1.5. **covariance** the covariance of two random variables X and Y with means/expected valyes of μ_X and μ_Y is $\sigma_{XY} = E(XY) - \mu_X\mu_Y$

4.13 **example** there are a number of blue refills X and the number of red refills Y. two refills for a ballpoint pen are selected at random from a certain box, and the following is the joint probability distribution

f(x, y)	x = 1	x = 1	x = 2	h(y)
y = 0	3/28	9/28	3/28	5/28
y = 1	3/14	3/14	0	3/7
y = 2	1/28	0	0	1/28
g(x)	5/14	14/28	3/28	1

find the covariance of X and Y the expected value E(XY) is = 3/14

$$E(XY) = \sum_{x=0}^{2}\sum_{y=0}^{2} xyf(x,y) = (0)(0)f(0,0) = (0)(1)f(0,1) + (1)(0)f(1,0) + (1)(1)f(1,1) + (2)(0)f(2,0)$$

$$E(XY) = f(1,1) = 3/14$$

now the covariance is therefore,

$$\mu_X = \sum_{x=0}^{2} xg(x) = (0)(5/14) + (1)(15/28) + (2)(3/28) = 3/4$$

$$\mu_Y = \sum_{y=0}^{2} yh(y) = (0)(15/28) + (1)(3/7) + (2)(1/28) = 1/2$$

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = (3/14) - (3/4)(1/2) = -9/56$$

4.14 **example** the fraction X of male runners anfd the fraction Y of feamle runners who compete in marathon races are described by the joint density function, find the covariance of X and Y

$$f(x,y) = \begin{cases} 8xy, & 0 \le y \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

first we compute the marginal density function, they are

$$g(x) = \begin{cases} 4x^3, & 0 \le x \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

$$h(x) = \begin{cases} 4y(1-y^2), & 0 \le x \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

then we compuete the joint density function

$$\mu_X = E(X) = \int_0^1 \int_y 18x^2y^2 dxdy = 4/9$$

$$\sigma_{XY} = E(XY) - \mu_X\mu Y = 4/9 - (4/5)(8/15) = 4/225$$

1.6. **correlation coefficient** of X and Y, let X and Y be random variables with covariance σ_{XY} and standard deviation σ_X and σ_Y, the correlation coefficient of X and Y

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y}$$

4.15. **example** find the correlation coefficient between X and Y in example 4.13

$$E(X^2) = (0^2)(5/14) + (1^2)(15/28) + (2^2)(3/28) = 27/28$$

$$E(X^2) = (0^2)(15/28) + (1^2)(3/7) + (2^2)(1/28) = 4/7$$

$$\sigma_X^2 = 27/28 - (3/4)^2 = 45/112$$

$$\sigma_Y^2 = 4/7 - (1/2)^2 = 9/28$$

therefore the correlation coefficient between X and Y are

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{-9/56}{\sqrt{(45/112)(9/28)}} = -\frac{1}{\sqrt{5}}$$

4.16 **example** find the correlation coefficient of X and Y in example 4.14, because

$$E(X^2) = \int_0^1 4x^5 dx = \frac{2}{3}$$

$$E(Y^2) = \int_0^1 4y^3(1-y^2)dy = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\rho_{XY} = \frac{2/225}{\sqrt{(2/75)(11/225)}} = \frac{4}{\sqrt{66}}$$

4.34. **exercise** let X be a random variable with the following probability distribution

X	-2	3	5
$f(x)$	0.3	0.2	0.5

find the standard deviation of X

$$\mu = (-2)(f(-2)) + (3)(f(3)) + (5)(f(5))$$

$$\mu = (-2)(0.3) + (3)(0.2) + (5)(0.5)$$

$$E(X^2) = (0)^2(0.4) + (1)^2(0.3) + (2)^2(0.2) + (3)^2(0.1) = 2.0$$

$$\sigma_2 = E(X^2) - \mu^2 = 9.25 \text{ and } \sigma = 3.041$$

(3.62) AN ISURANCE COMPANY OFFERS IT'S POLICYHOLDERS A NUMBER OF DIFFERENT PREMIUM PAYMENT OPTIONS. FOR A RANDOMLY SELECTED POLICYHOLDER, LET X BE THE NUMBER OF MONTHS BETWEEN SUCCESSIVE PAYMENTS. THE CUMULATIVE DISTRIBUTION FUNCTION OF X IS

$$F(x) = \begin{cases} 0, & \text{IF } x < 1 \\ 0.4, & \text{IF } 1 \le x < 3 \\ 0.6, & \text{IF } 3 \le x < 5 \\ 0.8, & \text{IF } 5 \le x < 7 \\ 1.0, & \text{IF } x \ge 7 \end{cases}$$

(A) WHAT IS THE PMF OF X?

