Friedmann–Lemaître–Robertson–Walker metric and Friedmann equations

Basics concepts of cosmology

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1 Introduction

General relativity, based upon the **principle of general relativity**, which states that the laws of physics are the same for all observers, describe gravitational interactions through the geometry of the universe.
The universe is modelled via a manifold, which charts represent different reference frames, and gravity is described by the metric tensor. Free-falling observers, that are not subject to forces like the electromagnetic one, following geodesics on the manifold will deviate their path, from the "euclidean" straight line, due to the metric of the manifold.
The metric tensor is determined by the **Einstein field equation**

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu},$$

which relates derivatives of the metric, contained in the components of $R_{\mu\nu}$ (Ricci tensor) and in R (Ricci scalar), to the energy-momentum tensor $T_{\mu\nu}$, describing the density of energy and momentum of the matter (but also radiation and other things) contained by the universe.

This deep connection between the geometry of spacetime and the physics of the universe, allows us to study the nature of it, what kind of manifold really it is, its curvature, its time evolution, knowing its content. This role is played by the **Friedmann equations**, which determine the cosmological parameters describing the universe. These will characterize the **Robertson Walker metric**, the metric that describe an isotropic and homogeneous universe.

In this short essay, we will first derive, from the **cosmological principle**, the Robertson Walker metric. We will describe the main features of it and the possible geometries that it can represent. Then, we will derive the Friedmann equation and describe the main cosmological parameters that are contained in it.

2 The geometry of the universe

Modern cosmology is based upon two basics principles:

- **the Copernican Principle**, or that there not exist a preferred observer in the universe;
- **the Cosmological Principle**, which states that the universe is homogeneous and isotropic.

These principles may not seem to be consistent with physical reality, clearly the core of a star is very different from the empty space or even from the intern of planets, but in order to describe the whole universe we need to make some assumptions that are valid on the largest scales. Observations, for example of the distribution of galaxies or of the cosmic microwave background radiation, show that at large scales, on average, the universe looks the same in all directions. From the Copernican Principle we than get that all observer should see an isotropic universe, thus we can claim that all points of the universe should also look the same. Again we should stress that these are just assumptions that, at some large scale, we think that can become adequate to approximate the description of spacetime.

We now have to translate the proprieties of isotropy and homogeneity to the language of differential geometry and manifolds.
Notice that the two above principles refer only to the universe, or better, to space at a fixed time, therefore it is space which is really isotropic and homogeneous, while time has no particular symmetries. Space is **maximally symmetric**, which means that it possesses the maximum number of killing vectors. In fact, homogeneity guarantees 3 killing vectors, associated to the 3 possible space translations, while isotropy guarantees other 3 killing vectors, associated to the 3 rotations around a point, and the maximum number of independent killing vectors for a $3D$ manifold is indeed 6.

2.1 The Robertson-Walker metric

We will now construct a set of charts that are the more convenient to describe the assumed geometry.
First, consider a space-like hypersurface Σ (a volume in this case), which is a slice of the spacetime manifold, corresponding to space (the universe) at a fixed time. On this hypersurface we chose one chart with coordinates $x^\mu = (0, x^1, x^2, x^3)$. At each point $P \in \Sigma$ we pick a vector $\vec{n}$ that is orthogonal to Σ (it should be orthogonal to each vector of the tangent space, in P, of the submanifold defined by Σ and our chart) such that those are normalized to -1, since they must be time-like. In each point P we can now build a unique geodesic, for which $\vec{n}$ is the tangent vector in P, from the following Chaucy problem

$$\begin{cases} \left(\nabla_{\vec{n}}\vec{n}\right)^\mu = \frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} = 0, \\ \frac{dx^\mu}{dt}\big|_P = n^\mu\big|_P, \\ x^\mu(0) = x^\mu\big|_P. \end{cases} \tag{2.1}$$

We can extend our initial chart, in a neighborhood of Σ, assigning to each point Q the coordinates $x^\mu = (t, x^1, x^2 x^3)$, where t is the value (in Q) of the parameter of one geodesic passing through Q, and $(x^1, x^2 x^3)$ are the coordinates of the point P, from which the geodesic starts. These coordinates will eventually fail once some geodesics, from our construction, will meet and intersect.

