NeuroMat

Time averages of a metastable system of spiking neurons

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Overview

1. **Metastability**

2. **The Model and the Result.**

3. **The Auxiliary Interacting Particle System.**

4. **Back to the Model.**

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Metastability

What is Metastability in Physics

In the context of physics, a metastable system is a system which is in a precarious equilibrium, with several available stable or quasi-stable states (e.g. liquid, solid and gas for water), and which can be taken out of its current pseudo-equilibrium, and pushed toward on of the other states, by some minor external or internal perturbation.

Paradigmatic examples are: Supercooling water, avalanche, radioactivity...

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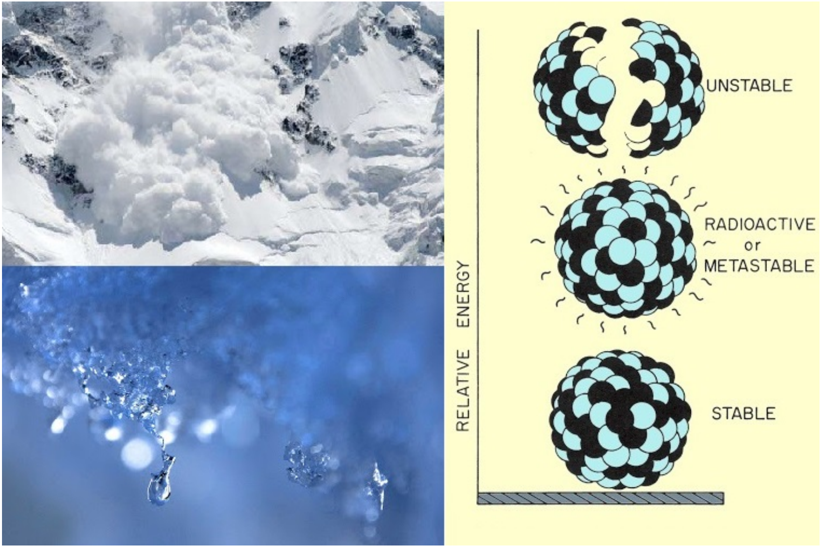
What is Metastability in Physics (some nice pictures)



What is Metastability in statistical physics

We take as a paradigm the characterization of metastability given in "Metastable behavior of stochastic dynamics: A pathwise approach" by M. Cassandro, A. Galves, E. Olivieri and M. E. Vares (1984).

In this paradigm metastability can be described as follows. We have a stochastic process with a unique stationary measure, but if the initial conditions are suitably chosen, then

- the system stays out of equilibrium for a long and unpredictable time,
- before reaching the actual equilibrium, the system is in a regime which resemble stationarity.

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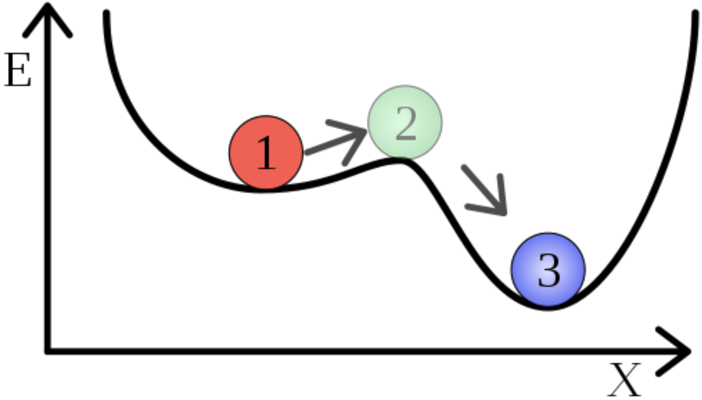

Metastable states can be seen as local minima of energy.

Why metastability is important in neurosciences

> “Metastability, a state in which signals (such as oscillatory waves) fall outside their natural equilibrium state but persist for an extended period of time, is a principle that describes the brain’s ability to make sense out of seemingly random environmental cues.”

