Differential geometry

A brief guide for general relativity

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Preface:

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Manifolds: vectors, 1-forms and tensors

The main idea that will guide us defining manifolds is that we want to generalize structures of $\mathbb{R}^n$ to a generic set of points. With this idea in mind, a manifold should be just the most basic set of points that we can use, combined with maps from it to $\mathbb{R}^n$.
If this construction is done in the right way, we will be able to transport backwards operation from $\mathbb{R}^n$ to our manifold.

2.1 Topological spaces

One of the most crucial properties of functions on $\mathbb{R}^n$ is *continuity*, therefore it is natural to consider as starting point some kind of set on which continuity is definable: those are **topological spaces**.

Definition 2.1 (Topological spaces)**.** *A topological space is the pair of a generic set $\mathcal{M}$ and a family of some of its subsets $\mathcal{T}$ (called opens) such that:*

- $\varnothing, \mathcal{M} \in \mathcal{T}$,
- $\bigcup_\alpha O_\alpha \in \mathcal{T}$ *with* $\{O_\alpha /\ O_\alpha \in \mathcal{T}\}$ *(finite collection or not),*
- $\bigcup_{i=1}^N O_i \in \mathcal{T}$ *with* $\{O_1, O_2, ..., O_N /\ O_i \in \mathcal{T}\}$.

Furthermore, we are interested in a specific family of topological spaces, **Hausdorff spaces**.

Definition 2.2. *A topological space $(\mathcal{M}, \mathcal{T})$ is a Hausdorff one if $\forall P, Q \in \mathcal{M}$ exist U and V, neighborhoods respectively of P and Q, such that $U \cap V = \varnothing$.*

On this kind of spaces we can define a topological continuity (which is actually equivalent to the one form calculus) and thus we will be able to define **continuous maps**.

Definition 2.3. *A map $\phi : \mathcal{M} \to \mathcal{N}$, with $\mathcal{M}$, $\mathcal{N}$ topological spaces, is continuous if, given $V \in \mathcal{N}$ open, then $\phi^{-1}(V) \in \mathcal{M}$ is an open set.*

2.2 Maps, charts and atlases

From the topological space we now want to build a connection to $\mathbb{R}^n$: this process is inspired by the process of building the map of a specific portion of land, since on the map we can evaluate distances and other quantities that otherwise we would have to determine by moving around that specific portion of land. In the same way we will do all the calculation on $\mathbb{R}^n$ instead of directly using points on the topological space, where we don't have coordinates and other constructions.

Definition 2.4 (Maps)**.** *An application $\phi : D \to \mathbb{R}^n$, with $D \subseteq \mathcal{M}$ open, is called a map.*

Not all the maps are *"good"* maps, more precisely we want maps that are continuous (near points get mapped to near points) and that allow us to go back from $\mathbb{R}^n$ to the topological space. Lastly, since usually the domain of a map doesn't correspond to the whole topological space, it is useful to know where the map is valid, and thus we want to encode the domain and the map together.

Definition 2.5 (Charts)**.** *A pair of a continuous invertible map $\phi : A \to \mathbb{R}^n$ and its domain, (A, ϕ), is called a chart.*

As we said, usually a chart is not enough to cover the whole topological space, however we need to map it all to $\mathbb{R}^n$. To do so we can use multiple charts, with some sort of notion of compatibility between them.

Definition 2.6 (Atlases)**.** *A collection of charts $\mathcal{A} = \{(A_i, \phi_i)\}$ is an atlas if:*

- $\bigcup_i A_i \supseteq \mathcal{M}$,
- *for each pair (A_i, ϕ_i), (A_j, ϕ_j) exist a function $\psi : \phi_i(A_i) \to \phi_j(A_j)$, invertible and such that $(\phi_j^{-1} \circ \psi \circ \phi_i) = id$, on $A_i \cap A_j$.*

Constructing the atlas of a topological space means (operationally) to build the **manifold**, which actually is the class of equivalence of all the atlases that can be put in bijection.
Furthermore, the functions that assure the compatibility of charts will characterize the manifold itself:

- since all ψ are invertible, all charts of an atlas will map to the same $\mathbb{R}^n$, and thus n is the **dimension** of the manifold;
- if all the $\psi \in C^p(\mathbb{R}^n)$, the manifold is said a **p-differentiable manifold**.

2.3 Functions, curves and vectors

Now, that we have built the full structure of a manifold, we can start to use it. On a topological space we can define function $f : \mathcal{M} \to \mathbb{R}$ and then determine if it is continuous, but we cannot define whether it is smooth, since we cannot take derivatives on the manifold. Instead, we can map the manifold to $\mathbb{R}^n$ and from there define differentiability.

Definition 2.7. *A function $f : \mathcal{M} \to \mathbb{R}$ is p-differentiable if*

$$f \circ \phi_i^{-1} \in C^p(\phi_i(A_i)),$$

for all the charts (A_i, ϕ_i) of the given manifold.

Functions represent scalars, since their value depends only on the point of the manifold and not from the coordinates of the charts. After scalars, we also need to define vectors, to do that we will rely on the intuition that tangent vectors, to a curve, are given by the derivative of the coordinates along it (in $\mathbb{R}^n$).

Definition 2.8 (Curves)**.** *A curve is a piece-wise continuous map $\gamma : I \subseteq \mathbb{R} \to \mathcal{M}$.*

Observation 2.1. *Derivative operators are linear and thus they form a vector space.
Let's consider a map $\phi : A \to \mathbb{R}^n$, a curve $\gamma : I \to \mathcal{M}$ and a function $f : \mathcal{M} \to \mathbb{R}$. Composing f with γ we can evaluate its derivative along the curve:*

$$\frac{d}{d\lambda}(f \circ \gamma) = \frac{d}{d\lambda} f(\gamma(\lambda)).$$

Composing now $f \circ \gamma$ with the map ϕ and using the Leibniz rule we obtain:

$$\frac{d}{d\lambda}[(f \circ \phi^{-1})(\phi \circ \gamma)] = \frac{\partial (f \circ \phi^{-1})}{\partial x^\mu} \frac{d(\phi \circ \gamma)}{d\lambda} = \frac{\partial f}{\partial x^\mu} \frac{dx^\mu}{d\lambda}.$$

This procedure gives us a way construct the vector space of derivatives, along a given curve, at each point of the manifold

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}.$$

Notice that the chart induces a basis for the vectors: $\frac{dx^\mu}{d\lambda} \in \mathbb{R}$, *and thus can be interpreted as components, while* $\frac{\partial}{\partial x^\mu}$ *are still derivatives from which (by linear combinations) we can construct all the others. For an* n*-dimensional manifold, each chart will induce* n *partial derivatives and thus it is natural to use those as basis for the tangent spaces.*

Definition 2.9 (Vectors)**.** *A vector in a point* $P \in \mathcal{M}$ *is defined as the differential operator along a curve* γ

$$\boxed{\vec{V}_\gamma = \frac{d}{d\lambda}\bigg|_P = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}\bigg|_P},$$

whose components are defined by the choice of a chart.
The vector space spanned by $\frac{\partial}{\partial x^\mu}\big|_P$ *is called the tangent space in* P, T_P.

