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Computational dynamics

Lecture 1: 1D Boundary Value Problems (BVP)

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Outline

1. **Preliminaries**
2. **Strong form (S)**
3. **Weak form (W)**
4. **Galerkin method (W^h)**
5. **Matrix form (M)**

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Preliminaries

For scalar-value u and scalar-valued functions $f(x)$ and $g(x)$ in 1D:

- Scalar derivative

$$u_{,x} = \frac{\mathrm{d}u}{\mathrm{d}x} \text{ or } \frac{\partial u}{\partial x}$$

- Fundamental theorem of calculus (extended as Divergence theorem in 3D)

$$\int_{x_i}^{x_j} f_{,x} \, \mathrm{d}x = [f]_{x_i}^{x_j}$$

- Integration by parts

$$\int_{x_i}^{x_j} f_{,x} g \, \mathrm{d}x = [fg]_{x_i}^{x_j} - \int_{x_i}^{x_j} f g_{,x} \, \mathrm{d}x$$

Preliminaries

Mathematical model of continuous system -**Strong form** ($\boldsymbol{S}$)

Weak form ($\boldsymbol{W}$)

Galerkin approximation ($\boldsymbol{W^h}$)

Matrix equations (FE-form) ($\boldsymbol{M}$)

Numerical implementation

Outline

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3. **Weak form (W)**
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Strong form

An initial boundary value problem (IBVP)

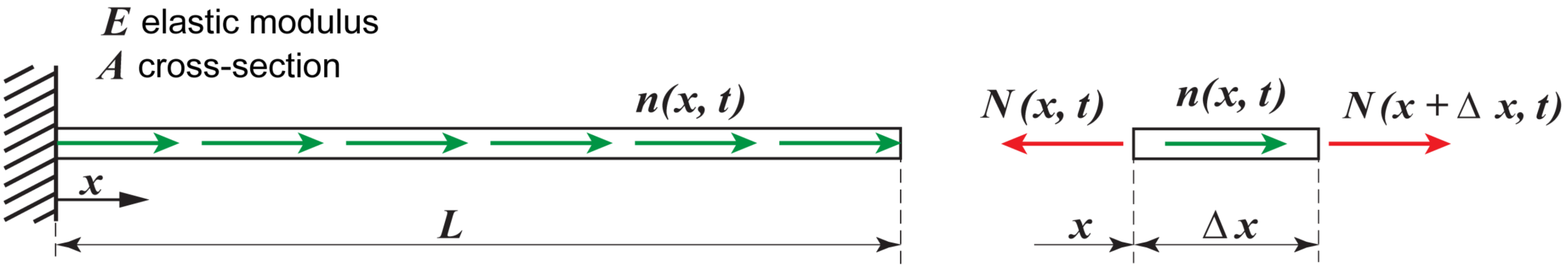
Involves imposing boundary conditions on the function u

It has a unique solution that depends continuously on the initial data and the boundary data.

Mathematical model of continuum system:

$\boxed{\text{Balance Eq.}}$ & $\boxed{\text{Constitutive Eq.}}$ & $\boxed{\text{Kinematics}}$

Strong form: (A) Elastic deformation of a truss



- **Definitions :**

  x position vector of a point.
  $\Omega =]0, L[$ open unit interval(internal domain of x)
  $\overline{\Omega} = [0, L]$ close unit interval (domain of x).
  $n\ (x, t)$ a distributed load in longitudinal direction varying in space and time.
  $N(x, t)$ internal forces.

Strong form: (A) Elastic deformation of a truss

- **Balance equation:**
  Second newton law of motion in x direction $\sum F_x = \Delta m\, a_x$

$$- N(x,t) + N(x+\Delta x, t) + n(x,t)\Delta x = \rho\, A\, \Delta x\, \frac{\partial^2 u}{\partial t^2}$$

- **Constitutive equation:**
  Hooke's law of linear elasticity which defines a linear relation between the stress and the strain. $\sigma(x) = E\,\varepsilon(x)$ $\quad \sigma(x) = \frac{N(x)}{A}$

$$\frac{\partial N(x,t)}{\partial x} = -n(x,t) + \rho\, A\, \frac{\partial^2 u}{\partial t^2} \qquad \Longrightarrow \qquad E\, A\, \frac{\partial \varepsilon(x)}{\partial x} = -n(x,t) + \rho\, A\, \frac{\partial^2 u}{\partial t^2}$$

