Differential geometry

A brief guide for general relativity

Luca Morelli

March 2024

Preface:

Contents

1

Introduction

2

Manifolds: vectors, 1-forms and tensors

The main idea that will guide us defining manifolds is that we want to generalize structures of $\mathbb{R}^n$ to a generic set of points. With this idea in mind, a manifold should be just the most basic set of points that we can use, combined with maps from it to $\mathbb{R}^n$.

If this construction is done in the right way, we will be able to transport backwards operation from $\mathbb{R}^n$ to our manifold.

2.1 Topological spaces

One of the most crucial properties of functions on $\mathbb{R}^n$ is *continuity*, therefore it is natural to consider as starting point some kind of set on which continuity is definable: those are **topological spaces**.

Definition 2.1 (Topological spaces)**.** *A topological space is the pair of a generic set $\mathcal{M}$ and a family of some of its subsets $\mathcal{T}$ (called opens) such that:*

- $\varnothing, \mathcal{M} \in \mathcal{T}$,

- $\bigcup_\alpha O_\alpha \in \mathcal{T}$ *with* $\{O_\alpha /\ O_\alpha \in \mathcal{T}\}$ *(finite collection or not),*

- $\bigcup_{i=1}^N O_i \in \mathcal{T}$ *with* $\{O_1, O_2, ..., O_N /\ O_i \in \mathcal{T}\}$.

Furthermore, we are interested in a specific family of topological spaces, **Hausdorff spaces**.

Definition 2.2. *A topological space $(\mathcal{M}, \mathcal{T})$ is a Hausdorff one if $\forall P, Q \in \mathcal{M}$ exist U and V, neighborhoods respectively of P and Q, such that $U \cap V = \varnothing$.*

On this kind of spaces we can define a topological continuity (which is actually equivalent to the one form calculus) and thus we will be able to define **continuous maps**.

Definition 2.3. *A map $\phi : \mathcal{M} \to \mathcal{N}$, with $\mathcal{M}$, $\mathcal{N}$ topological spaces, is continuous if, given $V \in \mathcal{N}$ open, then $\phi^{-1}(V) \in \mathcal{M}$ is an open set.*

2.2 Maps, charts and atlases

From the topological space we now want to build a connection to $\mathbb{R}^n$: this process is inspired by the process of building the map of a specific portion of land, since on the map we can evaluate distances and other quantities that otherwise we would have to determine by moving around that specific portion of land. In the same way we will do all the calculation on $\mathbb{R}^n$ instead of directly using points on the topological space, where we don't have coordinates and other constructions.

Definition 2.4 (Maps)**.** *An application $\phi : D \to \mathbb{R}^n$, with $D \subseteq \mathcal{M}$ open, is called a map.*

Not all the maps are "good" maps, more precisely we want maps that are continuous (near points get mapped to near points) and that allow us to go back from $\mathbb{R}^n$ to the topological space. Lastly, since usually the domain of a map doesn't correspond to the whole topological space, it is useful to know where the map is valid, and thus we want to encode the domain and the map together.

Definition 2.5 (Charts). *A pair of a continuous invertible map $\phi : A \to \mathbb{R}^n$ and its domain, (A, ϕ), is called a chart.*

As we said, usually a chart is not enough to cover the whole topological space, however we need to map it all to $\mathbb{R}^n$. To do so we can use multiple charts, with some sort of notion of compatibility between them.

Definition 2.6 (Atlases). *A collection of charts $\mathcal{A} = \{(A_i, \phi_i)\}$ is an atlas if:*

- $\bigcup_i A_i \supseteq \mathcal{M}$,

- *for each pair (A_i, ϕ_i), (A_j, ϕ_j) exist a function $\psi : \phi_i(A_i) \to \phi_j(A_j)$, invertible and such that $(\phi_j^{-1} \circ \psi \circ \phi_i) = id$, on $A_i \cap A_j$.*

Constructing the atlas of a topological space means (operationally) to build the **manifold**, which actually is the class of equivalence of all the atlases that can be put in bijection.
Furthermore, the functions that assure the compatibility of charts will characterize the manifold itself:

- since all ψ are invertible, all charts of an atlas will map to the same $\mathbb{R}^n$, and thus n is the **dimension** of the manifold;

- if all the $\psi \in C^p(\mathbb{R}^n)$, the manifold is said a **p-differentiable manifold**.