$f(0) = P(X=0) = P(X \le 0) = F(0) = 0$
$f(1) = P(X=1) = P(X \le 1) - P(X=0) = F(1) - P(X \le 0) = 0.4 - 0 = 0.4$
$f(2) = P(X=2) = P(X \le 2) - P(X=0) - P(X=1) = 0.4 - 0 - 0.4 = 0$
$f(3) = P(X=3) = P(X \le 3) - P(X=0) - P(X=1) - P(X=2) = F(3) - P(X \le 2) = F(3) - F(2) = 0.6 - 0.4 = 0.2$
$f(4) = P(X=4) = P(X \le 4) - P(X=0) - P(X=1) - P(X=2) = F(4) - P(X \le 3) = F(4) - F(3) = 0.6 - 0.6 = 0$
$f(5) = P(X=5) = P(X \le 5) - P(X=0) - \dots - P(X=3) = F(5) - P(X \le 4) = F(5) - F(4) = 0.8 - 0.6 = 0.2$
$f(6) = F(6) - F(5) = 0.8 - 0.8 = 0$
$f(7) = F(7) - F(6) = 0.2$

FOR EACH $x_0 \in \{8, 9, 10, \dots\}$ WE HAVE $f(x_0) = F(x_0) - F(x_0 - 1) = 1 - 1 = 0$

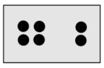
PMF FOR RV X IS

x	1	3	5	7
$f(x)$	0.4	0.2	0.2	0.2

(B) $P(4 < X \le 7) = P(X \le 7) - P(X \le 4) = F(7) - F(4) = 1 - 0.6 = 0.4$

$\boxed{P(4 < X \le 7) = 0.4}$

(3.26) FROM A BOX CONTAINING 4 BLACK BALLS & 2 GREEN BALLS, 3 BALLS ARE DRAWN IN SUCCESSION EACH BALL BEING REPLACED IN THE BOX BEFORE THE NEXT DRAW IS MADE. FIND THE PROBABILITY DISTRIBUTION FOR THE NUMBER OF GREEN BALLS.



$P(G)$ $\frac{2/6}{1}$ $\frac{2/6}{2}$ $\frac{2/6}{3}$ $P(B)$ $\frac{4/6}{1}$ $\frac{4/6}{2}$ $\frac{4/6}{3}$

$P(X=x) = \left(\frac{2}{6}\right)^x \cdot \left(\frac{4}{6}\right)^{3-x} = \frac{2^{3-x}}{27}$ OR $P(BBB) = P(B)P(B)P(B) = 4/6\ 4/6\ 4/6$
$P(BBG) = P(B)P(B)P(G) = 4/6\ 4/6\ 2/6$

$\frac{G \text{ OR } B}{}$ $\frac{G \text{ OR } B}{}$ $\frac{G \text{ OR } B}{}$
$P(B) \cdot P(B) \cdot P(B)$
$P(G) \cdot P(G) \cdot P(G)$

DENOTE X THE NUMBER OF GREEN BALLS IN THE 3 DRAWS.
LET G & B STAND FOR THE COLORS OF GREEN & BLACK
THE PROBABILITY MASS FUNCTION FOR X IS THEN

$f(0) = P(X=0)$

$f(1) = P(X=1)$

$f(2) = P(X=2)$

$f(3) = P(X=3)$

SIMPLE EVENT	X	$P(X=x) = 2^{3-x}/27$ OR $P(\_\ \_\ \_)$
B B B	0	$2^{3-0}/27 = 8/27$ OR $P(BBB) = (4/6)(4/6)(4/6) = 8/27$
B B G	1	$P(BBG) = (4/6)(4/6)(2/6) = 32/216 = 4/27$
B G B	1	$P(BGB) = (4/6)(2/6)(4/6) = 32/216 = 4/27$
G B B	1	$P(GBB) =$ " " " $= 4/27$
B G G	2	$P(BGG) = (4/6)(2/6)(2/6) = 2/27$
G B G	2	$P(GBG) =$ " " " $= 2/27$
G G B	2	$P(GGB) =$ " " " $=$
G G G	3	$P(GGG) = (2/6)(2/6)(2/6) = 1/27$

