Computer Systems 1
Lecture 2

Binary and Two's Complement Numbers

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Topics

Quiz

- There will be a quiz each week
- You can do it any time starting after the Thursday lecture, and must finish by Friday in the following week
- The quiz is on Moodle
- 10% of the total assessment comes from the quiz average
- You are encouraged to refer to the course documents as you do the quiz
- Read the course documents—don't ignore them and use random Google searches instead!

Number representation

There are several kinds of number: each has its own representation using bits

- **Integers**
  - Nonnegative integers use binary

    23   0   459   (must be at least 0)
  - Signed integers use two's complement

    48   -239 (can be negative)
- **Reals**
  - Approximate real numbers use floating point

    3.14   2.5e9   -351.02638134

These are the most common number representations, but computer hardware and modern programming languages support several others too.

Binary numbers

Binary numbers

- Binary representation uses a word of k bits to represent a nonnegative integer between 0 and $2^k - 1$
- Binary numbers cannot be negative — there are other ways to represent negative numbers (two's complement is most widely used)
- People often use terminology loosely, and say "binary" when they mean "word of bits"
- Binary representation is similar to decimal, but it uses base 2 instead of base 10

Decimal number representation

$$
\begin{aligned}
2053_{10} &= 2 \times 10^{3} + 0 \times 10^{2} + 5 \times 10^{1} + 3 \times 10^{0} \\
&= 2000 + 0 + 50 + 3 \\
&= 2053_{10}
\end{aligned}
$$

Column values are powers of 10

10^{0}	$=$	1	weight of rightmost digit
10^{1}	$=$	10	
10^{2}	$=$	100	
10^{3}	$=$	1000	weight of leftmost digit

Converting binary to decimal

Binary number representation

$$
\begin{array}{rcccccccc}
1001_2 & = & 1 \times 2^{3} & + & 0 \times 2^{2} & + & 0 \times 2^{1} & + & 1 \times 2^{0} \\
 & = & 8 & + & 0 & + & 0 & + & 1 \\
 & = & 9_{10} & & & & & &
\end{array}
$$

This is how to convert a binary number to decimal.

Column values are powers of 2

$$
\begin{array}{rcll}
2^{0} & = & 1 & \text{weight of rightmost bit} \\
2^{1} & = & 2 & \\
2^{2} & = & 4 & \\
2^{3} & = & 8 & \text{weight of leftmost bit}
\end{array}
$$

The powers of 2

It's useful to know the value of each bit position!

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
128	64	32	16	8	4	2	1

Tip

Don't memorise the table! Construct it whenever you need it. Just write down 1, and keep adding another value to the left by doubling the previous value.

Binary to decimal

Convert binary number 10011 to decimal

2^4	2^3	2^2	2^1	2^0
16	8	4	2	1
1	0	0	1	1

$16 + 2 + 1 = 19$

Converting decimal to binary

Convert 203_{10} to 8-bit binary

- When we convert a decimal number x to binary, we need to
  - Know the word size k of the result
  - Check that x will fit in the word: $0 \leq x \leq 2^k - 1$
- Example: we will convert 203 to an 8-bit binary number
- Check: $0 \leq 203 \leq 255$.
- This holds, so we can indeed represent 203 in an 8-bit word.

(1) Calculate the 128 column, remainder is 203

$203 \geq 128$ so enter 1

128	64	32	16	8	4	2	1
1							

The new remainder is $203 - 128 = 75$

(2) Calculate the 64 column, remainder is 75

$75 \geq 64$ so enter 1

128	64	32	16	8	4	2	1
1	1						

The new remainder is $75 - 64 = 11$

(3) Calculate the 32 column, remainder is 11

$11 \geq 32$ is false so enter 0

128	64	32	16	8	4	2	1
1	1	0					

The new remainder is still 11

(4) Calculate the 16 column, remainder is 11

$11 \geq 16$ is false so enter 0

128	64	32	16	8	4	2	1
1	1	0	0				

The new remainder is still 11

(5) Calculate the 8 column, remainder is 11

$11 \geq 8$ so enter 1

128	64	32	16	8	4	2	1
1	1	0	0	1			

The new remainder $11 - 8 = 3$

(6) Calculate the 4 column, remainder is 3

$3 \geq 4$ is false so enter 0

128	64	32	16	8	4	2	1
1	1	0	0	1	0		

The new remainder still 3

(7) Calculate the 2 column, remainder is 3

$3 \geq 2$ so enter 1

128	64	32	16	8	4	2	1
1	1	0	0	1	0	1	

The new remainder is $3 - 2 = 1$

(8) Calculate the 1 column, remainder is 1

$1 \geq 1$ so enter 1

128	64	32	16	8	4	2	1
1	1	0	0	1	0	1	1

The new remainder is $1 - 1 = 0$ and we're finished.