2.3 Functions, curves and vectors

Now, that we have built the full structure of a manifold, we can start to use it. On a topological space we can define function $f : \mathcal{M} \to \mathbb{R}$ and then determine if it is continuous, but we cannot define whether it is smooth, since we cannot take derivatives on the manifold. Instead, we can map the manifold to $\mathbb{R}^n$ and from there define differentiability.

Definition 2.7. *A function $f : \mathcal{M} \to \mathbb{R}$ is p-differentiable if*

$$f \circ \phi_i^{-1} \in C^p(\phi_i(A_i)),$$

for all the charts (A_i, ϕ_i) of the given manifold.

Functions represent scalars, since their value depends only on the point of the manifold and not from the coordinates of the charts. After scalars, we also need to define vectors, to do that we will rely on the intuition that tangent vectors, to a curve, are given by the derivative of the coordinates along it (in $\mathbb{R}^n$).

Definition 2.8 (Curves). *A curve is a piece-wise continuous map $\gamma : I \subseteq \mathbb{R} \to \mathcal{M}$.*

Observation 2.1. *Derivative operators are linear and thus they form a vector space. Let's consider a map $\phi : A \to \mathbb{R}^n$, a curve $\gamma : I \to \mathcal{M}$ and a function $f : \mathcal{M} \to \mathbb{R}$. Composing f with γ we can evaluate its derivative along the curve:*

$$\frac{d}{d\lambda}(f \circ \gamma) = \frac{d}{d\lambda}f(\gamma(\lambda)).$$

Composing now $f \circ \gamma$ with the map ϕ and using the Leibniz rule we obtain:

$$\frac{d}{d\lambda}[(f \circ \phi^{-1})(\phi \circ \gamma)] = \frac{\partial(f \circ \phi^{-1})}{\partial x^\mu}\frac{d(\phi \circ \gamma)}{d\lambda} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda}.$$

This procedure gives us a way construct the vector space of derivatives, along a given curve, at each point of the manifold

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \frac{\partial}{\partial x^\mu}.$$

Notice that the chart induces a basis for the vectors: $\frac{dx^\mu}{d\lambda} \in \mathbb{R}$, and thus can be interpreted as components, while $\frac{\partial}{\partial x^\mu}$ are still derivatives from which (by linear combinations) we can construct all the others. For an n-dimensional manifold, each chart will induce n partial derivatives and thus it is natural to use those as basis for the tangent spaces.

Definition 2.9 (Vectors). *A vector in a point $P \in \mathcal{M}$ is defined as the differential operator along a curve γ*

$$\boxed{\vec{V}_\gamma = \frac{d}{d\lambda}\bigg|_P = \frac{dx^\mu}{d\lambda} \frac{\partial}{\partial x^\mu}\bigg|_P},$$

whose components are defined by the choice of a chart.
The vector space spanned by $\frac{\partial}{\partial x^\mu}\big|_P$ is called the tangent space in P, T_P.

Notice that also vectors are really invariant objects; the components are not and from chain rule, changing chart, we get the components' transformation law:

$$\vec{V} = V^\mu \frac{\partial}{\partial x^\mu} = V^\mu \frac{\partial y^{\nu'}}{\partial x^\mu} \frac{\partial}{\partial y^{\nu'}} = V^{\nu'} \frac{\partial}{\partial y^{\nu'}}.$$

Definition 2.10. *Given an open subset of the manifold $U \in \mathcal{M}$, a vector field is an application that maps each point $P \in U$ to a vector $\vec{V}(P) \in T_P$.*

