Harmonischer Oszillator

$$\psi(x,t) = \alpha e^{-itE_0/\hbar}|0\rangle + \beta e^{-itE_1/\hbar}|1\rangle + \cdots$$
$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$$

Mit den Energien $E_n = \left(n + \frac{1}{2}\right)\omega\hbar$
Und den Eigenfunktionen $|n\rangle = \psi_n(x)$

Operatoren

Ortsoperator*	$\hat{x} = x = i\hbar\frac{\partial}{\partial p}$
Impulsoperator*	$\hat{p} = p = -i\hbar\frac{\partial}{\partial x}$
Abstiegsoperator	$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{p}$
Aufstiegsoperator	$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2\hbar m\omega}}\hat{p}$
Hamiltonoperator	
• harm. Oszil.	$\hat{H}_o = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \frac{\hat{p}^2}{2m} + V(\hat{x})$
• H_2-Atom	$\hat{H}_h = -\frac{\hbar^2}{2m}\vec{\nabla}^2 - \frac{e^2}{4\pi\varepsilon_0}\frac{1}{\vert r^2\vert}$
Damit gilt	$\hat{x} = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}(\hat{a} + \hat{a}^\dagger)$
	$\hat{p} = \frac{1}{2i}\sqrt{2\hbar m\omega}(\hat{a} - \hat{a}^\dagger)$
	$\hat{a}\vert n\rangle = \sqrt{n}\vert n-1\rangle \qquad \hat{a}\vert 0\rangle = 0$
	$\hat{a}^\dagger\vert n\rangle = \sqrt{n+1}\vert n+1\rangle$
	$\vert n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\vert 0\rangle$
Hermitescher Operator	$A^\dagger = (A^*)^{\mathrm{T}} = A$

$$\Leftrightarrow \int_{-\infty}^{\infty} f^*(x)P(g(x))\,dx = \int_{-\infty}^{\infty} P(f(x))g^*(x)\,dx$$

* = ist hermitesch	
Schrödingergleichung	$i\hbar\frac{\partial}{\partial t}\vert\,\psi(t)\rangle = \hat{H}\vert\,\psi(t)\rangle$
Unschärfe	$\Delta x\Delta p = \frac{\hbar}{2}$
$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$	$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

H_2-Atom

$$E_n = -\frac{E_{\mathrm{Ry}}}{n^2} \qquad E_{\mathrm{Ry}} = 13.6\text{ eV} \qquad \vec{r} = \vec{r}_e - \vec{r}_p$$

Pauli-Matrizen

$$S_0 = \frac{\hbar}{2}\sigma_0 = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
$$S_x = S_1 = \frac{\hbar}{2}\sigma_1 = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$
$$S_y = S_2 = \frac{\hbar}{2}\sigma_2 = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$
$$S_z = S_3 = \frac{\hbar}{2}\sigma_3 = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$
$$\vec{S} = \begin{pmatrix}S_x\\ S_y\\ S_z\end{pmatrix} = \frac{\hbar}{2}\sigma = \frac{\hbar}{2}\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{pmatrix}$$

Mittelwerte

$$\begin{aligned}\langle x\rangle &= \langle\psi|x|\psi\rangle\\ &= \int_{-\infty}^{\infty} x\,|\psi(x)|^2\,dx \qquad \text{nur falls } x \text{ kein Operator!}\\ &= \int_{-\infty}^{\infty}\psi(x)^*\hat{x}\,\psi(x)\,dx\\ &= \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}(\alpha^*\langle 0| + \beta^*\langle 1| + \ldots)(\hat{a} + \hat{a}^\dagger)(\alpha|0\rangle + \beta|1\rangle + \ldots)\\ \langle p\rangle &= \langle\psi|x|\psi\rangle\\ &= \int_{-\infty}^{\infty}\hat{p}\,|\psi(x)|^2\,dx = \text{ analog zu } \langle\psi|x|\psi\rangle\,.\end{aligned}$$

