**Esercizio 1.** Prove that if $T$ has exactly 2 maximal consistent extension $T_1$ and $T_2$ then there is a sentence $\varphi$ such that $T, \varphi \vdash T_1$ and $T, \neg\varphi \vdash T_2$.

**Esercizio 2.** Assume $L$ is countable and let $M \preceq N$ have arbitrary (large) cardinality. Let $A \subseteq N$ be countable. Prove that there is a countable model $K$ such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model). Assume $L$ is countable and let $M \subseteq N$ and $A \subseteq N$ be both countable. Prove that there is a countable model $K$ such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model).

Hint 1: adapt the construction used to prove the downward Löwenheim-Skolem.

Hint 2 (alternative construction): construct two countable chains of countable models such that $K_i \cap M \subseteq M_i \preceq N$ and $A \cup M_i \subseteq K_{i+1} \preceq N$. The required model is $K = \bigcup_{i\in\omega} K_i$. In fact, it is easy to check that $K \cap M = \bigcup_{i\in\omega} M_i \preceq N$.

**Esercizio 3.** Let $L$ be the language of strict orders augmented with countably many constants $\{c_i : i \in \omega\}$. Let $T$ be the theory that extends $T_{\mathrm{dlo}}$ with the axioms $c_i < c_{i+1}$ for all $i$. Exhibit 3 non isomorphic countable models of $T$, say $N_i$ for $i = 1, 2, 3$. (One could prove that $T$ is complete and up to isomorphism has exactly 3 countable models.) Say for which model $N_i$ an extension lemma similar to 5.1 holds with $N = N_i$ and $M \models T$.