

On the Torsion and rational Points of some Curves

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The Mordell conjecture 100 years later
MIT, July 8-12, 2024

Let C be an algebraic curve over \mathbb{Q} embedded in its Jacobian J_C .

- Task 1 : find C_{Tor} the torsion points of J_C that lie on C .
- Task 2 : find $C(\mathbb{Q})$ the set of rational points of C .

QUESTION: Can we do that?

ANSWER TO TASK 1: NO

ANSWER TO TASK 2: YES

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Explicit Manin Mumford Conjecture
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ANSWER TO TASK 2 : NO

OPTIMISTIC ANSWER : NOT YET, SOMETIMES

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Let C be an algebraic curve over \mathbb{Q} embedded in its Jacobian J_C (in an abelian variety A).

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Example to Task 1: Find the torsion (with R. Pengo)

Example

Consider $\mathbb{A}^2 \times \mathbb{A}^2$ with coordinates $(x_1, y_1) \times (x_2, y_2)$, and the family of curves (or their projective closures):

$$C_1(a, b): \begin{cases} y_1^2 &= x_1^3 - 16x_1 + 16 \\ y_2^2 &= x_2^3 - 16x_2 + 16 \\ ay_2 &= bx_1 \end{cases}$$

for every two non-zero integer numbers a, b . Then $C_1(a, b)$ is the affine part of a curve embedded in E^2 , where $E: y^2z = x^3 - 16xz^2 + 16z^3$.

The torsion subset $C_1(a, b)_{\text{Tor}}$ is exactly

$$C_1(a, b)_{\text{Tor}} = \{(0_E, P) : P \in E[2](\overline{\mathbb{Q}})\} \subset E \times E$$

If $\alpha = \sqrt[3]{\frac{8}{9}\sqrt{-111} - 8}$, then $x_P = -\frac{1}{2}\alpha(1 + \sqrt{-3}) - \frac{8(1 - \sqrt{-3})}{3\alpha}, -\frac{1}{2}\alpha(1 - \sqrt{-3}) - \frac{8(1 + \sqrt{-3})}{3\alpha}, \alpha + \frac{16}{3\alpha}$.

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Example

Consider in $\mathbb{A}^2 \times \mathbb{A}^2$ with coordinates $(x_1, y_1) \times (x_2, y_2)$, the family of curves

$$C_n = \begin{cases} y_1^2 &= x_1^3 - 3x_1 - 1 \\ y_2^2 &= x_2^3 - 3x_2 - 1 \\ y_2 &= x_1^n + x_1 + 1 \end{cases}$$

The rational points are exactly

$$C_n(\mathbb{Q}) = \begin{cases} \{(-1, \pm 1, -1, 1), (-1, \pm 1, 2, 1)\}, & \text{if } 2 \mid n \\ \{(-1, \pm 1, -1, -1), (-1, \pm 1, 2, -1)\}, & \text{if } 2 \nmid n \end{cases}$$

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Beyond task 2: (with F. Veneziano)

Find the $\mathbb{Q}(\sqrt{-3})$ -rational points

Example

Consider in $\mathbb{A}^2 \times \mathbb{A}^2$ with coordinates $(x_1, y_1) \times (x_2, y_2)$, the curve

$$C'_6 = \begin{cases} y_1^2 &= x_1^3 + 2 \\ y_2^2 &= x_2^3 + 2 \\ x_1^6 &= y_2 \end{cases}$$

The $\mathbb{Q}(\sqrt{-3})$ -rational points on C'_6 .

Let $\zeta = \frac{-1+\sqrt{-3}}{2}$. Then $C'_6(\mathbb{Q}(\sqrt{-3}))$ is given by:

$$\begin{aligned} & \{(-1, 1, -1, 1), (-1, -1, -1, 1), (-1, 1, -\zeta, 1), (-1, -1, -\zeta, 1), \\ & (-1, 1, \zeta+1, 1), (-1, -1, \zeta+1, 1), (-\zeta, 1, -1, 1), (-\zeta, -1, -1, 1), \\ & (-\zeta, 1, -\zeta, 1), (-\zeta, -1, -\zeta, 1), (-\zeta, 1, \zeta+1, 1), (-\zeta, -1, \zeta+1, 1), \\ & (\zeta+1, 1, -1, 1), (\zeta+1, -1, -1, 1), (\zeta+1, 1, -\zeta, 1), \\ & (\zeta+1, -1, -\zeta, 1), (\zeta+1, 1, \zeta+1, 1), (\zeta+1, -1, \zeta+1, 1)\} \end{aligned}$$

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The $\mathbb{Q}(\sqrt{-3})$ -rational points on the C'_n

Example

Consider in $\mathbb{P}^2 \times \mathbb{P}^2$ with coordinates $(x_1 : y_1 : z_1) \times (x_2 : y_2 : z_2)$, the projective closure of

$$C'_n = \begin{cases} y_1^2 = x_1^3 + 2 & \text{elliptic curve } E \text{ with CM} \\ y_2^2 = x_2^3 + 2 \\ x_1^n = y_2 \end{cases}$$

Let $g = (-1 : 1 : 1)$ and $\text{End}(E) = \mathbb{Z}[\zeta]$ for $\zeta = \frac{-1+\sqrt{-3}}{2}$. Then $C'_n(\mathbb{Q}(\sqrt{-3}))$ is given by:

$$\begin{aligned} C'_n(\mathbb{Q}(\sqrt{-3})) \setminus (0_E, 0_E) &= \\ &= \{([a]g, [b]g) \mid a = \pm 1, \pm \zeta, \pm \zeta^2 \text{ and } b = 1, \zeta, \zeta^2\} && \text{if } n \equiv 0 \pmod{6} \\ &= \{([a]g, [b]g) \mid a = \pm 1 \text{ and } b = -1, -\zeta, -\zeta^2\} && \text{if } n \equiv \pm 1 \pmod{6} \\ &= \{([a]g, [b]g) \mid a = \pm 1 \text{ and } b = 1, \zeta, \zeta^2\} && \text{if } n \equiv \pm 2 \pmod{6} \\ &= \{([a]g, [b]g) \mid a = \pm 1, \pm \zeta, \pm \zeta^2, b = -1, -\zeta, -\zeta^2\} && \text{if } n \equiv 3 \pmod{6}, \end{aligned}$$

Why is task 1 easier? Quantitative aspects

Question 1 Is C_{Tor} a finite set? (Manin-Mumford Conjecture).



