# Homework 1

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## 1 Proof by Contrapositive

We begin with some definitions.

**Definition 1.** An integer $n$ is said to be *even* if there is an integer $k$ such that $n = 2k$. Otherwise, $n$ is said to be *odd*.

**Remark:** It follows that odd integers can be written uniquely as $n = 2k + 1$ for some integer $k$.

**Definition 2.** We say that $x$ is a *rational number*, denoted $x \in \mathbb{Q}$, if there exist integers $p, q$ such that $x = \dfrac{p}{q}$.

**(Recommended) Problem 1.** Suppose that $a, b \in \mathbb{R}$ such that $ab \notin \mathbb{Q}$. Then we have $a \notin \mathbb{Q}$ or $b \notin \mathbb{Q}$.

*Proof by Contradiction.* Show that $\neg((a \notin \mathbb{Q}) \vee (b \notin \mathbb{Q})) \Rightarrow \neg(ab \notin \mathbb{Q})$ where $a, b \in \mathbb{R}$.
$\neg((a \notin \mathbb{Q}) \vee (b \notin \mathbb{Q})) \equiv (a \in \mathbb{Q}) \wedge (b \in \mathbb{Q})$ by DeMorgan's Law.
$\neg(ab \notin \mathbb{Q}) \equiv ab \in \mathbb{Q}$ by def. of negation.
Thus, we must show that if a and b are rational, then the product ab is also rational.
Symbolically: $(a \in \mathbb{Q}) \wedge (b \in \mathbb{Q}) \Rightarrow (ab \in \mathbb{Q})$.

By Definition 2, a and b can be written as $a = \frac{p}{q}$ and $b = \frac{m}{n}$, where $p, q, m, n \in \mathbb{Z}$.
For $a, b \in \mathbb{Q}$, $ab = \frac{p}{q} * \frac{m}{n} = \frac{pm}{qn}$ where $pm, qn \in \mathbb{Z}$ due to integer multiplication properties.
Therefore, $ab \in \mathbb{Q}$ since $\frac{pm}{qn}$ is a quotient of two integers, as per Definition 2.
As such, we have proven that $(a \in \mathbb{Q}) \wedge (b \in \mathbb{Q}) \Rightarrow (ab \in \mathbb{Q})$.

Hence, by the contrapositive proof, the statement if $ab \notin \mathbb{Q}$, then we have $a \notin \mathbb{Q}$ or $b \notin \mathbb{Q}$, holds. $\square$

**(Recommended) Problem 2.** Suppose $x, y \in \mathbb{Z}$ such that $xy$ is even. Then either $x$ is even or $y$ is even.

*Proof by Contrapositive.* Show that $\neg$(x is even $\vee$ y is even) $\Rightarrow \neg$(xy is even) where $x, y \in \mathbb{Z}$.
$\equiv$ (x is odd $\wedge$ y is odd) $\Rightarrow$ (xy is odd) by DeMorgan's Law and the def. of negation.

By Definition 1, x and y can be written as: $x = 2m + 1, y = 2n + 1$ where $m, n \in \mathbb{Z}$.
$xy = (2m + 1)(2n + 1)$
$= 4mn + 2m + 2n + 1$
$= 2(2mn + m + n) + 1$
$= 2k + 1$ where $k = 2mn + m + n$
Therefore, xy is odd, and as such, we have proven that (x is odd $\wedge$ y is odd) $\Rightarrow$ (xy is odd).

Hence, by the contrapositive proof, the statement if $xy$ is even, then either $x$ is even or $y$ is even. $\square$

**(Recommended) Problem 3.** Suppose $x, y \in \mathbb{Z}$ such that $x + y \geq 15$. Then either $x \geq 8$ or $y \geq 8$.

*Proof by Contrapositive.* Show that $\neg((x \geq 8) \vee (y \geq 8)) \Rightarrow \neg((x + y) \geq 15)$ where $x, y \in \mathbb{Z}$.
$\equiv ((x < 8) \wedge (y < 8)) \Rightarrow ((x + y) < 15)$ by DeMorgan's Law and def. of negation.

Since x and y are integers, we can rewrite the expression $(x < 8) \wedge (y < 8)$ as $(x \leq 7) \wedge (y \leq 7)$.
$x + y \leq 7 + 7 = 14$. Thus, 14 is the greatest possible sum of x and y.
Since $14 < 15$ is true, we see that it does follow that $(x + y) < 15$.

Hence, by the contrapositive proof, the statement if $x + y \geq 15$ then either $x \geq 8$ or $y \geq 8$, holds. $\square$

**(Recommended) Problem 4.** Suppose $n \in \mathbb{Z}$ such that $n^3$ is even. Then $n$ is even.

*Proof by Contrapositive.* Show that $\neg$(n is even) $\Rightarrow \neg$($n^3$ is even).
$\equiv$ (n is odd) $\Rightarrow$ ($n^3$ is odd) by the def. of negation.

Since n is odd, it can be represented generically as $n = 2k + 1$ where $k \in \mathbb{Z}$.
Thus, $n^3 = (2k + 1)^3$
$= (4k^2 + 4k + 1)(2k + 1)$
$= 8k^3 + 4k^2 + 8k^2 + 4k + 2k + 1$
$= 8k^3 + 12k^2 + 6k + 1$
$= 2(4k^3 + 6k^2 + 3k) + 1$
$= 2t + 1$ where $t = 4k^3 + 6k^2 + 3k$.
Thus, when n is odd, $n^3$ is odd.
Hence, by the contrapositive proof, the statement if $n^3$ is even, then n is even, holds. $\square$