Notice that also vectors are really invariant objects; the components are not and from chain rule, changing chart, we get the components' transformation law:

$$\vec{V} = V^\mu \frac{\partial}{\partial x^\mu} = V^\mu \frac{\partial y^{\nu'}}{\partial x^\mu}\frac{\partial}{\partial y^{\nu'}} = V^{\nu'}\frac{\partial}{\partial y^{\nu'}}.$$

Definition 2.10. *Given an open subset of the manifold* $U \in \mathcal{M}$, *a vector field is an application that maps each point* $P \in U$ *to a vector* $\vec{V}(P) \in T_P$.

2.4 Integral curves, exponential maps and commutators

Right now we have defined vectors from the notion of curves, we will now show how from a vector field we can construct a set of curves.
Consider a vector field $\vec{V} = \frac{d}{d\lambda}$, defined on some open subset U of a manifold, and let's pick point $P_0 \in U$. If we introduce coordinates, through a chart $\phi(P) = x^\mu(P)$, we can define a *Cauchy problem* from the composition of the first two objects and the chart:

$$\begin{cases} \frac{d}{d\lambda}x^\mu = V^\mu(\lambda), \\ x^\mu(\lambda_0) = x^\mu(P_0). \end{cases}$$

From calculus theorems, we know that there exist a unique solution $x^\mu(\lambda)$, if we now compose this function with the inverse of the chart we obtain a unique curve $\gamma(\lambda)$ on the manifold.

Definition 2.11 (Integral curves)**.** *Given a vector field* $\vec{V}$, *defined on* $U \in \mathcal{M}$, *and a curve* $\gamma : I \to U$, *we call* γ *the integral curve of* $\vec{V}$ *if:* $\frac{d}{d\lambda}\big|_P = \vec{V}(P) \quad \forall P \in \gamma(I)$. [1]

Another way, to better resume the previous result, is the *exponential map*: consider a vector field and its integral curve on some open set, composing this last one with a chart we can then Taylor expand it along the curve to get:

$$x^\mu(\lambda_0+\epsilon) = x^\mu(\lambda_0) + \epsilon\frac{dx^\mu}{d\lambda}\bigg|_{\lambda_0} + \frac{\epsilon^2}{2!}\frac{d^2x^\mu}{d\lambda^2}\bigg|_{\lambda_0} + \cdots = e^{\epsilon\frac{d}{d\lambda}}x^\mu\big|_{\lambda_0} = e^{\epsilon\vec{V}}x^\mu\big|_{\lambda_0}.$$

[1] Here $\frac{d}{d\lambda}$ is defined as the tangent vector to γ.

Notice that the exponential map is an implicit way to define a solution of the Cauchy problem we defined for integral curves and gives us an easy way (but most of the time just formal) to obtain connected points on a manifold, using just a vector field. Also notice that we could Taylor expand every smooth function defined on the manifold (along a curve), in the same way as we have just done, in this way the exponential map gives us the function in other points:

$$\boxed{f(\lambda+\epsilon) = f(\lambda_0) + \epsilon \frac{df}{d\lambda}\bigg|_{\lambda_0} + \frac{\epsilon^2}{2!}\frac{d^2 f}{d\lambda^2}\bigg|_{\lambda_0} + \cdots = e^{\epsilon\frac{d}{d\lambda}} f\big|_{\lambda_0} = e^{\epsilon\vec{V}} f\big|_{\lambda_0}}.$$

Having exponential maps, we can now discuss a rather unexpected object that we can define for vector, as we have constructed them.

Definition 2.12 (Commutators)**.** *Given two vectors $\vec{V} = \frac{d}{d\lambda}$, $\vec{W} = \frac{d}{d\sigma}$, in the tangent space T_P of a given manifold, we can define the commutator:*

$$[\vec{V}, \vec{W}] = \frac{d}{d\lambda}\frac{d}{d\sigma} - \frac{d}{d\sigma}\frac{d}{d\lambda}.$$

We are not used to define commutators of vectors, even thought now these are linear operators, therefore it could seem hard to interpret the meaning of this object.
Consider two vector fields $\frac{d}{d\lambda}$, $\frac{d}{d\sigma}$ and let's follow their integral maps from a point P to some other point. A priori we cannot know if moving along one vector field first will lead to the same point we would get starting first from the other.

$$\begin{aligned} x^\mu(A) &= e^{\epsilon_1\frac{d}{d\lambda}} e^{\epsilon_2\frac{d}{d\sigma}} x^\mu\big|_P = x^\mu(P) + \epsilon_1 \frac{dx^\mu}{d\lambda}\bigg|_P + \epsilon_2 \frac{dx^\mu}{d\sigma}\bigg|_P + \epsilon_1\epsilon_2 \frac{d}{d\lambda}\frac{dx^\mu}{d\sigma}\bigg|_P + \dots\ , \\ x^\mu(B) &= e^{\epsilon_2\frac{d}{d\sigma}} e^{\epsilon_1\frac{d}{d\lambda}} x^\mu\big|_P = x^\mu(P) + \epsilon_1 \frac{dx^\mu}{d\lambda}\bigg|_P + \epsilon_2 \frac{dx^\mu}{d\sigma}\bigg|_P + \epsilon_1\epsilon_2 \frac{d}{d\sigma}\frac{dx^\mu}{d\lambda}\bigg|_P + \dots\ . \end{aligned}$$

Subtracting the coordinates of the second point from the first we obtain:

$$x^\mu(A) - x^\mu(B) = \epsilon_1\epsilon_2 \left[\frac{d}{d\lambda}\frac{d}{d\sigma}\right] x^\mu\big|_P \cdots\ ,$$

clearly, from this calculation, we can see that in order to have the two path to end in the same point $A = B$, the commutator must vanish. Therefore, two commuting vector fields will generate "commuting" paths as integral curves.
In this way we can use the parametrization parameter, of the integral curves of n linearly independent[2] vector fields, as coordinates of some chart. In this case, the chart is well-defined, since the change of coordinate matrix, $\frac{\partial x^\mu}{\partial\lambda_i}$, is really just the coordinates of each vector, and thus it is invertible (we assumed linearly independent vectors).

