Dynamics and Optimal Control of a Legged Robot in Flight Phase

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Abstract

It is well known that the angular momentum of a legged robot in the flight phase is conserved. This paper discusses the control of the body orientation in flight using the internal motion of the leg. First, we use the angular momentum constraint (nonholonomic) to recast the problem into a nonholonomic motion planning problem. Then we apply Chow's Theorem to verify that the system is controllable and the concept of holonomy is introduced for constructing an optimal path. Finally, we use linearization control in the internal motion space to realize the planned path. This study provides an additional degree of control to dynamically balance a legged robot that run.

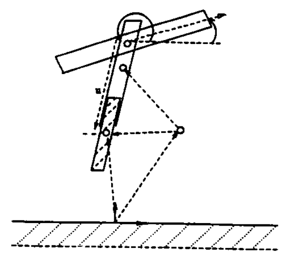

Figure 1

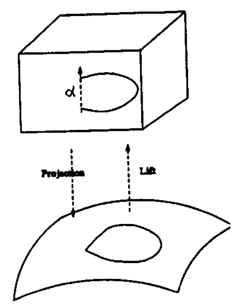

Figure 2

1 Introduction

Consider a model of a legged robot shown in Figure 1. A nominal running cycle of the system consists of a compression phase, in which the leg touches the ground and the leg spring compresses; a thrusting phase, in which the leg spring extends while additional energy is injected into the system; and a flight phase, in which the system takes off and then undergoes a parabolic trajectory and finally touches down again (see [Rai86]). Since the system in general has not been equipped with external gas jets and air resistance is negligible, the angular momentum of the system when it is in the flight phase is conserved. This paper studies the following two problems:

Problem 1.1 *Can the system, by swinging the leg and adjusting the leg length (i.e., using the internal motion of the system), rotate itself 360° (i.e., a forward somersault or a backward somersault) while in the air, with perhaps zero angular momentum?*

Problem 1.2 *Suppose that, the angular momentum at take-off is zero, the body is level and the leg is vertical. Can the system position its leg at an arbitrary angle relative to the vertical without change the levelness of the body at the end of the flight phase?*

Understanding these problems can provide an additional degree of freedom for the control of legged robots. Other applications are also found in robot gymnastics ([HRss]) and space robotics (e.g., repairing a satellite with a redundant robot arm).

The paper is structured as follows: In Section 2, we define notations and preliminary concepts needed; in Section 3, we discuss the notion of internal motion and its usage in solving these problems. We also show that the angular momentum constraint is nonholonomic, and the system is fully controllable. A "closed-loop" strategy that solves these problems is then proposed. In Section 4 we discuss the optimal control.

2 Preliminaries

A model of a one-legged hopping robot is given in Figure 1.

The generalized coordinates that describe motion of the system are $(\theta_b, \theta_l, u, r)$, where $\theta_b \in S^1$ is the pitch angle of the body (measured counter clockwise relative to the horizontal axis), $\theta_l \in S^1$ is the leg angle, $u \in \Re$ is the leg length (distance from the hip to the mass center of the lower leg) and $r \in \Re^2$ the position vector of the system's mass center relative to the inertial reference frame. Thus, the configuration space, Q, of the system is $Q = S^1 \times S^1 \times \Re \times \Re^2$. The shape space, M, consists of the set of hinge angle, $\psi_l = \theta_l - \theta_b \in S^1$ and the leg length $u \in \Re$. It is also called the control space or the internal motion space. Let (m_b, I_b) denote the mass and the moment of inertia of the body about the mass center. Similarly, (m_1, I_1) and (m_2, I_2) denote, respectively, the mass, the moment of inertia of the upper leg and the lower leg about the mass center. Furthermore, let d_1 denotes the distance from the hip to the mass center of the upper leg and d_2 the distance from the foot to the mass center of the lower leg; $m_l = m_1 + m_2$ the mass of the leg, and $m = (m_b + m_1 + m_2)$ the total mass of the system. $\epsilon_l = m_l/m$ is the mass ratio of the leg, and $\epsilon_2 = m_2/m$, and $\hat{\epsilon}_1 = m_1/m_l, \hat{\epsilon}_2 = m_2/m_l$.

