MATH 213 — Advanced Mathematics for Software Engineers

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1 Differential Equations

Differential equations are equations involving derivatives with respect to some independant variable. For example, Newton's Law states

$$M\ddot{x} = F$$

or

$$M\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F$$

In the **classical approach**, we suppose f is given as a function of time, and we solve for the dependant variable with respect to the independant one. For example,

$$F(x) \to x(t)$$

The **systems approach** has less of an emphasis on the response to a specific input and deals more with the overall relationships between the function and between the individual dependant variables.

1.1 Examples

The population of an organism (given abundant resources) or the growth of an economy (if the economy were to grow at a constant percentage rate) can be modelled as:

$$
\begin{aligned}
\dot{x} &= ax \\
\frac{\mathrm{d}x}{x} &= a \ \mathrm{d}t \\
\int \frac{\mathrm{d}x}{x} &= \int a \ \mathrm{d}t \\
\ln x + C_1 &= at + C_2 \\
\ln x &= at + C_3 \\
x(t) &= e^{at+C_3} \\
x(t) &= e^{at} \times e^{C_3} \\
x(t) &= C_4 \times e^{at} \\
x(t) &= x(0) \times e^{at}
\end{aligned}
$$

where the value of $x(0)$ is called the **initial condition**.

Given that this function assumes that the population growth is not limited by resources, etc., it is not very useful in the real world. More likely, we would find for large populations a limit of some sort must be included. For example, the **logistic equation** is modelled as:

$$\begin{aligned}
\dot{x} &= ax - bx^2 \\
\frac{\mathrm{d}x}{ax - bx^2} &= \mathrm{d}t \\
\int \frac{\mathrm{d}x}{ax - bx^2} &= \int \mathrm{d}t \\
\int \frac{\mathrm{d}x}{x(a - bx)} &= \int \mathrm{d}t \\
\int \mathrm{d}x(\frac{A}{x} + \frac{B}{a - bx}) &= \int \mathrm{d}t \\
\int \mathrm{d}x(\frac{1}{ax} + \frac{B}{a - bx}) &= \int \mathrm{d}t \\
\int \mathrm{d}x(\frac{1}{ax} + \frac{b}{a(a - bx)}) &= \int \mathrm{d}t \\
\int \frac{\mathrm{d}x}{ax} + \int \frac{b\ \mathrm{d}x}{a(a - bx)} &= \int \mathrm{d}t \\
\frac{\ln x}{a} + \frac{b}{a}\frac{-1}{b}\ln(a - bx) &= t + C_0 \\
\frac{1}{a}\left(\ln \frac{x}{a - bx}\right) &= t + C_0 \\
\frac{x}{a - bx} &= e^{at + aC_0} \\
x &= C_1 e^{at}(a - bx) \\
x &= \frac{aC_1 e^{at}}{1 + bC_1 e^{at}} \\
x &= \frac{a}{b}\left(\frac{1}{1 + C_2 e^{-at}}\right)
\end{aligned}$$

Where $C_2 = \frac{1}{bC_1}$. In this case, the population will "level out" at $ax = bx^2$ (i.e. have an asymptote). The solution to this specific **DE (differential equation)** is called the **logistic curve**.

1.2 Partial Differential Equations

PDEs (partial differential equations) arise when there is more than one independant variable.

If we were to model the vibration of a string, we would use a PDE. Assuming there is no length-wise vibration (i.e. horizontal motion), that the string has constant tension and mass per unit length, and that we are only considering small transverse displacements, we could write Newton's equation $F = ma$ as

$$F = (\rho \Delta x) \times \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}$$

Since all forces on this string are tension, we have

$$F = T \sin\theta_1 - T \sin\theta_2$$

Given that we have small displacements (which leads to small angles θ_1 and θ_2), we can replace all instances of sin with tan. Finally, this gives us

$$F \approx T \frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_{x+\frac{\Delta x}{2}} - T \frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_{x-\frac{\Delta x}{2}}$$

so therefore

$$\rho \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{T}{\Delta x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_{x+\frac{\Delta x}{2}} - \frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_{x-\frac{\Delta x}{2}} \right)$$

As Δx approaches zero, we see that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{T}{\rho} \frac{\mathrm{d}^2 y}{\mathrm{d}X^2}$$

We will not be solving PDEs in this course, but this equation is solveable to give us

$$y = A \sin k \left(x - t\sqrt{\frac{T}{\rho}} \right)$$

1.2.1 Boundary Conditions

This equation should additionally have several **boundary conditions** which represent the ends of the string being fixed in place.

$$y(0, t) = 0$$
$$y(L, t) = 0$$

We use these equations as well as the equation for a standing wave to get

$$y = A_+ \sin k \left(x - t\sqrt{\frac{T}{p}} \right) + A_- \sin k \left(L - t\sqrt{\frac{T}{\rho}} \right)$$

Since $y(0, t) = 0$,

$$0 = A_+ \sin k \left(0 - t\sqrt{\frac{T}{p}} \right) + A_- \sin k \left(0 - t\sqrt{\frac{T}{\rho}} \right)$$
$$A_+ = A_- = A$$

and since $y(L,t)=0$,

$$
\begin{aligned}
y &= A_+ \sin k\left(L - t\sqrt{\frac{T}{p}}\right) + A_- \sin k\left(L - t\sqrt{\frac{T}{\rho}}\right) \\
KL &= \pm n\pi
\end{aligned}
$$

Because waves can only have certain frequences we have

$$
\begin{aligned}
K\sqrt{\frac{T}{\rho}} &= \pm\frac{n\pi}{L}\sqrt{\frac{T}{\rho}} \\
f &= \pm\frac{n}{2L}\sqrt{\frac{T}{\rho}}
\end{aligned}
$$

where $n=1$ implies a fundamental frequency and $n \geq 2$ implies a harmonic frequency.