Lastly we evaluate in components the commutator of two vectors: fixing a chart

$$\begin{aligned} [\vec{V}, \vec{W}] &= \frac{d}{d\lambda}\frac{d}{d\sigma} - \frac{d}{d\sigma}\frac{d}{d\lambda} = \frac{dx^\nu}{d\lambda}\partial_\nu\left(\frac{dx^\mu}{d\sigma}\partial_\mu\right) - \frac{dx^\nu}{d\sigma}\partial_\nu\left(\frac{dx^\mu}{d\lambda}\partial_\mu\right) \\ &= \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\sigma}(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu) + \left(\frac{dx^\nu}{d\lambda}\partial_\nu\frac{dx^\mu}{d\sigma} - \frac{dx^\nu}{d\sigma}\partial_\nu\frac{dx^\mu}{d\lambda}\right)\partial_\mu \\ &= (V^\nu\partial_\nu W^\mu - W^\nu\partial_\nu V^\mu)\partial_\mu, \end{aligned}$$

it is now clear that the commutator of two vectors is still another one (this was trivial).

[2]Linearly independent here means that the coefficients of the linear combination are functions on $\mathcal{M}$, and in each point they make the vectors independents.

2.5 1-Forms and tensors

We have now introduced vectors and their formalism, actually what we have introduced is really, in components, what we usually refer as contravariant vectors, we now want to define co-vectors and tensors, which contain both.
On manifolds co-vectors are linear maps from tangent spaces to $\mathbb{R}$, which are the natural generalization of vectors of the dual space[3].

Definition 2.13 (1-forms)**.** *Given a specific point $P \in \mathcal{M}$, we can define a 1-form in that point as a linear map $\tilde{\omega} : T_P \to \mathbb{R}$.*
1-forms made up a vector space at each point of a manifold called cotangent space T_P^*.

Notice that also vectors, when applied to functions, result in a real number, therefore it is natural to define a 1-form associated to a function (in the opposite way a vector is associated to a curve) such that, when applied to a vector, it will result in the directional derivative of the function along the curve:

$$\tilde{df}(\vec{V}) = \frac{df}{d\lambda} = \vec{V}(f).$$

Imposing this consistency relation, we can define a dual basis: to better see this we just need to use the chain rule with composition with a chart, then we substitute every term containing the parameter of the curve with the supposed 1-form basis:

$$\tilde{df}(\vec{V}) = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda} \quad \Rightarrow \quad \boxed{\tilde{df} = \frac{\partial f}{\partial x^\mu}\tilde{dx}^\mu}.$$

We just now need to define the 1-form basis $\tilde{dx}^\mu$ such that $\tilde{dx}^\mu(\vec{V}) = \tilde{dx}^\mu\left(\frac{x^\nu}{d\lambda}\partial_\nu\right) = \frac{x^\nu}{d\lambda}\delta^\mu_\nu$.
This basis is thus defined by:

$$\boxed{\tilde{dx}^\mu(\partial_\nu) = \delta^\mu_\nu}.$$

Having defined 1-form, we can now introduce tensors, which are just combinations of vectors and 1-forms.

Definition 2.14 (Tensors)**.** *A (n,m) rank tensor, in a specific point $P \in \mathcal{M}$, is a multilinear map*

$$T : \underbrace{T_p^* \otimes \cdots \otimes T_p^*}_{m\ times} \otimes \underbrace{T_P \otimes \cdots \otimes T_P}_{n\ times} \longrightarrow \mathbb{R}.$$

Fixing a chart, and thus a basis in both T_P and T_P^*, we can expand T in components:

$$\boxed{T = T^{\mu_1,\dots,\mu_n}_{\nu_1\dots,\nu_m}\partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_n} \otimes \tilde{dx}^{\nu_1} \otimes \dots_\otimes \tilde{dx}^{\nu_m}}$$

$$\text{where} \quad T^{\mu_1,\dots,\mu_n}_{\nu_1\dots,\nu_m} = T(\partial_{\mu_1}, \dots \partial_{\mu_n}, \tilde{dx}^{\nu_1}, \dots, \tilde{dx}^{\nu_m}).$$

Both 1-forms and tensors can be generalized to fields that maps each point of the manifold to a tensor in that point.

Lastly we should concern about the change of basis for 1-forms, from which we can obtain the full transformation law of tensor components. Let's consider the action of $\tilde{dx}^\mu$ on the basis of T_P, then let's consider a new chart and a new associated basis, defined by:

$$\tilde{dy}^{\mu'}\left(\frac{\partial}{\partial y^{\nu'}}\right) = \delta^{\mu'}_{\nu'}.$$

[3] Actually co-vectors are really vectors of the dual space even in special relativity.

The transformation of the vector basis is given by the chain rule $\frac{\partial}{\partial y^{\nu'}} = \frac{\partial x^\mu}{\partial y^{\nu'}} \frac{\partial}{\partial x^\mu}$, inserting this in the above we get

$$\tilde{dy}^{\mu'} \left(\frac{\partial x^\rho}{\partial y^{\nu'}} \frac{\partial}{\partial x^\rho} \right) = \frac{\partial x^\rho}{\partial y^{\nu'}} \tilde{dy}^{\mu'} \left(\frac{\partial}{\partial x^\rho} \right) = \delta^{\mu'}_{\nu'},$$

to hold the Kronecker delta it must be $\tilde{dy}^{\mu'} = \frac{\partial y^{\mu'}}{\partial x^\nu} \tilde{dx}^\nu$, so that

$$\frac{\partial x^\rho}{\partial y^{\nu'}} \tilde{dy}^{\mu'} \left(\frac{\partial}{\partial x^\rho} \right) = \frac{\partial x^\rho}{\partial y^{\nu'}} \frac{\partial y^{\mu'}}{\partial x^\nu} \tilde{dx}^\nu \left(\frac{\partial}{\partial x^\rho} \right) = \frac{\partial x^\rho}{\partial y^{\nu'}} \frac{\partial y^{\mu'}}{\partial x^\nu} \delta^\nu_\rho = \delta^{\mu'}_{\nu'}.$$

In this way, we also obtain the general rule of transformation of components of tensors:

$$\boxed{T^{\mu'_1,\dots,\mu'_n}_{\nu'_1\dots,\nu'_m} = T^{\mu_1,\dots,\mu_n}_{\nu_i\dots,\nu_m} \frac{\partial x^{\nu_1}}{\partial y^{\nu'_1}} \cdots \frac{\partial x^{\nu_m}}{\partial y^{\nu'_m}} \frac{\partial y^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\mu'_n}}{\partial x^{\mu_n}}}\,.$$

2.6 Submanifolds

To end the general discussion on manifolds, we introduce the concept of submanifolds: the main idea, behind the definition, is that in $\mathbb{R}^n$ submanifolds are defined as the set of points that satisfy some equation as

$$f(\vec{x}) = \text{const}.$$

In the definition we will require this condition using a chart over a sub set of the manifold.

Definition 2.15. *Given a subset $\mathcal{S} \subset \mathcal{M}$ ($\mathcal{M}$ manifold with dimension n), it is a submanifold if it exists a chart over some open U, such that $U \cap \mathcal{S} \subset \mathcal{M}$, in which the coordinates read:*

$$x^{n-m+1} = x^{n-m+2} = \cdots = x^n = 0 \qquad \textit{for some } m \le n.$$

$\mathcal{S}$ is a submanifold of dimension m.