The total kinetic energy of the system in the flight phase is derived as ([LM90]):

$$K = \underbrace{\frac{1}{2}I_b\dot{\theta}_b^2 + \frac{1}{2}\tilde{I}_l\dot{\theta}_l^2 + \frac{1}{2}\hat{m}\dot{u}^2}_{rotation} + \underbrace{\frac{1}{2}m\|\dot{r}\|^2}_{translation}, \tag{1}$$

where

$$\tilde{I}_l = I_l + \{m_b\epsilon_l^2 + m_l(1-\epsilon_l)^2\}d_l^2, \tag{2}$$

is the effective inertia of the leg,

$$\hat{m} = m_2(1-\epsilon_2); \;\; d_l = \hat{\epsilon}_1 d_1 + \hat{\epsilon}_2(u - d_2),$$

are the effective mass of the lower leg and the the distance from the hip to the mass center of the leg, and

$$I_l = I_1 + I_2 + m_1(d_l - d_1)^2 + m_2(u - d_2 - d_l)^2$$

is the moment of inertia of the leg about its own mass center.

Note that the total kinetic energy is decoupled into a translational component ($\frac{1}{2}m\|\dot{r}\|^2$) of the system's mass center and a rotational component ($K_r = \frac{1}{2}(I_b\dot{\theta}_b^2 + \tilde{I}_l\dot{\theta}_l^2 + \tilde{m}\dot{u}^2)$) about the system's mass center. This enables us to study the translational motion and the rotational motion independently.

Let $v_z(0)$ be the vertical take-off velocity. Then, the flight time, T, is given by

$$T = \frac{2v_z(0)}{g}.$$

Problem 1.1 amounts to *rotating the body through* 360^o *degrees while the hinge angle and the leg length have the same value at the end of the flight phase as at take-off.* Similarly, Problem 1.2 amounts to *positioning the leg to a given angle while the body returns to the level position at the end of the flight phase.*

We discuss in the following sections how this can be done, in perhaps some optimal way.

3 Control Strategies

Consider now the rotational motion only. First, let

$$Q_{cm} = \{(\theta_b, \theta_l, u) \in S^1 \times S^1 \times \Re\}$$

be the reduced configuration space of the system. Formally, Q_{cm} is obtained from the original configuration space Q through *reduction* to the mass center of the system. The group action on Q in this case is the translation by $\Re^2$. The subject of reduction and various techniques for it have been discussed in ([AM78]), and excellent applications of the reduction techniques to coupled rigid body systems are in ([Pat88], [GKM88] and [Sre87]).

The rotational kinetic energy

$$K_r = \frac{1}{2}(I_b\dot{\theta}_b^2 + \tilde{I}_l\dot{\theta}_l^2 + \tilde{m}\dot{u}^2) = \frac{1}{2}[\dot{\theta}_b, \dot{\theta}_l, \dot{u}] \begin{bmatrix} I_b & 0 & 0 \\ 0 & \tilde{I}_l & 0 \\ 0 & 0 & \tilde{m} \end{bmatrix} \begin{bmatrix} \dot{\theta}_b \\ \dot{\theta}_l \\ \dot{u} \end{bmatrix} \stackrel{\triangle}{=} \frac{1}{2}\dot{q}^t J \dot{q} \tag{3}$$

can be viewed as a metric on the (reduced) configuration space Q_{cm}. In other words, we can define an inner product, $\langle\langle v_1,\ v_2 \rangle\rangle$, of two velocity vectors $v_1, v_2 \in T_q Q_{cm}$ by

$$\langle\langle v_1,\ v_2 \rangle\rangle = v_1^T J v_2.$$

(Q_{cm} equipped with this metric is an example of a Riemannian manifold.)

