

**Esercizio 1.** Prove that if  $T$  has exactly 2 maximal consistent extension  $T_1$  and  $T_2$  then there is a sentence  $\varphi$  such that  $T, \varphi \vdash T_1$  and  $T, \neg\varphi \vdash T_2$ .

**Esercizio 2.** Assume  $L$  is countable and let  $M \preceq N$  have arbitrary (large) cardinality. Let  $A \subseteq N$  be countable. Prove that there is a countable model  $K$  such that  $A \subseteq K \preceq N$  and  $K \cap M \preceq N$  (in particular,  $K \cap M$  is a model). Assume  $L$  is countable and let  $M \subseteq N$  and  $A \subseteq N$  be both countable. Prove that there is a countable model  $K$  such that  $A \subseteq K \preceq N$  and  $K \cap M \preceq N$  (in particular,  $K \cap M$  is a model).

Hint 1: adapt the construction used to prove the downward Löwenheim-Skolem.

Hint 2 (alternative construction): construct two countable chains of countable models such that  $K_i \cap M \subseteq M_i \preceq N$  and  $A \cup M_i \subseteq K_{i+1} \preceq N$ . The required model is  $K = \bigcup_{i \in \omega} K_i$ . In fact, it is easy to check that  $K \cap M = \bigcup_{i \in \omega} M_i \preceq N$ .

**Esercizio 3.** Let  $L$  be the language of strict orders augmented with countably many constants  $\{c_i : i \in \omega\}$ . Let  $T$  be the theory that extends  $T_{\text{dlo}}$  with the axioms  $c_i < c_{i+1}$  for all  $i$ . Find 3 non isomorphic countable models of  $T$ , say  $N_i$  for  $i = 1, 2, 3$ . (One could prove that  $T$  is complete and up to isomorphism has exactly 3 countable models.) Say for which model  $N_i$  an extension lemma similar to 5.1 holds with  $N = N_i$  and  $M \models T$ .