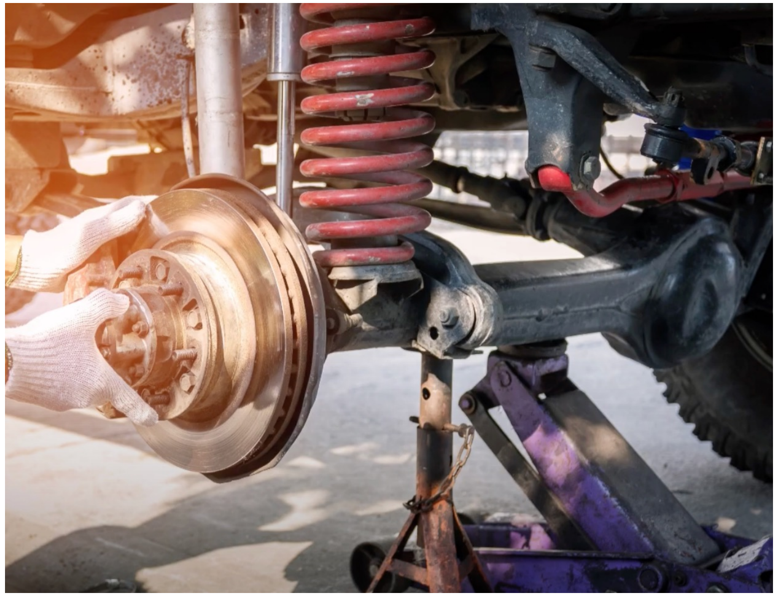
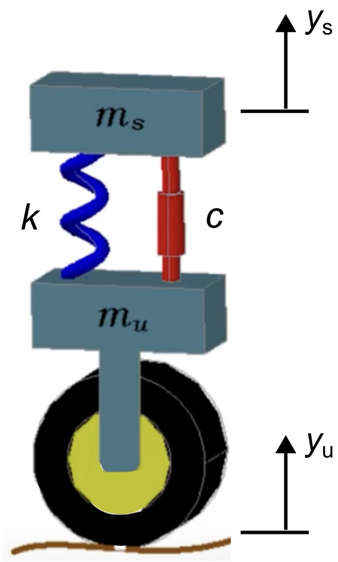
Analysis of Single Degree of Freedom Mass–Spring–Damper System: Free Vibration

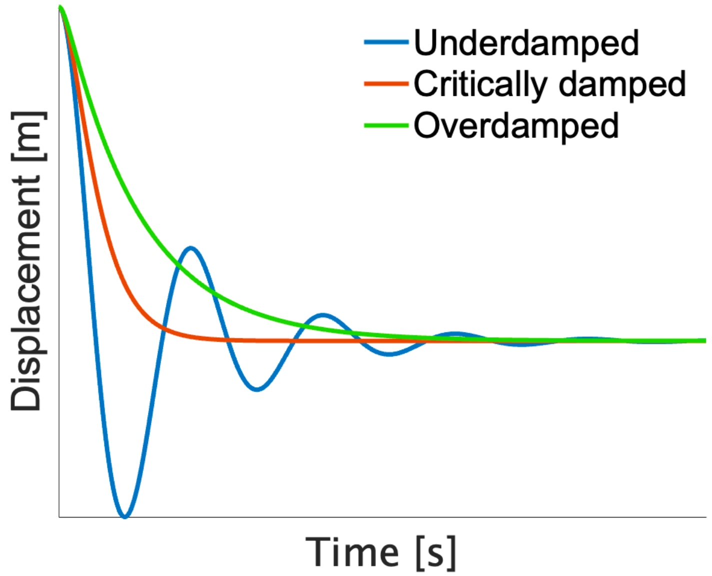
Dr. Miguel Ángel Moreno-Mateos

Automobile Suspension System



Mass + Spring + Damper





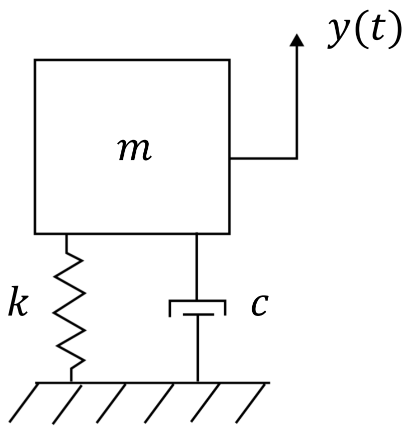
✅ **Critically damped**: Minimise oscillations & guarantee responsiveness (quick return to rest position).