PROBABILITY DISTRIBUTION OF X IS

x	0	1	2	3
$P(X=x)$	8/27	4/9	2/9	1/27

$P(X=3) = P(GGG) = 1/27$
$P(X=3) = 1/27$
$P(X=2) = P(BGG) + P(GBG) + P(GGB) = 2/27 + 2/27 + 2/27 = 6/27$
$P(X=2) = 2/9$
$P(X=1) = P(BBG) + P(BGB) + P(GBB) = 4/27 + 4/27 + 4/27 = 12/27 = 4/9$
$P(X=1) = 4/9$
$P(X=0) = P(BBB) = 8/27$
$P(X=0) = 8/27$

(3.38) IF THE JOINT PROBABILITY DISTRIBUTION OF X & Y IS GIVEN BY FIND

$f(x,y) = \frac{x+y}{30}$, FOR $x = 0, 1, 2, 3$
FOR $y = 0, 1, 2$

JOINT PROBABILITY DISTRIBUTION OF (X, Y) IS

$f(x,y)$	$x=0$	$x=1$	$x=2$	$x=3$
$y=0$	0	1/30	2/30	3/30
$y=1$	1/30	2/30	3/30	4/30
$y=2$	2/30	3/30	4/30	5/30

(A) $P(X \le 2, Y=1) = f(0,1) + f(1,1) + f(2,1)$
$= 1/30 + 2/30 + 3/30$
$= 6/30$
$\boxed{P(X \le 2, Y=1) = 1/5}$

(B) $P(X > 2, Y \le 1) = f(3,1) + f(3,0)$
$= 3/30 + 4/30$
$\boxed{P(X > 2, Y \le 1) = 7/30}$

(C) $P(X > Y) = f(1,0) + f(2,0) + f(3,0) + f(2,1) + f(3,1) + f(3,2)$
$= 1/30 + 2/30 + 3/30 + 1/30 + 4/30 + 5/30 = 16/30$
$\boxed{P(X > Y) = 8/15}$

(D) $P(X+Y=4) = f(2,2) + f(3,1)$
$= 4/30 + 4/30$
$= 8/30$
$\boxed{P(X+Y=4) = 4/15}$

(3.43) LET X DENOTE THE REACTION TIME, IN SECONDS, TO A CERTAIN STIMULUS & Y DENOTE THE TEMPERATURE (°F) AT WHICH A CERTAIN REACTION STARTS TO TAKE PLACE. SUPPOSE THAT TWO RANDOM VARIABLES X & Y HAVE THE JOINT DENSITY FIND

$$f(x,y) = \begin{cases} 4xy, & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{ELSEWHERE} \end{cases}$$

(A) $P(0 \le X \le 1/2,\ 1/4 \le Y \le 1/2) \Rightarrow \int_0^{1/2}\int_{1/4}^{1/2} 4xy\, dy\, dx = \int_0^{1/2} 4x \int_{1/4}^{1/2} y\, dy\, dx =$

$$= \int_0^{1/2} 4x \left.\frac{y^2}{2}\right|_{1/4}^{1/2} dx = \int_0^{1/2} 4x \left[\frac{1/2^2}{2} - \frac{1/4^2}{2}\right] dx = \int_0^{1/2} 4x\ \frac{1/4}{2} - \frac{1/16}{2}\ dx$$

$$= \int_0^{1/2} 4x\ \frac{4}{32} - \frac{1}{32}\ dx = \int_0^{1/2} 4x\ \frac{3}{32}\ dx = \frac{3}{8}\int_0^{1/2} 4x\, dx = \frac{3}{8}\left(\frac{4x^2}{2}\right)\Bigg|_0^{1/2} dx$$

$$= \frac{3}{8}\left(\frac{4(1/2)^2}{2}\right) = \frac{3}{8}\left(\frac{1}{2}\right) \boxed{= \frac{3}{16}}$$