According to the *Conservation Law of Angular Momentum*, the angular momentum of the system is conserved throughout the flight phase. When the system is in a configuration $q = (\theta_b, \theta_l, u) \in Q_{cm}$ with velocity $\dot{q} \in T_q Q_{cm}$, the angular momentum, μ, about the system's mass center is calculated as

$$\mu = I_b\dot{\theta}_b + \tilde{I}_l\dot{\theta}_l \stackrel{\triangle}{=} (1,1,0)J\dot{\theta} = \langle\langle e, \dot{\theta} \rangle\rangle, \tag{4}$$

where

$$e = (1,1,0)^t.$$

Note that the velocity, $\dot{u}$, of the leg length does not enter the angular momentum expression. Rather, the effective moment of inertia of the leg, $\tilde{I}_l$, can be adjusted using u (see Eq. (2)).

Remark 3.1 *Eq. (4) can be thought of as a constraint on the configuration space. An important question to ask is,* does this (velocity) constraint arise from a position constraint, or equivalently, is the constraint integrable (holonomic)? *That is, does there exist a constraint function of the position variables and time,*

$$f(\theta_b, \theta_l, u, t) = 0 \tag{5}$$

such that Eq. (4) is given by the differential of (5), i.e., the following equalities hold,

$$I_b = \frac{\partial f}{\partial \theta_b}, \tilde{I}_l = \frac{\partial f}{\partial \theta_l}, 0 = \frac{\partial f}{\partial u}, \text{and } -\mu = \frac{\partial f}{\partial t}?$$

Suppose there is. Then from the fact that

$$\frac{\partial^2 f}{\partial u \partial \theta_l} = \frac{\partial^2 f}{\partial \theta_l \partial u}$$

we conclude that $\tilde{I}_l$ is independent of u as $\frac{\partial f}{\partial u} = 0$, which is obviously false (see Eq. (2)). In other words, the angular momentum constraint is nonholonomic. □

Eq. (4) also shows that if the angular momentum μ were zero at take-off, then the system's velocity $\dot{q}$ is constrained to be orthogonal (under the metric) to the vector e.[1] A vector that is orthogonal to e is called a *horizontal vector* ([Mon89]) if we let e be the *vertical vector.* A path in Q_{cm} is called horizontal (or admissible when $\mu \neq 0$) if each of its velocity vectors along the path is horizontal (or satisfies Eq. (4)).

Let $q_0 \in Q_{cm}$ be the initial configuration and $q_f \in Q_{cm}$ the final configuration which we want to reach in these two problems. Then, in the case $\mu = 0$, both problems amount to *reaching q_f from q_0 along a horizontal path subject to the flight time constraint.* More generally, when $\mu \neq 0$, *we want to reach q_f from q_0 along a path $q(t) \in Q_{cm}$, $t \in [0,T]$ such that $q(0) = q_0, q(T) = q_f$ and $\dot{q}$ satisfies the angular momentum constraint (4) .*

It is not immediately clear how such a path can be constructed. Moreover, it is not obvious how the joints can be servoed to execute a given path in Q_{cm}. To address these problems, we introduce the shape space M, which is parameterized by $\psi_l = \theta_l - \theta_b$, the hinge angle, and u, the leg length. M has 2 dimensions, the same as the number of control inputs. The following results will help us with our problems.

Proposition 3.1 *(a) Define the projection P from the configuration space to the shape space by*

$$P : Q_{cm} \longmapsto M : (\theta_b, \theta_l, u) \longrightarrow \begin{bmatrix} \psi_l \\ u \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_b \\ \theta_l \\ u \end{bmatrix}. \tag{6}$$

Then, two configurations $q_1, q_2 \in Q_{cm}$ have the same projections,

[1] Viewed from a different angle, the conservation of angular momentum is a consequence of the system's rotational symmetry. That is, there exists a group action on Q_{cm} by the rotation group S^1, and e is the infinitesimal generator of the group action, see [AM78].

$P(q_1) = P(q_2)$, *if and only if they have the same shapes, i.e., they differ from each other by a rigid rotation. (b) Let* $s(t)$ *be a piecewise smooth path in* M, *and* $q_0 \in Q_{cm}$ *a configuration with shape* $s(0)$, *that is,* $P(q_0) = s(0)$. *Then, there is a unique piecewise smooth path* $q(t)$ *passing through* q_0 *at time* 0, *satisfying the angular momentum constraint (4), and having shape* $s(t)$, *that is,*

$$P(q(t)) = s(t).$$

Proof. (a) follows from the definition. For (b) we shall construct $q(t)$ and use the Fundamental Theorem of ODE to argue that the path is unique.