❌ **Underdamped**: undesired oscillations.

❌ **Overdamped**: unresponsive suspension system.

PHYSICAL SYSTEM

Single Degree of Freedom Spring-Mass-Damper System:



with:

- coordinate $y(t)$ with time derivatives $\dot{y}(t)$ and $\ddot{y}(t)$,

- inertia , mass m $[\mathrm{N\,m^{-1}s^2}]$,

- linear spring , coefficient k $[\mathrm{N\,m^{-1}}]$,

- viscous damper , coefficient c $[\mathrm{N\,m^{-1}\,s}]$,

Hypotheses:

- single coordinate describes motion;
- lumped mass system;
- linear elasticity for spring;
- linear viscous damping;
- time invariant parameters;
- no additional forces (e.g., friction).

Find:

Free motion of the mass with initial conditions y_0 and $\dot{y}_0$ → homogeneous solution.

Homogeneous solution: right side $f(t) = 0 \Longrightarrow y_{\mathrm{h}}(t)$

Particular solution: right side $f(t) \neq 0 \Longrightarrow y_{\mathrm{p}}(t)$

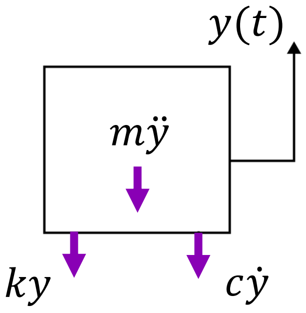
Solution of ODE:

Total solution: $y_{\mathrm{tot}}(t) = y_{\mathrm{h}}(t) + y_{\mathrm{p}}(t)$

Problem

a) **Equation of motion**: D'Alembert's principle

- Newton's second law, $\boxed{m\,\ddot{y}(t) + c\,\dot{y}(t) + k\,y(t) = 0}$



$m\,\ddot{y}(t)$: inertia force (opposes acceleration)

$c\,\dot{y}(t)$: damping force (dissipates energy)

$k\,y(t)$: spring force (restores position)

b) **Initial conditions:** $\boxed{y_0 = y(t_0);\ \dot{y}_0 = \dot{y}(t_0)}$

Normalised form:

$$\boxed{\ddot{y}(t) + 2\zeta\omega_\mathrm{n}\,\dot{y}(t) + \omega_\mathrm{n}^2\,y(t) = 0}$$

- undamped natural frequency,

$$\boxed{\omega_\mathrm{n} = \sqrt{k/m}} \quad [\mathrm{rad\ s^{-1}}]$$

- damping ratio,

$$\boxed{\zeta = \frac{c}{c_\mathrm{crit}}} \quad [-] \qquad \boxed{c_\mathrm{crit} = 2\sqrt{m\,k}} \quad [\mathrm{N\ m^{-1}\ s}]$$

Solution

1. Assume solution, substitute: $y(t) = Ce^{\lambda t}$; $\dot{y}(t) = C\lambda e^{\lambda t}$; $\ddot{y}(t) = C\lambda^2 e^{\lambda t}$

2. Characteristic equation: $\lambda^2 + 2\zeta\omega_\mathrm{n}\,\lambda + \omega_\mathrm{n}^2 = 0 \ \rightarrow\ \lambda_{1,2} = -\zeta\omega_\mathrm{n} \pm \omega_\mathrm{n}\sqrt{\zeta^2 - 1}.$

3. Two roots ($\lambda_{1,2}$), two solutions: $y_1(t) = C_1 e^{\lambda_1 t}$; $y_2(t) = C_2 e^{\lambda_2 t}$

4. General solution: $\boxed{y(t) = C_1 e^{\left[-\zeta\omega_\mathrm{n} + \omega_\mathrm{n}\sqrt{\zeta^2-1}\right]t} + C_2 e^{\left[-\zeta\omega_\mathrm{n} - \omega_\mathrm{n}\sqrt{\zeta^2-1}\right]t}}$ $\boxed{C_1;\ C_2}$ constants from initial conditions.