Strong form: (A) Elastic deformation of a truss

- **Kinematics:**
  relation between the strain ε and the displacement u

$$\varepsilon(x) = \frac{\partial u}{\partial x}$$

$$E\,A\,\frac{\partial \varepsilon(x)}{\partial x} = -n(x,t) + \rho\,A\,\frac{\partial^2 u}{\partial t^2} \qquad \Longrightarrow \qquad E\,A\,\frac{\partial^2 u}{\partial^2 x} = -n(x,t) + \rho\,A\,\frac{\partial^2 u}{\partial t^2}$$

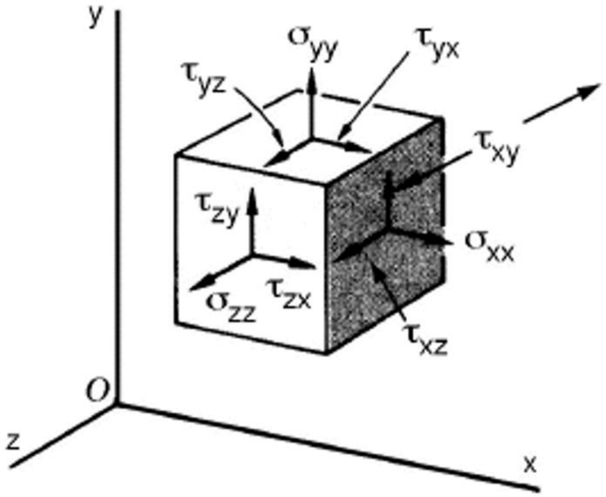
$$\boxed{\rho\,A\,\frac{\partial^2 u(x,t)}{\partial t^2} - E\,A\,\frac{\partial^2 u(x,t)}{\partial^2 x} - n(x,t) = 0 \qquad 0 < x < L}$$

Second order hyperbolic PDE

Boundary Conditions (BC): $u(0,t) = 0 \qquad u(x,0) = u_0$

… and how does the mechanical balance field equation in 3D look like?

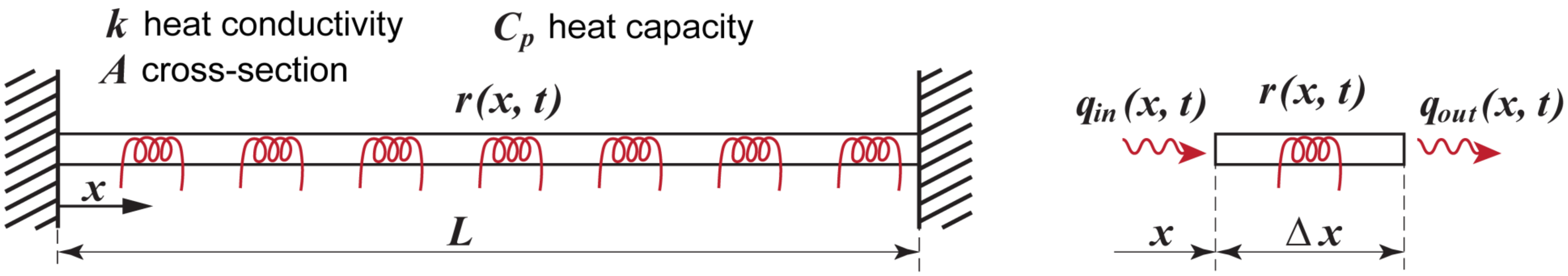
$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \mathbf{f} = \rho \ddot{\mathbf{u}}$$



$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

Prove: balance of forces on differential cube

Strong form: (B) Heat equation



- **Definitions:**

$q_{in}(x,t)$ heat flux, rate of energy entering the domain in $\mathrm{Js^{-1}m^{-2}}$

$q_{out}(x,t)$ heat flux, rate of energy outgoing the domain in $\mathrm{Js^{-1}m^{-2}}$

