Riemannian Geometry of a Paraboloid: Metric, Christoffel Symbols, and Curvatures

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1 Parametric surface

Let S be a surface in $\mathbb{R}^3$ defined by the paraboloid

$$f(x,y) = ax^2 + by^2 + cxy + dx + ey + f$$

Since f is a polynomial function of two variables, $f \in C^\infty(\mathbb{R}^2)$. Therefore, we can consider the differentiable parametrization $r : \mathbb{R}^2 \to \mathbb{R}^3$

$$r(u,v) = \langle u, v, f(u,v) \rangle$$

which is injective and has maximal rank at all points. This ensures that S is a 2-dimensional differentiable manifold (2-manifold) immersed in $\mathbb{R}^3$.

2 Tangent vectors

The tangent vectors to the surface, denoted as r_u and r_v, may be calculated as follows:

$$r_u(u,v) = \frac{\partial r}{\partial u} = \langle 1, 0, f_u(u,v) \rangle, \quad f_u(u,v) = \frac{\partial f}{\partial u} = 2au + cv + d$$
$$r_v(u,v) = \frac{\partial r}{\partial v} = \langle 0, 1, f_v(u,v) \rangle, \quad f_v(u,v) = \frac{\partial f}{\partial v} = 2bv + cu + e$$

Substituting these partial derivatives yields the explicit vector forms:

$$r_u(u,v) = \langle 1, 0, 2au + cv + d \rangle$$
$$r_v(u,v) = \langle 0, 1, 2bv + cu + e \rangle$$

Recall that for a point $p = (x, y, z)$ where the surface is defined by $z = f(x,y)$, we parameterize such that $u = x$ and $v = y$. Consequently, the vectors tangent to the surface S at the point p are defined by $r_u(u,v)$ and $r_v(u,v)$.

In differential geometry, it is standard practice to define the tangent space T_pM as the span of the coordinate basis vectors: $T_pM = \text{span}\left\{ \frac{\partial}{\partial x^\mu}\Big|_p, \frac{\partial}{\partial x^\nu}\Big|_p \right\}$. In this specific context, the manifold M corresponds to the surface S, and the tangent vectors are identified as $r_u = \frac{\partial}{\partial x^\mu}\Big|_p$ and $r_v = \frac{\partial}{\partial x^\nu}\Big|_p$.

Note that the unit normal vector is given by the cross product:

$$\hat{n}(u,v) = \frac{r_u(u,v) \times r_v(u,v)}{||r_u(u,v) \times r_v(u,v)||}$$

3 Metric Tensor - First Fundamental Form I

We calculate the metric tensor for each point $p \in S$ via the inner product of the tangent vectors as follows:

$$g_{\mu\nu}(p) = \left\langle \frac{\partial}{\partial x^\mu}\bigg|_p, \frac{\partial}{\partial x^\nu}\bigg|_p \right\rangle = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

Here, the components of the metric tensor correspond precisely to the coefficients of the first fundamental form:

$$\begin{aligned} E &= r_u \cdot r_u = \langle 1, 0, f_u \rangle \cdot \langle 1, 0, f_u \rangle = 1 + f_u^2 \\ F &= r_u \cdot r_v = \langle 1, 0, f_u \rangle \cdot \langle 0, 1, f_v \rangle = f_u f_v \\ G &= r_v \cdot r_v = \langle 0, 1, f_v \rangle \cdot \langle 0, 1, f_v \rangle = 1 + f_v^2 \end{aligned}$$

Consequently, the metric tensor takes the following matrix representation:

$$g_{\mu\nu}(p) = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 + f_u^2 & f_u f_v \\ f_u f_v & 1 + f_v^2 \end{pmatrix}$$

It is worth noting that in differential geometry, particularly in relativistic contexts, the indices run over $\mu, \nu \in \{0, 1\}$. In our parameterization, we identify the coordinates as $x^0 = u$ and $x^1 = v$.

3.1 Metric Tensor Derivatives

The derivatives of the metric tensor are defined by the tensor $\partial_\alpha g_{\mu\nu}(p)$, where $\alpha \in \{0, 1\}$. The differential operator is denoted as $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$. In our specific coordinate system, we define $x^0 = u$ and $x^1 = v$.

Recapitulation of f derivatives

The following partial derivatives constitute the components required for the metric tensor differentiation. It is sufficient to substitute these values into the corresponding matrices derived below.

$$\begin{aligned} f_u &= 2au + cv + d, & f_{uu} &= 2a, \\ f_v &= 2bv + cu + e, & f_{vv} &= 2b, \\ & & f_{vu} &= f_{uv} = c. \end{aligned}$$

Derivative with respect to u ($\alpha = 0$)

The first index corresponds to the variable u. The derivative of the metric tensor is given by:

$$\partial_0 g_{\mu\nu} = \frac{\partial}{\partial u} \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 2 f_u f_{uu} & f_{uu} f_v + f_u f_{vu} \\ f_{uu} f_v + f_u f_{vu} & 2 f_v f_{vu} \end{pmatrix}$$

Detailed component breakdown:

$$\begin{aligned} \frac{\partial E}{\partial u} &= \frac{\partial}{\partial u}(1 + f_u^2) = 2 f_u f_{uu} \\ \frac{\partial F}{\partial u} &= \frac{\partial}{\partial u}(f_u f_v) = f_{uu} f_v + f_u f_{vu} \\ \frac{\partial G}{\partial u} &= \frac{\partial}{\partial u}(1 + f_v^2) = 2 f_v f_{vu} \end{aligned}$$

