# Rationality in Arithmetic Dynamics

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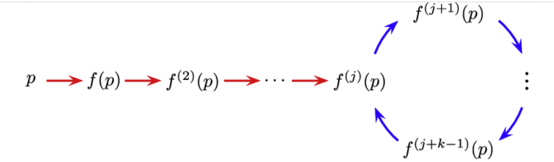
Let $\mathrm{Preper}(f, K) = \{P \in \mathbb{P}^N(K)$ preperiodic under f$\}$.

A preperiodic point $P \in \mathbb{P}^N(\bar{K})$ is one such that

$$|\{f^i(P)\}_{i=0}^{\infty}| < \infty.$$



# Forward Orbits

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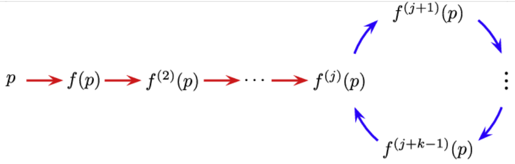


The *canonical height* of $P \in \mathbb{P}^N(\overline{K})$ is

$$\hat{h}_f(P) = \lim_{n\to\infty} \frac{1}{d^n} h(f^n(P)).$$

$$\hat{h}_f(P) = 0 \iff P \text{ is preperiodic under } f$$

# Forward Orbits

We'll focus on two kinds of instances of $K$-rationality:

1. $K$-rationality of points of small canonical height (especially preperiodic points)
2. $K$-rational points on higher genus curves and their connection to forward orbits

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- We say that $\mathcal{F}$ satisfies the **Uniform Boundedness Principle** (UBP) over $K$ if there is an $A = A(\mathcal{F}, K)$ such that for any $f \in \mathcal{F}(K)$,

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- We say that $\mathcal{F}$ satisfies the **Strong Uniform Boundedness Principle** (SUBP) over $K$ if for every $D \geqslant 1$ there is a $B = B(\mathcal{F}, D)$ such that for any extension $L/K$ of degree $\leqslant D$ and any $f \in \mathcal{F}(L)$,

$$|\mathrm{Preper}(f, L)| \leqslant B.$$

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- Mazur–Kamienny–Merel: Lattès maps over $\bar{\mathbb{Q}}$ (degree $> 1$ maps on $\mathbb{P}^1_{\bar{\mathbb{Q}}}$ descended from endomorphisms on elliptic curves over $\bar{\mathbb{Q}}$) satisfy the SUBP over $\mathbb{Q}$

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- Doyle–Poonen (2020): For $k$ a field, $K = k(t)$, and $d \geqslant 2$ with $\mathrm{char}(k) \nmid d$,
$$\mathcal{F} = \{z^d + c : c \in \bar{K} \backslash \bar{k}\}$$
satisfies the SUBP over $K$.

# Another example

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- Dvornicich–Zannier (2007): If $K$ is a number field, and $f \in K[z]$ is any polynomial of degree $d \geqslant 2$ not conjugate to $\pm z^d$ or $T_d(\pm z)$, where $T_d$ is the $d$th Chebyshev polynomial, then $f$ has only finitely many preperiodic points in $K^{\text{cyc}}$, the maximal cyclotomic extension of $K$.

  In other words, $\mathcal{F} = \{f\}$ satisfies the UBP over $K^{\text{cyc}}$.

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**Uniform Boundedness Conjecture (Morton–Silverman, 1994)**

Let $N \geqslant 1$, let $d \geqslant 2$, and let $K$ be a number field. Let $f : \mathbb{P}^N_K \to \mathbb{P}^N_K$ be a degree $d$ morphism defined over $K$. There is a $B = B(N, d, [K : \mathbb{Q}])$ such that $|\mathrm{Preper}(f, K)| \leqslant B$.

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**Uniform Boundedness Conjecture restated**

Let $K, N, d$ be as above. The family $\mathcal{F}$ of degree $d$ morphisms $\mathbb{P}^N_{\bar{K}} \to \mathbb{P}^N_{\bar{K}}$ satisfies the SUBP over $\mathbb{Q}$.

# Example Uniform Boundedness Results

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**Theorem 1 (L., 2021)**

Assume the *abcd*-conjecture. Let:

- $K$ be a number field
- $d \geqslant 2$
- $\mathcal{F}$ be the set of degree $d$ polynomials defined over $K$

Then $\mathcal{F}$ satisfies the UBP over $K$.

(A char. 0 function field analogue holds too.)

*Abcd* is a generalization of the *abc*-conjecture.

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**Step 2**: Use arithmetic info about pairwise differences to derive a contradiction of *abc* or *abcd* if too many of these differences lie in $K$.

