11.24 TRANSVERSE NATURE OF ELECTROMAGNETIC WAVES LO11

Transverse nature of electromagnetic waves can be proved with the help of Maxwell's equations div $\vec{E} = 0$ and div $\vec{H} = 0$ for free space. Using different relations as discussed in the previous section div $\vec{E}$ and div $\vec{H}$ can be calculated as follows.

$$\vec{\nabla}\cdot\vec{E} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot[(E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k})e^{i(k_x x + k_y y + k_z z - \omega t)}]$$

$$= \frac{\partial}{\partial x}[E_{0x}e^{i(k_x x + k_y y + k_z z - \omega t)}] + \frac{\partial}{\partial y}[E_{0y}e^{i(k_x x + k_y y + k_z z - \omega t)}] + \frac{\partial}{\partial z}[E_{0z}e^{i(k_x x + k_y y + k_z z - \omega t)}]$$

$$= (E_{0x}ik_x + E_{0y}ik_y + E_{0z}ik_z)e^{i(k_x x + k_y y + k_z z - \omega t)}$$

$$= i(k_x E_{0x} + k_y E_{0y} + k_z E_{0z})e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= i(\vec{k}\cdot\vec{E}_0)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= i\vec{k}\cdot\vec{E} \qquad (\because \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \vec{E})$$

$\therefore$ For free space, $\vec{\nabla}\cdot\vec{E} = 0$

$$i\vec{k}\cdot\vec{E} = 0 \quad \text{or} \quad \vec{k}\cdot\vec{E} = 0$$

It means the wave vector $\vec{k}$ is perpendicular to $\vec{E}$.

Similarly, from $\vec{\nabla}\cdot\vec{H} = 0$ it can be shown that $\vec{k}\cdot\vec{H} = 0$.

Hence, the wave vector $\vec{k}$ is perpendicular to $\vec{H}$. Therefore, the relations $\vec{k}\cdot\vec{E} = 0$ and $\vec{k}\cdot\vec{H} = 0$ indicate that the electromagnetic field vectors $\vec{E}$ and $\vec{H}$ (or $\vec{B}$, as $\vec{B} = \mu_0\vec{H}$) both are perpendicular to the direction of propagation vector $\vec{k}$. It means that the electromagnetic waves are transverse in nature.

11.25 MAXWELL'S EQUATIONS IN ISOTROPIC DIELECTRIC MEDIUM: EM WAVE PROPOGATION LO11

In an isotropic dielectric medium, the conduction or free current density $\vec{J}$ and volume charge density ρ are zero. Further, the displacement vector $\vec{D}$ and the magnetic field $\vec{B}$ are defined as $\vec{D} = \varepsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$. In fact $\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\vec{E} + \varepsilon_0\chi_e\vec{E} \equiv \varepsilon\vec{E}$ together with $\varepsilon = \varepsilon_0(1 + \chi_e)$ and $\vec{B} = \mu_0\vec{H} + \mu_0\vec{M} = \mu_0\vec{H} + \mu_0\chi_m\vec{H} \equiv \mu\vec{H}$ together with $\mu = \mu_0(1 + \chi_m)$ for the isotropic linear dielectric (polarizable and magnetic) medium. Here, the vectors $\vec{P}$ and $\vec{M}$ give respectively the measure of polarization and magnetization of the medium. However, for the dielectric medium, it would be sufficient to remember that ε_0 and μ_0 of free space have been simply replaced with ε and μ. Hence, for dielectric medium

$$J = 0 \quad (\text{or } \sigma = 0, f = 0, D = \varepsilon E \text{ and } \vec{B} = \mu\vec{H})$$

where ε and μ, which are respectively the absolute permittivity and permeability of the medium. Under this situation, we can express the Maxwell's equation as

$$\vec{\nabla}\cdot\vec{E} = 0 \tag{i}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \tag{ii}$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{iii}$$

$$\vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \tag{iv}$$

Taking curl of Eq. (iii), we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left[-\mu \frac{\partial \vec{H}}{\partial t} \right]$$

or
$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{H})$$

or
$$0 - \nabla^2 E = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t} \qquad \text{[Using Eqs (i) and (iv)]}$$

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \tag{v}$$

Similarly, taking curl of Eq. (iv) and using Eqs. (ii) and (iii), we get

$$\nabla^2 \vec{H} = \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} \tag{vi}$$

As discussed earlier, $\frac{1}{\sqrt{\mu\varepsilon}}$ gives the phase velocity of the wave in the medium. If we represent this as v, we obtain from Eqs. (v) and (vi)

$$\nabla^2 \vec{E} - \frac{1}{v^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

and
$$\nabla^2 \vec{H} - \frac{1}{v^2} \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

Eqs. (v) and (vi) are the wave equations in an isotropic linear dielectric medium.