We now want to describe the metric of our spacetime manifold using one of these charts. To do so we will take the chart induced basis of each tangent space $(\partial_t, \partial_1, \partial_2, \partial_3)$ and then label them:

$$\partial_t = \vec{n}, \qquad \partial_i = \vec{Y}_{(i)}, \tag{2.2}$$

where ∂_t is really the normal vector field we have defined, since they are parallel transported along the geodesics and defined by (2.1).
Using this basis, the first component of the metric reads, by our initial construction,

$$g_{tt} = g(\partial_t, \partial_t) = n^\mu n_\mu = -1. \tag{2.3}$$

On Σ, from our construction hypothesis $\vec{n} \perp \Sigma$, the time-spacial mixed components read

$$g_{ti} = g(\partial_t, \partial_i) = n_\mu Y^\mu_{(i)} = 0. \tag{2.4}$$

We can prove that this holds also outside Σ by evaluating its covariant derivative along one of the geodesics we constructed

$$\begin{aligned}
n^\nu \nabla_\nu (n_\mu Y^\mu_{(i)}) &= n^\nu n_\mu \nabla_\nu (Y^\mu_{(i)}) + Y^\mu_{(i)} n^\nu \nabla_\nu (n_\mu) \\
&= n^\nu n_\mu \nabla_\nu (Y^\mu_{(i)}) \\
&= Y^\nu_{(i)} n_\mu \nabla_\nu (n^\mu) \\
&= \frac{1}{2}\big(Y^\nu_{(i)} n_\mu \nabla_\nu (n^\mu) + Y^\nu_{(i)} n_\mu \nabla_\nu (n^\mu)\big) \\
&= \frac{1}{2}\big(Y^\nu_{(i)} n_\mu \nabla_\nu (n^\mu) + Y^\nu_{(i)} n_\mu \nabla_\nu (g^{\mu\lambda} n_\lambda)\big) \\
&= \frac{1}{2}\big(Y^\nu_{(i)} n_\mu \nabla_\nu (n^\mu) + Y^\nu_{(i)} n_\mu \nabla_\nu (g^{\mu\lambda}) n_\lambda + Y^\nu_{(i)} n_\mu g^{\mu\lambda} \nabla_\nu (n_\lambda)\big) \\
&= \frac{1}{2}\big(Y^\nu_{(i)} n_\mu \nabla_\nu (n^\mu) + Y^\nu_{(i)} n^\lambda \nabla_\nu (n_\lambda)\big) \\
&= \frac{1}{2} Y^\nu_{(i)} \nabla_\nu (n^\lambda n_\lambda) = 0,
\end{aligned}$$

in which we used (in order): the geodesic equation $n^\nu \nabla_\nu (n_\mu) = 0$, that coordinates vectors commute, so that $[\vec{n}, \vec{Y}_{(i)}]^\mu = n^\nu \nabla_\nu (Y^\mu_{(i)}) - Y^\nu_{(i)} \nabla_\nu (n^\mu) = 0$, the metric connection condition $\nabla g = 0$, and last that, being $n^\mu n_\mu = -1$, its derivative vanishes.
Summing up all the above results, we can write the metric, from (2.3) and (2.4), as

$$ds^2 = -dt^2 + g_{ij} dx^i dx^j.$$

In this expression the absence of the mixed terms $dt\, dx^i$ reflects that there exist a family of hypersurfaces, defined by $t =$const, that are all orthogonal to the vector field $\vec{n}$. These represent the evolved universe at different times.
The spacial components of the metric now depend on all the coordinates of the chart we have introduced. If we consider how time evolution could affect the spacial terms we can deduce that all the components g_{ij} should scale in the same way, otherwise we could have different scaling in different directions, which is against the idea that space is isotropic. We will write explicitly the time dependence as

$$ds^2 = -dt^2 + a^2(t) g_{ij} dx^i dx^j.$$

Let's now take into account that each space hypersurface is a maximally symmetric submanifold. As showed in Appendix, maximally symmetric manifolds have the peculiar propriety that, due to its high number of symmetries, the Riemann tensor reduces, in 3 dimensions, to

$$^{(3)}R_{ijkl} = \frac{^{(3)}R}{6}(g_{ik}g_{jl} - g_{il}g_{jk}), \tag{2.5}$$

in which the $^{(3)}$ is used to signal that these are tensor referred to the submanifold and $^{(3)}R$ is the Ricci scalar. The Ricci tensor reads:

$$^{(3)}R_{ij} = \frac{^{(3)}R}{6}(3g_{ij} - g^{lk}g_{il}g_{jk}) = \frac{^{(3)}R}{3}g_{ij}, \tag{2.6}$$

we want to use this relation to determine the metric, without the Einstein field equation. To simplify the metric, we can notice that being maximally symmetric, each space submanifold will also have spherical symmetry, which allows us to write the metric in spherical coordinates

$$ds^2 = -dt^2 + a(t)^2\left[e^{2\beta(r)}dr^2 + e^{2\gamma(r)}r^2(d\theta^2 + \sin^2\theta\ d\phi^2)\right]. \tag{2.7}$$