# Example Uniform Boundedness Results

**Theorem 2 (L., 2021)**

Let $\mathcal{F} = \{f\}$, where $f \in K[x]$ is a polynomial with a preperiodic critical point $\neq \infty$ and at least one place of bad reduction.* Then $\mathcal{F}$ satisfies the UBP over $K^{\text{ab}}$.

* bad reduction here means not potentially good reduction

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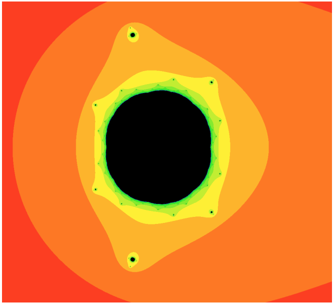

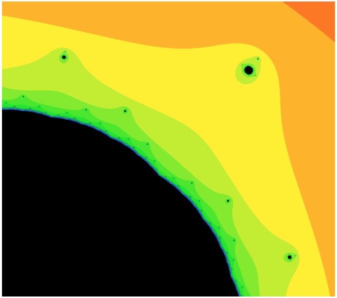

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Then there are explicit constants $c = c(K) > 0$ and $N = N(K)$, independent of $E$, such that there are at most $N$ points $P \in E(K)$ satisfying

$$\widehat{h_E}(P) \leqslant c \max\{h(j_E), 1\}.$$

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$$\log N_{K/\mathbb{Q}} \mathscr{D}_{E/K} \leqslant (6 + \epsilon) \log N_{K/\mathbb{Q}} \mathscr{F}_{E/K} + c,$$

where $\mathscr{D}_{E/K}$ is the minimal discriminant and $\mathscr{F}_{E/K}$ is the conductor of $E/K$.

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In other words, the valuations of $\mathscr{D}_{E/K}$ should not be too large on average.

# Outline of Hindry–Silverman

2. For non-archimedean places $v$, there are two cases:
    - $|j_E|_v \leqslant 1$ (i.e., potential good reduction at $v$)
    - $|j_E|_v > 1$. Tate uniformization gives maps

$$
\begin{array}{ccccc}
E(K_v) & \xrightarrow{\sim} & K_v^\times/q^{\mathbb{Z}} & \longrightarrow & \mathbb{R}/(\log|j_E|_v\mathbb{Z}) \\
 & & u & \longrightarrow & \log|u|_v
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where $q \in K_v$ with $|q|_v = |1/j_E|_v < 1$.

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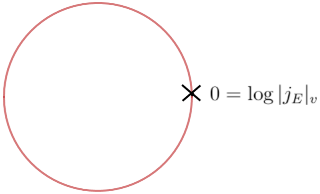

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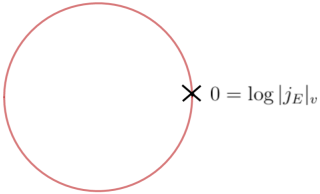


⋆ **If $P, Q \in E(K)$ map to distinct places on the circle, then their positions completely determine $\lambda_v(P - Q)$.**

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Within this bad reduction situation, two cases:

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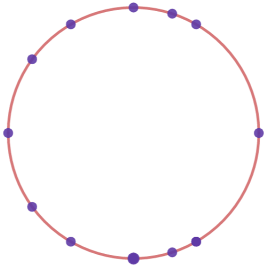
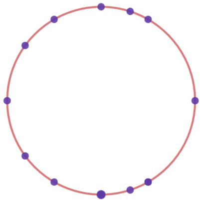
Case (1) tells us that points $P \in E(K)$ can only map to a restricted part of the circle:

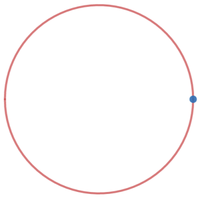


whereas (2) imposes no restriction on the position of the points.

# Outline of Hindry–Silverman

Upshot: If $P_1, \ldots, P_N \in E(K)$ are pairwise distinct and any significant "proportion" of bad places falls into Case (1), then replace with $Q_1 = [60]P_1, \ldots, Q_N = [60]P_N$





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If $N \gg 1$ and there are $N'$ distinct $Q_i$,

$$\frac{1}{N'(N'-1)} \sum_{v \in M_K^0} \sum_{Q_i \neq Q_j} \lambda_v(Q_i - Q_j) \geqslant C \sum_{v \in M_K^0} \log^+ |j_E|_v$$

for some explicit $C > 0$ independent of $E$.

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A separate combinatorial argument handles the archimedean places.

