I. Foundational knowledge.

1. $\mathbb{R}^n$ as a vector space : $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

1) $\vec{x}+\vec{y}$, $t\vec{x}$, $\vec{x}\cdot\vec{y}$

Notice : $\|\vec{x}\| = \sqrt{\vec{x}\cdot\vec{x}}$ $\quad \vec{x}=\vec{0}$ iff $\|\vec{x}\|=0$

$\vec{x} \perp \vec{y}$ iff $\vec{x}\cdot\vec{y}=0$.

2) $\vec{x} = \mathrm{Proj}_{\vec{v}}\vec{x} + \mathrm{Perp}_{\vec{v}}\vec{y}$, $\mathrm{Proj}_{\vec{v}}\vec{x} = \dfrac{\vec{x}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$.

Note : You shoud be cautious with the direction of $\vec{v}$ given in the problem to make sure whether you need $\mathrm{Prep}_{\vec{v}}\vec{x}$ or $\mathrm{Proj}_{\vec{v}}\vec{x}$.

3) The vector equation and scalar equation for a line or plane.

Line : vector $\vec{x} = t\vec{d}+\vec{p}$, $\vec{d}$ and $\vec{p}$ are fixed.

scalar $(\vec{x}-\vec{p})\cdot\vec{n}$, $\vec{n}$ and $\vec{p}$ are fixed.

$\vec{p}=0 \iff$ The line passed through the origin.

plane : vector : $\vec{x} = t_1\vec{a}+t_2\vec{b}+\vec{p}$, $\{\vec{a},\vec{b}\}$ are linearly independent.

scalar : $(\vec{x}-\vec{p})\cdot\vec{n}=0$ $\quad \vec{n}$ is the norm vector.

$\vec{p}=0 \iff$ The plane passes through the origin.

Ex: If the plane is described by $3x_1-2x_2+5x_3=0$ then $\vec{n} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$.

2. Matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \quad n\times m \text{ matrix}$$

1) $A+B$, tA , A^T ($(A^T)^T = A$)

2) AB Note : In general $AB \neq BA$. $\quad (AB)^T = B^T A^T$

3) $$A\vec{x} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{pmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_m\vec{a}_m$$

Here $(\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_m) = A$ $\quad \vec{a}_i$'s are the column vectors of A.

3. Solving the linear system of equations.

1) Any linear system of equations can be expressed as

$A\vec{x} = \vec{b}$. If A is $n \times m$, $\vec{x} \in \mathbb{R}^m$ and $\vec{b} \in \mathbb{R}^n$.

$\vec{b} = 0 \Rightarrow$ The system of equations is homogeneous.

2) Solving them always involves

i. $(A|\vec{b}) \overset{ERO}{\sim} (REF \mid \vec{c})$ for inhomogenous system, or.

$A \overset{ERO}{\sim} REF$ for homogeneous system,

(ERO = elementary row operations)

followed by the backward substitution. ⇐ Practice and get familiar with this.

ii) For inhomogeneous equations, know the criteria for the system to be consistent, i.e. there exists at least one solution.

Homogeneous equations are always consistent because $\vec{x} = \vec{0}$ is always a solution. But this solution is often not interesting and therefore is called the trivial solution.

iii) Counting the # of free parameters in the most general solution.

# of free parameters = # of unknowns − # of leading entries in the REF.

= # of columns of A − Rank A.

Note: Rank A = # of leading entries in its REF.

Note: Rank A ≤ min (# of columns , # of rows).

Note : Homogeneous system has nontrivial solution (i.e. $\vec{x} \neq 0$) iff # of free parameters > 0 , or Rank A < # of its columns.

One usually identify the free parameters in a general solution to be the unknowns variables to which the corresponding columns in the REF Contains NO leading entries.

II. Common sense knowledge.

1. Definition for vector spaces. (The Ten axioms, understand the idea.)

2. Definition for subspaces: subsets of vector spaces that are vector spaces themselves.

   3 criteria 1) non-empty (most easily checked by the existence of $\vec{0}$.)

   2) $\vec{x}, \vec{y} \in S \Rightarrow \vec{x} + \vec{y} \in S$.

   3) $\vec{x} \in S$, $t\vec{x} \in S$.

3. Definition for Span $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$

   Thm: Any Span is always a subspace.