2.4   Integral curves, exponential maps and commutators

Right now we have defined vectors from the notion of curves, we will now show how from a vector field we can construct a set of curves.
Consider a vector field $\vec{V} = \frac{d}{d\lambda}$, defined on some open subset U of a manifold, and let's pick point $P_0 \in U$. If we introduce coordinates, through a chart $\phi(P) = x^\mu(P)$, we can define a *Cauchy problem* from the composition of the first two objects and the chart:

$$\begin{cases} \frac{d}{d\lambda} x^\mu = V^\mu(\lambda), \\ x^\mu(\lambda_0) = x^\mu(P_0). \end{cases}$$

From calculus theorems, we know that there exist a unique solution $x^\mu(\lambda)$, if we now compose this function with the inverse of the chart we obtain a unique curve $\gamma(\lambda)$ on the manifold.

Definition 2.11 (Integral curves). *Given a vector field $\vec{V}$, defined on $U \in \mathcal{M}$, and a curve $\gamma : I \to U$, we call γ the integral curve of $\vec{V}$ if: $\frac{d}{d\lambda}\big|_P = \vec{V}(P) \quad \forall P \in \gamma(I)$.* [1]

Another way, to better resume the previous result, is the *exponential map*: consider a vector field and its integral curve on some open set, composing this last one with a chart we can then Taylor expand it along the curve to get:

$$x^\mu(\lambda_0 + \epsilon) = x^\mu(\lambda_0) + \epsilon \frac{dx^\mu}{d\lambda}\bigg|_{\lambda_0} + \frac{\epsilon^2}{2!} \frac{d^2 x^\mu}{d\lambda^2}\bigg|_{\lambda_0} + \cdots = e^{\epsilon \frac{d}{d\lambda}} x^\mu\big|_{\lambda_0} = e^{\epsilon \vec{V}} x^\mu\big|_{\lambda_0}.$$

---

[1]Here $\frac{d}{d\lambda}$ is defined as the tangent vector to γ.

Notice that the exponential map is an implicit way to define a solution of the Cauchy problem we defined for integral curves and gives us an easy way (but most of the time just formal) to obtain connected points on a manifold, using just a vector field. Also notice that we could Taylor expand every smooth function defined on the manifold (along a curve), in the same way as we have just done, in this way the exponential map gives us the function in other points:

$$f(\lambda + \epsilon) = f(\lambda_0) + \epsilon \frac{df}{d\lambda}\Big|_{\lambda_0} + \frac{\epsilon^2}{2!}\frac{d^2 f}{d\lambda^2}\Big|_{\lambda_0} + \cdots = e^{\epsilon \frac{d}{d\lambda}} f\big|_{\lambda_0} = e^{\epsilon \vec{V}} f\big|_{\lambda_0}.$$

Having exponential maps, we can now discuss a rather unexpected object that we can define for vector, as we have constructed them.

Definition 2.12 (Commutators). *Given two vectors $\vec{V} = \frac{d}{d\lambda}$, $\vec{W} = \frac{d}{d\sigma}$, in the tangent space T_P of a given manifold, we can define the commutator:*

$$[\vec{V}, \vec{W}] = \frac{d}{d\lambda}\frac{d}{d\sigma} - \frac{d}{d\sigma}\frac{d}{d\lambda}.$$

We are not used to define commutators of vectors, even thought now these are linear operators, therefore it could seem hard to interpret the meaning of this object.
Consider two vector fields $\frac{d}{d\lambda}$, $\frac{d}{d\sigma}$ and let's follow their integral maps from a point P to some other point. A priori we cannot know if moving along one vector field first will lead to the same point we would get starting first from the other.

$$x^\mu(A) = e^{\epsilon_1 \frac{d}{d\lambda}} e^{\epsilon_2 \frac{d}{d\sigma}} x^\mu\big|_P = x^\mu(P) + \epsilon_1 \frac{dx^\mu}{d\lambda}\Big|_P + \epsilon_2 \frac{dx^\mu}{d\sigma}\Big|_P + \epsilon_1\epsilon_2 \frac{d}{d\lambda}\frac{dx^\mu}{d\sigma}\Big|_P + \dots \ ,$$

$$x^\mu(B) = e^{\epsilon_2 \frac{d}{d\sigma}} e^{\epsilon_1 \frac{d}{d\lambda}} x^\mu\big|_P = x^\mu(P) + \epsilon_1 \frac{dx^\mu}{d\lambda}\Big|_P + \epsilon_2 \frac{dx^\mu}{d\sigma}\Big|_P + \epsilon_1\epsilon_2 \frac{d}{d\sigma}\frac{dx^\mu}{d\lambda}\Big|_P + \dots \ .$$

