Computational dynamics

Lecture 2: FE-Implementation

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Reminder

$$(S) = \begin{cases} \text{Given } g : [0,l] \to \mathbb{R} \text{ and constants } q \text{ and } h \text{, find } u : [0,l] \to \mathbb{R}, \text{ such that} \\ \quad u_{,xx} + g = 0 \quad \text{on} \quad]0,l[\\ \quad u(l) = q \\ \quad -u_{,x}(0) = h \end{cases}$$

$$(W) = \begin{cases} \text{let} \quad \mathcal{S} = \{u \,|\, u \in \mathcal{H}^1,\ u(l) = q\} \quad \text{and} \quad \mathcal{V} = \{w \,|\, w \in \mathcal{H}^1,\ w(l) = 0\}. \\ \text{Given } g : [0,l] \to \mathbb{R} \text{ and constants } q \text{ and } h \text{, find } u \in \mathcal{S}, \text{ such that for all } w \in \mathcal{V} \\ \quad \int_0^l w_{,x}\, u_{,x}\, dx = \int_0^l w\, g\, dx + w(0)\, h \end{cases}$$

$$(W^h) = \begin{cases} \text{let } \mathcal{V}^h \subset \mathcal{V} = \{w \,|\, w \in \mathcal{H}^1,\ w(l) = 0\} \text{ and } \mathcal{S}^h \subset \mathcal{S} = \{u \,|\, u \in \mathcal{H}^1,\ u(l) = q\}. \\ \text{Given } g : [0,l] \to \mathbb{R} \text{ and constants } q \text{ and } h \text{, find } u^h = v^h + q^h, \text{ where } v^h \in \mathcal{V}^h \\ \text{and } u^h \in \mathcal{S}^h, \text{ such that for all } w^h \in \mathcal{V}^h \\ \quad \int_0^l w^h_{,x}\, v^h_{,x}\, dx = \int_0^l w^h\, g\, dx + w^h(0)\, h - \int_0^l w^h_{,x}\, q^h_{,x}\, dx \end{cases}$$

$$(M) = \begin{cases} \text{Given the cofficient matrix } \boldsymbol{K} \text{ and vectro } \boldsymbol{F}, \text{ find } \boldsymbol{d} \text{ such that} \\ \quad \boldsymbol{K} \cdot \boldsymbol{d} = \boldsymbol{F} \end{cases}$$

$$(S) \;\Updownarrow\; (*) \;\; (W) \;\approx\; (W^h) \;\Updownarrow\; (M)$$

(*) $W \to S$ only if solution to W has sufficient regularity.

One more thing: a note on VARIATIONAL PRINCIPLES

- An alternative approach for developing the finite element equations is based on variational principles and is equivalent to the weak form.
- Principle of stationarity of a functional: the solution of the strong form ($u(x)$) is the minimizer of the potential energy of the system $W\big(u(x)\big)$.
- W is a functional, i.e., a "function of a function".
- Variation of a function ($\delta u(x)$): infinitesimal change in the function.
- **Variation of the functional ($\boldsymbol{\delta W}$)**: corresponding infinitesimal change in the functional.
- $w(x)$ must be smooth and vanish on the essential boundaries.
- $\delta W = W\big(u(x) + \xi\, w(x)\big) - W\big(u(x)\big) = W\big(u(x) + \delta u(x)\big) - W\big(u(x)\big)$
- **At the minimum of $\boldsymbol{W\big(u(x)\big)}$, the variation of the functional vanishes, i.e. , $\boldsymbol{\delta W = 0}$. This condition renders the weak form**.

SUMMARY OF APPROXIMATED METHODS

WEIGHTED RESIDUAL METHODS

Start with an estimate of the the solution and demand that its weighted average error is minimized

- The Galerkin Method
- The Least Square Method
- The Collocation Method
- The Subdomain Method
- Pseudo-spectral Methods

VARIATIONAL METHODS

Minimisation of a functional.

Outline

1. **Piecewise linear finite element space**
2. **The element point-of-view**
3. **Assemble process**
4. **Numerical example**
5. **Isoparametric element**
6. **Numerical Integration:** Gauss-quadrature & Newton-Cotes

Piecewise linear finite element space

We partition the domain, into n non-overlapping subintervals, as:

$$[x_A, x_{A+1}] \qquad \text{where} \quad x_A < x_{A+1}$$

$$\forall\, A = 1, 2, \ldots\ldots, n \qquad x_1 = 0 \qquad x_{n+1} = l$$

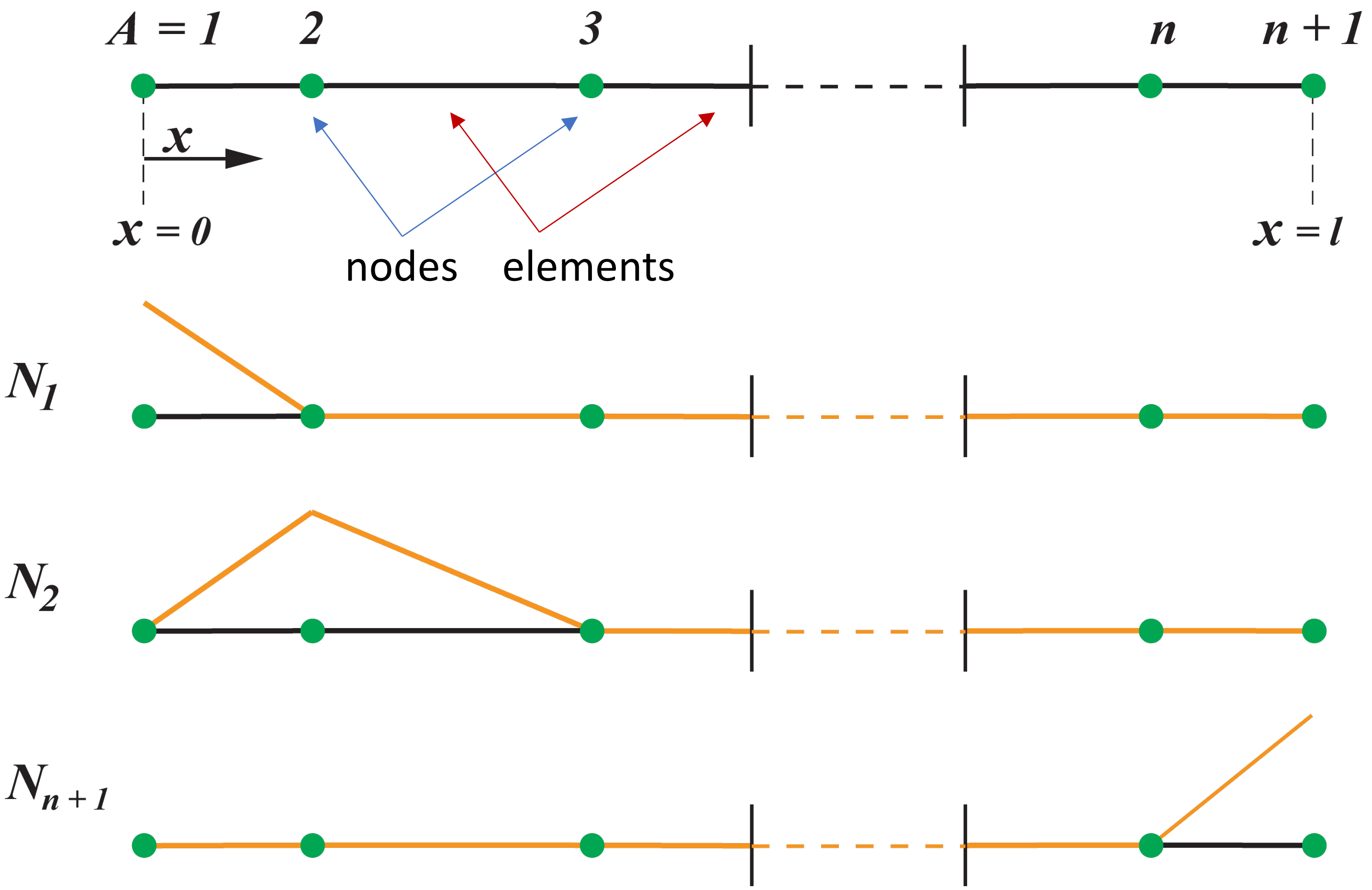
$$h_A = x_{A+1} - x_A \qquad\qquad h = \max h_A$$

Length of the elements: not required to be equal

Mesh parameter

If the subinterval length are equal, then $h = {}^{1}/_{n}$

The smaller h, the more refined the mesh.



Piecewise linear finite element space

- **GLOBAL POINT OF VIEW**:

  The shape functions are defined as:

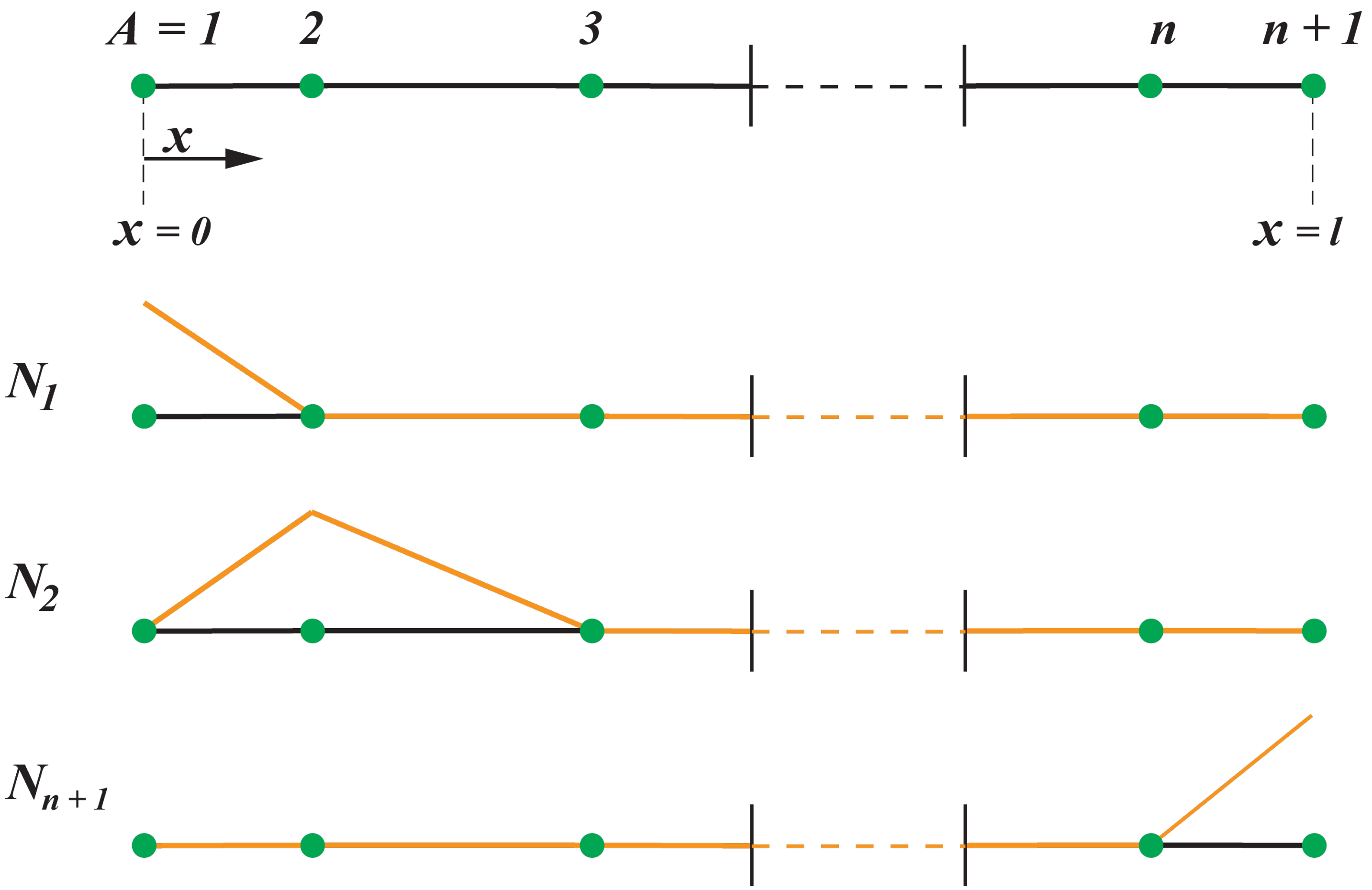
$$A = 1 \Rightarrow N_1(x) = \frac{x_2 - x}{h_1}, \qquad x_1 \le x \le x_2$$