Notice that also the dimension of the tangent spaces $T_P^{(\mathcal{S})}$ is equal to m.

3

The metric: lengths and integrals

We now want to introduce basics concepts of physics, such as length and angles. Usually we measure length and define angles, in vector spaces, using the scalar product.

3.1 The metric tensor

A scalar product is defined by a metric tensor, which we now introduce in the context of lengths but really posses a deeper physical meaning.

Definition 3.1 (Metric tensor). *The metric tensor, g, is a $(0,2)$ tensor, defined on a tangent space T_P, such that:*

- $g(\vec{V},\vec{W}) = g(\vec{w},\vec{V}) \qquad \forall\ \vec{V},\vec{W} \in T_P$;
- $g(\vec{V},\vec{W}) = 0 \quad \forall\ \vec{W} \in T_P \quad \Leftrightarrow \quad \vec{V} = 0.$

We call *scalar product* of two vectors $g(\vec{V},\vec{W}) \in \mathbb{R}$ and $g(\vec{V},\vec{V}) = |\vec{V}|^2$ defines the *norm* of a vector.
Notice that, this definition is the usual one of scalar product on regular vector spaces, here the metric tensor is really a tensor field: in each tangent space T_P we have precise metric tensor corresponding to the point P itself.

We can decompose this tensor in components, on the basis $\tilde{dx}^\mu$, induced by some chart, obtaining:

$$\boxed{g(x) = g_{\mu\nu}(x)\ \tilde{dx}^\mu \otimes \tilde{dx}^\nu}, \qquad \boxed{g_{\mu\nu} = g(\partial_\mu,\partial_\nu)}.$$

This allows us to show that the metric, as well as in special relativity flat space-time, lowers and rises the indices: given some vector $\vec{V} \in T_P$, we can define $\tilde{V} = g(\vec{V},\cdot) \in T_P^*$:

$$V_\nu = \tilde{V}(\partial_\nu) = g(V^\mu\partial_\mu,\partial_\nu) = V^\mu g_{\mu\nu}.$$

Lastly, we are going to show a peculiar propriety of the metric: from linear algebra we know that the metric can be always be put (via a change of basis) in canonical form[4], we should now adapt this result to the generalization of it as a tensor field on a manifold. Consider the metric in two different basis

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}} g_{\mu\nu}$$

and Taylor expand both sides, around some point P, to the first order,

$$\begin{aligned} g_{\mu'\nu'} &= g_{\mu'\nu'}\big|_P + \frac{\partial g_{\mu'\nu'}}{\partial y^{\lambda'}}\bigg|_P \delta y^{\lambda'} + \dots \\ &= \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}} g_{\mu\nu}\bigg|_P + \left[2\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial^2 x^\nu}{\partial y^{\lambda'}\partial y^{\nu'}} g_{\mu\nu} + \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}}\frac{\partial}{\partial y^{\lambda'}}(g_{\mu\nu})\right]\bigg|_P \delta y^{\lambda'} + \dots \end{aligned}$$

[4]The metric is diagonal with only ± 1 as entries.

now we should compare same order terms, since they will correspond.
At each order we will have free terms, defined by the change of basis, that are equals to those of the Taylor expansion in the new basis, thus defining the properties of the new metric components:

- the term $g_{\mu'\nu'}\big|_P$ is determined by $\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}}g_{\mu\nu}\big|_P$, the first posses (in an n-dimensional manifold) $(n^2-n)/2+n=(n^2+n)/2$ independent components (it is symmetric), while the second has n^2 independent components, in this way $(n^2+n)/2$ components of the change of basis can be used to reach the canonical form in P, the remaining are just free parameters[5];
- $\left.\frac{\partial g_{\mu'\nu'}}{\partial y^{\lambda'}}\right|_P$ is determined by $\left.\left[2\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial^2 x^\nu}{\partial y^{\lambda'}\partial y^{\nu'}}g_{\mu\nu}+\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}}\frac{\partial}{\partial y^{\lambda'}}(g_{\mu\nu})\right]\right|_P$, the second addend is fixed by the zeroth order expansion, the first one, $2\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial^2 x^\nu}{\partial y^{\lambda'}}\big|_P$, posses $n(n^2+n)/2$ independent parameters, while the derivatives of the new metric components has $(n^2+n)/2$ independent parameters, again we can use this abbundacy of parameter to set the value of $\frac{\partial g_{\mu'\nu'}}{\partial y^{\lambda'}}\big|_P=0$.

To sum up, we can always choose a change of basis such that the new metric tensor is in canonical form in P, its first derivatives vanish in P and, lastly, the components reads:

$$\boxed{g_{\mu'\nu'}=\pm\delta_{\mu'\nu'}+\frac{1}{2}\left.\frac{\partial g_{\mu\nu}}{\partial y^{\mu'}\partial y^{\nu'}}\right|_P\delta y^{\mu'}\delta y^{\nu'}+\dots}.$$

Physically, this corresponds to using a *locally inertial reference frame*, in which (only in P) we recover special relativity.

3.2 The length of a curve

Let's consider a curve on a manifold and its tangent vectors $\frac{d}{d\lambda}$. We can define a vector representing a small displacement, over the curve,

$$\Delta\vec{V}=\frac{d}{d\lambda}\Delta\lambda=\vec{V}\Delta\lambda,$$

from which we can evaluate its modulus

$$\Delta s^2=g\big(\Delta\vec{V},\Delta\vec{V}\big)=g\big(\vec{V},\vec{V}\big)\Delta\lambda^2.$$

In the limit of $\Delta\lambda\to 0$ and summing all the infinitesimal lengths Δs, by integration, we get the definition of the length of the curve:

$$\boxed{S(\lambda_1,\lambda_2)=\int_{\lambda_1}^{\lambda_2}\sqrt{g\big(\vec{V},\vec{V}\big)}d\lambda=\int_{\lambda_1}^{\lambda_2}\sqrt{g_{\mu\nu}(\lambda)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda}.$$

We have used that $\vec{V}=\frac{dx^\mu}{d\lambda}\partial_\mu$ and $g(\partial_\mu,\partial_\nu)=g_{\mu\nu}$.

[5]Notice that, for $n=4$, there will be $(n^2-n)/2=6$ free parameters, that we can think as not fixed by the freedom of choice of the inertial reference frame (3 rotations and 3 boosts).

3.3 n-forms

Now that we have introduced integrals to evaluate the length of a curve, we want to obtain a way to evaluate more general integrals over subsets of the manifold.
Notice that in $\mathbb{R}^3$ we can define the area of a parallelogram, which is determined by two vectors $\vec{V}$, $\vec{W}$ (used as sides), by their vector product. From this product we can obtain the volume of a prism using the scalar product and then, summing infinitesimal prisms or parallelograms, we can define integration.
Here we will follow the same idea but, instead of scalar or vector products, we are going to use a more general tool, the n-forms.

