MATH 239 — Combinatorics

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1 Combinatorics

Combinatorics is discrete math dealing with 'counting questions' and graph theory. For example:

"How many binary strings of length n are there?": 2^n

"How many binary strings of length n are there which do not contain the (continuous) substring '0101'?"

"How many ways can you make change for a dollar? (in Canada!)"

"How many k-element subsets are there in an n-element set?": n choose $k = \frac{n!}{k!(n-k)!}$

"How many ways are there to order the numbers from 1 to n with no constraints?": $n!$

"Given 123 letters addressed to the 123 students in this class and 123 associated envelopes, how many ways are there to put one letter in each envelope such that nobody gets the right letter?": $\approx \frac{123!}{e}$

"How many n-polyminos (tetris blocks) are there?": $2 \to 1$, $3 \to 2$, $4 \to 5|7$

"How many prime numbers p are there such that $p+2$ is also prime?": infinite?

"How many rooted binary trees are there with n vertices?"

"Can the vertices of a graph be colored by n colors in such a way that every adjacent vertex is given a different color?"

"Can a graph be drawn such that no two edges are crossing?"

"Which graphs can be drawn in 3d-space to correspond to a fair symmetrical die?"

1.1 Course Notation

- $[n] = \{1, 2, 3, \ldots, n\}$

- $\mathbb{N} = \{1, 2, 3, \dots\}$
- $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- $|A \cup B| = |A| + |B| - |A \cap B|$
- $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$
- $|A \times B| = |A||B|$
- $\mathbb{R} \times \mathbb{R}$ is the cartesian plane
- $\mathbb{Z} \times \mathbb{Z}$ is the cartesian latice
- $(a, b) \neq (b, a)$
- $\binom{n}{k}$ is the number of k-element subsets of n.

1.2 Functions

A **function** $f : A \to B$ is a subset of $A \times B$ such that each element of A occurs as the first element of exactly one pair in the subset.

Given the set $\{(dog, 4), (duck, 2), (cat, 3), (cow, 4)\}$, the mapping from first to second values in each tuple is a function.

A **bijective** ($\rightarrowtail\!\!\!\!\twoheadrightarrow$) function is a 'one-to-one' (no two elements of A map to the same element of B) and 'onto' (each element of B is mapped to by some element of A) function.

A function f is bijective if and only if it has an **inverse**: a function $g : B \to A$ such that $f(g(x)) = x$ and $g(f(y)) = y$. If A and B are finite and a bijection $f : A \rightarrowtail\!\!\!\!\twoheadrightarrow B$ exists, then $|A| = |B|$.

2 Combinatorial Proofs (Bijective Proofs)

Proposition: There are 2^n subsets of $[n]$.

Proof:
Let X_n be the set of all subsets of $[n]$.
Let Y_n be the set of binary strings of length $[n]$.
We know that $|Y_n| = 2^n$.

For each set $S \in X_n$, let $f(S) = a_1, a_2, \dots, a_n$ where $a_i = \begin{cases} 1, & i \in S \\ 0, & \text{otherwise.} \end{cases}$
For each string $a_1, a_2, \dots, a_n \in Y_n$, let $g(a_1, a_2, \dots, a_n) = \{i \in [n] : a_1 = 1\}$
Then, g is an inverse function of f, so f is a bijective function and thus $|X_n| = |Y_n| = 2^n$

Proposition: $\binom{n}{k} = \binom{n}{n-k}$ for all $0 \leq k \leq n$.

Proof:
Let $x = \{k\text{-element subsets of}[n]\}$.
Let $y = \{n-k\text{-element subsets of}[n]\}$.
We know that $|y| = \binom{n}{n-k}$.
For each set $S \in X$, let $f(S) = [n] \setminus S$ and let $g(S) = [n] \setminus S$.
Then f is a bijective function and thus $|x| = |y|$ and so $\binom{n}{k} = \binom{n}{n-k}$.