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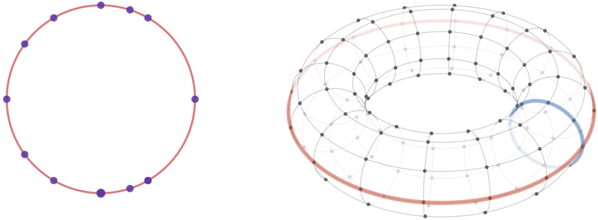
- Skeleton:



- Szpiro's conjecture analogue: there are analogous upper bounds on the average number of components of the Néron model at places of bad reduction, which follow from *abc*

# Uniform Boundedness in higher dimensions

Problem: normalized local heights don't sum to the global canonical height. Instead,

$$\hat{h}_\Theta(P) = \sum_{v \in M_K} \lambda_{v,\Theta}(P) + \kappa$$

for some $\kappa$.

Thus any higher-dimensional analogue of

$$\frac{1}{N'(N'-1)} \sum_{v \in M_K^0} \sum_{Q_i \neq Q_j} \lambda_v(Q_i - Q_j) \geqslant C \sum_{v \in M_K^0} \log^+ |j_E|_v$$

is not useful unless we can also prove this lower bound for $\frac{1}{N'(N'-1)} \sum_{v \in M_K^0} \sum_{Q_i \neq Q_j} \lambda_v(Q_i - Q_j) + \kappa$.

# Uniform Boundedness in higher dimensions

Solution: Replace $\mathrm{Avg}\lambda_v(P_i - P_j)$ with a generalized Vandermonde matrix evaluated at a basis $\{\eta_j\}$ of global sections of $\mathcal{L}^n$ for $\mathcal{L}$ very ample:

$$V_{m,v}(P_1, \ldots, P_m) = -\frac{1}{n} \log \left| \mathrm{Det}\left( \eta_j(\widetilde{P}_i) \right) \right|_v + \sum_i \hat{H}_v(\widetilde{P}_i),$$

where $m = h^0(\mathcal{L}^n)$ and $\hat{H}_v$ is a homogeneous escape-rate function.

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**Theorem (L., '24)**

*The functions $V_m$ satisfy an Elkies-type bound: There exists a $C$ such that for all $n \geqslant 2$, all $v$ and all $P_1, \ldots, P_m$ on the abelian variety,*

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Remarks:

- This result holds for general polarized dynamical systems.
- A Lehmer-style result on points of small canonical height on abelian varieties follows, over product formula fields having perfect residue fields. The bound has the form

$$\hat{h}_{\mathcal{L}}(P) \geqslant \frac{C'}{[K(P) : K]^{2\dim(A)+3+\epsilon}}.$$

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**Theorem (Height Uniformity)**

*Let $X$ be a nice algebraic curve of genus $\geqslant 2$ over a one-dimensional, characteristic $0$ function field $K$, and let $D \geqslant 1$. There are constants $C_1$ and $C_2$ depending on $X$, $K$, a chosen height $h$, and $D$ such that for all $P \in X(L)$ with $[L : K] \leqslant D$,*

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The constants $C_1$ and $C_2$ can be given very explicitly in the case of hyperelliptic curves $y^2 = f(x)$.

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1. Primitive prime divisors, and hence arboreal representations
2. Liminfs of the Néron–Tate height on curves embedded into their Jacobians (i.e., quantitative Bogomolov)

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## Definition

*We say that a prime $\mathfrak{p}$ of $K$ is a* **primitive prime divisor** *of $f^n(\alpha)$ if:*

- $v_{\mathfrak{p}}(f^n(\alpha)) > 0$, *and*
- $v_{\mathfrak{p}}(f^m(\alpha)) \leqslant 0$ *for all* $f^m(\alpha) \neq 0$ *with* $m < n$.

# Primitive prime divisors

We restrict our discussion to polynomial maps $f : \mathbb{P}^1 \to \mathbb{P}^1$.

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Example: $f(x) = x^2 - 7/4$, $\alpha = 0$

$0 \mapsto -7/4 \mapsto 21/16 \mapsto -7/256 \mapsto -114639/65536$

Here, $f^3(0)$ fails to have a primitive prime divisor.

# PPDs: specified multiplicities

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Example: the **Sylvester sequence** is given by the forward orbit of 2 under $f(x) = x^2 - x + 1$:

$$2 \mapsto 3 \mapsto 7 \mapsto 43 \mapsto 1807 = 13 \times 139 \cdots$$

It appears that each term in the sequence is squarefree, but this is not yet known to be true.

