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Spring Symposium

COLORING THE INTEGERS: PSEUDO PROGRESSIONS AND RAMSEY THEORY

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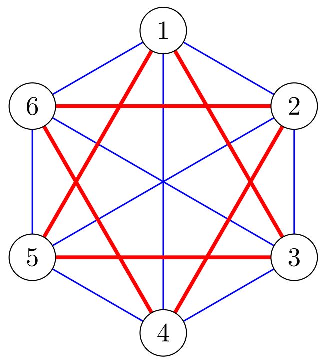
1 Summary

Suppose you are at a party with six people and you ask yourself the question, is it true that there are at least three people at that party who are either mutual strangers or mutual acquaintances? As it turns out, thanks to the research efforts of Frank Plumpton Ramsey in 1928, not only is this the case, but there is a mathematical reasoning behind it [2]. To visualize this, consider these people as six distinct vertices on a graph all connected to each other with a line, creating an edge, and then color each of these edges one of two distinct colors like the figure below.



Let's say a thin edge colored blue between two people represents an acquaintance and a thick edge colored red between two people represents two strangers. Notice the red triangles formed by vertices $1, 3, 5$ and $2, 4, 6$. These triangles represent three people who then are all mutual strangers, and this isn't a coincidence. What Ramsey proved in his theorem was that now matter how this graph is distinctly colored with two colors, there will always exist a monochromatic triangle between three vertices. Translated to our party problem: at a party with six people, you will always be able to find three people who are either mutual acquaintances or mutual strangers.

This problem is an example from a mathematical field of research known as Ramsey theory which essentially studies how order can be found in randomness. Our research specifically pertains to van der Waerden's theorem which states that if you colored the positive integers with r distinct colors, there exists a least positive integer n, denoted $W(k, r)$, such that every coloring of $1, 2, \ldots, n$ has a monochromatic k-term arithmetic progression [1]. An arithmetic progression can be defined as a list of numbers such that the difference between any two consecutive numbers is separated by one common distance. To illustrate this, let us consider $W(3,2)$. As proved by van der Waerden, $W(3,2) = 9$, that is, for any coloring of $1,\ldots,9$, there will always be a 3-term monochromatic arithmetic progression. Take for example the following coloring of $1,\ldots,8$. Note that there does not exist a 3-term monochromatic arithmetic progression, but by adding either a blue 9 or a red 9, we can find the following progressions.

$$\mathbf{1}\ 2\ 3\ \mathbf{4}\ \mathbf{5}\ 6\ 7\ \mathbf{8}$$

$$\mathbf{1}\ 2\ 3\ \mathbf{4}\ \mathbf{5}\ 6\ 7\ \mathbf{8}\ \mathbf{9} \qquad\qquad \mathbf{1}\ 2\ 3\ \mathbf{4}\ \mathbf{5}\ 6\ 7\ \mathbf{8}\ 9$$

On the left, if we color 9 blue, then we have the blue 3-term arithmetic progression $1, 5, 9$ where all consecutive integers are separated by a distance of 4. On the right, if we color 9 red, then we have the red 3-term arithmetic progression $3, 6, 9$ where all consecutive integers are separated by a distance of 3. Many other variations of this same problem have been studied and for our research we strive to study how the introduction of pseudo progressions affect this number n.

Since van der Waerden's theorem only applies to arithmetic progressions, we proposed to study what happens if we allow a single progression to have up to m common differences. Recall that in our previous example, the coloring of $1,\ldots,8$ did not contain a 3-term monochromatic arithmetic progression. Since an arithmetic progression has at most 1 distinct common difference, how would allowing for 2 distinct common differences have an impact on that same coloring? By observing that same coloring, we look for progressions that contain at most 2 distinct differences.

