On the convergence of the nth prime factor of the kth number with n prime factor

March 8, 2023

1 Introduction

This document has for aim to define an esoteric integer sequence that arises from the observation of the prime decomposition of the integers.

i apologise in advance if this document is messy or weirdly put together, this is the first time writing one like it.

In this document, i used notation for the number of prime factors, with or without repetition, that i found on this web page :
Prime Factorization

i do not know whether this sequence was defined anywhere else, but my assumption that it isn't is based on the fact that at time of writing, this sequence cannot be found on the Online Encyclopedia of Integer Sequences.

2 Definition

Definition 2.1 (Number of prime factors). *for any* $n \in \mathbb{N}_{\neq 0}$*, with* $\omega(n)$ *distinct prime factors, we have*

$$n = \prod_{i=1}^{\omega(n)} (p_i^{\alpha_i})$$

and we note $\Omega(n)$

$$\Omega(n) = \sum_{i=1}^{\omega(n)} \alpha_i$$

the number of prime factors of n*.*

Definition 2.2 (Numbers with at least n prime factors). *for any* $n \in \mathbb{N}_{\neq 0}$*,*
we note the set of integers with at least n *primes factors* $\mathbb{P}_n$

$$\mathbb{P}_n = \{k \in \mathbb{N}_{\neq 0}, \Omega(k) \geq n\}$$

.

Lemma 2.1. *for all $n \in \mathbb{N}_{\neq 0}$, $\mathbb{P}_n$ is countably infinite.*

Proof. for all $k \in \mathbb{N}$, $\Omega(2^{n+k}) = n + k >= k$, therefore $2^{n+k} \in \mathbb{P}_n$ so

$$\{2^{n+k}, k \in \mathbb{N}\} \subset \mathbb{P}_n \subset \mathbb{N}$$

and $\{2^{n+k}, k \in \mathbb{N}\}$ and $\mathbb{N}$ are countably infinite, therefore $\mathbb{P}_n$ is countably infinite. $\square$

Definition 2.3 (Sequence of numbers with at least n prime factors)**.** *for any $n \in \mathbb{N}_{\neq 0}$, given that $\mathbb{P}_n$ is a countably infinite subset of $\mathbb{N}$, we can define A_n*

$$A_n = (a_{n,i})_{i \in \mathbb{N}_{\neq 0}}$$

the sequence of numbers with at least n prime factors, ordered by the usual order over $\mathbb{N}$.

Definition 2.4 (nth prime factor of k)**.** *for any $k \in \mathbb{N}_{>1}$, with the usual prime decomposition $k = n = \prod_{i=1}^{\omega(k)}(p_i^{\alpha_i})$, for any $n \in [1, \Omega(n)]$ we define the nth prime factor of k*

$$f_n(k) = p_{\min(\{i \in [1,\omega(k)], n \leq \sum_{j=1}^{i} \alpha_j\})}$$

Lemma 2.2. *for any $n \in \mathbb{N}_{\neq 0}$, f_n is well-defined over $\mathbb{P}_n$, and for any $k \in \mathbb{N}_{>1}$,*

$$k = \prod_{i=1}^{\Omega(k)} f_i(k)$$

Proof. let $n \in \mathbb{N}_{\neq 0}$, and $k \in \mathbb{P}_n$. Then $\sum_{i=1}^{\omega(n)} \alpha_i = \Omega(k) \geq n$, therefore

$$\omega(k) \in \{i \in [1, \omega(k)], n \leq \sum_{j=1}^{i} \alpha_j\}$$

therefore this set is not empty. it is also finite, therefore it admits a min, which is smaller than $\omega(k)$, therefore $p_{\min(\{i \in [1,\omega(k)], n \leq \sum_{j=1}^{i} \alpha_j\})}$ is well-defined, and $f_n(k)$ is well-defined, and f_n is well-defined over $\mathbb{P}_n$.

let $k \in \mathbb{N}_{>1}$. Then k has at least one prime divisor, and

$$\prod_{i=1}^{\Omega(k)} f_i(k) = \prod_{j=1}^{\omega(k)} (\prod_{i=(1+\sum_{l=1}^{j-1} \alpha_l)}^{\sum_{l=1}^{j} \alpha_l} f_i(k))$$

or, given $j \in [1, \omega(k)]$, given $i \in [1 + \sum_{l=1}^{j-1} \alpha_l, \sum_{l=1}^{j} \alpha_l]$, we have $i \leq \sum_{l=1}^{j} \alpha_l$ and $i > \sum_{l=1}^{j-1} \alpha_l$, therefore $j \in \{m \in [1, \omega(k)], i \leq \sum_{l=1}^{m} \alpha_l\}$.