1.3 Linear Differential Equations

A **linear differential equation** has the form:

$$
a_n(t)\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + a_{n-1}(t)\frac{\mathrm{d}^{n-1} y(t)}{\mathrm{d}t^{n-1}} + \cdots + a_0(t)y(t) = b_n(t)\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n} + b_{n-1}(t)\frac{\mathrm{d}^{n-1} f(t)}{\mathrm{d}t^{n-1}} + \cdots + b_0(t)f(t)
$$

In this equation, we have two dependant variables: f and y. The 'classical' approach is to assume $f(t)$ is given, then solve for $y(t)$.

This equation is linear in the sense that it obeys the principle of superposition: if $y_1(t), y_2(t)$ are solutions corresponding to $f_1(t), f_2(t)$, then if $f(t) = k_1 f_1(t) + k_2 f_2(t)$, then $y(t) = k_1 y_1(t) + k_2 y_2(t)$.

Given constant coefficients, we would write this equation as

$$
a_n\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y(t)}{\mathrm{d}t^{n-1}} + \cdots + a_0 y(t) = b_n\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n} + b_{n-1}\frac{\mathrm{d}^{n-1} f(t)}{\mathrm{d}t^{n-1}} + \cdots + b_0 f(t)
$$

In shorthand, given

$$
\begin{aligned}
Dy &= \frac{\mathrm{d}y}{\mathrm{d}t} \\
D^2 y &= \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} \\
D^n y &= \frac{\mathrm{d}^n y}{\mathrm{d}t^n}
\end{aligned}
$$

we can write this as

$$
Q(D)y = P(D)y
$$

where $Q(x)$ and $P(x)$ are polynomials.

In general, we can assume $P(D)=1$.

Proof. Suppose $\tilde{y}(t)$ is a solution of $Q(D)y = f$ (e.g. $Q(D)\tilde{y} = f$). Now let $y = P(D)\tilde{y}$. Then

$$\begin{aligned} Q(D)y &= Q(D)P(D)\tilde{y} \\ &= P(D)Q(D)\tilde{y} \\ &= P(D)f \end{aligned}$$

So $y = Q(D)\tilde{y}$ solves the original equation. $\square$

Example 1.1. *Prove* $Dy = (D+1)f$

Proof. Suppose $Dy = (D+1)f$ and $f(t) = t, \forall t$. Let's find $\tilde{y}$ that solves

$$\begin{aligned} Q(D)\tilde{y} &= f \\ D\tilde{y} &= f \\ D\tilde{y} &= t \end{aligned}$$

So $\tilde{y} = \frac{1}{2}t^2 + C$. Let

$$\begin{aligned} y &= P(D)\tilde{y} \\ &= (D+1)\big(\frac{1}{2}t^2 + C\big) \\ &= t + \frac{1}{2}t^2 + C\frac{1}{2} \end{aligned}$$

Then

$$\begin{aligned} D\big(t + \frac{1}{2}t^2 + C\big) &= 1 + t \\ &= (D+1)t \end{aligned}$$

$\square$

Definition 1.1. *If $f(t)$ is continuous on an interval $a \leq t \leq b$, then there exists a solution $y(t)$ satisfying the above differential equation and also the "intial conditions" for $a \leq t_0 \leq b$*

$$\begin{aligned} y(t_0) &= P_0 \\ \frac{\mathrm{d}y(t_0)}{\mathrm{d}t} &= P_1 \\ \frac{\mathrm{d}^{n-1}y(t_0)}{\mathrm{d}t^{n-1}} &= P_{n-1} \end{aligned}$$

Moreover, this solution is unique.

Example 1.2. *For a falling block, given the mass and gravitational force, determine the distance fallen over a given time t.*

Proof.

$$\begin{aligned}
m\ddot{y} &= mg \\
\ddot{y} &= g \\
\dot{y} &= gt + C_1 \\
y &= \frac{1}{2}gt^2 + C_1 t + C_2
\end{aligned}$$

For an n-th degree differential equation, we will end up with n unknowns. Thus, we will need n sets of initial conditions to solve for a general solution. Let's assume $y(0) = 0$. This gives us

$$\begin{aligned}
y &= \frac{1}{2}gt^2 + C_1 t + C_2 \\
0 &= C_2
\end{aligned}$$

So we now have $y = \frac{1}{2}gt^2 + C_1 t$. Given a second initial condition $\dot{y}(0) = 0$, we have

$$\begin{aligned}
\dot{y} &= gt + C_1 \\
0 &= C_1
\end{aligned}$$

Thus

$$y = \frac{1}{2}gt^2$$

□

The **general solution** of the equation is an expression for y that solves the equation and contains n arbitrary constants.

Given a differential equation of order n and n initial conditions, we have an **initial value problem**.

To solve such problems, we find the general solution of the differential equation and plug in the initial conditions to evaluate the arbitrary constants.

Example 1.3. *We'll first find the general solution of the auxillary equation $q(D)y = 0$ (a "homogeneous equation" because the right-hand side is zero). Note: this contains n arbitrary constants and is called the complimentary solution y_c.*

Then, we'll find any solution of the original equation

$$Q(D)y = f$$

This is called a particular solution y_p.

To see why this works, let

$$y = y_c + y_p$$

. *Then*

$$\begin{aligned}Q(D)y &= Q(D)(y_c + y_p)\\ &= Q(D)y_c + Q(D)y_p\\ &= 0 + f\\ &= f\end{aligned}$$