CS32420: Computer Graphics & Games

Core Mathematics for 3D Graphics (2): Vectors and Matrices

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More on Vectors

Vectors

Vector: geometric (3D) objects that have magnitude and direction, and satisfy certain algebraic rules

Can be defined independent of a coordinate system,
generally work with components referred to a coordinate system

We have looked at: basic rules, addition and subtraction,
now: unit vectors, dot product, and cross product

Unit Vectors

Unit vector: a vector with a length (magnitude) of 1

Usually denoted with a hat,
e.g. $\hat{a}$

Any non-zero vector can be scaled into a unit vector by dividing it by the magnitude i.e. *normalization*

$$\text{e.g. } \hat{a} = \frac{a}{|a|} = \frac{a}{\sqrt{(a_x)^2 + (a_y)^2 + (a_z)^2}}$$

Note: a unit does not have any specific real-world correspondence – your unit could be millimetres, miles, kilometres, or light-years...

Operations: Dot Product

Dot product: multiply two vectors to obtain a scalar value

Also known as *scalar product* and *inner product*

Represented by a central dot,

e.g. $a \bullet b$

Some important laws:

Commutative: $a \bullet b = b \bullet a$

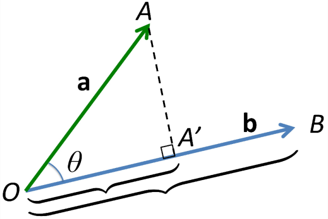
Associative[1]: $(\alpha a) \bullet b = \alpha(a \bullet b)$

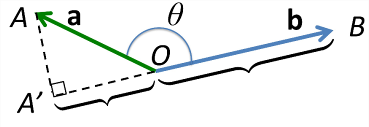
Distributive: $(a + b) \bullet c = a \bullet c + b \bullet c$

[1] with respect to scalar multiplication only

Dot Product Between Two Vectors

Here we have two examples of $a \bullet b$:



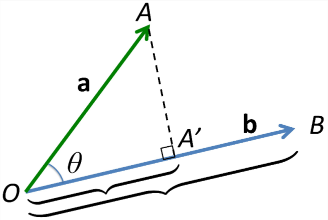


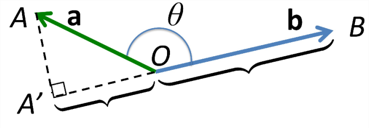
$$a \bullet b \equiv \overrightarrow{OA'} \text{ times } \overrightarrow{OB}$$

i.e. (the projection of a onto b) $\times$ magnitude of b

Dot Product Between Two Vectors

The angle (θ) between the two vectors affects
the sign of the result of $a \bullet b$ (i.e. whether it's +ve or −ve)



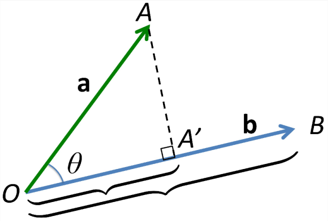


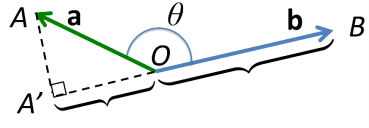
$$\text{sign}(a \bullet b) = \begin{cases} > 0 & \text{if } 0 < |\theta| < \frac{\pi}{2} \quad \text{(acute)} \\ 0 & \text{if } \theta = \pm\frac{\pi}{2} \quad \text{(right angles)} \\ < 0 & \text{if } \frac{\pi}{2} < |\theta| < \pi \quad \text{(obtuse)} \end{cases}$$

Dot Product Between Two Vectors

This is because the definition of dot product is:

$$\boxed{a \bullet b = |a||b| \cos \theta}$$





Remember: $|a| = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}$

Dot Product Between Two Vectors

If we have two fully-specified vectors, we can calculate the dot product just using vector components:

$$a = [a_x, a_y, a_z]$$
$$b = [b_x, b_y, b_z]$$

Can express this as a matrix row times a matrix column (we will come back to matrices later...)