"Metastability in the brain", Wikipedia

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The Model

The Model and the Results

- A countable set S representing the **neurons**.
- For each neuron $i \in S$, a set $\mathbb{V}_i \subset I$ of **presynaptic neurons**.
- For each $i \in S$, two point processes $\left(N_i^*(t)\right)_{t \geq 0}$ and $\left(N_i^\dagger(t)\right)_{t \geq 0}$ representing **spiking times** and **total leak times** respectively.
- For each $i \in S$, a process $\left(X_i(t)\right)_{t \geq 0}$ taking value in $\mathbb{N}$ representing the **membrane potential** of neuron i.
- A **spiking rate function** ϕ on $\mathbb{N}$.

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The Model

The point process $\left(N_i^{\dagger}(t)\right)_{t\geq 0}$ is a Poisson process of some rate $\gamma \geq 0$.

The point process $\left(N_i^{*}(t)\right)_{t\geq 0}$ has a fluctuating rate, given at time t by $\phi\big(X_i(t)\big)$.

The membrane potential at time t for neuron i is given by

$$X_i(t) = \sum_{j\in\mathbb{V}_i} \int_{]L_i(t),t[} dN_j^*(s),$$

and $L_i(t) = \sup\left\{s \leq t : N_i^*(\{s\}) + N_i^{\dagger}(\{s\}) = 1\right\}$.

(An instantiation of) The Model

- $S = \mathbb{Z}$,
- $\mathbb{V}_i = \{i-1, i+1\}$ for all $i \in S$,
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Previous results (Phase transition)

Theorem (Ferrari et al. (2018))

Assuming that $X_i(0) \geq 1$ *for all* $i \in \mathbb{Z}$*, there exists some* γ_c *(satisfying* $0 < \gamma_c < \infty$*) such that the following holds*

$$\mathbb{P}\left(N_i([0, +\infty[) < \infty\right) = 1 \text{ for all } i \in \mathbb{Z} \text{ if } \gamma > \gamma_c,$$

and

$$\mathbb{P}\left(N_i([0, +\infty[) < \infty\right) < 1 \text{ for all } i \in \mathbb{Z} \text{ if } \gamma < \gamma_c.$$

Previous results (first part of metastability)

Suppose that instead of $S = \mathbb{Z}$ we take $S = [\![-n, n]\!]$ for some $n \in \mathbb{N}$, and define the instant of the last spike

$$T_n = \inf\{t \geq 0 : N_i^*([t, \infty[) = 0 \text{ for all } i \in S\}.$$

Moreover let γ_c' be the critical value when $S = \mathbb{N}$.

Theorem (M. André (2019))

If $\gamma < \gamma_c'$ then the following holds

$$\frac{T_N}{\mathbb{E}(T_N)} \xrightarrow[N \to \infty]{\mathcal{D}} \mathcal{E}(1).$$

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Main result (second part of metastability)

Then take $S = [\![-n, n]\!]$ for some $n \geq 0$ and let $F \subset \mathbb{Z}$ be a subset of neurons satisfying $|F| < \infty$.

Theorem (Main theorem)

Suppose $0 < \gamma < \gamma_c$ *and let* $(R_n)_{n \geq 0}$ *be an increasing sequence of positive real numbers satisfying*

$$R_n \underset{n\to\infty}{\longrightarrow} +\infty \quad \textit{and} \quad \frac{R_n}{\mathbb{E}\left(T_n\right)} \underset{n\to\infty}{\longrightarrow} 0.$$

There exists some $0 < \rho < 1$ *(which depends only on* γ*) such that for any* $t \geq 0$

$$\frac{1}{R_n} \sum_{i \in F} N_i\left([t, t+R_n]\right) \xrightarrow[n\to\infty]{\mathbb{P}} |F| \cdot \rho.$$

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Sketch of the proof

Fix some $\epsilon > 0$. We aim to prove

$$\mathbb{P}\left(\left|\frac{1}{R_n}\sum_{i\in F} N_i\left([t, t+R_n]\right) - |F|\cdot\rho\right| > \epsilon\right) \underset{n\to\infty}{\longrightarrow} 0.$$

The main idea is to write

$$\begin{aligned}
&\mathbb{P}\left(\left|\frac{1}{R_n}\sum_{i\in F} N_i\left([t, t+R_n]\right) - |F|\cdot\rho\right| > \epsilon\right) \\
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The Auxiliary Interacting Particle System.