Definition 3.2 (n-forms)**.** *An n-form is $(0, n)$ tensor, defined on T_P, such that it is antisymmetric.*

A 2-form can be explicitly written as an antisymmetric linear combination of the basis element of $T_P^* \otimes T_P^*$

$$\tilde{\omega} = \sum_{\mu<\nu} \omega_{\mu\nu}(\tilde{dx}^\mu \otimes \tilde{dx}^\nu - \tilde{dx}^\nu \otimes \tilde{dx}^\mu),$$

to summarize this kind of linear combination we can define the **wedge product**, such that

$$\tilde{\omega} = \omega_{\mu\nu}\tilde{dx}^\mu \wedge \tilde{dx}^\nu.$$

In this way we can write the general form of an n-form as:

$$\boxed{\tilde{\omega} = \omega_{\mu_1,\dots,\mu_n}\ \tilde{dx}^{\mu_1} \wedge \cdots \wedge \tilde{dx}^{\mu_n}}.$$

3.4 Integration over manifolds

Now we can define the volume of an n-polyhedron using n-forms, notice that this is the beast way to do so, since the antisymmetry of n-forms guarantees that the sides of it must be n linear independent vectors, otherwise it will give a zero measure volume.
Considering n independent vectors $\{\Delta\vec{x}_i\}$, which are the sides of a polyhedron, we can define its volume through the n-form:

$$\begin{aligned}\tilde{\omega}(\Delta\vec{x}_1, \dots, \Delta\vec{x}_n) &= f\tilde{dx}^1(\Delta\vec{x}_1) \wedge \cdots \wedge \tilde{dx}^n(\Delta\vec{x}_n) \\ &= f\Delta x^1_{(1)} \dots \Delta x^n_{(n)},\end{aligned}$$

where we used that n-forms, in a n-dimensional manifold, form 1-dimensional vector spaces (since constructing the first antysimmetric base element we already used all the 1-forms).
In the limit of $\Delta x^\mu_{(i)} \to dx^\mu_{(i)}$ we obtain the definition of the infinitesimal element of integration

$$\tilde{\omega}(d\vec{x}_1, \dots, d\vec{x}_n) = f\ dx^1_{(1)} \dots dx^n_{(n)} = dV.$$

Summing all the polyhedra defined in some region $U \in \mathcal{M}$ we obtain the definition of the integral over the volume of U:

$$\boxed{V(U) = \int_U \tilde{\omega} = \int_{\phi(U)} f\ dx^1 dx^2 \dots dx^n}.$$

A key point of this definition is that the integral is actually defined ad an integral over some region of $\mathbb{R}^n$ by the n-form composed with a chart, that we know ho to evaluate.

We can generalize the above definition to submanifolds just by applying $\tilde{\omega}$ to all the linearly independent vectors that are not in the tangent space of the submanifold, for an $n-1$ submanifold we get

$$A(\Sigma) = \int_\Sigma \tilde{\omega}(\vec{dx}_1, \dots, \vec{dx}_{n-1}, \vec{v}) = \int_{\phi(\Sigma)} fv \; dx^1 \dots dx^{n-1}.$$

Until now the definition of volume is still arbitrary up to some constant, that we labelled f, thus the usefulness of this definition is somewhat unclear. To give the right meaning to this measure of volume we need to impose consistency between the metric (which defines length) and volume.
Choosing a chart in which g is in canonical form, we first define the n-form

$$\tilde{\omega}_g = \tilde{dx}^1 \wedge \dots \wedge \tilde{dx}^n,$$

a change of the chart will induce a change of basis, that reads for the n-form:

$$\tilde{\omega}_{g'} = J\tilde{dy}^1 \wedge \dots \wedge \tilde{dy}^n, \qquad J = \det\left(\frac{\partial y^\mu}{\partial x^\nu}\right).$$

To better understand the connection with the metric let's study how the determinant of the metric transforms

$$\det(g') = \det(\Lambda^t g' \Lambda) = \det(g)\det(\Lambda)^2 = \pm J^2$$

therefore we can write the integral over a volume in a generic chart as

$$\boxed{V(U) = \int_U \tilde{\omega}_g = \int_{\phi(U)} \sqrt{|\det(g')|} d^n y}.$$

4

The curvature of a manifold

In general relativity, curvature describe gravity, therefore we should now define the concept of curvature of a manifold. Intuitively we can understand that a sphere is curved differently with respect to a flat sheet of paper, but it is harder to describe this difference between various manifolds.

To have a better grasp behind the intuition that will guide us in this section let's further analyzed the comparison between a sphere and a flat sheet of paper. If we consider a vector in a tangent space of a point of the equator, we can move this vector (mapping it into other tangent spaces) on the surface of the sphere. We want to rigidly move it (as we were translating all the tangent space) to the north pole, then we slide down to the left, until we reach the equator. Lastly we go back to the initial point, passing along the equator. Doing so, we would obtain a new vector which hasn't the same orientation of the initial one, clearly this does not happen on the sheet of paper. We will use this strange behavior to describe curvature, since when it does not occur the manifold is flat as a sheet of paper.

4.1 Parallel transport and covariant derivatives

We first need to define a way to drag around the manifold a vector. Actually this drag can be whatever we want, thus the definition will be very general, later on we will restrict this concept to be adapted to the metric tensor (which, we remember, can be used to define angles).

Definition 4.1 (Parallel transport)**.** *Let's consider a curve $\gamma : I \to \mathcal{M}$ and a vector $\vec{W} \in T_{P_0}$, with $P_0 = \gamma(\lambda_0)$. We define a map, called parallel transport, defined from $T_{P_0} \to T_P$, with $P = \gamma(\lambda_0 + \Delta\lambda)$.*
The image of the parallel transport is denoted by $\vec{W}_{\Delta\lambda} \in T_P$.

Given a proper way to move vector around a manifold, we can define derivatives, indeed to define some "incremental ratio" we need to subtract two vectors that are defined into two different tangent spaces, but now we can map one in the tangent space of the other.

Definition 4.2 (Covariant derivatives)**.** *Given a curve γ, its tangent vector field $\vec{V} = \frac{d}{d\lambda}$ and a vector field $\vec{W}$, we can define the covariant derivative of $\vec{W}$ in a point $P = \gamma(\lambda_0)$*

$$\nabla_{\vec{V}} \vec{W} = \lim_{\Delta\lambda \to 0} \left. \frac{\vec{W}_{-\Delta\lambda} - \vec{W}}{\Delta\lambda} \right|_{\lambda_0} .$$

This definition applies only to vectors and without defining the precise rules to parallel transport objects, its is really not so useful. We will thus define parallel transport from the covariant derivative, after that we have defined describing its operatorial proprieties. We will require that for scalar function it reduces to derivatives along the curve, the Leibniz rule (possessed by all derivatives) and additional proprieties that define them as covariant derivatives.