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Thus, for any **non**-primitive prime divisor $\mathfrak{p}$ of $f^n(\alpha)$, either

$$\mathfrak{p} \mid f^j(0) \text{ for some } 0 \leqslant j \leqslant \lfloor n/2 \rfloor$$

or

$$\mathfrak{p} \mid f^j(\alpha) \text{ for some } 0 \leqslant j \leqslant \lfloor n/2 \rfloor.$$

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We have

$$d_n y_n^2 = f^3(f^{n-3}(\alpha)),$$

so $(f^{n-3}(\alpha), \sqrt{d_n} y_n)$ is a quadratic point on the higher genus curve

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$$\text{genus}(L) \geqslant \frac{1}{C_1} \left( h(f^{n-3}(\alpha)) - C_2 \right).$$

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In other words,

$$h(d_n) \gg h(f^{n-3}(\alpha)).$$

# Primitive prime divisors: specified multiplicities

OTOH, by our divisibility trick, the product of all of the non-primitive prime divisors is necessarily small:

$$h\left(\prod_{0\leqslant j\leqslant\lfloor n/2\rfloor} f^j(\alpha)f^j(0)\right) = O(d^{n/2})$$

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Remarks:

- Over function fields, this approach works well for *uniform* PPD results.
- For non-uniform results, can use *abc* to show that one has PPDs of multiplicity 1 for all but finitely many $n$.
- Odd multiplicity PPDs are crucial in large image results for arboreal reps.

# Canonical heights on Jacobians and points on higher-genus curves

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**Theorem (Zhang,'93)**

*Let $X/K$ be a nice curve of genus $g \geqslant 2$, and $j : X \hookrightarrow J := \mathrm{Jac}(X)$ an Abel-Jacobi embedding. Let $\omega_a$ be the admissible dualizing sheaf on $X$. Then*

$$\liminf_{P \in X(\overline{K})} h_{\mathrm{NT}}(j(P)) \geqslant \frac{\omega_a^2}{4(g-1)}.$$

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Hence $\omega_a^2 > 0$ implies the Bogomolov Conjecture.

Remark: $\omega_a$ is a more natural analogue of the Arakelov dualizing sheaf.

# Canonical heights on Jacobians and points on higher-genus curves

If $K/k(t)$ is a one-dimensional char. 0 function field, then $\omega_a^2$ is known to be commensurate to the total "badness" of the reduction of $X$.

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If $K/k(t)$ is a one-dimensional char. 0 function field, then $\omega_a^2$ is known to be commensurate to the total "badness" of the reduction of $X$.

If $\delta_v$ is the $v$-adic delta-invariant of $X$ for each $v \in M_K$, then there are nonzero constants $C_1, C_2, C_3, C_4$ depending only on $g(X)$ and $[K : k(t)]$ such that

$$C_1 \sum_{v \in M_K} n_v \delta_v \leqslant \omega_a^2 \leqslant C_2 \sum_{v \in M_K} n_v \delta_v + C_3 \cdot \text{genus}(K) + C_4. \qquad (\star)$$

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**Theorem (Moret-Bailly, '90)**

*The right-hand inequality in $(\star)$ implies the Height Uniformity Conjecture.*

# Canonical heights on Jacobians and points on higher-genus curves

Remark: Moret-Bailly also shows that the Height Uniformity Conjecture implies a weak form of *abc*, namely that (for each $K$) the *abc* conjecture is true for all sufficiently large $\epsilon$.

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Over $\mathbb{Q}$, the *abc* conjecture says:

**Conjecture (*abc*)**

*Let $\epsilon > 0$. There is a $C_\epsilon$ such that for any positive coprime integers $a, b, c$ satisfying $a + b = c$,*

$$c \leqslant C_\epsilon \left( \prod_{\text{primes } p | abc} p \right)^{1+\epsilon}.$$

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Many results conditioned on *abc* in fact only use its truth for all sufficiently large $\epsilon$.

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2. Unconditional (with or w/o uniformity, with or w/o multiplicities) PPD results over number fields? Aside from examples like:
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2. Unconditional (with or w/o uniformity, with or w/o multiplicities) PPD results over number fields? Aside from examples like:
   - $z^d + c \in \mathbb{Q}[z]$ with $\alpha = 0$ (Krieger, '13)
   - Polys over $\mathbb{Q}$ fixing 0 (Ingram–Silverman, '09)?
3. Upper bound in Conjecture ($\bigstar$) for Galois covers of $\mathbb{P}^1$? Namely

$$\omega_a^2 \leqslant C_2 \sum_{v \in M_K} n_v \delta_v + C_3 \cdot \mathrm{genus}(K) + C_4$$

Thank you!