Auxiliary IPS

We define the auxiliary process, denoted $\left(\xi(t)\right)_{t\geq 0}$, as follows

$$\forall t \geq 0,\ \forall i \in S, \quad \xi_i(t) \stackrel{\text{def}}{=} \mathbb{1}_{X_i(t)>0}.$$

This process is an **interacting particle system**. It is a continuous time Markov process taking value in $\{0,1\}^S$.

Depending on whether $\xi_i(t)$ is equal to 1 or 0 we will say that neuron i is **active** or **quiescent** respectively.

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Auxiliary IPS

The generator of the process $\big(\xi(t)\big)_{t\geq 0}$ is given by

$$\mathcal{L}f(\xi) = \gamma \sum_{i\in S} \Big(f(\pi_i^\dagger(\xi)) - f(\xi) \Big) + \sum_{i\in S} \xi_i \Big(f(\pi_i(\xi)) - f(\xi) \Big),$$

where the maps are given by

$$\pi_i^\dagger(\xi)_j = \begin{cases} 0 & \text{if } j = i, \\ \xi_j & \text{otherwise}, \end{cases}$$

and

$$\pi_i(\xi)_j = \begin{cases} 0 & \text{if } j = i, \\ \max(\xi_i, \xi_j) & \text{if } i \in \mathbb{V}_j, \\ \xi_j & \text{otherwise}. \end{cases}$$

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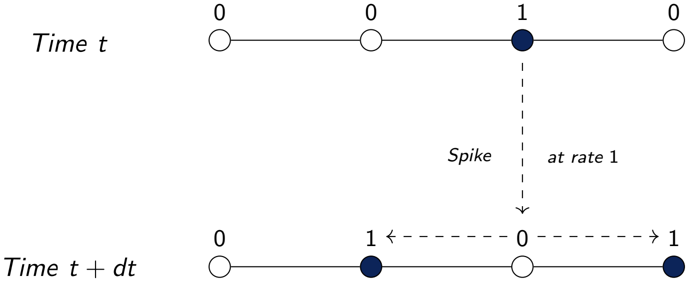
One-dimensional lattice with nearest-neighbours interaction



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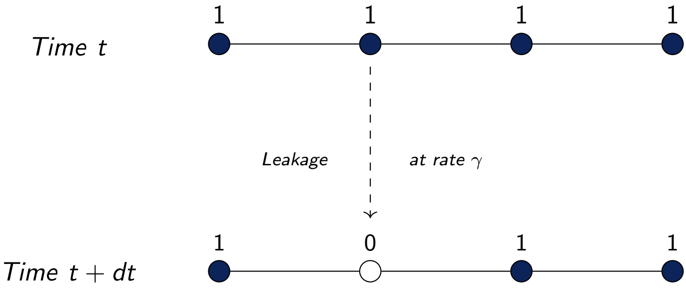
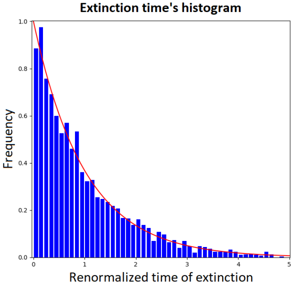
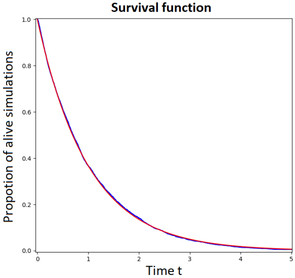
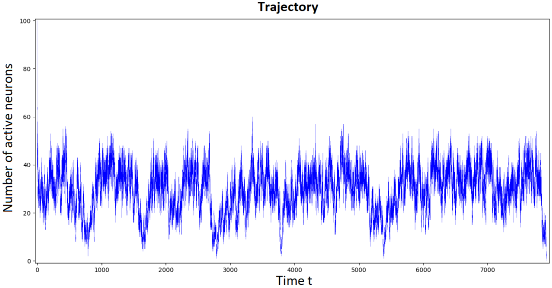


Simulations on the lattice for small γ.







Upper-invariant measure and Density of the Auxiliary Process

If $\gamma < \gamma_c$, then there exists a non-trivial invariant measure (in the sense that it doesn't give mass 1 to $\xi \equiv 0$) for $(\xi_t)_{t \geq 0}$, which corresponds to the weak limit of ξ_t when t diverges, and which we denote μ.