Check the result: convert it back to decimal

$$
\begin{aligned}
11001011_2 &= 2^7 + 2^6 + 2^3 + 2^1 + 2^0 \\
&= 128 + 64 + 8 + 2 + 1 \\
&= 203
\end{aligned}
$$

- It's easier to convert binary to decimal, so it's worth checking!
- Also, note that when you convert decimal to binary, the remainder in the 1 column must be 0. If not, you've made a mistake.

Binary addition

Binary addition

- You can add two binary numbers x and y the same way as adding decimal numbers
- Write one number above the other
- Work through each column, from right to left
- In each column, add the bit from x, the bit from y, and the carry from the column to the right.
- This gives the sum bit s for the column, and the carry output which goes to the left.

Adding bits

Calculate $x + y + z$ giving 2-bit result c, s (c is carry, s is sum)

x	y	z	c	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

- The sum is 1 if an odd number of inputs are 1
- The carry is 1 if two or more inputs are 1
- Think of carry, sum as a 2-bit binary number giving the result

Example

- We'll add $x + y$
- $x = 0010\ 1101 = 32 + 8 + 4 + 1 = 45$
- $y = 0100\ 1110 = 64 + 8 + 4 + 2 = 78$
- The correct answer is $x + y = 45 + 78 = 123$

Setting up the problem

		128	64	32	16	8	4	2	1
c									0
x		0	0	1	0	1	1	0	1
y		0	1	0	0	1	1	1	0
s									

(1) Add the weight 1 column

		128	64	32	16	8	4	2	1
c								0	0
x		0	0	1	0	1	1	0	1
y		0	1	0	0	1	1	1	0
s									1

(2) Add the weight 2 column

		128	64	32	16	8	4	2	1
c							0	0	0
x		0	0	1	0	1	1	0	1
y		0	1	0	0	1	1	1	0
s								1	1

(3) Add the weight 4 column

		128	64	32	16	8	4	2	1
c						1	0	0	0
x		0	0	1	0	1	1	0	1
y		0	1	0	0	1	1	1	0
s							0	1	1

(4) Add the weight 8 column

		128	64	32	16	8	4	2	1
c					1	1	0	0	0
x		0	0	1	0	1	1	0	1
y		0	1	0	0	1	1	1	0
s						1	0	1	1

(5) Add the weight 16 column

		128	64	32	16	8	4	2	1
c				0	1	1	0	0	0
x		0	0	1	0	1	1	0	1
y		0	1	0	0	1	1	1	0
s					1	1	0	1	1

(6) Add the weight 32 column

		128	64	32	16	8	4	2	1
c			0	0	1	1	0	0	0
x		0	0	1	0	1	1	0	1
y		0	1	0	0	1	1	1	0
s				1	1	1	0	1	1

(7) Add the weight 64 column

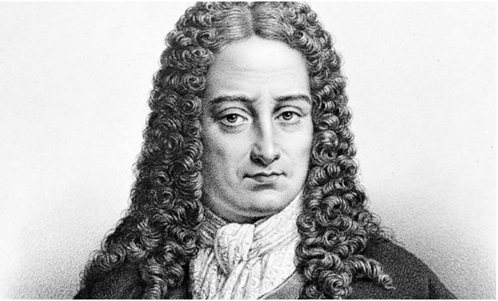
		128	64	32	16	8	4	2	1
c		0	0	0	1	1	0	0	0
x		0	0	1	0	1	1	0	1
y		0	1	0	0	1	1	1	0
s			1	1	1	1	0	1	1

(8) Add the weight 128 column

		128	64	32	16	8	4	2	1
c	0	0	0	0	1	1	0	0	0
x		0	0	1	0	1	1	0	1
y		0	1	0	0	1	1	1	0
s		0	1	1	1	1	0	1	1

The result is $0111\ 1011 = 64 + 32 + 16 + 8 + 2 + 1 = 123$ which is the right answer!