$$a \bullet b = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

This form is usually known as *inner product*:

$$\boxed{a \bullet b = a_x b_x + a_y b_y + a_z b_z}$$

Dot Product Between Two Vectors

We can rearrange the two formulae to tell us what the angle between the two vectors is:

$$\boxed{\begin{gathered} a \bullet b = |a||b| \cos\theta \\ a \bullet b = a_x b_x + a_y b_y + a_z b_z \\ \downarrow \\ \cos\theta = \frac{a \bullet b}{|a||b|} \end{gathered}}$$

Dot Product Between Two Vectors

If b is the same as a, then the angle is zero:

$$\boxed{\begin{gathered} a \bullet b = |a||b| \cos\theta \\[2ex] b = a, \quad \theta = 0, \quad \cos(0) = 1 \\ \downarrow \\ a \bullet a = |a||a| \times 1 \\ \downarrow \\ a \bullet a = |a|^2 \\ \downarrow \\ \sqrt{a \bullet a} = |a| \end{gathered}}$$

Method of getting size (magnitude) irrespective of direction,
i.e. converting the vector to a scalar value

Dot Product Examples

If $a = [1, 2, -2]$, what is $|a|$?

$$|a|^2 = a \bullet a$$

$$a \bullet b = a_x b_x + a_y b_y + a_z b_z$$

$$a \bullet a = (1 \times 1) + (2 \times 2) + (-2 \times -2)$$

$$|a|^2 = 9$$

$$|a| = \sqrt{9}$$

$$|a| = 3$$

Dot Product Examples

If $b = [2, 1, 2]$, what is $a \bullet b$?

$$a = [1, 2, -2], \quad b = [2, 1, 2]$$

$$a \bullet b = a_x b_x + a_y b_y + a_z b_z$$

$$a \bullet b = (1 \times 2) + (2 \times 1) + (-2 \times 2)$$

$$a \bullet b = 2 + 2 + (-4)$$

$$a \bullet b = 0$$

$\Rightarrow$ a and b are at right angles

Operations: Cross Product

Cross product: multiply two vectors to obtain a vector that is perpendicular to the inputs with a magnitude equal to the area of the parallelogram spanned by the inputs

Also known as *vector product*

Represented by a central cross,
e.g. $a \times b$

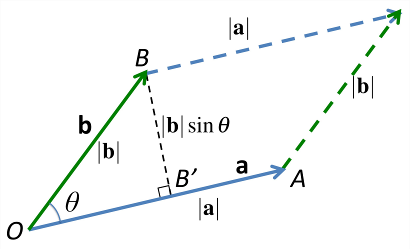
Some important laws:
Anti-commutative: $a \times b = -b \times a$
Associative: $(\alpha a) \times b = \alpha(a \times b)$
Distributive: $a \times (b + c) = a \times b + a \times c$

Cross Product Between Two Vectors

The magnitude of cross product is defined by:

$$\boxed{|a \times b| = |a||b| \sin\theta \qquad (0 \leq \theta \leq \pi)}$$

Which gives you the area of the parallelogram formed by a and b:

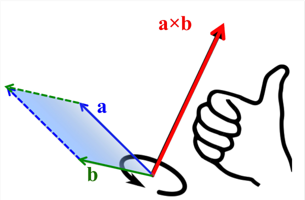


Remember: $|a| = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}$

Cross Product Between Two Vectors

The direction of the output vector is perpendicular to the plane containing the input vectors (i.e. normal)

The direction can be found using the **right hand rule**:



1. point your hand along the first vector
2. curl your fingers towards the second vector
3. stick out your thumb!

This can give you a useful estimate, but remember that the resulting vector will have a direction you can calculate from the origin to the coordinates specified

Cross Product Between Two Vectors

Some useful things to note:

- When vectors a and b are parallel, the area spanned is zero, giving $a \times b = 0$; in particular, $a \times a = 0$
- When vectors a and b are perpendicular, $\sin\theta = 1$, and the magnitude of the area is $|a||b|$. Vectors a, b, and $(a \times b)$ then are mutually orthogonal (i.e. all at 90° to each other)
- Vectors $a \times b$ and $b \times a$ have the same magnitude $|a||b|\sin\theta$, but by the right-hand rule will have opposite direction: $a \times b = -b \times a$

Cross Product Between Two Vectors

Another definition of cross product is:

$$a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \quad b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$\boxed{a \times b = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}}$$

Cross Product Example

If $a = [1, 2, -2]$ and $b = [3, 4, 12]$,
what is the cross product of vectors a and b?