and for any $o < j$, $o \leq j-1$ and $\sum_{l=1}^{o} \alpha_l \leq \sum_{l=1}^{j-1} \alpha_l < i$, therefore $o \notin \{m \in [1, \omega(k)], i \leq \sum_{l=1}^{m} \alpha_l\}$.

therefore,

$$j = \min(\{m \in [1, \omega(k)], i \leq \sum_{l=1}^{m} \alpha_l\}) \text{ and } f_i(k) = p_j$$

.

therefore

$$\prod_{i=1}^{\Omega(k)} f_i(k) = \prod_{j=1}^{\omega(k)} (\prod_{i=(1+\sum_{l=1}^{j-1} \alpha_l)}^{\sum_{l=1}^{j} \alpha_l} p_j) = \prod_{j=1}^{\omega(k)} p j^{\alpha_j} = k$$

□

Definition 2.5 (Sequence of the nth prime factors of the numbers with at least n prime factors)**.** *for all $n \in \mathbb{N}_{\neq 0}$, A_n is the sequence of numbers with at least n prime factors, therefore for all $i \in \mathbb{N}_{\neq 0}$, $f_n(a_{n,i})$ is well defined, and therefore*

$$B_n = (f_n(a_{n,i}))_{i \in \mathbb{N}_{\neq 0}} = (b_{n,i})_{i \in \mathbb{N}_{\neq 0}}$$

is well-defined. we call B_n the sequence of the nth prime factors of the numbers with at least n prime factors.

Definition 2.6 (index of 3^n)**.** *For for all $n \in \mathbb{N}_{\neq 0}$, we note $i_n \in \mathbb{N}_{\neq 0}$ the integer such that $a_{n,i_n} = 3^n$. it is well defined because $3^n \in \mathbb{P}_n$.*

Lemma 2.3. *$(i_n)_{n \in \mathbb{N}_{\neq 0}}$ is an increasing sequence, and for all $n \in \mathbb{N}_{\neq 0}$, $i_n > n$*

Proof. given $n \in \mathbb{N}_{\neq 0}$, we have

$$i_n - 1 = \#\{a_{n,k}, k < i_n\} = \#\{a_{n,k}, a_{n,k} < a_{n,i_n}\}$$

or

$$i_n = 1 + \#\{a_{n,k}, a_{n,k} < 3^n\}$$

and for all $a_{n,k}$ such that $a_{n,k} < 3^n$, $2 * a_{n,k} < 2 * 3^n < 3^n + 1$. and $2 * a_{n,k}$ has 1 more prime factor than $a_{n,k}$, therefore $(2 * a_{n,k}) \in \mathbb{P}_{n+1}$. therefore $(2 * a_{n,k}) \in \{a_{n+1,k}, a_{n+1,k} < 3^{n+1}\}$ and

$$\{2 * a_{n,k}, a_{n,k} < 3^n\} \subset \{a_{n+1,k}, a_{n+1,k} < 3^{n+1}\}$$

and

$$i_n = 1 + \#\{a_{n,k}, a_{n,k} < 3^n\} = 1 + \#\{2 * a_{n,k}, a_{n,k} < 3^n\} \leq 1 + \#\{a_{n+1,k}, a_{n+1,k} < 3^{n+1}\} = i_{n+1}$$

therefore $(i_n)_{n \in \mathbb{N}_{\neq 0}}$ is an increasing sequence. finally, $\forall n \in N_{\neq 0}, \forall k \in [1, n], 2^k * 3^{n-k} \in \mathbb{P}_n$ and $2^k * 3^{n-k} < 3^n$ therefore

$$n + 1 = 1 + \# 2^k * 3^{n-k}, k \in [1, n] \leq 1 + \#\{a_{n,k}, a_{n,k} < 3^n\} = i_n$$

therefore for all $n \in \mathbb{N}_{\neq 0}$, $i_n \geq n + 1 > n$ □

3 Proposition

proposition 3.1 (convergence of the nth prime factor of the kth number with n prime factor)**.**

$$\forall k \in \mathbb{N}_{\neq 0}, \exists l_k \in \mathbb{P}, b_{n,k} \underset{n \to +\infty}{\longrightarrow} l_k$$