First, using the definition of hinge angle, we rearrange Eq. (4) into the form

$$\mu = I_b\dot{\theta}_b + \tilde{I}_l(\dot{\theta}_b + \dot{\psi}_l) = (I_b + \tilde{I}_l)\dot{\theta}_b + \tilde{I}_l\dot{\psi}_l. \tag{7}$$

$I_o = (I_b + \tilde{I}_l)$ is called the locked-body inertia, i.e., the system's moment of inertia when the hinge angle and the leg length are frozen at (ψ_l, u) and the system is considered as a single rigid body. Eq. (7) can be further manipulated into the form

$$\dot{\theta}_b = \frac{\mu}{I_o} - \frac{\tilde{I}_l}{I_o}\dot{\psi}_l. \tag{8}$$

Combining Eq. (8) with the definition of hinge angle gives

$$\begin{bmatrix} \dot{\theta}_b \\ \dot{\theta}_l \\ \dot{u} \end{bmatrix} = \begin{bmatrix} \mu/I_o \\ \mu/I_o \\ 0 \end{bmatrix} + \begin{bmatrix} -\tilde{I}_l/I_o \\ 1-\tilde{I}_l/I_o \\ 0 \end{bmatrix} \dot{\psi}_l + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{u}. \tag{9}$$

Note that the u in the right hand side of Eq. (9) is considered as a variable parameterizing the shape space, while it is a variable parameterizing the configuration space in the left hand side. Eq. (9) is an ODE for the lifted path. Thus, for a given path $s(t) = (\psi_l(t), u(t))^t \in M$ and pre-specified initial condition q_0, the Fundamental Theorem of ODE guarantees that the lifted path $q(t) = (\theta_b(t), \theta_l(t), u(t))^t$ exists and is unique. Moreover, the lifted path satisfies the angular momentum constraint. □

Part (b) of Proposition 2.1 shows that *every horizontal (or admissible) path through a point* q_0 *is in fact the lift of a path in the shape space.* One can also consider Eq. (9) as a control system of the form

$$\dot{q} = f(q) + g_1(q)u_1 + g_2(q)u_2, \tag{10}$$

where Q_{cm} is the state space, $f(q) = (\mu/I_o, \mu/I_o, 0)^t$ the drifting vector field, $g_1(q) = (-\tilde{I}_l/I_o, (1-\tilde{I}_l/I_o), 0)^t$ and $g_2(q) = (0,0,1)^T$ the control vector fields and $(u_1, u_2) = (\dot{\psi}_l, \dot{u})$ the "control inputs" (see [SJ72] and [Bro81]). Note that $(\dot{\psi}_l, \dot{u})$ are not the real torque inputs to the joints. Rather, they are velocities.

An important question to ask is, *is the control system given by (9) fully controllable? In other words, given* q_0, $q_f \in Q_{cm}$, *does there exist a horizontal path (or an admissible path when* $\mu \neq 0$*) connecting them?*

This is a well addressed question in nonlinear control theory (see [HK77] and [Bro81]) and the answer is given by the so-called Chow's Theorem as follows:

Theorem 3.1 *(Chow's Theorem) Consider the control system given by Eq. (9) and suppose that* $\mu = 0$. *Then any two points in the configuration space can be connected by a horizontal path.*

Proof. According to Chow ([Cho40]), we compute the Lie algebra, ∇, generated by the two control vector fields, $\{g_1, g_2\}$. First, the Lie bracket, $[g_1, g_2]$, of g_1, g_2 is

$$[g_1, g_2] = Dg_2 \cdot g_1 - Dg_1 \cdot g_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{I_b}{I_o^2}\frac{d\tilde{I}_l}{du}.$$

Thus, $\nabla = \{g_1, g_2, [g_1, g_2]\}$ has determinant

$$\frac{I_b}{I_o^2}\frac{d\tilde{I}_l}{du},$$

which is clearly non-zero (see Eq. (15)). □

Once a horizontal (or an admissible) path that connects q_0 to q_f has been found[2], we use the control law given by the following theorem to realize it.