$$2 \le A \le n:$$

$$N_A(x) = \begin{cases} \dfrac{x - x_{A-1}}{h_{A-1}} & x_{A-1} \le x \le x_A \\ \dfrac{x_{A+1} - x}{h_A} & x_A \le x \le x_{A+1} \\ \boxed{0 \qquad \text{elsewhere}} \end{cases}$$

Compact-support functions

$$A = n+1 \Rightarrow N_{n+1}(x) = \frac{x - x_{n+1}}{h_n}, \qquad x_n \le x \le x_{n+1}$$



Piecewise linear finite element space

Note that,

Kronecker delta

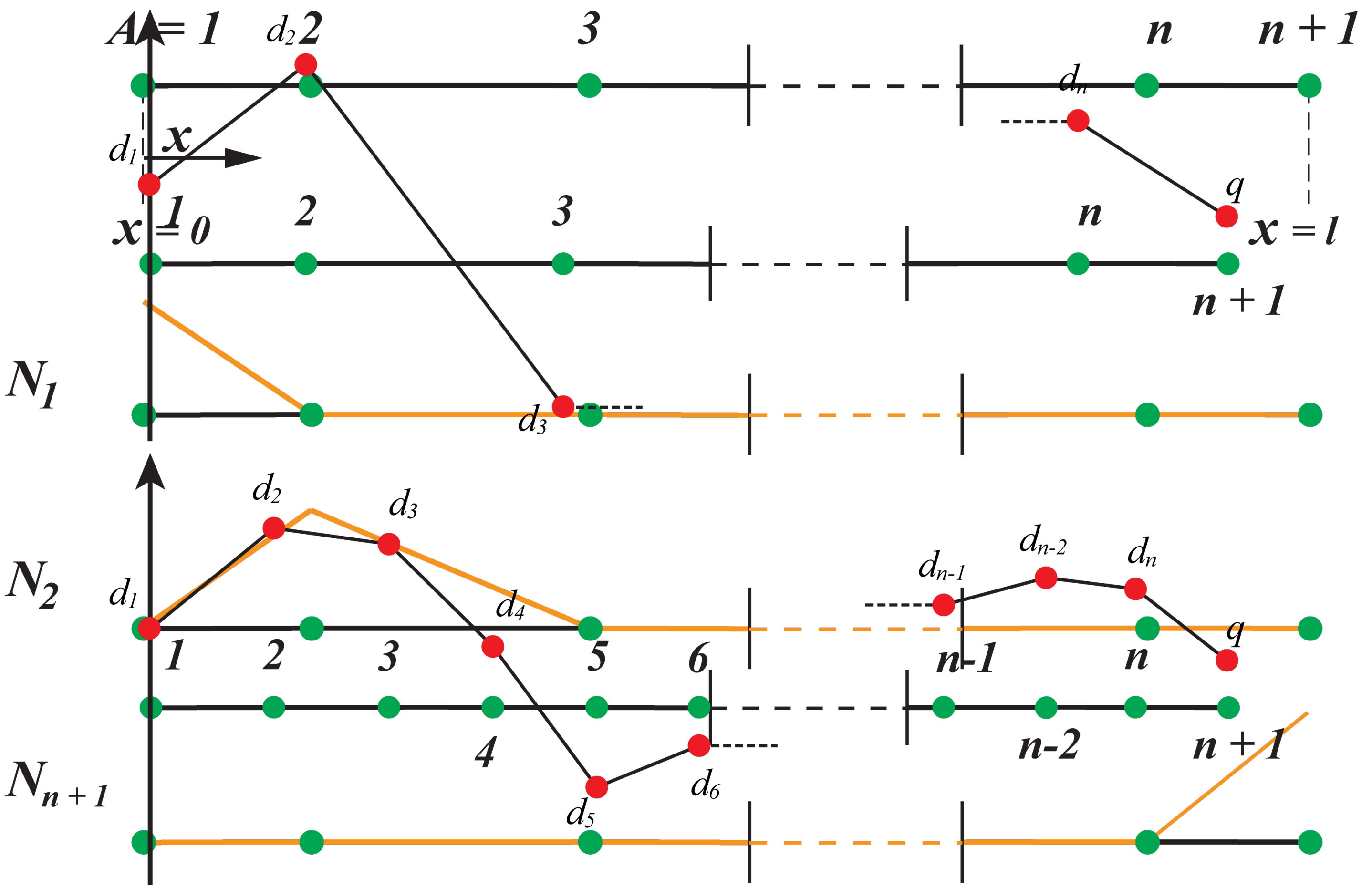
$$N_A(x_B) = \delta_{AB} = \begin{cases} 1 & A = B \\ 0 & A \neq B \end{cases}$$

i.e., **the shape function takes on the value 1 at the node assigned to it and 0 at all other nodes.**

Finite-dimensional approximation of functions:

$$\boxed{u^h = \sum_{A=1}^{n} d_A\, N_A + q\, N_{n+1} = \sum_{A=1}^{n+1} d_A\, N_A \quad \text{with} \quad d_{n+1} = q}$$

The approximate solution is continuous but has a discontinuous slope across each element boundary. Reminder: make u more amenable to **weak solutions without strong derivatives**.



Reminder: Matrix equations

- System of equations $\boldsymbol{K} \cdot \boldsymbol{d} = \boldsymbol{F}$

$$\boldsymbol{K} = [K_{AB}] = \begin{Bmatrix} K_{11} & K_{12} & . & . & . & K_{1n} \\ K_{21} & K_{22} & . & . & . & K_{2n} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ K_{n1} & K_{n2} & . & . & . & K_{nn} \end{Bmatrix} = \begin{Bmatrix} a(N_1, N_1) & a(N_1, N_2) & . & . & . & a(N_1, N_n) \\ a(N_2, N_1) & a(N_2, N_2) & . & . & . & a(N_2, N_n) \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ a(N_n, N_1) & a(N_n, N_2) & . & . & . & a(N_n, N_n) \end{Bmatrix}$$

$$\boldsymbol{F} = [F_B] = \begin{Bmatrix} F_1 \\ F_2 \\ . \\ . \\ . \\ F_n \end{Bmatrix} = \begin{Bmatrix} (N_1, g) + N_1(0)\,h - a(N_1, N_{n+1})\,q \\ (N_2, g) + N_2(0)\,h - a(N_2, N_{n+1})\,q \\ . \\ . \\ . \\ (N_n, g) + N_n(0)\,h - a(N_n, N_{n+1})\,q \end{Bmatrix} \qquad \boldsymbol{d} = [d_B] = \begin{Bmatrix} d_1 \\ d_2 \\ . \\ . \\ . \\ d_n \end{Bmatrix}$$

$\boldsymbol{K}$ Stiffness matrix

$\boldsymbol{d}$ Displacement vector

$\boldsymbol{F}$ Force vector

Piecewise linear finite element space

- For $n = 6$ (d.o.f.s) the stiffness matrix given as:

e.g., No real overlaping between N_2 and N_5 (since it is 1D).

with generally chosen finite elements

$$\boldsymbol{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & k_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix}$$

with linear finite elements

$$\boldsymbol{K} = \begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 & 0 & 0 \\ 0 & K_{32} & K_{33} & K_{34} & 0 & 0 \\ 0 & 0 & K_{43} & K_{44} & K_{45} & 0 \\ 0 & 0 & 0 & K_{54} & K_{55} & K_{56} \\ 0 & 0 & 0 & 0 & K_{65} & K_{66} \end{bmatrix}$$

Stiffness matrix:

- Symmetric $K_{AB} = K_{BA}$, $\boldsymbol{K} = \boldsymbol{K}^T$
- Banded

Why? $\leftrightarrow$ The trial space and the test space are chosen to be the same.

Piecewise linear finite element space

Stiffness matrix:

Symmetric $K_{AB} = K_{BA}$, $\boldsymbol{K} = \boldsymbol{K}^T$

- Positive-definite:
  - $\boldsymbol{c}^T \cdot \boldsymbol{K} \cdot \boldsymbol{c} \geq 0 \quad \forall\, \boldsymbol{c} \neq 0$; and
  - $\boldsymbol{c}^T \cdot \boldsymbol{K} \cdot \boldsymbol{c} = 0$ if and only if $\boldsymbol{c} = 0$.

  $\longrightarrow$ Unique inverse, unique solution of linear elastic problem.

  $\boldsymbol{c}$: arbitrary non-zero real column vector.

- Positive-definite matrices are at the basis of convex optimization. Given a function of several real variables that is twice differentiable, it will be convex at point P if its matrix of second partial derivatives (Hessian) is positive-definite.
- $\boldsymbol{K}$ is congruent with a diagonal matrix with positive real entries, $\boldsymbol{D} \cdot \boldsymbol{K} \cdot \boldsymbol{D}^{\mathrm{T}} = \boldsymbol{K}_2$.
- As a consequence, the eigenvalues of $\boldsymbol{K}$ are real and positive.