Question 2 Is $C(\mathbb{Q})$ a finite set? (Mordell Conjecture)



Why is task 1 easier? Quantitative aspects

Question 1 Is C_{Tor} a finite set? (Manin-Mumford Conjecture).

Answer 1 YES, if the genus of C is at least 2
(C non-torsion in A , i.e. not the translate of an elliptic curve by a torsion point.)
(Raynaud's Theorem).

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Question 2 Is $C(\mathbb{Q})$ a finite set? (Mordell Conjecture)

Answer 2 YES, if the genus of C is at least 2.
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- If we can bound the cardinality of C_{Tor} , then we can find C_{Tor} (in principle)



Question 2 Is $C(\mathbb{Q})$ a finite set? (Mordell Conjecture)

Answer 2 YES, if the genus of C is at least 2.
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Northcott's Theorem: there exists a procedure to find the points of bounded height and bounded degree in \mathbb{P}^n .

Can we say how many?

- Quantitative/Explicit Manin Mumford (Galateau & Martínez's Theorem):

$$|C_{\text{Tor}}| \leq 4^{(2c(A)+2)g} \deg(C)^2$$

where g is the dimension of A and $c(A)$ is a constant introduced by Serre (to be defined later).
Serre's constant is explicitly bounded for product of elliptic curves, CM abelian varieties and Jacobians.

- Quantitative Mordell Conjecture: how explicit should the constants be? What should they depend on?
 - explicit dependence on the degree of C , the dimension and height of A , and the rank of $A(\mathbb{Q})$.

$$|C(\mathbb{Q})| \leq (2^{34} \max(1, h_\theta(A)) \cdot \deg(C))^{(\text{rk}(A(\mathbb{Q}))+1) \dim(A)^{20}}. \quad (\text{Rémond + David-Philippon})$$

- for smooth C , dependence on the genus of C and rank of $J_C(\mathbb{Q})$, not explicit.

$$|C(\mathbb{Q})| \leq c(\dim(J_C))^{1+\text{rk}(J_C(\mathbb{Q}))} \quad (\text{Dimitrov, Gao, Habegger})$$

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Short sum up of explicit Manin Mumford \implies method for example 1

Let k be a field and $A \subseteq \mathbb{P}^n$ an abelian variety over k .

For $G_k := \text{Gal}(\bar{k}/k)$ and $\rho_A: G_k \rightarrow \text{Aut}_{\mathbb{Z}}(A_{\text{tors}}) \cong \text{GL}_{2g}(\hat{\mathbb{Z}})$, define Serre's constant as

$$c(A) := [\mathcal{H}_A(\hat{\mathbb{Z}}) : (\rho_A(G_k) \cap \mathcal{H}_A(\hat{\mathbb{Z}}))]$$

where with $\mathcal{H}_A(\hat{\mathbb{Z}}) \cong \hat{\mathbb{Z}}^\times$ we mean the homotheties.

Let V be an algebraic subvariety A and let $\delta(V)$ be the smallest d such that V is the intersection of hypersurfaces of degree at most d .

Define V_{tor}^j to be the equidimensional component of dimension j in V_{tor} . So

$$V_{\text{tor}} := \overline{V \cap A_{\text{tor}}} = \bigcup_{j=0}^{\dim V} V_{\text{tor}}^j.$$

Manin-Mumford for Varieties: The V_{tor}^j are torsion varieties, i.e. union of components of algebraic subgroups.

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Explicit Manin-Mumford

Work of **Galateau & Martínez (2017)** shows that:

$$\deg(V_{\text{tor}}^j) \leq ((2g+4)^3 16^{g(c(A)+2)})^{(g-j)\dim(V)} \cdot \deg(A) \cdot \delta(V)^{g-j},$$

where $c(A)$ is Serre's constant.

Estimates for $c(A)$:

- If E has CM and $j(E) \neq 0, 1728$, work of **Campagna & Pengo (2022)** gives explicit bounds. In particular, if E is defined over \mathbb{Q} then $c(E) = 2$.
- if E non-CM then work of **Lombardo (2015)** gives an explicit bound. If E is defined over \mathbb{Q} , then $c(E) \leq e^{1.910^{10}} \max\{1, h(E)\}^{12395}$.
- If A is a CM abelian variety then work of **Eckstein (2005)** gives $c(A) \leq [k : \mathbb{Q}] 3^{5g^2}$.
- For any Jacobian $A = J_C$, work of **Buium (1996)** gives a method to find an explicit bound.
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Our example

- Choose a *Serre curve* E , that means an elliptic curve with $c(E) = 1$. (Example $E: y^2 = x^3 - 16x + 16$)
- Compute $\delta(C_1(a, b)) \leq \deg(C_1(a, b)) = \deg(C_1) = 15$ and $g(C_1(a, b)) = 6$.
- Go through Galateau-Martinez's proof to keep constants as small as possible.
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Note that the general bound is independent of $h(C)$ thus this bound holds for all curves $C_1(a, b)$

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In conclusion:

- in many cases (such as CM abelian varieties, Jacobians and products of elliptic curves), Serre's constant $c(A)$ can be bounded explicitly;
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Irreducible curve C in an abelian variety A of dimension N

Given a set $S \subseteq A$ with some properties, when is $S \cap C$ finite ?

⋮

Recall: Weak-transverse: irreducible $V \not\subseteq B$ for any proper algebraic subgroup $B \subsetneq A$.
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 C non-torsion $\Rightarrow S_0 \cap C$ finite

Weak-transverse case
(see previous discussion)