Theorem 3.2 *Let* $s^d(t) \in M, t \in [0,T]$, *be a desired path in the shape space,* $\tau = (\tau_1, \tau_2)^t$ *the torque/force inputs to the two joints,* [3] *and* μ *the angular momentum at take-off. Then, there exists a choice of torque inputs* τ *that drives the trajectory tracking error*

$$e_p(t) = s(t) - s^d(t)$$

asymptotically to zero, where $s(t)$ *is the actual trajectory in the shape space.*

Proof. The equations describing motion in the configuration space are

$$J \begin{bmatrix} \ddot{\theta}_b \\ \ddot{\theta}_l \\ \ddot{u} \end{bmatrix} + N = \begin{bmatrix} -\tau_1 \\ \tau_1 \\ \tau_2 \end{bmatrix} \triangleq P^T\tau, \tag{11}$$

where $N = [0, 0, \frac{\partial K_r}{\partial u}]^t$ and

$$\frac{\partial K_r}{\partial u} = \{[m_b\epsilon_l^2 + m_l(1-\epsilon_l)^2]d_l\hat{\epsilon}_2 + m_2(u - d_l)\}\dot{\theta}_l^2.$$

Since J is nonsingular, we multiply both sides of Eq. (11) by J^{-1} and then project the resulting equations to the shape space. This gives

$$\begin{bmatrix} \ddot{\psi}_l \\ \ddot{u} \end{bmatrix} + PJ^{-1}N = PJ^{-1}P^T\tau \triangleq J_M^{-1}\tau. \tag{12}$$

Because P has full rank, $PJ^{-1}P^T = J_M^{-1}$ is invertible. In fact, J_M is the inertia matrix in the shape space. Now let $s^d(t)$ be a desired trajectory and

$$e_p(t) = s(t) - s^d(t)$$

the trajectory tracking error. Clearly, the following feedback law for τ drives $e_p(t)$ asymptotically to zero.

$$\tau = J_M\left\{\ddot{s}^d(t) - K_v\dot{e}_p - K_pe_p\right\} + J_MPJ^{-1}N \tag{13}$$

where $K_v, K_p \in \Re^{2\times 2}$ are velocity and position gains chosen so that the solutions e_p to the equations are damped exponentials. □

It may seem that by combining Proposition 3.1, Theorem 3.1 and 3.2, a solution to Problem 1.1 and 1.2 can be obtained. That is, the following steps may seem to work:

[2] According to part (b) of Proposition (3.1), we really have a path in the shape space.

[3] We assume that the system is driven by ideal actuators, and this assumption may not be satisfied by a physical system.

(a) Pick a path in the shape space and lift it to the configuration space and see if it connects q_0 to q_f.

(b) If it does then let it be the desired path, and use the control law specified in Theorem 3.2 to execute it.

The above strategy will work provided that the actual trajectory (in the shape space) agrees with the desired trajectory for all time. Since the error between the actual and the desired landing configurations is the integral of the shape space trajectory tracking error, Theorem 3.2 does not guarantee that this error will go to zero. In order to make sure that the system will land properly, knowledge of the configuration variables will be needed to adjust the desired shape space trajectory. This also explains that a blindfolded cat can not land on her feet (T. Kane, 1989, private communication with the second author).

We thus propose the following "close-loop" strategy for these problems.

Data: q_0, q_f and measurement of $q(t)$.

Step A: Divide the flight phase into n subintervals, $0 = t_0 < t_1 < \ldots < t_n = T$.

Step B: At time t_i, take $q(t_i)(q(0) = q_0)$ as the initial configuration, and plan a desired path $s_i^d(t)$ whose lift connects $q(t_i)$ to q_f. (Hope this can be done in zero time.)

Step C: During the time interval $[t_i, t_{i+1}]$, execute the path $s_i^d(t)$.

