e. yRb. We are given aRb, and thus bRa by symmetry. Transitivity implies yRa, i.e. $y \in [a]$. Hence $[a] = [b]$.

($\Leftarrow$) Suppose $[a] = [b]$. We have seen that $a \in [a]$. But $[a] = [b]$, so $a \in [b]$. Therefore aRb. □

4. Two elements from the same equivalence class are equivalent.

Proposition

Let R be an equivalence relation on A and $a, x, y \in A$. If $x, y \in [a]$, then xRy.

Proof.

Homework. □

5. Equivalence classes are either disjoint or coincide.

Proposition

Let R be an equivalence relation on A and suppose $[a] \cap [b] \neq \emptyset$. Then $[a] = [b]$.

Proof.

Let R be an equivalence relation on A and suppose $[a]$ and $[b]$ are equivalence classes with $[a] \cap [b] \neq \emptyset$. Hence $\exists x \in [a] \cap [b]$. So xRa and xRb. By symmetry, we have aRx and therefore by transitivity aRb. We have seen that this implies $[a] = [b]$. $\square$

Corollary

The equivalence classes of an equiv. rel R are nonempty, pairwise disjoint subsets of A whose union is A.

Outline

Definition of a partition

Theorem

Let R be an equivalence relation on a set A. The equivalence classes of R are nonempty, pairwise disjoint subsets of A whose union is A.

In other words we say that the equivalence classes of R from a partition of A.

Partition

Let A be a set. A **partition** of A is a set of nonempty, pairwise disjoint sets whose union is A.

- A partition is a subset of 2^A. Its members are called **parts**.
- The parts of the partition are non-empty.
- The parts are pairwise disjoint.
- The union of all the parts is the original set.

Example

Let $A = \{1, 2, 3, 4, 5, 6\}$ and let

$$\mathcal{P} = \{\{1, 2\}, \{3\}, \{4, 5, 6\}\}$$

This is a partition of A into three parts.

Two trivial partitions:

$$\{\{1, 2, 3, 4, 5, 6\}\}, \qquad \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

Practice: Do Problem 5 on the worksheet.

Equivalence relations and partitions

Theorem

Let R be an equivalence relation on a set A. The equivalence classes of R form a partition of the set A.

We can also go the other way.

Definition

Let $\mathcal{P}$ be a partition of a set A. We define an equivalence relation $\stackrel{\mathcal{P}}{\equiv}$ as follows. For $a, b \in A$,

$$a \stackrel{\mathcal{P}}{\equiv} b, \quad \Leftrightarrow \exists P \in \mathcal{P} : a, b \in P$$

In other words, a and b are equivalent under the partition $\mathcal{P}$ provided they belong to the same part $P \in \mathcal{P}$.

Proposition

The relation $\overset{\mathcal{P}}{\equiv}$ is an equivalence relation on A.

Proof.

We will show that $\overset{\mathcal{P}}{\equiv}$ is reflexive, symmetric and transitive.

- We show that $\overset{\mathcal{P}}{\equiv}$ is reflexive. Let a be an arbitrary element of A. Since $\mathcal{P}$ is a partition, there must be a part $P \in \mathcal{P}$ that contains a since the union of all parts is the entire set. Since $a, a \in P \in \mathcal{P}$, we have $a \overset{\mathcal{P}}{\equiv} a$.
- We show that $\overset{\mathcal{P}}{\equiv}$ is symmetric. Suppose $a \overset{\mathcal{P}}{\equiv} b$ for some $a, b \in A$. Then there is a $P \in \mathcal{P}$ such that $a, b \in P$. This also implies $b \overset{\mathcal{P}}{\equiv} a$.

$\square$

Proposition

The relation $\overset{\mathcal{P}}{\equiv}$ is an equivalence relation on A.

Proof.

We will show that $\overset{\mathcal{P}}{\equiv}$ is reflexive, symmetric and transitive.
[...]

- We show that $\overset{\mathcal{P}}{\equiv}$ is transitive. Let $a, b, c \in A$ and suppose $a \overset{\mathcal{P}}{\equiv} b$, and $b \overset{\mathcal{P}}{\equiv} c$. Since $a \overset{\mathcal{P}}{\equiv} b$, there is a part $P \in \mathcal{P}$ containing both a and b. Since $b \overset{\mathcal{P}}{\equiv} c$, there is a part $Q \in \mathcal{P}$ with $b, c \in Q$. Notice that b is in both P and Q. Since the parts are pairwise disjoint, this is only possible if $P = Q$. Therefore $a, c \in P$, which implies $a \overset{\mathcal{P}}{\equiv} c$.

□

Proposition

The relation $\overset{\mathcal{P}}{\equiv}$ is an equivalence relation on A.

What are the equivalence classes?

Proposition

The equivalence classes of $\overset{\mathcal{P}}{\equiv}$ are exactly the parts of $\mathcal{P}$.