4. Linear dependency of $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$:

   If the only solution to the equations $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = 0$ is $a_1 = a_2 = \cdots = a_k = 0$, we say $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ is linearly independent. If there exist solutions of $a_1, a_2, \cdots, a_k$ not all zero, we say $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ is linearly dependent.

   Since $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = A\vec{a}$ where $A = (\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_k)$ is a matrix.

   Linear dependency $\Leftrightarrow$ Existence of non-trivial solution for $A\vec{a} = 0$

   $$\Updownarrow$$

   $$\text{Rank } A < k.$$

   Note: If $\vec{x} \in \text{Span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\} \Rightarrow \{\vec{x}, \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ is linearly dependent.

   If $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ is linearly dependent, alway exist some $\vec{v}_i$ that can be expressed as linear combination of others.

5. Basis for any vector space / subspaces.

   $B = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ is a basis for a vector space S iff :

i) B is linearly independent.
ii) Span $B = S$.

Note: Basis for any given S is NOT unique. but all bases consist of equal # of vectors. $\Rightarrow$

Defn: $\dim S$ = # of vectors of its basis.

Note: If B is a basis for S, for all $\vec{x} \in S$, there exists a unique expansion $\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k$, $\vec{v}_1 \vec{v}_2, \cdots, \vec{v}_k \in B$.

Note: $\{\hat{e}_1, \hat{e}_2, \cdots, \hat{e}_n\}$ is called the standard basis for $\mathbb{R}^n$.

6. The true face of matrices:

1) $f: \mathbb{R}^m \to \mathbb{R}^n$ is a linear mapping if
$f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y})$, $f(t\vec{x}) = tf(\vec{x})$.

2) Any linear mapping can be represented by a matrix (the standard matrix $[f]$).
$f(\vec{x}) = [f]\vec{x}$

$[f] = (f(\hat{e}_1) \;\; f(\hat{e}_2) \;\; \cdots \;\; f(\hat{e}_m))$, a $n \times m$ matrix.

Note: rotation, reflection in a line or plane, projection in a line or plane are all linear mappings.
Know how to find their standard matrices.

3) $Null(f) = \{\vec{x} \mid f(\vec{x}) = 0\}$.
$Range(f) = \{\vec{x} \mid f(\vec{y}) = \vec{x}$ for some $\vec{y}\}$

$Null(A) = \{\vec{x} \mid A\vec{x} = 0\}$ is just the solution space for $A\vec{x} = 0$

$Range(A) = \{\vec{x} \mid A\vec{y} = \vec{x}$ for some $\vec{y}\}$
$= Span\{$column vectors of $A\} = Col(A)$

$Row(A) = Span\{$row vectors of $A\}$.
These are all subspaces. (Prove it!)

Note: Know how to find a basis for Null (A), Row (A) and Col(A).

It always start with reducing $A \overset{ERO}{\sim} REF$.

$Row(A) = Span\{$ Non-zero row vectors in the REF$\}$

$Col(A) = Span\{$ the column vectors in A in the same column as those in the REF that contain the leading entries. $\}$

$Null(A) = \{$ the most general solution to $A\vec{x} = 0 \}$.

Thm:

Rank A + dim Null (A) = # of columns of A

Rank A = dim Col (A) = dim Row (A).

$\Rightarrow$ This proves Rank A $\leq$ min(# of rows , # of columns).

Note: Finding a basis for $Span\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is the same as finding a basis for Col(A) where $A = (\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_k)$.

7. More abstract vector spaces:

Polynomial space, matrix space. function space and etc.

Know how they are defined, how to find a basis for them, how to identify subspaces and find a basis for them, and how to examine linearly dependency among abstract objects.

Ex: $S = \{a + bx + cx^2 \mid a + c - 2b = 0\}$. Find a basis.

For any $p \in S$, $a - 2b = -c$ and $c = -a + 2b$

$\Rightarrow p = a + bx + (-a+2b)x^2 = a(1-x^2) + b(x+2x^2)$

$\Rightarrow S = Span\{1-x^2, x+2x^2\}$.

III Advanced materials.

All about square matrices. Let us assume it's $n \times n$.

1. Inverse matrix. $A^{-1}A = AA^{-1} = I$.

$$I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

1): Inverse exists iff $A\vec{x}=0$ implies $\vec{x}=0$. This is because if A^{-1} exists,

$A\vec{x}=0 \Rightarrow A^{-1}A\vec{x}=A^{-1}0 \Rightarrow I\vec{x}=0 \Rightarrow \vec{x}=0$.