Here we introduced some unknown functions $\beta(r)$, $\gamma(r)$, that depend only on the radial coordinate due to spherical symmetry, and can be expressed as an exponential because we want to preserve the signature. The angular part, $d\Omega^2 = d\theta^2 + \sin\theta\ d\phi^2$, scale with the same factor $e^{2\gamma}$, in order to maintain sphere to be perfectly round.
We can simplify this metric even more by scaling the radial coordinate

$$r \to e^{-\gamma(r)}r, \qquad dr \to \left(1 - r\frac{d\gamma}{dr}\right)e^{-\gamma(r)}dr, \tag{2.8}$$

in this way the metric becomes

$$ds^2 = -dt^2 + a^2(t)\left[\left(1 - r\frac{d\gamma}{dr}\right)^2 e^{2(\beta(r)-\gamma(r))}dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)\right], \tag{2.9}$$

since g_{rr} must be positive, we can define a function $\alpha(r)$, such that $e^{2\alpha} = \left(1-r\frac{d\gamma}{dr}\right)^2 e^{2(\beta(r)-\gamma(r))}$, so that the metric reads

$$ds^2 = -dt^2 + a^2(t)\left[e^{2\alpha(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)\right]. \tag{2.10}$$

Now, we can evaluate the Christoffel symbols of the metric on the universe submanifold:

$$\begin{aligned} &^{(3)}\Gamma^r_{rr} = \frac{d\alpha}{dr}, \qquad ^{(3)}\Gamma^\theta_{r\theta} = \frac{1}{r}, \qquad ^{(3)}\Gamma^r_{\theta\theta} = -re^{-2\alpha}, \qquad ^{(3)}\Gamma^r_{rr} = \frac{\cos\theta}{\sin\theta}, \\ &^{(3)}\Gamma^\phi_{r\phi} = \frac{1}{r}, \qquad ^{(3)}\Gamma^r_{\phi\phi} = -re^{-2\alpha}\sin^2\theta, \qquad ^{(3)}\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \end{aligned} \tag{2.11}$$

all the others are zero or deducible from the symmetries of the above.
Therefore, the non-vanishing components of the Riemann tensor are:

$$\begin{aligned} ^{(3)}R^r_{\theta r\theta} &= re^{-2\alpha}\frac{d\alpha}{dr}, \\ ^{(3)}R^r_{\phi r\phi} &= re^{-2\alpha}\sin^2\theta\frac{d\alpha}{dr}, \\ ^{(3)}R^\theta_{\phi\theta\phi} &= (1 - e^{-2\alpha})\sin^2\theta. \end{aligned} \tag{2.12}$$

Lastly, we can get the Ricci tensor:

$$^{(3)}R_{rr} = \frac{2}{r}\frac{d\alpha}{dr}, \qquad ^{(3)}R_{\theta\theta} = e^{-2\alpha}\left[r\frac{d\alpha}{dr} - 1\right] + 1, \qquad ^{(3)}R_{\phi\phi} = \sin^2\theta R_{\theta\theta}. \tag{2.13}$$

Combining the expression for the Ricci tensor of maximally symmetric space (2.6) and the one above (2.13) we get 2 differential equation that we can solve to determine the metric

$$\begin{aligned} ^{(3)}R_{rr} &= \frac{^{(3)}R}{3} g_{tt} \quad \Rightarrow \quad \boxed{\frac{2}{r}\frac{d\alpha}{dr} = \frac{^{(3)}R}{3} e^{2\alpha}} \\ ^{(3)}R_{ij} &= \frac{^{(3)}R}{3} g_{ij} \quad \Rightarrow \quad \boxed{e^{-2\alpha}\left[r\frac{d\alpha}{dr} - 1\right] + 1 = \frac{^{(3)}R}{3} r^2}. \end{aligned}$$

Substituting the first one into the second, we can get an initial condition for the first

$$\frac{d\alpha}{dr} = \frac{^{(3)}R}{6} r e^{2\alpha}, \qquad e^{-2\alpha}\left[\frac{^{(3)}R}{6} r^2 e^{2\alpha} - 1\right] + 1 = \frac{^{(3)}R}{3} r^2. \tag{2.14}$$

To solve this differential equation we start by defining $k = \frac{^{(3)}R}{6}$, and then we can integrate

$$\int e^{-2\alpha}\ d\alpha = \int kr\ dk \Rightarrow e^{-2\alpha} = -kr^2 + C, \tag{2.15}$$

then, to determine C we plug this solution into the initial condition

$$\begin{aligned} 2kr^2 &= e^{-2\alpha}\left[kr^2 e^{2\alpha} - 1\right] + 1 = kr^2 - e^{-2\alpha} + 1 \\ &= kr^2 + kr^2 - C + 1 = 2kr^2 - C + 1, \quad \Rightarrow \quad C = 1. \end{aligned}$$

In this way we have obtained the **Robertson Walker metric**

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)\right], \tag{2.16}$$

notice that, to obtain this metric, we never used the Einstein field equation, but only geometrical proprieties of spacetime, deduced from the cosmological principle, therefore this metric is totally generic once we assume that.
The coordinates $(t, r, \theta.\phi)$ are called *comoving coordinates*, since these precise choice makes manifest the isotropy and homogeneity of the universe, that wouldn't be manifest in a moving reference frame with respect to the universe (the cosmic fluid that we will use to model it).
In the metric appear two parameters

- $a(t)$, the **cosmic scale factor**, which measure how the "size" of the universe change with time;
- k, the **curvature constant**, that is proportional to the Ricci scalar of each universe submanifold and thus measures the curvature of space.