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Effective/Explicitly Bounded Height
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Torsion Anomalous Conjecture (TAC)
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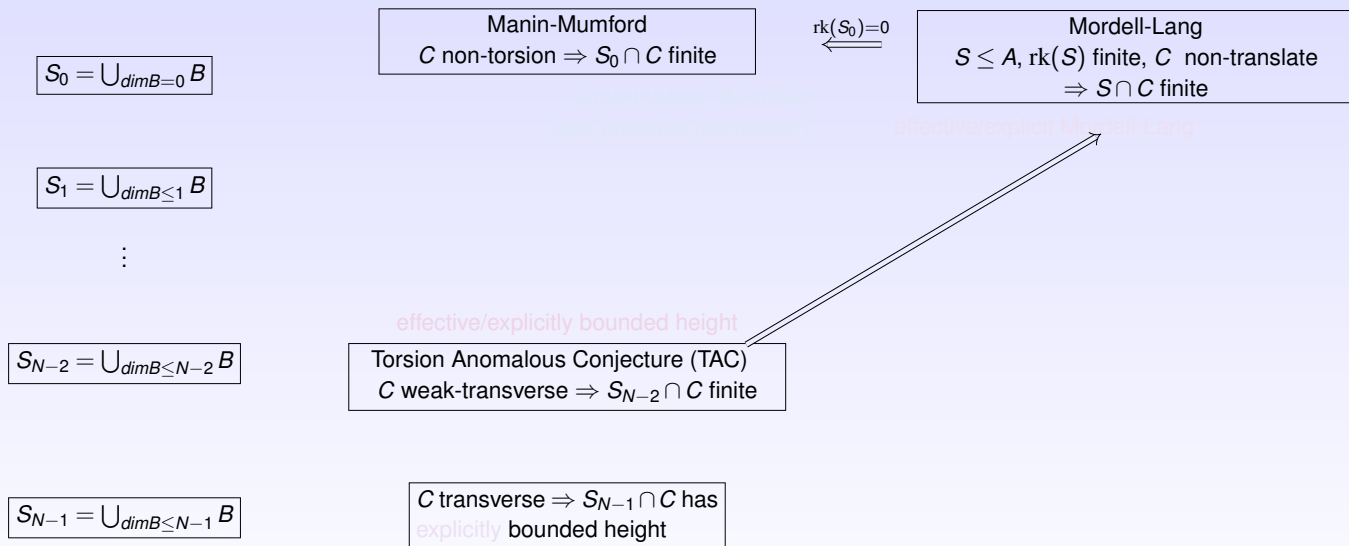
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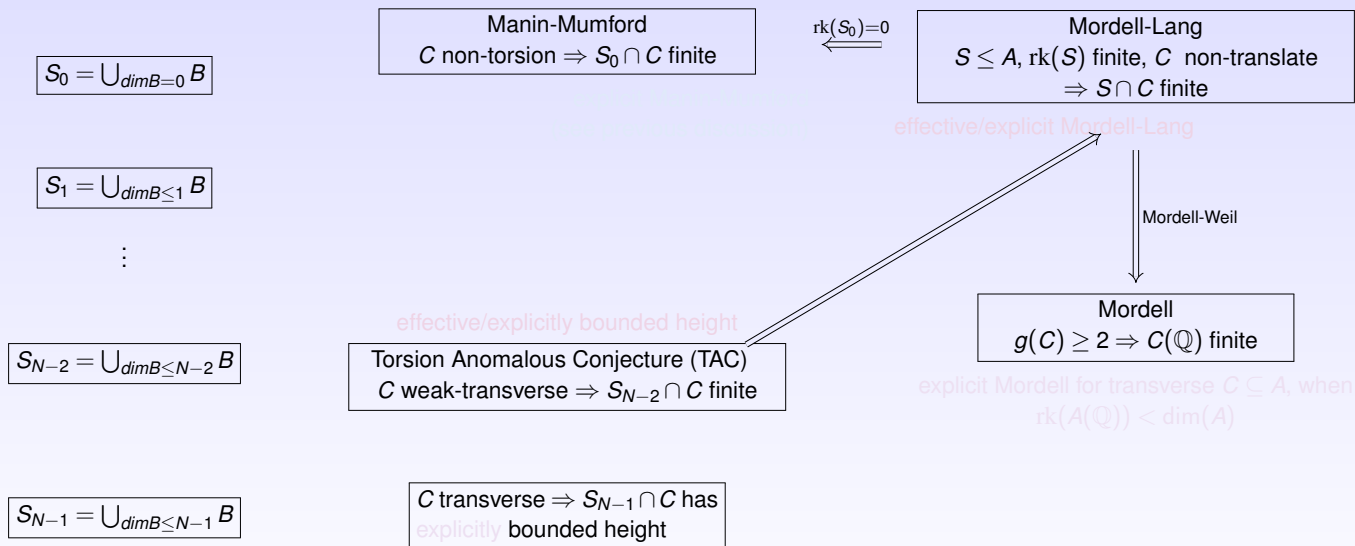
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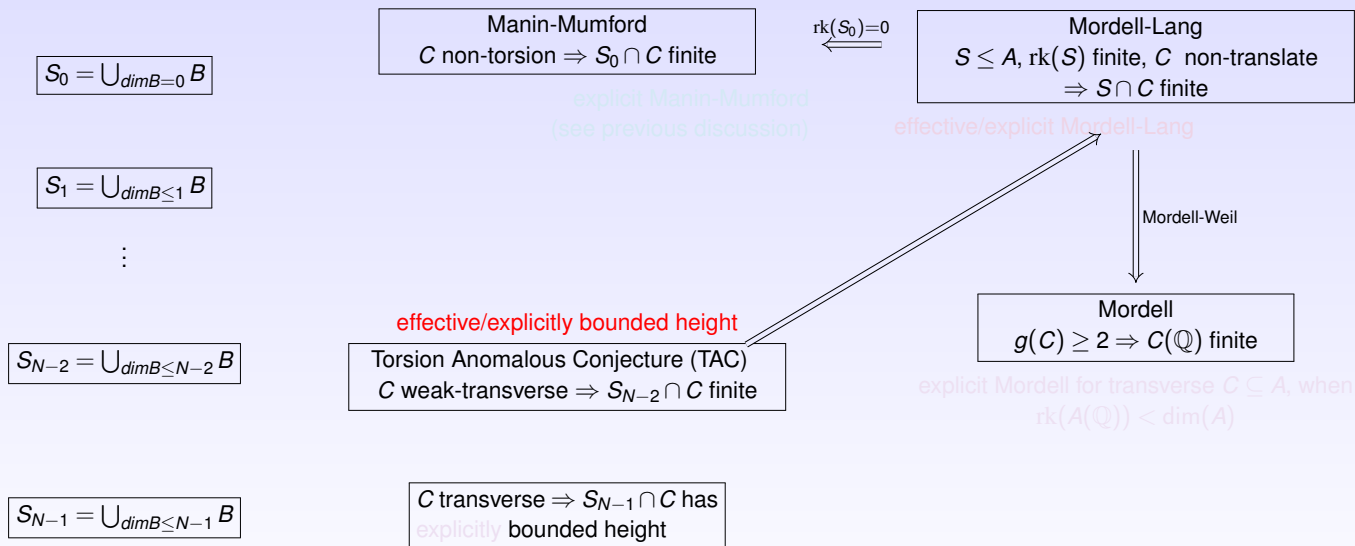