(B) $\boxed{P(X < Y) = \int_0^1\int_0^y 4xy\, dx\, dy = 2\int_0^1 y^3 dy = \frac{1}{2}}$

probability and stochastic processes

statistics

1. sample mean:

$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

2. sample median if n is even: $\tilde{x} = \frac{x_{\frac{n+1}{2}} + x_{\frac{n+2}{2}}}{2}$
3. sample median if n is odd $\tilde{x} = x_{\frac{n+1}{2}}$
4. sample variance: $X_1, X_2, \cdots, X_n$ is denoted by σ^2

$$\sigma^2 = s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

5. sample standard deviation: $s = \sqrt{s^2}$

probability

0. event operations:

- $A \cap \emptyset = \emptyset$
- $A \cup \emptyset = A$
- $A \cap A' = \emptyset$, $A \cup A' = S$
- $S' = \emptyset$, $\emptyset' = S$
- $(A \cap B)' = A' \cup B'$
- $(A \cup B)' = A' \cap B'$

1. $(A \cup B)^c = (A^c \cap B^c)$
2. mutually exclusive events A_1 and A_2, $P[A_1 \cup A_2] = P[A_1] + P[A_2]$
3. if $A = A_1 \cup A_2 \cup \cdots \cup A_m \text{ and } A_i \cap A_j = \emptyset \text{ for all } i \neq j \text{, then }$ $ P[A] = \sum_{i=1}^m P[A_{i}] $
4. $P[.]$, $P[\emptyset] = 0$, $P[A^c] = 1 - P[A]$, $P[A \cup B] = P[A] + P[B] - P[A \cap B]$, and $A \subset B, P[A] \leq P[B]$
5. $B = s_1, s_2, \cdots, s_m$ is the sum of the probabilities of the outcomes contained in the event:

$$P[B] = \sum_{i=1}^{m} P[s_i]$$

6. $S = s_1, s_2, \cdots, s_n$ in which each outcomes s_i is equally likely,

$$P[s_i] = \frac{1}{n} \quad 1 \leq i \leq n$$

7. conditional probability measure $P[A|B]$

- $P[A|B] \geq 0$
- $P[B|B] = 1$
- $A = A_1 \cup A_2 \cup \cdots \cup A_m$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$, then, $P[A|B] = P[A_1|B] + P[A_2|B] + \cdots + P[A_m|B]$

8. partition $B = B_1, B_2, \cdots, B_m$ and any event A in the sample space, let $C_i = A \cap B_i$ For $i \neq j$, the events C_i and C_j are mutually exclusive and $A = C_1 \cup C_2 \cup \cdots$
9. A and partition $B_1, B_2, \cdots, B_m$,

$$P[A] = \sum_{i=1}^{m} P[A \cap B_i]$$

10. law of total probability $B_1, B_2, \cdots, B_m$ with $P[B_i] > 0$ for all i,

$$P[A] = \sum_{i=1}^{m} P[A|B_i]P[B_i]$$

11. conditional probability

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \text{if } P(A) > 0$$

12. bayes' theorem

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i|A)P(B_i)}$$

$$P[B|A] = \frac{P[A|B]P[B]}{P[A]}$$

13. independent events two events A and B are independent if

$$P[AB] = P[A]P[B]$$

3. Discrete Random Variables

1. **probability mass function** is the set of ordered pars $(x, f(x))$ is a probability function, probability mass function, or probability distribution of the discrete random variable X if for each possible outcome x.
2. $f(x) \geq 0$
3. $\sum f(x) = 1$
4. $P(X = x) = f(x)$

let X be the random variable: number of heads in 3 tosses of a fair coin.

sample space	x
TTT	0
TTH	1
THT	1
THH	2
HTT	1
HTH	2
HHT	2
HHH	3

the probability $P(X = x)$ that the outcome is a specific x value is the probability that the number of heads is x.

x	0	1	2	3
$P(X = x)$	1/8	3/8	3/8	1/8

2. **the cumulative distribution function** $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \text{ for } -\infty \leq x \leq \infty$$