Remark 3.2 *In step B, if the current configuration $q(t_i)$ agrees with the expected value (i.e., the lift of $s_{i-1}^d(t_i)$), then, $s_i^d(t)$ remains the same as $s_{i-1}^d(t)$. Otherwise, correction terms will be incorporated into the new trajectory, $s_i^d(t)$.* □

4 Holonomy and Optimal Control

Consider again Figure 2 and assume that $\mu = 0$. Let $s(t), t \in [0,T]$, be a closed path in the shape space, i.e., $s(0) = s(T)$, and lift it to a path $q(t)$ in the configuration space. From our early discussions we know that, (1) $q(t)$ is horizontal, and (2), $q(t)$ is in general not closed, i.e., $q(0) \neq q(T)$. This follows from the constraint being nonholonomic. Since $q(T)$ and $q(0)$ project to the same point, they must differ by a constant. That is, if $q(0) = (\theta_b(0), \theta_l(0), u(0))^t$, then $q(T) = (\theta_b(0) + \alpha, \theta_l(0) + \alpha, u(0))^t$, where α is expressed in radians. *The value of α depends only on the path in the shape space and is called the holonomy of $s(t)$.* It measures the degree of nonholonomy of the system. If the system were holonomic, then the holonomy of any closed path in the shape space is zero.

In the famous *falling cat problem* studied extensively by ([KS69], [Pat88] and [Mon89]), the landing configuration differs from the releasing configuration by a constant rotation, π radians, and the cat is entitled to find a path, perhaps a path of shortest distance that gives a holonomy of π radians. One may also take Problem 1.1 as two falling-cat problems. First, rotate itself to the upside down configuration within $T/2$ and then come to the landing configuration within the remaining half period.

We now wish to calculate the holonomy of a closed path in M. For this let Ω be the region enclosed by $s(t)$, or s be the boundary of Ω. Then, integrating Eq. (8) along s and applying Green's Theorem[4] yields

$$\begin{aligned}\theta_b(T) - \theta_b(0) &= \int_s \frac{\mu}{I_o} dt - \int_s \frac{\tilde{I}_l}{I_o} d\psi_l \\ &= \int_s \frac{\mu}{I_o} dt + \iint_\Omega \frac{d}{du}\left(\frac{\tilde{I}_l}{I_o}\right) d\psi_l du.\end{aligned} \tag{14}$$

The integrand in the second term is given by

$$\frac{d}{du}\left(\frac{\tilde{I}_l}{I_o}\right) = \frac{I_b}{I_o^2}\frac{d\tilde{I}_l}{du} = \frac{2I_b}{I_o^2}\{m_2(u-d_l) + [m_b\epsilon_l^2 + m_l(1-\epsilon_l)^2]d_l\bar{\epsilon}_2\}. \tag{15}$$

In other words, the holonomy of a closed path is given by the area integral of the function $\frac{d}{du}(\frac{\tilde{I}_l}{I_o})$ over the region enclosed by the path. The final destination depends on the history of the path!

Suppose that the enclosed region is a rectangle with four corner points

$$\{(\psi_{l,min}, u_{min}), (\psi_{l,max}, u_{min}), (\psi_{l,max}, u_{max}), (\psi_{l,min}, u_{max})\},$$

then the holonomy is given by

$$\alpha = \left[\frac{\tilde{I}_l(u_{max})}{I_b + \tilde{I}_l(u_{max})} - \frac{\tilde{I}_l(u_{min})}{I_b + \tilde{I}_l(u_{min})}\right](\psi_{l,max} - \psi_{l,min}). \tag{16}$$

To perform a forward somersault, this rectangle must be traversed $2\pi/\alpha$ times, since each circuit provides a change α of the body orientation.

Definition 4.1 *Let $q(t), t \in [0,T]$, be a path in the configuration space. Then, the integral*

$$E(q) = \int_0^T \dot{q}^t J \dot{q} dt \tag{17}$$

measures the length, or the energy of the path.

