Calculus Quiz 3 ATMO

1. (5 *point*) Define $f(0,0)$ in a way that extends
$$f(x,y) = x^2y\frac{x^2-y^2}{x^2+y^2}$$
to be continuous at the origin.
Sol. Observe that
$$\begin{aligned} |x^2y(x^2-y^2)| &= x^2|y||x^2-y^2| = x^2\sqrt{y^2}|x^2-y^2| \\ &\le x^2\sqrt{y^2}(x^2+y^2) \le (x^2+y^2)\sqrt{x^2+y^2}(x^2+y^2) \\ &= (x^2+y^2)^{\frac{5}{2}} \end{aligned}$$
Thus, we have that
$$0 \le \left| x^2y\frac{x^2-y^2}{x^2+y^2} \right| \le \frac{(x^2+y^2)^{\frac{5}{2}}}{x^2+y^2} = (x^2+y^2)^{\frac{3}{2}}$$
Since $\lim\limits_{(x,y)\to(0,0)}(x^2+y^2) = 0$. By sandwich theorem, we have that
$$\lim_{(x,y)\to(0,0)} \left| x^2y\frac{x^2-y^2}{x^2+y^2} \right| = 0 \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y(x^2-y^2)}{x^2+y^2} = 0$$
Hence we define $f(0,0) = 0$, then $f(x,y)$ is continuous at the origin. $\square$

2. (5 *point*) Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ when $u = v = 0$ if $w = \ln\sqrt{1+x^2} - \tan^{-1}x$ and $x = 2e^u \sin v$.
Sol. Since $x(u,v) = 2e^u\sin v$, $w(x) = \ln\sqrt{1+x^2} - \tan^{-1}x$. When $u = v = 0$, we have that $x(0,0) = 0$. Observe that
$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{dw}{dx}\frac{\partial x}{\partial u} = \left(\frac{1}{\sqrt{1+x^2}}\frac{x}{\sqrt{1+x^2}} - \frac{1}{1+x^2}\right)\cdot(2e^u\sin v) = 2e^u\sin v\cdot\frac{x-1}{1+x^2} \\ \frac{\partial w}{\partial v} &= \frac{dw}{dx}\frac{\partial x}{\partial v} = \left(\frac{1}{\sqrt{1+x^2}}\frac{x}{\sqrt{1+x^2}} - \frac{1}{1+x^2}\right)\cdot(2e^u\cos v) = 2e^u\cos v\cdot\frac{x-1}{1+x^2} \end{aligned}$$
Hence,
$$\begin{aligned} \left.\frac{\partial w}{\partial u}\right|_{u=v=0} &= \left.\frac{dw}{dx}\right|_{x=0}\cdot\left.\frac{\partial x}{\partial u}\right|_{u=v=0} = 0 \\ \left.\frac{\partial w}{\partial v}\right|_{u=v=0} &= \left.\frac{dw}{dx}\right|_{x=0}\cdot\left.\frac{\partial x}{\partial v}\right|_{u=v=0} = -2 \end{aligned}$$