Subtracting the coordinates of the second point from the first we obtain:

$$x^\mu(A) - x^\mu(B) = \epsilon_1\epsilon_2 \left[\frac{d}{d\lambda}\frac{d}{d\sigma}\right] x^\mu\big|_P \cdots \ ,$$

clearly, from this calculation, we can see that in order to have the two path to end in the same point $A = B$, the commutator must vanish. Therefore, two commuting vector fields will generate "commuting" paths as integral curves.
In this way we can use the parametrization parameter, of the integral curves of n linearly independent[2] vector fields, as coordinates of some chart. In this case, the chart is well-defined, since the change of coordinate matrix, $\frac{\partial x^\mu}{\partial \lambda_i}$, is really just the coordinates of each vector, and thus it is invertible (we assumed linearly independent vectors).

Lastly we evaluate in components the commutator of two vectors: fixing a chart

$$\begin{aligned}
[\vec{V}, \vec{W}] &= \frac{d}{d\lambda}\frac{d}{d\sigma} - \frac{d}{d\sigma}\frac{d}{d\lambda} = \frac{dx^\nu}{d\lambda}\partial_\nu\left(\frac{dx^\mu}{d\sigma}\partial_\mu\right) - \frac{dx^\nu}{d\sigma}\partial_\nu\left(\frac{dx^\mu}{d\lambda}\partial_\mu\right) \\
&= \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\sigma}(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu) + \left(\frac{dx^\nu}{d\lambda}\partial_\nu\frac{dx^\mu}{d\sigma} - \frac{dx^\nu}{d\sigma}\partial_\nu\frac{dx^\mu}{d\lambda}\right)\partial_\mu \\
&= (V^\nu\partial_\nu W^\mu - W^\nu\partial_\nu V^\mu)\partial_\mu,
\end{aligned}$$

it is now clear that the commutator of two vectors is still another one (this was trivial).

---

[2]Linearly independent here means that the coefficients of the linear combination are functions on $\mathcal{M}$, and in each point they make the vectors independents.

2.5   1-Forms and tensors

We have now introduced vectors and their formalism, actually what we have introduced is really, in components, what we usually refer as contravariant vectors, we now want to define co-vectors and tensors, which contain both.

On manifolds co-vectors are linear maps from tangent spaces to $\mathbb{R}$, which are the natural generalization of vectors of the dual space[3].

Definition 2.13 (1-forms)**.** *Given a specific point $P \in \mathcal{M}$, we can define a 1-form in that point as a linear map $\tilde{\omega} : T_P \to \mathbb{R}$.*
1-forms made up a vector space at each point of a manifold called cotangent space T_P^*.

Notice that also vectors, when applied to functions, result in a real number, therefore it is natural to define a 1-form associated to a function (in the opposite way a vector is associated to a curve) such that, when applied to a vector, it will result in the directional derivative of the function along the curve:

$$\tilde{d}f(\vec{V}) = \frac{df}{d\lambda} = \vec{V}(f).$$

Imposing this consistency relation, we can define a dual basis: to better see this we just need to use the chain rule with composition with a chart, then we substitute every term containing the parameter of the curve with the supposed 1-form basis:

$$\tilde{d}f(\vec{V}) = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda} \quad \Rightarrow \quad \boxed{\tilde{d}f = \frac{\partial f}{\partial x^\mu}\tilde{d}x^\mu}.$$

We just now need to define the 1-form basis $\tilde{d}x^\mu$ such that $\tilde{d}x^\mu(\vec{V}) = \tilde{d}x^\mu\left(\frac{x^\nu}{d\lambda}\partial_\nu\right) = \frac{x^\nu}{d\lambda}\delta^\mu_\nu$.
This basis is thus defined by:

$$\boxed{\tilde{d}x^\mu(\partial_\nu) = \delta^\mu_\nu}.$$

Having defined 1-form, we can now introduce tensors, which are just combinations of vectors and 1-forms.