Definition 4.3 (Covariant derivatives)**.** *The covariant derivative of a tensor, along a curve which tangent vector field is $\vec{V} = \frac{d}{d\lambda}$, must satisfy:*

- *given a function f and a vector field $\vec{W}$:* $\nabla_{\vec{V}}(f\vec{W}) = \frac{df}{d\lambda}\vec{W} + f\nabla_{\vec{V}}(\vec{W})$
- $\nabla_{\vec{V}}(\vec{W} \otimes \vec{U}) = \nabla_{\vec{V}}(\vec{W}) \otimes \vec{U} + \vec{W} \otimes \nabla_{\vec{V}}(\vec{U})$
- $\nabla_{\vec{V}}\big(\tilde{\omega}(\vec{W})\big) = \nabla_{\vec{V}}(\tilde{\omega})(\vec{W}) + \tilde{\omega}\nabla_{\vec{V}}(\vec{W})$
- *given two curves and two functions f, g:* $\nabla_{g\vec{V}+h\vec{W}}(\vec{U}) = g\nabla_{\vec{V}}(\vec{U}) + h\nabla_{\vec{W}}(\vec{U})$

With this definition in our hands we can evaluate the covariant derivative of a vector, in components:

$$\begin{aligned}
\nabla_{\vec{V}}(\vec{W}) &= V^\mu \nabla_{\partial_\mu}(W^\nu \partial_\nu) \\
&= V^\mu \nabla_{\partial_\mu}(W^\nu)\partial_\nu + V^\mu W^\nu \nabla_{\partial_\mu}(\partial_\nu) \\
&= V^\mu \partial_\mu(W^\nu)\partial_\nu + V^\mu W^\nu \nabla_{\partial_\mu}(\partial_\nu),
\end{aligned}$$

in this expression we have obtained a strange object $\nabla_{\partial_\mu}(\partial_\nu)$ which is responsible for the difference between covariant derivatives and the regular ones.

Definition 4.4. *We define the Christoffel symbols, or affine connections*

$$\nabla_{\partial_\mu}(\partial_\nu) = \Gamma^\lambda_{\mu\nu}\partial_\lambda.$$

Note that these **are not tensors**, as partial derivatives are not.
We can now obtain the components of the covariant derivative of a vector

$$\begin{aligned}
\nabla_{\vec{V}}(\vec{W}) &= V^\mu \partial_\mu(W^\nu)\partial_\nu + V^\mu W^\nu \nabla_{\partial_\mu}(\partial_\nu) \\
&= V^\mu \partial_\mu(W^\nu)\partial_\nu + V^\mu W^\nu \Gamma^\lambda_{\mu\nu}\partial_\lambda \\
&= V^\mu [\partial_\mu(W^\lambda) + W^\nu \Gamma^\lambda_{\mu\nu}]\partial_\lambda,
\end{aligned}$$

in general we write

$$\boxed{W^\lambda_{;\mu} = \nabla_\mu W^\lambda = \partial_\mu(W^\lambda) + W^\nu \Gamma^\lambda_{\mu\nu}}.$$

Now, using the proprieties of covariant derivatives, we can evaluate the covariant derivative of 1-forms, and this, of tensors

$$\begin{aligned}
\nabla_\mu(W_\nu V^\nu) &= \partial_\mu(W_\nu V^\nu) = W_\nu \partial_\mu V^\nu + V^\nu \partial_\mu W_\nu \\
&= \nabla_\mu(W_\nu)V^\nu + W_\nu \nabla_\mu(V^\nu) \\
&= \nabla_\mu(W_\nu)V^\nu + W_\nu \partial_\mu V^\nu + W_\lambda V^\nu \Gamma^\lambda_{\mu\nu} \\
\Rightarrow &\ \boxed{\nabla_\mu(W_\nu) = \partial_\mu W_\nu - W_\lambda \Gamma^\lambda_{\mu\nu}}, \\
\Rightarrow &\ \boxed{\nabla_\mu(T^\nu{}_\rho) = \partial_\mu T^\mu{}_\nu + T^\lambda{}_\rho \Gamma^\nu_{\mu\lambda} - T^\nu{}_\lambda \Gamma^\lambda_{\mu\rho}}.
\end{aligned}$$

4.2 Geodesics and geodesic maps

With covariant derivatives, and parallel transport, in our hands, we can define **geodesics**. On a flat sheet of paper geodesics are straight lines, onto a sphere are maximum circles: in both cases tangent vectors to geodesics, when moved along them, are always tangent to the geodesic. This means that their "rate of change" lies (or better, is tangent) on the geodesic. With this in mind we can proceed to define geodesics.

Definition 4.5 (Geodesics)**.** *A curve $\gamma : I \to \mathcal{M}$ is a geodesic if its tangent vector field $\vec{V}$ satisfies*

$$\nabla_{\vec{V}}\vec{V} = \alpha\vec{V},$$

with α real function along the curve.

Note that we can change parametrization in order to set $\alpha = 0$: indeed

$$\lambda \to \mu = k(\lambda)\lambda \quad \Rightarrow \vec{V} = \frac{d}{d\lambda} = \frac{d\mu}{d\lambda}\frac{d}{d\mu} = k\vec{W}$$
$$\Rightarrow \nabla_{\vec{W}}\vec{W} = \frac{1}{k^2}\nabla_{\vec{V}}\vec{V} + \vec{V}\frac{1}{k}\frac{d}{d\lambda}\frac{1}{k} = \left(\frac{1}{k^2}\alpha - \frac{1}{k^3}\frac{dk}{d\lambda}\right)\vec{V} = \left(\frac{1}{k}\alpha - \frac{1}{k^2}\frac{dk}{d\lambda}\right)\vec{W},$$

thus, imposing $\frac{dk}{d\lambda} = k\alpha$, we may obtain some parametrization for which $\alpha' = 0$.
Once the geodesic definition has this form, we can obtain, through composition with a chart, the *geodesic equation*

$$\begin{aligned}(\nabla_{\vec{V}}\vec{V})^\mu &= V^\nu(\partial_\nu V^\mu + \Gamma^\mu_{\rho\nu}V^\rho)\\ &= \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\rho\nu}V^\rho V^\nu\\ &= \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\nu}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} = 0.\end{aligned}$$

Note that, given a basis of a tangent space, we can solve the geodesic equation for each vector, obtaining n geodesics (n dimension of the manifold) and we can use (since the basis vectors are linearly independent) their parameters as coordinates. In this way, in the original point, geodesic equations implies that the Christoffel symbols vanish, thus in that point covariant derivatives are equivalent to the partial ones. This charts constitute what we call a **normal frame**, and physically is the reference frame in which, locally, we recover special relativity.

Lastly, let's define the *geodesic map*, which is the application that parallel transport vectors,

$$\vec{W}_{\Delta\lambda} = e^{\Delta\lambda\nabla_{\vec{V}}}\vec{W}\Big|_P,$$

which is a generic solution of the equation, $\nabla_{\vec{V}}\vec{W} = \vec{W}$, that implies that $\vec{W}(P_0)$ is parallel transported along $\vec{V}$.