Now, $v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0 \mu_r \varepsilon_0 \varepsilon_r}}$ $\qquad$ ($\because \mu = \mu_0 \mu_r$ and $\varepsilon = \varepsilon_0 \varepsilon_r$)

or
$$v = \frac{c}{\sqrt{\mu_r \varepsilon_r}} \qquad \left[\because \quad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \right] \tag{vii}$$

Eq. (vii) shows that the propagation velocity of an electromagnetic wave in a dielectric medium is less than that in free space.

Also, refractive index $= \frac{c}{v} = \sqrt{\mu_r \varepsilon_r}$ (viii)

For non-magnetic dielectric medium $\mu_r \approx 1$. Hence, refractive index $= \sqrt{\varepsilon_r}$ or Refractive index $= \sqrt{\text{Relative permittivity}}$

11.26 MAXWELL'S EQUATIONS IN CONDUCTING MEDIUM: EM WAVE PROPAGATION AND SKIN DEPTH

LO11

We consider a linear and isotropic conducting medium whose permeability is μ, permittivity is ε and the conductivity is σ. Under this situation, we can write the Maxwell's equation as

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{i}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \tag{ii}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{iii}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

or
$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad [\because \vec{J} = \sigma \vec{E}] \tag{iv}$$

Taking curl of Eq. (iii), we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left[-\mu \frac{\partial \vec{H}}{\partial t} \right]$$

$$= -\mu \left[\vec{\nabla} \times \frac{\partial \vec{H}}{\partial t} \right]$$

$$= -\mu \frac{\partial}{\partial t} [\vec{\nabla} \times \vec{H}]$$

$$= -\mu \frac{\partial}{\partial t} \left[\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \right] \qquad \text{[using Eq. (iv)]}$$

or
$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Also, $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{E} = -\nabla^2 \vec{E}$ $\qquad [\therefore \vec{\nabla} \cdot \vec{E} = 0$ from Eq. (i)]

$\therefore$
$$-\nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

or
$$\nabla^2 \vec{E} = +\mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \tag{v}$$

Eq. (v) is the electromagnetic wave equation for the electric field $\vec{E}$ in a conducting medium.

In case of non-conducting medium $\sigma = 0$. Hence, from Eq. (v)

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \tag{vi}$$

Eqs (v) and (vi) show that the term $\mu\sigma \frac{\partial \vec{E}}{\partial t}$ is the dissipative term which allows the current to flow through the medium due to the appearance of conductivity σ.

Now, by taking curl of Eq. (iv), we obtain

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\times\left[\sigma\vec{E}+\varepsilon\frac{\partial\vec{E}}{\partial t}\right]$$

$$= \sigma(\vec{\nabla}\times\vec{E}) + \varepsilon\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

$$= \sigma\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) + \varepsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) \qquad \text{[Using Eq. (iii)]}$$

$$= -\mu\sigma\frac{\partial\vec{H}}{\partial t} - \varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2}$$

$\because$ $$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{H}) - (\vec{\nabla}\cdot\vec{\nabla})\vec{H}$$

$$= 0 - \nabla^2\vec{H} \qquad [\therefore\ \vec{\nabla}\cdot\vec{H} = 0 \text{ from Eq. (ii)}]$$

$\therefore$ $$-\nabla^2\vec{H} = -\mu\sigma\frac{\partial\vec{H}}{\partial t} - \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}$$

or $$\nabla^2\vec{H} = \mu\sigma\frac{\partial\vec{H}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} \qquad \text{(vii)}$$

Equations (vii) represents the electromagnetic wave equation for magnetic field ($\vec{H}$) in conducting medium.