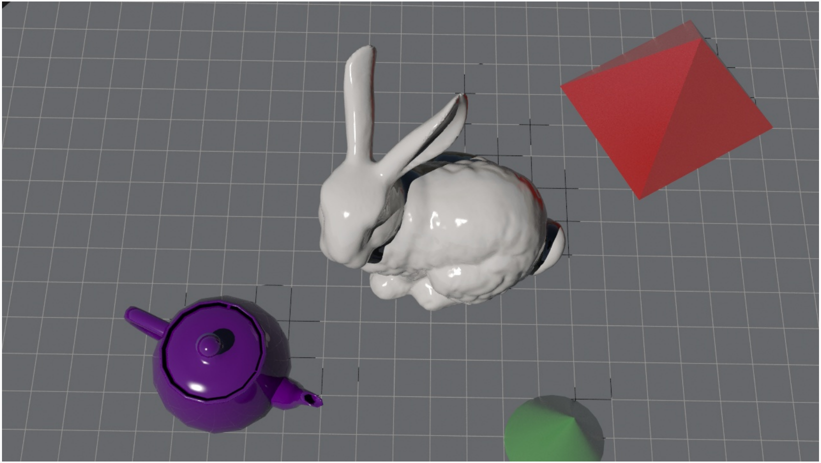
$$a \times b = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

$$a \times b = \begin{bmatrix} (2 \times 12) - (-2 \times 4) \\ (-2 \times 3) - (1 \times 12) \\ (1 \times 4) - (2 \times 3) \end{bmatrix} = \begin{bmatrix} (24) - (-8) \\ (-6) - (12) \\ (4) - (6) \end{bmatrix} = \begin{bmatrix} 32 \\ -18 \\ -2 \end{bmatrix}$$

Matrices for Transformations

Techniques for Transformations

The transformations act on the objects coordinates,
keeping the structural information unaltered



Transformations

We can use individual expressions for these transformations, e.g.

Translation

$$x' = x + T_a$$

Rotation

$$x' = x \cos\theta - y \sin\theta$$

Scaling

$$x' = x \times S_a$$

Transformations

It is very difficult to chain these expressions...

e.g. I want to translate and scale an object,
but must do operations for every point in the same/correct order

$$\boxed{x + T_a \times S_a \quad \neq \quad x \times S_a + T_a \quad \neq \quad x \times (S_a + T_a)}$$

Matrices to the Rescue!

We have a matrix of coordinates already

$$\begin{bmatrix} -4 & 4 & 4 & -2 & -4 & -4 & -2 & 4 & 4 & -4 & -3 & -2 & -2 & -3 & 4 & 4 & 4 & 4 \\ -3 & -3 & -3 & -3 & -3 & 3 & 3 & 3 & 3 & 3 & -3 & -3 & -3 & -3 & -2 & 2 & 2 & -2 \\ 0 & 0 & 3 & 5 & 3 & 3 & 5 & 3 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 2 & 2 \end{bmatrix}$$

We can multiply the object matrix by a transformation matrix

Matrix Multiplication: Scalars

To multiply a matrix by a scalar,
you simply multiply each element by the scalar:

$$\boxed{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = a \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}}$$

$$x' = ax$$

$$y' = ay$$

$$z' = az$$

Matrix Multiplication: Matrices

You can only multiply matrices of the *same length*

(row to column: $abc \equiv xyz$)

$$\boxed{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}}$$

$$x' = ax + by + cz$$

$$y' = dx + ey + fz$$

$$z' = gx + hy + iz$$

Matrix Multiplication: Matrices

You can only multiply matrices of the *same length*
(row to column: $abc \equiv xyz$)

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \times \begin{bmatrix} o & r & u \\ p & s & v \\ q & t & w \end{bmatrix} =$$

$$\begin{bmatrix} (ao + bp + cq) & (ar + bs + ct) & (au + bv + cw) \\ (do + ep + fq) & (dr + es + ft) & (du + ev + fw) \\ (go + hp + iq) & (gr + hs + it) & (gu + hv + iw) \end{bmatrix}$$