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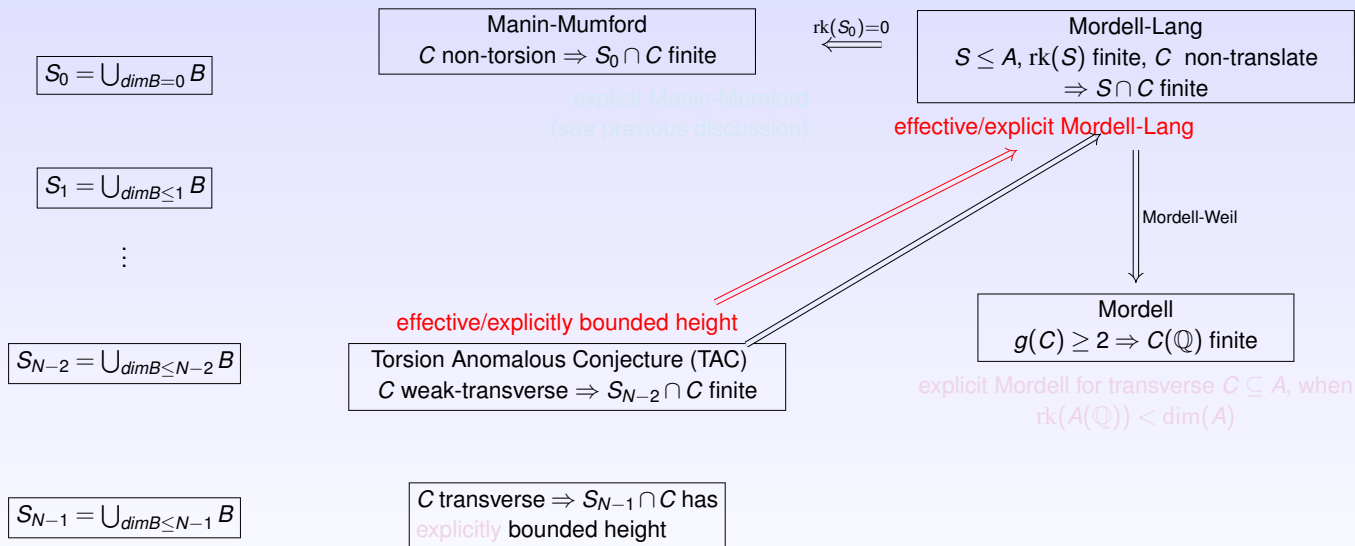
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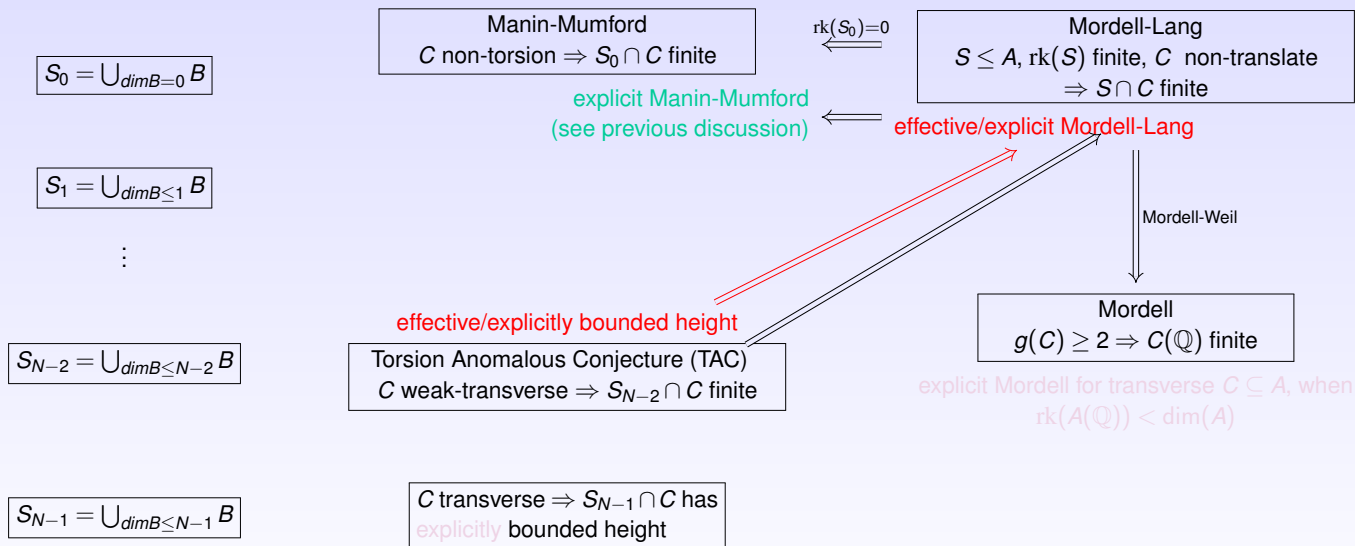
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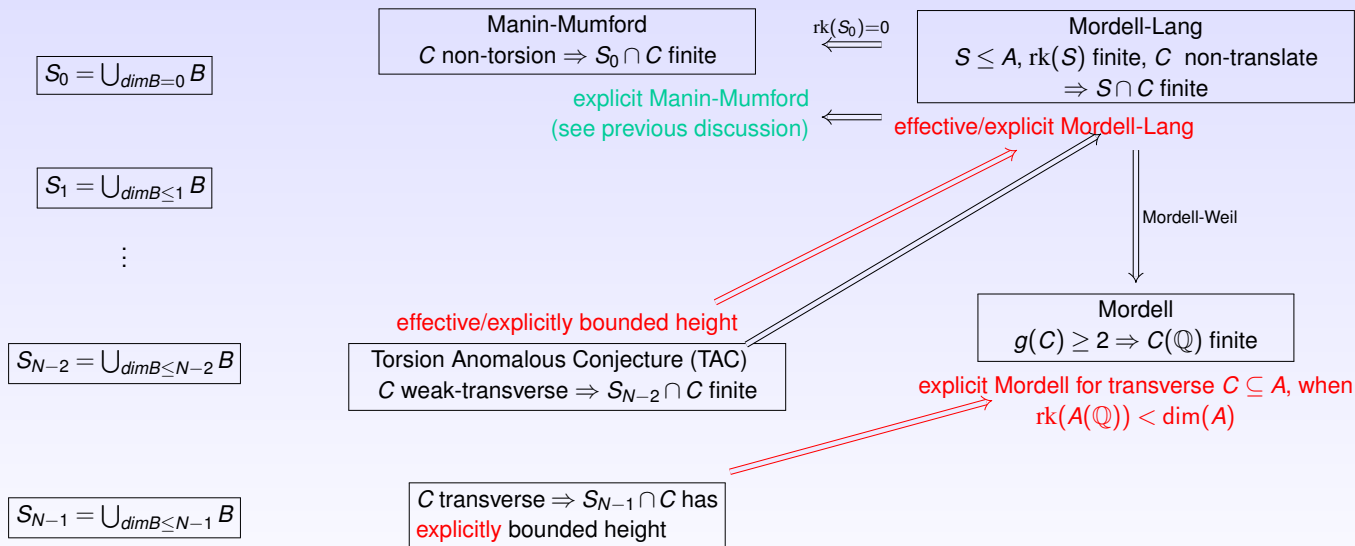
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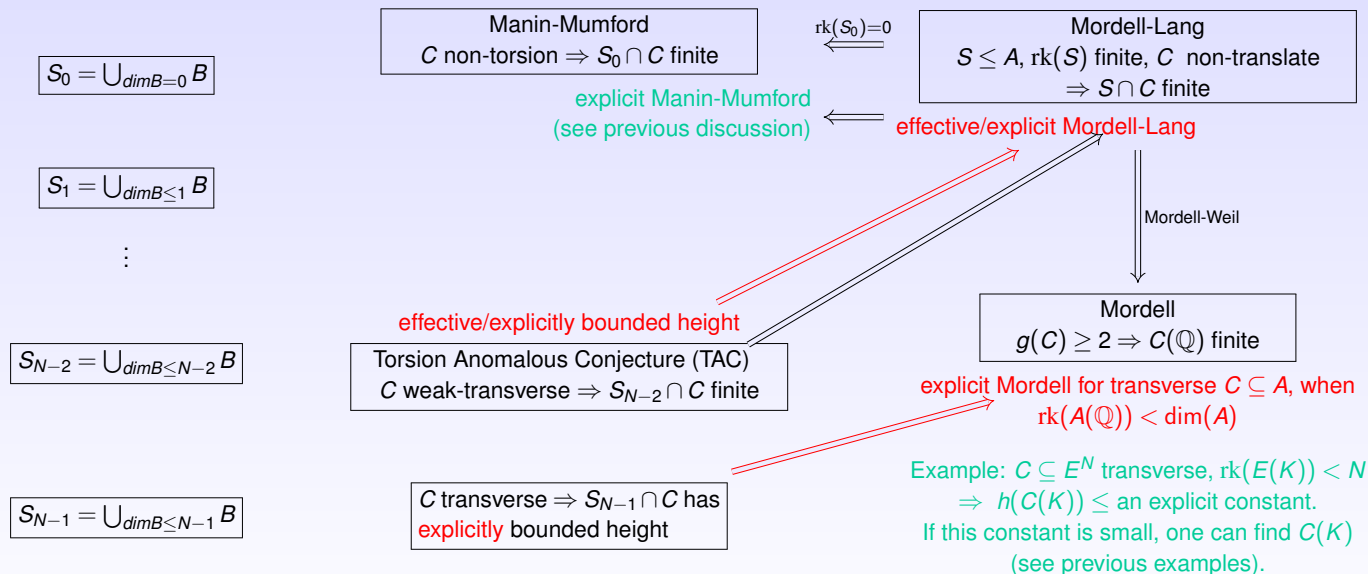
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Effective Methods for Faltings Theorem/ Mordell Conjecture

