Translations and momentum operator in quantum mechanics

From commutators to rappresentations

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1 The translation operator

We want to define an operator that translate the position of the state of a system. This should be analogous to the translation of a classical particle and from this analogy we will deduce its proprieties. For simplicity, we will consider a system for which his position is defined on just one dimension.

Definition 1.1. *Given an arbitrary autoket of the position operator $|x\rangle$, the translation, of a distance Δx, operator $\hat{T}(\Delta x)$ is defined such as:*

$$\hat{T}(\Delta x)\,|x\rangle = |x + \Delta x\rangle \tag{1.1}$$

Now, from the assumption that $\hat{T}(\Delta x)$ should behave as a classical translation, we impose that this operator satisfies a set of proprieties:

- classical translations can be composed $\Rightarrow$ $\hat{T}(\Delta x)\hat{T}(\Delta y) = \hat{T}(\Delta x + \Delta y)$,
- classical translations commute $\Rightarrow$ $[\hat{T}(\Delta x), \hat{T}(\Delta y)] = 0$,
- as $\Delta x \to 0$ a classical particle doesn't move $\Rightarrow$ $\hat{T}(0) = \hat{1}$.

From these proprieties we can get another one:

$$\hat{T}(\Delta x)\hat{T}(-\Delta x) = \hat{T}(\Delta x - \Delta x) = \hat{T}(0) = \hat{1} \;\Rightarrow\; \hat{T}^{-1}(\Delta x) = \hat{T}(-\Delta x).$$

Lastly we need to evaluate the commutator of this new operator with the position operator $\hat{x}$, Given an arbitrary autoket $|x\rangle$:

$$\begin{aligned}
[\hat{x}, \hat{T}(\Delta x)]\,|x\rangle &= \hat{x}\hat{T}(\Delta x)\,|x\rangle - \hat{T}(\Delta x)\hat{x}\,|x\rangle \\
&= \hat{x}\,|x + \Delta x\rangle - \hat{T}(\Delta x)\,|x\rangle\,x = \hat{x}\,|x + \Delta x\rangle - \hat{T}(\Delta x)\,|x\rangle\,x \\
&= |x + \Delta x\rangle\,(x + \Delta x) - |x + \Delta x\rangle\,x = |x + \Delta x\rangle\,\Delta x.
\end{aligned}$$

Using the definition of the translation operator we get:

$$[\hat{x}, \hat{T}(\Delta x)] = \hat{T}(\Delta x)\Delta x \tag{1.2}$$

1.1 Infinitesimal translations

From the translation operator, that we have just defined, we can obtain another, perhaps more useful, operator. Let's consider an infinitesimal translation dx, its operator counterpart will be $\hat{T}(dx)$ such that:

$$\hat{T}(dx)\,|x\rangle = |x + dx\rangle\,.$$

It is clear that being still a translation $\hat{T}(dx)$ has the same proprieties of a finite translation, we will ask furthermore that every infinitesimal translation will be same kind of expression of the first order in dx. In this way, every expression of higher order will be considered negligible.

We will also think the infinitesimal translation as the limit for $\Delta x \to 0$ of a finite translation, in this way we can calculate the commutator $[\hat{x}, \hat{T}(dx)]$ using a limit procedure.
Firstly we can express (in this limit) the translated state as $|x + dx\rangle = |x\rangle + O(dx)$, the equation (1.2) reads:

$$[\hat{x}, \hat{T}(dx)] \, |x\rangle = |x + dx\rangle \, dx = |x\rangle \, dx + O[(dx)^2].$$

Discarding higher order terms in dx we get that the commutator for infinitesimal translations is then:

$$[\hat{x}, \hat{T}(dx)] = \hat{1} dx. \tag{1.3}$$