probability and statistics

morgan bergen

04 mathematical expectation

- mean if a random variable
- variance and covariance of a random variable
- means and variances of linear combinations of random variables
- chebyshev's theorem

4.4 a coin is biased such that a head is three times as likely to occur as a tail. find the expected number of tails when this coin is tossed.

4.10 two tire-quality experts examine stacks of tires and assign a quality rating to each tire on a 3-point scale.

let X denote the rating given by expert A and Y denote the rating given by B. the following table gives the joint distribution for X and Y find the expected value of x and the expected value of y

4.12 if a dealer's profit, in units of $5,000, on a new automobile can be looked upon as a random variable X having the density function, shown below find the average profit per automobile.

f(x) = { 2(1 - x), 0 < x < 1,

0, elsewhere

4.20 a continuous random variable X has the density function shown below, find the expect value o0f g(X) = e^{2X/3}

f(x) = { e^{-x}, x > 0

0, elsewhere

4.34 let X be a random variable with the following probability distributions. Find the standard deviation of X

x	2	3	4	5	6
f(x)	0.01	0.25	0.4	0.3	0.04

4.39 the total number of hours in units of 100 hours that a family runs a vacuum cleaner over a period of 1 year is a random variable X having the density function, find the variance of X.

f(x) = { x, 0 < x < 1

2 - x, 1 ≤ x < 2

0, elsewhere

4.50 for a laboratory assignment, if the equipment is working, the density function of the observed outcome X is below, find the variance and standard deviation of X

f(x) = { 2(1 - x), 0 < x < 1,

0, otherwise

4.52 random variables X and Y follow a joint distribution shown below. determine the correlation coefficient between X and Y

f(x, y) = { 2, 0 < x ≤ y < 1,

0, otherwise

4.57 let X be a random variable with the following probability distribution shown below, find E(X) and E(X^{2}) and then using these values evaluate E[(2X + 1)^{2}]

x	-3	6	9
f(x)	1/6	1/2	1/3

4.62 if X and Y are independent random variables with variances σ_X^2 = 5 and σ_Y^2 = 3, find the variance of the random variable Z = -2X + 4Y - 3

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4.4 a coin is biased such that a head is three times as likely to occur as a tail. find the expected number of tails when this coin is tossed.

- T is the number of tails in 2 tosses of a biased coin
- let X & Y be the rv that rep the number of tails in the first & second toss respectively of the same coin

T can be 0, 1, 2

X & Y can be 0, 1

$P(X=0) = 3P(X=1)$

$P(X=0) + P(X=1) = 1$

$3P(X=1) + P(X=1) = 1$

$P(X=1) = 0.25$ & $P(X=0) = 0.75$

$P(Y=1) = 0.25$ & $P(Y=0) = 0.75$

<u>toss zero</u> : $P(T=0) = P(X=0, Y=0)$

$= P(X=0)P(Y=0)$

$= (0.75)(0.75)$

$= 0.5625$

<u>first toss</u> : $P(T=1) = P(X=1, Y=0) + P(X=0, Y=1)$

$= P(X=1)P(Y=0) + P(X=0)P(Y=1)$

$= (0.25)(0.75) + (0.75)(0.25)$

$= 0.375$

<u>second toss</u> : $P(T=2) = P(X=1, Y=1)$

$= P(X=1)P(Y=1)$

$= (0.25)(0.25)$

$= 0.0625$

$P(X=0) = 0.5625$

$P(X=1) = 0.375$

$P(X=2) = 0.0625$

$$E(T) = \sum_t t f(t) = \sum_{t=0}^{2} t \cdot P(T=t)$$

$$= 0 \cdot (0.5625) + 1(0.375) + 2(0.0625)$$

$$\boxed{\therefore E(T) = 0.5}$$

4.10 Two tire-quality experts examine stacks of tires and assign a quality rating to each tire on a 3-point scale. Let X denote the rating given by expert A and Y denote the rating given by B.