$r(x,t)$ heat source, rate of energy generation in $\mathrm{Js^{-1}m^{-3}}$

ΔE rate of energy accumulation in $\mathrm{Js^{-1}}$

Strong form: (B) Heat equation

- **Balance equation:**

Balance of energy,

rate of energy accumulation = rate of energy entering the domain – rate of energy outgoing the domain + rate of energy generation in the domain

$$\Delta E = q_{in}(x,t)\,A - q_{out}(x,t)\,A + r(x,t)\,A\,\Delta x$$

$$\dot{E} = q(x,t)\,A - q(x,t)\,A - dq(x,t)\,A + r(x,t)\,A\,dx$$

$$\boxed{\rho\,C_p\frac{\partial T}{\partial t} = -\frac{\partial q(x,t)}{\partial x} + r(x,t)}$$

$$\boxed{\dot{E} = \rho\,C_p\frac{\partial T}{\partial t}\,A\,dx}$$

Strong form: (B) Heat equation

- **Constitutive equation:**
  Fourier's law

$$q = -k\frac{\partial T}{\partial x}$$

k : heat conductivity

$$\rho\, C_p \frac{\partial T}{\partial t} = -\frac{\partial q(x,t)}{\partial x} + r(x,t) \quad \Longrightarrow \quad \rho\,\frac{C_p}{k}\frac{\partial T}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2} + \frac{r(x,t)}{k}$$

$$\boxed{\rho\,\frac{C_p}{k}\frac{\partial T}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2} + \frac{r(x,t)}{k}}$$

Second order parabolic PDE

Boundary Conditions (BC): $T(0,t) = 0$

Initial Conditions (BC): $T(x,0) = T_0$

… and how does the heat balance field equation in 3D look like?

$$\rho C \frac{\partial \Theta}{\partial t} + \nabla \cdot \mathbf{q} = f$$

Prove: conservation of energy in differential cube

Input + Source = Output + Rate of accumulation

Θ: temperature
t: time
f: heat source
$\mathbf{q}$: heat vector
ρ: density $[\mathrm{ML}^{-3}]$
C: heat capacity $[\mathrm{L}^2\ \mathrm{M}\ \mathrm{T}^{-2}\ \Theta^{-1}]$

Strong form

- Two solution approaches

FDM FVM FEM

requires different formulation

ANALYTICAL

Advantage:

- Accurate.

Disadvantage:

- Difficult to implement (sometimes impossible).
- Solvable by hand.

NUMERICAL

Advantage:

- Applicable to complex problems.
- Solvable by computer.

Disadvantage:

- Approximated approach (less accurate).
- The accuracy depends on numerous factors.

Outline

Motivation Weak form

- Relax the requirement of strong differentiability for the solution field u.
- Integration by parts → transfer derivatives from u (solution) to a weighting function w (test function).
- **The solution of the weak form** (W) **needs not be as smooth as the solution of the strong form** (S), i.e., it has weaker continuity requirements. → Make the solution u more amenable to weak solutions that might not have strong derivatives.
  - A strong solution automatically satisfies the weak form, because it is regular enough to perform all the necessary integrations and manipulations, i.e., $\boldsymbol{S} \rightarrow \boldsymbol{W}$.
  - However, a solution to the weak form does not always have enough continuity to have strong derivatives to satisfy the strong solution unless it has sufficient regularity to make the weak derivatives classical (strong) derivatives. Only then, $\boldsymbol{W} \rightarrow \boldsymbol{S}$.
- Other names: Principle of virtual works.

Weak form

- The weak form (W) is the variational counterpart of the strong form (S).
- Before deriving the weak form, we introduce a test function w.
- **Arbitrary** $w \leftrightarrow$ strong form and weak form are equivalent. [Approximation: later with the Galerkin method].
- Then, we need to characterize two classes of functions:

$$\mathcal{S} = \{u \,|\, u \in \mathcal{H}^1 \,,\, u(l) = q\}$$

trial space

Homogeneous BCs for test function

$$\mathcal{V} = \{w \,|\, w \in \mathcal{H}^1 \,,\, w(l) = 0\}$$

test space

the same continuity requirements hold for u and w

$$\mathcal{H}^1 = \left\{ u : [0,l] \to \mathbb{R} \,\middle|\, \int_0^l u^2 + (u_{,x})^2 \, dx < \infty \right\}$$

Sobolev space

Square-integrable derivatives

Weak form

$$\mathcal{H}^1 = \left\{ u : [0,l] \rightarrow \mathbb{R} \,\middle|\, \int_0^l u^2 + (u_{,x})^2 \, dx < \infty \right\}$$

<u>Sobolev spaces</u>:

- In real problems, functions may not be smooth and have well-defined derivatives. Sobolev spaces provide a way to extend the concept of derivatives to functions that are not necessarily smooth in the traditional sense.
- Functions and their derivatives up to a certain order are “well-behaved” according to the norm used. The metric of the space requires them to have finite integrals.
- $\mathcal{H}^1$ functions are smooth enough to ensure that the weak derivatives used in the variational formulation exist and are well behaved but not so smooth as to make the numerical approximation computationally expensive.