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The Edge of Dual of the Auxiliary Process

The process $(\xi_t)_{t\geq 0}$ admits a dual, that is a process $(\eta_t)_{t\geq 0}$ which is such that for any states A and B the following holds

$$\mathbb{P}\Big(\xi^A(t) \cap B \neq \emptyset\Big) = \mathbb{P}\Big(\eta^B(t) \cap A \neq \emptyset\Big)$$

For any set $A \in \mathcal{P}(\mathbb{Z})$ and $t \geq 0$ define $r_t^A = \max\left\{i \in \eta_t^A\right\}$.

Then the following important result holds

Proposition

Suppose $\gamma < \gamma_c$. Then there exists a constant $\alpha(\gamma) > 0$ such that the following holds

$$\frac{r_t^-}{t} \underset{t\to\infty}{\longrightarrow} \alpha(\gamma) \text{ almost surely.}$$

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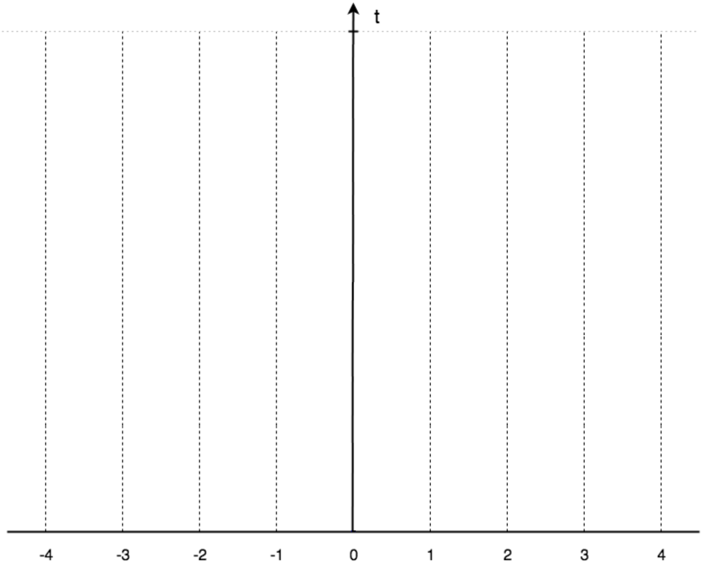

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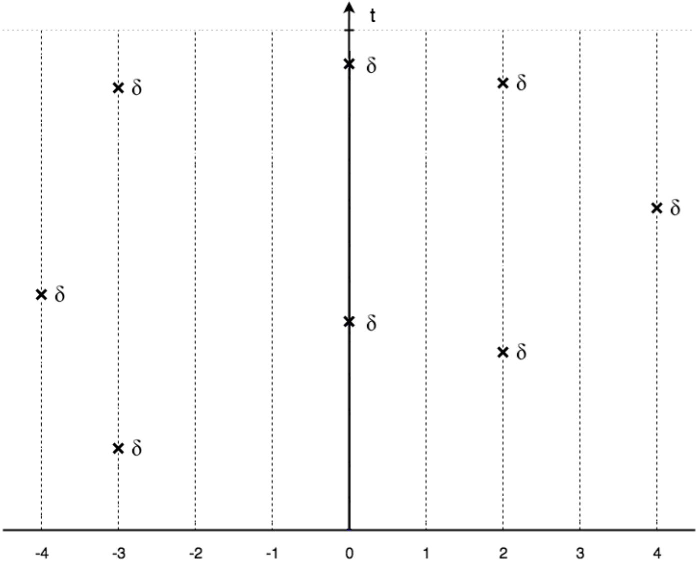

