Discrete Mathematics, 2016 Spring - HW 11

November 27, 2016

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To get full credit in all of the problems, use rigorous justification and unless otherwise indicated, make sure that your solution reads as a perfect English sentence. You should only assume integers, operations and order relations as given. If you use a statement or a definition from the textbook, make sure to indicate it.

Section 37

8) Prove Proposition 37.4 in the textbook. Why is this proposition restricted to $n \geq 2$?

3) Solve the following equations for x in the $\mathbb{Z}_n$ specified. Make sure to find all solutions.

   (a) $342 \otimes x = 73$ in $\mathbb{Z}_{1003}$.

   (b) $9 \otimes x = 4$ in $\mathbb{Z}_{12}$.

6) Prove that for all $a, b \in \mathbb{Z}_n$, $(a \ominus b) \oplus (b \ominus a) = 0$.

10) For ordinary integers, the following is true. If $ab = 0$, then $a = 0$ or $b = 0$. Note, however that in $\mathbb{Z}_n$,

$$2 \otimes 5 = 0, \qquad \text{but} \qquad 2 \neq 0, 5 \neq 0.$$

For which values of $n \geq 2$ does the implication

$$a \otimes b = 0 \Leftrightarrow a = 0 \text{ or } b = 0$$

hold in $\mathbb{Z}_n$?

Section 38

1) Solve $100x \equiv 74 \pmod{127}$. Make sure to find all solutions.

3) Solve the following system of congruences

$$3x \equiv 8 \ (10) \qquad 2x + 4 \equiv 9 \ (11)$$

Section 39

10) Let a and b be integers. A *common multiple* of a and b is an integer n for which $a|n$ and $b|n$. We call an integer m the *least common multiple* of n provided (1) m is positive, (2) m is a common multiple of a and b, and (3) if n is any other positive common multiple of a and b, then $n \geq m$. For example, $lcm(24, 30) = 120$.

    (a) Develop a formula for the least common multiple of two positive integers in terms of their prime factorization.

    (b) Use your formula to show that if a and b are positive integers, then

$$ab = gcd(a, b)lcm(a, b)$$

14) Let n be a positive integer and suppose we factor n into primes as follows:

$$n = p_1^{e_1} p_2^{e_2} \dots p_t^{e_t},$$

where the p_j-s are distinct primes and the e_j-s are natural numbers.

    (a) Find a formula for the number of positive divisors of n.

    (b) Recall that an integer n is called *perfect* if it equals the sum of all its divisors d with $1 \leq d < n$. For example 28 is perfect as $28 = 1 + 2 + 4 + 7 + 14$. Let a be a positive integer and prove that if $2^a - 1$ is prime, then $n = 2^{a-1}(2^a - 1)$ is perfect.

22) Prove that $\log_2 3$ is irrational.