Definition 2.14 (Tensors)**.** *A (n,m) rank tensor, in a specific point $P \in \mathcal{M}$, is a multilinear map*

$$T : \underbrace{T_p^* \otimes \cdots \otimes T_p^*}_{m\ times} \otimes \underbrace{T_P \otimes \cdots \otimes T_P}_{n\ times} \longrightarrow \mathbb{R}.$$

Fixing a chart, and thus a basis in both T_P and T_P^*, we can expand T in components:

$$\boxed{T = T^{\mu_1,\ldots,\mu_n}_{\nu_1\ldots,\nu_m}\partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_n} \otimes \tilde{d}x^{\nu_1} \otimes \ldots_\otimes \tilde{d}x^{\nu_m}}$$

where $T^{\mu_1,\ldots,\mu_n}_{\nu_1\ldots,\nu_m} = T(\partial_{\mu_1},\ldots\partial_{\mu_n},\tilde{d}x^{\nu_1},\ldots,\tilde{d}x^{\nu_m}).$

Both 1-forms and tensors can be generalized to fields that maps each point of the manifold to a tensor in that point.

Lastly we should concern about the change of basis for 1-forms, from which we can obtain the full transformation law of tensor components. Let's consider the action of $\tilde{d}x^\mu$ on the basis of T_P, then let's consider a new chart and a new associated basis, defined by:

$$\tilde{d}y^{\mu'}\left(\frac{\partial}{\partial y^{\nu'}}\right) = \delta^{\mu'}_{\nu'}.$$

---

[3]Actually co-vectors are really vectors of the dual space even in special relativity.

The transformation of the vector basis is given by the chain rule $\frac{\partial}{\partial y^{\nu'}} = \frac{\partial x^{\mu}}{\partial y^{\nu'}} \frac{\partial}{\partial x^{\mu}}$, inserting this in the above we get

$$\tilde{d}y^{\mu'} \left(\frac{\partial x^{\rho}}{\partial y^{\nu'}} \frac{\partial}{\partial x^{\rho}} \right) = \frac{\partial x^{\rho}}{\partial y^{\nu'}} \tilde{d}y^{\mu'} \left(\frac{\partial}{\partial x^{\rho}} \right) = \delta^{\mu'}_{\nu'},$$

to hold the Kronecker delta it must be $\tilde{d}y^{\mu'} = \frac{\partial y^{\mu'}}{\partial x^{\nu}} \tilde{d}x^{\nu}$, so that

$$\frac{\partial x^{\rho}}{\partial y^{\nu'}} \tilde{d}y^{\mu'} \left(\frac{\partial}{\partial x^{\rho}} \right) = \frac{\partial x^{\rho}}{\partial y^{\nu'}} \frac{\partial y^{\mu'}}{\partial x^{\nu}} \tilde{d}x^{\nu} \left(\frac{\partial}{\partial x^{\rho}} \right) = \frac{\partial x^{\rho}}{\partial y^{\nu'}} \frac{\partial y^{\mu'}}{\partial x^{\nu}} \delta^{\nu}_{\rho} = \delta^{\mu'}_{\nu'}.$$

In this way, we also obtain the general rule of transformation of components of tensors:

$$\boxed{T^{\mu'_1,\dots,\mu'_n}_{\nu'_1\dots,\nu'_m} = T^{\mu_1,\dots,\mu_n}_{\nu_i\dots,\nu_m} \frac{\partial x^{\nu_1}}{\partial y^{\nu'_1}} \cdots \frac{\partial x^{\nu_m}}{\partial y^{\nu'_m}} \frac{\partial y^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\mu'_n}}{\partial x^{\mu_n}}} .$$

2.6 Submanifolds

To end the general discussion on manifolds, we introduce the concept of submanifolds: the main idea, behind the definition, is that in $\mathbb{R}^n$ submanifolds are defined as the set of points that satisfy some equation as

$$f(\vec{x}) = \text{const.}$$

In the definition we will require this condition using a chart over a sub set of the manifold.