These parameters can be rescaled without affecting the metric (2.16) in the following way

$$r \to \lambda r, \qquad a \to \lambda^{-1} a, \qquad k \to \lambda^{-2} k, \tag{2.17}$$

this allows us to give dimensions of a length arbitrarily to r or to a.
Lastly we should stress that the distance between the origin of our reference frame and a point is given by

$$R = \int_0^{r^*} \frac{a(t)\ dr}{\sqrt{1-kr^2}}, \tag{2.18}$$

while $a(t)r$ should really be interpreted as an areal radius, which scales distances on different concentric spheres.

2.2 The curvature of the universe

We will now give some interpretation to the curvature constant of the Robertson Walker metric (2.16).
First, it is useful to use the scale invariance of the metric to reduce the possible values of this parameter just to three, we will see that the actual value of this constant isn't really relevant, while it is just its sign to determine the curvature. Rescaling

$$r \to \sqrt{|k|}r, \qquad a \to \frac{a}{\sqrt{|k|}}, \qquad k \to \frac{k}{|k|}, \tag{2.19}$$

k can only be $\{-1, 0, +1\}$.
Let's discuss each case studying just the spacial metric $d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)$.

- **Flat universe**, for $k = 0$, the metric reduces to usual metric of $\mathbb{R}^3$ in spherical coordinates

  $$d\sigma^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)$$

  which correspond to a flat universe.

- **Closed universe**, for $k = +1$, the metric can be reduced to a more familiar one introducing

  $$d\chi = \frac{dr}{\sqrt{1-r^2}} \quad \Rightarrow \quad r = \sin\chi,$$

  $$d\sigma^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\ d\phi^2),$$

  which clearly shows that the radial coordinate is bounded[1] ($r \in [0, +1]$) and the metric is really the one of a 3-dimensional sphere.

- **Open universe**, for $k = -1$, the metric can be better interpreted by introducing

  $$d\chi = \frac{dr}{\sqrt{1+r^2}} \quad \Rightarrow \quad r = \sinh\chi,$$

  $$d\sigma^2 = d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\ d\phi^2),$$

  which shows that r is not bounded, and the metric takes the form of the one of a 3-dimensional hyperboloid.

The value of k will be determined by the energy content of the universe, through the Einstein field equations, this will be the goal of the next section.

[1]This behavior is signaled by the fact that in the previous chart the metric was singular for $r = 1$.

2.3 Christoffel symbol of the R-W metric

Since in the following sections we will need the metric connection and the Ricci tensor, we are going just to calculate them now.
The Christoffel symbols of the Robertson Walker metric (2.16) are

$$\begin{aligned}
\Gamma^0_{11} &= \frac{a\dot{a}}{1-kr^2}, & \Gamma^1_{11} &= \frac{kr}{1-kr^2}, \\
\Gamma^0_{22} &= a\dot{a}r^2, & \Gamma^0_{33} &= a\dot{a}r^2\sin^2\theta, \\
\Gamma^1_{01} &= \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a}, & \Gamma^1_{22} &= -r(1-kr^2), \\
\Gamma^1_{33} &= -r(1-kr^2)\sin^2\theta, & \Gamma^2_{12} &= \Gamma^3_{13} = \frac{1}{r}, \\
\Gamma^2_{33} &= -\sin\theta\cos\theta, & \Gamma^2_{23} &= \cot\theta,
\end{aligned} \tag{2.20}$$

the ones that are not listed are zero or obtainable from the symmetry of the connection. From the above Christoffel symbols, the non-zero components of the Ricci tensor are

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a}, \\
R_{11} &= \frac{a\ddot{a}-2\dot{a}+2k}{1-kr^2}, \\
R_{22} &= r^2(a\ddot{a}-2\dot{a}+2k), \\
R_{33} &= r^2(a\ddot{a}-2\dot{a}+2k)\sin^2\theta.
\end{aligned} \tag{2.21}$$

3 The Friedmann equations

We now want to develop a model that can predict the values of the parameters of the Robertson Walker metric knowing the energy content of the universe. The connection between the metric and the energy is given by the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{3.1}$$

where it appears the energy-momentum tensor $T^{\mu\nu}$. Therefore, we should define the form of this tensor, and so we have to find a way to model the content of the universe.