The result of Faltings is not effective, in the sense that it does not give any method for finding the rational points on C . This is due to the non existence of an effective bound for the height of the points in $C(\mathbb{Q})$ in general.

Effective Methods

- The method of **Chabauty-Coleman** and the **quadratic Chabauty and Kim** program.
A significant number of examples of curves in general of small genus and restricted to the condition that the \mathbb{Q} -rank of the Jacobian is strictly smaller than the genus of the curve (Bruin, Flynn, Poonen, Stoll, ...). Exception for the split Cartan modular curve of level 13, and other modular curves whose rank is equal to the genus of the curve (Balakrishnan, Dogra, Müller, Tuitman, Vonk, ...).
- The **Manin-Dem'janenko** method (our method relies on the same starting principle).
Families of examples of genus 2 or 3 and \mathbb{Q} -rank 1 or 2 and some other condition for instance a factor of the Jacobian given by $y^2 = x^3 + a^2x$, with a square-free integer. (Kulesz, Girard, Matera, Silverman, Schost...)

Explicit TAC \Rightarrow method for Example 2

Theorem (Torsion Anomalous Conjecture for Curves)

Let C be weak-transverse in E^N . Then the set

$$C \cap S_{N-2} = C \cap \bigcup_{\dim B \leq N-2} B \quad \text{is finite.}$$

Let C be transverse in E^N . Then the set

$$C \cap S_{N-1} = C \cap \bigcup_{\dim B \leq N-1} B \quad \text{has explicitly bounded height.}$$

Remark

The original proof of **Bombieri-Masser-Zannier (1999)** in \mathbb{G}_m^n for transverse curves can be adapted to E^N for E with CM, due to the use of a Lehmer Type bound. For E without CM one can use a Bogomolov Type bound and Vojta's inequality in a non effective way. To get the examples we make explicit the second part with a different approximation method that keeps the constants small. This implies the effective Mordell Conjecture for C transverse in E^N and $\text{rk} E(\mathbb{Q}) \leq N-1$. Thus we can in principle find the rational points of such curves.

For C weak-transverse in A , the finiteness of $C \cap S_{N-2}$ is proven by **Habegger-Pila (2016)** using O -minimality.

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Algebraic subgroups of E^N

- Let $\phi_B \in \text{Mat}_{r,n}(\text{End}(E))$ be a matrix of rank r

$$\phi_B = \begin{pmatrix} b_{11} & \dots & b_{1N} \\ \vdots & \vdots & \vdots \\ b_{r1} & \dots & b_{rN} \end{pmatrix} : E^N \rightarrow E^r$$
$$\phi_B : (x_1, \dots, x_N) \mapsto (b_{11}x_1 +_E \dots +_E b_{1N}x_N, \\ \vdots \\ b_{r1}x_1 +_E \dots +_E b_{rN}x_N)$$

- $B = \ker \phi_B$ is an algebraic subgroup of $\text{codim} B = r$
Minkowski reduction of ϕ_B gives $\deg B \approx ||b_1||^2 \dots ||b_r||^2$.