4 Demonstration

Lemma 4.1. *Given* $n \in \mathbb{N}_{\neq 0}$*,*

$$\forall k \in [1, i_n[, 2 \mid a_{n,k}$$

Proof. let $a_{n,k}, k \in [1, i_n[$ be such a number. then $a_{n,k} < 3^n$.

let us call $p = f_1 a_{n,k}$ the smallest prime factor of $a_{n,k}$.
Then $\forall j \in [1, \Omega(a_{n,k})], f_j(a_{n,k}) \geq p > 1$, and

$$3^n > a_{n,k} = \prod_{j=1}^{\Omega(a_{n,k})} f_j(a_{n,k}) \geq \prod_{j=1}^{n} f_j(a_{n,k}) \geq \prod_{j=1}^{n} p = p^n$$

therefore $3^n > p^n$, and $p = 2$. so $f_1(a_{n,k}) = 2$ and $2 \mid a_{n,k}$

□

Lemma 4.2. *Given* $n \in \mathbb{N}_{\neq 0}$*, ,*

$$\forall k \in [1, i_{n+1}[, a_{n+1,k} = 2a_{n,k}$$

Proof. let $n \in \mathbb{N}_{\neq 0}$. let us prove by induction that $\forall k \in [1, i_{n+1}[, a_{n+1,k} = 2a_{n,k}$
base case : $k = 1$.
let us show that for all $i \in \mathbb{N}_{\neq 0}$, $a_{1,i} = 2^i$. $\Omega(2^i) = i$ therefore $2^i \in \mathbb{P}_i$. furthermore, for all $j \in \mathbb{P}_i$,

$$j = \prod_{l=1}^{\Omega(j)} f_l(j) \geq \prod_{l=1}^{i} f_l(j) \geq \prod_{l=1}^{i} 2 = 2^i$$

therefore for all $l \in \mathbb{N}_{\neq 0}$, $a_{l,i} \geq 2^i$, therefore $a_{1,i} = 2^i$.
thus $2 * a_{1,k} = 2 * 2^k = 2^{k+1} = a_{1,k+1}$
induction : let $k \in [1, i_{n+1} - 1[$ such that $a_{n+1,k} = 2a_{n,k}$. let us show that $a_{n+1,k+1} = 2a_{n,k+1}$
because A_n and A_{n+1} respectively contain all the elements of $\mathbb{P}_n$ and $\mathbb{P}_{n+1}$ in increasing order, we have $a_{n,k+1} = \min(\{l \in \mathbb{P}_n, l > a_{n,k}\})$ and $a_{n+1,k+1} = \min(\{l \in \mathbb{P}_{n+1}, l > a_{n+1,k}\})$.
moreover $k < i_{n+1} - 1$, therefore $k + 1 < i_{n+1}$ and $2 \mid a_{n+1,k+1}$ (by 4.1). thus $(a_{n+1,k+1})/2$ is an integer and element of $\mathbb{P}_n$. and by induction hypothesis, $2a_{n,k} = a_{n+1,k} < a_{n+1,k+1}$.
thus $(a_{n+1,k+1})/2 > a_{n,k}$, and element of $\mathbb{P}_n$, thus $(a_{n+1,k+1})/2 \geq a_{n,k+1}$, or $a_{n+1,k+1} \geq 2a_{n,k+1}$.

Symmetrically, $2a_{n,k+1}$ is element of $\mathbb{P}_{n+1}$ and $2a_{n,k+1} > 2a_{n,k} = a_{n+1,k}$, thus $2a_{n,k+1} \geq a_{n+1,k+1}$. therefore

$$2a_{n,k+1} \geq a_{n+1,k+1} \geq 2a_{n,k+1} \text{ and } a_{n+1,k+1} = 2a_{n,k+1}$$

therefore, by induction, for all $\forall k \in [1, i_{n+1}[, a_{n+1,k} = 2a_{n,k}$

□

Lemma 4.3. *given $n \in \mathbb{N}_{\neq 0}$, given $k \in \mathbb{P}_n$, for all $i \in [1, n]$, $f_i(k) = f_{i+1}(2k)$*

Proof. let $n \in \mathbb{N}_{\neq 0}$ and $k \in \mathbb{P}_n$.
if $2 \mid k$: then

$$k = 2^{\alpha_1} * \prod_{j=2}^{\omega(k)} (p_j^{\alpha_j}) \text{ and } 2k = 2^{\alpha_1+1} * \prod_{j=2}^{\omega(k)} (p_j^{\alpha_j})$$