3.1 Cosmic fluids

The simplest model of the content of the universe is that of a perfect fluid of energy and matter. A perfect fluid, in general, is described by an energy-momentum tensor given by

$$T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p g^{\mu\nu}, \tag{3.2}$$

where ρ is the energy density of the fluid, p the pressure and U^μ the 4-velocity of a particle of the fluid.
When we described the coordinate, used in the Robertson Walker metric, we introduced that those where comoving coordinate, with respect to the content of the universe, so that in that reference frame the metric would be manifestly isotropic and homogeneous. Now that we added some information about what is the content of the universe, we understand that, in the reference frame associated to those coordinates, the fluid will be at rest, otherwise we wouldn't have manifest isotropy and homogeneity.
Being at rest, the energy-momentum tensor of the fluid takes the form

$$U^\mu = (1,0,0,0), \quad \Rightarrow \quad T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij} p & \\ 0 & & & \end{pmatrix}, \quad T^\mu{}_\nu = \mathrm{diag}(-\rho, p, p, p) \tag{3.3}$$

Even before than plugging everything in the Einstein equations, we can study the energy conservation of this fluid, which reads

$$\begin{aligned} 0 &= \nabla_\mu T^\mu{}_0 \\ &= \partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu\lambda} T^\lambda{}_0 - \Gamma^\lambda_{\mu 0} T^\mu{}_\lambda \\ &= \partial_0 T^0{}_0 + \Gamma^\mu_{\mu 0} T^0{}_0 - \Gamma^\lambda_{\mu 0} T^\mu{}_\lambda \\ &= -\dot{\rho} - 3\frac{\dot{a}}{a}\rho - 3\frac{\dot{a}}{a} p \\ &= -\dot{\rho} - 3\frac{\dot{a}}{a}(\rho + p), \end{aligned} \tag{3.4}$$

in which we used that $T^\mu{}_\nu$ is diagonal, and the Christoffel symbols (2.20) of the Robertson Walker metric.
From here we should assume that the fluid follows some simple kind of equation of state, like

$$p = \omega \rho, \qquad \omega = \text{constant}. \tag{3.5}$$

Inserting this into the conservation of energy equation, we have just obtained, we get

$$\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a}, \tag{3.6}$$

that can be solved to obtain how the energy density of the fluid scales with respec to $a(t)$

$$\int \frac{d\rho}{\rho} = -3(1+\omega) \int \frac{da}{a} \quad \Rightarrow \quad \boxed{\rho = \rho_0 a^{-3(1+\omega)}}. \tag{3.7}$$

It can be useful to study some simple cases of fluids.

- **Dust**: this kind of fluid is defined as a set of collisionless, non-relativistic particles, that therefore will have zero pressure:

  $$p_d = \omega_d \rho_d = 0, \quad \Rightarrow \quad \omega_d = 0 \quad \Rightarrow \quad \rho_d = \frac{E}{V} = \rho_0 a^{-3}.$$

  We can appreciate how, for dust, the energy density scales with the volume ($V \propto a^3$), keeping constant the total energy. This sort of fluid can be used to model stars and galaxies, for the pressure is negligible, compared to the energy density.

- **Radiation**: in this case we want to describe massless particles or ultra-relativistic ones, which can be approximated to be massless. We can obtain an equation of state for this fluid by first observing that the $T^{\mu\nu}$ is traceless for E-M fields

  $$T^{\mu}{}_{\mu} = F^{\mu\lambda} F_{\mu\lambda} - \frac{1}{4} g^{\mu}{}_{\mu} F^{\lambda\sigma} F_{\lambda\sigma} = 0,$$

  at the same time the (3.3) shows that

  $$T^{\mu}{}_{\mu} = -\rho + 3P, \quad \Rightarrow \quad P_r = \frac{1}{3}\rho_r,$$

  which implies $\omega_r = \frac{1}{3}$. Therefore, the energy density of radiation scales as

  $$\rho_r = \rho_0 a^{-4},$$

  that means that for radiation the total energy is not conserved, we interpret this as the fact that, while the universe expands, radiation gets redshifted.

- **Vacuum or dark energy**: this last type of cosmic fluid is quite a strange one, the equation of state for this fluid is

  $$p_v = -\rho_v, \quad \Rightarrow \quad \omega_v = -1.$$

  This implies that the energy density, as well as the pressure, is a constant, that we can set to be proportional to the *cosmological constant* Λ:

  $$\rho_v = \frac{\Lambda}{8\pi G}.$$

Initially, it was thought that the universe could be described just but dust and radiation, a radiation dominated universe that then transitioned into a matter dominated universe. This was supported by the fact that ρ_r decrease faster than ρ_d, when the universe expands. Nowadays, we know that the expansion of the universe is accelerating, and led to the introduction of the dark energy.