In case of non-conducting medium $\sigma = 0$, the wave equation takes the form

$$\nabla^2\vec{H} = \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} \qquad \text{(viii)}$$

Equations (vii) show that the term $\mu\sigma\frac{\partial\vec{H}}{\partial t}$ is the dissipative term which allows the current to flow through the conducting medium.

11.26.1 Solution of Wave Equation

Equations (v) and (vii) are called inhomogeneous wave equation due to the presence of dissipative term. These equations in one-dimension (along z-axis) are written as

$$\frac{\partial^2\vec{E}}{\partial z^2} = \mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \qquad \text{(ix)}$$

$$\frac{\partial^2\vec{H}}{\partial z^2} = \mu\sigma\frac{\partial\vec{H}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} \qquad \text{(x)}$$

We assume the following plane wave solutions to the above equations

$$\vec{E}(z, t) = \vec{E}_0\, e^{i(kz - \omega t)} \qquad \text{(xi)}$$

and $$\vec{H}(z, t) = \vec{H}_0\, e^{i(kz - \omega t)} \qquad \text{(xii)}$$

The use of Eq. (xi) in Eq. (ix) and Eq. (xii) in Eq. (x) leads

$$k^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega \qquad \text{(xiii)}$$

This relation is called dispersion relation that governs the electromagnetic wave propagation in a conducting medium. Equation (xiii) suggests that the quantity k, i.e., wave number, will be a complex quantity. So, we assume

$$k = k_r + ik_i \qquad \text{(xiv)}$$

With this the fields $\vec{E}$ and $\vec{H}$ become

$$\vec{E}(z,t) = \vec{E}_0 \bar{e}^{k_i z} \cdot e^{i(k_r z - \omega t)} \tag{xv}$$

and
$$\vec{H}(z,t) = \vec{H}_0 \bar{e}^{k_i z} e^{i(k_r z - \omega t)} \tag{xvi}$$

11.26.2 Skin Depth

The expressions (xv) and (xvi) follow that the amplitude of the electric field $\vec{E}$ is $E_0 \bar{e}^{k_i z}$ and that of the magnetic field $\vec{H}$ is $H_0 \bar{e}^{k_i z}$. Hence the amplitude of electromagnetic wave will decrease exponentially as it propagates through the conductor. This is called the attenuation of the wave and the distance through which the amplitude is reduced by a factor of $1/e$ is called the skin depth or penetration depth δ. At $z = \delta$, the amplitude is E_0/e. Hence

$$E_0 \bar{e}^{k_i \delta} = E_0/e \tag{i}$$

This gives the skin depth as

$$\delta = \frac{1}{k_i} \tag{ii}$$

Equation (ii) shows that the imaginary part of the wave number k is the measure of the skin depth. However, the real part k_r of k determines the wave propagation characteristics in the following manner.

Wavelength $\lambda = 2\pi/k_r$ (iii)

Phase velocity $v = \omega/k_r$ (iv)

Refractive index $n = \frac{c}{v} = \frac{ck_r}{\omega}$ (v)

By putting $k = k_r + i\,k_i$ in Eq. (xiii) we obtain

$$k_r = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]} \tag{vi}$$

and
$$k_i = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} \tag{vii}$$

For good conductors, $\sigma >> \omega\varepsilon$. This condition when put in Eqs. (vi) and (vii) gives

$$k_r = k_i = \omega\sqrt{\frac{\mu\varepsilon}{2}\frac{\sigma}{\omega\varepsilon}}$$

$\Rightarrow$
$$k_r = k_i = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma} \tag{viii}$$

Hence, the skin depth is given by

$$\delta = \frac{1}{\sqrt{\pi f \mu\sigma}} \tag{ix}$$

Since δ is inversely proportional to f, which is the frequency of electromagnetic wave, high frequency waves are found to penetrate less into the conductor. Also, the penetration will be less in the medium having high conductivity σ. Ideally an electromagnetic wave will not penetrate into a perfect conductor as $\sigma = \infty$.

