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Tensor Perturbations

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Abstract

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1.1 The geometry of the universe

Modern cosmology is based upon two basics principles:

- **the Copernican Principle**, or that all the observers are on equal footing;
- **the Cosmological Principle**, which states that the universe, at the largest scales, is homogeneous and isotropic.

These principles may not seem to be consistent with physical reality: clearly the core of a star is very different from empty space or even from the interior of planets, but in order to describe the whole universe we need to make some simplifying assumptions. Observations, for example of the distribution of galaxies or of the cosmic microwave background radiation, show that at large scales, on average, the universe looks the same in all directions. From the Copernican Principle we then get that all observers should see an isotropic universe, thus we can claim that all points of the universe should also look the same. Again we should stress that these are just assumptions that, at some large scale, we think can become adequate to approximate the description of space, allowing us to reduce significantly the degrees of freedom that we have to study.

We now have to translate the proprieties of isotropy and homogeneity to the language of General Relativity, namely differential geometry and manifolds.
Notice that the two above principles refer only to the universe, or better, to space at a fixed time, therefore it is space which is really isotropic and homogeneous, while time has no particular symmetries.
Hence, we will assume that space is **maximally symmetric**, which means that it possesses the maximum number of independent Killing vectors. In fact, homogeneity guarantees 3 Killing vectors, associated to the 3 possible space translations, while isotropy guarantees other 3 Killing vectors, associated to the 3 rotations around a point, and the maximum number of independent Killing vectors for a 3D manifold is indeed 6 (this is proven in Appendix A.1).

In the next sections we will study first how to describe a spacetime with the above proprieties, then we will work out the dynamics that the Einstein field equations give to it.

1.1.1 The Robertson-Walker metric

We will now proceed constructing charts (coordinates) that are the more convenient to describe the assumed geometry. The main goal of this section is to find the most general form of the metric of an isotropic and homogeneous universe.

To start, consider a space-like hypersurface Σ (a volume in this case), which is a slice of the spacetime manifold, corresponding to space (the universe) at a fixed time. On this hypersurface we choose one chart with coordinates $x^\mu = (0, x^1, x^2, x^3)$.
For each point $P \in \Sigma$ we pick a vector $\vec{n}$ that is orthogonal to Σ (it should be orthogonal to each vector of the tangent space, in P, of the submanifold defined by Σ) such that those are normalized to -1 (since they must be time-like).
In each point P, the following Cauchy problem defines a unique geodesic, for which $\vec{n}$ is the tangent vector,

$$\begin{cases} \left(\nabla_{\vec{n}}\vec{n}\right)^\mu = \frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} = 0, \\ \frac{dx^\mu}{dt}\Big|_P = n^\mu\Big|_P, \\ x^\mu(0) = x^\mu\Big|_P. \end{cases} \tag{1.1}$$

We can extend our initial chart, in a neighborhood of Σ, assigning to each point Q the coordinates $x^\mu = (t, x^1, x^2, x^3)$, where t is the value (in Q) of the parameter of the geodesic passing through Q, and $(0, x^1, x^2, x^3)$ are the coordinates of the point P, from which the geodesic starts. These coordinates will eventually fail once some geodesics, from our construction, will meet and intersect.

We now want to describe the metric of our spacetime manifold using one of these charts. To do so we will take the chart induced basis of each tangent space $(\partial_t, \partial_1, \partial_2, \partial_3)$ and then label them:

$$\partial_t = \vec{n}, \qquad \partial_i = \vec{Y}_{(i)},$$

where ∂_t is by construction the normal vector field we have defined, since $\vec{n}$ is tangent to each geodesic by (1.1) and then is parallel transported along them.
Using this basis, the first component of the metric reads, by our initial construction and because scalar products of parallel transported vectors is preserved by metric connection,

$$g_{tt} = g(\partial_t, \partial_t) = n^\mu n_\mu = -1. \tag{1.2}$$

On Σ, from our construction hypothesis $\vec{n} \perp \Sigma$, the time-spacial mixed components read

$$g_{ti} = g(\partial_t, \partial_i) = n_\mu Y^\mu_{(i)} = 0. \tag{1.3}$$

We can prove that this holds also outside Σ by evaluating its covariant derivative along one of the geodesics we constructed

$$\begin{aligned}
n^\nu\nabla_\nu(n_\mu Y^\mu_{(i)}) &= n^\nu n_\mu \nabla_\nu(Y^\mu_{(i)}) + Y^\mu_{(i)} n^\nu \nabla_\nu(n_\mu) \\
&= n^\nu n_\mu \nabla_\nu(Y^\mu_{(i)}) \\
&= Y^\nu_{(i)} n_\mu \nabla_\nu(n^\mu) \\
&= \frac{1}{2}\big(Y^\nu_{(i)} n_\mu \nabla_\nu(n^\mu) + Y^\nu_{(i)} n_\mu \nabla_\nu(n^\mu)\big) \\
&= \frac{1}{2}\big(Y^\nu_{(i)} n_\mu \nabla_\nu(n^\mu) + Y^\nu_{(i)} n_\mu \nabla_\nu(g^{\mu\lambda} n_\lambda)\big) \\
&= \frac{1}{2}\big(Y^\nu_{(i)} n_\mu \nabla_\nu(n^\mu) + Y^\nu_{(i)} n_\mu \nabla_\nu(g^{\mu\lambda}) n_\lambda + Y^\nu_{(i)} n_\mu g^{\mu\lambda} \nabla_\nu(n_\lambda)\big) \\
&= \frac{1}{2}\big(Y^\nu_{(i)} n_\mu \nabla_\nu(n^\mu) + Y^\nu_{(i)} n^\lambda \nabla_\nu(n_\lambda)\big) \\
&= \frac{1}{2} Y^\nu_{(i)} \nabla_\nu(n^\lambda n_\lambda) = 0,
\end{aligned}$$

in which we used (in order): the geodesic equation $n^\nu\nabla_\nu(n_\mu) = 0$, that coordinates vectors commute, so that $[\vec{n}, \vec{Y}_{(i)}]^\mu = n^\nu\nabla_\nu(Y^\mu_{(i)}) - Y^\nu_{(i)}\nabla_\nu(n^\mu) = 0$[1], the metric connection condition $\nabla g = 0$, and last that, being $n^\mu n_\mu = -1$, its derivative vanishes.

Summing up the above results, we can write the metric, from (1.2) and (1.3), as

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j.$$

In this expression the absence of the mixed terms $dt\ dx^i$ reflects that there exist a family of hypersurfaces, defined by $t = \text{const}$, that are all orthogonal to the vector field $\vec{n}$. These represent the evolved universe at different times.
The spatial components of the metric now depend on all the coordinates of the chart we have introduced. If we consider how time evolution could affect the spatial terms we can deduce that all the components g_{ij} should scale in the same way, otherwise we could have different scaling in different directions, which is against the idea that space is isotropic. We will write explicitly the time dependence as

$$ds^2 = -dt^2 + a^2(t) g_{ij}dx^i dx^j.$$

Let's now take into account that each space hypersurface is a maximally symmetric submanifold. As showed in Appendix A.1, maximally symmetric manifolds have the peculiar propriety that, due to their high number of symmetries, the Riemann tensor reduces, in 3 dimensions, to

$${}^{(3)}R_{ijkl} = \frac{{}^{(3)}R}{6}(g_{ik}g_{jl} - g_{il}g_{jk}),$$

in which the ${}^{(3)}$ is used to signal that these are tensors referred to the submanifold Σ and ${}^{(3)}R$ is the Ricci scalar. The Ricci thus tensor reads:

$${}^{(3)}R_{ij} = \frac{{}^{(3)}R}{6}(3g_{ij} - g^{lk}g_{il}g_{jk}) = \frac{{}^{(3)}R}{3} g_{ij}. \tag{1.4}$$

[1] The Christoffel symbols cancel out, due to symmetric connection, leaving only partial derivatives.

With this relation, we want to determine the metric without the Einstein field equations. To simplify the metric, we can note that, being maximally symmetric, each space submanifold will also have spherical symmetry. This allows us to write the metric in spherical coordinates as

$$ds^2 = -dt^2 + a(t)^2\left[e^{2\beta(r)}dr^2 + e^{2\gamma(r)}r^2(d\theta^2 + \sin^2\theta\; d\phi^2)\right],$$

where $\beta(r)$, $\gamma(r)$ are some unknown functions, that depend only on the radial coordinate due to spherical symmetry. Note that we exploited the exponential in order to preserve the signature. Lastly, the angular part, $d\Omega^2 = d\theta^2 + \sin\theta\; d\phi^2$, must scales with an overall factor $e^{2\gamma}$, in order to maintain sphere to be perfectly round. We can simplify this metric even more by scaling the radial coordinate

$$r \to e^{-\gamma(r)}r, \qquad dr \to \left(1 - r\frac{d\gamma}{dr}\right)e^{-\gamma(r)}dr,$$

in this way the metric becomes

$$ds^2 = -dt^2 + a^2(t)\left[\left(1 - r\frac{d\gamma}{dr}\right)^2 e^{2(\beta(r)-\gamma(r))}dr^2 + r^2(d\theta^2 + \sin^2\theta\; d\phi^2)\right].$$

Since g_{rr} must always be non-negative, we can define a function $\alpha(r)$, such that $e^{2\alpha} = \left(1 - r\frac{d\gamma}{dr}\right)^2 e^{2(\beta(r)-\gamma(r))}$, so that the metric reads

$$ds^2 = -dt^2 + a^2(t)\left[e^{2\alpha(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\; d\phi^2)\right].$$

Now, we can evaluate the Christoffel symbols of the metric restitricted to the universe submanifold (because with this construction the time component has no dynamics):

$$\begin{aligned}
&{}^{(3)}\Gamma^r_{rr} = \frac{d\alpha}{dr}, \qquad {}^{(3)}\Gamma^\theta_{r\theta} = \frac{1}{r}, \qquad {}^{(3)}\Gamma^r_{\theta\theta} = -re^{-2\alpha}, \qquad {}^{(3)}\Gamma^r_{rr} = \frac{\cos\theta}{\sin\theta},\\
&{}^{(3)}\Gamma^\phi_{r\phi} = \frac{1}{r}, \qquad {}^{(3)}\Gamma^r_{\phi\phi} = -re^{-2\alpha}\sin^2\theta, \qquad {}^{(3)}\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta,
\end{aligned} \tag{1.5}$$

all the others are zero or deducible from the symmetries of the above.
We then obtain the non-vanishing components of the Riemann tensor are:

$$\begin{aligned}
{}^{(3)}R^r_{\theta r\theta} &= re^{-2\alpha}\frac{d\alpha}{dr},\\
{}^{(3)}R^r_{\phi r\phi} &= re^{-2\alpha}\sin^2\theta\frac{d\alpha}{dr},\\
{}^{(3)}R^\theta_{\phi\theta\phi} &= (1 - e^{-2\alpha})\sin^2\theta.
\end{aligned} \tag{1.6}$$

Lastly, we can get the Ricci tensor:

$${}^{(3)}R_{rr} = \frac{2}{r}\frac{d\alpha}{dr}, \qquad {}^{(3)}R_{\theta\theta} = e^{-2\alpha}\left[r\frac{d\alpha}{dr} - 1\right] + 1, \qquad {}^{(3)}R_{\phi\phi} = \sin^2\theta R_{\theta\theta}. \tag{1.7}$$

Combining the expression for the Ricci tensor (1.4) and the one above (1.7), we end up with two differential equations that can be solved to determine the metric

$$\begin{aligned}
{}^{(3)}R_{rr} &= \frac{{}^{(3)}R}{3}g_{tt} \quad \Rightarrow \quad \boxed{\frac{2}{r}\frac{d\alpha}{dr} = \frac{{}^{(3)}R}{3}e^{2\alpha}}\\
{}^{(3)}R_{ij} &= \frac{{}^{(3)}R}{3}g_{ij} \quad \Rightarrow \quad \boxed{e^{-2\alpha}\left[r\frac{d\alpha}{dr} - 1\right] + 1 = \frac{{}^{(3)}R}{3}r^2}.
\end{aligned}$$

Since we have two equations for one unknown, substituting the first equation into the second one, we can obtain an initial condition for the former

$$\frac{d\alpha}{dr} = \frac{{}^{(3)}R}{6} r e^{2\alpha}, \qquad\qquad e^{-2\alpha}\left[\frac{{}^{(3)}R}{6} r^2 e^{2\alpha} - 1\right] + 1 = \frac{{}^{(3)}R}{3} r^2.$$

To solve this differential equation we start by defining $k = \frac{{}^{(3)}R}{6}$, and then we integrate

$$\int e^{-2\alpha}\ d\alpha = \int kr\ dr \Rightarrow e^{-2\alpha} = -kr^2 + C,$$

then, to determine C we plug this solution into the initial condition

$$\begin{aligned} 2kr^2 &= e^{-2\alpha}\left[kr^2 e^{2\alpha} - 1\right] + 1 = kr^2 - e^{-2\alpha} + 1 \\ &= kr^2 + kr^2 - C + 1 = 2kr^2 - C + 1, \quad \Rightarrow \quad C = 1. \end{aligned}$$

In this way we have obtained the **Robertson Walker metric**

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)\right], \tag{1.8}$$

notice that, to obtain this metric, we never used the Einstein field equation, but only geometrical proprieties of spacetime, deduced from the cosmological principle, therefore this metric is totally generic once we assume the cosmological principle.
The coordinates (t, r, θ, ϕ) are called **comoving coordinates**, since these precise choice makes manifest the isotropy and homogeneity of the universe, that wouldn't be manifest in a moving reference frame with respect to the universe content (the cosmic fluid that we will use to model it).
In the metric appear two parameters

- $a(t)$, the **cosmic scale factor**, which measure how the "size" of the universe change with time;

- k, the **curvature constant**, that is proportional to the Ricci scalar of each universe submanifold and thus measures the curvature of space.