3.2 Derivation of the equations

Now that we characterized the main types of fluids that we can use to model the content of the universe, we can proceed to derive, from Einstein field equations, the equation governing the evolution of the universe.
First we want to simplify a bit Einstein equations (3.1): from the trace of both sides of the equation we get

$$R - \frac{4}{2}R = 8\pi GT \quad \Rightarrow \quad R = -8\pi GT, \tag{3.8}$$

where $T = T^{\mu}{}_{\mu}$, plugging this result in the field equations, we can remove the Ricci scalar:

$$R_{\mu\nu} = 8\pi G\biggl(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\biggr). \tag{3.9}$$

From the Ricci tensor components of the Robertson Walker metric (2.21) and the energy momentum tensor (3.3) we can obtain two equations:

- the $\mu\nu = 00$ component leads to

$$\begin{aligned} -3\frac{\ddot{a}}{a} &= 8\pi G\biggl[-\rho - \frac{1}{2}(-\rho + 3p)\biggr] \\ &= 4\pi G(\rho + 3p); \end{aligned}$$

- the $\mu\nu = ij$ components lead to

$$\begin{aligned} \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{a^2}g_{ij} &= 8\pi G\biggl[pg_{ij} - \frac{1}{2}g_{ij(-\rho+3p)}\biggr] \\ &= 4\pi G(\rho - p)g_{ij}. \end{aligned}$$

Substituting the first into the second we obtain

$$\begin{aligned} -\frac{4}{3}\pi G(\rho + 3p) + \frac{2\dot{a}^2 + 2k}{a^2} &= 4\pi G(\rho - p) \\ \frac{2\dot{a}^2 + 2k}{a^2} &= 4\pi G\frac{4}{3}\rho \end{aligned}$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}}, \tag{3.10}$$

which is the **first Friedmann equation**, while from the 00 component alone we get the **second Friedmann equation**

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)}. \tag{3.11}$$

The first, which is the one that is usually referred as the Friedmann equation, will determine the time evolution of the scale factor $a(t)$. To solve it, it is enough to know the dependence $\rho(a)$, that we previously discussed.

3.3 Universe geometry and its density

Usually, the first Friedmann equation (3.10) is expressed in terms of some cosmological parameters:

- the **Hubble parameter**, $H = \frac{\dot{a}}{a}$, which measure the rate of expansion,
- the **critical density**, $\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$
- the **density parameter**, $\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{\text{crit}}}$.

In this way (3.10) explicitly tells us whether the universe is flat or not, indeed it reads

$$\boxed{\Omega - 1 = \frac{k}{H^2 a^2}}, \tag{3.12}$$

from which we can distinguish 3 distinct cases:

- $\rho < \rho_{\text{crit}} \quad \Leftrightarrow \quad \Omega < 1 \quad \Leftrightarrow \quad k < 0 \quad \Leftrightarrow$ *open universe*,
- $\rho = \rho_{\text{crit}} \quad \Leftrightarrow \quad \Omega = 1 \quad \Leftrightarrow \quad k = 0 \quad \Leftrightarrow$ *flat universe*,
- $\rho > \rho_{\text{crit}} \quad \Leftrightarrow \quad \Omega > 1 \quad \Leftrightarrow \quad k > 0 \quad \Leftrightarrow$ *closed universe.*

Observations suggest that now, for our universe, $k \approx 0$. Therefore, we will solve the Friedmann equation for flat geometry

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho.$$

- **Matter dominated universe**: in this case is approximated to containing only dust, therefore $\rho = \rho_0 a^{-3}$, plugging this into the above differential equation we get

  $$\dot{a} = H_0 a^{-\frac{1}{2}} \quad \Rightarrow \quad a(t) = \left(\frac{3}{2}H_0 t\right)^{2/3},$$

  where we introduced $H_0 = H(t_0) = \sqrt{\frac{8\pi G}{3}\rho_0}$ and imposed $a(0) = 0$.
  This kind of universe is expanding but at a slower and slower rate ($\ddot{a} \leq 0$).

- **Radiation dominated universe**: now the universe is approximately filled only by radiation, therefore $\rho = \rho_0 a^{-4}$, and the above differential equation now reads

  $$\dot{a} = H_0 a^{-1} \quad \Rightarrow \quad a(t) = \sqrt{2H_0 t},$$

  where again $H_0 = H(t_0) = \sqrt{\frac{8\pi G}{3}}$ and we imposed $a(0) = 0$.
  Again, this universe is expanding at a slower and slower rate ($\ddot{a} \leq 0$).

- **Empty universe**: lastly we consider an empty universe, in which only vacuum energy plays the role, therefore $\rho = \frac{\Lambda}{8\pi G}$, from which we get

  $$\dot{a} = a\sqrt{\frac{\Lambda}{3}} \quad \Rightarrow \quad a(t) = a_0 e^{\sqrt{\frac{\Lambda}{3}}(t-t_0)},$$

  in which we imposed $a(t_0) = a_0$.
  Note that, among the 3 cases, this universe is the only one that has an accelerating expansion ($\ddot{a} \geq 0$).