$$\overbrace{\mathbf{1}\ 2\ 3\ \mathbf{4}}^{3}\!\overbrace{\ \mathbf{5}}^{1}\ 6\ 7\ \mathbf{8}$$

$$\mathbf{1}\ \underbrace{2\ 3}_{1}\ \underbrace{\mathbf{4}\ \mathbf{5}\ 6}_{3}\ 7\ \mathbf{8}$$

Above we have the blue 3-term 2-pseudo progression $1, 4, 5$ where consecutive integers are separated by a distance of 1 or 3 and below we have the red 3-term 2-pseudo progression $2, 3, 6$ where consecutive integers are separated by a distance of 1 or 3. Thus let us define a k-term m-pseudo progression as a list of increasing integers $a_1, a_2, \ldots, a_k$ from $\mathbb{N}$ for which there exists a set $\{d_1, d_2, \ldots, d_m\}$ such that $a_{i+1} - a_i \in \{d_1, d_2, \ldots, d_m\}$ for all $i \in \mathbb{N}$. We will then define $B_m(k, r)$ to be the smallest positive integer n for which every r-coloring contains a k-term m-pseudo progression. It is worth noting that the set of all $(m-1)$-pseudo progressions are a subset of the set of all m-pseudo progressions since we do not require that the progression have exactly m distinct differences—we require that it contains at most m distinct differences. Since van der Waerden proved that $W(k, r)$ exists and by our example we must have $B_m(k, r) \leq W(k, r)$, we know that $B_m(k, r)$ exists. Since we know that this exists, our research goal is to prove several precise values of $B_m(k, r)$ for small values of k, m, and r and to find general bounds for all values of k, m, and r.

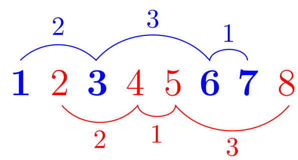
Over a century of research suggests that finding an explicit formula for $B_m(k, r)$ is nearly impossible, so we look to prove small cases and general bounds by using integer partitions. Included below is a table of some of the known values of of $B_m(k, 2)$ that we have found during our research. Currently we have restricted our research to looking at colorings with at most two distinct colors. Note that the first row with $m = 1$ are simply van der Waerden's numbers of arithmetic progressions with two distinct colors.

$m\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1	3	9	35	178	1132	>3703	>11,495	$> 41,265$	>103,474
2	1	3	5	9	14	> 13	> 13	> 14	> 16	> 18
3	1	3	5	7	9	11	> 14	> 14	> 16	> 18
4	1	3	5	7	9	11	13	15	17	19
5	1	3	5	7	9	11	13	15	17	19
6	1	3	5	7	9	11	13	15	17	19

As the table suggests, as we decrease the restrictions on the progression by allowing for an increase in m distinct differences, $B_m(k, 2)$ becomes much easier to find. The majority of our approach has involved the use of integer partitions combined with targeted elements of brute force that require checking individual colorings for various m-pseudo progressions. To demonstrate how we used integer partitions to look for small values, we include the following proof of $B_2(4, 2) = 9$.

Theorem 1. $B_2(4, 2) = 9$

Proof. First let us consider the following coloring:



We use this as a counter example to show that $B_2(4, 2) \neq 8$. Since we are looking for monochromatic 4-term progressions with at most 2 distinct common differences and both 4-term monochromatic progressions in the given coloring have 3 distinct differences, it must be that $B_2(4, 2) > 8$.

So, let us choose to color all the integers in $1, \ldots, 9$ either red or blue and define the number of integers colored blue as b and the number of integers colored red as r and assume without loss of generality $r \leq b$. The first case we must consider is $b = 5$ and $r = 4$. From this we will use integer partitions to partition the 5 blue colors into 5 distinct parts on the left and the 4 red colors into 4 distinct parts on the right.