Normierung $\quad |\langle\psi\rangle|^2 \overset{!}{=} 1$

$$\vec{j} = \frac{\hbar}{2mi}(\psi^*\vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi) \text{ Wahrscheinlichkeitsstromdichte}$$
$$0 = \frac{\partial\rho}{\partial t} + \vec{\nabla}\vec{j} \quad \text{Kontinuitäsgleichung, } \|\psi\| = \text{const}$$

Relativitätstheorie

Lorentzfaktor	$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$
Bewegte Masse	$m = \gamma m_0 \qquad$ (Ruhemasse m_0)
Impuls	$p = mv = \gamma m_0 v$
Energie	$E = mc^2 = \gamma m_0 c^2$
	$E^2 = m_0^2c^4 + p^2c^2$
Kinetische Energie	$E_{\text{kin}} = \frac{1}{2}mv^2 = \frac{\gamma}{2}m_0v^2 = mc^2 - m_0c^2$
Photoeffekt	$E_\gamma = W_A + E_{\text{el,kin}}$
$v_1 \oplus v_2$	$= \frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}}$

Konstanten

$$m_e = 9.10938291 \cdot 10^{-31}\text{ kg}$$
$$m_p = 1.67262178 \cdot 10^{-27}\text{ kg}$$
$$m_n = 1.67492735 \cdot 10^{-27}\text{ kg}$$
$$h = 6.62606957 \cdot 10^{-34}\text{Js}$$
$$\hbar = \frac{h}{2\pi}$$
$$E_\gamma = h\nu \quad \text{(Energie des Photons)}$$
$$p_\gamma = \frac{E_\gamma}{c} \quad \text{(Impuls des Photons)}$$

De Broglie

$$\lambda = \frac{h}{p} = \frac{h}{m\cdot v}$$

Kommutatoren

$$\left[\hat{L}_i, \hat{x}_j\right] = i\hbar\epsilon_{ijk}\hat{x}_k$$
$$\left[\hat{L}_i, \hat{p}_j\right] = i\hbar\epsilon_{ijk}\hat{p}_k$$
$$\left[\hat{A}, \hat{B}\hat{C}\right] = \left[\hat{A}, \hat{B}\right]\hat{C} + \hat{B}\left[\hat{A}, \hat{C}\right]$$
$$\left[\hat{A}\hat{B}, \hat{C}\right] = \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B}$$
$$\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$\Rightarrow \left[\cdot,\cdot\right]$ antisymmetrisch, linear.

Drehimpuls

$$\vec{r} = \begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{pmatrix}, \quad \vec{p} = \begin{pmatrix}\hat{p}_x\\ \hat{p}_y\\ \hat{p}_z\end{pmatrix}, \quad \vec{L} = \vec{p}\times\vec{r}$$
$$\text{Ł}_x = yp_z - zp_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$
$$\text{Ł}_y = zp_x - xp_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$
$$\text{Ł}_z = xp_y - yp_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$
$$\left[L_i, \hat{f}\right] = 0, \text{ für Skalaroperatoren } \hat{f}$$
$$\left[L_i, \hat{f}_j\right] = -i\hbar\epsilon_{ijk}\hat{f}_k, \text{ für Vektoroperatoren } \hat{f}$$
$$\left[L_i, \hat{f}_{kl}\right] = i\hbar(\epsilon_{ikp}\delta_{nl} + \epsilon_{iln}\delta_{kp})\hat{f}_{pn}$$
$$L_\pm = L_x \pm iL_y\text{(Leiteroperator)}$$
$$L_+|m\rangle = c\,|m+1\rangle$$
$$L^2 = \vec{L}^2 = L_x^2 + L_y^2 + L_z^2$$
$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$$
$$L_+|l,n\rangle = \hbar\sqrt{(l-m)(l+m+1)}\,|l,n+1\rangle$$
$$L_-|l,n\rangle = \hbar\sqrt{(l+m)(l-m+1)}\,|l,n-1\rangle$$
$$[]L_i, L_j] = i\hbar L_k \quad i\neq j\neq k\neq i, \quad i,j,k\in\{x,y,z\}$$

