Discrete Mathematics, 2016 Spring - Worksheet 10

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1. Please state the contrapositive of each of the following statements:
   (a) If x is odd, then x^2 is odd.
   If x^2 is not odd then x is not odd.
   (b) If x is non-zero, then x^2 is positive.
   If x^2 is not positive, then x is zero.

2. Prove by the contrapositive method that if a does not divide b, then the equation $ax^2 + bx + b - a = 0$ has no positive integer solution for x.

   Proof. We prove the contrapositive, i. e. that if $ax^2 + bx + b - a = 0$ has a positive integer solution then a divides b.

   Assume that there is a positive integer solution $ax^2 + bx + b - a = 0$, denote this by k. Since k is an integer solution, we have $k \in \mathbb{Z}$ and $ak^2 + bk + b - a = 0$. Rearranging this gives $a(1 - k^2) = b(k + 1)$, which can also be written as $b(k + 1) = a(1 - k)(1 + k)$. Since k is a positive integer, $k + 1 \neq 0$ and we can simplify by it to get $b = a(1 - k)$. Since $1 - k \in \mathbb{Z}$, this means $a|b$. □

3. For each of the following statements, write the first sentences of a proof by contradiction (do not attempt to complete the proofs). Please use the phrase "for the sake of contradiction".
   (a) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

   Proof. Let $A \subseteq B$, $B \subseteq C$ and for the sake of contradiction assume $A \not\subseteq C$. [...] □

   (b) The sum of two negative integers is a negative integer.

   Proof. Let x and y be two negative integers and for the sake of contradiction assume $x + y$ is positive. [...] □

   (c) If the square of a rational number is an integer, then the rational number must also be an integer.

   Proof. Let q be a rational number such that $q^2\mathbb{Z}$ and for the sake of contradiction assume that q is not an integer. [...] □

4. Prove the following statements by contradiction.

   (a) Consecutive integers cannot be both even.

   Proof. Let x be an integer and assume for the sake of contradiction that x and $x+1$ are both even. Then there are integers k_1 and k_2 such that $x = 2k_1$ and $x+1 = 2k_2$. Combining the two, we get $2k_1 + 1 = 2k_2$ which implies $k_1 - k_2 = 1/2$. Since k_1 and k_2 are integers, this is impossible. $\square$

   (b) Consecutive integers cannot be both odd.

   Proof. Let x be an integer and assume for the sake of contradiction that x and $x+1$ are both odd. In particular, this means that there is an integer b such that $x+1 = 2c+1$. Substracting 1 from both sides, we get $x = 2c$ which means that x is even. Since we have proved in class that a number cannot be both even and odd, this contradicts the assumption that x is odd. This proves the claim. $\square$

   (c) If the sum of two primes is prime, then one of the primes must be 2 (you may assume that every integer is either even or odd, but never both.)

   Proof. Let p_1 and p_2 be two primes such that $p_1 + p_2$ is also a prime and assume for the sake of contradiction that neither p_1 nor p_2 equals 2. Since $p_1 \neq 2$, p_1 must not be divisible by 2 and therefore it is odd and there is an integer $k_1 \in \mathbb{Z}$ such that $p_1 = 2k_1 + 1$. Similarly $p_2 = 2k_2 + 1$ for some integer $k_2 \in \mathbb{Z}$. Then $p_1 + p_2 = 2(k_1 + k_2 + 1)$ which means $2|(p_1 + p_2)$. Since $p_1 + p_2$ is also a prime, this means $p_1 + p_2 = 2$. But as $p_1, p_2 \geq 2$, this is impossible. $\square$

   (d) Suppose n is an integer that is divisible by 4. Then $n+2$ is not divisible by 4.

   Proof. Let n be an integer divisible by 4. For the sake of contradiction, assume that $n+2$ is divisible by 4. This means that there is an integer $k \in \mathbb{Z}$, such that $n+2 = 4k$, which means $n = 4k-2$, which contradicts the assumption that n is divisible by 4. $\square$

   (e) Let A and B be sets. Then $(A - B) \cap (B - A) = \emptyset$.

   Proof. Let A and B be sets and for the sake of contradiction suppose $(A - B) \cap (B - A) \neq \emptyset$. This means that there is an element $x \in (A - B) \cap (B - A)$. On one hand, this means $x \in A - B$ which in turn implies that $x \in A$ but $x \notin B$. On the other hand $x \in B - A$ which implies $x \in B$ but $x \notin A$. These two are in contradiction to each other, which proves the claim. $\square$

5. Prove by the method of smallest counterexample that $1 + 2 + 3 + \cdots + n = n(n+1)/2$ for all positive integer n.

Proof. For the sake of contradiction assume that there is a positive integer such that the claim is not true. Let $n*$ be the smallest such number. Note that $n^* \neq 1$ as $1 = 1 \cdot 2/2$. This means that $n^* - 1$ is a positive integer and

$$1 + 2 + 3 + \cdots + n^* - 1 = \frac{(n^* - 1)n^*}{2}.$$

Adding n^* to both side of this equation gives

$$1 + 2 + 3 + \cdots + n^* = \frac{(n^* - 1)n^*}{2} + n^* = \frac{(n^*)^2 - n^* + 2n^*}{2} = \frac{(n^*)^2 + n^*}{2} = \frac{n^*(n^* + 1)}{2}$$

which contradicts n^* being a counterexample.

$\square$

6. Prove by the method of smallest counterexample that $n < 2^n$ for all $n \in \mathbb{N}$.

   Proof. For the sake of contradiction, assume that there is a $k \in \mathbb{N}$ such that $k > 2^k$ and by the well ordering principle assume that it is the smallest such natural. As $0 < 2^0 = 1$, we now that $k \neq 0$. This implies that $k - 1 \in \mathbb{N}$ and since $k - 1 < k$, we have by assumption that $k - 1 < 2^{k-1}$. By adding 1 to both sides of this, we get

   $$k < 2^{k-1} + 1 < 2^{k-1} + 2^{k-1} = 2 \cdot 2^{k-1} = 2^k,$$

   using that for $k \geq 1$, we have $1 \leq 2^{k-1}$. This contradicts the assumption that k is a counterexample which proves the claim. $\square$

7. Prove by the method of smallest counterexmaple that when $a \neq 0, 1$, then

   $$a^0 + a^1 + a^2 + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}, \qquad \forall n \in \mathbb{N}.$$

   Proof. p131 in the textbook. $\square$

8. For all integers $n \geq 5$, we have $2^n > n^2$.

   Proof. p132 in the textbook. $\square$