These parameters can be rescaled as follows, without affecting the metric (1.8),

$$r \to \lambda r, \qquad a \to \lambda^{-1} a, \qquad k \to \lambda^{-2} k,$$

this allows to give dimensions of a length arbitrarily to r or to a.
Lastly, we should stress that the distance between the origin of our reference frame and a point is given by

$$L = \int_0^{r^*} \frac{a(t)\ dr}{\sqrt{1 - kr^2}},$$

while $a(t)r$ should really be interpreted as an areal radius, which scales distances on different concentric spheres.

1.1.2 The curvature of the universe

We will now give some interpretation to the curvature constant that appears in the Robertson Walker metric (1.8).
First, it is useful to use the scale invariance of the metric to reduce the possible values of this parameter just to three, so that it is just its sign to determine the curvature. Rescaling as follows

$$r \to \sqrt{|k|}r, \qquad a \to \frac{a}{\sqrt{|k|}}, \qquad k \to \frac{k}{|k|},$$

k can now only assume the following values $\{-1, 0, +1\}$.
Let's now discuss the geometry associated to each value of k, we will focus just on the spatial metric $d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)$.

- **Flat universe**, for $k = 0$, the metric reduces to usual metric of $\mathbb{R}^3$ in spherical coordinates
$$d\sigma^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)$$
which correspond to a flat universe.

- **Closed universe**, for $k = +1$, the metric can be reduced to a more familiar one introducing
$$d\chi = \frac{dr}{\sqrt{1-r^2}} \qquad \Rightarrow \qquad r = \sin\chi,$$
$$d\sigma^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\ d\phi^2),$$
which clearly shows that the radial coordinate is bounded[2] ($r \in [0, +1]$) and the metric is the one of a 3-dimensional sphere.

- **Open universe**, for $k = -1$, the metric can be better understood by introducing
$$d\chi = \frac{dr}{\sqrt{1+r^2}} \qquad \Rightarrow \qquad r = \sinh\chi,$$
$$d\sigma^2 = d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\ d\phi^2),$$
which shows that r is not bounded, and the metric takes the form of the one of a 3-dimensional hyperboloid.

The value of k will be determined by the energy content of the universe, through the Einstein field equations, this will be the goal of the next section.

1.1.3 Christoffel symbol of the R-W metric

Since in the following sections we will need the metric connection and the Ricci tensor, we are going just to calculate them now.

[2]This behavior is signaled by the fact that in the previous chart the metric was singular for $r = 1$.

The Christoffel symbols of the Robertson Walker metric (1.8) are

$$
\begin{aligned}
\Gamma^0_{11} &= \frac{a\dot{a}}{1-kr^2}, & \Gamma^1_{11} &= \frac{kr}{1-kr^2},\\
\Gamma^0_{22} &= a\dot{a}r^2, & \Gamma^0_{33} &= a\dot{a}r^2\sin^2\theta,\\
\Gamma^1_{01} &= \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a}, & \Gamma^1_{22} &= -r(1-kr^2),\\
\Gamma^1_{33} &= -r(1-kr^2)\sin^2\theta, & \Gamma^2_{12} &= \Gamma^3_{13} = \frac{1}{r},\\
\Gamma^2_{33} &= -\sin\theta\cos\theta, & \Gamma^2_{23} &= \cot\theta,
\end{aligned}
\tag{1.9}
$$

the ones that are not listed are zero or obtainable from the symmetry of the connection.
From the above Christoffel symbols, the non-zero components of the Ricci tensor are

$$
\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a},\\
R_{11} &= \frac{a\ddot{a}-2\dot{a}+2k}{1-kr^2},\\
R_{22} &= r^2(a\ddot{a}-2\dot{a}+2k),\\
R_{33} &= r^2(a\ddot{a}-2\dot{a}+2k)\sin^2\theta.
\end{aligned}
\tag{1.10}
$$

1.2 The Friedmann equations

We now want to determine the dynamics of the parameters appearing in the Robertson Walker (1.8) metric knowing the energy content of the universe. The connection between the metric and the energy is given by the *Einstein field equations*

$$
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.11}
$$

where it appears the energy-momentum tensor $T^{\mu\nu}$.

1.2.1 Cosmic fluids

The simplest model for the content of the universe is a *perfect fluid* of energy and matter. A perfect fluid, in general, is described by an energy-momentum tensor given by

$$
T^{\mu\nu} = (\rho+p)U^\mu U^\nu + pg^{\mu\nu}, \tag{1.12}
$$

where ρ is the energy density of the fluid, p the pressure and U^μ the 4-velocity of a particle of the fluid.
When we described the coordinates appearing in the Robertson Walker metric, we anticipated that those were comoving coordinates with respect to the content of the universe (so that in that reference frame the metric would be manifestly isotropic and homogeneous). In the reference frame associated to those coordinates, the fluid is at rest, thus its energy-momentum tensor takes the form

$$
U^\mu = (1,0,0,0), \quad \Rightarrow \quad T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0\\ 0 & & & \\ 0 & & g_{ij}p & \\ 0 & & & \end{pmatrix}, \quad T^\mu{}_\nu = \mathrm{diag}(-\rho,p,p,p) \tag{1.13}
$$

Even before plugging everything in the Einstein equations, we can study the energy conservation of this fluid, which reads

$$\begin{aligned} 0 &= \nabla_\mu T^\mu{}_0 \\ &= \partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu\lambda} T^\lambda{}_0 - \Gamma^\lambda_{\mu 0} T^\mu{}_\lambda \\ &= \partial_0 T^0{}_0 + \Gamma^\mu_{\mu 0} T^0{}_0 - \Gamma^\lambda_{\mu 0} T^\mu{}_\lambda \\ &= -\dot\rho - 3\frac{\dot a}{a}\rho - 3\frac{\dot a}{a}p \\ &= -\dot\rho - 3\frac{\dot a}{a}(\rho + p), \end{aligned} \tag{1.14}$$

in which we used that $T^\mu{}_\nu$ is diagonal, and the Christoffel symbols (1.9).

From now, we assume that the fluid follows some simple equation of state, as

$$p = \omega\rho, \qquad \omega = \text{constant}. \tag{1.15}$$

Inserting this into the conservation of energy equation (1.14) we find

$$\frac{\dot\rho}{\rho} = -3(1+\omega)\frac{\dot a}{a},$$

that can be solved to obtain how the energy density of the fluid scales as the universe expands:

$$\int \frac{d\rho}{\rho} = -3(1+\omega)\int \frac{da}{a} \quad \Rightarrow \quad \boxed{\rho = \rho_0 a^{-3(1+\omega)}}.$$

To better grasp the physics of our construction let's study the evolution of some simple cases of fluids.

- **Dust**: this kind of fluid is defined as a set of collisionless, non-relativistic particles, that therefore will have zero pressure:

  $$p_d = \omega_d \rho_d = 0, \quad \Rightarrow \quad \omega_d = 0 \quad \Rightarrow \quad \rho_d = \frac{E}{V} = \rho_0 a^{-3}.$$

  We can appreciate how, for dust, the energy density scales with the volume ($V \propto a^3$), keeping constant the total energy. This sort of fluid can be used to model groups of stars and galaxies, for which the pressure is negligible, compared to the energy density.

- **Radiation**: in this case we want to describe massless particles or ultra-relativistic ones, which can be approximated to be massless. We can obtain an equation of state for this fluid by first observing that the $T^{\mu\nu}$ is traceless for E-M fields

  $$T^\mu{}_\mu = F^{\mu\lambda}F_{\mu\lambda} - \frac{1}{4}g^\mu{}_\mu F^{\lambda\sigma}F_{\lambda\sigma} = 0,$$

  at the same time the (1.13) gives that

  $$T^\mu{}_\mu = -\rho + 3P, \quad \Rightarrow \quad P_r = \frac{1}{3}\rho_r,$$

  which implies $\omega_r = \frac{1}{3}$. Therefore, the energy density of radiation scales as

  $$\rho_r = \rho_0 a^{-4},$$

  that means that for radiation the total energy is not conserved. We interpret this as the fact that, while the universe expands, radiation gets redshifted.

- **Vacuum or dark energy**: this last type of cosmic fluid is quite a strange one, the equation of state for this fluid is

$$p_v = -\rho_v, \quad \Rightarrow \quad \omega_v = -1.$$

This means that the energy density, as well as the pressure, as the universe expands, remains constant. Sometimes this is not considered a content of the universe, and it is referred as the *cosmological constant* Λ:

$$\rho_v = \frac{\Lambda}{8\pi G}.$$

Initially, it was thought that the universe could be described just but dust and radiation: a radiation dominated universe that then transitioned into a matter dominated. This was supported by the fact that $\rho_r \propto a^{-4}$ decrease faster than $\rho_d \propto a^{-3}$, as the universe expands.
Nowadays, we know that the expansion of the universe is accelerating, and this led to the introduction of the dark energy, to account for this behavior.

1.2.2 Firedmann equations

Now that we characterized the main types of fluids that we can use to model the content of the universe, we can proceed to derive the equations governing the time evolution of the universe.

First we want to simplify a bit Einstein equations (1.11): from the trace of both sides of the equation we get

$$R - \frac{4}{2}R = 8\pi GT \quad \Rightarrow \quad R = -8\pi GT,$$

where $T = T^{\mu}{}_{\mu}$, plugging this result in the field equations (1.11), we can remove the Ricci scalar:

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right).$$

From the Ricci tensor components of the Robertson Walker metric (1.10) and the energy momentum tensor (1.13) we can obtain two equations:

- the $\mu\nu = 00$ component leads to

$$\begin{aligned} -3\frac{\ddot{a}}{a} &= 8\pi G\left[-\rho - \frac{1}{2}(-\rho + 3p)\right] \\ &= 4\pi G(\rho + 3p); \end{aligned}$$

- the $\mu\nu = ij$ components lead to

$$\begin{aligned} \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{a^2}g_{ij} &= 8\pi G\left[pg_{ij} - \frac{1}{2}g_{ij}(-\rho + 3p)\right] \\ &= 4\pi G(\rho - p)g_{ij}. \end{aligned}$$

Substituting the former into the latter we find

$$-\frac{4}{3}\pi G(\rho+3p)+\frac{2\dot{a}^2+2k}{a^2}=4\pi G(\rho-p)$$
$$\frac{2\dot{a}^2+2k}{a^2}=4\pi G\frac{4}{3}\rho$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G}{3}\rho-\frac{k}{a^2}}, \tag{1.16}$$

which is the **first Friedmann equation**, while from the 00 component alone we get the **second Friedmann equation**

$$\boxed{\frac{\ddot{a}}{a}=-\frac{4\pi G}{3}(\rho+3p)}. \tag{1.17}$$

The first, which is the one that is usually referred as the Friedmann equation, will determine the time evolution of the scale factor $a(t)$. To solve it, it is enough to know the dependence $\rho(a)$, that we previously discussed.

1.2.3 Universe geometry and its density

Usually, the first Friedmann equation (1.16) is expressed in terms of some cosmological parameters:

- the **Hubble parameter**, $H=\frac{\dot{a}}{a}$, which measure the rate of expansion,
- the **critical density**, $\rho_{\text{crit}}=\frac{3H^2}{8\pi G}$
- the **density parameter**, $\Omega=\frac{8\pi G}{3H^2}\rho=\frac{\rho}{\rho_{\text{crit}}}$.

In this way (1.16) explicitly relates the matter content of the universe with its geometry (flat. open or closed). Indeed, inserting the above parameters in (1.16) it reads

$$\boxed{\Omega-1=\frac{k}{H^2a^2}}, \tag{1.18}$$

from which we can distinguish 3 distinct cases:

- $\rho<\rho_{\text{crit}} \quad\Leftrightarrow\quad \Omega<1 \quad\Leftrightarrow\quad k<0 \quad\Leftrightarrow$ *open universe*,
- $\rho=\rho_{\text{crit}} \quad\Leftrightarrow\quad \Omega=1 \quad\Leftrightarrow\quad k=0 \quad\Leftrightarrow$ *flat universe*,
- $\rho>\rho_{\text{crit}} \quad\Leftrightarrow\quad \Omega>1 \quad\Leftrightarrow\quad k>0 \quad\Leftrightarrow$ *closed universe*.

Observations suggest that now, for our universe, $k\approx 0$. Therefore, we will always consider flat geometry. In this case the dynamics of the universe is determined by

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G}{3}\rho.$$

Depending on the different cosmic fluids the universe evolves mainly in three different ways.

- **Matter dominated universe**: in this case, the universe is approximated to contain only dust, therefore $\rho = \rho_0 a^{-3}$. Plugging this energy density into the above differential equation we get

$$\dot{a} = H_0 a^{-\frac{1}{2}} \quad \Rightarrow \quad a(t) = \left(\frac{3}{2} H_0 t\right)^{2/3},$$

  where we introduced $H_0 = H(t_0) = \sqrt{\frac{8\pi G}{3}\rho_0}$ and imposed $a(0) = 0$.
  This kind of universe is expanding but at a slower and slower rate ($\ddot{a} \leq 0$).

- **Radiation dominated universe**: now the universe is approximately filled only by radiation, therefore $\rho = \rho_0 a^{-4}$. The above differential equation now reads

$$\dot{a} = H_0 a^{-1} \quad \Rightarrow \quad a(t) = \sqrt{2H_0 t},$$

  where again $H_0 = H(t_0) = \sqrt{\frac{8\pi G}{3}}$ and we imposed $a(0) = 0$.
  Again, this universe is expanding at a slower and slower rate ($\ddot{a} \leq 0$).