We finally apply a result of R. Montgomery ([Mon89]) to study the following problem:

Problem 4.1 *Consider two configurations q_o and q_f such that $P(q_o) = P(q_f)$. Find a* shortest *path, $q(t) \in Q_{cm}$, that joins q_o to q_f and satisfies the angular momentum constraint (4).*

Remark 4.1
- *Problem 1.1 is equivalent to this problem, with a holonomy angle of 2π.*
- *In Problem 1.2, let $q_0 = (0, \theta_l(0), u(0))^t$, $q_f = (0, \theta_l(0)+\delta, u(0))^t$ be the initial and the final configurations, and $s_0 = P(q_0), s_f = P(q_f)$. Then, q_f can be reached from q_0 with first an arbitrary path from s_0 to s_f and then followed by a closed path based at s_f.* □

Theorem 4.1 *(Minimal Energy Path, R. Montgomery([Mon89])) Let $p_q = (p_{\theta_b}, p_{\theta_l}, p_u)^t$ be the generalized momentum conjugate to q, i.e., $P_{\theta_b} = \frac{\partial K_r}{\theta_b}$. Then, a path $q(t) \in Q_{cm}$ is an extremal of (17)*

[4] Green's Theorem says, $\int_{\partial\Omega}(g_1(x,y)dx + g_2(x,y)dy) = \iint_\Omega(\frac{\partial g_2(x,y)}{\partial x} - \frac{\partial g_1(x,y)}{\partial y})dxdy$, where $g_1(x,y), g_2(x,y)$ are functions of (x,y) and $\partial\Omega$ is the boundary of the compact region Ω.

subject to the angular momentum constraint if and only if there exists a smooth generalized momentum vector p_q conjugate to $q(t)$ such that $(q(t), p_q(t))$ satisfies Hamilton's differential equation for the Hamiltonian

$$H_o(q,p_q) = \frac{1}{2}\left[\frac{1}{\tilde{I}_l}p_{\theta_l}^2 + \frac{1}{I_b}p_{\theta_b}^2 + \frac{1}{\tilde{m}}p_u^2\right] - \frac{1}{2I_o}(p_{\theta_b}+p_{\theta_l})^2 + \frac{\mu}{I_o}(p_{\theta_b}+p_{\theta_l}). \tag{18}$$

Remark 4.2 • *The Hamilton's differential equation of a Hamiltonian $H(q,p)$, $q = \{q^i\}$, $p = \{p_i\}$, is given by*

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i}; \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i}. \end{aligned} \tag{19}$$

- *The optimal path in the shape space is obtained from the optimal path in the configuration space by projection.*
- *The first component of Eq. (18) is the original kinetic energy, and the second component is the vertical kinetic energy. Thus, the first two components give the horizontal energy.* □

Proof. This follows from the Lagrange multiplier tecchnique. See R. Montgomery ([Mon89]). □

According to the above theorem, the equations of motion for the optimal path are

$$\begin{cases} \dot{\theta}_b = \frac{1}{I_b}p_{\theta_b} - \frac{1}{I_o}(p_{\theta_b} + p_{\theta_l}) + \frac{\mu}{I_o}; \\ \dot{\theta}_l = \frac{1}{\tilde{I}_l}p_{\theta_l} - \frac{1}{I_o}(p_{\theta_b} + p_{\theta_l}) + \frac{\mu}{I_o}; \\ \dot{u} = \frac{1}{\tilde{m}}p_u; \;\; \dot{p}_{\theta_b} = 0; \;\; \dot{p}_{\theta_l} = 0; \\ \dot{p}_u = \frac{1}{2}\frac{d\tilde{I}_l}{du}\left\{\frac{I_o^2 p_{\theta_l}^2 - \tilde{I}_l^2(p_{\theta_b}+p_{\theta_l})^2 + 2\tilde{I}_l^2\mu(p_{\theta_b}+p_{\theta_l})}{\tilde{I}_l^2 I_o^2}\right\}. \end{cases} \tag{20}$$

It is straightforward to verify that, using the first two equations of (20), the angular momentum constraint is not violated, i.e., $I_b\dot{\theta}_b + \tilde{I}_l\dot{\theta}_l = \mu$.

The fourth and the fifth equations of (20) indicate that $p_{\theta_b}, p_{\theta_l}$ are both constants. Combining the equations for the leg length variable yields

$$\ddot{u} = \frac{1}{2\tilde{m}}\frac{d\tilde{I}_l}{du}\left\{\frac{I_o^2 p_{\theta_l}^2 - \tilde{I}_l^2(p_{\theta_b} + p_{\theta_l})^2 + 2\tilde{I}_l^2\mu(p_{\theta_b} + p_{\theta_l})}{\tilde{I}_l^2 I_o^2}\right\}. \tag{21}$$

Eq. (21) can be numerically integrated to solve for $u(t)$.