Weak form

- To derive the weak form we follow the steps:

1. **Multiply the PDE by test w function** and **integrate over the domain**

$$-\int_0^l w(\,u_{,xx} + g)\,dx = 0 \qquad\qquad \int_0^l -\,w\,u_{,xx}\,dx = \int_0^l w\,g\,dx$$

2. Apply integration by parts

$$-w\,u_{,x}\big|_0^l + \int_0^l w_{,x}\,u_{,x}\,dx = \int_0^l w\,g\,dx$$

3. Rearrange and use the BCs on the boundary term

$$\overbrace{-w(l)\,u(l)_{,x}}^{\partial\Omega_\mathrm{D}} + \overbrace{w(0)\,u(0)_{,x}}^{\partial\Omega_\mathrm{N}} + \int_0^l w_{,x}\,u_{,x}\,dx = \int_0^l w\,g\,dx$$

$0 \qquad\qquad -h$

Test function vanishes at Dirichlet boundary

$$\int_0^l w_{,x}\,u_{,x}\,dx = \int_0^l w\,g\,dx + w(0)\,h$$

Due to the second derivative of u(x), very smooth trial solutions would be needed. Furthermore, the resulting stiffness matrix would not be symmetric. Thus, **we transform it into a form containing only first derivatives, allowing less smooth solutions.**

Another advantage: **Traction boundary conditions emanate naturally from the weak form** due to integration by parts.

Weak form

Note: $\partial\Omega_\mathrm{D} \cap \partial\Omega_\mathrm{N} = \emptyset$: either essential or natural BCs.

essential BCs

$$(W) = \begin{cases} \text{let} \quad \mathcal{S} = \{u \,|\, u \in \mathcal{H}^1 \,,\, u(l) = q\} \quad \text{and} \quad \mathcal{V} = \{w \,|\, w \in \mathcal{H}^1 \,,\, w(l) = 0\}\,. \\ \text{Given } g : [0,l] \to \mathbb{R} \text{ and constants } q \text{ and } h \text{, find } u \in \mathcal{S}, \text{ such that for all } w \in \mathcal{V} \\ \displaystyle\int_0^l w_{,x}\, u_{,x}\, dx = \int_0^l w\, g\, dx + w(0)\, h \end{cases}$$

natural BCs

- ***Essential boundary conditions***: the primary field, e.g., displacement, is essential for the trial solutions. The trial solution $u(x)$ must satisfy the essential BCs.
- ***Natural boundary conditions***: these emanate naturally from the weak form, so trial solutions need not be constructed to satisfy traction BC. Related to first derivative of the field (e.g., mechanical traction forces).

- Admissible trial solution: smooth and **satisfies essential (Dirichlet) boundary conditions, i.e., in $\partial\Omega_\mathrm{D}$**.
- Admissible test function: smooth and **vanishes on essential (Dirichlet) boundaries, i.e., in $\partial\Omega_\mathrm{D}$**.

Weak form

The strong and weak forms are equivalent

$$(S) \overset{}{\underset{(*)}{\iff}} (W)$$

If u satisfies the strong form, then it satisfies the weak form and vice versa.

(*) $W \rightarrow S$ only if solution to W has sufficient regularity.

Weak form

$$a(w,u) = \int_0^l w_{,x}\, u_{,x}\, dx \qquad (w,g) = \int_0^l w\, g\, dx \qquad \boxed{a(w,u) = (w,g) + w(0)\, h}$$

***Symmetric* - bilinear form**

$$a(w,u) = a(u,w) \qquad a(c_1\, u + c_2\, v, w) = c_1\, a(u,w) + c_2\, a(v,w)$$

$$(w,v) = (v,w) \qquad (c_1\, u + c_2\, v, w) = c_1\, (u,w) + c_2\, (v,w)$$

Where c_1 and c_2 are consant, and u , v , and w are scalar-valued functions.

Outline

Galerkin methods – introduction & motivation

They **convert a continuous operator problem**, such as differential equation, commonly in a weak formulation, **to a (spatial) discrete problem** by applying linear constraints determined by **finite sets of basis functions**.