- **Empty universe**: lastly we consider an empty universe or in which vacuum energy dominates, therefore $\rho = \frac{\Lambda}{8\pi G}$, from which we get

$$\dot{a} = a\sqrt{\frac{\Lambda}{3}} \quad \Rightarrow \quad a(t) = a_0 e^{\sqrt{\frac{\Lambda}{3}}(t-t_0)},$$

  in which we imposed $a(t_0) = a_0$.
  Note that, among the cases, this universe is the only one that has an accelerating expansion ($\ddot{a} \geq 0$). It is worth noting also that the first two cases admit a finite time (in our calculations $t = 0$) for which the universe has no spatial extension ($a(0) = 0$ generates a singularity).
  The empty universe does not admit it.

1.2.4 Matter-radiation universe

asd

1.2.5 The ΛCDM model

asd

II

The inhomogeneous universe

III

CMB physic

Chapter 2

Anisotropies of the CMB

2.1 Angular power spectrum

When we observe the *CMB* in the sky, we are interested to measure the temperature of photons coming from a specific direction $\hat{\mathbf{n}}$ to us. In general, this temperature is not perfectly uniform from all directions, we describe these anisotropies in the following way:

$$T(\hat{\mathbf{n}}) = \bar{T}\left[1 + \Theta(\hat{\mathbf{n}})\right] \qquad \text{with } \Theta(\hat{\mathbf{n}}) \stackrel{\text{def}}{=} \frac{\delta T(\hat{\mathbf{n}})}{\bar{T}}$$

and where $\bar{T}$ is the average temperature in the sky and $\delta T(\hat{\mathbf{n}})$ is the temperature fluctuation in the direction $\hat{\mathbf{n}}$. To compare the temperature at two district points in the sky we define the *two point correlation function*:

$$C(\hat{\mathbf{n}}, \hat{\mathbf{n}}') \stackrel{\text{def}}{=} \langle \Theta(\hat{\mathbf{n}})\Theta(\hat{\mathbf{n}}')\rangle, \tag{2.1}$$

here the angle brackets denote an average over an ensemble of universes (It will be discussed later in this section how we can approximate this averaging process).

The most appropriate way to describe the temperature fluctuations, given that these are observed from the sky, is to expand Θ in spherical harmonics

$$\Theta(\hat{\mathbf{n}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\hat{\mathbf{n}}), \tag{2.2}$$

where the coefficients $a_{\ell m}$, also called **multipole moments**, are given by

$$a_{\ell m} = \int d\Omega\ \Theta(\hat{\mathbf{n}}) Y^*_{\ell m}(\hat{\mathbf{n}}).$$

Also, for the multipole moments we can define a two point correlation function

$$\langle a_{\ell m} a^*_{\ell' m'}\rangle = \delta_{\ell\ell'}\delta_{mm'}\ C_\ell, \tag{2.3}$$

where C_ℓ is the **angular power spectrum** and again the angle brackets represent an ensemble average. Sometimes it is also used $\mathcal{D}_\ell \stackrel{\text{def}}{=} \frac{\ell(\ell+1)}{2\pi}\bar{T}^2 C_\ell$.

Note that combining (2.2) and (2.1) we obtain

$$\begin{aligned}
C(\hat{\mathbf{n}}, \hat{\mathbf{n}}') &= \langle \Theta(\hat{\mathbf{n}})\Theta(\hat{\mathbf{n}}')\rangle \\
&= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} \langle a_{\ell m} a_{\ell' m'}\rangle Y_{\ell m}(\hat{\mathbf{n}}) Y_{\ell' m'}(\hat{\mathbf{n}}') \\
&= \sum_{\ell=0}^{\infty} C_\ell \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\mathbf{n}}) Y^*_{\ell m}(\hat{\mathbf{n}}') \\
&\Big\downarrow \quad \text{using} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\mathbf{n}}) Y^*_{\ell m}(\hat{\mathbf{n}}') = \frac{2\ell+1}{4\pi} P_\ell(\cos\theta) \\
&= \sum_{\ell=0}^{\infty} C_\ell \frac{2\ell+1}{4\pi} P_\ell(\cos\theta),
\end{aligned}$$

where P_ℓ are the Legendre polynomials and θ is the angle between $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$. Invoking the orthogonality of the Legendre polynomials we can write

$$C_\ell = 2\pi \int_{-1}^{1} d(\cos\theta)\; P_\ell(\cos\theta) C(\hat{\mathbf{n}}, \hat{\mathbf{n}}'), \tag{2.4}$$

this shows that the angular power spectrum encodes the same information as the two point correlation function (2.1), hence it measures the correlation between the temperature fluctuations at two points in the sky separated by an angle θ.

2.1.1 Estimate the angular power spectrum

We now want to understand how can we estimate the average over the ensemble of universes in the previous definitions. Note that, fixed ℓ, we still get $2\ell + 1$ different values of $a_{\ell m}$, this allows us to estimate the angular power spectrum as

$$\hat{C}_\ell = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2. \tag{2.5}$$

One can show that this estimator is unbiased[1], however its variance is non-zero:

$$\Delta \hat{C}_\ell \stackrel{\text{def}}{=} \sqrt{\langle (C_\ell - \hat{C}_\ell)^2 \rangle} = \sqrt{\frac{2}{2\ell+1}} C_\ell, \tag{2.6}$$

this error that systematically appears in this estimate is usually called **cosmic variance**. Cosmic variance will result in a larger error for smaller values of ℓ, which corresponds to larger angular scales. This can be understood as a consequence of the fewer number of modes $a_{\ell m}$ available at lower ℓ.

2.1.2 Multipole expansion

In the previous section we considered the temperature fluctuations observed by us in the sky, therefore it was natural to assume that these were functions of the direction

[1]An estimator is said to be unbiased if its expected value is equal to the true value of the parameter being estimated.

of observation $\hat{\mathbf{n}}$.
In general however, we should consider that these anisotropies varies also with the position of the observer in spacetime. This broader view is needed since to predict the observations we will need to describe the evolution of the anisotropies throughout the whole universe. Therefore, we will now consider

$$\Theta(t, \mathbf{x}, \hat{\mathbf{p}}) \quad \text{with} \begin{cases} t & \text{cosmic time,} \\ \mathbf{x} & \text{position of the anisotropy in space,} \\ \hat{\mathbf{p}} & \text{direction of motion of the photons.} \end{cases} \tag{2.7}$$

To come back to the observed anisotropies we just fix t at the present day, $\mathbf{x}$ on the earth and we consider the direction of motion of the photons as the direction of observation (since it is the direction from which they come from).

In section 2.2 we will see that the evolution of the anisotropies is described by a linear differential equation (since we are working with first order perturbations). It is therefore useful to introduce here some expansions that will simplify these equations.

First of all, we can simplify the spacial dependence moving to Fourier space

$$\Theta(t, \mathbf{x}, \hat{\mathbf{p}}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{\Theta}(\mathbf{k}, t, \hat{\mathbf{p}}), \tag{2.8}$$

where $\tilde{\Theta}$ is the Fourier transform of Θ. In this way, we obtained a decomposition on plane waves that leaves $\tilde{\Theta}$ depending on two vectors, $\mathbf{k}$ and $\hat{\mathbf{p}}$. However, since the background spacetime is homogeneous and isotropic, the real useful information is encoded in one of these two vectors and in the angle between them. This allows us to define

$$\mu = \frac{\mathbf{k}\cdot\hat{\mathbf{p}}}{k} \qquad \Rightarrow \qquad \tilde{\Theta}(t, \mathbf{k}, \mu) \quad \text{with } \mu \in [-1, 1].$$

This suggests us to that another useful expansion is the **Legendre polynomial expansion**:

$$\begin{aligned} \tilde{\Theta}(t, \mathbf{k}, \mu) &= \sum_{\ell=0}^{\infty} \frac{2\ell+1}{i^\ell} \tilde{\Theta}_\ell(t, \mathbf{k}) P_\ell(\mu), \\ \tilde{\Theta}_\ell(t, \mathbf{k}) &= \frac{i^\ell}{2} \int_{-1}^{1} d\mu \; P_\ell(\mu) \tilde{\Theta}(t, \mathbf{k}, \mu), \end{aligned} \tag{2.9}$$

where P_ℓ are the Legendre polynomials and $\tilde{\Theta}_\ell$ are the **multipoles**.
The Legendre polynomials can be computed recursively using the Bonnet's formula

$$(\ell+1) P_{\ell+1}(\mu) = (2\ell+1)\mu P_\ell(\mu) - \ell P_{\ell-1}(\mu), \tag{2.10}$$

and knowing that $P_0 = 1$, $P_1 = \mu$ and $P_2 = \frac{3\mu^2-1}{2}$.

2.1.3 From perturbations to anisotropies

It is now time to discuss how in general we connect the perturbations of the FRW universe to the power spectrum of the anisotropies that we observe in the *CMB*.

The following sections will be devoted to understand how to get all the physical quantities that are needed to relate perturbations to the anisotropies.

We are interested in evaluating $\langle\tilde{\Theta}(\mathbf{k},\hat{\mathbf{n}})\tilde{\Theta}(\mathbf{k}',\hat{\mathbf{n}}')\rangle$. This quantity is determined by two phenomena:

1. the initial amplitude of the perturbations generated during inflation, which from our point of view are random variables generated by vacuum fluctuations;

2. the evolution of these perturbations that turns these perturbations into the anisotropies that we observe today, this process is clearly deterministic.

This consideration allows us to proceed in the following way: considering the initial perturbations $\mathcal{R}(\mathbf{k})$ we can decompose $\tilde{\Theta}(\mathbf{k},\hat{\mathbf{n}}) = \mathcal{R}(\tilde{\Theta}/\mathcal{R})$, now the ratio $\tilde{\Theta}/\mathcal{R}$ is completely independent of the initial amplitude of the perturbation and won't contribute to the ensemble average.
In this way we get

$$\begin{aligned}\langle\tilde{\Theta}(\mathbf{k},\hat{\mathbf{n}})\tilde{\Theta}^*(\mathbf{k}',\hat{\mathbf{n}}')\rangle &= \langle\mathcal{R}(\mathbf{k})\mathcal{R}(\mathbf{k}')\rangle\frac{\tilde{\Theta}(\mathbf{k},\hat{\mathbf{n}})}{\mathcal{R}(\mathbf{k})}\frac{\tilde{\Theta}^*(\mathbf{k}',\hat{\mathbf{n}}')}{\mathcal{R}^*(\mathbf{k}')}\\ &= (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\mathcal{P}_{\mathcal{R}}(k)\frac{\tilde{\Theta}(\mathbf{k},\hat{\mathbf{n}})}{\mathcal{R}(\mathbf{k})}\frac{\tilde{\Theta}^*(\mathbf{k}',\hat{\mathbf{n}}')}{\mathcal{R}^*(\mathbf{k}')},\end{aligned}$$

where we used the definition of the perturbation power spectrum. In this expression the last two factors now depend only on the magnitude of $\mathbf{k}$ and $\mathbf{k}'$
Now, by inserting this result in the expression for the C_ℓ (2.4) we find

$$\begin{aligned}C_\ell &= 2\pi\int_{-1}^{1}d\mu P_\ell(\mu)\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\langle\tilde{\Theta}(\mathbf{k},\hat{\mathbf{n}})\tilde{\Theta}^*(\mathbf{k}',\hat{\mathbf{n}}')\rangle\\ &= 2\pi\int\frac{d^3k}{(2\pi)^3}\mathcal{P}_{\mathcal{R}}(k)\int_{-1}^{1}d\mu P_\ell(\mu)\frac{\tilde{\Theta}(\mathbf{k},\hat{\mathbf{n}})}{\mathcal{R}(\mathbf{k})}\frac{\tilde{\Theta}^*(\mathbf{k},\hat{\mathbf{n}}')}{\mathcal{R}^*(\mathbf{k})}\\ &= 2\pi\int\frac{dk\ k^2}{(2\pi)^3}\mathcal{P}_{\mathcal{R}}(k)\int_{-1}^{1}d\mu P_\ell(\mu)\sum_{\ell',\ell''}\frac{\tilde{\Theta}_{\ell'}}{\mathcal{R}}\frac{\tilde{\Theta}^*_{\ell''}}{\mathcal{R}^*}(2\ell'+1)(2\ell''+1)i^{\ell'-\ell''}\times\\ &\qquad\qquad\times\int_0^{2\pi}d\phi\int_{-1}^{1}d\cos\theta\ P_{\ell'}(\hat{\mathbf{n}}\cdot\hat{\mathbf{k}})P_{\ell''}(\hat{\mathbf{n}}'\cdot\hat{\mathbf{k}})\\ &\Big\downarrow\text{using}\int_0^{2\pi}d\phi\int_{-1}^{1}d\cos\theta\ P_\ell(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}})P_{\ell'}(\hat{\mathbf{k}}'\cdot\hat{\mathbf{n}}') = \frac{4\pi}{2\ell+1}P_\ell(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}')\delta_{\ell\ell'}\\ &= 8\pi^2\int\frac{dk\ k^2}{(2\pi)^3}\mathcal{P}_{\mathcal{R}}(k)\sum_{\ell'=0}^{\infty}(2\ell'+1)\left|\frac{\tilde{\Theta}_{\ell'}(\mathbf{k},\hat{\mathbf{n}})}{\mathcal{R}(\mathbf{k})}\right|^2\int_{-1}^{1}d\mu P_\ell(\mu)P_{\ell'}(\mu)\\ &\Big\downarrow\text{orthogonality}\int_{-1}^{+1}d\mu\ P_\ell(\mu)P_{\ell'}(\mu) = \frac{2}{2\ell+1}\delta_{\ell\ell'}\\ &= 16\pi^2\int\frac{dk\ k^2}{(2\pi)^3}\mathcal{P}_{\mathcal{R}}(k)\left|\frac{\tilde{\Theta}_\ell(\mathbf{k},\hat{\mathbf{n}})}{\mathcal{R}(\mathbf{k})}\right|^2 = \frac{2}{\pi}\int dk\ k^2\mathcal{P}_{\mathcal{R}}(k)\left|\frac{\tilde{\Theta}_\ell(\mathbf{k},\hat{\mathbf{n}})}{\mathcal{R}(\mathbf{k})}\right|^2,\end{aligned}\tag{2.11}$$

where $\mu = \cos(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}')$ and we used the orthogonality of the Legendre polynomial and that Θ is real.

