Homework 1

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1 Proof by Contrapositive

We begin with some definitions.

Definition 1. An integer n is said to be *even* if there is an integer k such that $n = 2k$. Otherwise, n is said to be *odd*.

Remark: It follows that odd integers can be written uniquely as $n = 2k + 1$ for some integer k.

Definition 2. We say that x is a *rational number*, denoted $x \in \mathbb{Q}$, if there exist integers p, q such that $x = \dfrac{p}{q}$.

(Recommended) Problem 1. Suppose that $a, b \in \mathbb{R}$ such that $ab \notin \mathbb{Q}$. Then we have $a \notin \mathbb{Q}$ or $b \notin \mathbb{Q}$.

Proof by Contradiction. Show that $\neg((a \notin \mathbb{Q}) \vee (b \notin \mathbb{Q})) \Rightarrow \neg(ab \notin \mathbb{Q})$ where $a, b \in \mathbb{R}$.
$\neg((a \notin \mathbb{Q}) \vee (b \notin \mathbb{Q})) \equiv (a \in \mathbb{Q}) \wedge (b \in \mathbb{Q})$ by DeMorgan's Law.
$\neg(ab \notin \mathbb{Q}) \equiv ab \in \mathbb{Q}$ by def. of negation.
Thus, we must show that if a and b are rational, then the product ab is also rational.
Symbolically: $(a \in \mathbb{Q}) \wedge (b \in \mathbb{Q}) \Rightarrow (ab \in \mathbb{Q})$.

By Definition 2, a and b can be written as $a = \frac{p}{q}$ and $b = \frac{m}{n}$, where $p, q, m, n \in \mathbb{Z}$.
For $a, b \in \mathbb{Q}$, $ab = \frac{p}{q} * \frac{m}{n} = \frac{pm}{qn}$ where $pm, qn \in \mathbb{Z}$ due to integer multiplication properties.
Therefore, $ab \in \mathbb{Q}$ since $\frac{pm}{qn}$ is a quotient of two integers, as per Definition 2.
As such, we have proven that $(a \in \mathbb{Q}) \wedge (b \in \mathbb{Q}) \Rightarrow (ab \in \mathbb{Q})$.

Hence, by the contrapositive proof, the statement if $ab \notin \mathbb{Q}$, then we have $a \notin \mathbb{Q}$ or $b \notin \mathbb{Q}$ is true. □

(Recommended) Problem 2. Suppose $x, y \in \mathbb{Z}$ such that xy is even. Then either x is even or y is even.

Proof by Contrapositive. Show that $\neg$(x is even $\vee$ y is even) $\Rightarrow$ $\neg$(xy is even) where $x, y \in \mathbb{Z}$.
$\equiv$ (x is odd $\wedge$ y is odd) $\Rightarrow$ (xy is odd) by DeMorgan's Law and the def. of negation.

By Definition 1, x and y can be written as: $x = 2m + 1, y = 2n + 1$ where $m, n \in \mathbb{Z}$.
$xy = (2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1 = 2k + 1$ where $k = 2mn + m + n$.
Therefore, xy is odd, and as such, we have proven that (x is odd $\wedge$ y is odd) $\Rightarrow$ (xy is odd).

Hence, by the contrapositive proof, the statement if xy is even, then either x is even or y is even. □

(Recommended) Problem 3. Suppose $x, y \in \mathbb{Z}$ such that $x + y \geq 15$. Then either $x \geq 8$ or $y \geq 8$.

Proof by Contrapositive. Show that $\neg((x \geq 8) \vee (y \geq 8)) \Rightarrow \neg((x + y) \geq 15)$ where $x, y \in \mathbb{Z}$.
$\equiv ((x < 8) \wedge (y < 8)) \Rightarrow ((x + y) < 15)$ by DeMorgan's Law and def. of negation.

Since x and y are integers, we can rewrite the expression $(x < 8) \wedge (y < 8)$ as $(x \leq 7) \wedge (y \leq 7)$.
$x + y \leq 7 + 7 = 14$. Thus, 14 is the greatest possible sum of x and y.
Since $14 < 15$ is true, we see that it does follow that $(x + y) < 15$.

Hence, by the contrapositive proof, the statement if $x + y \geq 15$ then either $x \geq 8$ or $y \geq 8$. □

(Recommended) Problem 4. Suppose $n \in \mathbb{Z}$ such that n^3 is even. Then n is even.

Proof by Contrapositive. Show that $\neg$(n is even) $\Rightarrow \neg$(n^3 is even).
$\equiv$ (n is odd) $\Rightarrow$ (n^3 is odd) by the def. of negation.

Since n is odd, it can be represented generically as $n = 2k + 1$ where $k \in \mathbb{Z}$.
Thus, $n^3 = (2k + 1)^3$
$= (4k^2 + 4k + 1)(2k + 1)$
$= 8k^3 + 4k^2 + 8k^2 + 4k + 2k + 1$
$= 8k^3 + 12k^2 + 6k + 1$
$= 2(4k^3 + 6k^2 + 3k) + 1$
$= 2t + 1$ where $t \in \mathbb{Z}$.
Thus, when n is odd, n^3 is odd.
Hence, by the contrapositive proof, the statement if n^3 is even, then n is even. $\square$