Outline

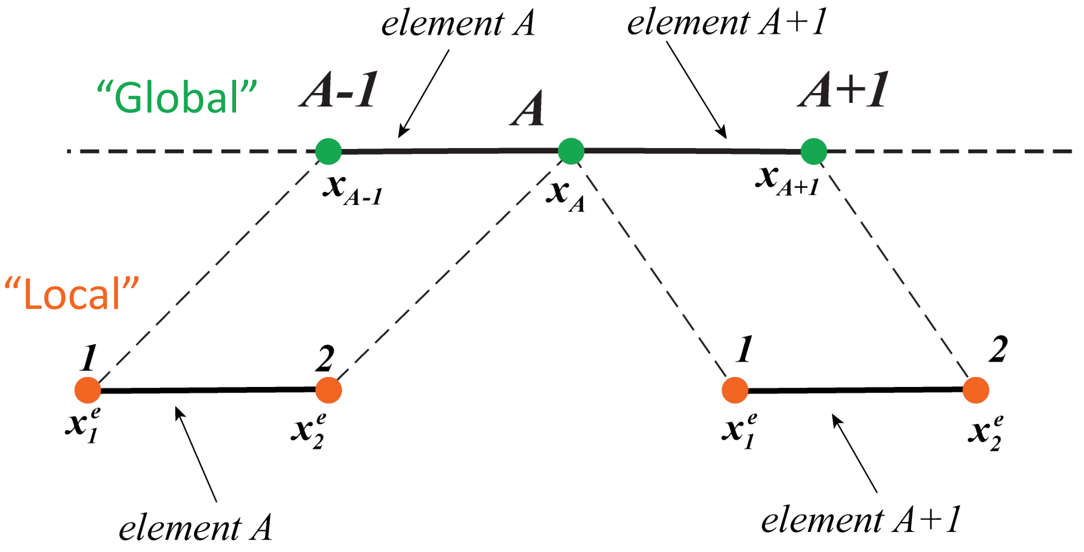
1. **Piecewise linear finite element space**
2. **The element point-of-view**
3. **Assemble process**
4. **Numerical example**
5. **Isoparametric element**
6. **Numerical Integration:** Gauss-quadrature & Newton-Cotes

The element point-of-view

Why?:

- To ease the split of the integrals according to the finite elements and the compactly supported bases, consistently defined according to the element point of view.

The domain is subdivided into n_{el} equal elements with element length $h^e = l/n_{el}$.



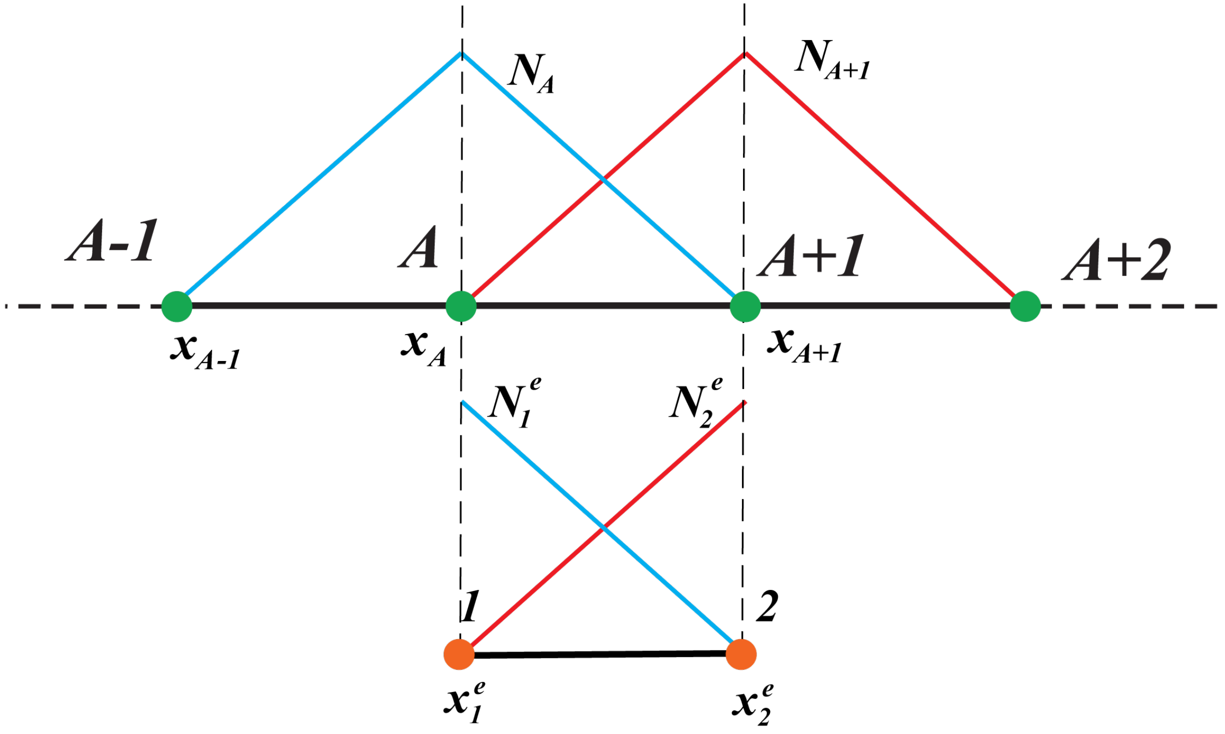
For each and every liner finite element

Point of view	global	local
Nodes	$\{A, A+1\}$	$\{1, 2\}$
Domain	$[x_A, x_{A+1}]$	$[x_1^e, x_2^e]$
Nodes coordinates	$\{x_A, x_{A+1}\}$	$\{x_1^e, x_2^e\}$
Degrees of freedom	$\{d_A, d_{A+1}\}$	$\{d_1^e, d_2^e\}$
Shape functions	$\{N_A, N_{A+1}\}$	$\{N_1^e, N_2^e\}$
Interpolation function	$u^h(x) = N_A\, d_A + N_{A+1}\, d_{A+1}$	$u^h(x) = N_1^e\, d_1^e + N_2^e\, d_2^e$

The element point-of-view

Point of view	global	local
Nodes	$\{A, A+1\}$	$\{1, 2\}$
Domain	$[x_A, x_{A+1}]$	$[x_1^e, x_2^e]$
Nodes coordinates	$\{x_A, x_{A+1}\}$	$\{x_1^e, x_2^e\}$
Degrees of freedom	$\{d_A, d_{A+1}\}$	$\{d_1^e, d_2^e\}$
Shape functions	$\{N_A, N_{A+1}\}$	$\{N_1^e, N_2^e\}$
Interpolation function	$u^h(x) = N_A\, d_A + N_{A+1}\, d_{A+1}$	$u^h(x) = N_1^e\, d_1^e + N_2^e\, d_2^e$

$$N_1^e(x) = \frac{x_2^e - x}{h^e} \Rightarrow (N_1^e)_{,x}(x) = \frac{-1}{h^e}$$
$$N_2^e x(x) = \frac{x - x_1^e}{h^e} \Rightarrow (N_2^e)_{,x}(x) = \frac{1}{h^e}$$



The element point-of-view

- Compute the stiffness matrix by considering the element point-of-view:

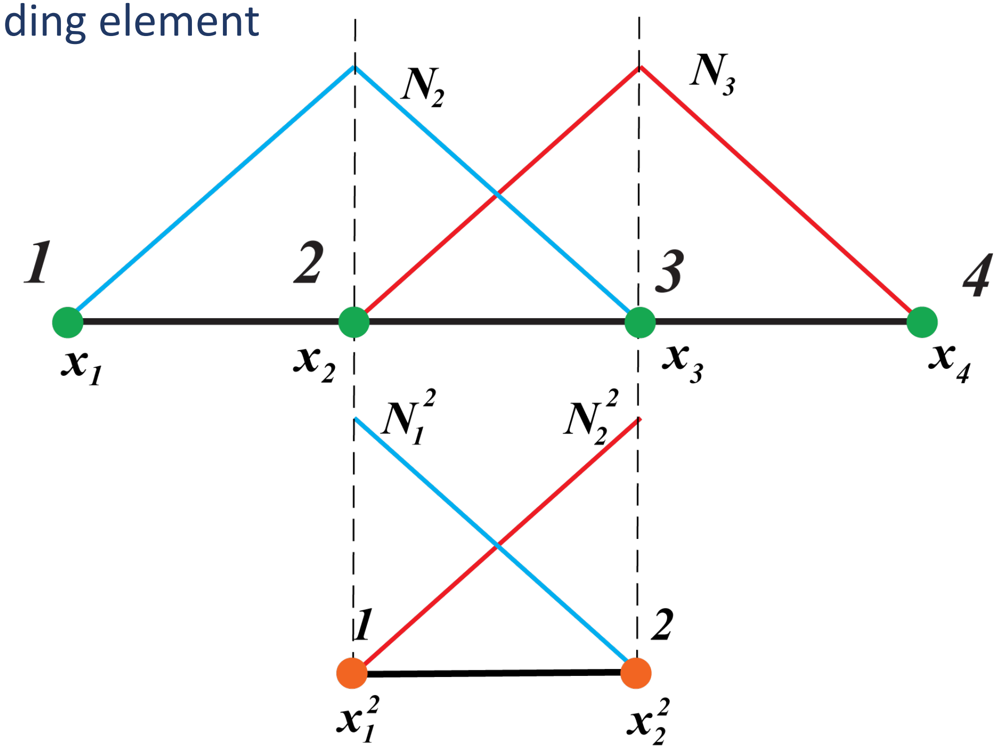
$$K_{23} = k^2_{12}$$

Global node

Local node

Corresponding element

$n = 4; \quad e = 3$



$$K_{23} = k^2_{12} = a(N_2, N_3) = a(N^2_1, N^2_2)$$
$$= \boxed{\int_0^l (N_2)_{,x}(x)\,(N_3)_{,x}(x)\,dx}$$
$$= \boxed{\int_{x^2_1}^{x^2_2} (N^2_1)_{,x}(x)\,(N^2_2)_{,x}(x)\,dx}$$
$$= \int_{x^2_1}^{x^2_2} \frac{-1}{h^e} \times \frac{1}{h^e}\,dx = -\frac{1}{(h^e)^2}\,x\Big|_{x^2_1}^{x^2_2} = -\frac{1}{(h^e)^2}\,\underbrace{(x^2_2 - x^2_1)}_{h^e} = -\frac{1}{h^e}$$