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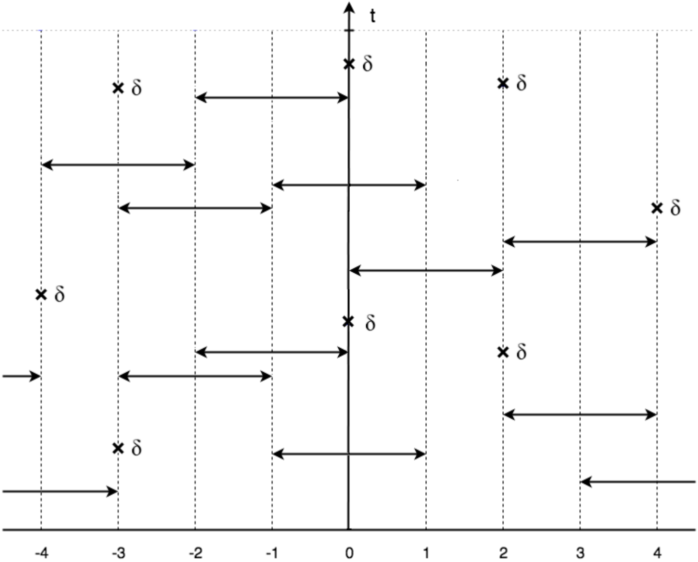

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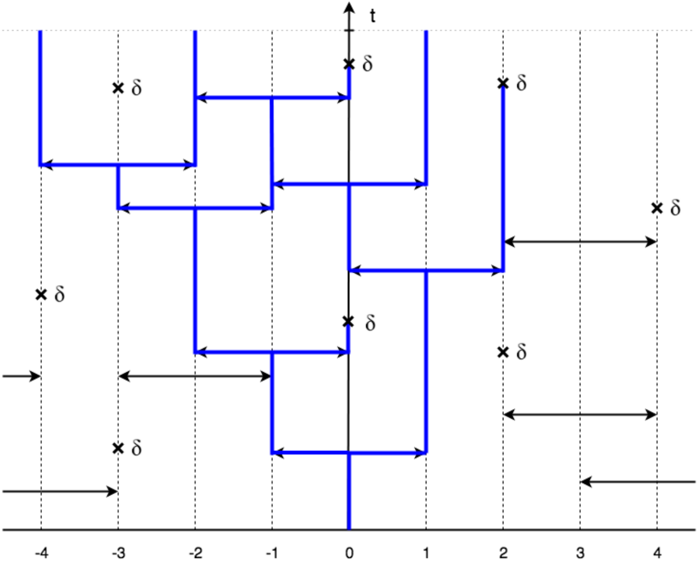
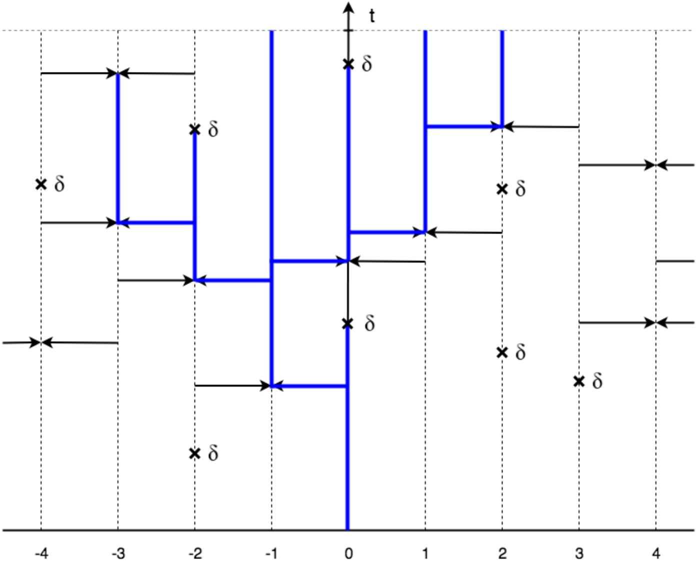


Graphical construction of the dual



Exponential estimates

Then from the previous proposition, using classical arguments relating nearest-neighbours interacting particles system to one-dependent percolation we obtain

Proposition

If $\gamma < \gamma_c$ then for any $a < \alpha$ there exists positive constants C_1 and C_2 (depending on a) such that for any $t \geq 0$

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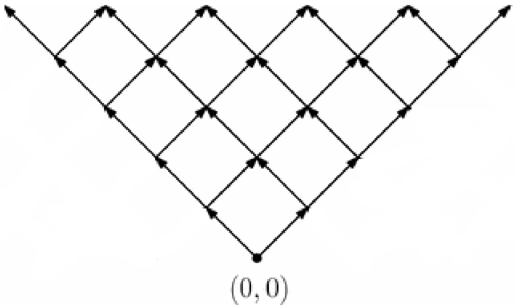