Galerkin methods – introduction & motivation

- Finite-dimensional approximation of the trial and test function spaces.
- In the FEM, the weight functions and trial solutions are constructed by subdividing the domain of the problem into elements and constructing functions within each elements.
- The functions within the elements are carefully constructed so that the method is convergent, i.e., the accuracy improves with mesh refinement.
- The **trial solutions and weight functions are linear combinations of nodal values and a set of basis functions in the subspaces $\mathcal{S}^h$ and $\mathcal{V}^{\mathrm{h}}$**.
- Reducing the problem to finite-dimensional vector subspaces allows to numerically compute u^{h} as a finite linear combination of the basis vectors in $\mathcal{S}^{\mathrm{h}}$.

Galerkin method

- Let us consider a collection of a finite-number of functions $\mathcal{S}^h$ and $\mathcal{V}^h$ that belong to $\mathcal{S}$ and $\mathcal{V}$, respectively.

finite dimensional functional spaces

$$u^h \in \mathcal{S}^h \subset \mathcal{S} \qquad w^h \in \mathcal{V}^h \subset \mathcal{V}$$

Referes to the mesh or discretization of the domain

Linear space

$$w_1 \text{ and } w_2 \in \mathcal{V} \Rightarrow w_3 = w_1 + w_2 \in \mathcal{V}$$

$$u_1 \text{ and } u_2 \in \mathcal{S} \Rightarrow u_3 = u_1 + u_2 \notin \mathcal{S}$$

$$u_3(l) = u_1(l) + u_2(l) = 2q$$

Due to inhomogeneous BCs

$$\mathcal{S} = \{u \,|\, u \in \mathcal{H}^1 \,,\, u(l) = q\}$$

$$\mathcal{V} = \{w \,|\, w \in \mathcal{H}^1 \,,\, w(l) = 0\}$$

Infinite dimensional functional spaces

Galerkin method

- Galerkin solution: reconstruct the trial solution as

$$\boxed{u^h \in \mathcal{S}^h} \qquad \boxed{w^h \in \mathcal{V}^h}$$

$$u^h = v^h + q^h \qquad \text{where} \quad q^h(l) = q \Rightarrow q^h \in \mathcal{S}^h$$

unknown   known (essential BCs)

Further, homogeneous BCs for test function w^h

$$u^h(l) = v^h(l) + q^h(l) = 0 + q = q$$

Trial solution (u): must satisfy (Dirichlet) essential BCs.

The key point to observe that, up to the function q^h , $\mathcal{S}^h$ **and** $\mathcal{V}^h$ **are composed of identical collections of functions.**

Galerkin method

- Write the Weak form in terms of $u^h \in \mathcal{S}^h$ and $w^h \in \mathcal{V}^h$

$$a(w,u) = (w,g) + w(0)\,h \qquad \equiv \qquad \int_0^l w_{,x}\,u_{,x}\,dx = \int_0^l w\,g\,dx + w(0)\,h$$

$$a(w^h,u^h) = (w^h,g) + w^h(0)\,h$$

$$a(w^h,v^h+q^h) = (w^h,g) + w^h(0)\,h$$

Bilinear property

$$a(w^h,v^h) + a(w^h,q^h) = (w^h,g) + w^h(0)\,h$$

$$a(w^h,v^h) = (w^h,g) + w^h(0)\,h - a(w^h,q^h)$$

Known

Galerkin method

$$
(W^h) = \begin{cases} \text{let } \mathcal{V}^h \subset \mathcal{V} = \{w \,|\, w \in \mathcal{H}^1\,,\ w(l) = 0\} \text{ and } \mathcal{S}^h \subset \mathcal{S} = \{u \,|\, u \in \mathcal{H}^1\,,\ u(l) = q\}\,. \\ \text{Given } g : [0,l] \to \mathbb{R} \text{ and constants } q \text{ and } h\,, \text{ find } u^h = v^h + q^h, \text{where } v^h \in \mathcal{V}^h \\ \text{and } u^h \in \mathcal{S}^h, \text{ such that for all } w^h \in \mathcal{V}^h \\ \displaystyle\int_0^l w^h_{,x}\, v^h_{,x}\, dx = \int_0^l w^h\, g\, dx + w^h(0)\, h - \int_0^l w^h_{,x}\, q^h_{,x}\, dx \end{cases}
$$

An **approximated** version of (W) posed in terms of finite-dimensional collections of functions, namely, $\mathcal{V}^h$

$$
(S) \overset{(*)}{\Longleftrightarrow} (W) \approx (W^h) \qquad u^h \in \mathcal{S}^h \subset \mathcal{S} \qquad w^h \in \mathcal{V}^h \subset \mathcal{V}
$$

(*) $W \to S$ only if solution to W has sufficient regularity.