Lastly, introducing the dimensionless power spectrum $\Delta^2_{\mathcal{R}}(k) \stackrel{\text{def}}{=} \frac{k^3}{2\pi^2}\mathcal{P}_{\mathcal{R}}(k)$ we obtain:

$$\boxed{C_\ell = 4\pi \int \frac{dk}{k}\ \Delta^2_{\mathcal{R}}(k) \left| \frac{\tilde{\Theta}_\ell(\mathbf{k}, \hat{\mathbf{n}})}{\mathcal{R}(\mathbf{k})} \right|^2.} \tag{2.12}$$

We ended with a formula that relates the angular power spectrum to the power spectrum of the perturbations via the so-called **transfer function** $|\frac{\Theta_\ell(\mathbf{k},\hat{\mathbf{n}})}{\mathcal{R}(\mathbf{k})}|$, which describes how the perturbations generates anisotropies and that we have to find in the next sections.

2.2 Time evolution of anisotropies

In this section we want to develop the machinery needed to understand how the anisotropies of the *CMB* observed today evolved from the anisotropies at recombination.

To tackle this problem we need to study the evolution of the phase space of photons in perturbed spacetime. Imposing the *newtonian gauge* we can write the metric as

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(t)(1 - 2\Phi)\delta_{ij}dx^i dx^j,$$

in appendix B.1 we show that the *Liouville operator* can be expressed as (B.3):

$$\hat{L}[f] = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\hat{p}^i}{a} - p\left(H - \frac{\partial \Phi}{\partial t} + \frac{\partial \Psi}{\partial x^i}\frac{\hat{p}^i}{a}\right)\frac{\partial f}{\partial p},$$

where $p^i = \hat{p}^i p$ is the local 3-momentum.
In the above $f = f(x^\mu, p^i)$, however we know that at the background level the phase space distribution should depend only on (t, p) (due to homogeneity and isotropy of the universe). For this reason we should also decompose the distribution in:

$$f(x^\mu, \mathbf{p}) = \bar{f}(t, p) + \Upsilon(x^\mu, \mathbf{p}), \tag{2.13}$$

where Υ is the perturbation of the phase space distribution function.
We can get an expression for this perturbation considering a *blackbody radiation* distribution with a fluctuating temperature $T(x^\mu, \hat{\mathbf{p}}) = \bar{T}(1 + \Theta(x^\mu, \hat{\mathbf{p}}))$.
Note that now Θ depends on the time ($t = x^0$) and position (x^i) of observation, other than the direction of motion of the photons, which corresponds to the direction of observation $\hat{\mathbf{n}}$ of section 2.1, where the position and time of observation were fixed by the Earth position in spacetime.
In this way, expanding in Θ we find

$$\begin{aligned} f(x^\mu, p^i) &= \left[\exp\left\{\frac{p}{k_B\bar{T}(1+\Theta)}\right\} - 1\right]^{-1} \\ &\approx \frac{1}{e^{\frac{p}{k_B\bar{T}}} - 1} + \frac{e^{\frac{p}{k_B\bar{T}}}}{(e^{\frac{p}{k_B\bar{T}}} - 1)^2}\frac{p}{k_B\bar{T}}\Theta = \bar{f} - \Theta p\frac{\partial \bar{f}}{\partial p} \\ \implies \Upsilon &= -\Theta p\frac{\partial \bar{f}}{\partial p}. \end{aligned}$$

Expanding the distribution function also in the Liouville operator we get, at the first order

$$\hat{L}[\Upsilon] = -p\frac{\partial \bar{f}}{\partial p}\left[\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} - \frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Phi}{\partial x^i}\right], \tag{2.14}$$

where the first two terms describe free streaming (free motion of photons without scatterings) while the last two terms account for the effect of gravity.

To complete the Boltzmann equation we need to consider the first order collision term describing Compton scatterings (as explained by Dodelson [1]):

$$C[\Upsilon]|_{\mathbf{CS}} = -p\frac{\partial \bar{f}}{\partial p} n_e \sigma_T \left[\Theta_0 - \Theta + \hat{\mathbf{p}} \cdot \mathbf{v}_b\right], \tag{2.15}$$

where $\mathbf{v}_b$ is the **electron bulk velocity** and Θ_0 is the **anisotropy monopole**, defined as

$$\Theta_0(x^\mu) = \frac{1}{4\pi}\int d\Omega_{\hat{\mathbf{p}}}\Theta(x^\mu, \hat{\mathbf{p}}).$$

Let's appreciate that the collision term, assuming $\mathbf{v}_b = 0$, will vanish, and thus give equilibrium, if the anisotropies $\Theta(\hat{\mathbf{p}}) = \Theta_0$.
Equating the Liouville operator (2.14) with the collision term (2.15) we obtain

$$\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} - \frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Phi}{\partial x^i} = n_e \sigma_T \left[\Theta_0 - \Theta + \hat{\mathbf{p}} \cdot \mathbf{v}_b\right] \tag{2.16}$$

which is the equation describing the dynamics of the CMB anisotropies.
Since this equation is a linear partial differential equation, it can be reduced to an ordinary differential equation by Fourier transforming the spatial coordinates.
Introducing

$$\Theta(x^\mu) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}}\tilde{\Theta}(\mathbf{k}, t), \qquad \mu \overset{\text{def}}{=} \cos\theta = \frac{\mathbf{k}\cdot\hat{\mathbf{p}}}{k},$$

respectively the Fourier transform of Θ and the cosine of the angle between $\mathbf{k}$ and $\hat{\mathbf{p}}$, and assuming that $\mathbf{v}_b$ is irrotational ($\tilde{\mathbf{v}}_b = \hat{\mathbf{k}}\tilde{v}_b$) we can write the Boltzmann equation as

$$\frac{\partial \tilde{\Theta}}{\partial t} + \frac{ik\mu}{a}\tilde{\Theta} + \frac{\partial \tilde{\Psi}}{\partial t} + \frac{ik\mu}{a}\tilde{\Phi} = n_e \sigma_T \left[\tilde{\Theta}_0 - \tilde{\Theta} + \mu \tilde{v}_b\right].$$

If we were to account also for the angular dependence of Compton scatterings (as explained in [1]) we would have obtained:

$$\frac{\partial \tilde{\Theta}}{\partial t} + \frac{ik\mu}{a}\tilde{\Theta} + \frac{\partial \tilde{\Psi}}{\partial t} + \frac{ik\mu}{a}\tilde{\Phi} = n_e \sigma_T \left[\tilde{\Theta}_0 - \tilde{\Theta} + \mu \tilde{v}_b - \frac{3\mu^2 - 1}{4}\tilde{\Theta}_2\right], \tag{2.17}$$

where $\tilde{\Theta}_2 \overset{\text{def}}{=} -\frac{1}{2}\int_{-1}^{+1} d\mu \frac{3\mu^2-1}{2}\tilde{\Theta}$ is the **anisotropy quadrupole**.

2.2.1 Polarization of light

In the previous section we studied how the phase space of photons evolve in a perturbed spacetime. However, we have not yet considered that photons are spin 1 particles, and thus, to fully describe them, we also need to know their polarization.

To better understand how polarization works, let's consider a monochromatic plane wave (which we could consider as a Fourier component of a generic wave). The electric and magnetic fields of such a wave, in empty space, are not independent, due to Maxwell equations, and thus we can just focus on the electric field.
If the wave is propagating along the $\hat{\mathbf{z}}$ axis, its electric field can be written as

$$\mathbf{E}(z,t) = \mathrm{Re}\left\{(E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}})e^{ik(z-t)}\right\},$$

where E_x and E_y are the components of the electric field in complex space. Since they are complex number we can decompose them in $E_x = |E_x|e^{i\phi_x}$, $E_y = |E_y|e^{i\phi_y}$, in this way the monochromatic wave reads:

$$\mathbf{E}(z,t) = |E_x|\cos[k(z-t)]\hat{\mathbf{x}} + |E_y|\cos[k(z-t)+\phi]\hat{\mathbf{y}} \qquad \text{with } \phi = \phi_y - \phi_x.$$

This shows that the electric field, at a fixed $z = z_0$, evolves drawing an ellipse in the xy plane. Note that this ellipse can degenerate depending on the values of $|E_x|$, $|E_y|$ and ϕ:

- if $\phi = 0, \pi$ or if one of the components E_x, E_y vanishes, the ellipse degenerates into a line, we call this case **linear polarization**;
- if $\phi = \pm\frac{\pi}{2}$ and $E_x = E_y$, the ellipse degenerates into a circle, we call this case **circular polarization**.

In general, we can describe the state of a photon by the **Stokes parameters**

$$\begin{aligned} I &\stackrel{\text{def}}{=} |E_x|^2 + |E_y|^2, & Q &\stackrel{\text{def}}{=} |E_x|^2 - |E_y|^2, \\ U &\stackrel{\text{def}}{=} 2|E_x||E_y|\cos\phi, & V &\stackrel{\text{def}}{=} 2|E_x||E_y|\sin\phi, \end{aligned} \tag{2.18}$$

where I is the intensity of the light while Q, U and V describe the polarization. These last three parameters could also be interpreted as the difference of the intensity of the electric field components along different orthogonal axis.
Circular polarization is not produced in the early universe, therefore we will set $V = 0$, so that we are describing only linearly polarized or unpolarized light.

Before proceeding, we should note that under rotations in the xy plane

$$E_x \to E_x\cos\theta - E_y\sin\theta, \qquad E_y \to E_x\sin\theta + E_y\cos\theta,$$

the Stokes parameters will transform as

$$I \to I, \qquad Q \pm iU \to e^{\pm 2i\theta}(Q \pm iU).$$

This transformation shows that the combination $(Q \pm iU)$ transforms as a *spin-2 tensor* while I as a scalar. This observation will be crucial when will need to decompose these modes.

2.2.2 Polarization from Compton scattering

The main mechanism that influence the evolution of CMB is *Compton scattering*. This process can also induce polarization in the photons: indeed, if we consider an electron, on which light can be scatter off, the interaction can absorb some components of the electric field modifying the polarization of the photon.

For example, an unpolarized photon moving along the x axis and deflected along the z axis, in the end, will have a polarization along the y axis. This is due to the simple fact that $\mathbf{E}$ and $\mathbf{B}$ must be orthogonal to the direction of motion and therefore any component along the z axis will be absorbed by the electron.

Consider now some incoming radiation with polarization $\boldsymbol{\epsilon}'_i$[2] which gets scattered off by an electron. The deflected radiation will instead have a polarization $\boldsymbol{\epsilon}_i$. Without loss of generality, we can orient our coordinate axis such that the outgoing radiation is travelling along the z axis and the polarization $\boldsymbol{\epsilon}_1 = \hat{\mathbf{x}}$ and $\boldsymbol{\epsilon}_2 = \hat{\mathbf{y}}$.

The parameter Q, after the scattering, can be estimated decomposing the incoming polarization on the outgoing ones and then averaging over all possible incoming photons:

$$Q \propto \int d\Omega_{in} f_{\text{in}}(\hat{\mathbf{n}}') \sum_{i=1}^{2} \Big[|\boldsymbol{\epsilon}'_i \cdot \hat{\mathbf{x}}|^2 - |\boldsymbol{\epsilon}'_i \cdot \hat{\mathbf{y}}|^2 \Big],$$

where f_{in} is the phase space distribution of the incoming photons.
As a function of the polar incoming angles, the incoming polarization can be written as

$$\begin{aligned}
\boldsymbol{\epsilon}'_1(\theta', \phi') &= (\cos\theta' \cos\phi', \cos\theta' \sin\phi', -\sin\theta'),\\
\boldsymbol{\epsilon}'_2(\theta', \phi') &= (-\sin\phi', \cos\theta', 0).
\end{aligned}$$

Once inserted in the previous integral we find

$$\begin{aligned}
Q &\propto \int d\Omega_{in} f_{\text{in}}(\hat{\mathbf{n}}') \Big[\cos^2\theta' \cos^2\phi' + \sin^2\phi' - \cos^2\theta' \sin^2\phi' - \cos^2\phi' \Big] \\
&\propto \int d\Omega_{in} f_{\text{in}}(\hat{\mathbf{n}}') (\sin^2\theta' \cos 2\phi') \propto \int d\Omega_{\text{in}} f_{\text{in}}(\hat{\mathbf{n}}') \Big[Y_{2,2}(\hat{\mathbf{n}}') + Y_{2,-2}(\hat{\mathbf{n}}') \Big],
\end{aligned}$$

where we recognized, in the last step, that $\sin^2\phi' \cos 2\phi'$ is proportional to the sum of two spherical harmonics[3].
Now, considering perturbations of the temperature in f_{in}, as in (2.13), we discover that, since the integral picks the modes with $\ell = 2$, polarization will be generated through Compton scatterings by the quadrupole anisotropy Θ_2. Similar calculations can lead to the same conclusion for the parameter U.

Following Dodelson [1], we will derive the Boltzmann equation for the polarization. To begin, we define the polarization anisotropy, $\Theta_P(\hat{\mathbf{n}}, \mathbf{k})$: consider polarized light with polarization vectors aligned with the x and y axes (thus propagating in the z direction), the stokes parameter Q will therefore measure the difference of the intensity associated with each polarization. We can then associate to each intensity a temperature (recall $\rho \propto T^4$) so that the Stokes parameter Q can be interpreted as

[2] $\boldsymbol{\epsilon}'_i$ are the versos onto which the $\mathbf{E}$ decomposes.