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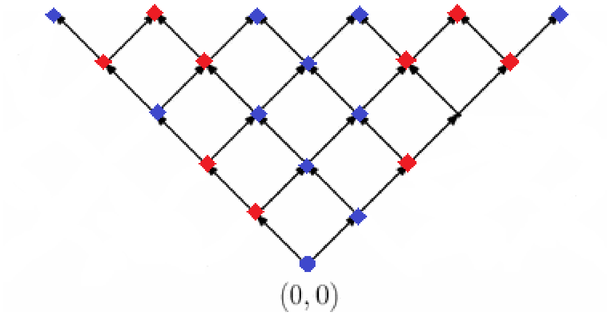

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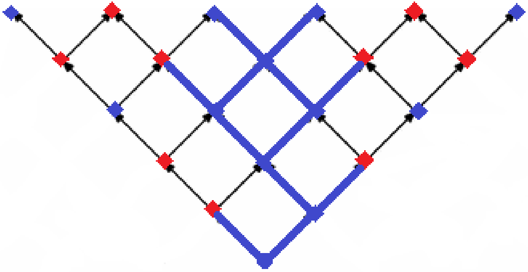

$(0, 0)$

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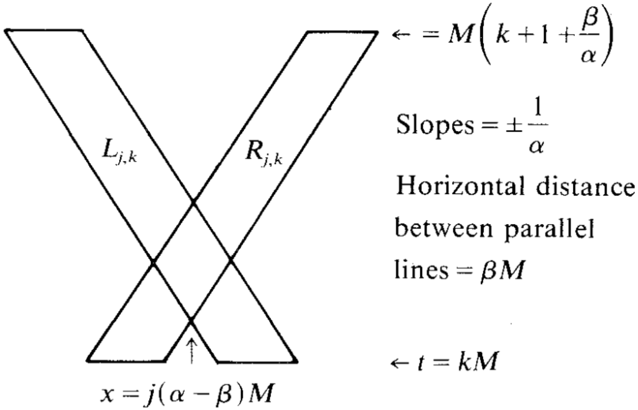

$\leftarrow = M\left(k+1+\frac{\beta}{\alpha}\right)$
$L_{j,k}$
$R_{j,k}$
Slopes $= \pm\frac{1}{\alpha}$
Horizontal distance
between parallel
lines $= \beta M$
$\leftarrow t = kM$
$x = j(\alpha - \beta)M$

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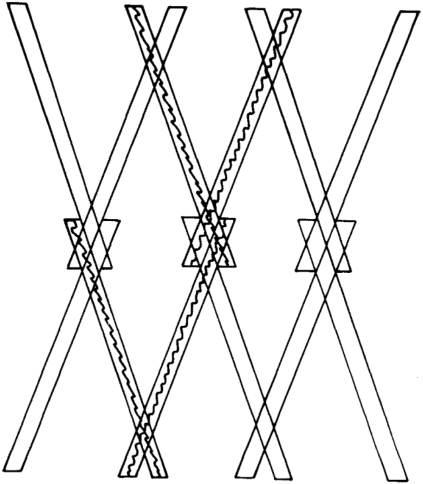

Important side effect $\gamma_c = \gamma_c'$

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And then

Theorem (Exponentially decaying time correlations)

Let $_sigma^A$ be the time of extinction of the infinite dual process starting from A. Let $f : \mathcal{P}(\mathbb{Z}) \to \mathbb{R}$ be a cylinder function. Then there exists positive constants C_1 and C_2 (depending on f) such that for any $s, t \in \mathbb{R}^+$

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$$\mathbb{P}\left(\frac{1}{R_n}\left|\int_t^{t+R_n} f(\xi_s)ds - \mathbb{E}\left[\int_t^{t+R_n} f(\xi_s)ds\right]\right| > \epsilon\right) \underset{n\to\infty}{\longrightarrow} 0$$