$5 =$ ○○○○○	$4 =$ ○○○○
$2 + 3 =$ ○○ ○○○	$2 + 2 =$ ○○ ○○
$1 + 4 =$ ○ ○○○○	$1 + 3 =$ ○ ○○○
$1 + 2 + 2 =$ ○ ○○ ○○	$1 + 1 + 2 =$ ○ ○ ○○
$1 + 1 + 3 =$ ○ ○ ○○○	$1 + 1 + 1 + 1 =$ ○ ○ ○ ○
$1 + 1 + 1 + 2 =$ ○ ○ ○ ○○	
$1 + 1 + 1 + 1 + 1 =$ ○ ○ ○ ○ ○	

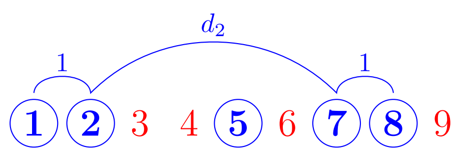
We view the red and blue circles as colorings of a list of numbers, but in no particular order. Our goal will be to interweave the blue parts with corresponding red parts and then consider all possible permutations that will give us distinct colorings. We will first begin by looking for blue 2-pseudo progressions because if we can find a blue 2-pseudo progression within every coloring for $b = 5$ and

$r = 4$, every other coloring will contain a blue 2-pseudo progression with $r \leq b$. If we cannot find a blue 2-pseudo progression for every coloring such that $b = 5$ and $r = 4$, we will have to look at the subsequent cases for $b > 5$ and $r < 4$.

Our next step will be to eliminate partitions that will always contain a blue 2-pseudo progression and to do so we will introduce a lemma.

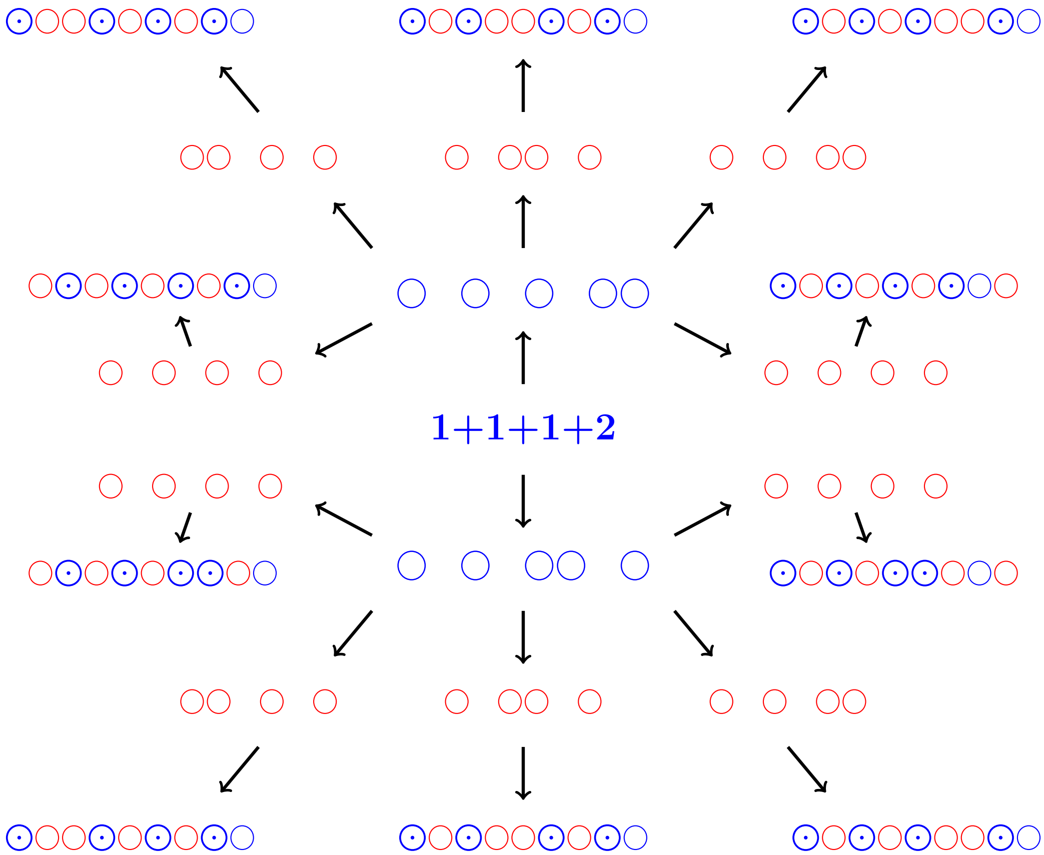
Lemma 1. *Suppose you are looking at any integer partition. If for some k-length m-pseudo progression, the sum of m parts is greater than or equal to k, then you are guaranteed a monochromatic m-pseudo progression of length k.*

