probability and stochastic processes

statistics

1. sample mean:

$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

2. sample median if n is even: $\tilde{x} = \frac{x_{\frac{n+1}{2}} + x_{\frac{n+2}{2}}}{2}$
3. sample median if n is odd $\tilde{x} = x_{\frac{n+1}{2}}$
4. sample variance: $X_1, X_2, \cdots, X_n$ is denoted by σ^2

$$\sigma^2 = s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

5. sample standard deviation: $s = \sqrt{s^2}$

probability

0. event operations:

- $A \cap \emptyset = \emptyset$
- $A \cup \emptyset = A$
- $A \cap A' = \emptyset$, $A \cup A' = S$
- $S' = \emptyset, \emptyset' = S$
- $(A \cap B)' = A' \cup B'$
- $(A \cup B)' = A' \cap B'$

1. $(A \cup B)^c = (A^c \cap B^c)$
2. mutually exclusive events A_1 and A_2, $P[A_1 \cup A_2] = P[A_1] + P[A_2]$
3. if $A = A_1 \cup A_2 \cup \cdots \cup A_m \text{ and } A_i \cap A_j = \emptyset \text{ for all } i \neq j \text{, then }$ `$ P[A] = \sum_{i=1}^m P[A_{i}] $`
4. $P[.]$, $P[\emptyset] = 0$, $P[A^c] = 1 - P[A]$, $P[A \cup B] = P[A] + P[B] - P[A \cap B]$, and $A \subset B, P[A] \leq P[B]$
5. $B = s_1, s_2, \cdots, s_m$ is the sum of the probabilities of the outcomes contained in the event:

$$P[B] = \sum_{i=1}^{m} P[s_i]$$

6. $S = s_1, s_2, \cdots, s_n$ in which each outcomes s_i is equally likely,

$$P[s_i] = \frac{1}{n} \quad 1 \leq i \leq n$$

7. conditional probability measure $P[A|B]$

- $P[A|B] \geq 0$
- $P[B|B] = 1$
- $A = A_1 \cup A_2 \cup \cdots \cup A_m$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$, then, $P[A|B] = P[A_1|B] + P[A_2|B] + \cdots + P[A_m|B]$

8. partition $B = B_1, B_2, \cdots, B_m$ and any event A in the sample space, let $C_i = A \cap B_i$ For $i \neq j$, the events C_i and C_j are mutually exclusive and $A = C_1 \cup C_2 \cup \cdots$
9. A and partition $B_1, B_2, \cdots, B_m$,

$$P[A] = \sum_{i=1}^{m} P[A \cap B_i]$$

10. law of total probability $B_1, B_2, \cdots, B_m$ with $P[B_i] > 0$ for all i,

$$P[A] = \sum_{i=1}^{m} P[A|B_i]P[B_i]$$

11. conditional probability

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \text{if } P(A) > 0$$

12. bayes' theorem

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i|A)P(B_i)}$$

$$P[B|A] = \frac{P[A|B]P[B]}{P[A]}$$

13. independent events two events A and B are independent if

$$P[AB] = P[A]P[B]$$

3. Discrete Random Variables

1. **probability mass function** is the set of ordered pars $(x, f(x))$ is a probability function, probability mass function, or probability distribution of the discrete random variable X if for each possible outcome x.
2. $f(x) \geq 0$
3. $\sum f(x) = 1$
4. $P(X = x) = f(x)$

let X be the random variable: number of heads in 3 tosses of a fair coin.

sample space	x
TTT	0
TTH	1
THT	1
THH	2
HTT	1
HTH	2
HHT	2
HHH	3

the probability $P(X = x)$ that the outcome is a specific x value is the probability that the number of heads is x.

x	0	1	2	3
$P(X = x)$	1/8	3/8	3/8	1/8

2. **the cumulative distribution function** $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \text{ for } -\infty \leq x \leq \infty$$