A Maximally symmetric spaces

Consider $\mathbb{R}^n$, this space is highly symmetric: clearly it is isotropic and homogeneous, or, in a simpler way, every point and every direction "look" the same.
This means that $\mathbb{R}^n$ is symmetric under every type of rotation and translations: in n-dimensions there are n possible translations (along the n axes) and $n\frac{n-1}{2}$ possible rotations (for each axis we can rotate it towards $n-1$ other axes and to avoid double counting $x \to y$ and $y \to x$ we divide by 2), for a total number of symmetries equals to

$$n + n\frac{n-1}{2} = n\frac{n+1}{2}.$$

An n-dimensional manifold is said to be **maximally symmetric** if it possesses the same number of symmetries of $\mathbb{R}^n$. In the differential geometry language, a symmetry is defined through isometries, that are diffeomorphisms under which the metric tensor is invariant. For each symmetry of the metric we can define a **Killing vector**, which satisfies the Killing equation

$$0 = (\pounds_{\vec{K}}g)_{\mu\nu} = \nabla_\mu K_\nu + \nabla_\nu K_\mu, \tag{A.1}$$

where $\pounds_{\vec{K}}$ is the Lie derivative along $\vec{K}$, which is the Killing vector.

We now want to show that a maximally symmetric space really possesses the maximum number of symmetry, namely the maximum number of independent Killing vectors. Consider the defining equation of the Riemann tensor applied to a 1-form

$$R^\mu_{\nu\rho\sigma}K_\mu = -[\nabla_\rho, \nabla_\sigma]k_\nu, \tag{A.2}$$

this equation combined with the algebraic Bianchi identity ($R^\mu_{\nu\rho\sigma} + R^\mu_{\rho\sigma\nu} + R^\mu_{\sigma\nu\rho} = 0$) implies that each Killing vector must satisfy

$$\nabla_\rho\nabla_\sigma K_\nu - \nabla_\sigma\nabla_\rho K_\nu + \nabla_\sigma\nabla_\nu K_\rho - \nabla_\nu\nabla_\sigma K_\rho + \nabla_\nu\nabla_\rho K_\sigma - \nabla_\rho\nabla_\nu K_\sigma = 0.$$

This equation can be simplified by the Killing equation (A.1), using this relation we can sum pairs of terms obtaining

$$2(\nabla_\rho\nabla_\sigma K_\nu - \nabla_\sigma\nabla_\rho K_\nu - \nabla_\nu\nabla_\sigma K_\rho) = 0,$$

that using (A.2) turns out to be the following

$$R^\mu_{\nu\rho\sigma}K_\mu = \nabla_\nu\nabla_\sigma K_\rho. \tag{A.3}$$

This equation show that the second covariant derivatives acts on Killing vectors just as a linear application. In this way we can determine every derivative of a killing vector in a specific point, just by knowing its value and the one of its first covariant derivative at the same point.
If we now Taylor expand the Killing vector around a point P, we will obtain some kind of expansion that depends on all orders covariant derivatives in P, however we showed that we can evaluate those just knowing $K_\mu(P)$ and $\nabla_\nu K_\mu(P)$. This means that we can express the Killing vector field as a combination of two functions that do not depend on the killing vector itself or its derivatives:

$$K_\mu(x) = A_\mu{}^\lambda(x,P)K_\lambda(P) + B_\mu{}^{\lambda\nu}(x,P)\nabla_\nu K_\lambda(P),$$

note that these functions depend only on x, the point P, and the metric, through the Riemann tensor. For this reason these must be the same functions for all Killing vectors:

$$K_\mu^{(n)}(x) = A_\mu{}^\lambda(x,P)K_\lambda^{(n)}(P) + B_\mu{}^{\lambda\nu}(x,P)K_\lambda^{(n)}(P). \tag{A.4}$$

The above equation tells us that a given killing vector is determined by $K_\lambda^{(n)}(P)$, which has N possible independent values, and by $K_\lambda^{(n)}(P)$, which has $N\frac{N-1}{2}$ independent values, due to its antisymmetry (which is a consequence of the Killing equation (A.1)).
In this way we have shown that the maximum number of independent Killing vectors in an N-dimensional manifold is exactly the same number that possesses $\mathbb{R}^N$

$$N + N\frac{N-1}{2} = N\frac{N+1}{2}.$$

We want to conclude deriving the form that has the Riemann tensor in a maximally symmetric space.
In general, equation (A.3) must hold for every Killing vector, this means that it also must be consistent with the commutator of covariant derivatives (A.2). This requirement and the fact that we have the maximum number of linearly independent Killing vectors will determine the form of $R^\mu_{\nu\rho\sigma}$. Consider (A.2) applied to the two indices tensor

$$[\nabla_\sigma, \nabla_\nu]\nabla_\mu K_\rho = -R^\lambda_{\mu\sigma\nu}\nabla_\lambda K_\rho - R^\lambda_{\rho\sigma\nu}\nabla_\mu K_\lambda,$$