- No damping: $\zeta = 0$

- Solution for complex conjugate roots without real part:

$$y(t) = C_1 e^{\left[-\zeta\omega_\mathrm{n} + i\omega_\mathrm{n}\sqrt{1-\zeta^2}\right]t} + C_2 e^{\left[-\zeta\omega_\mathrm{n} - i\omega_\mathrm{n}\sqrt{1-\zeta^2}\right]t}$$

Note(*): $e^{i\alpha} = \cos\alpha + i\sin\alpha$

- Express as harmonic oscillation (Euler's equation*):

$$\boxed{y_h(t) = C_1' \cos(\omega_\mathrm{n}\, t) + C_2' \sin(\omega_\mathrm{n}\, t)}$$

$$\begin{aligned} y_0 &= y(t_0); \\ \dot{y}_0 &= \dot{y}\,(t_0) \end{aligned} \quad \rightarrow \quad C_1' = y_0\;;\; C_2' = \frac{\dot{y}_0}{\omega_\mathrm{n}}$$

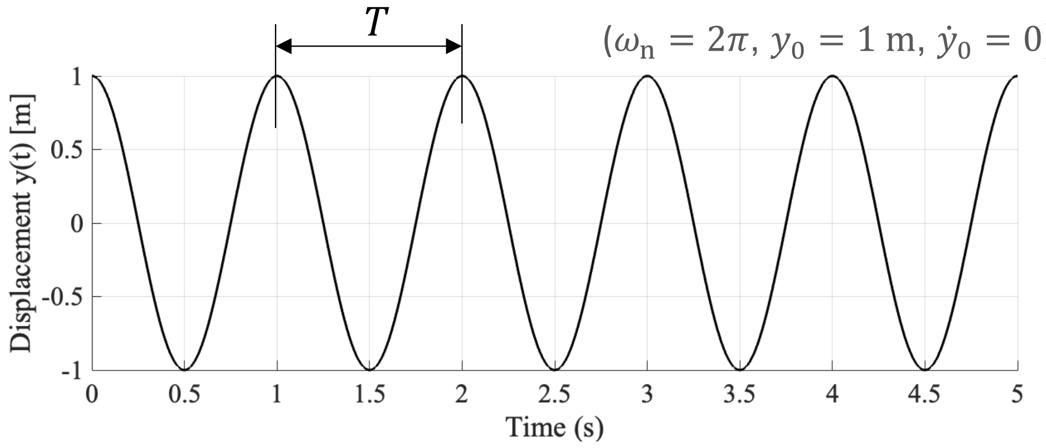
- Natural frequency without damping:

$$\boxed{\omega_\mathrm{n} = \sqrt{k/m}} \qquad T = \frac{2\pi}{\omega_\mathrm{n}}$$

$$\uparrow k \implies \uparrow \omega_\mathrm{n}$$

$$\uparrow m \implies \downarrow \omega_\mathrm{n}$$

$(\omega_\mathrm{n} = 2\pi,\ y_0 = 1\ \mathrm{m},\ \dot{y}_0 = 0)$



Application:
- baseline for more complex systems (with damping and forcing).

- Underdamped case: $0 < \zeta < 1$ $\quad c < c_{\mathrm{crit}} = 2\sqrt{m\,k}$

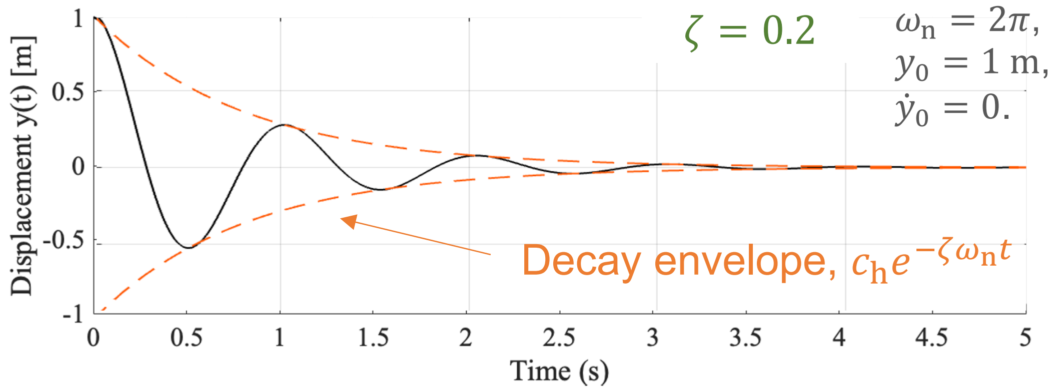
- Solution for complex conjugate roots, negative real part:

$$y(t) = C_1 e^{[-\zeta\omega_{\mathrm{n}} + i\omega_{\mathrm{n}}\sqrt{1-\zeta^2}]\,t} + C_2 e^{[-\zeta\omega_{\mathrm{n}} - i\omega_{\mathrm{n}}\sqrt{1-\zeta^2}]\,t}$$

- Express as harmonic oscillation (Euler's equation):

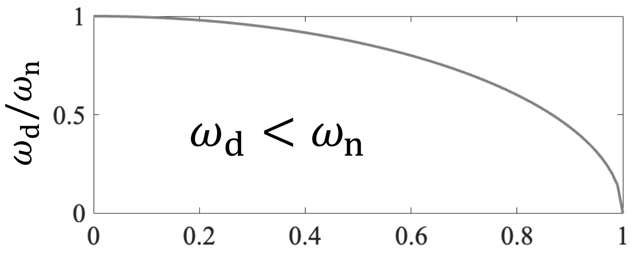
$$\boxed{y(t) = e^{-\zeta\omega_{\mathrm{n}} t}[C_1' \cos(\omega_{\mathrm{d}} t) + C_2' \sin(\omega_{\mathrm{d}} t)]}$$

$$\begin{matrix} y_0 = y(t_0); \\ \dot{y}_0 = \dot{y}\,(t_0) \end{matrix} \quad \rightarrow \quad C_1' = y_0\,;\; C_2' = \frac{\dot{y}_0 + \zeta\omega_{\mathrm{n}} y_0}{\omega_{\mathrm{d}}}$$



- Frequency of damped vibration:

$$\boxed{\omega_{\mathrm{d}} = \omega_{\mathrm{n}}\sqrt{1-\zeta^2}}$$



$$\zeta = \frac{c}{2\,\sqrt{m\,k}}$$

- Damping → lowers frequency, reduces amplitude.

Applications:

- most automobile suspension,
- vibrating mechanical systems,
- musical instruments.

- Critically damped case: $\zeta = 1$

$$c = c_{\text{crit}} = 2\sqrt{m\,k}$$

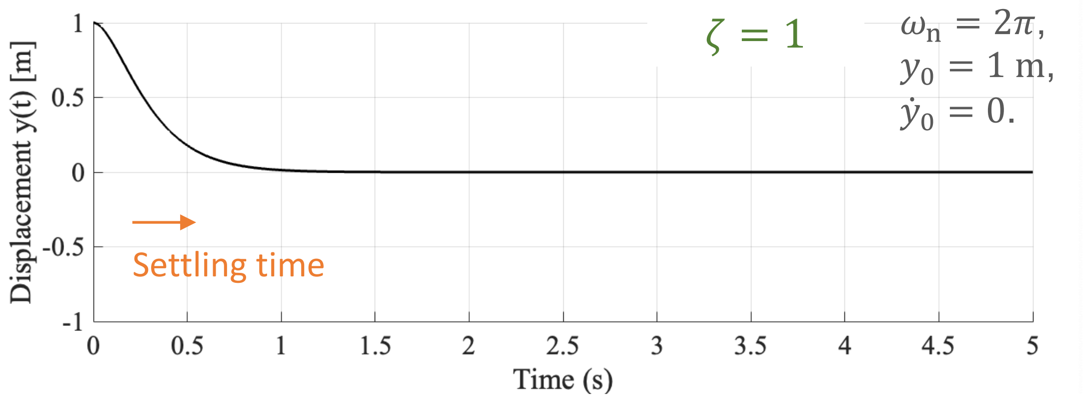
- Solution for two real equal roots:

$$\boxed{y(t) = C_1' e^{-\omega_n t} + C_2' t e^{-\omega_n t}}$$

$$\begin{aligned} y_0 &= y(t_0); \\ \dot{y}_0 &= \dot{y}\,(t_0) \end{aligned} \rightarrow \quad C_1' = y_0\;;\;\; C_2' = \dot{y}_0 + \omega_n y_0$$

- Minimum damping to avoid oscillations.
- Mass returns to rest position in shortest time (responsiveness) without oscillations.