3. continuous random variable has a probability of zero assuming exactly any of its values $P(a < X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a \leq X \leq b)$
4. the function $f(x)$ is a **probability density function** (pdf) for the continuous random variable X, defined over the set of real numbers, if
5. $f(x) \geq 0$ for all $x \in R$
6. $\int_{-\infty}^{\infty} f(x)dx = 1$
7. $P(a < X < b) = \int_a^b f(x)dx$
8. definition of joint probability distribution / probability mass function $f(x, y) = P(X = x, Y = y)$ the function $f(x, y)$ is a joint probability distribution or **probability mass function** of the discrete random variables X and Y if
9. $f(x, y) \geq 0$ for all (x, y)
10. $\sum_x \sum_y f(x, y) = 1$
11. $P(X = x, Y = y) = f(x, y)$

for any region A in the xy plane, $P[(X, Y) \in A] = \sum \sum_A f(x, y)$

mathematical expectation

1.1. **expeted value** let X be a random variable with probability distribution $f(x)$, the **mean** or **expected value** of X is

if X is discrete

$$\mu = E(X) = \sum_x x f(x) dx$$

if X is continuous

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

4.1. **example** a lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. a sample of 3 is taken by the inspector. find the expected value of the number of good components in this sample.

let X represent the number of good components in the sample. the probability distribution of X is

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, x = 0, 1, 2, 3$$

$f(0) = 1/35$, $f(1) = 12/35$, $f(2) = 18/35$, $f(3) = 4/35$

$$\mu = E(X) = \sum_x x f(x) dx = (0)\frac{1}{35} + (1)\frac{12}{35} + (2)\frac{18}{35} + (3)\frac{4}{35} = 1.7$$

therefore if a sample size 3 is selected at random over and over again from a lot of 4 good components and 3 defective components, it will contain on average 1.7 good components

4.4. **example** a coin is biased such that a head is three times as likely to occur as a tail. Find the expected number of tails when this coin is tossed twice.

assigning weights of $3w$ and w for a head and tail respectively. we obtain $P(H) = 3/4$ and $P(T) = 1/4$. the sample space for the experiment is $S = HH, HT, TH, TT$. now if X represents the number of trials that occur in two tosses of coins, we have

$P(X = 0) = P(HH) = (3/4)(3/4) = 9/16$

$P(X = 1) = P(HT) + P(TH) = (3/4)(1/4) + (1/4)(3/4) = 3/8$

$P(X = 2) = P(TT) = (1/4)(1/4) = 1/16$

the probability distribution for X is then

x	0	1	2
$f(x)$	9/16	3/8	1/16

expected value is $\mu = E(X) = (0)(9/16) + (1)(3/8) + (2)(1/16) = 1/2$

1.2. **expected value** let X and Y be random variables with joint probability distribution $f(x, y)$. the mean or expected value of the random variable $g(X, Y)$ is

if X and Y are discrete

$$\mu_{g(X,Y)} = E[g(X, Y)] = \sum_x \sum_y g(x, y) f(x, y)$$

if X and Y are continuous

$$\mu_{g(X,Y)} = E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy$$

1.3. **standard deviation** let X be a random variable with probability distribution $f(x)$ and mean μ. the variance of X is σ^2. the variance σ^2 is called the standard deviation.

if X is discrete

$$\sigma^2 = E[(X - \mu)^2] = \sum_x (x - \mu)^2 f(x) dx$$

if X is continuous

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) ds$$

1.4. the variance of the random variable X is $\sigma^2 = E(X^2) - \mu^2$

1.5. let X be a random variable with probability distribution $f(x)$ the variance of the random variable $g(X)$ is

if X is discrete

$$\sigma^2_{g(X)} = E[g(X) - \mu^2_{g(X)} = \sum_x [g(x) - \mu_{g(X)}]^2 f(x)$$

if X is continuous

$$\sigma^2_{g(X)} = E[g(X) - \mu_{g(X)}]^2 = \int_{-\infty}^{\infty} [g(x) - \mu_{g(X)}]^2 f(x) dx$$

1.6. **covariance** let X and Y be random variables with joint probability distribution $f(x, y)$ the covariance of X and Y is if X and Y are discrete

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \sum_x sum_y (x - \mu_X)(y - \mu_y) f(x, y)$$

if X and Y are continuous

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_y) f(x, y) dx dy$$