$\Rightarrow$ A^{-1} exists is equivalent to all below: (we assume A is $n \times n$.)

i) $Null(A) = \{\vec{0}\}$

ii) $Rank\ A = n$

iii) $Col(A) = Row(A) = \mathbb{R}^n$.

iv) $\det A \neq 0$

v) 0 is not an eigenvalue of A.

Notice: Since $\det(AB\cdots D) = \det A \det B \cdots \det D$, product of any matrices is invertible iff all the factor matrices are!

2) Finding the inverse:

$(A \mid I) \overset{ERO}{\rightsquigarrow} (RREF \mid B)$. If the RREF $= I$, A is invertible and

$B = A^{-1}$, otherwise A is not invertible.

3) $(A^{-1})^{-1} = A$. $(A^T)^{-1} = (A^{-1})^T$ $\quad (tA)^{-1} = \frac{1}{t}A^{-1}$.

$(AB)^{-1} = B^{-1}A^{-1}$

4) Know the definition for elementary matrices and their usage.

i) How are they constructed.

ii) What do they do when multiplying a matrix on the left.

5) Understand the identity mapping and the inverse mapping. If $f \circ g = id$, $[f][g] = I$.

2. Determinants.

1) The recursive definiton by the cofactor expansion:

$C_{ij} = (-1)^{i+j} \det A(i,j)$. $A(i,j)$ is the submatrix obtained from A by crossing out the i-th row and j-th column.

Note: Be careful with $(-1)^{i+j}$. For 3x3, we can visualise it as

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\begin{aligned} \det A &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + \cdots + a_{2n}C_{2n} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + \cdots + a_{n3}C_{n3} \\ &\ \ \vdots \end{aligned}$$

Note: Cofactor expansion can be done along any row or column. So pick the one with the most 0's.

2) i) $\det A \to -\det A$ upon exchanging two rows or columns.

ii) $\det A \to r \det A$ if any row or column is multiplied by r.

iii) $\det A$ is unchanged if any multiples of one row or column are added into or subtracted from any other row or column

Note: $\det(rA) = r^n \det A$ (A is $n \times n$).

Note: $\det A = 0$ if any two rows or columns are proportional.

In fact, one can state more strongly that $\det A \neq 0$ iff all its columns or row vectors are linearly Independent. (Prove this!)

Note: One should always use there properties to calculate the determinants much more quickly.

3) $\det(AB) = \det A \det B$

$\det A^T = \det A$

$\det A^{-1} = (\det A)^{-1}$.

4) Upper/lower triangular matrices or digonal matrices:

$\det A = a_{11}a_{22}a_{33}\cdots a_{nn}$, i.e. the product of its diagonal entries.

5) When $\det A \neq 0$.

$$A^{-1} = \frac{1}{\det}(\mathrm{Cof}\, A)^T.$$

$$\mathrm{Cof}\, A = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$$

3. Eigenvalue and eigenvectors.

1) $A\vec{v} = \lambda\vec{v}$, $\vec{v} \neq 0$. $\vec{v}$: eigenvector. λ: the corresponding eigenvalue.
λ may be zero. But if $\lambda = 0$ is an eigenvalue of A, it means
$A\vec{v} = 0$ for some $\vec{v} \neq 0$. Therefore we find $\dim \mathrm{Null}(A) \geq 1$,

$\mathrm{Rank}\, A \leq n-1$, $\det A = 0$ and A is not invertible.

2) $E_\lambda(A) = \{\vec{x} \mid A\vec{x} = \lambda\vec{x}\}$, the eigenspace for eigenvalue λ.
It is a subspace.

3) Eigenvectors corresponding to distinct eigenvalues are linearly independent.

4) Finding eigenvalues: $\det(A - \lambda I) = 0$ The Characteristic Polynomial.

$\det(A - \lambda I) = (\lambda_1 - \lambda)^{n_1}(\lambda_2 - \lambda)^{n_2} \cdots (\lambda_k - \lambda)^{n_k}$. Here $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \cdots \neq \lambda_k$.
and $n_1 + n_2 + \cdots + n_k = n$.