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Matrix form

- Let $\mathcal{V}^h$ consist of all linear combinations of given functions denoted by

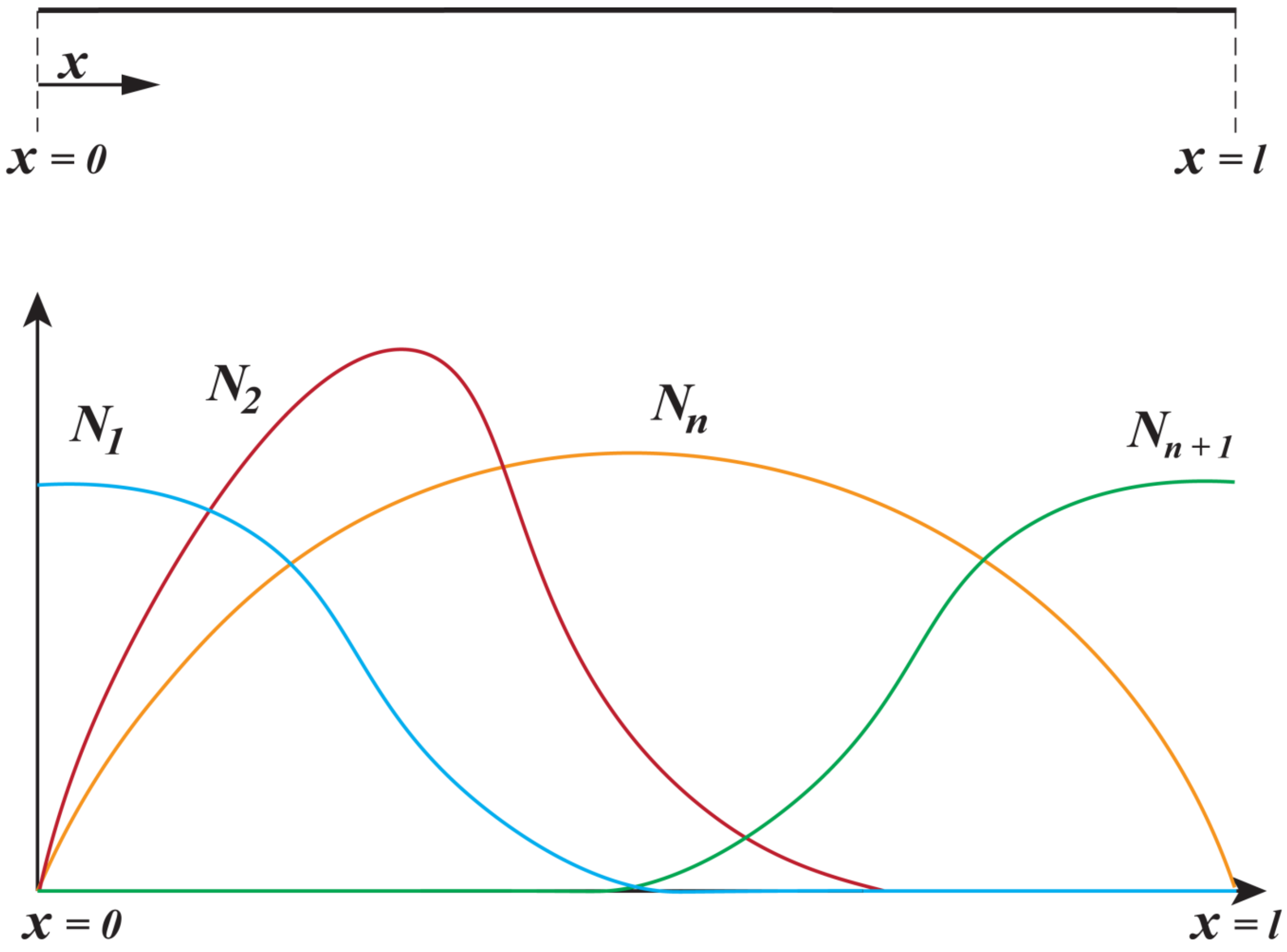
$$N_A : \overline{\Omega} \to \mathbb{R} \; ([0,l] \to \mathbb{R}) \quad \text{where} \quad A = 1, 2, \ldots\ldots, n, n+1$$