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From TAC to Mordell

Case $N = 2$

An algebraic subgroup of E^2 of dimension 1 is given, up to some torsion, by

$$B : \{b_1 X_1 + b_2 X_2 = 0\}$$

Consider an algebraic curve $\mathcal{C} \subset E^2$, and suppose that $\text{rk}(E(\mathbb{Q})) = 1$, i.e. $E(\mathbb{Q}) = \langle g_1 \rangle$. Then a point $P = (P_1, P_2) \in \mathcal{C}(\mathbb{Q}) \subset E(\mathbb{Q})^2$ has the form

$$P = (a_1 g_1, a_2 g_1)$$

and therefore P is a point in

$$P \in B_P : \{a_2 X_1 - a_1 X_2 = 0\}$$

So

$$\forall P \in \mathcal{C}(\mathbb{Q}) \text{ then } P \in \mathcal{C} \cap B_P \Rightarrow \mathcal{C}(\mathbb{Q}) \subset \mathcal{C} \cap \bigcup_{\dim B=1} B.$$

The Theorem tells us that the set

$$\mathcal{C} \cap \bigcup_{\dim B=1} B \text{ has height explicitly bounded.}$$

Thus if $E(\mathbb{Q})$ has rank 1 and $\mathcal{C} \in E^2$ has genus ≥ 2 then $\mathcal{C}(\mathbb{Q})$ has explicitly bounded height.

The constraint on the rank is unavoidable with this method.

If $C \subset E^2$ and E has rank 2 with $E(\mathbb{Q}) = \langle g_1, g_2 \rangle_{\mathbb{Z}}$ and $P \in C(\mathbb{Q})$ then $P = (a_1 g_1 + a_2 g_2, b_1 g_1 + b_2 g_2)$ but you do not have any subgroup B_P that contains P .



To remove the hypothesis on the rank is equivalent to prove that for weak-transverse curves C the set $C \cap (\cup_{\dim B \leq N-2} B)$ has explicitly bounded height (Explicit TAC implies Explicit Mordell).

If C_0 is transverse in E^2 and $E(\mathbb{Q}) = \langle g_1, \dots, g_r \rangle$ then $P = (P_1, P_2) \in C(\mathbb{Q})$ is given by $(P_1, P_2) = (a_1 g_1 + \dots + a_r g_r, b_1 g_1 + \dots + b_r g_r)$. Consider the curves $C = C_0 \times g_1 \times \dots \times g_r$ weak-transverse in E^{2+r} . Thus $(P_1, P_2, g_1, \dots, g_r)$ is a point in C and in

$$B_P = \begin{cases} X_1 &= a_1 Y_1 + \dots + a_r Y_r \\ X_2 &= b_1 Y_1 + \dots + b_r Y_r \end{cases}$$

Main Ingredients for the Bound on the Height

Let E be an elliptic curve given in the form

$$y^2 = x^3 + Ax + B.$$

with A, B algebraic integers.

Let $C(E) = \frac{h_W(\Delta) + 3h_W(j)}{4} + \frac{h_W(A) + h_W(B)}{2} + 4$.

Let \hat{h} be the Néron-Tate height on E^N .

For a curve C let $h(C)$ be the normalised height of C .

Theorem (Arithmetic Bézout Theorem, explicit version of Philippon)

Let V and W be irreducible subvarieties of \mathbb{P}^m . Let Z_1, \dots, Z_n be the irreducible components of $V \cap W$. Then

$$\sum_{i=1}^n h(Z_i) \leq \deg V \, h(W) + h(V) \, \deg W + c(m) \deg V \deg W.$$

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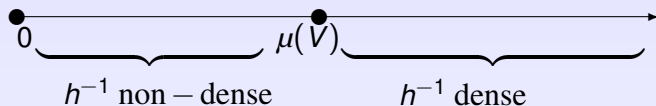
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Essential Minimum

Definition:

$$h : V(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}^+$$



Essential Minimum

$$\mu(V) = \sup\{\varepsilon : h^{-1}[0, \varepsilon) \text{ non - dense in } V\}$$

Theorem (Zhang Inequality)

Let V be an irreducible subvariety of \mathbb{P}^m , then

$$\frac{1}{(1 + \dim V)} \frac{h(V)}{\deg V} \leq \mu(V) \leq \frac{h(V)}{\deg V}.$$

Theorem (Minkowski Convex Body Theorem)

Let Λ be a lattice of volume Δ in \mathbb{R}^n and $S \subset \mathbb{R}^n$ a convex body, symmetric with respect to the origin. If the volume of S is $> 2^n \Delta$ then S contains at least one lattice point other than the origin.

Main results on bounded Height

Theorem (Veneziano, V. 2022 (generalization of N=2 Checcoli, Veneziano, V. 2018))

Let E be a non-CM elliptic curve and let C be a curve transverse in E^N . Then all the points P in $C \cap S_{N-1}$ have Néron-Tate height explicitly bounded as follows:

$$\hat{h}(P) \leq D_1(N) \cdot h(C)(\deg C)^{N-1} + D_2(N, E)(\deg C)^N + D_3(N, E).$$

The constants are given by:

$$D_1(N) = 4N! \left(\frac{N^2(N-1)^2 3^N}{4^{N-3}} N! (N-1)!^4 \right)^{N-1},$$

$$D_2(N, E) = D_1(N) (N^2 C(E) + 3^N \log 2),$$

$$D_3(N, E) = (N+1)C(E) + 1, C(E)$$

Theorem (Veneziano, V. 2022)

Let E be an elliptic curve with Complex Multiplication by the field K and let C be a curve transverse in E^N . Then all the points P in $C \cap S_{N-1}$ have Néron-Tate height explicitly bounded as follows:

$$\hat{h}(P) \leq C_1(N, E) \cdot h(C)(\deg C)^{N-1} + C_2(N, E)(\deg C)^N + C_3(N, E).$$

The constants are given by:

$$\begin{aligned} c(N) &= N! (N \cdot N! \cdot (2N)!^2)^{N-1}, \\ C_1(N, E) &= c(N) f^N |D_K|^{N^2 - \frac{3}{2}N + 1} + 1, \\ C_2(N, E) &= c(N) f^N |D_K|^{N^2 - \frac{3}{2}N + 1} (N^2 C(E) + 3^N \log 2 + 1), \\ C_3(N, E) &= N(N+1)C(E) + 3^N \log 2 + 1. \end{aligned}$$

where D_K is the discriminant of the field of complex multiplication and f is the conductor of $\text{End}(E)$.

