EXAM 1

MATH 526 SPRING 2023(4232-41285)

So

Mar.1, 2023

INSTRUCTIONS: Print your name, student ID and signature in the space below. Read the questions carefully and completely. Answer each question and show work in the space provided. Partial credit will be awarded for useful work. There are 5 questions and 85 points in total and You will have approximately 50 minutes to write the exam.

Name:

Student Number:

Signature:

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Question 1:(10 points) Consider the sample space $S = \{0, 1, 2, 3, 4, 5\}$ and the event $A = \{0, 2, 4\}$ and $B = \{1, 4, 5\}$. List the elements of the following events: $A \cup B$, A' and $A' \cap B$.

$A \cup B = \{0, 1, 2, 4, 5\}$

$A' = \{1, 3, 5\}$

$A' \cap B = \{1, 5\}$

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Question 2:(15 points) Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} k(x+1)^2 & -1 < x < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

(a)What is k;
(b)Find the cumulative distribution function $F(x)$.
(c) Use $F(x)$ to evaluate $P(X > \frac{1}{2})$.

(a) $\int_{-1}^{1} k(x+1)^2 dx = 1 \Rightarrow k\int_{-1}^{1} (x^2+2x+1)dx = 1$

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$\Rightarrow k\left[\frac{x^3}{3} + 2\cdot\frac{x^2}{2} + x\right]_{-1}^{1} = 1$

$\Rightarrow \frac{8}{3}k = 1 \Rightarrow k = \frac{3}{8}$

(b). $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-1}^{x} \frac{3}{8}(x+1)^2 dx = \frac{3}{8}\left[\frac{x^3}{3} + x^2 + x\right]_{-1}^{x}$

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$= \frac{3}{8}\left(\frac{x^3}{3} + x^2 + x + \frac{1}{3}\right) = \frac{x^3}{8} + \frac{3x^2}{8} + \frac{3x}{8} + \frac{1}{8}$

Thus $F(x) = \begin{cases} 0 & x \le -1 \\ \frac{x^3}{8} + \frac{x^2}{8} + \frac{x}{8} + \frac{1}{8} & -1 < x < 1 \\ 1 & x \ge 1 \end{cases}$

(c). $P(X > \frac{1}{2}) = 1 - P(X \le \frac{1}{2}) = 1 - F(\frac{1}{2}) = 1 - \left[\frac{(\frac{1}{2})^3}{8} + \frac{3(\frac{1}{2})^2}{8} + \frac{3\cdot\frac{1}{2}}{8} + \frac{1}{8}\right]$

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$= 1 - \frac{27}{64} = \frac{37}{64} \approx 0.5781$

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Question 3:(14 points) A rare disease exists with which only 1 in 500 is affected. A test for the disease exists, but of course it is not infallible. A correct positive result (patient actually has the disease) occurs 95% of the time, while a false positive result (patient does not have the disease) occurs 1% of the time. If a randomly selected individual is tested and the result is positive, what is the probability that the individual has the disease?

Let $D :=$ A rare disease with only 1 in 500.

$+ :=$ positive result

$- :=$ negative result.

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$P(D) = \frac{1}{500}$ $\quad P(D') = \frac{499}{500}$ $\quad P(+|D) = 0.95$. $\quad P(-|D) = 1 - 0.95 = 0.05$

$P(+|D') = 0.01$ $\quad P(-|D') = 1 - 0.01 = 0.99$

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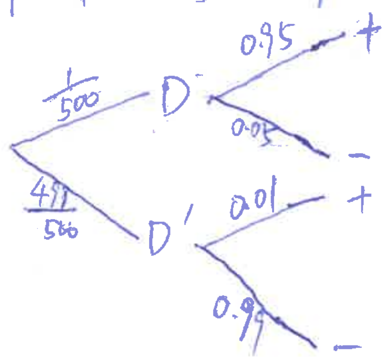

$\frac{1}{500} \times 0.95 = P(D \cap +)$

$\frac{1}{500} \times 0.05 = P(D \cap -)$

$\frac{499}{500} \times 0.01 = P(D' \cap +)$

$\frac{499}{500} \times 0.99 = P(D' \cap -)$

$$P(D|+) = \frac{P(D \cap +)}{P(+)} = \frac{P(D \cap +)}{P(D \cap +) + P(D' \cap +)}$$

$$= \frac{\frac{1}{500} \times 0.95}{\frac{1}{500} \times 0.95 + \frac{499}{500} \times 0.01} = \frac{0.0019}{0.0019 + 0.00998}$$

$$\approx 0.1599$$

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Question 4:(16 points) A lot containing 7 components is sampled by a quality inspector. The lot contains 4 good components and 3 defective components. Three components are selected at random without replacement. Let X be the number of good components which are selected.
(a) Find the probability function of X;
(b) Find the probability that at least one selected component is good.