Matrix form

$$N_A : \overline{\Omega} \to \mathbb{R} \; ([0, l] \to \mathbb{R})$$

where $\quad A = 1, 2, \ldots\ldots, n, n+1$

Chosen Shape, basis, interpolation functions associated with node A



- Let $\mathcal{V}^h$ consist of all linear combinations of given functions denoted by

$$N_A : \overline{\Omega} \to \mathbb{R} \; ([0, l] \to \mathbb{R}) \quad \text{where} \quad A = 1, 2, \ldots\ldots, n, n+1$$

$$w^h = \sum_{A=1}^{n} c_A \, N_A$$

Unknown constants

$$\left.\begin{aligned} v^h &= \sum_{A=1}^{n} d_A \, N_A \\ q^h &= q \, N_{n+1} \end{aligned}\right\} \; u^h = \sum_{A=1}^{n} d_A \, N_A + q \, N_{n+1} = \sum_{A=1}^{n+1} d_A \, N_A \quad \text{with} \quad d_{n+1} = q$$

Known BCs

- **Same shape (basis) functions N_A for both test and trial functions** → symmetric stiffness matrix.
- Also other alternative choices.

Matrix form

- Define the finite-dimensional spaces $\mathcal{S}^h$ and $\mathcal{V}^h$ in terms of N_A

$$\mathcal{V}^h = \left\{ w^h \,\middle|\, w^h = \sum_{A=1}^{n} c_A\, N_A\,,\; w^h(l) = 0,\, N_A : [0,l] \to \mathbb{R},\; \forall\, c_A \in \mathbb{R}^n \right\}$$

$$\mathcal{S}^h = \left\{ u^h \,\middle|\, u^h = \sum_{A=1}^{n+1} d_A\, N_A\,,\; u^h(l) = q,\, N_A : [0,l] \to \mathbb{R},\; \forall\, d_A \in \mathbb{R}^{n+1} \right\}$$

$$N_{n+1} \in \mathcal{S}^h \quad \text{and} \quad N_{n+1} \notin \mathcal{V}^h$$

$$u^h \in \mathcal{S}^h \subset \mathcal{S} \qquad\qquad w^h \in \mathcal{V}^h \subset \mathcal{V}$$

Matrix form

- Write the approximated Weak form in terms of N_A

$$w^h = \sum_{A=1}^{n} c_A \, N_A \qquad v^h = \sum_{A=1}^{n} d_A \, N_A \qquad q^h = q \, N_{n+1}$$

$$a(w^h, v^h) = (w^h, g) + w^h(0)\, h - a(w^h, q^h)$$

$$a(N_A, N_B) = \int_0^l (N_A)_{,x}(x)\,(N_B)_{,x}(x)\,dx$$

$$(N_A, g) = \int_0^l N_A(x)\, g\, dx$$

Matrix form

- Write the approximated Weak form in terms of N_A

$$\sum_{A=1}^{n}\sum_{B=1}^{n} c_A\, a(N_A, N_B) d_B = \sum_{A=1}^{n} c_A\, (N_A, g) + \sum_{A=1}^{n} c_A\, N_A(0)\, h - \sum_{A=1}^{n} c_A\, a(N_A, N_{n+1})\, q$$

Since the c_A 's are arbitrary. This equation must to hold for all c_A , $A = 1, 2, \ldots\ldots, n, n+1$.

$$\sum_{B=1}^{n} a(N_A, N_B) d_B = (N_A, g) + N_A(0)\, h - a(N_A, N_{n+1})\, q \qquad \forall\; A = 1, 2, 3, \ldots\ldots, n$$

Solve for unknown constants (dofs) d_B

System of n equations with n unknowns

Matrix form

- System of equations

$$\sum_{B=1}^{n} \underbrace{a(N_A, N_B)}_{K_{AB}} d_B = \underbrace{(N_A, g) + N_A(0)\, h - a(N_A, N_{n+1})\, q}_{F_A} \qquad \forall\, A = 1, 2, 3, \ldots.., n$$

$$\sum_{B=1}^{n} K_{AB}\, d_B = F_A \qquad \forall\, A = 1, 2, 3, \ldots.., n$$

$$\boldsymbol{K} \cdot \boldsymbol{d} = \boldsymbol{F}$$

Matrix form

- System of equations $\boldsymbol{K} \cdot \boldsymbol{d} = \boldsymbol{F}$

$$\boldsymbol{K} = [K_{AB}] = \begin{Bmatrix} K_{11} & K_{12} & . & . & . & K_{1n} \\ K_{21} & K_{22} & . & . & . & K_{2n} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ K_{n1} & K_{n2} & . & . & . & K_{nn} \end{Bmatrix} = \begin{Bmatrix} a(N_1, N_1) & a(N_1, N_2) & . & . & . & a(N_1, N_n) \\ a(N_2, N_1) & a(N_2, N_2) & . & . & . & a(N_2, N_n) \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ a(N_n, N_1) & a(N_n, N_2) & . & . & . & a(N_n, N_n) \end{Bmatrix}$$