Matrix Multiplication

Special case: identity matrix

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$x' = x$$

$$y' = y$$

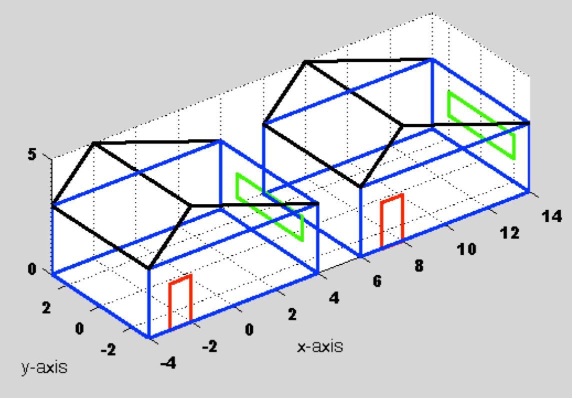
$$z' = z$$

Awesome Website Time!

`http://matrixmultiplication.xyz`

Let's Look at an Example

Translate object by $+10$ units along the x-axis



$$(x', y', z') = (x + 10, y, z)$$

A Complication...

Translation is problematic...

Translating vertex-by-vertex uses a constant term which conflicts with matrix multiplication:

$$x' = ax + by + cz$$

$$x' = x + T_x$$

A Solution

To deal with the constant term, we move to a 4×4 matrix:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} ax & + & by & + & cz & + & (d \times 1) \\ ex & + & fy & + & gz & + & (h \times 1) \\ ix & + & jy & + & kz & + & (l \times 1) \\ mx & + & ny & + & oz & + & (p \times 1) \end{bmatrix}$$

A Solution

Let's look at how this works with an identity matrix:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} (1 \times x) & + & (0 \times y) & + & (0 \times z) & + & (T_x \times 1) \\ (0 \times x) & + & (1 \times y) & + & (0 \times z) & + & (T_y \times 1) \\ (0 \times x) & + & (0 \times y) & + & (1 \times z) & + & (T_z \times 1) \\ (0 \times x) & + & (0 \times y) & + & (0 \times z) & + & (1 \times 1) \end{bmatrix}$$

A Solution

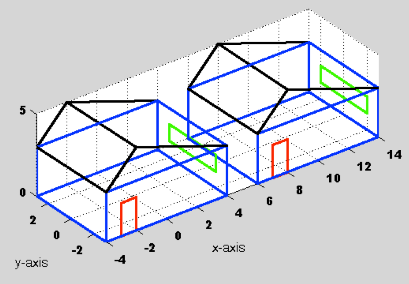
Let's look at how this works with an identity matrix:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} x & + & 0 & + & 0 & + & T_x \\ 0 & + & y & + & 0 & + & T_y \\ 0 & + & 0 & + & z & + & T_z \\ 0 & + & 0 & + & 0 & + & 1 \end{bmatrix} = \begin{bmatrix} x + T_x \\ y + T_y \\ z + T_z \\ 1 \end{bmatrix}$$

Back to the Example

Example: translate object by $+10$ units along the x-axis



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + 10 \\ y + 0 \\ z + 0 \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

Homogeneous Coordinates

Another reason to use a 4×4 matrix:

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ W' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}$$

Introduce a fourth dimension W

Note the use of caps for **homogeneous coordinates**

Homogeneous Coordinates

Define physical coordinates to be ratios of
pairs of homogeneous coordinates

$$x = X/W$$

$$y = Y/W$$

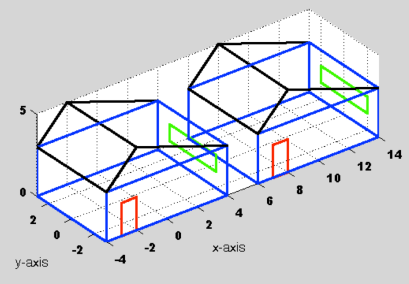
$$z = Z/W$$

$$\textbf{where } W \neq 0$$

if $W = 1$, coord values don't change,
as W approaches 0, the coords approach infinity

Back to the Example (again...)

