Discrete Mathematics, 2016 Fall - Worksheet 8

October 2, 2016

Instructor: Zsolt Pajor-Gyulai, CIMS

In all of the above problems explain your answer in full English sentences.

1. Evaluate the following without doing any writing or arithmetic.
   (a) $\binom{9}{0}$
   (b) $\binom{9}{1}$
   (c) $\binom{9}{8}$
   (d) $\binom{9}{6} - \binom{9}{3}$
   (e) $\binom{0}{0}$
   (f) $\binom{0}{2}$

2. Write out all 3-element subsets of $\{1, 2, 3, 4, 5, 6\}$ to verify that $\binom{6}{3} = 20$.

3. Twenty people attend a party. If everyone shakes everyone else's hand exactly once, how many handshakes take place?

4. Prove that

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

   One easy way is to substitute $x = y = 1$ into the binomial theorem. However, please give a combinatorial proof.

5. (a) Use the binomial theorem to prove

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \pm \binom{n}{n} = 0$$

   (b) Move all the negative terms over to the right hand side to give

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$$

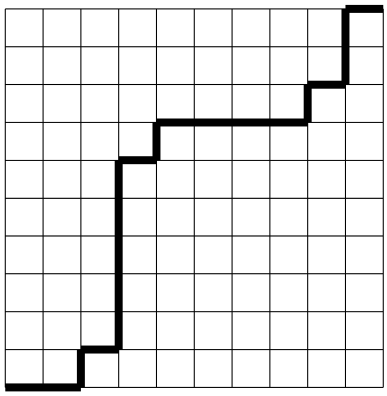
   Give a combinatorial interpretation of this identity.

6. Consider the following formula. Give two different proofs one using the factorial formula, the other being a combinatorial proof.

$$\binom{2n+2}{n+1} = \binom{2n}{n+1} + 2\binom{2n}{n} + \binom{2n}{n-1}$$

7. What is the coefficient of x^3 in $(2x-3)^6$?

8. (Counting lattice paths) Consider a grid such as the one shown in the figure. We want to count the number of paths from the lower left corner to the upper right corner in which each step of the path either goes one unit to the right or one unit upwards (e.g. the bold one on the picture). How many such path are there? (If you are stuck, you can look at p97 of the textbook.)



9. Evaluate the following
   (a) $\binom{3}{1\ 1\ 1}$
   (b) $\binom{10}{1\ 2\ 5}$
   (c) $\binom{5}{0\ 5\ 0}$
   (d) $\binom{10}{7\ 3\ 0}$
   (e) $\binom{10}{5\ 2\ 3} - \binom{10}{2\ 3\ 5}$

10. A coach must choose two teams of 5 from a team of 12 players. How many different ways can the coach choose the teams?

Computational problems:

1. (EXTRA PROBLEM: Computational cost of binomial coefficient) To compute $\binom{n}{k}$ by generating Pascal's triangle, it is not necessary to generate the entire triangle down to row n; you need only the part of the triangle in a 90 degree wedge above $\binom{n}{k}$. Estimate how many additions you would need to perform to calculate $\binom{100}{30}$ by this method.

PE15) How many lattice paths are there for a 50×50 grid?