Definition 2.15. *Given a subset $\mathcal{S} \subset \mathcal{M}$ ($\mathcal{M}$ manifold with dimension n), it is a submanifold if it exists a chart over some open U, such that $U \cap \mathcal{S} \subset \mathcal{M}$, in which the coordinates read:*

$$x^{n-m+1} = x^{n-m+2} = \cdots = x^n = 0 \qquad \text{for some } m \le n.$$

$\mathcal{S}$ is a submanifold of dimension m.

Notice that also the dimension of the tangent spaces $T_P^{(\mathcal{S})}$ is equal to m.

3

The metric: lengths and integrals

We now want to introduce basics concepts of physics, such as length and angles. Usually we measure length and define angles, in vector spaces, using the scalar product.

3.1 The metric tensor

A scalar product is defined by a metric tensor, which we now introduce in the context of lengths but really posses a deeper physical meaning.

Definition 3.1 (Metric tensor). *The metric tensor, g, is a $(0,2)$ tensor, defined on a tangent space T_P, such that:*

- $g(\vec{V}, \vec{W}) = g(\vec{w}, \vec{V}) \qquad \forall \ \vec{V}, \vec{W} \in T_P$;

- $g(\vec{V}, \vec{W}) = 0 \quad \forall \ \vec{W} \in T_P \quad \Leftrightarrow \quad \vec{V} = 0$.

We call *scalar product* of two vectors $g(\vec{V}, \vec{W}) \in \mathbb{R}$ and $g(\vec{V}, \vec{V}) = |\vec{V}|^2$ defines the *norm* of a vector.

Notice that, this definition is the usual one of scalar product on regular vector spaces, here the metric tensor is really a tensor field: in each tangent space T_P we have precise metric tensor corresponding to the point P itself.

We can decompose this tensor in components, on the basis $\tilde{dx}^{\mu}$, induced by some chart, obtaining:

$$\boxed{g(x) = g_{\mu\nu}(x) \ \tilde{dx}^{\mu} \otimes \tilde{dx}^{\nu}}, \qquad \boxed{g_{\mu\nu} = g(\partial_\mu, \partial_\nu)}.$$

This allows us to show that the metric, as well as in special relativity flat space-time, lowers and rises the indices: given some vector $\vec{V} \in T_P$, we can define $\tilde{V} = g(\vec{V}, \cdot) \in T_P^*$:

$$V_\nu = \tilde{V}(\partial_\nu) = g(V^\mu \partial_\mu, \partial_\nu) = V^\mu g_{\mu\nu}.$$

Lastly, we are going to show a peculiar propriety of the metric: from linear algebra we know that the metric can be always be put (via a change of basis) in canonical form[4], we should now adapt this result to the generalization of it as a tensor field on a manifold. Consider the metric in two different basis

$$g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial y^{\mu'}} \frac{\partial x^{\nu}}{\partial y^{\nu'}} g_{\mu\nu}$$

and Taylor expand both sides, around some point P, to the first order,

$$g_{\mu'\nu'} = g_{\mu'\nu'}\big|_P + \frac{\partial g_{\mu'\nu'}}{\partial y^{\lambda'}}\bigg|_P \delta y^{\lambda'} + \dots$$

$$= \frac{\partial x^{\mu}}{\partial y^{\mu'}} \frac{\partial x^{\nu}}{\partial y^{\nu'}} g_{\mu\nu}\bigg|_P + \left[2\frac{\partial x^{\mu}}{\partial y^{\mu'}} \frac{\partial^2 x^{\nu}}{\partial y^{\lambda'} \partial y^{\nu'}} g_{\mu\nu} + \frac{\partial x^{\mu}}{\partial y^{\mu'}} \frac{\partial x^{\nu}}{\partial y^{\nu'}} \frac{\partial}{\partial y^{\lambda'}} (g_{\mu\nu}) \right]\bigg|_P \delta y^{\lambda'} + \dots$$

---

[4]The metric is diagonal with only ± 1 as entries.

now we should compare same order terms, since they will correspond.

