Esercizio 1. Let $p(x) \subseteq L(A)$, with $|x| < \omega$. Prove that if $p(\mathcal{U})$ is infinite then it has cardinality κ. Show that this may not be true for all $p(x) \subseteq L(\mathcal{U})$.

Esercizio 2. Let $\varphi(x, y) \in L(\mathfrak{U})$. Prove that the following are equivalent

1. there is a sequence $\langle a_i : i \in \omega \rangle$ such that $\varphi(\mathfrak{U}, a_i) \subset \varphi(\mathfrak{U}, a_{i+1})$ for every $i < \omega$;
2. there is a sequence $\langle a_i : i \in \omega \rangle$ such that $\varphi(\mathfrak{U}, a_{i+1}) \subset \varphi(\mathfrak{U}, a_i)$ for every $i < \omega$.

Esercizio 3. Suppose $|L| \leq \omega$ and let M be an infinite structure. Then for every non-principal ultrafilter F on ω the structure M^ω/F is a ω_1-saturated elementary superstructure of M.

Suggerimento: si espanda il suggerimento sulle dispense.

Esercizio 1. Let $a \in \mathcal{U}$ be such that $\mathcal{O}(a/A) = \{a\}$. Prove that there is a formula $\varphi(x) \in L(A)$ such that $\varphi(a) \wedge \exists^{=1} x\, \varphi(x)$.

Suggerimento: si ricordi che $\mathcal{O}(a/A) = p(\mathcal{U})$ per $p(x) = \mathrm{tp}(a/A)$.

Esercizio 2. Let $\varphi(x) \subseteq L(\mathfrak{U})$ be such that $\{f[\varphi(\mathfrak{U})] \ : \ f \in \mathrm{Aut}(\mathfrak{U}/A)\}$ contains exactly 2 elements. Prove $\varphi(\mathfrak{U})$ is definable over *any* model M containing A.

Esercizio 3. Let $p(x) \subseteq L(A)$, with $|x| < \omega$. Prove that if $p(\mathcal{U})$ is infinite then it has cardinality κ. Show that this may not be true if x is an infinite tuple.

Suggerimento: è sufficiente ci sia un insieme definibile con esattamente due elementi.

Esercizio 1. Let $p(x) \subseteq L$ be such that $p(\mathcal{U})$ contains just one element. Prove that there is a formula $\varphi(x)$, a conjunctions of formulas in $p(x)$, such that $p(\mathcal{U}) = \varphi(\mathcal{U})$.

Esercizio 2. Let $p(x) \subseteq L(A)$ be such that $\neg p(x) \leftrightarrow q(x)$ for some $q(x) \subseteq L(B)$. Prove that $p(x)$ is equivalent to some conjunction of formulas in $p(x)$.

Esercizio 3. Let $\varphi(x, y) \in L(\mathfrak{U})$. Prove that the following are equivalent

1. there is a sequence $\langle a_i : i \in \omega \rangle$ such that $\varphi(\mathfrak{U}, a_i) \subset \varphi(\mathfrak{U}, a_{i+1})$ for every $i < \omega$;
2. there is a sequence $\langle a_i : i \in \omega \rangle$ such that $\varphi(\mathfrak{U}, a_{i+1}) \subset \varphi(\mathfrak{U}, a_i)$ for every $i < \omega$.

Esercizio 1. Let M and N be elementarily homogeneous structures of the same cardinality λ. Suppose that $M \models \exists x\, p(x) \Leftrightarrow N \models \exists x\, p(x)$ for every $p(x) \subseteq L$ such that $|x| < \lambda$. Prove that the two structures are isomorphic.

Suggerimento: vedi dimostrazione di *universale* + *homogeneo* $\Rightarrow$ *ricco*

Esercizio 2. Let $\varphi(x;z) \in L$. Prove that if the set $\{\varphi(a;\mathcal{U}) \ : \ a \in \mathcal{U}^{|x|}\}$ is infinite then it has cardinality κ. Does the claim remains true with a type $p(x;z) \subseteq L$ for $\varphi(x;z)$?

Suggerimento: potrebbe esserci un controesempio in $\mathcal{U} \equiv \mathbb{N}$ nel linguaggio degli ordini.

Esercizio 3. Let $\varphi(x) \subseteq L(\mathcal{U})$ be such that $\{f[\varphi(\mathcal{U})] \ : \ f \in \mathrm{Aut}(\mathcal{U}/A)\}$ contains exactly 2 elements. Prove $\varphi(\mathcal{U})$ is definable over *any* model M containing A.