To find the optimal hinge motion, we combine the first two equations of (20) to get

$$\dot{\psi}_l = \frac{1}{\tilde{I}_l}p_{\theta_l} - \frac{1}{I_b}p_{\theta_b}. \tag{22}$$

Eqs. (21) and (22) determine the optimal snape space trajectory. The final question is to choose the constants $p_{\theta_b}, p_{\theta_l}$ such that the optimal path connects q_0 to q_f. In general, one has to do this by "shooting".

5 Simulations

Using the results of the previous sections, we have simulated a one legged model to perform a forward somersault with *zero angular momentum*. The monoped parameters used in the simulation are given in the following table (from [Rai89]).

Parameter	Values	Parameter	Values
m_b	11.45 kg	m_1	2×1.055 kg
m_2	2×0.608 kg	I_b	$0.40\ kg - m^2$
I_1	$2 \times 0.0204\ kg - m^2$	I_2	$2 \times 0.0237\ kg - m^2$
d_1	0.0838 m	d_2	0.317 m

The other parameters can be calculated using data from the above table. The leg length and the hinge angle are constrained to the following intervals.

$$u_{min} \le u \le u_{max}, \;\; \psi_{l,min} \le \psi_l \le \psi_{l,max}.$$

For simplicity, we have chosen a rectangular path in the shape space with

$$\psi_{l,max} - \psi_{l,min} = 2.875 = 165^o(\text{degrees})$$

and

$$u_{max} = 0.505\text{m}, \; u_{min} = 0.075\text{m}.$$

Thus, from Eq. (16) the holonomy of the path is given by

$$\alpha = 0.27314;$$

and the number of loops that need to be repeated is 8 $(= \frac{2\pi}{.27314 * 2.875})$. Suppose that the take-off velocity is 4.9 m/s, which gives 1 second of flight time, the machine has to complete one loop within one eighth of a second.

The nominal trajectory of the monoped is a hopping motion, and the z-trajectory is shown in Figure 3, while Figures 4, 5 and 6 display the pitch, the hinge and the leg length trajectories. The gravity is $g/3$.

6 Conclusion

This paper studied an important problem related to legged locomotion system control. The results are appplicable to space robotics and satellite control.

Acknowledgement

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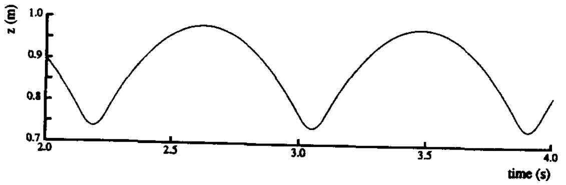

Figure 3: Trajectory of the vertical motion

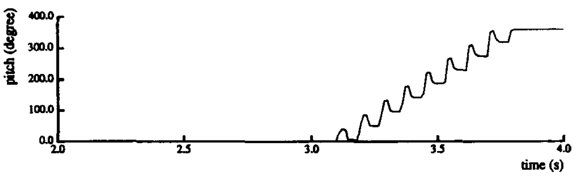

Figure 4: Trajectory of the pitch motion

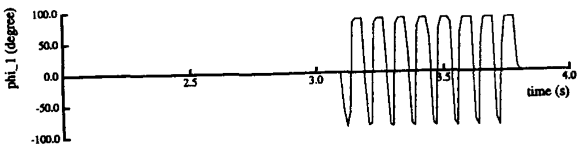

Figure 5: Trajectory of the hinge motion

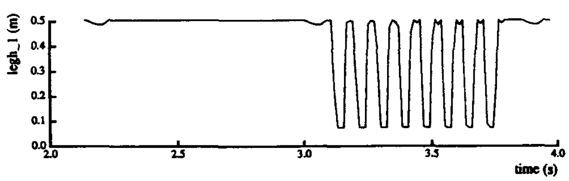

Figure 6: Trajectory of the leg length

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