The element point-of-view

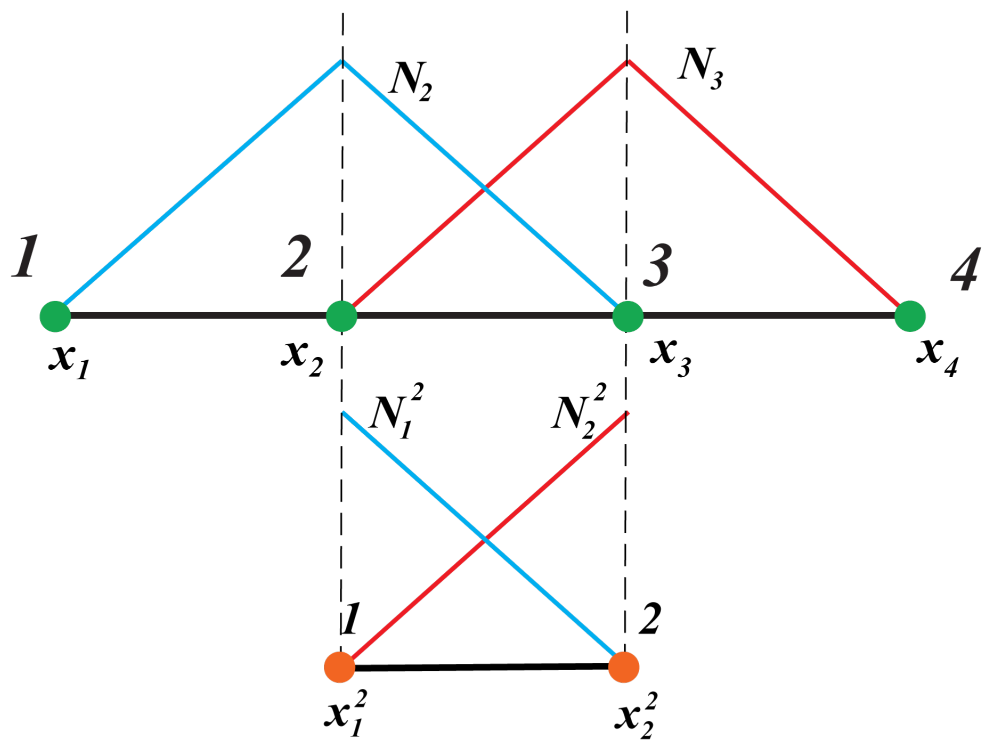
- Compute the stiffness matrix by considering the element point-of-view:

$$k_{11}^2 = a(N_1^2, N_1^2) = \int_{x_1^2}^{x_2^2} \frac{-1}{h^e} \times \frac{-1}{h^e}\, dx = \frac{1}{h^e}$$

$$k_{21}^2 = a(N_2^2, N_1^2) = \int_{x_1^2}^{x_2^2} \frac{1}{h^e} \times \frac{-1}{h^e}\, dx = \frac{-1}{h^e}$$

$$k_{22}^2 = a(N_2^2, N_2^2) = \int_{x_1^2}^{x_2^2} \frac{1}{h^e} \times \frac{1}{h^e}\, dx = \frac{1}{h^e}$$

$n = 4; \quad e = 3$



The element point-of-view

- Compute the stiffness matrix by considering the element point-of-view:

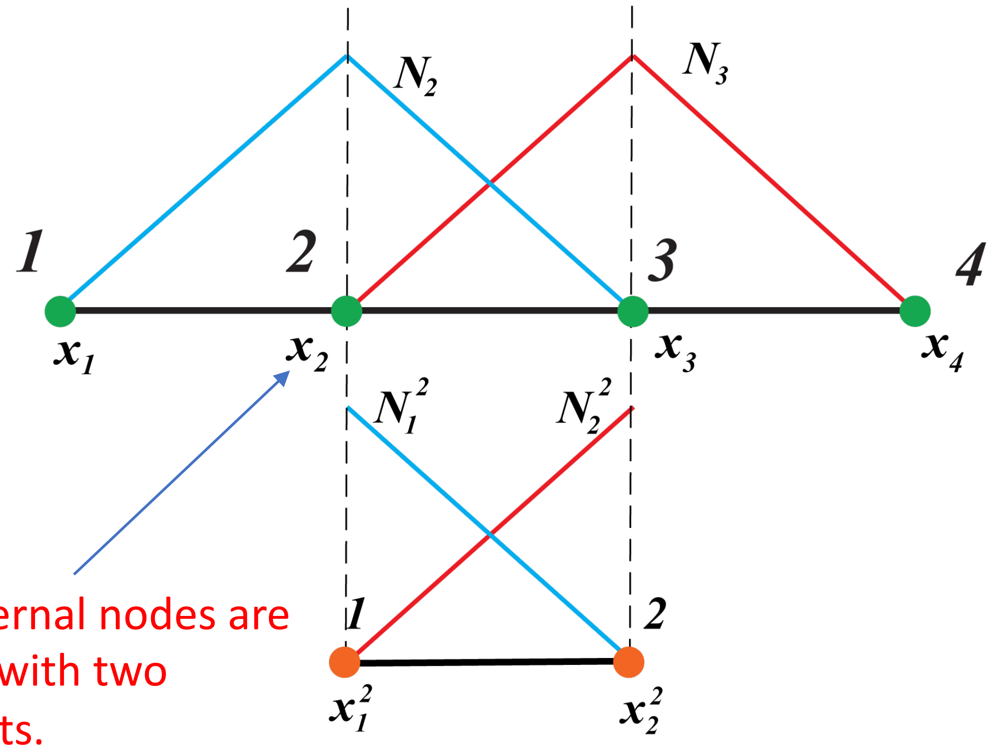
local contribution of the global node 2 as a second node of the local element 1

local contribution of the global node 2 as a first node of the local element 2

Global (total) contribution of the global node 2

$$K_{22} = k_{22}^1 + k_{11}^2$$
$$K_{33} = k_{22}^2 + k_{11}^3$$

$n = 4; \quad e = 3$



The internal nodes are shared with two elements.

The element point-of-view

- Compute the stiffness matrix by considering the element point-of-view:

To compute the local element stiffness matrix:

1. Define the element shape function and its derivative

$$N_1^e(x) = \frac{x_2^e - x}{h^e} \Rightarrow (N_1^e)_{,x}(x) = \frac{-1}{h^e}$$
$$N_2^e x(x) = \frac{x - x_1^e}{h^e} \Rightarrow (N_2^e)_{,x}(x) = \frac{1}{h^e}$$

2. Calculate the local stiffness matrix entries

$$k_{ij}^e = a(N_i^e, N_j^e) = \int_{x_1^e}^{x_2^e} (N_i^e)_{,x}(x)\; (N_j^e)_{,x}(x)\; dx$$

$$\boldsymbol{k}_{2\times 2}^e = [k_{ij}^e] = \begin{bmatrix} k_{11}^e & k_{12}^e \\ k_{21}^e & k_{22}^e \end{bmatrix} = \frac{1}{h^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- The element stiffness matrices (for the linear shape function and equal elements length) are the same, so it is enough to compute it once.

- To compute the global stiffness matrix, we do an operation called an assembly.

The element point-of-view

- Compute the force vector by considering the element point-of-view:

$$F_A = (N_A, g) + N_A(0)\, h - a(N_A, N_{n+1})\, q$$

$$F_2 = (N_2, g) + N_2(0)\, h - a(N_2, N_5)\, q$$

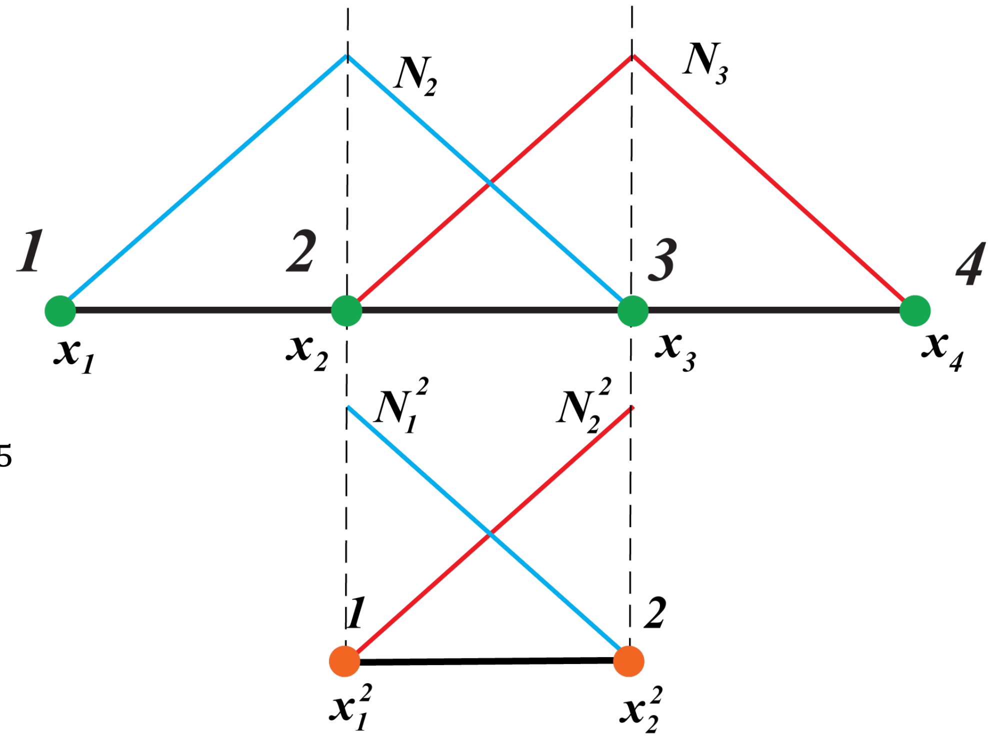
= 0

No real overlaping between N_2 and N_5

$$F_2 = f_2^1 + f_1^2 = (N_2^1, g) + (N_1^2, g)$$

local contribution of the global node 2 as a second node of the element 1

local contribution of the global node 2 as a first node of the element 2



The element point-of-view

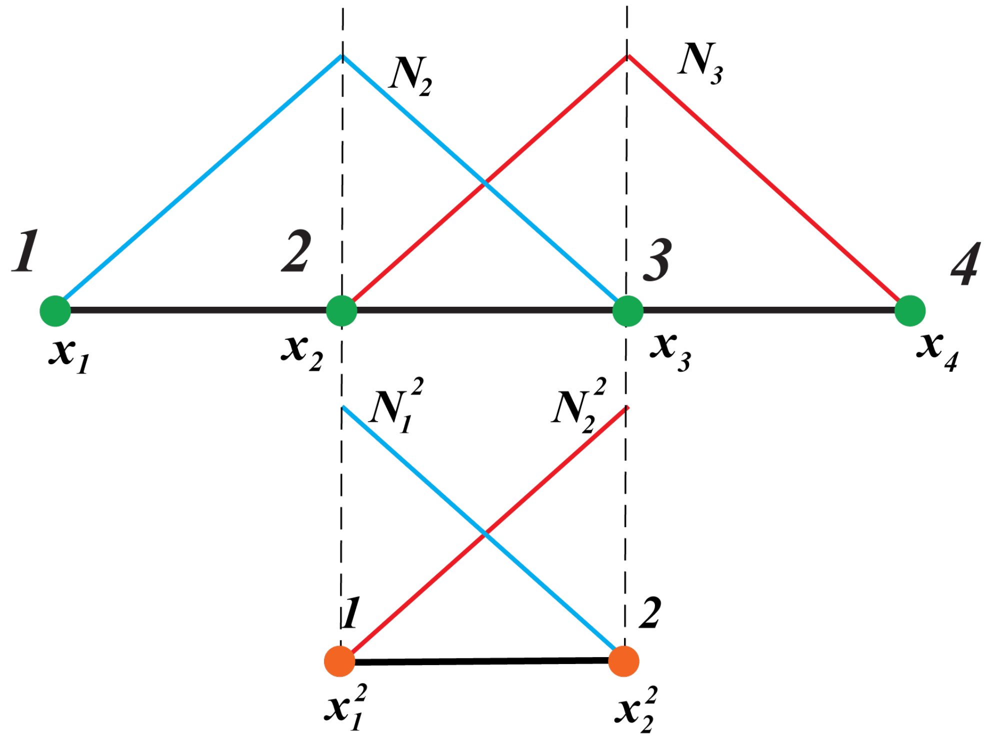
- Compute the force vector by considering the element point-of-view:

$$F_A = (N_A, g) + N_A(0)\,h - a(N_A, N_{n+1})\,q$$

$$F_2 = (N_2, g) + N_2(0)\,h - a(N_2, N_5)\,q$$

$$F_2 = f_2^1 + f_1^2 = (N_2^1, g) + (N_1^2, g)$$

$$= \int_{x_1^1}^{x_2^1} \frac{x - x_1^1}{h^e} g + \int_{x_1^2}^{x_2^2} \frac{x_2^2 - x}{h^e}\, g\,\mathrm{dx}$$