Estimates for the height of curves

Proposition (with Riccardo Pengo)

Let $C \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ be the curve

$$C: \begin{cases} y_1^2 z_1 &= x_1^3 + Ax_1 z_1^2 + Bz_1^3 \\ y_2^2 z_2 &= x_2^3 + Ax_2 z_2^2 + Bz_2^3 \\ f(x_1: y_1: z_1, x_2: y_2: z_2) &= 0, \end{cases}$$

where $f \in \mathbb{Z}[x_1: y_1: z_1, x_2: y_2: z_2]$ is a bi-homogeneous polynomial of bi-degree $(\delta_1(f), \delta_2(f))$. Then

$$h(C) \leq 9 \cdot \left((\delta_1(f) + \delta_2(f)) \left(\frac{1}{2} \log \left(\frac{|A|^2 + 3|B|^2 + 4}{3} \right) + \frac{15}{4} \right) + \frac{1}{2} \log \left(\sum_{v,w} \frac{|a_{v,w}(f)|^2}{(v)(w)} \right) \right)$$

where

$$f = \sum_{\substack{v_1+v_2+v_3=\delta_1(f) \\ w_1+w_2+w_3=\delta_2(f)}} a_{v,w}(f) \cdot x_1^{v_1} y_1^{v_2} z_1^{v_3} x_2^{w_1} y_2^{w_2} z_2^{w_3} \in \mathbb{Z}[x_1: y_1: z_1, x_2: y_2: z_2]$$

and we denote by $(b) := \frac{(b_1 + \dots + b_k)!}{b_1! \dots b_k!}$ for $b \in \mathbb{N}^k$.

Producing Families of Curves of increasing Genus s.t. we can find $C(\mathbb{Q})$

Choose an elliptic curve E such that $\text{rk } E(\mathbb{Q}) = 1$.

For example

$$E_1: y^2 = x^3 - x + 1 \quad E_4: y^2 = x^3 + 3x + 1$$

$$E_2: y^2 = x^3 + x - 1 \quad E_5: y^2 = x^3 - 3x - 1$$

$$E_3: y^2 = x^3 + 2x + 1 \quad E_6: y^2 = x^3 + 4x + 1:$$

Remark

The method works for any number field K such that $\text{rk } E(K) = 1$.

We do not need to have the generator of $E(K)$ even if this speeds up the implementation.

The Family C_n

Cut in ExE a (family) of curves with an extra polynomial.

For example

$$C_{n,1} = \begin{cases} y_1^2 &= x_1^3 - x_1 + 1 \\ y_2^2 &= x_2^3 - x_2 + 1 \\ y_2 &= x_1^n + x_1 + 1 \end{cases}$$

For other E_i , we can define $C_{n,i}$ to be the curve in $E_i \times E_i$ cut by $y_2 = x_1^n + x_1 + 1$.

Compute the invariants $h(C)$ and $\deg C$

The C_n have genus $n+5$, degree $\deg C_n = 6n+9$ and height:

$$h((C_n)) \leq 9 \left((n+1) \left(\frac{1}{2} \log \left(\frac{|A|^2 + 3|B|^2 + 4}{3} \right) + \frac{15}{4} \right) + \frac{1}{2} \log \left(3 + \frac{1}{n} \right) \right).$$

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Plug the invariants in our non-CM Theorem
to get $h(P) \leq \text{Number}$

For every point $P \in C_n(\mathbb{Q})$ we have

$$\hat{h}(P) \leq 81193n^2 + 238012n + 174343.$$

For a family we need a result of Stoll

This shows that it suffices to check the curves for $n \leq \text{Number}$ and the integral points.
For our family only the curves with $n \leq 19$ need to be checked.

Use Belabas Algorithm to make a computer search and obtain $C(\mathbb{Q})$

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The affine rational points on the families $C_{n,i}(\mathbb{Q})$ are

$$C_{n,i}(\mathbb{Q}) = \begin{cases} \left\{ \begin{aligned} &((-1, \pm 1), (0, -1)), ((-1, \pm 1), \\ &(\pm 1, -1)), ((0, \pm 1), (0, 1)), ((0, \pm 1), (\pm 1, 1)), \\ &((5, 11), (5, 11)) \end{aligned} \right\}, & \text{if } i = 1 \text{ and } n = 1 \\ \left\{ \begin{aligned} &((0, \pm 1), (0, 1)), ((0, \pm 1), (\pm 1, 1)), \\ &((-1, \pm 1), (0, 1)), ((-1, \pm 1), (\pm 1, 1)) \end{aligned} \right\}, & \text{if } i = 1 \text{ and } 2 \mid n \\ \left\{ \begin{aligned} &((0, \pm 1), (0, 1)), ((0, \pm 1), (\pm 1, 1)), \\ &((-1, \pm 1), (0, -1)), ((-1, \pm 1), (\pm 1, -1)) \end{aligned} \right\}, & \text{if } i = 1, n \geq 3 \text{ and } 2 \nmid n \\ \{((1, \pm 1), (2, 3))\}, & \text{if } i = 2 \\ \{((0, \pm 1), (0, 1))\}, & \text{if } i = 3, 4 \\ \{((-1, \pm 1), (-1, 1)), ((-1, \pm 1), (2, 1))\}, & \text{if } i = 5 \text{ and } 2 \mid n \\ \{((-1, \pm 1), (-1, -1)), ((-1, \pm 1), (2, -1))\}, & \text{if } i = 5 \text{ and } 2 \nmid n \\ \{((0, \pm 1), (0, 1)), ((4, \pm 9), (4, 9))\}, & \text{if } i = 6 \text{ and } n = 1 \\ \{((0, \pm 1), (0, 1))\}, & \text{if } i = 6 \text{ and } n \neq 1 \end{cases}$$

The CM case (with Veneziano)

Choose an elliptic curve E with CM by K such that $\text{rk}_{\mathbb{Q}} E(K) = 2$.

For example

$$E : y^2 = x^3 + 2.$$

$E(\mathbb{Q}(\sqrt{-3})) = \langle (-1 : 1 : 1), (-\zeta : 1 : 1) \rangle_{\mathbb{Z}}$ has rank 2 as an abelian group.