$\lambda_1, \lambda_2, \cdots, \lambda_k$ are eigenvalues of A and $n_1, n_2, \cdots, n_k$ are their algebraic multiplicities.

If $\dim E_{\lambda_i} < n_i$, we say λ_i is deficient.

Finding the eigenvectors for the eigenvalue λ_i, solve
$(A - \lambda_i I)\vec{x} = 0$. Eigenvectors are never unique so choose the free parameters any way one prefers.

5) $\det A = \lambda_1^{n_1} \lambda_2^{\lambda_2} \cdots \lambda_k^{n_k}$ for any square matrix A.

6) Diagonalization: if exists a P, invertible, such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$, we say A is diagonalizable.

Note: A is diagonalizable iff none of its eigenvalues are deficient.

To diagonalize A, find n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ (know how!) and arrange them in the columns of P i.e.
$P = (\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n)$

i) Solve $\det(A-\lambda I)=0$ for eigenvalues and identify the algebraic multiplicities for each solution.

ii) For each λ_i found, solve $(A-\lambda_i I)\vec{x}=0$ for the eigenvectors.

iii) If for all λ_i, the # of free parameters for the solution to $(A-\lambda_i I)\vec{x}=0$ matches its algebraic multiplicity, A is diagonalizable. Group all the n linearly independent eigenvectors, one finds P.

If for any λ_i, the # of free parameters is smaller than its algebraic multiplicity, the eigenvalue is deficient and A is not diagonalizable.

7) If A is diagonalizable $P^{-1}AP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \Rightarrow A = P\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}P^{-1}$.

It is easy to calculate $A^k = P\begin{pmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{pmatrix}P^{-1}$

6. Othogonal transformation, orthonormal basis and orthogonal matrices.

1) Definition for orthonormal basis and their usage as defining a new coordinate system.

2) Coordinate transformation rule: $(\vec{x})_B = P^T\vec{x}$, where

$P = (\hat{v}_1, \hat{v}_2 \cdots \hat{v}_n)$, orthonormal.

The standard matrix for any linear mapping transform as
$[f]_B = P^T [f] P$.

3) When $(\hat{v}_1\ \hat{v}_2 \cdots \hat{v}_n)$ orthonormal, $P = (\hat{v}_1\ \hat{v}_2 \cdots \hat{v}_n)$ satisfy
$P^TP = PP^T = I$. We say P is orthogonal.
Note: $P^TAP = B \Rightarrow A = PBP^T$.
Note: When P is orthogonal, $P^{-1} = P^T$.
Note: Orthogonal matrices satisfying $\det P = +1$ are called special orthogonal matrices and they are 1-1 correspondent to the standard matrices for all rotations.

4) Construct the orthonormal basis. Gram-Schmidt procedure allows one to construct an orthogonal set based on any given linearly independent vectors. This procedure allows one to find an orthogonal (or orthonormal if one further normalizes the vectors) from any basis.

5) S is a subspace of $\mathbb{R}^n$. Know the definition for $S^\perp$, and that
$\dim S + \dim S^\perp = n$.

Know how to find the orthogonal complement, and extend an orthonormal basis for S to an orthonormal basis for $\mathbb{R}^n$.

Note: Just solve $\vec{x}\cdot\vec{v}_i = 0$ for all $i=1,\cdots,k$, assuming $\{\vec{v}_1, \vec{v}_2 \cdots \vec{v}_k\}$ is a basis for S.

Ex: In $\mathbb{R}^3$ the orthogonal complement to a plane defined by $\vec{n}\cdot\vec{x} = 0$ is just $\mathrm{Span}\{\vec{n}\}$ and is a line.

7. Orthogonally diagonalization.

1) If there exists $P^TAP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, where $P^TP = PP^T = I$ (P is orthogonal),

we say A is orthogonally diagonalizable.

Note: If $P^TAP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, $A = P\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}P^T$, assuming P orthogonal.

2) A is orthogonally diagonalizable iff $A = A^T$, i.e. A is symmetric.

3) Finding P.

i) Follow the standard procedure to find all n linearly independent eigenvectors.

ii) For eigenvectors corresponding to the same eigenvalue, apply Gram-Schmidt to make them orthogonal.

iii) Normalize each eigenvector to make it unit-length.

Group the vectors found above in the columns to form P.

Note: And symmetric real matrix only has real eigenvalues.