The element point-of-view

- Compute the force vector by considering the element point-of-view:

$$\boldsymbol{f}^e_{2\times1} = [f^e_i]$$

To compute the local force vector:

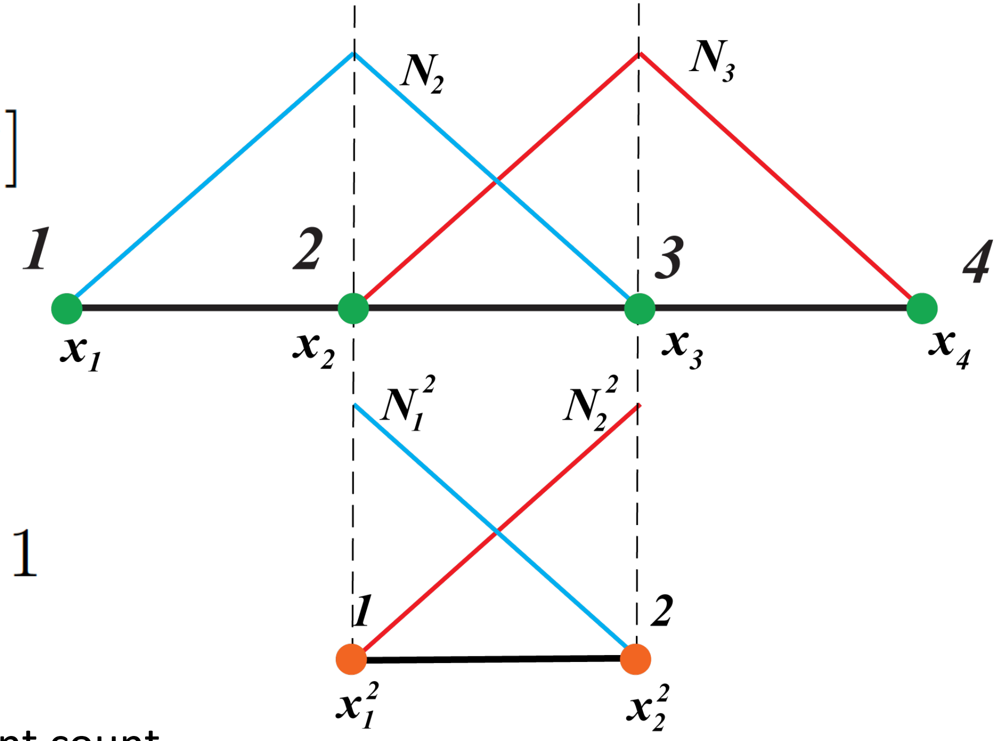
Only for the first node of the first element (or Neumann BCs node)

$$f^e_i = (N^e_i, g) + \begin{cases} \delta_{i1}\, h & \text{if} \quad e = 1 \\ 0 & \text{if} \quad e = 2, 3, \ldots.., n_{el} - 1 \\ -k^e_{i2}\, q & \text{if} \quad e = n_{el} \end{cases}$$

Local node index

Only for the second node of the last element (or Dirichlet BCs node)

Element count



The element point-of-view

Why element point-of-view is needed?

- To ease the split of the integrals according to the finite elements and the compactly supported bases, consistently defined according to the element point of view.

- For the computation of the bilinear form:

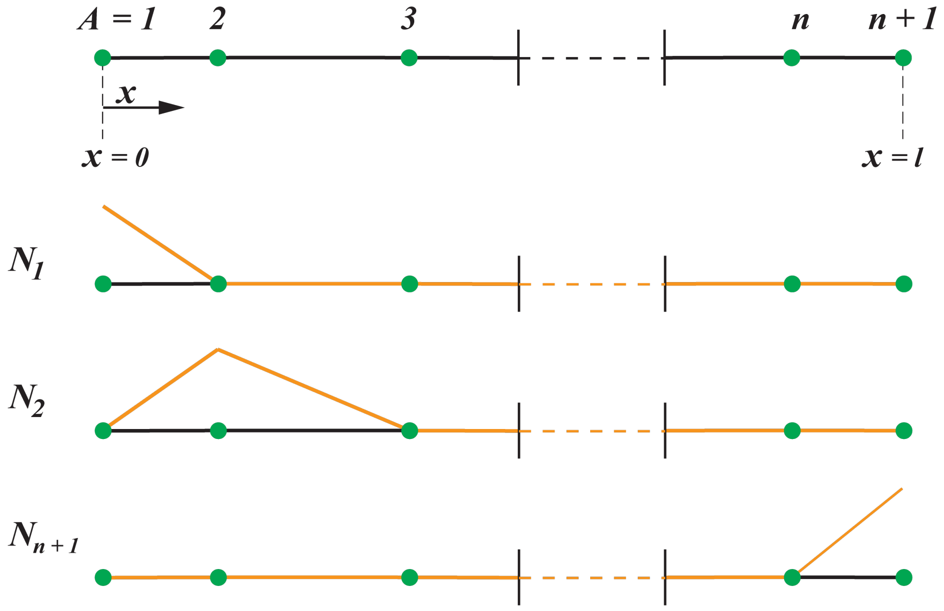
- $K_{AB} \propto a(N_A, N_B) = \int_0^l (N_A)_{,x}(x)(N_B)_{,x}(x)\,\mathrm{d}x$ [Global p.o.v.]
- $k_{ij}^e \propto a\left(N_i^e, N_j^e\right) = \int_{x_i}^{x_j} (N_i^e)_{,x}(x)\left(N_j^e\right)_{,x}(x)\,\mathrm{d}x.$ [Local p.o.v.]

- For the computation of the force vector:

- $F_A = f_2^{e_1} + f_1^{e_2} \propto \left(N_2^{e_1}, g\right) + \left(N_1^{e_2}, g\right) =$

$= \int_{x_1^{e_1}}^{x_2^{e_1}} N_2^{e_1}(x)\, g\,\mathrm{d}x + \int_{x_1^{e_2}}^{x_2^{e_2}} N_1^{e_2}(x)\, g\,\mathrm{d}x$ [Local p.o.v.]

with e_1 and e_2 the finite elements adjacent to global node A.



Outline

Assemble process

- The local stiffness matrices and force vectors are added to the appropriate location in the global stiffness matrix and force vector, respectively.

local node indices

$$\boldsymbol{k}^e_{2\times2} = [k^e_{ij}] = \begin{bmatrix} k^e_{11} & k^e_{12} \\ k^e_{21} & k^e_{22} \end{bmatrix} = \frac{1}{h^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{f}^e_{2\times1} = [f^e_i]$$

The total number of elemets

$$\boldsymbol{K} = \underset{el=1}{\overset{n_{el}}{\mathrm{A}}} \boldsymbol{k}^e \qquad \boldsymbol{F} = \underset{el=1}{\overset{n_{el}}{\mathrm{A}}} \boldsymbol{f}^e$$

Assembly operation

Assemble process

- The local stiffness matrices and force vectors are added to the appropriate location in the global stiffness matrix and force vector, respectively.

$$\boldsymbol{K} = \mathop{\mathrm{A}}_{el=1}^{n_{el}} \boldsymbol{k}^e$$

First element

Second element

Third element

$$\boldsymbol{K} = \begin{bmatrix} K_{11} & K_{12} & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 \\ 0 & K_{32} & K_{33} & K_{34} \\ 0 & 0 & K_{43} & K_{44} \end{bmatrix} = \begin{bmatrix} k^1_{11} & k^1_{12} & 0 & 0 \\ k^1_{21} & k^1_{22} + k^2_{11} & k^2_{12} & 0 \\ 0 & k^2_{21} & k^2_{22} + k^3_{11} & k^3_{12} \\ 0 & 0 & k^3_{21} & k^3_{22} \end{bmatrix}$$

Assemble process

- The local stiffness matrices and force vectors are added to the appropriate location in the global stiffness matrix and force vector, respectively.

$$\boldsymbol{F} = \mathop{\mathrm{A}}_{el=1}^{n_{el}} \boldsymbol{f}^e$$

First element

Second element

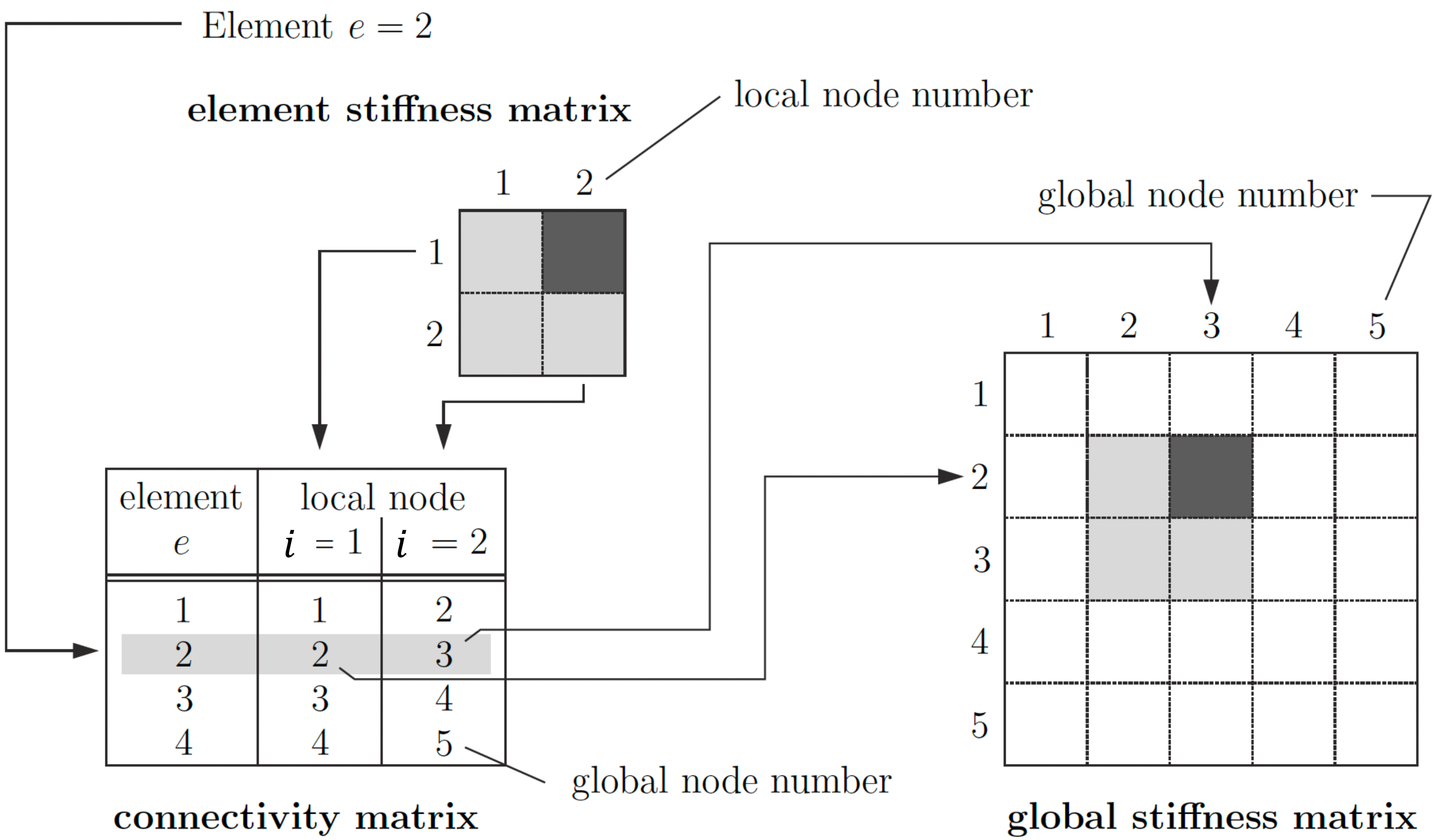
Third element

$$\boldsymbol{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ f_2^3 \end{bmatrix}$$

