Harmonischer Oszillator

$$\psi(x,t) = \alpha e^{-itE_0/\hbar}\,|0\rangle + \beta e^{-itE_1/\hbar}\,|1\rangle + \cdots$$

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$$

Mit den Energien $E_n = \left(n + \frac{1}{2}\right)\omega\hbar$
Und den Eigenfunktionen $|n\rangle = \psi_n(x)$

Operatoren

Ortsoperator*	$\hat{x} = x = i\hbar\frac{\partial}{\partial p}$
Impulsoperator*	$\hat{p} = p = -i\hbar\frac{\partial}{\partial x}$
Abstiegsoperator	$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{p}$
Aufstiegsoperator	$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2\hbar m\omega}}\hat{p}$
Hamiltonoperator	
• harm. Oszil.	$\hat{H}_o = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \frac{\hat{p}^2}{2m} + V(\hat{x})$
• H_2-Atom	$\hat{H}_h = -\frac{\hbar^2}{2m}\vec{\nabla}^2 - \frac{e^2}{4\pi\varepsilon_0}\frac{1}{\lvert r^2\rvert}$
Damit gilt	$\hat{x} = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}(\hat{a} + \hat{a}^\dagger)$
	$\hat{p} = \frac{1}{2i}\sqrt{2\hbar m\omega}(\hat{a} - \hat{a}^\dagger)$
	$\hat{a}\,\lvert n\rangle = \sqrt{n}\,\lvert n-1\rangle \qquad \hat{a}\,\lvert 0\rangle = 0$
	$\hat{a}^\dagger\,\lvert n\rangle = \sqrt{n+1}\,\lvert n+1\rangle$
	$\lvert n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\,\lvert 0\rangle$
Hermitescher Operator	$A^\dagger = (A^*)^{\mathrm{T}} = A$

$$\Leftrightarrow \int_{-\infty}^{\infty} f^*(x)P(g(x))\,dx = \int_{-\infty}^{\infty} P(f(x))g^*(x)\,dx$$

* = ist hermitesch	
Schrödingergleichung	$i\hbar\frac{\partial}{\partial t}\lvert\,\psi(t)\rangle = \hat{H}\lvert\,\psi(t)\rangle$
Unschärfe	$\Delta x\Delta p = \frac{\hbar}{2}$
$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$	$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

H_2-Atom

$$E_n = -\frac{E_{\mathrm{Ry}}}{n^2} \qquad E_{\mathrm{Ry}} = 13.6\ \mathrm{eV} \qquad \vec{r} = \vec{r}_e - \vec{r}_p$$

Pauli-Matrizen

$$S_0 = \frac{\hbar}{2}\sigma_0 = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$S_x = S_1 = \frac{\hbar}{2}\sigma_1 = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$S_y = S_2 = \frac{\hbar}{2}\sigma_2 = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$S_z = S_3 = \frac{\hbar}{2}\sigma_3 = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\vec{S} = \begin{pmatrix}S_x\\ S_y\\ S_z\end{pmatrix} = \frac{\hbar}{2}\sigma = \frac{\hbar}{2}\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{pmatrix}$$

Mittelwerte

$$\begin{aligned}
\langle x\rangle &= \langle\psi|x|\psi\rangle\\
&= \int_{-\infty}^{\infty} x\,|\psi(x)|^2\,dx \qquad \text{nur falls } x \text{ kein Operator!}\\
&= \int_{-\infty}^{\infty} \psi(x)^*\,\hat{x}\,\psi(x)\,dx\\
&= \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}(\alpha^*\,\langle 0| + \beta^*\,\langle 1| + \ldots)(\hat{a} + \hat{a}^\dagger)(\alpha\,|0\rangle + \beta\,|1\rangle + \ldots)\\
\langle p\rangle &= \langle\psi|x|\psi\rangle\\
&= \int_{-\infty}^{\infty} \hat{p}\,|\psi(x)|^2\,dx = \ \text{analog zu } \langle\psi|x|\psi\rangle\,.
\end{aligned}$$

Normierung $\quad |\langle\psi\rangle|^2 \overset{!}{=} 1$

$$\vec{j} = \frac{\hbar}{2mi}(\psi^*\vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi) \ \text{ Wahrscheinlichkeitsstromdichte}$$

$$0 = \frac{\partial\rho}{\partial t} + \vec{\nabla}\vec{j} \quad \text{Kontinuitäsgleichung, } \|\psi\| = \text{const}$$

Relativitätstheorie

Lorentzfaktor	$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$
Bewegte Masse	$m = \gamma m_0 \qquad$ (Ruhemasse m_0)
Impuls	$p = mv = \gamma m_0 v$
Energie	$E = mc^2 = \gamma m_0 c^2$
	$E^2 = m_0^2 c^4 + p^2 c^2$
Kinetische Energie	$E_{\mathrm{kin}} = \frac{1}{2}mv^2 = \frac{\gamma}{2}m_0 v^2 = mc^2 - m_0 c^2$
Photoeffekt	$E_\gamma = W_A + E_{\mathrm{el,kin}}$
$v_1 \oplus v_2$	$= \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$

Konstanten

$$\begin{aligned}
m_e &= 9.10938291\cdot 10^{-31}\ \mathrm{kg}\\
m_p &= 1.67262178\cdot 10^{-27}\ \mathrm{kg}\\
m_n &= 1.67492735\cdot 10^{-27}\ \mathrm{kg}\\
h &= 6.62606957\cdot 10^{-34}\mathrm{Js}\\
\hbar &= \frac{h}{2\pi}\\
E_\gamma &= h\nu \qquad \text{(Energie des Photons)}\\
p_\gamma &= \frac{E_\gamma}{c} \qquad \text{(Impuls des Photons)}
\end{aligned}$$