And we can use Chebyshev's inequality

$$\mathbb{P}\left(\left|\int_t^{t+R_n} f(\xi_s)ds - \mathbb{E}\left[\int_t^{t+R_n} f(\xi_s)ds\right]\right| > \epsilon R_n\right)$$
$$\leq \frac{1}{\epsilon^2 R_n^2}\mathrm{Var}\left(\int_t^{t+R_n} f(\xi_s)ds\right)$$

Time averages for the auxiliary process

We want to show that $\mathbb{P}\left(\left|\frac{1}{R_n}\int_t^{t+R_n} f(\xi_s)ds - \mu(f)\right| > \epsilon\right) \underset{n\to\infty}{\longrightarrow} 0.$

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Then

$$\frac{1}{\epsilon^2 R_n^2}\mathrm{Var}\left(\int_t^{t+R_n} f(\xi_s)ds\right)$$
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And using the exponential decays it gives

$$\frac{1}{\epsilon^2 R_n^2}\mathrm{Var}\left(\int_t^{t+R_n} f(\xi_s)ds\right) \leq \frac{C}{R_n}$$

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Back to the Model.

Sketch of the proof

Fix some $\epsilon > 0$. We aim to prove

$$\mathbb{P}\left(\left|\frac{1}{R_n}\sum_{i\in F} N_i\left([t,t+R_n]\right) - |F|\cdot\rho\right| > \epsilon\right) \underset{n\to\infty}{\longrightarrow} 0.$$

The main idea is to write

$$\begin{aligned}
&\mathbb{P}\left(\left|\frac{1}{R_n}\sum_{i\in F} N_i\left([t,t+R_n]\right) - |F|\cdot\rho\right| > \epsilon\right) \\
&\leq \mathbb{P}\left(\left|\frac{1}{R_n}\sum_{i\in F} N_i\left([t,t+R_n]\right) - \frac{1}{R_n}\sum_{i\in F}\int_t^{t+R_n} \mathbb{1}_{X_i(s)>0} ds\right| > \frac{\epsilon}{2}\right) \\
&+ \mathbb{P}\left(\left|\frac{1}{R_n}\sum_{i\in F}\int_t^{t+R_n} \mathbb{1}_{X_i(s)>0} ds - |F|\cdot\rho\right| > \frac{\epsilon}{2}\right)
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Fix some $\epsilon > 0$. We aim to prove

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&\leq \mathbb{P}\left(\left|\frac{1}{R_n}\sum_{i\in F} N_i\left([t,t+R_n]\right) - \frac{1}{R_n}\sum_{i\in F}\int_t^{t+R_n} \mathbb{1}_{X_i(s)>0}ds\right| > \frac{\epsilon}{2}\right)\\
&+\mathbb{P}\left(\left|\frac{1}{R_n}\sum_{i\in F}\int_t^{t+R_n} \mathbb{1}_{X_i(s)>0}ds - |F|\cdot\rho\right| > \frac{\epsilon}{2}\right)
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Back to the Model

$$\mathbb{P}\left(\left|\frac{1}{R_n}\sum_{i\in F} N_i\left([t,t+R_n]\right) - \frac{1}{R_n}\sum_{i\in F}\int_t^{t+R_n} \mathbb{1}_{X_i(s)>0}ds\;\right| > \epsilon\right)$$

$$\leq \mathbb{P}\left(\frac{1}{R_n}\sum_{i\in F}\left|N_i([t,t+R_n]) - \int_t^{t+R_n}\xi_{n,i}(s)ds\right| > \epsilon\right)$$

$$= \mathbb{P}\left(\frac{1}{R_n}\sum_{i\in F}|N_i(I_{n,i}) - \lambda(I_{n,i})| > \epsilon\right)$$

Where

$$I_{n,i} \stackrel{\text{def}}{=} \{s:\ t \leq s \leq t+R_n \text{ and } \xi_{n,i}(s)=1\}.$$

Back to the Model

$$\mathbb{P}\left(\left|\frac{1}{R_n}\sum_{i\in F} N_i\left([t,t+R_n]\right) - \frac{1}{R_n}\sum_{i\in F}\int_t^{t+R_n} \mathbb{1}_{X_i(s)>0}ds\ \right| > \epsilon\right)$$
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Back to the Model