What this means for us is if the sum of any two parts of any blue partition adds up to 4, by our lemma that partition will contain a 2-pseudo progression. Consider the following example of the partition $2 + 2 + 1$ permuted in the order $2 + 1 + 2$.

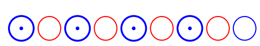


While we will not always know the distance each part is separated by, taking one "jump" from one 2-term arithmetic progression to another 2-term arithmetic progression gives us a 4-term 2-pseudo progression where the difference between every consecutive pair of integers is an element of the set $\{1, d_2\}$ for some unknown $d_2 \in \mathbb{N}$. By using this lemma, we will only need to consider the blue partitions $1+1+1+1+1+1$ and $1+1+1+2$ and all matching red partitions. Assuming $r \leq b$, for any blue partition with d distinct parts, a red partition is a *match* if $d-1 \leq c \leq d+1$ for any red partition with c distinct parts. Thus we have that the blue partition $1+1+1+1+1$ matches with the red partition $1+1+1+1$ and the blue partition $1+1+1+2$ matches with the red partitions $1+1+1+1$ and $1+1+2$. Next we will choose to fix blue and consider all distinct permutations of blue. Since we only care about finding a monochromatic 2-pseudo progression, any reflection of blue partitions will be disregarded as it is not considered distinct. By choosing to fix blue, however, we must consider all possible permutations of red partitions. Below, we demonstrate how we start with the blue partition 1+1+1+2 and match it with the following red partitions 1+1+2 and 1+1+1+1.

Around the outside of the map are all of the possible distinct colorings for the partitions, each marked with a dot inside the circle to show the existence of a 4-term blue 2-pseudo progression.



Finally, by checking the last possible blue partition $1 + 1 + 1 + 1 + 1$ with the only matching red partition $1 + 1 + 1 + 1$ we obtain the following coloring:



Since we have found a blue 4-term 2-pseudo progression within every distinct possible coloring of the integers $1, \ldots, 9$ for $b = 5$ and $r = 4$, by Lemma 1 we conclude that every other coloring must contain a 4-term blue 2-pseudo progression which completes the proof. $\square$

This proof of $B_2(4, 2)$ briefly illustrates the approach we have been taking to solve other small values of $B_m(k, r)$. As our research has progressed, we are finding other various patterns that emerge from partitioning the integers and have added many theorems and lemmas that we hope to aid us in finding rigorous upper and lower bounds. Currently we are in the process of developing a program that will help us check colorings for m-pseudo progressions and hope to utilize that program to find as many small values as we possibly can. Also, since we have solely focused on finding m-pseudo progressions for integers colored up to two distinct colors, we expect to eventually increase the number of distinct colors and study how that impacts the values for $B_m(k, r)$. In addition, this research is an extension of a paper we wrote last semester in which we define how to count m-pseudo progressions within $1, 2, \ldots, n$, which may play a role later in our research.

2 Acknowledgements

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References

[1] Landman, Bruce M., and Aaron Robertson. *Ramsey Theory on the Integers*. Providence, RI: American Mathematical Society, 2004.

[2] Graham, Ronald L, and Joel H. Spencer. *Ramsey Theory*. Scientific American 263. 1 (1990): 112-117. PDF.