At each order we will have free terms, defined by the change of basis, that are equals to those of the Taylor expansion in the new basis, thus defining the properties of the new metric components:

- the term $g_{\mu'\nu'}\big|_P$ is determined by $\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}}g_{\mu\nu}\big|_P$, the first posses (in an n-dimensional manifold) $(n^2 - n)/2 + n = (n^2 + n)/2$ independent components (it is symmetric), while the second has n^2 independent components, in this way $(n^2 + n)/2$ components of the change of basis can be used to reach the canonical form in P, the remaining are just free parameters[5];

- $\frac{\partial g_{\mu'\nu'}}{\partial y^{\lambda'}}\bigg|_P$ is determined by $\left[2\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial^2 x^\nu}{\partial y^{\lambda'}\partial y^{\nu'}}g_{\mu\nu} + \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}}\frac{\partial}{\partial y^{\lambda'}}\left(g_{\mu\nu}\right)\right]\bigg|_P$, the second addend is fixed by the zeroth order expansion, the first one, $2\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial^2 x^\nu}{\partial y^{\lambda'}}\big|_P$, posses $n(n^2 + n)/2$ independent parameters, while the derivatives of the new metric components has $(n^2 + n)/2$ independent parameters, again we can use this abbundacy of parameter to set the value of $\frac{\partial g_{\mu'\nu'}}{\partial y^{\lambda'}}\big|_P = 0$.

To sum up, we can always choose a change of basis such that the new metric tensor is in canonical form in P, its first derivatives vanish in P and, lastly, the components reads:

$$g_{\mu'\nu'} = \pm\delta_{\mu'\nu'} + \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial y^{\mu'}\partial y^{\nu'}}\bigg|_P \delta y^{\mu'}\delta y^{\nu'} + \dots.$$

Physically, this corresponds to using a *locally inertial reference frame*, in which (only in P) we recover special relativity.

3.2   The length of a curve

Let's consider a curve on a manifold and its tangent vectors $\frac{d}{d\lambda}$. We can define a vector representing a small displacement, over the curve,

$$\Delta\vec{V} = \frac{d}{d\lambda}\Delta\lambda = \vec{V}\Delta\lambda,$$

from which we can evaluate its modulus

$$\Delta s^2 = g\left(\Delta\vec{V}, \Delta\vec{V}\right) = g\left(\vec{V}, \vec{V}\right)\Delta\lambda^2.$$

In the limit of $\Delta\lambda \to 0$ and summing all the infinitesimal lengths Δs, by integration, we get the definition of the length of the curve:

$$S(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} \sqrt{g(\vec{V}, \vec{V})}d\lambda = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{\mu\nu}(\lambda)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda.$$

We have used that $\vec{V} = \frac{dx^\mu}{d\lambda}\partial_\mu$ and $g(\partial_\mu, \partial_\nu) = g_{\mu\nu}$.

---

[5]Notice that, for $n = 4$, there will be $(n^2 - n)/2 = 6$ free parameters, that we can think as not fixed by the freedom of choice of the inertial reference frame (3 rotations and 3 boosts).

3.3   n-forms

Now that we have introduced integrals to evaluate the length of a curve, we want to obtain a way to evaluate more general integrals over subsets of the manifold.

Notice that in $\mathbb{R}^3$ we can define the area of a parallelogram, which is determined by two vectors $\vec{V}$, $\vec{W}$ (used as sides), by their vector product. From this product we can obtain the volume of a prism using the scalar product and then, summing infinitesimal prisms or parallelograms, we can define integration.

Here we will follow the same idea but, instead of scalar or vector products, we are going to use a more general tool, the n-forms.