Assemble process

Assemble process

- In the FE- codes, the assembly information is stored in a matrix named ***Location matrix “LM”*** or ***connectivity matrix***



Outline

Numerical example

Total number of nodes: $n+1 = 5$

Number of unknown: $n = 4$

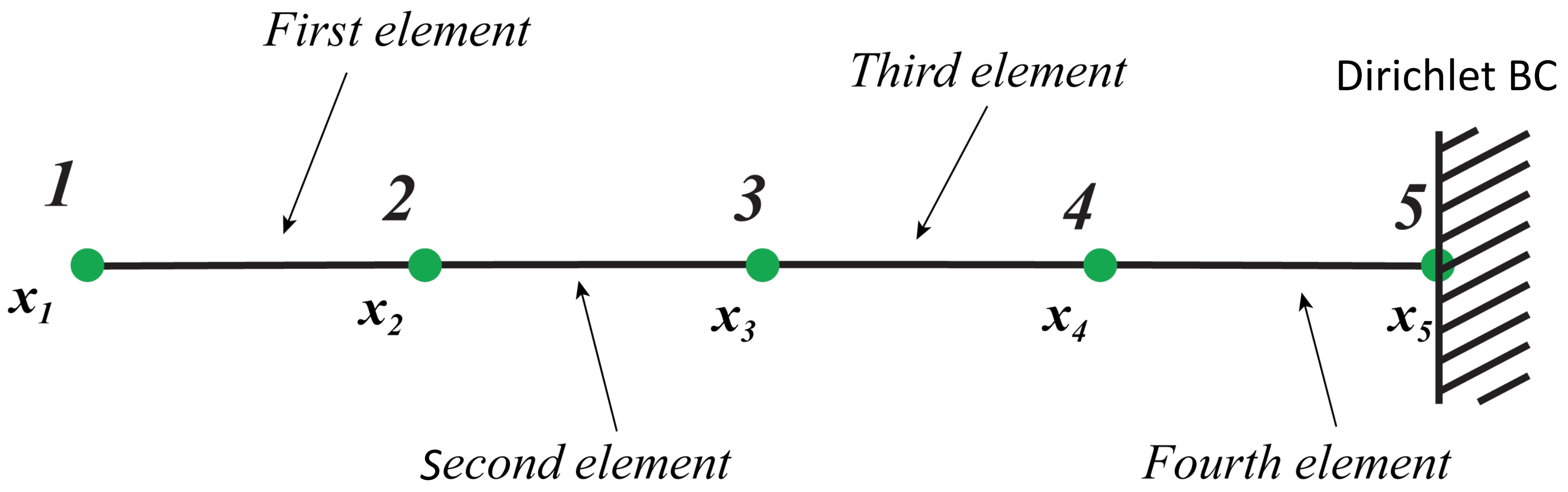
Known node: 5

Number of elements: 4

Equal elements lengths.

The mesh parameter: h=1 cm

Homogeneous BCs.



$$x_1 = 1\text{ cm},\ x_2 = 2\text{ cm},\ x_3 = 3\text{ cm},\ x_4 = 4\text{ cm},\ x_5 = 5\text{ cm}$$

Numerical example

- Derive shape functions (global point of view):

$$N_1(x) = \begin{cases} \dfrac{2-x}{h}, & 1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$N_2(x) = \begin{cases} \dfrac{x-1}{h} & 1 \le x \le 2 \\ \dfrac{3-x}{h} & 2 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$N_3(x) = \begin{cases} \dfrac{x-2}{h} & 2 \le x \le 3 \\ \dfrac{4-x}{h} & 3 \le x \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

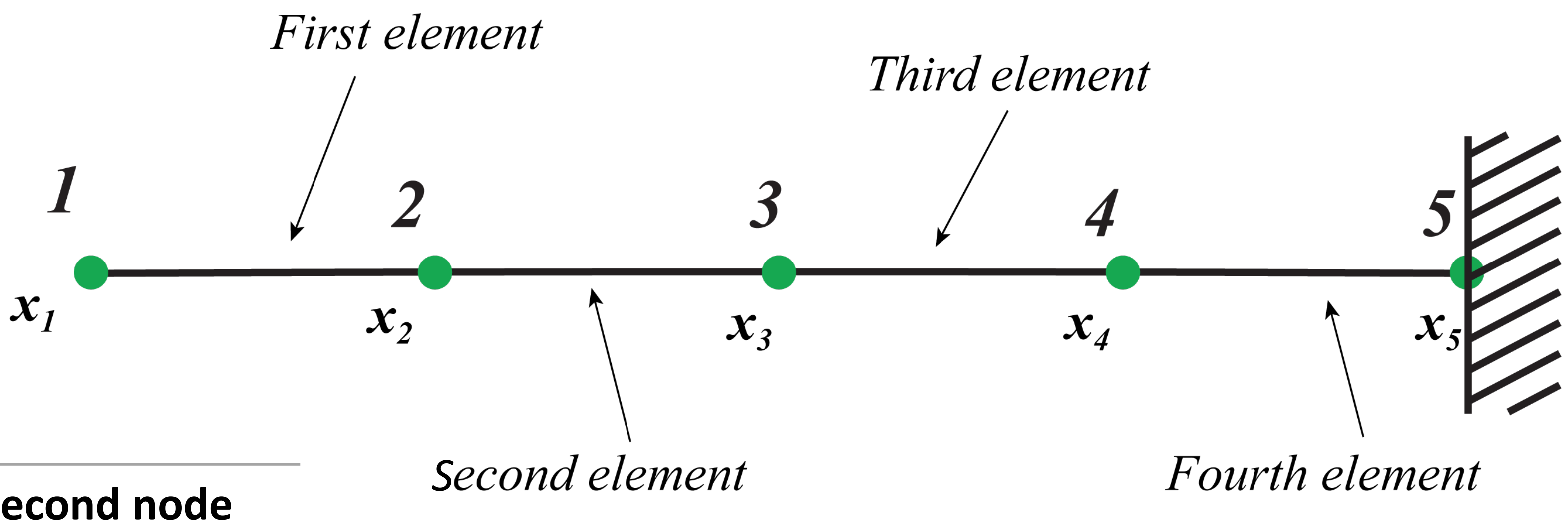
$$N_4(x) = \begin{cases} \dfrac{x-3}{h} & 3 \le x \le 4 \\ \dfrac{5-x}{h} & 4 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

$$N_5(x) = \begin{cases} \dfrac{x-5}{h}, & 4 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

$x_1 = 1$ cm , $x_2 = 2$ cm ,
$x_3 = 3$ cm, $x_4 = 4$ cm,
$x_5 = 5$ cm

Numerical example

- Connectivity matrix:



Element number	First node coordinate	Second node coordinate
First element	$x_1^1 = 1$	$x_2^1 = 2$
Second element	$x_1^2 = 2$	$x_2^2 = 3$
Third element	$x_1^3 = 3$	$x_2^3 = 4$
Fourth element	$x_1^4 = 4$	$x_2^4 = 5$

$$x_1 = 1\ cm\ , x_2 = 2\ cm\ ,$$
$$x_3 = 3\ cm,\ x_4 = 4cm,$$
$$x_5 = 5\ cm$$

Numerical example

- Shape functions in element point-of-view

$$\text{first element}: \begin{cases} N_1^1(x) = \dfrac{2-x}{h} \\ N_2^1(x) = \dfrac{x-1}{h} \end{cases} \qquad \text{third element}: \begin{cases} N_1^3(x) = \dfrac{4-x}{h} \\ N_2^3(x) = \dfrac{x-3}{h} \end{cases}$$

$$\text{second element}: \begin{cases} N_1^2(x) = \dfrac{3-x}{h} \\ N_2^2(x) = \dfrac{x-2}{h} \end{cases} \qquad \text{fourth element}: \begin{cases} N_1^4(x) = \dfrac{5-x}{h} \\ N_2^4(x) = \dfrac{x-4}{h} \end{cases}$$

Numerical example

- Calculate the derivatives of shape functions in element point-of-view

$$\text{first element}: \begin{cases} (N_1^1)_{,x}(x) = \dfrac{-1}{h} \\ (N_2^1)_{,x}(x) = \dfrac{1}{h} \end{cases}$$

$$\text{third element}: \begin{cases} (N_1^3)_{,x}(x) = \dfrac{-1}{h} \\ (N_2^3)_{,x}(x) = \dfrac{1}{h} \end{cases}$$

$$\text{second element}: \begin{cases} (N_1^2)_{,x}(x) = \dfrac{-1}{h} \\ (N_2^2)_{,x}(x) = \dfrac{1}{h} \end{cases}$$

$$\text{fourth element}: \begin{cases} (N_1^4)_{,x}(x) = \dfrac{-1}{h} \\ (N_2^4)_{,x}(x) = \dfrac{1}{h} \end{cases}$$

Numerical example

- Calculate the element stiffness matrix

(Local stiffness matrices for each elements)

$$k_{11}^1 = a(N_1^1, N_1^1) = \int_{x_1^1}^{x_2^1} (N_1^1)_{,x}(x)\,(N_1^1)_{,x}(x)\,dx$$

$$= \int_1^2 \frac{-1}{h}\frac{-1}{h}\,dx = \frac{x}{h^2}\Big|_1^2 = 1$$

$$k_{12}^1 = a(N_1^1, N_2^1) = \int_1^2 \frac{-1}{h}\frac{1}{h}\,dx = -\frac{x}{h^2}\Big|_1^2 = -1$$

$$k_{21}^1 = a(N_2^1, N_1^1) = \int_1^2 \frac{1}{h}\frac{-1}{h}\,dx = -\frac{x}{h^2}\Big|_1^2 = -1$$

$$k_{22}^1 = a(N_2^1, N_2^1) = \int_1^2 \frac{1}{h}\frac{1}{h}\,dx = -\frac{x}{h^2}\Big|_1^2 = 1$$

$$\boldsymbol{k}_{2\times2}^1 = [k_{ij}^1] = \begin{bmatrix} k_{11}^1 & k_{12}^1 \\ k_{21}^1 & k_{22}^1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