the equation (A.3) can be used to obtain

$$\begin{aligned}&\nabla_\sigma(R^\lambda_{\nu\rho\mu}K_\lambda) - \nabla_\nu(R^\lambda_{\sigma\rho\mu}K_\lambda) = \\ &\quad = \nabla_\sigma R^\lambda_{\nu\rho\mu}K_\lambda - \nabla_\nu R^\lambda_{\sigma\rho\mu}K_\lambda + \ R^\lambda_{\nu\rho\mu}\nabla_\sigma K_\lambda - R^\lambda_{\sigma\rho\mu}\nabla_\nu K_\lambda = -R^\lambda_{\mu\sigma\nu}\nabla_\lambda K_\rho - R^\lambda_{\rho\sigma\nu}\nabla_\mu K_\lambda.\end{aligned}$$

Now, killing equation (A.1) allows us to move the index λ to the covariant derivative in each term, then, using a bunch of Kronecker deltas we get

$$(\nabla_\sigma R^\lambda_{\nu\rho\mu} - \nabla_\nu R^\lambda_{\sigma\rho\mu})K_\lambda = (R^\lambda_{\nu\rho\mu}\delta_\sigma{}^\alpha - R^\lambda_{\sigma\rho\mu}\delta_\nu{}^\alpha + R^\lambda_{\mu\sigma\nu}\delta_\rho{}^\alpha - R^\lambda_{\rho\sigma\nu}\delta_\mu{}^\alpha)\nabla_\alpha K_\lambda.$$

This relation must hold for every Killing vector, but we have the maximum number of independent Killing vector, thus we can generate every killing vector from these. The general expansion of a Killing vector field (A.4) shows that a Killing vector field that vanishes in P while its derivatives does not can exists, and we surely can obtain it from a linear combination of the others. The above equation holds also for this one in P only if the right-hand side vanishes too, this can happen only if the term in parentheses is symmetric in λ α (so that it vanishes when contracted with $\nabla_\alpha K_\lambda$ that is antisymmetric)

$$R^\lambda_{\nu\rho\mu}\delta_\sigma{}^\alpha - R^\lambda_{\sigma\rho\mu}\delta_\nu{}^\alpha + R^\lambda_{\mu\sigma\nu}\delta_\rho{}^\alpha - R^\lambda_{\rho\sigma\nu}\delta_\mu{}^\alpha = R^\alpha_{\nu\rho\mu}\delta_\sigma{}^\lambda - R^\alpha_{\sigma\rho\mu}\delta_\nu{}^\lambda + R^\alpha_{\mu\sigma\nu}\delta_\rho{}^\lambda - R^\alpha_{\rho\sigma\nu}\delta_\mu{}^\lambda.$$

Contracting μ and α, recalling that $R^\mu_{\nu\mu\rho} = R_{\nu\rho}$ and $R^\mu_{\mu\nu\rho} = 0$, we find

$$R^\lambda_{\nu\rho\sigma} - R^\lambda_{\sigma\rho\nu} + R^\lambda_{\rho\sigma\nu} - NR^\lambda_{\rho\sigma\nu} = -R_{\nu\rho}\delta_\sigma{}^\lambda + R_{\sigma\rho}\delta_\nu{}^\lambda - R^\lambda_{\rho\sigma\nu},$$

here we recognize that, from the algebraic Bianchi identity,

$$R^\lambda_{\sigma\rho\nu} = -R^\lambda_{\sigma\nu\rho} = R^\lambda_{\nu\rho\sigma} + R^\lambda_{\rho\sigma\nu},$$

which cancels two terms in the previous equation, that now reads, after having lowered one index,

$$(N-1)R_{\lambda\rho\sigma\nu} = R_{\nu\rho}g_{\sigma\lambda} - R_{\sigma\rho}g_{\nu\lambda}. \tag{A.5}$$

Notice that the above equation must be antisymmetric in λ ρ (due to the proprieties of the Riemann tensor),

$$R_{\nu\rho}g_{\sigma\lambda} - R_{\sigma\rho}g_{\nu\lambda} = -R_{\nu\lambda}g_{\sigma\rho} + R_{\sigma\lambda}g_{\nu\rho},$$

contracting λ ν, this relation becomes

$$R_{\sigma\rho} - NR_{\sigma\rho} = -Rg_{\sigma\rho} + R_{\sigma\rho}, \quad \Rightarrow \quad \boxed{R_{\sigma\rho} = \frac{R}{N}g_{\sigma\rho}}, \tag{A.6}$$

inserting this one into the (A.5) we get our final result

$$\boxed{R_{\lambda\rho\sigma\nu} = \frac{R}{N(N-1)}(g_{\nu\rho}g_{\lambda\sigma} - g_{\sigma\rho}g_{\lambda\nu})}. \tag{A.7}$$