Allgemein

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\operatorname{div} \vec{X} = \vec{\nabla}\vec{X} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} X = \begin{pmatrix} \frac{\partial \vec{X}}{\partial x} \\ \frac{\partial \vec{X}}{\partial y} \\ \frac{\partial \vec{X}}{\partial z} \end{pmatrix}$$
$$\operatorname{rot} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix}$$
$$\frac{\partial^2}{\partial x^2} e^{-ax^2} = 2ae^{-ax^2}\left(2ax^2 - 1\right)$$
$$\vec{\nabla}(\vec{a} \cdot \vec{b}) = (\vec{a} \cdot \vec{\nabla})\vec{b} + (\vec{b} \cdot \vec{\nabla})\vec{a} + \vec{a} \times \vec{\nabla} \times \vec{b} + \vec{b} \times \vec{\nabla} \times \vec{a}$$

Maxwell

Elektrisches Feld $\vec{E}$, magnetisches Feld $\vec{B}$, Ladungsdichte ρ, elektrische Stromdichte $\vec{j}$

$$\operatorname{div} \vec{E} = \vec{\nabla}\vec{D} = \frac{\rho}{\varepsilon_0}$$
$$\operatorname{div} \vec{B} = \vec{\nabla}\vec{B} = 0$$
$$\operatorname{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\operatorname{rot} \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Hamilton - Lagrange

$$L = T - V = -mc^2 \cdot \sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}} - U(\vec{r}, t)$$
$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i} L = \frac{\partial}{\partial q_i} L \qquad \frac{\partial}{\partial \dot{q}_i} L = p_i$$
$$H = \sum_i p_i \dot{q}_i(q, t) - L$$
$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$
$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \text{ falls } = 0 \Rightarrow \text{ Energie ist erhalten}$$

3N-s Gleichungen bei s Zwangsbedingungen und N Teilchen.
Noether: Falls eine Schar existiert mit $q(t, \alpha)$ sodass gilt

$$L(q(t, \alpha), \dot{q}(t, \alpha), t) = L(q(t), \dot{q}(t), t)$$

dann gilt

$$\frac{d}{d\alpha} L(q(t, \alpha), \dot{q}(t, \alpha), t) = 0$$

Erweitertes Noether: Falls L nicht invariant, dann

$$\frac{d}{d\alpha} L(q(t, \alpha), \dot{q}(t, \alpha), t)|_{\alpha=0} = \frac{d}{dt} L(q(t), \dot{q}(t), t)$$

$\Rightarrow \sum_i \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} \frac{\partial q_i}{\partial \alpha} - f$ ist erhalten, falls L zeitinvariant.

Potentialbarriere

$$k_1 = k_3 = \sqrt{\frac{2mE}{\hbar^2}}, \qquad k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$
$$r = A + B - 1, \qquad \psi_{\mathrm{I}}(x) = e^{ik_1 x} + re^{-ik_1 x}$$
$$\psi_{\mathrm{II}}(x) = Ae^{ik_2 x} + Be^{-ik_2 x}, \qquad \psi_{\mathrm{III}}(x) = te^{-ik_3(x-a)}$$
$$|t|^2 = \frac{4k_1^2 k_2^2}{4k_1^2 k_2^2 + (k_1^2 - k_2^2)\sin^2(k_2 a)} = \frac{4E(E-V_0)}{4E(E-V_0) + V_0^2 \sin^2(k_2 a)} (E > V_0)$$
$$|t|^2 = \frac{4k_1^2 k_2^2}{4k_1^2 k_2^2 + (k_1^2 + \kappa)\sinh^2(\kappa a)} = \frac{4E(E-V_0)}{4E(E-V_0) + V_0^2 \sinh^2(\kappa a)} (E < V_0)$$
$$|t^2| \approx \frac{E(V_0 - E)}{V_0^2} e^{-2\kappa a} \qquad \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

Dabei ist t bzw. r der Transmissions- bzw. Reflektionskoeffizent.