3. continuous random variable has a probability of zero assuming exactly any of its values $P(a < X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a \leq X \leq b)$
4. the function $f(x)$ is a **probability density function** (pdf) for the continuous random variable X, defined over the set of real numbers, if
5. $f(x) \geq 0$ for all $x \in R$
6. $\int_{-\infty}^{\infty} f(x)dx = 1$
7. $P(a < X < b) = \int_a^b f(x)dx$
8. definition of joint probability distribution / probability mass function $f(x, y) = P(X = x, Y = y)$ the function $f(x, y)$ is a joint probability distribution or **probability mass function** of the discrete random variables X and Y if
9. $f(x, y) \geq 0$ for all (x, y)
10. $\sum_x \sum_y f(x, y) = 1$
11. $P(X = x, Y = y) = f(x, y)$

for any region A in the xy plane, $P[(X, Y) \in A] = \sum \sum_A f(x, y)$

mathematical expectation

1.1. **expeted value** let X be a random variable with probability distribution $f(x)$, the **mean** or **expected value** of X is

if X is discrete

$$\mu = E(X) = \sum_x x f(x) dx$$

if X is continuous

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

4.1. **example** a lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. a sample of 3 is taken by the inspector. find the expected value of the number of good components in this sample.

let X represent the number of good components in the sample. the probability distribution of X is

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, x = 0, 1, 2, 3$$

$f(0) = 1/35$, $f(1) = 12/35$, $f(2) = 18/35$, $f(3) = 4/35$

$$\mu = E(X) = \sum_x x f(x) dx = (0)\frac{1}{35} + (1)\frac{12}{35} + (2)\frac{18}{35} + (3)\frac{4}{35} = 1.7$$

therefore if a sample size 3 is selected at random over and over again from a lot of 4 good components and 3 defective components, it will contain on average 1.7 good components

4.4. **example** a coin is biased such that a head is three times as likely to occur as a tail. Find the expected number of tails when this coin is tossed twice.

assigning weights of $3w$ and w for a head and tail respectively. we obtain $P(H) = 3/4$ and $P(T) = 1/4$. the sample space for the experiment is $S = HH, HT, TH, TT$. now if X represents the number of trials that occur in two tosses of coins, we have

$P(X = 0) = P(HH) = (3/4)(3/4) = 9/16$

$P(X = 1) = P(HT) + P(TH) = (3/4)(1/4) + (1/4)(3/4) = 3/8$

$P(X = 2) = P(TT) = (1/4)(1/4) = 1/16$

the probability distribution for X is then

x	0	1	2
$f(x)$	9/16	3/8	1/16

expected value is $\mu = E(X) = (0)(9/16) + (1)(3/8) + (2)(1/16) = 1/2$

1.2. **expected value** let X and Y be random variables with joint probability distribution $f(x, y)$. the mean or expected value of the random variable $g(X, Y)$ is

if X and Y are discrete

$$\mu_{g(X,Y)} = E[g(X, Y)] = \sum_x \sum_y g(x, y) f(x, y)$$

if X and Y are continuous

$$\mu_{g(X,Y)} = E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy$$

1.3. **standard deviation** let X be a random variable with probability distribution $f(x)$ and mean μ. the variance of X is σ^2. the variance σ^2 is called the standard deviation.

if X is discrete

$$\sigma^2 = E[(X - \mu)^2] = \sum_x (x - \mu)^2 f(x) dx$$

if X is continuous

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) ds$$

1.4. the variance of the random variable X is $\sigma^2 = E(X^2) - \mu^2$

1.5. let X be a random variable with probability distribution $f(x)$ the variance of the random variable $g(X)$ is

if X is discrete

$$\sigma^2_{g(X)} = E[g(X) - \mu^2_{g(X)} = \sum_x [g(x) - \mu_{g(X)}]^2 f(x)$$

if X is continuous

$$\sigma^2_{g(X)} = E[g(X) - \mu_{g(X)}]^2 = \int_{-\infty}^{\infty} [g(x) - \mu_{g(X)}]^2 f(x) dx$$

1.6. **covariance** let X and Y be random variables with joint probability distribution $f(x, y)$ the covariance of X and Y is if X and Y are discrete

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \sum_x sum_y (x - \mu_X)(y - \mu_y) f(x, y)$$

if X and Y are continuous

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_y) f(x, y) dx dy$$