Prerequisites

- Syntax and semantics of first-order languages.
- Elementary equivalence and elementary substructures.
- Isomorphisms and embeddings between structures.

Exercise 1. Let M be an L-structure and let $\psi(x), \varphi(x,y) \in L$. For each of the following conditions, write a sentence true in M exactly when the condition holds:

a. $\psi(M) \in \{\varphi(a,M) : a \in M\}$;

b. $\{\varphi(a,M) : a \in M\}$ contains at least two elements;

c. $\{\varphi(a,M) : a \in M\}$ contains only sets that are pairwise disjoint.

Remark. Recall that $\varphi(a,M) = \{b \ : \ M \models \varphi(a,b)\}$. Do not confuse definable subsets of M with *sets* of definable subsets of M. Beware that the relation between parameters and the sets they define is not one-to-one.

Exercise 2. Let M be a structure in a signature with only a binary relation symbol r. Write a sentence φ such that, for every structure M,

a. $M \models \varphi$ if and only if there is an $A \subseteq M$ such that $r^M \subseteq A \times \neg A$.

Remark. Note that φ asserts that r^M is a bipartite directed graph. As motivation, it is interesting to note that a is an asymmetric version of

b. $M \models \psi$ if and only if there is an $A \subseteq M$ such that $r^M \subseteq (A \times \neg A) \cup (\neg A \times A)$.

The sentence ψ says that the (undirected) graph naturally associated to r^M is a *bipartite graph*, or equivalently that its *chromatic number is* 2 (its vertices are colorable with 2 colors so that any two adjacent vertices have different colors). The compactness theorem can be used to prove that there is no such sentence ψ (not required).

Exercise 3. Prove that $M \equiv_A N$ if and only if $M \equiv_B N$ for every finite $B \subseteq A$.

Remark. This is called the finite nature (or character) of elementary equivalence.

Exercise 4. Prove that in the language of orders $(0,1) \not\equiv (0,2]$ and $(0,1) \preceq (0,2)$.

Remark. Assume the intervals above are all rational, or all real. Give a short proof without assuming elimination of quantifiers. Exercise 3 above may be useful.

Exercise 5. Let $M \preceq N$ and let $\varphi(x,z) \in L$. Suppose there are finitely many sets of the form $\varphi(a,N)$ for some $a \in N^{|x|}$. Prove that all these sets are definable over M.

Hint. Exercise 1 above may be useful.

Exercise 6. Prove that if $A_1, \ldots, A_n \subseteq \{1, \ldots, m\}$ are distinct sets such that $A_i \cap A_j \neq \varnothing$ for all i, j, then $n \leq 2^{m-1}$.

Hint. The proof is two lines.

Remark. This exercise is not directly related to model theory. However, basic combinatorial principles are pivotal in model theory, as they are used to distinguish first-order structures according to their complexity.

(Péter Frankl credits this exercise to Katona who used it to advertise his combinatorics seminar by saying: "if you can solve this exercise, come to the seminar".)