The discoverer of binary numbers



- Gottfried Wilhelm Leibniz (1646–1716)
- German mathematician and philosopher
- Invented the calculus (Isaac Newton also invented calculus independently around the same time) and the binary number system (about 1680)

Two's complement

Two's complement

- Binary cannot represent negative numbers!
- Two's complement is a method for representing integers that can be negative or positive
- Recall that a k-bit word has 2^k distinct values
  - In binary, all those values represent nonnegative numbers from 0 to $2^k - 1$
  - In two's complement, half of those values represent negative integers, and half represent nonnegative integers
  - The range is -2^{k-1} to $2^{k-1} - 1$

Sign bit

- The sign bit is the lefmost bit of a two's complement number
  - If the sign bit is 1, the number is negative (< 0)
  - If the sign bit is 0, the number is nonnegative (≥ 0)
  - If all the bits are 0, the number is 0
- 0101 1100 is positive (> 0)
- 1001 1001 is negative (< 0)
- 0000 0000 is the integer 0

How to interpret a two's complement number

- There are many ways to convert a two's complement word to/from decimal
- Our approach is based on how computers actually work, and is the easiest for humans to use:
  - We have an algorithm to negate *any* two's complement number
  - If a two's complement number is nonnegative, it acts just like a binary number
  - If it is negative, just negate it and then use binary conversion

Negating a number

Negating a two's complement number x

Two steps:

1. Invert each bit (replace 0 by 1, replace 1 by 0)
2. Add 1

The result is the representation of $-x$

Example: -36 in two's complement

x	0010 0100
invert:	1101 1011
add 1:	1101 1100

Two's complement to decimal

Decoding a two's complement number

- If the sign bit is 0, then treat it just like a binary number
- If the sign bit is 1, then negate it and treat the result like a binary number

0010 0110 (is nonnegative)
= 32 + 4 + 2
= 38

1011 1000 (is negative)
0100 0111 (invert)
0100 1000 (add 1)
= 64 + 8 = 72
so 1011 1000 = -72

Hexadecimal: easier notation for writing words

- When working with machine language and assembly language, we frequently need to write down the values of words
- This is normally done using hexadecimal notation
- It's base 16 (while binary is base 2, and decimal is base 10)
- Why use base 16?
- You can break a long word of bits into groups of 4 bits, and replace each group by the corresponding hex digit

Table of 4-bit numbers

word	hex value	bin value	tc value
0000	0	0	0
0001	1	1	1
0010	2	2	2
0011	3	3	3
0100	4	4	4
0101	5	5	5
0110	6	6	6
0111	7	7	7
1000	8	8	-8
1001	9	9	-7
1010	a	10	-6
1011	b	11	-5
1100	c	12	-4
1101	d	13	-3
1110	e	14	-2
1111	f	15	-1

Why use use hex?

- Here's a 16 bit word: 0011110000101111
- That's bad enough, but we'll usually have about 20 of these to look at at a time
- And if you look at current commercial computers, the words are 64 bits; you would need to work with a couple dozen of these at a time: 1000111010100111010011110000101111110110001010001010010101010
- Hex representation of 0011110000101111 is 3c2f
- It's easier with hex!

Arithmetic with hex

- It's easy to add hex numbers
- It is extremely rare to multiply or divide them — you will probably never need to do this
- To add two hex numbers, write them one above the other, and add by columns
- Just remember what each hex digit means: $c + 2$ means $12 + 2$, which is 14, and that's hex digit e
- If the sum in a column is greater than 16, you add a carry of 1 to the column to the left: $004a + 0009 = 0053$

A couple of tips

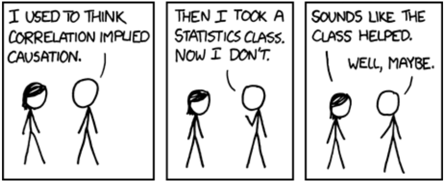
- We will write hex numbers with a dollar sign in front:
  - 23 is decimal: $2 \times 10 + 3$, pronounced "twenty three"
  - \$0023 is hex: $2 \times 16 + 3 = 35$, pronounced "zero zero two three"
  - Professionals pronounce hex numbers by saying every digit, including leading zeros, and never use teens, twenty, hundreds, etc for hex

A word has many meanings

- There are many ways to interpret the meaning of a word of bits
- Binary, two's complement, floating point, character, and many more
- A word of bits has no inherent meaning
- It has one meaning if interpreted as binary, another if interpreted as two's complement, and so on
- It is meaningless to ask "what does 1010 represent?"
- We can ask
  - "what does 1010 represent as a binary number?" (10)
  - "what does 1010 represent as a two's complement number?" -6)

To do

- Review the slides and work through the examples
- Quiz 1 on Moodle. This is assessed. Deadline: Friday next week
- No lab this week; the first lab is next week
- Check Moodle for schedule, documents, announcements
- Over the weekend, the lab sheet for next week will be posted on Moodle
- It contains problems about the first two lectures
- Solve the problems
- Discuss these at your lab next week



https://xkcd.com/552/