$\boldsymbol{K}$ Stiffness matrix

$\boldsymbol{d}$ Displacement vector

$\boldsymbol{F}$ Force vector

$$\boldsymbol{F} = [F_B] = \begin{Bmatrix} F_1 \\ F_2 \\ . \\ . \\ . \\ F_n \end{Bmatrix} = \begin{Bmatrix} (N_1, g) + N_1(0)\, h - a(N_1, N_{n+1})\, q \\ (N_2, g) + N_2(0)\, h - a(N_2, N_{n+1})\, q \\ . \\ . \\ . \\ (N_n, g) + N_n(0)\, h - a(N_n, N_{n+1})\, q \end{Bmatrix} \qquad \boldsymbol{d} = [d_B] = \begin{Bmatrix} d_1 \\ d_2 \\ . \\ . \\ . \\ d_n \end{Bmatrix}$$

Matrix form

$$(M) = \begin{cases} \text{Given the cofficient matrix } \boldsymbol{K} \text{ and vectro } \boldsymbol{F}\text{, find } \boldsymbol{d} \text{ such that} \\ \qquad\qquad \boldsymbol{K} \cdot \boldsymbol{d} = \boldsymbol{F} \end{cases}$$

$$(S) \iff (W) \approx (W^h) \iff (M)$$

The only apparent approximation made thus far is in approximately solving (W) via (W^h)

However, encountered in practice, the number of approximations increases.

For example, the data g , q , and h maybe approximated, as well as the domain, calculation of integration, and so on.

$$(S) = \begin{cases} \text{Given } g:[0,l]\to\mathbb{R} \text{ and constants } q \text{ and } h\text{, find } u:[0,l]\to\mathbb{R}\text{, such that} \\ \qquad u_{,xx}+g=0 \qquad \text{on} \quad]0,l[\\ \qquad u(l)=q \\ \qquad -u_{,x}(0)=h \end{cases}$$

$$(W) = \begin{cases} \text{let} \quad \mathcal{S}=\{u\,|\,u\in\mathcal{H}^1\,,\,u(l)=q\} \quad \text{and} \quad \mathcal{V}=\{w\,|\,w\in\mathcal{H}^1\,,\,w(l)=0\}\,. \\ \text{Given } g:[0,l]\to\mathbb{R} \text{ and constants } q \text{ and } h\text{, find } u\in\mathcal{S}\text{, such that for all } w\in\mathcal{V} \\ \qquad \int_0^l w_{,x}\,u_{,x}\,dx = \int_0^l w\,g\,dx + w(0)\,h \end{cases}$$

$$(W^h) = \begin{cases} \text{let } \mathcal{V}^h\subset\mathcal{V}=\{w\,|\,w\in\mathcal{H}^1\,,\,w(l)=0\} \text{ and } \mathcal{S}^h\subset\mathcal{S}=\{u\,|\,u\in\mathcal{H}^1\,,\,u(l)=q\}\,. \\ \text{Given } g:[0,l]\to\mathbb{R} \text{ and constants } q \text{ and } h\text{, find } u^h=v^h+q^h\text{, where } v^h\in\mathcal{V}^h \\ \text{and } u^h\in\mathcal{S}^h\text{, such that for all } w^h\in\mathcal{V}^h \\ \qquad \int_0^l w^h_{,x}\,v^h_{,x}\,dx = \int_0^l w^h\,g\,dx + w^h(0)\,h - \int_0^l w^h_{,x}\,q^h_{,x}\,dx \end{cases}$$

$$(M) = \begin{cases} \text{Given the cofficient matrix } \boldsymbol{K} \text{ and vectro } \boldsymbol{F}\text{, find } \boldsymbol{d} \text{ such that} \\ \qquad \boldsymbol{K}\cdot\boldsymbol{d}=\boldsymbol{F} \end{cases}$$

$$(S) \quad \Updownarrow \text{(*)} \quad (W) \quad \approx \quad (W^h) \quad \Updownarrow \quad (M)$$

(*) $W \to S$ only if solution to W has sufficient regularity.

Bibliography:

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