Definition 3.2 (n-forms). *An n-form is $(0, n)$ tensor, defined on T_P, such that it is antisymmetric.*

A 2-form can be explicitly written as an antisymmetric linear combination of the basis element of $T_P^* \otimes T_P^*$

$$\tilde{\omega} = \sum_{\mu < \nu} \omega_{\mu\nu} (\tilde{dx}^\mu \otimes \tilde{dx}^\nu - \tilde{dx}^\nu \otimes \tilde{dx}^\mu),$$

to summarize this kind of linear combination we can define the **wedge product**, such that

$$\tilde{\omega} = \omega_{\mu\nu} \tilde{dx}^\mu \wedge \tilde{dx}^\nu.$$

In this way we can write the general form of an n-form as:

$$\boxed{\tilde{\omega} = \omega_{\mu_1, \ldots, \mu_n} \ \tilde{dx}^{\mu_1} \wedge \cdots \wedge \tilde{dx}^{\mu_n}}.$$

3.4   Integration over manifolds

Now we can define the volume of an n-polyhedron using n-forms, notice that this is the beast way to do so, since the antisymmetry of n-forms guarantees that the sides of it must be n linear independent vectors, otherwise it will give a zero measure volume.

Considering n independent vectors $\{\vec{\Delta x_i}\}$, which are the sides of a polyhedron, we can define its volume through the n-form:

$$\tilde{\omega}(\vec{\Delta x_1}, \ldots, \vec{\Delta x_n}) = f \tilde{dx}^1(\vec{\Delta x_1}) \wedge \cdots \wedge \tilde{dx}^n(\vec{\Delta x_n})$$
$$= f \Delta x^1_{(1)} \ldots \Delta x^n_{(n)},$$

where we used that n-forms, in a n-dimensional manifold, form 1-dimensional vector spaces (since constructing the first antysimmetric base element we already used all the 1-forms).

In the limit of $\Delta x^\mu_{(i)} \rightarrow dx^\mu_{(i)}$ we obtain the definition of the infinitesimal element of integration

$$\tilde{\omega}(\vec{dx_1}, \ldots, \vec{dx_n}) = f \ dx^1_{(1)} \ldots dx^n_{(n)} = dV.$$

Summing all the polyhedra defined in some region $U \in \mathcal{M}$ we obtain the definition of the integral over the volume of U:

$$\boxed{V(U) = \int_U \tilde{\omega} = \int_{\phi(U)} f \ dx^1 dx^2 \ldots dx^n}.$$

A key point of this definition is that the integral is actually defined ad an integral over some region of $\mathbb{R}^n$ by the n-form composed with a chart, that we know ho to evaluate.

We can generalize the above definition to submanifolds just by applying $\tilde{\omega}$ to all the linearly independent vectors that are not in the tangent space of the submanifold, for an $n-1$ submanifold we get

$$A(\Sigma) = \int_\Sigma \tilde{\omega}(\vec{dx}_1, \dots, \vec{dx}_{n-1}, \vec{v}) = \int_{\phi(\Sigma)} fv \; dx^1 \dots dx^{n-1}.$$

Until now the definition of volume is still arbitrary up to some constant, that we labelled f, thus the usefulness of this definition is somewhat unclear. To give the right meaning to this measure of volume we need to impose consistency between the metric (which defines length) and volume.
Choosing a chart in which g is in canonical form, we first define the n-form

$$\tilde{\omega}_g = \tilde{dx}^1 \wedge \cdots \wedge \tilde{dx}^n,$$

a change of the chart will induce a change of basis, that reads for the n-form:

$$\tilde{\omega}_{g'} = J\tilde{dy}^1 \wedge \cdots \wedge \tilde{dy}^n, \qquad J = \det\left(\frac{\partial y^\mu}{\partial x^\nu}\right).$$

To better understand the connection with the metric let's study how the determinant of the metric transforms

$$\det(g') = \det(\Lambda^t g' \Lambda) = \det(g)\det(\Lambda)^2 = \pm J^2$$

therefore we can write the integral over a volume in a generic chart as

$$\boxed{V(U) = \int_U \tilde{\omega}_g = \int_{\phi(U)} \sqrt{|\det(g')|}d^n y.}$$