[3] Recall that $Y_{\ell,\pm\ell}(\theta,\phi) = \frac{(\mp)^\ell}{2^\ell \ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}} \sin^\ell\theta \; e^{\pm i\ell\phi}$ and therefore $Y_{2,2} + Y_{2,-2} \propto sin^2\theta \cos 2\phi$.

a measure of the difference of the temperature associated with the two polarization states. In this way we can define (using $\delta\rho/\rho \propto 4\delta T/T$)

$$\Theta_P \stackrel{\text{def}}{=} \frac{Q}{4I} = \left.\frac{\delta T}{T}\right|_{\text{Polarization}} . \tag{2.19}$$

To consider a generic polarization state we just need to rotate the coordinate system, in this way we can write the Stokes parameters as

$$\frac{Q}{4I} = \Theta_P \cos 2\phi_k, \qquad \frac{U}{4I} = \Theta_P \sin 2\phi_k,$$

where ϕ_k is the angle between $\mathbf{k}$ and the x axis if we assume the motion of the photon along the z axis.

Θ_P will then evolve with its own Boltzmann equation: all the physics described in the previous section is unchanged, however we need to account for Compton scattering effect con polarization. Indeed, we already discussed that the quadrupole Θ_2 will polarize scattered photons, therefore a collision term proportional to Θ_2 must be added to the Boltzmann equation (2.16). However, if polarization is not sourced, through Compton scattering the radiation will gradually become unpolarized. This means that now a term proportional to $-\Theta_P$ must be added as a collision contribution. The final result (Bond and efstathiou 1987) is the Boltzmann equation for the polarization anisotropy:

$$\frac{\partial \tilde{\Theta}_P}{\partial t} + \frac{ik\mu}{a}\tilde{\Theta}_P = -n_e\sigma_T\left[\Theta_P + \frac{1}{2}\Big(1 - P_2(\mu)\Big)\Pi\right], \tag{2.20}$$

where $\Pi = \Theta_2 + \Theta_{P,2} + \Theta_{P,0}$ and $P_2(\mu) = \frac{3\mu^2-1}{2}$ is the order 2 Legendre polynomial.

As we saw Compton scattering is influenced by the polarization of photons, taking into account its effect also the Boltzmann equation (2.16) must be corrected:

$$\frac{\partial \tilde{\Theta}}{\partial t} + \frac{ik\mu}{a}\tilde{\Theta} + \frac{\partial \tilde{\Psi}}{\partial t} + \frac{ik\mu}{a}\tilde{\Phi} = n_e\sigma_T\left[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v}_b - \frac{1}{4}P_2(\mu)\Pi\right]. \tag{2.21}$$

2.2.3 Multipole expansion of the Boltzmann equation

In section 2.2.2 we obtained the differential equations (2.21) and (2.20) governing the time evolution of the anisotropies in the CMB. To end our discussion of the time evolution of the anisotropies we want to expand these equations in multipoles.

Since the CMB is observed in the sky, spherical harmonics are the natural basis to use to project the anisotropies. The fact that the equations (2.21) and (2.20) depend only on $\mu = \hat{\mathbf{p}} \cdot \hat{\mathbf{k}}$ corresponds to a rotational symmetry, of the system, around one of these two vectors. By using spherical polar coordinates, such that the vector $\hat{\mathbf{k}}$ lies on the z axis, the above rotational symmetry corresponds to a rotational symmetry of the azimuthal angle ϕ. Considering that $Y_{\ell m} \propto e^{im\phi}$, we immediately recognize that such symmetry is respected only by spherical harmonics with $m = 0$ and these precisely corresponds to the Legendre polynomials. Therefore, for scalar perturbation, we can limit ourselves to a multipole expansion on the Legendre polynomials without worrying of all the spherical harmonics.

By multiplying the (2.21) by the order ℓ Legendre polynomial $P_\ell(\mu)$ and integrating over μ we can exploit the orthogonality of the Legendre polynomials as follows.

- $\frac{\partial\tilde{\Theta}}{\partial t}$ and $n_e\sigma_T\tilde{\Theta}$ depending on μ in this expansion will give contributions corresponding respectively to $\frac{\partial\tilde{\Theta}_\ell}{\partial t}$ and $n_e\sigma_T\tilde{\Theta}_\ell$.
- $\frac{\partial\tilde{\Psi}}{\partial t}$ and $n_e\sigma_T\tilde{\Theta}_0$ have no μ dependence, which corresponds to the zeroth order Legendre polynomial $P_0(\mu) = 1$, and thus they only contribute to the $\ell = 0$ equation.
- $\tilde{\Phi}$ and $n_e\sigma_T\tilde{v}_b$ are multiplied by $P_1(\mu) = \mu$, giving contributions only to $\ell = 1$ equation, while Π is multiplied by $P_2(\mu) = \frac{3\mu^2-1}{2}$, contributing only to $\ell = 2$ equation. Note that these terms must also be multiplied by a factor corresponding to the integral of their respective Legendre polynomial, since they don't contain any $\tilde{\Theta}$ function to be expanded
- $\frac{ik\mu}{a}\tilde{\Theta}$ is instead more complicated since it is the product of two functions depending on μ. Bonnet's formula (2.10) allows us to simplify the corresponding integral

$$\frac{i^\ell}{2}\int_{-1}^{+1} d\mu\ \mu P_\ell(\mu)\tilde{\Theta} = \frac{i^\ell}{2}\int_{-1}^{+1} d\mu\left[\frac{\ell+1}{2\ell+1}P_{\ell+1}(\mu) + \frac{\ell}{2\ell+1}P_{\ell-1}(\mu)\right]\tilde{\Theta},$$

  in this way this will give contributions to all the equations coupling them together.

Putting all of this together we obtain the following coupled system of differential equations

$$\dot{\tilde{\Theta}}_0 = -\frac{k}{a}\tilde{\Theta}_1 + \dot{\tilde{\Psi}} \tag{2.22a}$$

$$\dot{\tilde{\Theta}}_1 = \frac{k}{3a}\tilde{\Theta}_0 - \frac{2k}{3a}\tilde{\Theta}_2 + \frac{k}{3}\tilde{\Phi} - n_e\sigma_T\left[\tilde{\Theta}_1 + \frac{\tilde{v}_b}{3}\right] \tag{2.22b}$$

$$\dot{\tilde{\Theta}}_\ell = \frac{\ell k}{(2\ell+1)a}\tilde{\Theta}_{\ell-1} - \frac{(\ell+1)k}{(2\ell+1)a}\tilde{\Theta}_{\ell+1} - n_e\sigma_T\left[\tilde{\Theta}_\ell - \frac{\delta_{\ell,2}}{10}\Pi\right] \qquad \ell \geq 2, \tag{2.22c}$$

Similarly, equation (2.20) will result in the following system of differential equations

$$\dot{\tilde{\Theta}}_{P0} = -\frac{k}{a}\tilde{\Theta}_{P1} - n_e\sigma_T\left[\tilde{\Theta}_{P0} - \frac{1}{2}\Pi\right] \tag{2.23a}$$

$$\dot{\tilde{\Theta}}_{P\ell} = \frac{\ell k}{(2\ell+1)a}\tilde{\Theta}_{P\ell-1} - \frac{(\ell+1)k}{(2\ell+1)a}\tilde{\Theta}_{P\ell+1} - n_e\sigma_T\left[\tilde{\Theta}_{P\ell} - \frac{\delta_{\ell,2}}{10}\Pi\right] \qquad \ell \geq 1,. \tag{2.23b}$$

Equations (2.22) and (2.23) are not the full system of coupled equations, indeed in these equations depend on the potential $\tilde{\Psi}$ and $\tilde{\Phi}$ and on the electron bulk velocity $\tilde{v}_b$. The differential equations governing these quantities must then be added to the ones above and solved all toghther.

2.2.4 Polarization power spectrum

We already discussed that in order to completely describe photons (thus the CMB) we also need to account for polarization. It is therefore natural to define a power spectrum for the polarization, which can be done similarly as for the temperature.

We want to expand in spherical harmonics the Stokes parameters Q and U(2.18), however we showed, in section 2.2.1 that under a rotation the combination $Q \pm iU$ will transform as a spin 2 fields. This means that we cannot resort to the usual spherical harmonics decomposition, instead we must use **spin-weighted spherical harmonics** $Y^{\pm 2}_{\ell m}$. In this way we get

$$Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}}) = \sum_{\ell m} a^{\pm 2}_{\ell m} Y^{\pm 2}_{\ell m}(\hat{\mathbf{n}}).$$

It is then common to use modes that are projectable on the regular spherical harmonics: we start defining

$$a^E_{\ell m} \stackrel{\text{def}}{=} -\frac{a^2_{\ell m} + a^{-2}_{\ell m}}{2}, \qquad a^B_{\ell m} \stackrel{\text{def}}{=} \frac{a^2_{\ell m} - a^{-2}_{\ell m}}{2i},$$

then we can recompose these modes as

$$E(\hat{\mathbf{n}}) = \sum_{\ell m} a^E_{\ell m} Y_{\ell m}(\hat{\mathbf{n}}), \qquad B(\hat{\mathbf{n}}) = \sum_{\ell m} a^B_{\ell m} Y_{\ell m}(\hat{\mathbf{n}}). \tag{2.24}$$

The power spectra of the polarization can then be defined, as usual, as

$$\langle a^E_{\ell m} a^{E*}_{\ell' m'} \rangle \stackrel{\text{def}}{=} C^{EE}_\ell \delta_{\ell\ell'} \delta_{mm'}, \tag{2.25}$$

$$\langle a^B_{\ell m} a^{B*}_{\ell' m'} \rangle \stackrel{\text{def}}{=} C^{BB}_\ell \delta_{\ell\ell'} \delta_{mm'}, \tag{2.26}$$

$$\langle a_{\ell m} a^{E*}_{\ell' m'} \rangle \stackrel{\text{def}}{=} C^{TE}_\ell \delta_{\ell\ell'} \delta_{mm'}. \tag{2.27}$$

2.3 Tensor perturbations effects on the CMB

In the previous sections we focused on the effect of scalar perturbations on the anisotropies of the CMB. We will instead now study how tensor perturbations affects the evolution of these anisotropies.

In appendix B.3 we showed that, considering perturbed metric

$$ds^2 = -dt^2 + a^2(\delta_{ij} h_{ij}) dx^d x^j,$$

the *Liouville operator* reads as in (B.6)

$$\hat{\mathbf{L}}[f] = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - \frac{1}{2}\frac{\partial f}{\partial t}\dot{h}_{ij}\hat{p}^i\hat{p}^j,$$

where $p^i = p\, \hat{p}^i$ is the local 3-momentum of a photon and f the phase space distribution.

To obtain the equation describing the evolution of anisotropies we must expand the photon phase space distribution on a blackbody radiation background $\bar{f}(t, p) + \Upsilon(x^\mu, \mathbf{p})$, assuming that the temperature is perturbed as $T(x^\mu, \hat{\mathbf{p}}) = \bar{T}(1+\Theta(x^\mu, \hat{\mathbf{p}}))$.

In section 2.2 we showed that, in this way, the first order contribution to the phase space distribution reads $\Upsilon = -\Theta p \frac{\partial \bar{f}}{\partial p}$. This expansion allows to obtain a Liouville operator contribution at first order corresponding to

$$\hat{\mathbf{L}}[\Upsilon] = -p\frac{\partial \bar{f}}{\partial p}\left[\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{1}{2}\dot{h}_{ij}\hat{p}^i\hat{p}^j\right].$$

At this point we need to add the first order collision term associated to Compton scattering. For now let's consider the simplified form (2.15). Using Boltzmann equation and canceling out the common factor $-p\frac{\partial \bar{f}}{\partial t}$ from both sides, as in section 2.2, we get the differential equation that describes the time evolution of the CMB anisotropies in presence of tensor perturbations

$$\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{1}{2}\dot{h}_{ij}\hat{p}^i\hat{p}^j = n_e\sigma_T\left[\Theta_0 - \Theta + \hat{\mathbf{p}}\cdot\mathbf{v}_b\right]. \tag{2.28}$$

2.3.1 Coupling of tensors to anisotropies

Previously, to solve equation (2.16), Fourier transformed $\Theta(t, \mathbf{x}, \hat{\mathbf{p}})$ and then expanded in Legendre polynomial. However, this last step was justified, in section 2.2.3, by noting that the equations depended only on the cosine of angle between the direction of motion of the photon $\hat{\mathbf{p}}$ and the wave number vector of the Fourier transform $\mathbf{k}$. This corresponded to a rotational symmetry around one of the above vectors that implied that Legendre polynomials were the appropriate basis to expand on. We shall now study equation (2.28), and its symmetries, to understand what will now be the right basis to use for this expansion.

To begin, let's recall that, being traceless transverse, in Fourier space, h_{ij} can be separated in two independent polarizations

$$0 = \partial^i h_{ij} = \int \frac{d^3k}{(2\pi)^3} e^{\mathbf{k}\cdot\mathbf{x}}\tilde{h}_{ij}k^i \xrightarrow[\text{Symmetric}]{\text{Traceless}} \tilde{h}_{ij} = \tilde{h}_\times \mathbf{e}^\times_{ij} + \tilde{h}_+\mathbf{e}^+_{ij} = \begin{pmatrix} \tilde{h}_\times & \tilde{h}_+ & 0 \\ \tilde{h}_+ & -\tilde{h}_\times & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The only term in (2.28) that can induce more complicated angular dependence is $\dot{h}_{ij}\hat{p}^i\hat{p}^j$: considering spherical polar coordinates (r, θ, ϕ) with $\mathbf{k} \parallel \hat{\mathbf{z}}$, once Fourier transformed, this term will be proportional to

$$\hat{\mathbf{p}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \quad \Rightarrow \quad (\mathbf{e}^\times_{ij} + \mathbf{e}^+_{ij})\hat{p}^i\hat{p}^j = \sin^2\theta(\cos 2\phi + \sin 2\phi).$$

This clearly shows that anisotropies coupled to tensor perturbations can no longer be decomposed on Legendre polynomials, since the azimuthal symmetry is now spoiled by the explicit dependence on ϕ.