- But E has CM by $K = \mathbb{Q}(\sqrt{-3})$.
- $\text{End}(E) = \mathbb{Z}[\zeta]$ for $\zeta = \frac{-1+\sqrt{-3}}{2}$ a primitive cube root of 1
- $E(\mathbb{Q}(\sqrt{-3})) = \langle (-1 : 1 : 1) \rangle_{\mathbb{Z}[\zeta]}$ has $\mathbb{Z}[\zeta]$ -rank 1 with generator $g = (-1 : 1 : 1)$.
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Define a Family in E^2

Example

Consider the family

$$C'_n = \begin{cases} y_1^2 &= x_1^3 + 2 \\ y_2^2 &= x_2^3 + 2 \\ x_1^n &= y_2 \end{cases}$$

- Compute the invariants: the C'_n have genus $4n+2$ and

$$\deg C'_n = 6n+9,$$

$$h(C'_n) \leq 6(2n+3) \log(3 + |A| + |B|).$$

- Plug all invariants in our CM theorem to get that for $P \in C'_n(\mathbb{Q}(\sqrt{-3}))$ the height

$$\hat{h}(P) \leq 644391 \cdot (2n+3)^2 + 28$$

- Generalize Stoll's result to number fields to obtain that for $n \geq 21$ then $C'_n(K) = C'_n(O_K)$.
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The $\mathbb{Q}(\sqrt{-3})$ -rational points on the C'_n

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Let $g = (-1 : 1 : 1)$. Our explicit bound on the height of $C'_n(\mathbb{Q}(\sqrt{-3}))$ implies:

$$\begin{aligned} C'_n(\mathbb{Q}(\sqrt{-3})) \setminus \{(O, O)\} = & \\ = \{(ag, bg) \mid a = \pm 1, \pm \zeta, \pm \zeta^2 \text{ and } b = 1, \zeta, \zeta^2\} & \text{if } n \equiv 0 \pmod{6} \\ = \{(ag, bg) \mid a = \pm 1 \text{ and } b = -1, -\zeta, -\zeta^2\} & \text{if } n \equiv \pm 1 \pmod{6} \\ = \{(ag, bg) \mid a = \pm 1 \text{ and } b = 1, \zeta, \zeta^2\} & \text{if } n \equiv \pm 2 \pmod{6} \\ = \{(ag, bg) \mid a = \pm 1, \pm \zeta, \pm \zeta^2, b = -1, -\zeta, -\zeta^2\} & \text{if } n \equiv 3 \pmod{6}, \end{aligned}$$

Examples in E^3 : find $C(\mathbb{Q})$ for $C \subset E^3$

- Choose E such that $\text{rk} E(\mathbb{Q}) = 2$ and cut C on E^3 with two extra polynomials.
- Choose two polynomials that cut a transverse $C \subset E^N$ and so that $\deg C$ and $h(C)$ are small.
- Estimate $h(C)$ and $\deg C$
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In general the bound is not implementable

Try to improve the bound choosing a special E , for instance such that the generators of $E(\mathbb{Q})$ are almost orthogonal, sharp comparison of $h(P)$ and $\hat{h}(P)$, some new ideas to improve the bound to

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Then it is implementable.

This gives infinitely many possible master/PhD projects 😊

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Irreducible variety V embedded in A

Manin-Mumford

V non-torsion $\Rightarrow S_0 \cap V$ non-dense



Mordell-Lang

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Remark: $\text{codim}(V \cap B)$ is expected to be $\text{codim}(V) + \text{codim}(B)$.

If C is a weak-transverse curve, then $C \not\subseteq B$

$\dim(B) = N - 1 \Rightarrow$ expect points in $C \cap B$

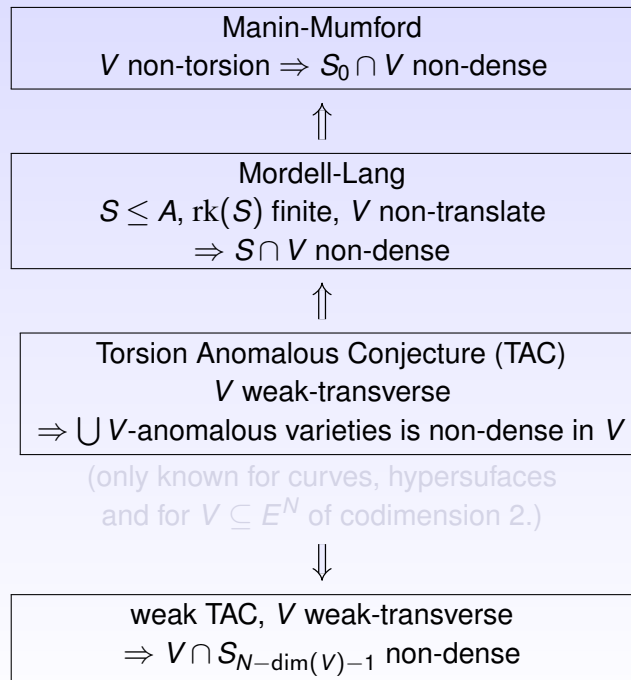
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If V is a weak-transverse variety, we expect that for every component Y of $V \cap B$

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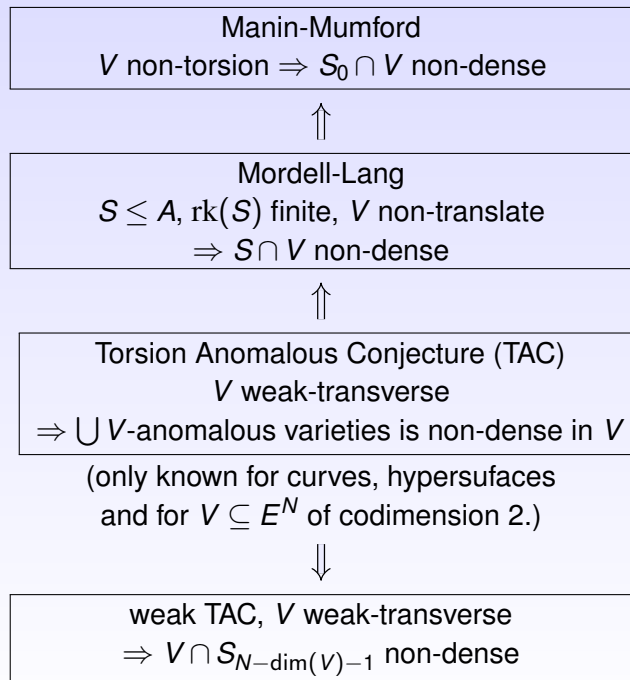
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Exercise

Define a curve C of genus ≥ 2 in E^2 with E an elliptic curve of rank 1 such that C has a rational point of large height. Large height means for instance $\geq 10(h(C) + \deg C)$, or $\gg (h(C) + \deg C)^\alpha$ with $1 < \alpha < 2$.

THANK YOU