11.26.3 Phase Relationship of $\vec{E}$ and $\vec{B}$ Fields

In view of the imaginary wave number k_i, we can also make another observation with regard to the phase difference between $\vec{E}$ and $\vec{H}$ vectors. If we take the direction of $\vec{E}$ field along the x-axis, then $\vec{\nabla}\times\vec{E} = -\mu\dfrac{\partial\vec{H}}{\partial t}$ gives

$$\vec{H}(z,t) = \hat{j}\frac{kE_0}{\omega\mu}e^{-k_i z}e^{i(k_r z-\omega t)} \tag{x}$$

Since k is the complex quantity, it can be represented as $k = k'e^{i\theta_k}$. Here $k' = \sqrt{k_r^2 + k_i^2}$ and $\theta_k = \tan^{-1}\left(\dfrac{k_i'}{k_r}\right)$. Then the expression of $\vec{H}$ becomes

$$\vec{H}(z,t) = \hat{j}\frac{k'E_0}{\omega\mu}e^{-k_i z}e^{i(k_r z-\omega t+\theta_k)} \tag{xi}$$

A comparison of Eq. (xi) with $\vec{E}(z,t) = \hat{i}E_0 e^{-k_i z}e^{i(k_r z-\omega t)}$ reveals that the electric field and magnetic field vectors do not remain in phase when electromagnetic wave propagates in a conducing medium. This is in contrast to the cases of vacuum and dielectrics.

11.27 ELECTROMAGNETIC ENERGY DENSITY LO11

It can be proved that the work done in assembling a static charge distribution (number n) against the Coulomb repulsion of like charges is

$$W_E = \frac{1}{2}\sum_{j=1}^{n} q_j V(\vec{r}_j) \tag{i}$$

For a volume charge density ρ, this equation takes the form

$$W_E = \frac{1}{2}\int \rho V dX \tag{ii}$$

Here dX is the volume element. Now the above equation can be written in terms of the resulting electric field $\vec{E}$ if we apply Gauss's law $\varepsilon_0\vec{\nabla}\cdot\vec{E} = \rho$ and mention the potential V in terms of $\vec{E}$. This yields the following relation where the integration is over all the space containing the whole charge distribution.

$$W_E = \frac{\varepsilon_0}{2}\int E^2 dX \tag{iii}$$

The same way we can derive an expression for the work done on a unit charge against the back emf in one trip around the circuit, as follows

$$W_B = \frac{1}{2\mu_0}\int B^2 dX \tag{iv}$$

Here B is the resulting magnetic field. Eqs. (iii) and (iv) suggest that the total energy stored in electromagnetic field would be

$$W_{EM} = \frac{1}{2}\int\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)dX \tag{v}$$

Therefore, the electromagnetic energy density can be obtained as

$$U_{EM} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) \tag{vi}$$

For a monochromatic plane electromagnetic wave, $B = \frac{E}{c}$, where $c \left(\equiv \frac{1}{\sqrt{\mu_0 \varepsilon_0}}\right)$ is the speed of light. Hence, it can be seen that the contribution of magnetic field $\vec{B}$ to the energy density is the same as that of the electric field $\vec{E}$.

11.28 POYNTING VECTOR AND POYNTING THEOREM — LO12

The electromagnetic waves carry energy when they propagate and there is an energy density associated with both the electric and magnetic fields.

The amount of energy flowing through unit area, perpendicular to the direction of energy propagation per unit time, i.e., the rate of energy transport per unit area, is called the *poynting vector*. It is also termed as instantaneous energy flux density and is represented by $\vec{S}$ (or $\vec{P}$, sometimes). Mathematically it is defined

$$\vec{S} = \vec{E} \times \vec{H}$$

where $\vec{E}$ and $\vec{H}$ represent the instantaneous values of the electric and magnetic field vectors. This is clear that the rate of energy transport $\vec{S}$ is perpendicular to both $\vec{E}$ and $\vec{H}$ and is in the direction of propagation of the wave, as $\vec{E} \times \vec{H}$ is in the direction of $\vec{k}$. Since the poynting vector represents the rate of energy transport per unit area, its units are W/m^2.