- Exponential decay:



Applications:

- automatic / hydraulic doors,
- some automobile suspensions,
- safety systems in brakes or shock absorbers.

ANALYSIS ζ: OVERDAMPED SYSTEM

- Overdamped case: $\zeta > 1$

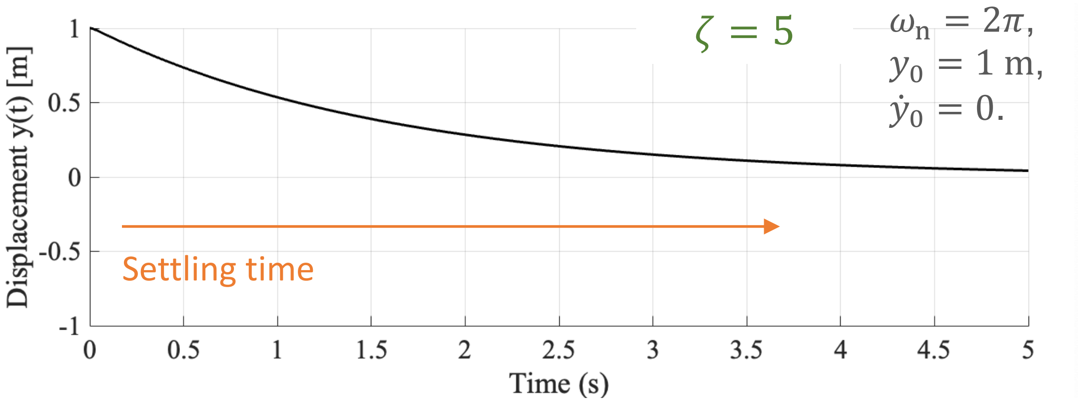
$$c > c_{\text{crit}} = 2\sqrt{m\,k}$$

- Solution for two real distinct roots:

$$\boxed{y(t) = C_1 e^{\left[-\zeta\omega_{\text{n}} + \omega_{\text{n}}\sqrt{\zeta^2-1}\right]\,t} + C_2 e^{\left[-\zeta\omega_{\text{n}} - \omega_{\text{n}}\sqrt{\zeta^2-1}\right]\,t}}$$

$$\begin{aligned} y_0 &= y(t_0); \\ \dot{y}_0 &= \dot{y}\,(t_0) \end{aligned} \quad \rightarrow \quad \begin{aligned} C_1 &= \frac{y_0\omega_{\text{n}}\left[\zeta + \sqrt{\zeta^2-1}\right] + \dot{y}_0}{2\omega_{\text{n}}\sqrt{\zeta^2-1}}; \\ C_2 &= \frac{-y_0\omega_{\text{n}}\left[\zeta - \sqrt{\zeta^2-1}\right] - \dot{y}_0}{2\omega_{\text{n}}\sqrt{\zeta^2-1}}. \end{aligned}$$

- Exponential decay:

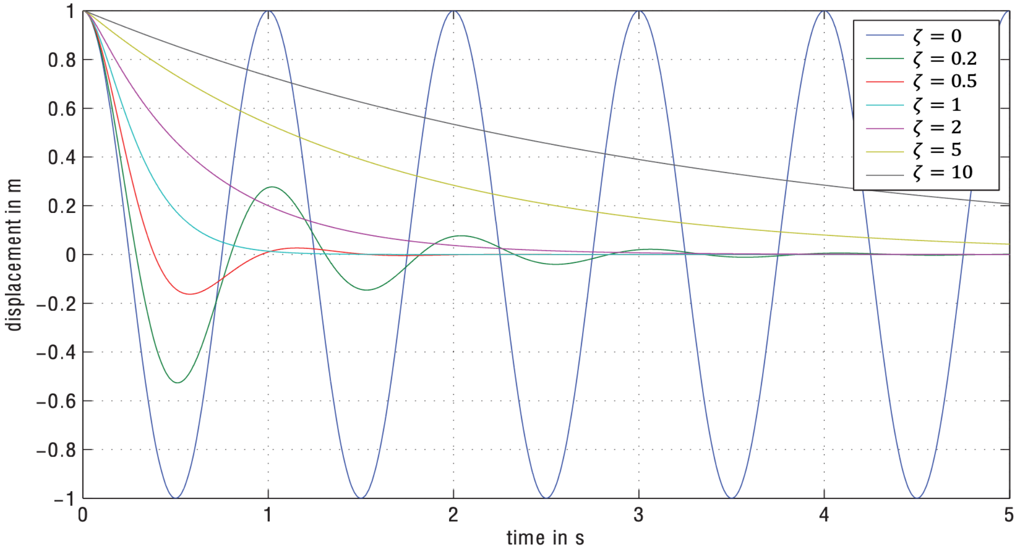


- Equilibrium reached without oscillations, slower than in critically damped case.

Applications:
- systems where oscillations are unacceptable,
- seismic dampers,
- aircraft landing gear.

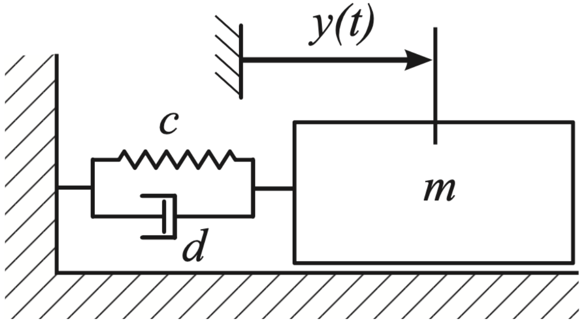
damping:	Solution free vibration:	oscillation:	settling time:	applications
no damping: $\zeta = 0$	$y(t) = C_1 \cos(\omega_\mathrm{n}\, t) + C_2 \sin(\omega_\mathrm{n}\, t)$	Yes	No	Baseline
underdamped: $0 < \zeta < 1$	$y(t) = e^{-\zeta\omega_\mathrm{n} t}[C_1' \cos(\omega_\mathrm{d} t) + C_2' \sin(\omega_\mathrm{d} t)]$	Yes	Medium	Suspensions, machinery, musical instruments
critically damped: $\zeta = 1$	$y(t) = C_1' e^{-\omega_\mathrm{n} t} + C_2' t\, e^{-\omega_\mathrm{n} t}$	No	Minimum possible	Suspensions, measuring instruments, hydraulic doors
overdamped: $\zeta > 1$	$y(t) = C_1 e^{-\zeta\omega_\mathrm{n} + \omega_\mathrm{n}\sqrt{\zeta^2 - 1} t} + C_2 e^{-\zeta\omega_\mathrm{n} - \omega_\mathrm{n}\sqrt{\zeta^2 - 1} t}$	No	Slower than critical	Seismic dampers, aircraft landings

- Damping → reduces amplitude, lowers frequency with respect to undamped free vibration.



Statement: Derive the differential equation for the depicted spring-damper-mass system, determine the homogeneous solution $y_\mathrm{h}(t)$ and adapt it to the given initial conditions y_0 and $\dot{y}_0$. Consider the four different values for the damping c.

given: $\mathrm{c} = 0\ \mathrm{N\,s\,mm^{-1}}$ | $0.1\mathrm{N\,s\,mm^{-1}}$| $0.2\ \mathrm{N\,s\,mm^{-1}}$| $0.3\ \mathrm{N\,s\,mm^{-1}}$
$k = 10\ \mathrm{N\,cm^{-1}}$; $m = 10\ \mathrm{kg}$; $t_0 = 0\ \mathrm{s}$; $y_0 = 1\ \mathrm{mm}$; $\dot{y}_0 = -3\ \mathrm{mm\,s^{-1}}$



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- Moreno-Mateos, M.A. (2025). MorenoMiguelES/CD-computdynamics: Elastodynamics Computer Laboratory, Moreno-Mateos (v1.1). Zenodo. (*https://doi.org/10.5281/zenodo.15771587*).