To individuate the appropriate basis for the spherical harmonics expansion, let's use the basis introduced by Hu and White in [3]

$$h_{ij} = -\sqrt{\frac{3}{2}}(h^{(+)}\mathbf{e}^{(+)}_{ij} + h^{(-)}\mathbf{e}^{(-)}_{ij}) \quad \text{with } \mathbf{e}^{(+)} = \begin{pmatrix} 1 & +i & 0 \\ +i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ \mathbf{e}^{(-)} = \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.29}$$

Comparing this definition and the previous basis, we can easily find the transformation between the two polarizations

$$h_+ = -\sqrt{\frac{3}{2}}(h^{(+)} + h^{(-)}), \qquad h^{(+)} = \frac{1}{\sqrt{6}}(h_+ - ih_-),$$
$$h_+ = -i\sqrt{\frac{3}{2}}(h^{(+)} - h^{(-)}), \qquad h^{(-)} = -\frac{1}{\sqrt{6}}(h_+ + ih_-).$$

Let's now project the versor $\hat{\mathbf{p}}$, defined as above, onto $\mathbf{e}^{(\pm)}$,

$$\mathbf{e}^{(\pm)}_{ij}\hat{p}^i\hat{p}^j = \sin^2\theta[\cos^2\phi - \sin^2\phi \pm 2i\sin\phi\cos\phi] = \sin^2\theta e^{\pm i2\phi},$$

immediately we should recognize that this term is proportional to the spherical harmonics $Y_{2,\pm 2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{\pm i2\phi}$.
This shows that the appropriate basis for the expansion of the anisotropies are the spherical harmonics $Y_{\ell m}$ with $m = \pm 2$, since they all posses the same azimuthal symmetry $Y_{\ell m} \propto e^{\pm i2\phi}$ as the tensor term in (2.28).

Following Hu and White [3] convention, we will use the following multipole expansion for the anisotropies

$$\Theta(t,\mathbf{x},\hat{\mathbf{p}}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{\ell m} (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}(\hat{\mathbf{p}}) \tilde{\Theta}^{(m)}_\ell(t,\mathbf{k}), \tag{2.30}$$

in which we will only use the multipoles with $m = \pm 2$, for the reasons we just explained.
Note that for $m = 0$, since $Y_{\ell 0} = \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta)$, we recover the Legendre polynomials expansion used for scalar perturbations. In this case the sum over the $2\ell+1$ different values of m will result in the correspondence $\tilde{\Theta}^{(0)}_\ell = (2\ell+1)\tilde{\Theta}_\ell$.
Lastly, we shall note that the normalization of the new polarization (2.29) is such that

$$\begin{aligned}\frac{1}{2}\dot{\tilde{h}}_{ij}\hat{p}^i\hat{p}^j &= \frac{1}{2}\left(-\sqrt{\frac{3}{2}}\right)\left[\dot{\tilde{h}}^{(+)}\sin^2\theta e^{i2\phi} + \dot{\tilde{h}}^{(-)}\sin^2\theta e^{-i2\phi}\right] \\ &= -\sqrt{\frac{4\pi}{5}}\left[\dot{\tilde{h}}^{(+)}Y_{2,2}(\hat{\mathbf{p}}) + \dot{\tilde{h}}^{(-)}Y_{2,-2}(\hat{\mathbf{p}})\right],\end{aligned}$$

where the factor $-\sqrt{\frac{4\pi}{5}}$ precisely corresponds to $i^\ell\sqrt{\frac{4\pi}{2\ell+1}}|_{\ell=2}$, which is also the factor that get all the others $\tilde{\Theta}^{(2)}_2$ in the expansion. In this way the contribution of the tensor perturbation to the multipole expansion of equation (2.28), with $\ell = 2$ and $m = \pm 2$, will be exactly $\dot{\tilde{h}}^{(\pm)}$.

2.3.2 Multipole expansion of tensor induced anisotropies

Knowing that tensor perturbations are projected in the sky onto spherical harmonics with $m = \pm 2$ while scalar perturbations are projected onto spherical harmonics with $m = 0$, by their orthogonality, we can expand Θ in multipoles using only $Y_{\ell,\pm 2}$. In this way we effectively decoupled the multipoles generated by tensor perturbations and the scalar ones.

First, we move to Fourier space where the equation describing anisotropies (2.28) reads

$$\frac{\partial\tilde{\Theta}}{\partial t}+i\frac{k\cos\theta}{a}\tilde{\Theta}-\sqrt{\frac{4\pi}{5}}\left[\dot{\tilde{h}}^{(+)}Y_{2,2}(\hat{\mathbf{p}})+\dot{\tilde{h}}^{(-)}Y_{2,-2}(\hat{\mathbf{p}})\right]=n_e\sigma_T\left[\tilde{\Theta}_0-\tilde{\Theta}+k\tilde{v}_b\cos\theta\right],$$

in which we assumed that $\mathbf{v}_b$ is irrotational.
Now, the expansion in spherical harmonics (2.30) yields

$$\tilde{\Theta}_\ell^{(m)}(\mathbf{k})=i^\ell\sqrt{\frac{2\ell+1}{4\pi}}\int d\Omega_{\hat{\mathbf{p}}}\tilde{\Theta}(\mathbf{x},\hat{\mathbf{p}})Y^*_{\ell,m}(\theta,\phi),$$

therefore, upon integration of the equation above we can decompose it in a set of ordinary differential equations, one for each multipole $\Theta_\ell^{(m)}$.
However, similarly to what happened in section 2.2.3, we obtain from the second term on the left hand side a contribution proportional to $Y^*_{\ell m}(\theta,\phi)\cos\theta$. The proprieties of spherical harmonics allows to simplify this term as follows

$$\cos\theta Y_{\ell m}(\theta,\phi)=\sqrt{\frac{4\pi}{3}}Y_{10}Y_{\ell m}=\sqrt{\frac{\ell^2-m^2}{2\ell-1}}Y_{\ell-1,m}+\sqrt{\frac{\ell^2-m^2}{2\ell+3}}Y_{\ell+1,m}$$

and upon integration we therefore also get contributions from other multipoles.
In this way equation (2.28) will be decomposed into

$$\dot{\tilde{\Theta}}_\ell^{(\pm2)}=\frac{k}{a}\left[\frac{\sqrt{\ell^2-m^2}}{2\ell-1}\Theta_{\ell-1}^{(\pm2)}-\frac{\sqrt{\ell^2-m^2}}{2\ell+3}\Theta_{\ell+1}^{(\pm2)}\right]-n_e\sigma_T\tilde{\Theta}_\ell^{(\pm2)}-\dot{\tilde{h}}^{(\pm)}\quad \ell\geq 2, \tag{2.31}$$

where the contributions from $\tilde{\Theta}_0$ and $\tilde{v}_b$ are not appearing since they are multiplied by $m=0$ spherical harmonics.

As we discussed in section 2.2.2 the Compton scattering is influenced by the polarization of incoming and outgoing photons. For this reason we must add corrections to the above equations. To add the proper corrections we need also to describe the dynamics of the polarization.
Expanding the polarization in spherical harmonics as

$$(\Theta_Q\pm i\Theta_U)(\mathbf{x},\hat{\mathbf{p}})=\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}\sum_{\ell m}(-i)^\ell\sqrt{\frac{4\pi}{2\ell+1}}Y^{\pm2}_{\ell m}(\hat{\mathbf{p}})(\tilde{E}_\ell^{(m)}\pm i\tilde{B}_\ell^{(m)}), \tag{2.32}$$

where $Y^{\pm2}_{\ell m}$ are again the spin weighted spherical harmonics. Again, arguing that the free streaming of photons is not influenced by the polarization while polarization is sourced by the quadrupole of the anisotropies, we can write

$$\dot{\tilde{E}}_\ell^{(\pm2)}=\frac{k}{a}\left[\frac{\ell^2-2}{\ell^2(2\ell-1)}\tilde{E}_{\ell-1}^{(\pm2)}-\frac{2m}{\ell(\ell+1)}\tilde{B}_\ell^{(\pm2)}-\frac{\ell^2-2}{\ell^2(2\ell+3)}\tilde{E}_{\ell+1}^{(\pm2)}\right]+ \\ -n_e\sigma_T\left[\tilde{E}_\ell^{(\pm2)}+\sqrt{6}\Pi^{(\pm2)}\delta_{\ell,2}\right], \tag{2.33a}$$

$$\dot{\tilde{B}}_\ell^{(\pm2)}=\frac{k}{a}\left[\frac{\ell^2-2}{\ell^2(2\ell-1)}\tilde{B}_{\ell-1}^{(\pm2)}-\frac{2m}{\ell(\ell+1)}\tilde{E}_\ell^{(\pm2)}+\frac{\ell^2-2}{\ell^2(2\ell+3)}\tilde{B}_{\ell+1}^{(\pm2)}\right]+ \\ -n_e\sigma_T\tilde{B}_\ell^{(\pm2)}, \tag{2.33b}$$

$$\text{with}\quad \Pi^{(\pm2)}=\frac{1}{10}\left[\tilde{\Theta}_2^{(\pm2)}-\sqrt{6}\tilde{E}_2^{(\pm2)}\right].$$

These are the equations governing the time evolution of the polarization in the plasma of photons considering only the multipoles coupled to the tensor perturbations.

In a similar way, also the equations for the anisotropies must be corrected, giving

$$\dot{\tilde{\Theta}}_\ell^{(\pm 2)} = \frac{k}{a}\left[\frac{\sqrt{\ell^2 - m^2}}{2\ell - 1}\Theta_{\ell-1}^{(\pm 2)} - \frac{\sqrt{\ell^2 - m^2}}{2\ell + 3}\Theta_{\ell+1}^{(\pm 2)}\right] + \\ + n_e\sigma_T\left[\Pi^{(\pm 2)}\delta_{\ell,2} - \tilde{\Theta}_\ell^{(\pm 2)}\right] - \dot{\tilde{h}}^{(\pm)} \quad \ell \geq 2. \qquad (2.34)$$

A full detailed derivation of these equations can be found in [3].

Let's stop for a second to appreciate that the equations (2.34), (2.33a) and (2.33b) do not mix the ± 2 modes. Furthermore, for both values of m the equations read the same, this means that from now on we can study only the $m = 2$ modes and then use the results to obtain the $m = -2$ ones.

2.4 Approximate solutions for the dynamics of the anisotropies

All the differential equations we derived are pretty hare to solve analytically since they are strongly coupled and in theory they are an infinite number (as much as the number of multipoles). Usually we resort to numerical methods to obtain exact results, however some approximations can be useful to understand the general behavior of the CMB or even to simplify some numerical calculations.

2.4.1 The tight coupling approximation

At early times, when the plasma was denser and hotter, the mean free path of photons was very small and the rate of Compton scattering was very high. We will show that in this regime only the first two multipoles are relevant to describe fully the plasma. This is somewhat similar to a fluid that can be fully described by its density and velocity field.

The guiding idea behind the tight coupling approximations is that the scatterings between baryons and photons, in this limit, is the only relevant interaction that determines the dynamics of the anisotropies. This is equivalent to consider the limit in which $n_e\sigma_T \gg 1$, which means that the mean free path ($\propto \frac{1}{n_e\sigma_T}$) is very small. Starting from scalar perturbations, in equation (2.22c) we can drop the time derivative, since it is negligible with respect to the terms multiplied by $n_e\sigma_T$. In this way we are left with

$$\frac{\ell k}{(2\ell+1)a}\tilde{\Theta}_{\ell-1} - n_e\sigma_T\left[\tilde{\Theta}_\ell - \frac{\delta_{\ell,2}}{10}\Pi\right] = -\frac{(\ell+1)k}{(2\ell+1)a}\tilde{\Theta}_{\ell+1},$$

from which we can note that the term $\Theta_{\ell+1}$ is small compared to $\Theta_{\ell-1}$. This essentially prove that only the first few moments are relevant while higher multipoles are always smaller and smaller as ℓ increases. In this limit we are left with two only differential equations, (2.22a) and (2.22b) (still coupled to the rest of the plasma), to be solved.

Similar considerations are valid for the polarization equations (2.23a) and (2.23b), which can be simplified in the same way.

Also for tensor perturbations the tight coupling limit significantly simplifies the equations of motion. Reasoning as we have just done, from equations (2.34), (2.33a) and (2.33b) we conclude that only the multipoles with $\ell = 2$ are relevant. In this way, after ahving dropped the time derivatives, we are left with

$$n_e\sigma_T\left[\frac{1}{10}\Pi^{(\pm 2)} - \tilde{\Theta}_2^{(\pm 2)}\right] \approx \dot{\tilde{h}}^{(\pm 2)}, \quad \tilde{E}_2^{(\pm 2)} \approx -\sqrt{6}\Pi^{(\pm 2)}, \quad \tilde{B}_2^{(\pm 2)} \approx 0,$$

that using the definition of $\Pi^{(\pm 2)} = \frac{1}{10}[\tilde{\Theta}_2^{(\pm 2)} - \sqrt{6}\tilde{E}_2^{(\pm 2)}]$ gives

$$\tilde{\Theta}_2^{(\pm 2)} \approx -\frac{4\dot{\tilde{h}}^{(\pm 2)}}{3n_e\sigma_T}, \qquad \tilde{E}_2^{(\pm 2)} \approx -\frac{\sqrt{6}}{4}\tilde{\Theta}_2^{(\pm 2)}, \qquad \tilde{B}_2^{(\pm 2)} \approx 0. \tag{2.35}$$

This approximation will be particularly useful in the next chapter to study the spectral distortions associated to the dissipation of gravitational waves.