Derivation: We can calculate the energy density carried by electromagnetic waves with the help of Maxwell's equations given below.

$$\text{div } \vec{D} = 0 \tag{i}$$

$$\text{div } \vec{B} = 0 \tag{ii}$$

$$\text{curl } \vec{E} = \frac{-\partial \vec{B}}{\partial t} \tag{iii}$$

$$\text{curl } \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{iv}$$

Take scalar (dot) product of Eq. (iii) and Eq. (iv) with $\vec{H}$ and $\vec{E}$ respectively, i.e.,

$$\vec{H} \cdot \text{curl } \vec{E} = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \tag{v}$$

and

$$\vec{E} \cdot \text{curl } \vec{H} = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \tag{vi}$$

Subtract Eq. (vi) from Eq. (v), i.e.,

$$\begin{aligned}\vec{H} \cdot \text{curl } \vec{E} - \vec{E} \cdot \text{curl } \vec{H} &= -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \vec{J} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \\ &= -\left(\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right) - \vec{E} \cdot \vec{J}\end{aligned}$$

or
$$\operatorname{div}(\vec{E}\times\vec{H}) = -\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right) - \vec{E}\cdot\vec{J} \tag{vii}$$

$$[\because \operatorname{div}(\vec{A}\times\vec{B}) = \vec{B}\cdot\operatorname{curl}\vec{A} - \vec{A}\cdot\operatorname{curl}\vec{B}]$$

Using the relations $\vec{B} = \mu\vec{H}$ and $\vec{D} = \varepsilon\vec{E}$, we can get

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \vec{E}\cdot\frac{\partial}{\partial t}(\varepsilon\vec{E}) = \frac{1}{2}\varepsilon\frac{\partial}{\partial t}(E)^2 = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{E}\cdot\vec{D}\right) \qquad [\because E^2 = \vec{E}\cdot\vec{E}]$$

$$\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} = \vec{H}\cdot\frac{\partial}{\partial t}(\mu\vec{H}) = \frac{1}{2}\mu\frac{\partial}{\partial t}(H)^2 = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{H}\cdot\vec{B}\right) \qquad [\because H^2 = \vec{H}\cdot\vec{H}]$$

Now Eq. (vii) can be written as

$$\operatorname{div}(\vec{E}\times\vec{H}) = \frac{\partial}{\partial t}\left[\frac{1}{2}(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D})\right] - \vec{E}\cdot\vec{J}$$

or
$$\vec{E}\cdot\vec{J} = \frac{\partial}{\partial t}\left[\frac{1}{2}(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D})\right] - \operatorname{div}(\vec{E}\times\vec{H}) \tag{viii}$$

Integrating Eq. (viii) over a volume V enclosed by a surface S, we get

$$\int_v \vec{E}\cdot\vec{J}dV = -\int_v\left[\frac{\partial}{\partial t}\left\{\frac{1}{2}(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D})\right\}\right]dV - \int_v \operatorname{div}(\vec{E}\times\vec{H})\,dV$$

or
$$\int_v \vec{E}\cdot\vec{J}dV = -\frac{\partial}{\partial t}\int_v\left[\left(\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right)\right]dV - \oint_s(\vec{E}\times\vec{H})\cdot d\vec{S} \tag{ix}$$

$$[\because \quad \vec{B} = \mu\vec{H}, \vec{D} = \varepsilon\vec{E} \quad \text{and} \quad \int_v \operatorname{div}(\vec{E}\times\vec{H})dV = \oint_s(\vec{E}\times\vec{H})\cdot d\vec{S}]$$

Eq. (ix) can also be written as

$$\int_v \vec{E}\cdot\vec{J}\,dV = -\frac{\partial}{\partial t}\int_v\left[\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right]dV - \oint_s(\vec{E}\times\vec{H})\cdot d\vec{S} \tag{x}$$

Interpretation

(a) $\int_v(\vec{E}\cdot\vec{J})\,dV$: This term represents the rate of energy transferred into the electromagnetic field through the motion of charges in the volume V, i.e., the total power dissipated in a volume V.

(b) $\frac{\partial}{\partial t}\int_v\left[\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right]dV$: The terms $\frac{1}{2}\mu H^2$ and $\frac{1}{2}\varepsilon E^2$ represent the energy stored in electric and magnetic fields respectively and their sum will be equal to the total energy stored in electromagnetic field. Therefore, this total expression represents the rate of decrease of energy stored in volume V due to electric and magnetic fields.

(c) $\int_s(\vec{E}\times\vec{H})\cdot d\vec{S}$: This term represents the amount of electromagnetic energy crossing the closed surface per second or the rate of flow of outward energy through the surface S enclosing volume V. The vector $(\vec{E}\times\vec{H})$ is known as the *poynting vector* $\vec{S}$ or $\vec{S} = (\vec{E}\times\vec{H})$.