, $\omega(k) = \omega(2k)$, and one could write $2k$ as

$$\prod_{j=1}^{\omega(k)} (p_j^{\alpha_j^{'}}) \text{ where } \alpha_j^{'} = \alpha_j + 1 \text{ if } j = 1 \text{ else } \alpha_j$$

therefore

$$\min(\{j \in [1, \omega(2k)], i + 1 \leq \sum_{l=1}^{j} \alpha_l^{'}\})$$

$$= \min(\{j \in [1, \omega(k)], i + 1 \leq 1 + \sum_{l=1}^{j} \alpha_l\})$$

$$= \min(\{j \in [1, \omega(k)], i \leq \sum_{l=1}^{j} \alpha_l\})$$

and

$$f_{i+1}(2k) = f_i(k)$$

if $2 \nmid k$: then

$$k = \prod_{j=1}^{\omega(k)} (p_j^{\alpha_j}) \text{ and } 2k = 2 * \prod_{j=1}^{\omega(k)} (p_j^{\alpha_j}) \text{ where } p_1 \neq 2$$

therefore

$$2k = \prod_{j=1}^{\omega(2k)} (p'^{\alpha'_j}_j) \text{ where}$$

$$\alpha'_j = 1 \text{ if } j = 1 \text{ else } \alpha_{j-1}$$

$$p'_j = 2 \text{ if } j = 1 \text{ else } p_{j-1}$$

and $\omega(2k) = \omega(k) + 1$. therefore

$$\min(\{j \in [1, \omega(2k)], i + 1 \leq \sum_{l=1}^{j} \alpha'_l\})$$

$$= \min(\{j \in [1, \omega(k) + 1], i + 1 \leq 1 + \sum_{l=2}^{j} \alpha'_l\})$$

$$= \min(\{j \in [1, \omega(k) + 1], i \leq \sum_{l=1}^{j-1} \alpha_l\})$$

$$= \min(\{j' \in [0, \omega(k)], i \leq \sum_{l=1}^{j'} \alpha_l\}) + 1$$

and

$$f_i(k) = p_{\min(\{j \in [1,\omega(k)], i \leq \sum_{l=1}^{j} \alpha_l\})} = p'_{\min(\{j \in [1,\omega(k)], i \leq \sum_{l=1}^{j} \alpha_l\})+1}$$

$$= p'_{\min(\{j \in [1,\omega(2k)], i+1 \leq \sum_{l=1}^{j} \alpha'_l\})} = f_{i+1}(2k)$$

thus, for all $k \in \mathbb{P}_n$, for all $i \in [1, n]$, $f_i(k) = f_{i+1}(2k)$ □

Lemma 4.4. *Given $n \in \mathbb{N}_{\neq 0}$ given an integer $m \geq n$,*

$$\forall k \in [1, i_{n+1}[, b_{m,k} = b_{n,k}$$

Proof. proof by induction
Case $(m = n)$: $b_{n,k} = b_{n,k}$
induction: given $m \geq n$, such that $b_{m,k} = b_{n,k}$.
$k \leq i_{n+1}$, therefore $k \leq i_{m+1}$ (by 2.2) and $a_{m+1,k} = 2 * a_{m,k}$ (by 4.2).
and $f_m(a_{m,k}) = f_{m+1}(2 * a_{m,k})$ (by 4.3) therefore

$$b_{m,k} = f_m(a_{m,k}) = f_{m+1}(a_{m+1,k}) = b_{m+1,k} = b_{n,k}$$

so $\forall m \geq n, (b_{m,k} = b_{n,k}) \Rightarrow (b_{m+1,k} = b_{n,k})$, and

$$\forall m \geq n, b_{m,k} = b_{n,k}$$

□

5 Final proof

. given $k \in \mathbb{N}_{\neq 0}$, $i_{k+1} > k + 1 > k$ (by 2.2), therefore $k \in [1, i_{k+1}[$, and for all $m \geq k$, $b_{m,k} = b_{k,k}$ (by 4.4) therefore $(b_{i,k})_{i \in \mathbb{N}_{\neq 0}}$ is constant after the kth element and

$$b_{n,k} \underset{n \to +\infty}{\longrightarrow} b_{k,k}$$

□