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$$f(x) = P(X=x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}} \qquad x = 0, 1, 2, 3.$$

$$f(0) = P(X=0) = \frac{\binom{4}{0}\binom{3}{3}}{\binom{7}{3}} = \frac{1}{35}$$

$$f(1) = P(X=1) = \frac{\binom{4}{1}\binom{3}{2}}{\binom{7}{3}} = \frac{12}{35}$$

$$f(2) = P(X=2) = \frac{\binom{4}{2}\binom{3}{1}}{\binom{7}{3}} = \frac{18}{35}$$

$$f(3) = P(X=3) = \frac{\binom{4}{3}\binom{3}{0}}{\binom{7}{3}} = \frac{4}{35}$$

Thus the probability function of X is

x	0	1	2	3
$f(x)$	$\frac{1}{35}$	$\frac{12}{35}$	$\frac{18}{35}$	$\frac{4}{35}$
	0.0285	0.342	0.514	0.114

(b). $P(X \geq 1) = 1 - P(X=0) = 1 - \frac{1}{35} = \frac{34}{35} \approx 0.9714$

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$$\rho_{XY}=\frac{\sigma_{XY}}{\sigma_X\cdot\sigma_Y}=\frac{-\frac{1}{144}}{\sqrt{\frac{11}{144}}\cdot\sqrt{\frac{11}{144}}}=-\frac{1}{11}\approx-0.0909$$

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Question 5:(30 points) Consider the random variables X and Y with joint density function

$$f(x,y)=\begin{cases}x+y, & 0<x<1, 0<y<1,\\ 0, & \text{elsewhere.}\end{cases}$$

(a) Find the marginal distribution of X and Y;
(b)Are X and Y independent? Verify your results by calculation;
(c)Find $P(0<X<\frac{1}{2}|Y=\frac{1}{2})$;
(d)Find the correlation coefficient of two random variables X and Y

(a) 6 $g(x)=\int_0^1 (x+y)dy=(xy+\frac{1}{2}y^2)\Big|_0^1=x+\frac{1}{2}$, $0<x<1$.

$h(y)=\int_0^1 (x+y)dx=(\frac{1}{2}x^2+xy)\Big|_0^1=y+\frac{1}{2}$, $0<y<1$.

(b). 6 $g(x)\cdot h(y)=(x+\frac{1}{2})(y+\frac{1}{2})\neq x+y$, for $0<x<1, 0<y<1$

So X and Y are NOT independent.

6 (c). $f(x|y)=\frac{f(x,y)}{h(y)}=\frac{x+y}{y+\frac{1}{2}}$ $0<x<1$, $0<y<1$.

So $P(0<X<\frac{1}{2}|Y=\frac{1}{2})=\int_0^{\frac{1}{2}}\frac{(x+\frac{1}{2})}{\frac{1}{2}+\frac{1}{2}}dx=\int_0^{\frac{1}{2}}(x+\frac{1}{2})dx$

$=(\frac{1}{2}x^2+\frac{1}{2}x)\Big|_0^{\frac{1}{2}}=\frac{3}{8}$

12 (d). $\mu_X=E(X)=\int_0^1 x(x+\frac{1}{2})dx=(\frac{x^3}{3}+\frac{x^2}{4})\Big|_0^1=\frac{7}{12}$, $\mu_Y=\frac{7}{12}$.

$E(X^2)=\int_0^1 x^2(x+\frac{1}{2})dx=(\frac{x^4}{4}+\frac{1}{2}\cdot\frac{1}{3}x^3)\Big|_0^1=\frac{5}{12}$, $E(Y^2)=\frac{5}{12}$.

$\sigma_X^2=E(X^2)-\mu_X^2=\frac{5}{12}-(\frac{7}{12})^2=\frac{11}{144}$, $\sigma_Y^2=\frac{11}{144}$.

$E(XY)=\int_0^1\int_0^1 (xy)(x+y)dxdy=\int_0^1\int_0^1(x^2y+xy^2)dx\,dy=\int_0^1(\frac{y}{3}x^3+\frac{y^2}{2}x^2)\Big|_{x=0}^{x=1}dy$

$=\int_0^1(\frac{y}{3}+\frac{y^2}{2})dy=(\frac{1}{3}\cdot\frac{y^2}{2}+\frac{1}{2}\cdot\frac{y^3}{3})\Big|_{y=0}^{y=1}=\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$

$\sigma_{XY}=E(XY)-\mu_X\cdot\mu_Y=\frac{1}{3}-(\frac{7}{12})(\frac{7}{12})=-\frac{1}{144}$