4.3 Measuring curvature

Recalling the introduction we developed in the beginning, we can develop formally that procedure for a generic manifold.
Consider two commuting vector fields $\vec{V} = \frac{d}{d\lambda}$, $\vec{W} = \frac{d}{d\mu}$, since they commute they generate curves on which we can move along and come back to the initial point (closed loops).
Consider a third vector $\vec{A}$, and let's parallel transport along one of these loops using the geodesic map:

$$\begin{aligned}\vec{A}^* &= e^{\delta\lambda\nabla_{\vec{V}}}e^{\delta\mu\nabla_{\vec{W}}}e^{-\delta\lambda\nabla_{\vec{V}}}e^{-\delta\mu\nabla_{\vec{W}}}\vec{A}\\ &= \vec{A} + [\nabla_{\vec{V}}, \nabla_{\vec{W}}]\delta\lambda\delta\mu + o(3),\end{aligned}$$

in this way we can see that, in order to have the vector transported to be equal to the original, the commutator of its covariant derivative must vanish.

Definition 4.6 (Riemann tensor)**.** *The (1,1) tensor R, such that*

$$R(\vec{V},\vec{W})\vec{A} = [\nabla_{\vec{V}}, \nabla_{\vec{W}}]\vec{A} - \nabla_{[\vec{V},\vec{W}]}\vec{A},$$

is called the Riemann tensor. Componentwise it reads

$$R^{\mu}_{\nu\rho\sigma} = R(\partial_\rho, \partial_\sigma)^{\mu}_{\nu}, \qquad R = R^{\mu}_{\nu\rho\sigma} \tilde{dx^{\nu}} \otimes \partial_\mu.$$

In this way we can interpret the Riemann tensor as measuring the difference between a parallel transported vector, along a closed loop, and the original one, or as some sort of second derivative

$$\delta\vec{A} \approx R(\vec{V},\vec{W})\vec{A}, \quad \Leftrightarrow \quad \frac{\delta^2 A^\mu}{\delta\lambda\delta\mu} \approx R^{\mu}_{\nu\rho\sigma} V^\rho W^\sigma A^\nu.$$

4.4 Metric connection

Notice that, until now, we really haven't defined what really means *to be curved*: the Riemann tensor is defined using covariant derivatives that can be defined using parallel transport, or can be used to define parallel transport. Either way we need to define what parallel means, or to define parallel transport or to define what means that the covariant derivative of a vector field is zero.
The notion of parallel is can be defined in lots of ways, but we will focus only on the notion that is induced by the metric tensor (through the scalar product), this approach leads to the **metric connection.**

Consider two vector fields $\vec{U}$, $\vec{W}$, such that $\nabla_{\vec{V}}\vec{U} = \nabla_{\vec{V}}\vec{W} =$, so that they are parallel transported along integral curves of $\vec{V}$. If we want to impose that "parallel" is defined by the metric, we must impose that the scalar product (so the angles between vectors) is preserved when we parallel transport vectors along $\vec{V}$. Thus, we impose

$$0 = \nabla_{\vec{V}}(g(\vec{U},\vec{W})) = \nabla_{\vec{V}}(g)(\vec{U},\vec{W}) + g(\nabla_{\vec{V}}(\vec{U}),\vec{W}) + g(\vec{U},\nabla_{\vec{V}}(\vec{W})), \Rightarrow \boxed{\nabla_{\vec{V}}(g)(\vec{U},\vec{W}) = 0}.$$

Definition 4.7 (Metric connection)**.** *Given the metric tensor g, the metric connection condition is that*

$$\boxed{\nabla_{\vec{V}} g = 0} \quad \forall\vec{V}.$$

Imposing this condition, we can evaluate the Christoffel symbols, and thus we can see that the covariant derivatives are well-defined, as well parallel transport. To do that we should also impose **symmetric connection**, or that $\Gamma^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\rho\nu}$.

Consider the metric connection condition in components

$$\nabla_\mu g_{\rho\sigma} = \partial_\mu g_{\rho\sigma} - \Gamma^{\lambda}_{\mu\rho} g_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma} g_{\rho\lambda} = 0,$$

cycling indices

$$\nabla_\sigma g_{\mu\rho} = \partial_\sigma g_{\mu\rho} - \Gamma^{\lambda}_{\sigma\mu} g_{\lambda\rho} - \Gamma^{\lambda}_{\sigma\rho} g_{\mu\lambda} = 0,$$
$$\nabla_\rho g_{\sigma\mu} = \partial_\rho g_{\sigma\mu} - \Gamma^{\lambda}_{\rho\sigma} g_{\lambda\mu} - \Gamma^{\lambda}_{\rho\mu} g_{\sigma\lambda} = 0,$$

subtracting the last two rows from the first we get

$$2\Gamma^{\lambda}_{\rho\sigma} g_{\lambda\mu} = \partial_\mu g_{\rho\sigma} - \partial_\sigma g_{\mu\rho} - \partial_\rho g_{\sigma\mu}.$$

Multiplying by the inverse metric $g^{\mu\nu}$ and $1/2$ we get the explicit form of the metric connection

$$\boxed{\Gamma^{\nu}_{\rho\sigma} = \frac{g^{\mu\nu}}{2}(\partial_\mu g_{\rho\sigma} - \partial_\sigma g_{\mu\rho} - \partial_\rho g_{\sigma\mu})}.$$

4.5 The length of a geodesic

Knowing the precise meaning of what is a geodesic, we can try to find a different interpretation. To do so, we will try to understand what is the shortest path between two points, this can be accomplished using Euler-Lagrange equation on the length functional

$$S = \int_A^B ds = \int_{\lambda_A}^{\lambda_B} \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}d\lambda.$$

If we define the Lagrangian $L = \frac{g_{\mu\nu}}{2}\dot{x}^\mu\dot{x}^\nu$, and we parametrize the curve with its own length ($\lambda = s$), from the above equation we read the normalization $L = \frac{1}{2}$.
If we now evaluate the first variation of this functional we get

$$\delta S = \int_{s_A}^{s_B} \frac{\delta L}{\sqrt{2L}}ds = \delta \int_{s_A}^{s_B} L\ ds,$$

this show that minimizing the length is equal to minimizing our Lagrangian (it will be easier). Using Euler-Lagrange

$$\begin{aligned} 0 = \frac{d}{ds}\frac{\partial L}{\partial \dot{x}^\lambda} - \frac{\partial L}{\partial x^\lambda} &= \frac{d}{ds}(g_{\mu\lambda}\dot{x}^\mu) - \frac{\partial_\lambda g_{\mu\nu}}{2}\dot{x}^\mu\dot{x}^\nu \\ &= g_{\mu\lambda}\ddot{x}^\mu + \partial_\nu g_{\mu\lambda}\dot{x}^\nu\dot{x}^\mu - \frac{\partial_\lambda g_{\mu\nu}}{2}\dot{x}^\mu\dot{x}^\nu, \end{aligned}$$

since $\dot{x}^\nu\dot{x}^\mu$ is symmetric, we can symmetrize $\partial_\nu g^{\mu\lambda}$ and multiplying by $g^{\lambda\rho}$ we get the geodesic equation

$$\ddot{x}^\rho + \frac{g^{\lambda\rho}}{2}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})\dot{x}^\mu\dot{x}^\nu = \ddot{x}^\rho + \Gamma^\rho_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0.$$

This shows that geodesics are also the shortest path between two points.
These calculations are fundamentals for the study of motion in curved spacetime, since this is the way we obtain geodesics and conserved quantities, since we can use Lagrange formalism, as we showed.

