Discrete Laplace-Beltrami

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1 Scalar Fields on Discrete Manifolds

Let $G = (V, E)$ be a graph representing a discrete manifold M. The set of vertices $V = \{p_1, p_2, \ldots, p_n\}$ represents points sampled from the manifold in the ambient space $\mathbb{R}^3$. The edges $E \subseteq V \times V$ represent the connectivity or local neighborhood structure of the manifold. We define a scalar field on the nodes of the graph as a function $\phi : V \to \mathbb{R}$, which assigns a real value ϕ_i to each node p_i.

To accurately reflect the geometry of the underlying manifold, we introduce a weight matrix W, where $w_{ij} > 0$ if $(i, j) \in E$ and $w_{ij} = 0$ otherwise. These weights depend on the Euclidean distance between points.

1.1 Laplace-Beltrami Operator

The Graph Laplacian is a discrete linear operator that serves as an approximation of the continuous Laplace-Beltrami operator Δ_M on a manifold. For a weighted graph, the Laplacian acting on a scalar field ϕ at a specific node p_i is defined as:

$$(\Delta\phi)_i = \sum_{j \in \mathcal{N}(i)} w_{ij}(\phi_i - \phi_j)$$

where $\mathcal{N}(i)$ denotes the set of neighbors of node i. This operator captures the local difference between the value at p_i and the weighted average of its neighbors. In a discrete setting, this is fundamental for solving heat diffusion problems, performing mesh smoothing, and spectral manifold analysis.

1.2 Discrete Gradient

The discrete gradient is an approximation of the surface gradient. Unlike the Laplacian, which is node-based, the gradient is defined over the edges of the graph. It represents the variation of the scalar field along the direction established by each edge (i, j):

$$(\nabla\phi)_{ij} = \sqrt{w_{ij}}(\phi_j - \phi_i)$$

This formulation ensures that the gradient captures the rate of change of ϕ scaled by the edge weight. By defining the gradient on edges, we can view it as a differential 1-form, providing a link between the scalar values at nodes and the flow across the discrete structure.

1.3 Gradient Norm and Dirichlet Energy

The squared norm of the gradient provides a measure of the "smoothness" of the scalar field. When integrated (or summed) over the entire manifold, it coincides with the Dirichlet Energy, which quantifies the total spatial variation of the function. For a specific node p_i, the local squared norm is defined as:

$$\|\nabla\phi\|_i^2 = \frac{1}{2} \sum_{j \in \mathcal{N}(i)} w_{ij}(\phi_j - \phi_i)^2$$

A low Dirichlet energy indicates that the scalar field is smooth and changes slowly across the manifold, while a high energy suggests significant oscillations. This functional is a cornerstone in variational problems, such as image denoising on surfaces and minimal surface computations.