The following table gives the joint distribution for X and Y

FIND μ_X & μ_Y

- X RATING BY A 3 PT SCALE
- Y RATING BY A 3 PT SCALE

$f(x,y)$		y = 1	2	3	ROW TOTALS
x	1	0.10	0.05	0.02	0.17
	2	0.10	0.35	0.05	0.50
	3	0.03	0.10	0.20	0.33
COL TOTALS		0.23	0.50	0.27	1

$1(0.17) + 2(0.5) + 3(0.33) = 2.16$

$1(0.23) + 2(0.50) + 3(0.27) = 2.04$

DERIVING EXPECTED VALUE μ

LET X & Y BE RV'S WITH JOINT PROB DISTRIBUTIONS $f(x,y)$. THE MEAN, OR EXPECTED VALUE, OF RANDOM VARIABLE $g(X,Y)$ IS WHERE $g(x)$ IS THE MARGINAL DISTRIBUTION

$$\boxed{\begin{aligned}\mu_x &= 2.16\\ \mu_Y &= 2.04\end{aligned}}$$

$$\mu_x = \sum_x \sum_y x f(x,y) = \sum_x g(x)$$

$$g(1) = \sum_y f(1,y) = f(1,1) + f(1,2) + f(1,3) = 0.10 + 0.05 + 0.02 = 0.17$$

$$g(2) = \sum_y f(2,y) = f(2,1) + f(2,2) + f(2,3) = 0.10 + 0.35 + 0.05 = 0.50$$

$$g(3) = \sum_y f(3,y) = f(3,1) + f(3,2) + f(3,3) = 0.03 + 0.10 + 0.20 = 0.33$$

$$\mu_x = \sum_x g(x) = \sum_{x=1}^{3} x\, g(x) = 1(0.17) + 2(0.50) + 3(0.33) = 2.16$$

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4.12 if a dealer's profit, in units of $5,000, on a new automobile can be looked upon as a random variable X having the density function, shown below find the average profit per automobile.

$$f(x) = \begin{cases} 2(1 - x), & 0 < x < 1, \\ 0, & \text{elsewhere} \end{cases}$$

THE EXPECTED VALUE / MEAN IS SHOWN AS LETTING X BE A RANDOM VARIABLE WITH PROBABILITY DISTRIBUTION $f(x)$

$$\mu = E(X) = \sum_x x f(x) \quad \text{IFF} \quad X \text{ IS DISCRETE}$$

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) \partial x \quad \text{IFF} \quad Y \text{ IS CONTINUOUS}$$

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0 & \text{ELSEWHERE} \end{cases}$$

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) \partial x = \int_0^1 x\, 2(1-x) \partial x = 2\int_0^1 (x - x^2) \partial x$$

$$= 2\left(\int_0^1 x \,\partial x - \int_0^1 x^2 \partial x\right)$$

$$= 2\left(\frac{x^2}{2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1\right)$$

$$= 2\left(1/2 - 1/3\right)$$

$$\mu = 1/3$$

$$\boxed{\mu(5000) = 1/3(5000) = \$1666.67}$$

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4.20 a continuous random variable X has the density function shown below, find the expect value of g(X) = e^{2X/3}

f(x) = { e^{-x}, x > 0

0, elsewhere

SINCE X IS A CONTINUOUS RANDOM VARIABLE WE WILL USE THE FOLLOWING FORMULA,

$$\mu_g(X) = E(g(X)) = \int_x g(x)f(x)\,\partial x = \int_x g(x)f(x)\,\partial x$$

$$= \int_{x=0}^{\infty} e^{2x/3} \cdot e^{-x}\,\partial x$$

$$= \lim_{t\to\infty} \int_{x=0}^{t} e^{-x/2}\,\partial x$$

$$= \lim_{t\to\infty} \left(-3e^{-x/3}\right)\Big|_0^t$$

$$= \lim_{t\to\infty} \left(-3e^{-t/3} + 3\right)$$

$$\therefore \boxed{\mu_g(X) = 3}$$

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4.34 let X be a random variable with the following probability distributions. find the standard deviation of X

x	-2	3	5
f(x)	0.3	0.2	0.5

PDF $\rightarrow \mu_X \rightarrow \sigma^2_X \rightarrow \sigma_X$

$\mu = E(X) =$ MEAN / EXPECTED VALUE

$\sigma^2_X = VAR(X) = \sigma^2_X =$ VARIANCE OF RANDOM VARIABLE X

TO FIND THE STANDARD DEVIATION WE WILL FIND THE VARIANCE & THEN TAKE IT'S + SQUARE ROOT

FIRST WE WILL FIND THE MEAN OR EXPECTED VALUE

$$\mu = E(X) = \sum_x x f(x) = -2(0.3) + 3(0.2) + 5(0.5) = 2.5$$

$$\mu_X = 2.5$$

$$\sigma_Y^2 = E[(X - \mu^2)] = \sum_x (x-\mu)^2 f(x) = (-2-2.5)^2(0.3) + (3-2.5)^2(0.2) + (5-2.5)^2(0.5) = 9.25$$

$$\sigma_X^2 = 9.25$$

$$\therefore \boxed{\sigma_X = \sqrt{9.25} = 3.04}$$

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4.39 the total number of hours in units of 100 hours that a family runs a vacuum cleaner over a period of 1 year is a random variable X having the density function, find the variance of X.