Derivative with respect to v ($\alpha = 1$)

The second index corresponds to the variable v. The derivative of the metric tensor is given by:

$$\partial_1 g_{\mu\nu} = \frac{\partial}{\partial v}\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 2f_u f_{uv} & f_{uv}f_v + f_u f_{vv} \\ f_{uv}f_v + f_u f_{vv} & 2f_v f_{vv} \end{pmatrix}$$

Detailed component breakdown:

$$\begin{aligned} \frac{\partial E}{\partial v} &= \frac{\partial}{\partial v}(1 + f_u^2) = 2f_u f_{uv} \\ \frac{\partial F}{\partial v} &= \frac{\partial}{\partial v}(f_u f_v) = f_{uv}f_v + f_u f_{vv} \\ \frac{\partial G}{\partial v} &= \frac{\partial}{\partial v}(1 + f_v^2) = 2f_v f_{vv} \end{aligned}$$

4 Christoffel Symbols

With the metric tensor and its derivatives established, the Christoffel symbols are determined by the following relation:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right)$$

where $g^{\sigma\lambda}$ denotes the inverse of the metric tensor. It is important to note that the Einstein summation convention applies to the index λ.

The inverse metric tensor $g^{\sigma\lambda}(p)$ is given by:

$$g^{\sigma\lambda}(p) = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

5 Second Fundamental Form II

The Second Fundamental Form can be expressed in matrix notation as follows:

$$\mathbf{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

Based on the tangent and normal vectors calculated in Section 2:

$$r_u = (1, 0, f_u), \quad r_v = (0, 1, f_v) \quad \text{and} \quad \hat{n} = \frac{r_u \times r_v}{\|r_u \times r_v\|}$$

Then, the coefficients are defined by the following inner products:

$$\begin{aligned} L &= r_{uu} \cdot \hat{n} \qquad & r_{uu} &= (0, 0, f_{uu}) \\ M &= r_{uv} \cdot \hat{n} \qquad & r_{uv} &= (0, 0, f_{uv}) \\ N &= r_{vv} \cdot \hat{n} \qquad & r_{vv} &= (0, 0, f_{vv}) \end{aligned}$$

6 Curvature

In this section, the different types of curvature to be computed will be briefly defined. Since we are working on a surface, some of them are significantly simplified.

6.1 Gaussian curvature

Gaussian curvature of a surface in $\mathbb{R}^3$ can be expressed as the ratio of the determinants of the second and first fundamental forms **II** and **I**:

$$K = \frac{\det(\mathbf{II})}{\det(\mathbf{I})} = \frac{LN - M^2}{EG - F^2}$$

6.2 Scalar curvature

Scalar curvature is the trace of the Ricci tensor (the final contraction). It is a single number assigned to every point on the surface. However, the following relationship with Gaussian curvature exists (exclusively in 2-manifolds):

$$R = 2K$$

6.3 Ricci curvature tensor

The Ricci tensor is the contraction (a form of averaging) of the Riemann tensor. In 2-manifolds, the Ricci tensor provides no new information; it is simply proportional to the metric scaled by the curvature.

$$R_{\mu\nu} = K g_{\mu\nu}$$

The formal definition of the Ricci tensor is given in terms of the Christoffel symbols:

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\lambda\rho}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\lambda\nu}\Gamma^\lambda_{\mu\rho}$$

Recall that:

$$-\partial_\nu \Gamma^\rho_{\mu\rho} = \partial_\mu \ln \sqrt{\det(g)}$$

All of this is equivalent to contracting the first upper index (ρ) with the second lower index (μ) of the Riemann tensor ($R^\lambda_{\mu\lambda\nu}$).

6.4 Riemann curvature tensor

In higher dimensions (such as in 4D General Relativity), the Riemann tensor possesses 20 independent components and is highly complex. However, in 2D, due to the symmetries of the tensor, it has only one independent component (which is essentially K).

$$R_{\rho\sigma\mu\nu} = K(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$$

Due to the skew-symmetry properties of the tensor ($R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\nu\mu}$), the complete set of non-zero components is:

$$R_{1212} = -R_{1221} = -R_{2112} = R_{2121} = K \det(g)$$

Given the covariant components, the mixed tensor is obtained by

$$R^\rho{}_{\sigma\mu\nu} = g^{\rho\lambda} R_{\lambda\sigma\mu\nu}$$

The formal definition of the Riemann tensor is given in terms of the Christoffel symbols:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

6.5 Geodesics

Geodesics are the generalization of straight lines to curved Riemannian manifolds. They represent the locally shortest path between two points and are defined as curves whose tangent vectors remain parallel to themselves when transported along the curve (autoparallel curves).

Given a parametrization $x^\mu(\lambda)$, where λ is an affine parameter (such as proper time or arc length), the geodesic equation is given by:

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

Arc Length

The length L of a geodesic curve (or any differentiable curve) connecting two points parametrized by $\lambda \in [\lambda_1, \lambda_2]$ is determined by the metric tensor $g_{\mu\nu}$. The arc length formula is:

$$L = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \, d\lambda$$

In the specific case where the parameter is the arc length itself ($\lambda = s$), the tangent vector is normalized such that $g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 1$.