2.4.2 Improved tight coupling approximation

To conclude we want to present a simple way to improve the tight coupling approximation relaxing the approximation of stationary solutions, without spoiling the reduced number of multipoles excited.
For the purpose of this work, we are going to focus primarily on tensor perturbations as illustrated in Consider the equations (2.34), (2.33a) and (2.33b) with conformal time and considering only the quadrupole we have

$$\begin{aligned}
\partial_\tau\tilde{\Theta}_2^{(\pm 2)} &= n_e\sigma_T a\left[\frac{9}{10}\tilde{\Theta}_2^{(\pm 2)} + \frac{\sqrt{6}}{10}\tilde{E}_2^{(\pm 2)}\right] - \partial_\tau\tilde{h}^{(\pm)},\\
\partial_\tau\tilde{E}_2^{(\pm 2)} &= n_e\sigma_T a\left[\frac{2}{5}\tilde{E}_2^{(\pm 2)} + \frac{\sqrt{6}}{10}\tilde{\Theta}_2^{(\pm 2)}\right] - k\frac{2}{3}\tilde{B}_2^{\pm 2},\\
\partial_\tau\tilde{B}_2^{(\pm 2)} &= n_e\sigma_T a\tilde{B}_2^{(\pm 2)} + k\frac{2}{3}\tilde{E}_2^{\pm 2}.
\end{aligned}$$

To solve these equations we should proceed with an ansatz: assume that the solution has the form $\tilde{\Theta}_2^{(2)} = A_\Theta e^{ik\tau}$, $\tilde{E}_2^{(2)} = A_E e^{ik\tau}$ and $\tilde{B}_2^{(2)} = A_B e^{ik\tau}$ and that the gravitational perturbation is $\tilde{h}^{(\pm)} = A_h e^{ik\tau}$, where we dropped the $\pm$ since the equations are the same for both cases.
In this way the above system of differential equations reduces to a system of linear equations for the coefficients

$$\begin{aligned}
ikA_\Theta &= n_e\sigma_T a\left[\frac{9}{10}A_\Theta + \frac{\sqrt{6}}{10}A_E\right] - ikA_h,\\
ikA_E &= n_e\sigma_T a\left[\frac{2}{5}A_E + \frac{\sqrt{6}}{10}A_\Theta\right] - k\frac{2}{3}A_B,\\
ikA_B &= n_e\sigma_T a A_B + k\frac{2}{3}A_E.
\end{aligned}$$

Once solved this system we find

$$\begin{aligned}
\frac{|A_\Theta|}{\frac{4}{3}\frac{|A_h|}{n_e\sigma_T a}} &= \sqrt{\frac{1+\frac{341}{36}\xi^2+\frac{625}{324}\xi^4}{1+\frac{142}{9}\xi^2+\frac{1649}{82}\xi^4+\frac{2500}{729}\xi^6}}, \\
\tan\phi_\Theta &= -\frac{11}{6}\xi\frac{1+\frac{697}{99}\xi^2+\frac{1250}{891}\xi^4}{1+\frac{197}{18}\xi^2+\frac{125}{54}\xi^4}, && \text{(2.36a)} \\
\frac{|A_E|}{\frac{4}{3}\frac{|A_h|}{n_e\sigma_T a}} &= \frac{\sqrt{6}}{4}\sqrt{\frac{1+\xi^2}{1+\frac{142}{9}\xi^2+\frac{1649}{82}\xi^4+\frac{2500}{729}\xi^6}}, \\
\tan(\phi_E-\pi) &= -\frac{13}{3}\xi\frac{1+\frac{121}{117}\xi^2}{1-\xi^2-\frac{50}{27}\xi^4}, && \text{(2.36b)} \\
\frac{|A_B|}{\frac{4}{3}\frac{|A_h|}{n_e\sigma_T a}} &= \frac{\xi}{\sqrt{6}}\sqrt{\frac{1}{1+\frac{142}{9}\xi^2+\frac{1649}{82}\xi^4+\frac{2500}{729}\xi^6}} \\
\tan(\phi_B-\pi) &= -\frac{16}{3}\xi\frac{1+\frac{121}{117}\xi^2}{1-\frac{19}{3}\xi^2}. && \text{(2.36c)}
\end{aligned}$$

In the above $\xi \stackrel{\text{def}}{=} \frac{k}{n_e\sigma_T a}$, note that in the tight coupling limit $\xi \ll 1$ and the equations above reduce to the previous discussed approximations with $\phi_E = \phi_B = \phi_\Theta = 0$. This approximation also shows that as we exit the thigh coupling regimes and photons start to free stream ($\xi \gg 1$) the anisotropies start to decay as ξ^{-1}, while the polarization decays even faster as ξ^{-2}.

Chapter 3

Spectral Distortions

As the *CMB* is composed of photons, it is reasonable to that it should be described as some black body radiation. Indeed, the measurements from COBE/FIRAS satellite [2] showed that the CMB spectrum is very compatible with a black body radiation. However, many physical processes could affect the spectrum of the CMB generating some **spectral distortions** that were not detectable by the previous experiments.

In the next sections we will discuss the theory of spectral distortions and the physical processes that could generate them. Our goal is to describe a particular process called **Silk dumping** that allows for dissipation of the primordial perturbation to generate the distortions

3.1 Production of spectral distortion

3.2 Silk dumping

Primordial perturbations, when reentering the Hubble horizon after inflation, they can generate small fluctuation in the temperature of the plasma. In this way, photons from the hotter and denser regions will diffuse in the blackbody radiation of the plasma and produce distortions in the *CMB spectrum*. This effect is called *Silk damping* and it mainly consists of two processes: **mixing of blackbodies** and **comptonization**.
We will now show that the first one produces *y-distortions* while the latter will turn these distortions in *μ-distortions*.

3.2.1 Mixing of blackbodies

To start we recall the main formulae that describe a blackbody radiation: given the phase space distribution of a black body

$$f(\nu, T) = \left(\exp\left\{\frac{\nu}{k_B T}\right\} - 1\right)^{-1} = \left(e^x - 1\right)^{-1}, \qquad \text{with } x = \frac{\nu}{k_B T},$$

we can evaluate the energy, the entropy and the number of photons in the radiation

$$E = a_R T^4, \qquad S = \frac{4}{3} a_R T^3, \qquad N = b_R T^3.$$

Appendix A

Differential geometry tools

A.1 Maximally symmetric spaces

Consider $\mathbb{R}^n$, this space is highly symmetric: it is isotropic and homogeneous, or, in a simpler way, every point and every direction "look" the same.
This means that $\mathbb{R}^n$ is symmetric under every rotation and translation: in n-dimensions there are n possible translations (along the n axes) and $n\frac{n-1}{2}$ possible rotations (for each axis we can rotate it towards $n-1$ other axes and to avoid double counting $x \to y$ and $y \to x$ we divide by 2), for a total number of symmetries equals to

$$n + n\frac{n-1}{2} = n\frac{n+1}{2}.$$

An n-dimensional manifold is said to be **maximally symmetric** if it possesses the same number of symmetries of $\mathbb{R}^n$. In the differential geometry language, a symmetry is defined through isometries, that are diffeomorphisms under which the metric tensor is invariant.
For each symmetry of the metric we can define a **Killing vector**, which satisfies the Killing equation

$$0 = (\pounds_{\vec{K}} g)_{\mu\nu} = \nabla_\mu K_\nu + \nabla_\nu K_\mu, \tag{A.1}$$

where $\pounds_{\vec{K}}$ is the Lie derivative along $\vec{K}$, which is the Killing vector.

We now want to show that a maximally symmetric space really possesses the maximum number of symmetries, namely the maximum number of independent[1] Killing vectors. Consider the defining equation of the Riemann tensor applied to a 1-form

$$R^\mu_{\nu\rho\sigma} K_\mu = -[\nabla_\rho, \nabla_\sigma] K_\nu, \tag{A.2}$$

this definition, combined with the algebraic Bianchi identity, ($R^\mu_{\nu\rho\sigma} + R^\mu_{\rho\sigma\nu} + R^\mu_{\sigma\nu\rho} = 0$) implies that each Killing vector must satisfy

$$\nabla_\rho \nabla_\sigma K_\nu - \nabla_\sigma \nabla_\rho K_\nu + \nabla_\sigma \nabla_\nu K_\rho - \nabla_\nu \nabla_\sigma K_\rho + \nabla_\nu \nabla_\rho K_\sigma - \nabla_\rho \nabla_\nu K_\sigma = 0.$$

[1] Linearly independent here means that $\nexists$ a set of constants c_n such that

$$\sum_n c_n K^{(n)}_\mu (P) = 0 \qquad \forall P \in \mathcal{M}.$$

This equation can be simplified by the Killing equation (A.1): using this relation we can sum pairs of terms obtaining

$$2(\nabla_\rho\nabla_\sigma K_\nu - \nabla_\sigma\nabla_\rho K_\nu - \nabla_\nu\nabla_\sigma K_\rho) = 0,$$

that using (A.2) turns out to be the following

$$R^\mu_{\nu\rho\sigma}K_\mu = \nabla_\nu\nabla_\sigma K_\rho. \tag{A.3}$$

This equation shows that the second covariant derivative acts on Killing vectors just as a linear application. In this way we can determine every derivative of a Killing vector in a specific point, just by knowing its value and the value of its first covariant derivative, at the same point.
If we now Taylor expand the Killing vector around a point P, we will obtain some kind of expansion that depends on the value in P of all covariant derivatives of all orders, however we showed that we can evaluate those just knowing $K_\mu(P)$ and $\nabla_\nu K_\mu(P)$. This means that we can express the Killing vector field as a combination of two functions that do not depend on the Killing vector itself or on its derivatives:

$$K_\mu(x) = A_\mu{}^\lambda(x,P)K_\lambda(P) + B_\mu{}^{\lambda\nu}(x,P)\nabla_\nu K_\lambda(P),$$

these functions depend only on x, the point P, and the metric, through the Riemann tensor. For this reason these must be the same functions for all Killing vectors:

$$K^{(n)}_\mu(x) = A_\mu{}^\lambda(x,P)K^{(n)}_\lambda(P) + B_\mu{}^{\lambda\nu}(x,P)\nabla_\nu K^{(n)}_\lambda(P). \tag{A.4}$$

The above equation tells us that a given Killing vector is determined by $K^{(n)}_\lambda(P)$, which has N possible independent values, and by $\nabla_\nu K^{(n)}_\lambda(P)$, which has $N\frac{N-1}{2}$ independent values, due to its antisymmetry (which is a consequence of the Killing equation (A.1)).
In this way we have shown that the maximum number of independent Killing vectors in an N-dimensional manifold is exactly the same number that possesses $\mathbb{R}^N$

$$N + N\frac{N-1}{2} = N\frac{N+1}{2}.$$

We want to conclude deriving the form that has the Riemann tensor in a maximally symmetric space.
In general, equation (A.3) must hold for every Killing vector, furthermore it also must be consistent with the commutator of covariant derivatives (A.2). This requirement and the fact that we have the maximum number of linearly independent Killing vectors will determine the form of $R^\mu_{\nu\rho\sigma}$. Consider (A.2) applied to the two indices tensor

$$[\nabla_\sigma,\nabla_\nu]\nabla_\mu K_\rho = -R^\lambda_{\mu\sigma\nu}\nabla_\lambda K_\rho - R^\lambda_{\rho\sigma\nu}\nabla_\mu K_\lambda,$$

the equation (A.3) can be used to obtain

$$\begin{aligned}&\nabla_\sigma(R^\lambda_{\nu\rho\mu}K_\lambda) - \nabla_\nu(R^\lambda_{\sigma\rho\mu}K_\lambda) = \\ &\quad = \nabla_\sigma R^\lambda_{\nu\rho\mu}K_\lambda - \nabla_\nu R^\lambda_{\sigma\rho\mu}K_\lambda + R^\lambda_{\nu\rho\mu}\nabla_\sigma K_\lambda - R^\lambda_{\sigma\rho\mu}\nabla_\nu K_\lambda = -R^\lambda_{\mu\sigma\nu}\nabla_\lambda K_\rho - R^\lambda_{\rho\sigma\nu}\nabla_\mu K_\lambda.\end{aligned}$$

Now, Killing equation (A.1) allows us to move the index λ to the covariant derivative in each term, then, using a bunch of Kronecker deltas we get

$$(\nabla_\sigma R^\lambda_{\nu\rho\mu} - \nabla_\nu R^\lambda_{\sigma\rho\mu})K_\lambda = (R^\lambda_{\nu\rho\mu}\delta_\sigma{}^\alpha - R^\lambda_{\sigma\rho\mu}\delta_\nu{}^\alpha + R^\lambda_{\mu\sigma\nu}\delta_\rho{}^\alpha - R^\lambda_{\rho\sigma\nu}\delta_\mu{}^\alpha)\nabla_\lambda K_\alpha.$$

This relation must hold for every Killing vector. We have the maximum number of independent Killing vectors, thus we can generate any other Killing vector from a combination of these. The general expansion (A.4) shows that a Killing vector field that vanishes in P, while its derivatives does not, can exists, and we surely can obtain it from a linear combination of the others. The above equation holds also for this one in P only if the right-hand side vanishes too, this can happen only if the term in parentheses is symmetric in λ α (so that it vanishes when contracted with $\nabla_\lambda K_\alpha$ that is antisymmetric)

$$R^\lambda_{\nu\rho\mu}\delta_\sigma{}^\alpha - R^\lambda_{\sigma\rho\mu}\delta_\nu{}^\alpha + R^\lambda_{\mu\sigma\nu}\delta_\rho{}^\alpha - R^\lambda_{\rho\sigma\nu}\delta_\mu{}^\alpha = R^\alpha_{\nu\rho\mu}\delta_\sigma{}^\lambda - R^\alpha_{\sigma\rho\mu}\delta_\nu{}^\lambda + R^\alpha_{\mu\sigma\nu}\delta_\rho{}^\lambda - R^\alpha_{\rho\sigma\nu}\delta_\mu{}^\lambda.$$