Counting parts

Question

How many ways can the letters in the word WORD be rearranged?

Answer

4 letters to first place, 3 choices for second, $\cdots \rightarrow \quad 4! = 24$.

What happens if a letter occurs more than once?

Question

How many different ways can the letters in the word HELLO be rearranged?

Counting parts

Question

How many different ways can the letters in the word HELLO be rearranged?

- If there were no repeated letters $\rightarrow$ $5! = 120$.
- But this counts

$$HEL_1L_2O, \qquad HEL_2L_1O$$

as different.

- Guess?

Counting parts

Question

How many different ways can the letters in the word HELLO be rearranged?

- Let

$$A = \{\text{All rearrangements of } H, E, L_1, L_2, O\}, \qquad |A| = 120.$$

- Next define a relation R such that for $a, b \in A$,

$$aRb \quad \Leftrightarrow \quad a \text{ and } b \text{ differ only by } L_1 \leftrightarrow L_2$$

- Check that this is an equivalence relation.

$$[HL_1EOL_2] = \{HL_1EOL_2, HL_2EOL_1\}$$

- We need to count the number of equivalence classes!

Counting parts

Question

How many different ways can the letters in the word HELLO be rearranged?

- $A = \{\text{All rearrangements of } H, E, L_1, L_2, O\}, \qquad |A| = 120.$
- Define **an equivalence relation** R such that for $a, b \in A$,

$$aRb \quad \Leftrightarrow \quad a \text{ and } b \text{ differ only by } L_1 \leftrightarrow L_2$$

- Every class has to elements: $|[a]| = 2$ for all $a \in A$.
- Therefore there are

$$|A|/|[a]| = 120/2 = 60$$

equivalence classes which are the possible different rearrangements when the two L's are not distinguished.

Counting parts

Question

How many different ways can the letters in the word AARDVARK be rearranged?

Again, the problem boils down to figuring out how big the equivalence classes

$$[R_1 A_1 D A_2 K R_2 A_3 V].$$

- 2 choices for first R.

Counting parts

Question

How many different ways can the letters in the word AARDVARK be rearranged?

Again, the problem boils down to figuring out how big the equivalence classes

$$[R_1 A_1 D A_2 K R_2 A_3 V].$$

- 2 choices for first R.

$$2 \cdot$$

Counting parts

Question

How many different ways can the letters in the word AARDVARK be rearranged?

Again, the problem boils down to figuring out how big the equivalence classes

$$[R_1 A_1 D A_2 K R_2 A_3 V].$$

- 3 choices for the first A.

$$2 \cdot 3 \cdot$$

Counting parts

Question

How many different ways can the letters in the word AARDVARK be rearranged?

Again, the problem boils down to figuring out how big the equivalence classes

$$[R_1 A_1 D A_2 K R_2 A_3 V].$$

- D is fixed.

$$2 \cdot 3 \cdot 1 \cdot$$

Counting parts

Question

How many different ways can the letters in the word AARDVARK be rearranged?

Again, the problem boils down to figuring out how big the equivalence classes

$$[R_1 A_1 D A_2 K R_2 A_3 V].$$

- 2 choices for second A.

$$2 \cdot 3 \cdot 1 \cdot 2 \cdot$$

Counting parts

Question

How many different ways can the letters in the word AARDVARK be rearranged?

Again, the problem boils down to figuring out how big the equivalence classes

$$[R_1 A_1 D A_2 K R_2 A_3 V].$$

- K is fixed.

$$2 \cdot 3 \cdot 1 \cdot 2 \cdot 1 \cdot$$

Counting parts

Question

How many different ways can the letters in the word AARDVARK be rearranged?

Again, the problem boils down to figuring out how big the equivalence classes

$$[R_1 A_1 D A_2 K R_2 A_3 V].$$

- 1 choice for second R.

$$2 \cdot 3 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot$$

Counting parts

Question

How many different ways can the letters in the word AARDVARK be rearranged?

Again, the problem boils down to figuring out how big the equivalence classes

$$[R_1 A_1 D A_2 K R_2 A_3 V].$$

- 1 choice for last *A*.

$$2 \cdot 3 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot$$

Counting parts

Question

How many different ways can the letters in the word AARDVARK be rearranged?

Again, the problem boils down to figuring out how big the equivalence classes

$$[R_1 A_1 D A_2 K R_2 A_3 V].$$

- V is fixed.

$$2 \cdot 3 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 12$$

Counting parts

Question

How many different ways can the letters in the word AARDVARK be rearranged?

Again, the problem boils down to figuring out how big the equivalence classes

$$[R_1 A_1 D A_2 K R_2 A_3 V].$$

- V is fixed.

$$2 \cdot 3 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 12$$

And therefore the number of rearrangements is

$$\frac{8!}{3!2!} = \frac{40320}{12} = 3360.$$