Example: translate object by +10 units along the x-axis



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + 10 \\ y + 0 \\ z + 0 \\ 1 \end{bmatrix}$$

Example

The right hand expression achieves the goal of isolating all the coordinates as a multiplicative factor

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ W' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

We still have to choose the value of W

Can use $W = 1$
to make $x = X/1$, etc ...

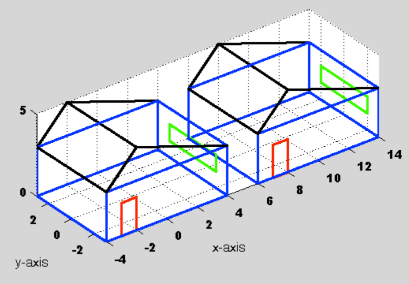
Example

Usually, we are happy for W to be 1, so you will commonly see lower case used directly, e.g.

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Example

The procedure is:



(1) convert physical coords to homogeneous coords

(2) carry out the transformation

(3) convert back to physical coords

(1) convert physical coords to homogeneous coords

$$\begin{bmatrix} -4 & 4 & 4 & -2 & -4 & -4 & -2 & 4 & 4 & -4 & -3 & -2 & -2 & -3 & 4 & 4 & 4 & 4 \\ -3 & -3 & -3 & -3 & -3 & 3 & 3 & 3 & 3 & 3 & -3 & -3 & -3 & -3 & -2 & 2 & 2 & -2 \\ 0 & 0 & 3 & 5 & 3 & 3 & 5 & 3 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 2 & 2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} -4 & 4 & 4 & -2 & -4 & -4 & -2 & 4 & 4 & -4 & -3 & -2 & -2 & -3 & 4 & 4 & 4 & 4 \\ -3 & -3 & -3 & -3 & -3 & 3 & 3 & 3 & 3 & 3 & -3 & -3 & -3 & -3 & -2 & 2 & 2 & -2 \\ 0 & 0 & 3 & 5 & 3 & 3 & 5 & 3 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(2) carry out the transformation

$$\begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} -4 & 4 & 4 & -2 & -4 & \cdots & 4 \\ -3 & -3 & -3 & -3 & -3 & \cdots & -2 \\ 0 & 0 & 3 & 5 & 3 & \cdots & 2 \\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 6 & 14 & 14 & 8 & 6 & 6 & 8 & 14 & 14 & 6 & 7 & 8 & 8 & 7 & 14 & 14 & 14 & 14 \\ -3 & -3 & -3 & -3 & -3 & 3 & 3 & 3 & 3 & 3 & -3 & -3 & -3 & -3 & -2 & 2 & 2 & -2 \\ 0 & 0 & 3 & 5 & 3 & 3 & 5 & 3 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(3) convert back to physical coords

$$\begin{bmatrix} 6 & 14 & 14 & 8 & 6 & 6 & 8 & 14 & 14 & 6 & 7 & 8 & 8 & 7 & 14 & 14 & 14 & 14 \\ -3 & -3 & -3 & -3 & -3 & 3 & 3 & 3 & 3 & 3 & -3 & -3 & -3 & -3 & -2 & 2 & 2 & -2 \\ 0 & 0 & 3 & 5 & 3 & 3 & 5 & 3 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 6 & 14 & 14 & 8 & 6 & 6 & 8 & 14 & 14 & 6 & 7 & 8 & 8 & 7 & 14 & 14 & 14 & 14 \\ -3 & -3 & -3 & -3 & -3 & 3 & 3 & 3 & 3 & 3 & -3 & -3 & -3 & -3 & -2 & 2 & 2 & -2 \\ 0 & 0 & 3 & 5 & 3 & 3 & 5 & 3 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 2 & 2 \end{bmatrix}$$

Note that this easily generalises to arbitrary shifts,
not just in the x-direction

Wrapping Up

Summary

- Vectors and trigonometry
  - We can calculate angles using trigonometric rules; we can apply these in vector multiplication (dot and cross product)
- Matrices
  - Transformations are much easier to manage using matrices to process coordinates
- Homogeneous Coordinates
  - By introducing a fourth term W, we can easily add translations to any axis

Next time...

Core Mathematics for 3D Graphics (3):
Matrices for Transformations