4.6 Proprieties of Riemann tensor

To study the proprieties of the Riemann tensor, consider its components, defined by[6]

$$\begin{aligned} R^\rho_{\sigma\mu\nu}V^\sigma &= [\nabla_\mu, \nabla_\nu]V^\rho = \nabla_\mu\nabla_\nu V^\rho - \nabla_\nu\nabla_\mu V^\rho \\ &= \partial_\mu\nabla_\nu V^\rho - \cancel{\Gamma^\lambda_{\mu\nu}\nabla_\lambda V^\rho} + \Gamma^\rho_{\lambda\nu}\nabla_\mu V^\lambda - (\partial_\nu\nabla_\mu V^\rho - \cancel{\Gamma^\lambda_{\mu\nu}\nabla_\lambda V^\rho} + \Gamma^\rho_{\lambda\mu}\nabla_\nu V^\lambda) \\ &= \partial_\mu(\cancel{\partial_\nu V^\rho} + \Gamma^\rho_{\nu\sigma}V^\sigma) + \Gamma^\rho_{\lambda\nu}(\partial_\mu V^\lambda + \Gamma^\lambda_{\mu\sigma}V^\sigma) + \\ &\qquad - (\partial_\nu(\cancel{\partial_\mu V^\rho} + \Gamma^\rho_{\mu\sigma}V^\sigma) + \Gamma^\rho_{\lambda\mu}(\partial_\nu V^\lambda + \Gamma^\lambda_{\nu\sigma}V^\sigma)) \\ &= \partial_\mu\Gamma^\rho_{\nu\sigma}V^\sigma + \cancel{\Gamma^\rho_{\nu\sigma}\partial_\mu V^\sigma} + \Gamma^\rho_{\lambda\nu}(\cancel{\partial_\mu V^\lambda} + \Gamma^\lambda_{\mu\sigma}V^\sigma) + \\ &\qquad - (\partial_\nu\Gamma^\rho_{\mu\sigma}V^\sigma + \cancel{\Gamma^\rho_{\mu\sigma}\partial_\nu V^\sigma} + \Gamma^\rho_{\lambda\mu}(\cancel{\partial_\nu V^\lambda} + \Gamma^\lambda_{\nu\sigma}V^\sigma)) \\ &= \partial_\mu\Gamma^\rho_{\nu\sigma}V^\sigma + \Gamma^\rho_{\lambda\nu}\Gamma^\lambda_{\mu\sigma}V^\sigma - \partial_\nu\Gamma^\rho_{\mu\sigma}V^\sigma - \Gamma^\rho_{\lambda\mu}\Gamma^\lambda_{\nu\sigma}V^\sigma. \\ &\Rightarrow \boxed{R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\lambda\nu}\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\lambda\mu}\Gamma^\lambda_{\nu\sigma}} \end{aligned}$$

[6]The full expression $R(\vec{V}, \vec{W}) = [\nabla_{\vec{V}}, \nabla_{\vec{W}}] - \nabla_{[\vec{V},\vec{W}]}$ reduces to $R(\partial_\mu, \partial_n u) = [\nabla_\mu, \nabla_\nu]$, since coordinates vector commute.

From this expression we can note that:

- having defined the metric connection to be symmetric is crucial, otherwise we wouldn't have had all the cancellations;
- the Riemann tensor is constructed from non-tensorial objects, arranged in such a way that it is a tensor;
- the last two indices are antisymmetric, as the commutator is;
- the Riemann tensor depends entirely on the connection, and not directly from the metric, during the derivation we never used it, therefore this expression is valid for all symmetric connections, not only metric ones.

Given a normal reference all Cristoffel symbols vanish in a point, although not its derivatives, thus in P it reads (using $\Gamma^{\nu}_{\rho\sigma} = \frac{g^{\mu\nu}}{2}(\partial_\mu g_{\rho\sigma} - \partial_\sigma g_{\mu\rho} - \partial_\rho g_{\sigma\mu})$)

$$\begin{aligned} R_{\rho\sigma\mu\nu} &= g_{\rho\lambda} R^{\lambda}_{\sigma\mu\nu} = g_{\rho\lambda}(\partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma}) \\ &= \frac{1}{2} g_{\rho\lambda} g^{\lambda\delta}(\cancel{\partial_\mu \partial_\nu g_{\sigma\delta}} + \partial_\mu \partial_\sigma g_{\delta\nu} - \partial_\mu \partial_\delta g_{\nu\sigma} - \cancel{\partial_\mu \partial_\nu g_{\sigma\delta}} - \partial_\nu \partial_\sigma g_{\delta\mu} + \partial_\nu \partial_\delta g_{\mu\sigma}) \\ &= \frac{1}{2}(\partial_\mu \partial_\sigma g_{\rho\nu} - \partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\nu \partial_\sigma g_{\rho\mu} + \partial_\nu \partial_\rho g_{\mu\sigma}), \end{aligned}$$

in which we used that we can always choose a reference frame in which the first derivatives of the metric vanish in a precise point.
From this expression we can deduce important proprieties of this tensor:

- it is antisymmetric with respect the exchange of the first two indices or the second

$$\boxed{R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\nu\mu}};$$

- it is symmetric for the exchange of the first pair of indices with the second one

$$\boxed{R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}};$$

- it satisfies the algebraic **Bianchi identity**

$$\boxed{R_{\rho[\sigma\mu\nu]} = 0};$$

These 3 symmetries lower down the number of independent components of the tensors: from n^4 we go down to $n^2 \frac{n^2-1}{12}$, which is 20 in 4 dimensions.
Lastly, from a direct calculation from the equation derived above, one can prove that the Riemann tensor satisfies the **Bianchi identity**

$$\boxed{\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0}.$$

We now define other tensors that are pressnt in the Einstein field equations:

- **Ricci tensor**, $R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$;
- **Curvature scalar**, $R = R^{\mu}{}_{\mu}$;
- **Einstein tensor**, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$.

From the above proprieties we can obtain the **Bianchi identity** for the Einstein tensor

$$\boxed{\nabla_\mu G^{\mu\nu} = 0}.$$