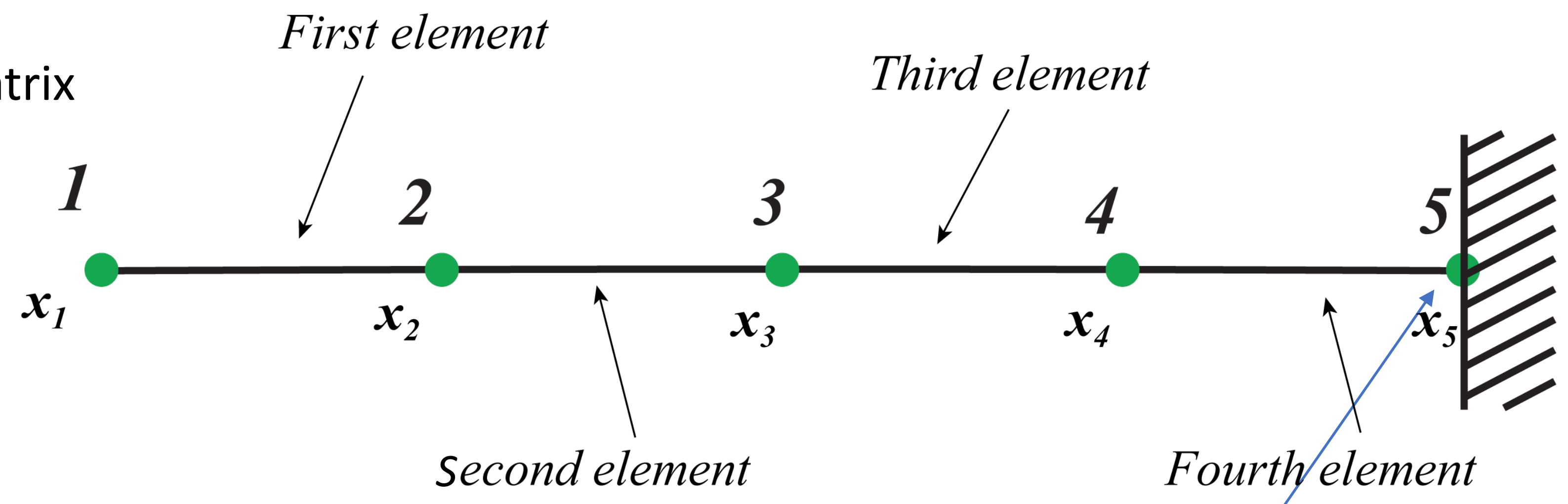
$$\boldsymbol{k}_{2\times2}^2 = [k_{ij}^2] = \begin{bmatrix} k_{11}^2 & k_{12}^2 \\ k_{21}^2 & k_{22}^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{k}_{2\times2}^3 = [k_{ij}^3] = \begin{bmatrix} k_{11}^3 & k_{12}^3 \\ k_{21}^3 & k_{22}^3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{k}_{2\times2}^4 = [k_{ij}^4] = \begin{bmatrix} k_{11}^4 & k_{12}^4 \\ k_{21}^4 & k_{22}^4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Numerical example

- Assembly the global stiffness matrix



Element	First node	Second node
1	1	2
2	2	3
3	3	4
4	4	5

Location matrix (LM)

This node is part of the Dirichlet boundary, the displacement is known $(n+1)$, so it is Not part of global matrix

Numerical example

- Assembly the global stiffness matrix

Element	First node	Second node
1	1	2
2	2	3
3	3	4
4	4	5

Not part of global matrix

$$\boldsymbol{K} = \begin{bmatrix} K_{11} & K_{12} & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 \\ 0 & K_{32} & K_{33} & K_{34} \\ 0 & 0 & K_{43} & K_{44} \end{bmatrix} = \begin{bmatrix} k_{11}^1 & k_{12}^1 & 0 & 0 \\ k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 & 0 \\ 0 & k_{21}^2 & k_{22}^2 + k_{11}^3 & k_{12}^3 \\ 0 & 0 & k_{21}^3 & k_{22}^3 + k_{11}^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+1 & -1 & 0 \\ 0 & -1 & 1+1 & -1 \\ 0 & 0 & -1 & 1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\boldsymbol{k}_{2\times 2}^1 = [k_{ij}^1] = \begin{bmatrix} k_{11}^1 & k_{12}^1 \\ k_{21}^1 & k_{22}^1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{k}_{2\times 2}^2 = [k_{ij}^2] = \begin{bmatrix} k_{11}^2 & k_{12}^2 \\ k_{21}^2 & k_{22}^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{k}_{2\times 2}^3 = [k_{ij}^3] = \begin{bmatrix} k_{11}^3 & k_{12}^3 \\ k_{21}^3 & k_{22}^3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{k}_{2\times 2}^4 = [k_{ij}^4] = \begin{bmatrix} k_{11}^4 & k_{12}^4 \\ k_{21}^4 & k_{22}^4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Outline

Isoparametric element

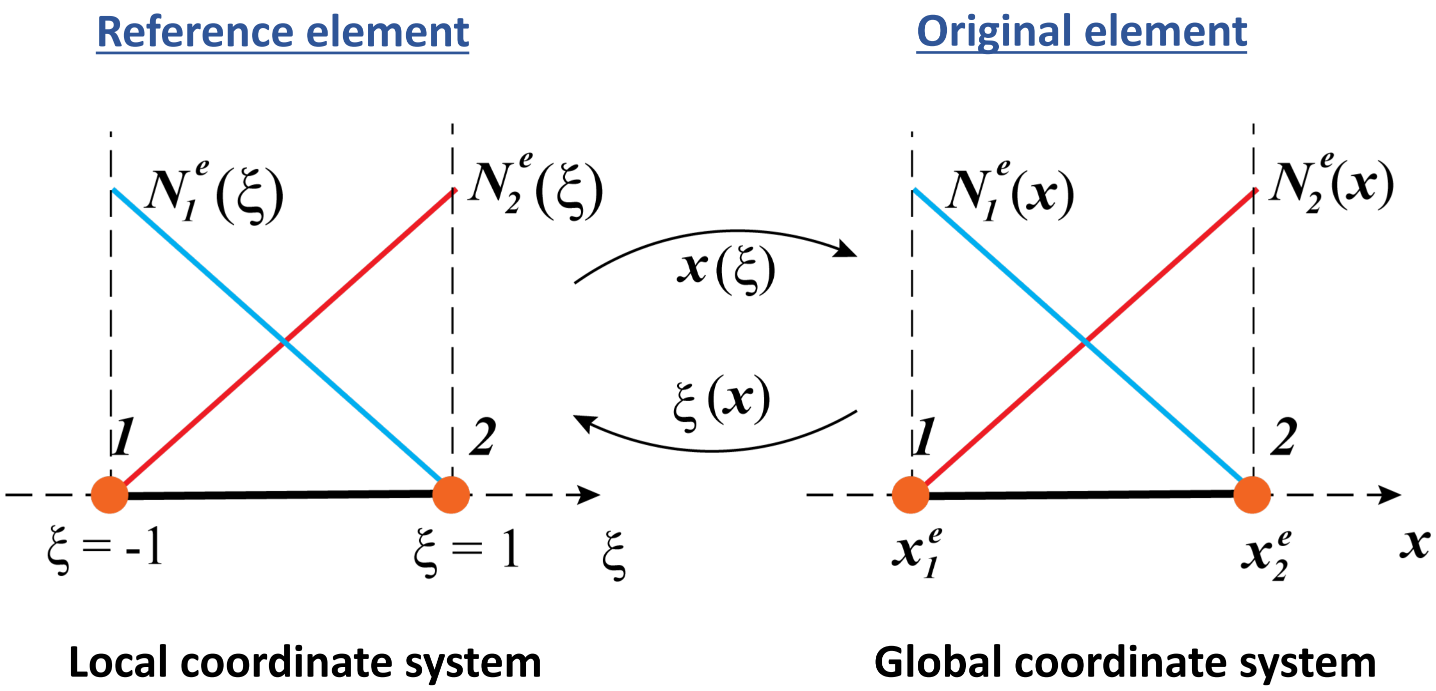
- Let us consider the element local coordinate system

$$\xi \in [-1, 1]$$

The reference element has the same size (shape) across elements regardless of the element's original size (shape).

The shape functions in the local coordinate system are written as:

$$N_1^e(\xi) = \frac{1}{2}[1-\xi] \qquad N_2^e(\xi) = \frac{1}{2}[1+\xi]$$



Isoparametric element

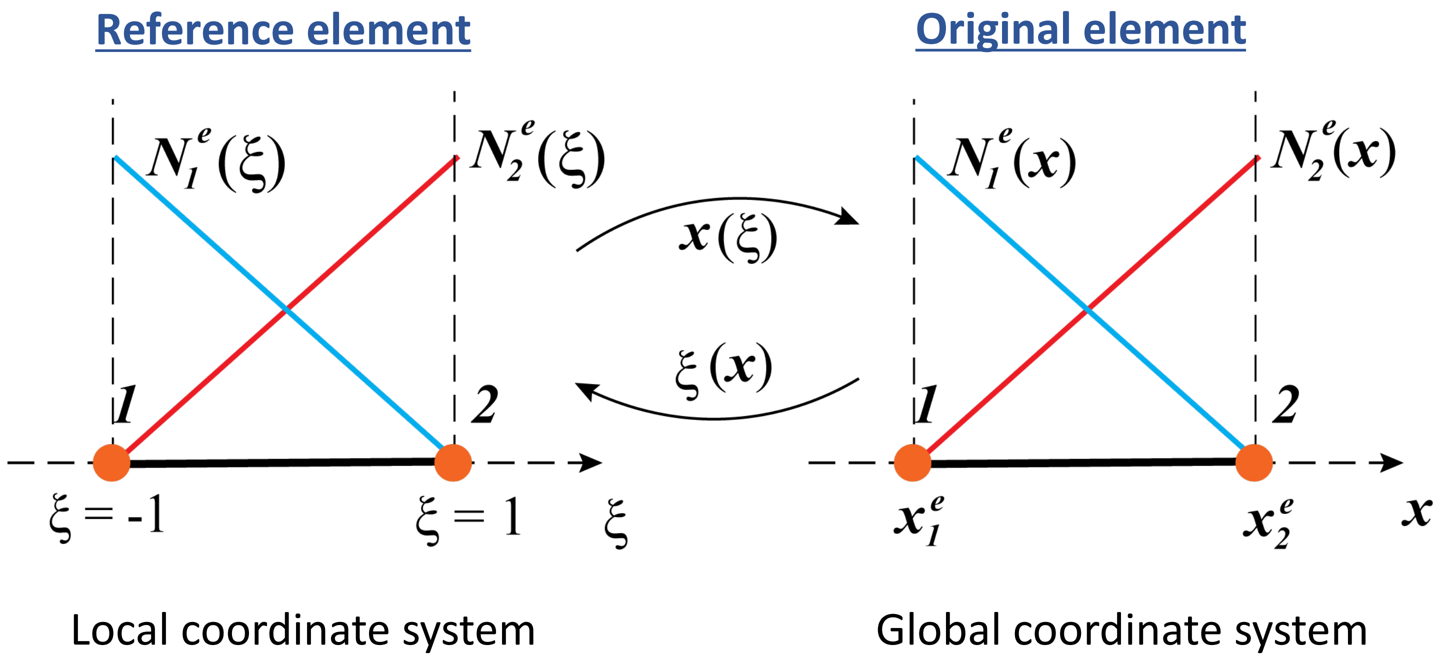
Mapping between global and local coordinates:

$$x : \xi \in [-1, 1] \rightarrow [x_1^e, x_2^e]$$

$$\boxed{x(\xi) = \sum_{a=1}^{2} N_a^e(\xi) x_a^e} = N_1^e(\xi)\, x_1^e + N_2^e(\xi)\, x_2^e$$

$$= \frac{1}{2}[1-\xi]\, x_1^e + \frac{1}{2}[1+\xi]\, [x_1^e + h^e]$$

$$= \frac{h^e}{2}[1+\xi] + x_1^e$$



isoparametric concept:

means that the geometry is approximated in the same way as the solution and test functions

$$\boxed{u^h(\xi) = \sum_{a=1}^{2} N_a^e(\xi) d_a^e}$$

Isoparametric element

- In order to formulate the stiffness matrix and the force vector also in local coordinates, we need a relation between $d\,x$ and $d\,\xi$

$$\frac{dx(\xi)}{d\xi} = \frac{d}{d\xi}\left(\sum_{a=1}^{2} N_a^e(\xi) x_a^e\right) = \underbrace{\sum_{a=1}^{2} \frac{d\,N_a^e(\xi)}{d\xi} x_a^e}_{=:\det J^e} = \det J^e(\xi) = J^e(\xi)$$

Jacobian (matrix)

In 1D case only

$$\boxed{d\,x(\xi) = J^e(\xi)\,d\xi}$$

$$J^e(\xi) = \sum_{a=1}^{2} \frac{d\,N_a^e(\xi)}{d\xi} x_a^e = -\frac{1}{2}x_1^e + \frac{1}{2}[x_1^e + h^e] = \frac{h^e}{2}$$

$$N_1^e(\xi) = \frac{1}{2}[1-\xi]$$

$$(N_a^e)_{,x}(\xi) = \frac{\partial N_a^e(\xi)}{\partial x} = \frac{\partial N_a^e(\xi)}{\partial \xi}\underbrace{\frac{\partial \xi}{\partial x}}_{J^{e-1}} = \frac{2}{h^e}\frac{\partial N_a^e(\xi)}{\partial \xi}$$

Using chain rule

$$N_2^e(\xi) = \frac{1}{2}[1+\xi]$$

Isoparametric element

- Introduce the change of coordinates in the element stiffness matrix:

$$k^e_{ij} = a(N^e_i, N^e_j) = \int_{x^e_1}^{x^e_2} (N^e_i)_{,x}(x)\,(N^e_j)_{,x}(x)\,dx \qquad \boldsymbol{k}^e_{2\times 2} = [k^e_{ij}] = \begin{bmatrix} k^e_{11} & k^e_{12} \\ k^e_{21} & k^e_{22} \end{bmatrix}$$

With $(N^e_a)_{,x}(\xi) = \dfrac{\partial N^e_a(\xi)}{\partial \xi} J^{e-1}$ Give:

$$k^e_{ij} = a(N^e_i(\xi), N^e_j(\xi)) = \int_{-1}^{1} (N^e_i)_{,\xi}(\xi)\,(N^e_j)_{,\xi}(\xi)\,(J^e)^{-1}\,d\xi$$

Only the Jacobian has to be computed for each element

Outline

Numerical Integration

- **Gauss-Legendre quadrature**

$$\int_{x_1^e}^{x_2^e} f(x)\,\mathrm{d}x = \int_{-1}^{+1} \widehat{f}(\xi)\,J^e\,\mathrm{d}\xi = \sum_{l=1}^{n_{\text{int}}} \widehat{f}(\xi_l)\,J^e(\xi_l)\,w_l$$

weights

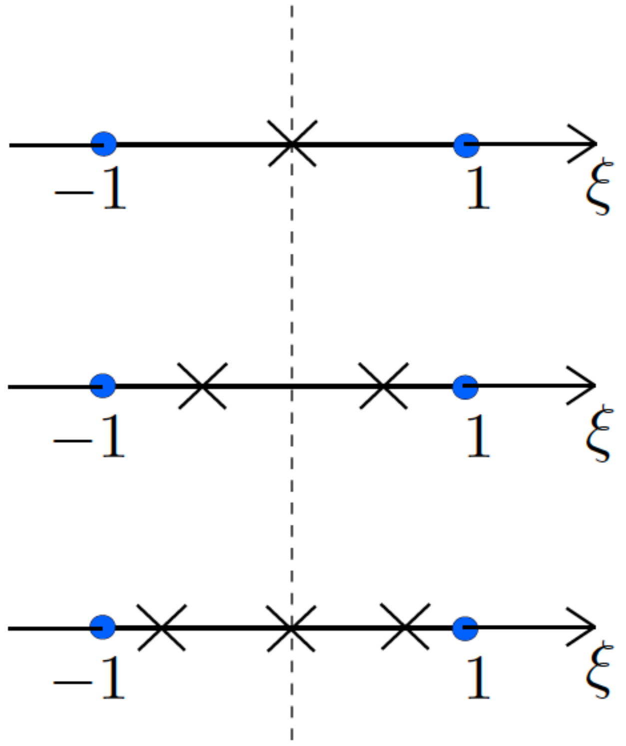
quadrature points (Gauss point)

a weighted sum of function values at specified points within the domain of integration

Constructed to be exact for polynomials of degree $2n_{\mathrm{el}} - 1$

In 1D

n_{int}	ξ_l	w_l
1	0	2
2	$-\frac{1}{\sqrt{3}}$	1
	$\frac{1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$



Numerical Integration

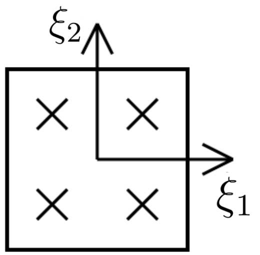
- **Gauss-Legendre quadrature**

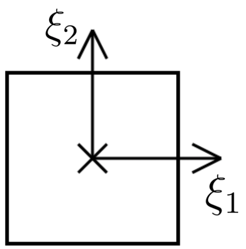
$$k^e_{pq} = \int_{\widehat{\Omega}^e} [\widehat{\nabla}\phi^e_p(\boldsymbol{\xi}) \cdot \boldsymbol{J}^{e-1}(\boldsymbol{\xi})] \cdot \boldsymbol{k} \cdot [\widehat{\nabla}\phi^e_q(\boldsymbol{\xi}) \cdot \boldsymbol{J}^{e-1}(\boldsymbol{\xi})] \det\boldsymbol{J}^e(\boldsymbol{\xi}) \mathrm{d}\widehat{v}$$

$$k^e_{pq} = \sum_{l=1}^{n_{\mathrm{int}}} [\widehat{\nabla}\phi^e_p(\boldsymbol{\xi}_l) \cdot \boldsymbol{J}^{e-1}(\boldsymbol{\xi}_l)] \cdot \boldsymbol{k} \cdot [\widehat{\nabla}\phi^e_q(\boldsymbol{\xi}_l) \cdot \boldsymbol{J}^{e-1}(\boldsymbol{\xi}_l)] \det\boldsymbol{J}^e(\boldsymbol{\xi}_l) w_l$$

$$f^e_p = \int_{\widehat{\Omega}^e} \phi^e_p(\boldsymbol{\xi}) \widehat{f}(\boldsymbol{\xi}) \det\boldsymbol{J}^e(\boldsymbol{\xi}) \, \mathrm{d}\widehat{v}$$

$$f^e_p = \sum_{l=1}^{n_{\mathrm{int}}} \phi^e_p(\boldsymbol{\xi}_l) \widehat{f}(\boldsymbol{\xi}_l) \det\boldsymbol{J}^e(\boldsymbol{\xi}_l) \, w_l$$





n_{int}	$(\xi_1)_l$	$(\xi_2)_l$	w_l
1	0	0	$2 \times 2 = 4$
4	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$1 \times 1 = 1$
	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$1 \times 1 = 1$
	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$1 \times 1 = 1$
	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$1 \times 1 = 1$

Numerical Integration

In the **Gauss–Legendre quadrature** The integral is approximated by

$$\int_a^b f(\xi)\,\mathrm{d}\xi \approx \sum_{i=1}^{n} f(\xi_i) w_i$$

where n is the number of integration points in the element. These are **not equally spaced**, instead the so–called **Gauss points** and **Gauss weights** are optimized such that polynomials of **degree** $\mathbf{2}n-\mathbf{1}$ are exactly integrated with n Gauss points.

Some exemplary Gauss points and Gauss weights for the one dimensional integration domain $[-1,1]$ are (note that the number of integration points is n):

degree	**points**	**weights**
$n=1$	$\xi=0$	$w=2$
$n=2$	$\xi=\left\{-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\}$	$w=\{1,1\}$
$n=3$	$\xi=\left\{-\sqrt{\frac{3}{5}},0,\sqrt{\frac{3}{5}}\right\}$	$w=\left\{\frac{5}{9},\frac{8}{9},\frac{5}{9}\right\}$

Numerical Integration

- **Newton-Cotes formulas**

$$\int_a^b f(x)dx \approx [a-b]\sum_{i=0}^{n} f(x_i)\,w_i = [a-b]\sum_{i=0}^{n} f(x_i)\frac{1}{[b-a]}\int_a^b L_i(x)dx$$

i denotes the integration points and w_i the weights for each integration point.

$L_i(x)$ is the Lagrange polynomial for i :

$$L_i(x) = \prod_{k=0,k\neq i}^{n} \frac{x - x_k}{x_i - x_k}$$

Numerical Integration

In the closed form, the **Newton–Cotes formulas** approximate the integral by

$$\int_a^b f(x)\,\mathrm{d}x \approx [b-a]\sum_{i=0}^{n} f(x_i)w_i,$$

where the integration points are equally spaced, and $a = x_0 < x_1 < \cdots < x_n = b$.

A Newton-Cotes formula is exact for

- polynomials of degree $n+1$, if n is **even**,
- polynomials of degree n, if n is **odd**.

Some well-known Newton-Cotes formulas are (note that the number of integration points is $n+1$):

degree	**name**	**formula**	**exact for**
$n=1$	Trapezoid rule	$\frac{b-a}{2}[f(x_0)+f(x_1)]$	$p(1)$
$n=2$	Simpson's rule	$\frac{b-a}{6}[f(x_0)+4f(x_1)+f(x_2)]$	$p(3)$
$n=3$	$\frac{3}{8}$ rule	$\frac{b-a}{8}[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)]$	$p(3)$

References:

- Hughes, Thomas J. R. *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*. Prentice-Hall, 1987.

- This material builds upon previous course content developed by D. Davydov, S. Zarzor, and colleagues.