De Broglie

$$\lambda = \frac{h}{p} = \frac{h}{m\cdot v}$$

Kommutatoren

$$\begin{aligned}
\left[\hat{L}_i, \hat{x}_j\right] &= i\hbar\epsilon_{ijk}\hat{x}_k\\
\left[\hat{L}_i, \hat{p}_j\right] &= i\hbar\epsilon_{ijk}\hat{p}_k\\
\left[\hat{A}, \hat{B}\hat{C}\right] &= \left[\hat{A}, \hat{B}\right]\hat{C} + \hat{B}\left[\hat{A}, \hat{C}\right]\\
\left[\hat{A}\hat{B}, \hat{C}\right] &= \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B}\\
\left[\hat{A}, \hat{B}\right] &= \hat{A}\hat{B} - \hat{B}\hat{A}
\end{aligned}$$

$\Rightarrow \left[\cdot,\cdot\right]$ antisymmetrisch, linear.

Drehimpuls

$$\vec{r} = \begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{pmatrix}, \quad \vec{p} = \begin{pmatrix}\hat{p}_x\\ \hat{p}_y\\ \hat{p}_z\end{pmatrix}, \quad \vec{L} = \vec{p}\times\vec{r}$$

$$\text{Ł}_x = yp_z - zp_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\text{Ł}_y = zp_x - xp_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\text{Ł}_z = xp_y - yp_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$\left[L_i, \hat{f}\right] = 0, \text{ für Skalaroperatoren } \hat{f}$$

$$\left[L_i, \hat{f}_k\right] = -i\hbar\epsilon_{ijk}\hat{f}_k, \text{ für Vektoroperatoren } \hat{f}$$

$$\left[L_i, \hat{f}_{kl}\right] = i\hbar(\epsilon_{ikp}\delta_{nl} + \epsilon_{iln}\delta_{kp})\hat{f}_{pn}$$

Potentialbarriere

$$k_1 = k_3 = \sqrt{\frac{2mE}{\hbar^2}}, \qquad k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

$$r = A + B - 1, \qquad \psi_{\mathrm{I}}(x) = e^{ik_1x} + re^{-ik_1x}$$

$$\psi_{\mathrm{II}}(x) = Ae^{ik_2x} + Be^{-ik_2x}, \qquad \psi_{\mathrm{III}}(x) = te^{-ik_3(x-a)}$$

$$|t|^2 = \frac{4k_1^2k_2^2}{4k_1^2k_2^2 + (k_1^2 - k_2^2)\sin^2(k_2a)} = \frac{4E(E-V_0)}{4E(E-V_0) + V_0^2\sin^2(k_2a)}\,(E > V_0)$$

$$|t|^2 = \frac{4k_1^2k_2^2}{4k_1^2k_2^2 + (k_1^2 + \kappa)\sinh^2(\kappa a)} = \frac{4E(E-V_0)}{4E(E-V_0) + V_0^2\sinh^2(\kappa a)}\,(E < V_0)$$

$$|t^2| \approx \frac{E(V_0 - E)}{V_0^2}e^{-2\kappa a} \qquad \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

Dabei ist t bzw. r der Transmissions- bzw. Reflektionskoeffizent.

Allgemein

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\operatorname{div} \vec{X} = \vec{\nabla}\vec{X} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} X = \begin{pmatrix} \frac{\partial \vec{X}}{\partial x} \\ \frac{\partial \vec{X}}{\partial y} \\ \frac{\partial \vec{X}}{\partial z} \end{pmatrix}$$
$$\operatorname{rot} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix}$$
$$\frac{\partial^2}{\partial x^2} e^{-ax^2} = 2ae^{-ax^2}\left(2ax^2 - 1\right)$$

Maxwell
Elektrisches Feld $\vec{E}$, magnetisches Feld $\vec{B}$, Ladungsdichte ρ, elektrische Stromdichte $\vec{j}$

$$\operatorname{div} \vec{E} = \vec{\nabla}\vec{D} = \frac{\rho}{\varepsilon_0}$$
$$\operatorname{div} \vec{B} = \vec{\nabla}\vec{B} = 0$$
$$\operatorname{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\operatorname{rot} \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Hamilton - Lagrange

$$L = T - V = -mc^2 \cdot \sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}} - U(\vec{r}, t)$$
$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i} L = \frac{\partial}{\partial q_i} L \qquad \frac{\partial}{\partial \dot{q}_i} L = p_i$$
$$H = \sum_i p_i \dot{q}_i(q,t) - L$$
$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$
$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \text{ falls} = 0 \Rightarrow \text{ Energie ist erhalten}$$

3N-s Gleichungen bei s Zwangsbedingungen und N Teilchen.
Noether: Falls eine Schar existiert mit $q(t, \alpha)$ sodass gilt

$$L(q(t,\alpha), \dot{q}(t,\alpha), t) = L(q(t), \dot{q}(t), t)$$

dann gilt

$$\frac{d}{d\alpha} L(q(t,\alpha), \dot{q}(t,\alpha), t) = 0$$

Erweitertes Noether: Falls L nicht invariant, dann

$$\frac{d}{d\alpha} L(q(t,\alpha), \dot{q}(t,\alpha), t)|_{\alpha=0} = \frac{d}{dt} L(q(t), \dot{q}(t), t)$$

$\Rightarrow \sum_i \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} \frac{\partial q_i}{\partial \alpha} - f$ ist erhalten, falls L zeitinvariant.