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

VARIANCE $\sigma^2 = E(x^2) - \mu$

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x(x)\,dx + \int_1^2 x(2-x)\,dx$$
$$= \int_0^1 x^2 dx + \int_1^2 (2x - x^2)\,dx$$
$$= \frac{x^3}{3}\Big|_0^1 + \int_1^2 2x\,dx - \int_1^2 x^2 dx$$
$$= \frac{1}{3} + \frac{2x^2}{2}\Big|_1^2 - \frac{x^3}{3}\Big|_1^2$$
$$= \frac{2}{6} + \left(\frac{2(2)^2}{2} - \frac{2(1)^2}{2}\right) - \left(\frac{2^3}{3} - \frac{1^3}{3}\right)$$
$$= \frac{6}{6}$$

$\mu = 1$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) = \int_0^1 x^2(x)\,dx + \int_1^2 x^2(2-x)\,dx$$
$$= \int_0^1 x^3 dx + \int_1^2 2x^2 - x^3\,dx$$
$$= \frac{x^4}{4} + \int_1^2 2x^2 dx - \int_1^2 x^3\,dx$$
$$= \frac{x^4}{4}\Big|_0^1 + \frac{2x^3}{3}\Big|_1^2 - \frac{x^4}{4}\Big|_1^2$$
$$= \frac{1}{4} + \frac{2(2)^2}{3} - \frac{2(1)^2}{3} - \frac{x^4}{4}\Big|_1^2$$
$$= \frac{7}{6}$$

$E(X^2) = \frac{7}{6}$

$$\sigma^2 = E(X^2) - \mu$$
$$\sigma^2 = 7/6 - 6/6$$

$\therefore \boxed{\sigma^2 = 1/6}$

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4.50 for a laboratory assignment, if the equipment is working, the density function of the observed outcome X is below, find the variance and standard deviation of X

f(x) = { 2(1 - x), 0 < x < 1,

0, otherwise

VARIANCE $\sigma_X^2 = E(X^2) - \mu_X^2 = 1/6 - (1/3)^2 = 1/18$

STD DEV. $\sigma_X = \sqrt{\sigma_X^2} = \sqrt{1/18} = \sqrt{2}/6$

$\therefore \boxed{\sigma_X^2 = 1/18 \quad \sigma_X = \sqrt{2}/6}$

$$\mu_X = E(X) = \int_x x f(x)\,dx = \int_0^1 x\, 2(1-x)\,dx = \int_0^1 (2x - 2x^2)\,dx = \int_0^1 2x\,dx - 2\int_0^1 x^2\,dx = \frac{2x^2}{2}\Big|_0^1 - \frac{2x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$E(X^2) = \int_x x^2 f(x)\,dx = \int_0^1 x^2(2(1-x))\,dx = \int_0^1 x^2(2-2x)\,dx = \int_0^1 2x^2 - 2x^3\,dx = 2\left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1 = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6}$$

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- chebyshev's theorem

4.52 random variables X and Y follow a joint distribution shown below. determine the correlation coefficient between X and Y

f(x, y) = { 2, 0 < x ≤ y < 1,
0, otherwise

CORRELATION COEFFICIENT $\rho_{XY} = \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y}$

$$\sigma_X^2 = E(X^2) - \mu_X^2 = \int_x \int_y x^2 (2)\, dy\, dx - \left(\int_x\int_y x\, 2\, dy\, dx\right)^2$$
$$= \int_0^1 \int_x^1 2x^2\, dy\, dx - \left(\int_0^1\int_x^1 2x\, dy\, dx\right)^2$$
$$= \int_0^1 2x^2 y \Big|_x^1 dx - \left(\int_0^1 2xy\Big|_x^1 dx\right)^2$$
$$= \int_0^1 2x^2(1) - 2x^2(x)\, dx - \left(\int_0^1 2x(1) - 2xx\, dx\right)^2$$
$$= \int_0^1 2x^2 - 2x^3\, dx - \left(\int_0^1 2x - 2x^2\, dx\right)^2$$
$$= \left(\frac{2x^3}{3} - \frac{2x^4}{4}\right)\Big|_0^1 - \left(\left(\frac{2x^2}{2} - \frac{2x^3}{3}\right)\Big|_0^1\right)^2$$
$$= \frac{2}{3} - \frac{2}{4} - \left(\frac{2}{2} - \frac{2}{3}\right)^2$$
$$= \frac{1}{18}$$

