CS 341 — Alorithms

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Contents

1 Solving Recurrences

The "Guess-and-Check" (Substitution) method involves guessing the form of the solution $T(n) \leq x$. We then verify our guess by induction proof and fill in any constants.

Example:

$$\begin{aligned}
T(n) &= 2T\left(\frac{n}{2}\right) + n^2, 7 \\
T(n) &\leq cn^2 \\
n = 1 : T(n) &= 7 \leq cn^2, c \geq 7 \\
T\left(\frac{n}{2}\right) &\leq c\left(\frac{n}{2}\right)^2 \\
T(n) &= 2T\left(\frac{n}{2}\right) + n^2 \\
&\leq 2c\left(\frac{n}{2}\right) + n^2 \\
&= \frac{2cn^2}{4} + n^2 \\
&= \left(\frac{c}{2} + 1\right)n^2 \\
&\leq cn^2, \frac{c}{2} + 1 \leq c, c \geq 2
\end{aligned}$$

We can then pick $c = 7$ and solve as

$$T(n) \leq 7n^2 \implies T(n) \in O(n^2)$$

We can also note that $T(n) \geq n^2$, so $T(n) \in \Theta(n^2)$.

Example:

$$\begin{aligned}
T(n) &= 3T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + 4T\left(\left\lfloor\frac{n}{4}\right\rfloor\right) + 1, 1 \\
T(n) &\leq cn^2 \\
n = 1 : T(n) &= 1 \leq cn^2, c \geq 1 \\
T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) &\leq c\left\lfloor\frac{n}{2}\right\rfloor^2 \\
T(n) &= 3T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + 4T\left(\left\lfloor\frac{n}{4}\right\rfloor\right) + 1 \\
&\leq 3c\left\lfloor\frac{n}{2}\right\rfloor^2 + 4c\left\lfloor\frac{n}{4}\right\rfloor + 1 \\
&\leq 3c\left(\frac{n}{2}\right)^2 + 4c\left(\frac{n}{2}\right)^2 + 1 \\
&= \left(\frac{3}{4} + \frac{4}{16}\right)cn^2 + 1
\end{aligned}$$

$$= cn^2 + 1$$

but since we could not get rid of the constant, we try

$$\begin{aligned}
T(n) &\le cn^2 - c_0 \\
n = 1 : T(n) &= 1 \le cn^2 - c_0, c \ge 1 + c_0 \\
T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) &\le c\left\lfloor \frac{n}{2} \right\rfloor^2 - c_0 \\
T(n) &= 3T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 4T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + 1 \\
&\le 3\left(c\left\lfloor \frac{n}{2} \right\rfloor^2 - c_0\right) + 4\left(c\left\lfloor \frac{n}{4} \right\rfloor - c_0\right) + 1 \\
&\le 3c\left(\frac{n}{2}\right)^2 - 3c_0 + 4c\left(\frac{n}{2}\right)^2 - 4c_0 + 1 \\
&= \left(\frac{3}{4} + \frac{4}{16}\right)cn^2 - 7c_0 + 1 \\
&= cn^2 - c_0, -7c_0 + 1 \le -c_0, c_0 \ge \frac{1}{6}
\end{aligned}$$

Pick $c_0 = \frac{1}{6}, c = \frac{7}{6}$ and we see that

$$T(n) \le \frac{7}{6}n^2 - \frac{1}{6} \implies T(n) \in O(n^2)$$

2 Algorithm Design Techniques

2.1 Divide and Conquer

Divide your problem into subproblems of the same type, then use recursion to solve each problem and combine the results.

Problem: Maxima