Contracting μ and α, recalling that $R^\mu_{\nu\mu\rho} = R_{\nu\rho}$ and $R^\mu_{\mu\nu\rho} = 0$, we find

$$R^\lambda_{\nu\rho\sigma} - R^\lambda_{\sigma\rho\nu} + R^\lambda_{\rho\sigma\nu} - NR^\lambda_{\rho\sigma\nu} = -R_{\nu\rho}\delta_\sigma{}^\lambda + R_{\sigma\rho}\delta_\nu{}^\lambda - R^\lambda_{\rho\sigma\nu},$$

here we recognize that, from the algebraic Bianchi identity,

$$R^\lambda_{\sigma\rho\nu} = -R^\lambda_{\sigma\nu\rho} = R^\lambda_{\nu\rho\sigma} + R^\lambda_{\rho\sigma\nu},$$

which cancels two terms in the previous equation, that now reads, after having lowered one index,

$$(N-1)R_{\lambda\rho\sigma\nu} = R_{\nu\rho}g_{\sigma\lambda} - R_{\sigma\rho}g_{\nu\lambda}. \tag{A.5}$$

Notice that the above equation must be antisymmetric in λ ρ (due to the proprieties of the Riemann tensor),

$$R_{\nu\rho}g_{\sigma\lambda} - R_{\sigma\rho}g_{\nu\lambda} = -R_{\nu\lambda}g_{\sigma\rho} + R_{\sigma\lambda}g_{\nu\rho},$$

contracting λ ν, this relation becomes

$$R_{\sigma\rho} - NR_{\sigma\rho} = -Rg_{\sigma\rho} + R_{\sigma\rho}, \quad \Rightarrow \quad \boxed{R_{\sigma\rho} = \frac{R}{N}g_{\sigma\rho}}, \tag{A.6}$$

inserting this one into the (A.5) we get our final result

$$\boxed{R_{\lambda\rho\sigma\nu} = \frac{R}{N(N-1)}(g_{\nu\rho}g_{\lambda\sigma} - g_{\sigma\rho}g_{\lambda\nu})}. \tag{A.7}$$

Appendix B

Thermodynamics tools

B.1 Scalar perturbed Liouville operator

In this appendix we will show how to obtain the perturbed Liouville operator, at first order, for the photon phase space in the presence of scalar perturbations. Since we are interested only in first order perturbations, in the next calculations, we will always neglect higher order contributions by Taylor expanding every function of the perturbations. We will work with the perturbed metric in the conformal Newtonian gauge, which is given by

$$ds^2 = a^2(\eta)\left[-(1+2\Psi)d\tau^2 + (1-2\Phi)\delta_{ij}dx^i dx^j\right].$$

The Liouville operator, defined in , reads

$$\hat{\mathbf{L}}[f] = \frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \frac{\partial f}{\partial x^i}\frac{dx^i}{d\tau} + \frac{\partial f}{\partial p}\frac{dp}{d\tau} + \frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{d\tau},$$

where $p^i = p\ \hat{p}^i$ (with $\hat{p}^i\hat{p}^j\delta_{ij} = 1$) is the local 3-momentum of the photon, and we already considered that the local energy and the 3-momentum are not independent due to the mass-shell condition. We also assume that f can also be expanded on a background, which corresponds to the black body radiation, plus a first order perturbation (see section 2.2 for more on this expansion). Note that, since the blackbody radiation is isotropic and homogeneous (it does not depend on x^i or $\hat{p}^i$) the two factors $\frac{\partial f}{\partial x^i}$ and $\frac{\partial f}{\partial \hat{p}^i}$ are only first order contributions. This observation will simplify our calculations later on since it implies that $\frac{dx^i}{d\tau}$ and $\frac{d\hat{p}^i}{d\tau}$ are needed only at order zero.

Let's spend some time discussing local energy and momentum. The local energy is defined as the energy of a photon in the local rest frame of an observer, thus for a static observer ($U^\mu = (\frac{1-\Psi}{a}, 0, 0, 0)$) it reads

$$E = -U_\mu P^\mu = a(1+2\Psi)P^0(1-\Psi) \approx aP^0(1+\Psi).$$

The local momentum defined in the same way, therefore it must satisfy the usual Minkowskian mass-shell relation $E = \sqrt{p^i p^j \delta_{ij}}$. Using the mass-shell condition for the 4-momentum of the photon $P^\mu P_\mu = 0$, we can write

$$
\begin{aligned}
&P^\mu P_\mu = -a^2(1+2\Psi)(P^0)^2 + a^2(1+2\Phi)P^iP^j\delta_{ij} = 0,\\
&\Rightarrow \quad P^0 = \sqrt{\frac{1+2\Phi}{1+2\Psi}P^iP^j\delta_{ij}} \approx (1+\Phi-\Psi)\sqrt{P^iP^j\delta_{ij}},\\
&E = \sqrt{p^ip^j\delta_{ij}} = aP^0(1+\Psi) = a(1+\Phi)\sqrt{P^iP^j\delta_{ij}}.
\end{aligned}
$$

In this way we identify $p^i = a(1+\Phi)P^i$ as the local 3-momentum. Note that it follows from $E = \sqrt{p^ip^j\delta_{ij}}$ that decomposing $p^i = p\,\hat{p}^i$ then $p = E$, as we expect in the local reference frame.

We are now ready to determine all the contribution to the Liouville operator. First, from the above discussion on the local energy and momentum, we recognize that

$$
\frac{dx^i}{d\tau} = \frac{P^i}{P^0} = \frac{(1-\Phi)p^i}{(1+\Psi)E} \approx \hat{p}^i(1-\Psi-\Phi). \tag{B.1}
$$

Then we have to evaluate

$$
\frac{dp}{d\tau} = \frac{d}{d\tau}aP^0(1+\Psi) = \mathcal{H}p + a(1+\Psi)\frac{dP^0}{d\tau} + p\Psi',
$$

therefore we need to compute $\frac{dP^0}{d\tau}$. This can be accomplished by using the geodesic equation

$$
\frac{dP^0}{d\tau} = \frac{dP^0}{d\lambda}\frac{1}{P^0} = -\frac{\Gamma^0_{\mu\nu}}{P^0}P^\mu P^\nu,
$$

in the conformal Newtonian gauge the relevant Christoffel symbols are

$$
\Gamma^0_{00} = \mathcal{H} + \Psi', \quad \Gamma^0_{0i} = \Psi_{,i}\,, \quad \Gamma^0_{ij} = \Big[\mathcal{H}(1+2\Phi-2\Psi) + \Phi'\Big]\delta_{ij}.
$$

In this way we get

$$
\begin{aligned}
\frac{dp}{d\tau} &= (\mathcal{H}+\Psi')p - a(1+\Psi)\times\\
&\qquad\qquad \times\left[(\mathcal{H}+\Psi')P^0 + P^i\Psi_{,i} + \Big(\mathcal{H}(1+2\Phi-2\Psi)+\Phi'\Big)\frac{P^iP^j\delta_{ij}}{P^0}\right]\\
&\approx \mathcal{H}p - \mathcal{H}p + \Psi'p - \Psi'p - p^i\Psi_{,i} - \mathcal{H}p - \Phi'p\\
&= -\mathcal{H}p - \Phi'p - p^i\Psi_{,i}
\end{aligned} \tag{B.2}
$$

We now have to obtain

$$
\frac{d\hat{p}^i}{d\tau} = \frac{d}{d\tau}\frac{p^i}{p} = \frac{dp^i}{d\tau}\frac{1}{p} - \frac{dp}{d\tau}\frac{p^i}{p^2},
$$

in which we can get $\frac{dp^i}{d\tau}$ by (B.2) noting

$$
\frac{dp}{d\tau} = \frac{d}{d\tau}\sqrt{p^ip^j\delta_{ij}} = \frac{p^j}{p}\frac{dp^i}{d\tau}\delta_{ij}.
$$

These two simple calculations, with equation (B.2), show that $\frac{d\hat{p}^i}{d\tau}$ has no zeroth order contributions, but only first order ones. Therefore, when multiplied by $\frac{\partial f}{\partial \hat{p}^i}$ a second order, in perturbations, term is generated, and for this reason we will neglect its contributions.

Inserting equations (B.1) and (B.2) into the Liouville operator we end up with

$$\hat{\mathbf{L}}[f] = \frac{\partial f}{\partial \tau} + \hat{p}^i \frac{\partial f}{\partial x^i} - p\left(\mathcal{H} - \frac{\partial \Phi}{\partial \tau} + \frac{\partial \Phi}{\partial x^i}\hat{p}^i\right)\frac{\partial f}{\partial p},$$

$$\Big\downarrow \quad \text{moving to cosmic time } dt = a\ d\tau,$$

$$\hat{\mathbf{L}}[f] = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\left(H - \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x^i}\frac{\hat{p}^i}{a}\right)\frac{\partial f}{\partial p} \tag{B.3}$$

B.2 Collision term

B.3 Tensor perturbed Liouville operator

Previously, we considered how scalar perturbations of the metric (which are in general the most studied) contributes to the evolution of the phase space associated to photons. Now, we are going to follow the same steps to study instead the effects of tensor perturbations. Again keep in mind that we will only consider first order perturbations and higher contributions will be neglected.

Tensor perturbations are described by the transverse traceless tensor h_{ij}, which happens to be gauge invariant (section). At first order in perturbation theory the tensor perturbed metric thus reads

$$ds^2 = a^2(-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j).$$

The Liouville operator, as usual, is defined as

$$\hat{\mathbf{L}}[f] = \frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \frac{\partial f}{\partial x^i}\frac{dx^i}{d\tau} + \frac{\partial f}{\partial p}\frac{dp}{d\tau} + \frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{d\tau},$$

where $p^i = p\ p^i$ is again the local 3-momentum, note that even though its definition (as the momentum observed by a static observer) won't change, its relation to the 4-momentum is changed due to the different metric considered. Now, indeed the local energy is

$$U^\mu = (a^{-1}, 0, 0, 0) \ \Rightarrow\ E = -U_\mu P^\mu = aP^0$$

and then requiring that $E = \sqrt{p^i p^j \delta_{ij}} = p$ we observe that the form of the local 3-momentum should be

$$P^\mu P_\mu = -a^2(P^0)^2 + a^2(\delta_{ij} + h_{ij})P^i P^j = 0 \quad \Rightarrow \quad (P^0)^2 = (\delta_{ij} + h_{ij})P^i P^j$$

$$E^2 = p_i p_j \delta_{ij} = (aP^0)^2 = a^2(\delta_{ij} + h_{ij})P^i P^j \quad \Rightarrow \quad p_i = a(\delta_{ij} + \frac{1}{2}h_{ij})P^j,$$

where we used that there is no difference between covariant e contravariant vectors for the local momentum since in the local reference frame the spatial metric is the identity.

We can now proceed and evaluate the first contribution to the Liouville operator $\frac{dx^i}{d\tau}$, keeping in mind that (as in appendix B.1) we only need order zero contributions.

$$\frac{dx^i}{d\tau} = \frac{P^i}{P^0} = \frac{p_j}{E}(\delta^{ij} - \frac{1}{2}h^{ij}) \approx \frac{p^i}{E}. \tag{B.4}$$

The second factor is instead needed up to the first order. We start by evaluating

$$\frac{dp}{d\tau} = \frac{d}{d\tau}aP^0 = \mathcal{H}p + a\frac{dP^0}{d\tau} = \mathcal{H}p - \frac{a}{P^0}\Gamma^0_{\mu\nu}P^\mu P^\nu,$$

where in the last step we used the geodesic equation. With the metric in consideration the relevant Christoffel symbols read:

$$\Gamma^0_{00} = \mathcal{H}, \qquad \Gamma^0_{0i} = 0, \qquad \Gamma^0_{ij} = \mathcal{H}(\delta_{ij} + h_{ij}) + \frac{1}{2}h'_{ij}.$$

With these, recalling that $p^2 = a^2(\delta_{ij} + h_{ij})P^iP^j$ and that at order zero $p^\mu = aP^\mu$, we finally get

$$\frac{dp}{d\tau} = -\frac{a}{P^0}\left[\mathcal{H}(\delta_{ij} + h_{ij}) - \frac{1}{2}h'_{ij}\right]P^iP^j = -\mathcal{H}p - \frac{1}{2}h'_{ij}\hat{p}^i\hat{p}^j. \tag{B.5}$$

We are left with $\frac{d\hat{p}^i}{d\tau}$ to be evaluated, however as in appendix B.1, we can show that this gives no zeroth order contribution, generating in the Liouville operator a second order term (since $\frac{\partial f}{\partial \hat{p}^i}$ has to be of first order too since the unperturbed distribution is isotropic) that can be neglected.

Summing all up we get that the Liouville operator now reads

$$\hat{\mathbf{L}}[f] = \frac{\partial f}{\partial \tau} + \hat{p}^i\frac{\partial f}{\partial x^i} - \frac{1}{2}\frac{\partial f}{\partial \tau}h'_{ij}\hat{p}^i\hat{p}^j$$

$$\Big\downarrow \quad \text{moving to cosmic time } dt = a\ d\tau,$$

$$\hat{\mathbf{L}}[f] = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - \frac{1}{2}\frac{\partial f}{\partial t}\dot{h}_{ij}\hat{p}^i\hat{p}^j. \tag{B.6}$$

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