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Back to the Model

$$\mathbb{P}\left[\left|\mathbb{E}\left[N_i(I_{n,i})\right] - \lambda(I_{n,i})\right| > \frac{\epsilon R_n}{2|F|}\right]$$

$$\leq \frac{4|F|^2}{\epsilon^2 R_n^2}\mathrm{Var}(\lambda(I_{n,i}))$$

$$= \frac{4|F|^2}{\epsilon^2 R_n^2}\int_t^{t+R_n}\int_t^{t+R_n}\mathrm{Cov}\left(\xi_{n,i}(x), \xi_{n,i}(y)\right) dxdy.$$

Back to the Model

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Back to the Model

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&\mathbb{P}\left[\left|\mathbb{E}\left[N_i(I_{n,i})\right] - \lambda(I_{n,i})\right| > \frac{\epsilon R_n}{2|F|}\right] \\
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\end{aligned}
$$

Back to the Model

Let $\tau_n = \inf\{t \geq 0 : \xi_n(t) = \emptyset\}$.

Lemma

Let $s, r \in \mathbb{R}_+$ and let $i \in \mathbb{Z}$. Then there exists two positive constants C_1 and C_2 such that for any $n \geq 0$

$$\left|\mathrm{Cov}\left(\mathbb{1}_{\xi_n(s)\cap\{i\}\neq\emptyset}, \mathbb{1}_{\xi_n(t)\cap\{i\}\neq\emptyset}\right)\right| \leq C_1 e^{-C_2|t-s|} + \mathbb{P}\left(\tau_n < \max(s,t)\right) + \epsilon_n,$$

where ϵ_n is some positive quantity satisfying $\epsilon_n \underset{n\to\infty}{\longrightarrow} 0$.

$$\mathbb{P}\left[\left|\mathbb{E}\left[N_i(I_{n,i})\right] - \lambda(I_{n,i})\right| > \frac{\epsilon R_n}{2|F|}\right]$$
$$\leq \frac{4|F|^2}{\epsilon^2 R_n^2} \mathrm{Var}(\lambda(I_{n,i}))$$
$$= \frac{4|F|^2}{\epsilon^2 R_n^2} \int_t^{t+R_n} \int_t^{t+R_n} \mathrm{Cov}\left(\xi_{n,i}(x), \xi_{n,i}(y)\right) dxdy.$$
$$\leq \frac{C}{R_n} + \frac{4|F|^2}{\epsilon^2}\left(\mathbb{P}(\tau_n < t + R_n) + \epsilon_n\right)$$

$$
\begin{aligned}
&\mathbb{P}\left[\left|\mathbb{E}\left[N_i(I_{n,i})\right] - \lambda(I_{n,i})\right| > \frac{\epsilon R_n}{2|F|}\right] \\
&\leq \frac{4|F|^2}{\epsilon^2 R_n^2} \mathrm{Var}(\lambda(I_{n,i})) \\
&= \frac{4|F|^2}{\epsilon^2 R_n^2} \int_t^{t+R_n} \int_t^{t+R_n} \mathrm{Cov}\left(\xi_{n,i}(x), \xi_{n,i}(y)\right) dxdy. \\
&\leq \frac{C}{R_n} + \frac{4|F|^2}{\epsilon^2} \big(\mathbb{P}(\tau_n < t + R_n) + \epsilon_n\big)
\end{aligned}
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Back to the Model

$$\mathbb{P}\left[\frac{1}{R_n}\sum_{i\in F}|N_i(I_{n,i})-\lambda(I_{n,i})|>\epsilon\right]$$

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Back to the Model

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For the remaining part we can use the law of total variance

$$\mathrm{Var}\big(N_i(I_{n,i})\big) = \mathbb{E}\left[\mathrm{Var}\big(N_i(I_{n,i})|\lambda(I_{n,i})\big)\right] + \mathrm{Var}\big(\mathbb{E}\left[N_i(I_{n,i})|\lambda(I_{n,i})\right]\big).$$

But this part has already been taken care of.

For this part we have $\mathrm{Var}\left(N_i(I_{n,i})|\lambda(I_{n,i})\right) = \lambda(I_{n,i})$

so that

$$\mathbb{E}\left[\mathrm{Var}\big(N_i(I_{n,i})|\lambda(I_{n,i})\big)\right] \leq R_n.$$

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Thank you for your attention!