$\sqrt{\sigma_X^2} = \sqrt{1/18}$

$\sigma_X = \dfrac{1}{\sqrt{18}}$

$$\sigma_Y^2 = E(Y^2) - \mu_Y^2 = \int_x\int_y y^2 f(x,y)\, dy\, dx - \left(\int_x\int_y y\, f(x,y)\, dy\, dx\right)^2$$
$$= \int_0^1\int_x^1 y^2\, 2\, dy\, dx - \left(\int_0^1\int_x^1 y\, 2\, dy\, dx\right)^2$$
$$= \int_0^1 \frac{2y^3}{3}\Big|_x^1 dx - \left(\int_0^1 y^2\Big|_x^1 dy\, dx\right)^2$$
$$= \int_0^1 \frac{2}{3} - \frac{2x^3}{3}\, dx - \left(\int_0^1 1^2 - x^2\, dx\right)^2$$
$$= \frac{2}{3}x - \frac{2x^4}{3(4)}\Big|_0^1 - \left(x - \frac{x^3}{3}\Big|_0^1 dx\right)^2$$
$$= \frac{2}{3} - \frac{2}{12} - \left(1 - \frac{1}{3}\right)^2$$
$$= \frac{1}{18}$$

$\sigma_Y^2 = 1/18$

$\sigma_Y = \dfrac{1}{\sqrt{18}}$

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \int_0^1\int_x^1 xy\, 2\, dy\, dx - \left(\frac{1}{3}\cdot\frac{2}{3}\right)$$
$$= \int_0^1 2x \int_x^1 y\, dy\, dx - \frac{2}{9}$$
$$= \int_0^1 2x \frac{y^2}{2}\Big|_x^1 dx - \frac{2}{9}$$
$$= \int_0^1 x - x^3\, dx - \frac{2}{9}$$
$$= \frac{x^2}{2} - \frac{x^4}{4}\Big|_0^1 - \frac{2}{9}$$
$$= \frac{1}{2} - \frac{1}{4} - \frac{2}{9}$$
$$= \frac{1}{36}$$

$\sigma_{XY} = 1/36$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{1/36}{\sqrt{(1/18)(1/18)}} = \frac{1/36}{1/18} = \frac{1}{2}$$

$\therefore \boxed{\rho_{XY} = \frac{1}{2}}$

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4.57 let X be a random variable with the following probability distribution shown below, find E(X) and E(X^{2}) and then using these values evaluate E[(2X + 1)^{2}]

x	-3	6	9
f(x)	1/6	1/2	1/3

$$E(X) = \mu = -3(1/6) + 6(1/2) + 9(1/3) = 5.5$$

$$E(X^2) = \mu = (-3)^2(1/6) + (6)^2(1/2) + (9)^2(1/3) = 93/2$$

$$E(aX^2 + bX + c) = aE(X^2) + bE(X^2) + c$$

$$E\left[(2X+1)^2\right] = E(4X^2 + 4X + 1) = 4E(X^2) + 4E(X) + 1$$

$$= 4(93/2) + 4(11/2) + 1$$

$$= 209$$

$$\therefore \boxed{E\left[(2X+1)^2\right] = 209}$$

4.62 if X and Y are independent random variables with variances $\sigma_X^2 = 5$ and $\sigma_Y^2 = 3$, find the variance of the random variable Z = -2X + 4Y - 3

$X_1, X_2, \ldots, X_N \quad \sigma^2_{a_1X_1 + a_2X_2 + \cdots + a_NX_N} = a_1^2\sigma^2_{X_1} + a_2^2\sigma^2_{X_2} + \cdots + a_N^2\sigma^2_{X_N}$

$$\sigma_Z^2 = \sigma^2_{-2X+4Y-3} = \sigma^2_{-2X+4Y} = (-2)^2\sigma_X^2 + 4^2\sigma_Y^2) = 4 \cdot 5 + 16 \cdot 3 = 68$$

$\therefore \boxed{\sigma_Z^2 = 68}$