Circle lattice: Example and other motivations

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General Formula

From the report:

This idea can be summed up in terms of a formula known as the "Sum of Squares Function". The formula is commonly denoted as:

$$r_2(n) = 4(d_1(n) - d_3(n)) \tag{1}$$

Where $n \in \mathbb{N}$ is the number we are interested in, r_2 is a function which gives the amount of combinations of reaching our chosen number n with the sum of 2 numbers, the 4 comes from the $\pm$ combinations and $d_1(n)$ and $d_3(n)$ are the number of divisors of n congruent to 1 modulo 4 and the number of divisors of n congruent to 3 modulo 4 respectively. The reason this formula works is due to the prime decomposition's of the natural numbers other than 2 being of the form $4n+1$ or $4n+3$ for $n = 0, 1, 2, \ldots$. Another interesting observation of the prime decomposition is for any prime factor of n congruent to 3 modulo 4 with an odd numbered power then the total number of lattice points of the circle will be 0.

Example: Find the number of lattice points on the circumference of circles with radius $a = \sqrt{3}, b = \sqrt{46}$ and $c = \sqrt{27}$.

Solution: The prime decomposition of $3 = 3^1$, since $3 \equiv 3(\text{mod}4)$ and the exponent 1 is odd, we can immediately conclude a circle with radius a has no lattice points.
The divisors of 90 are; $1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90$. We can check which numbers are congruent to 1 modulo 4 and 3 modulo 4 and use the formula 1 giving : $r_2(90) = 4(4-2) = 8$. Using the python code from the supplementary material we have the coordinates of the lattice points also which are; $(3, 9), (3, -9), (-3, 9), (-3, -9), (9, 3), (9, -3), (-9, 3), (-9, -3)$.
For c we can write $27 = 3^3$ as $3 \equiv 3(\text{mod}4)$ and the exponent 3 is odd we can conclude this circle will have 0 lattice point.

Motivations

Gauss's circle problem+ generalisations for differently shaped lattices. N dimensions, $0 = 2^0, 1 = 2^1, 2 = 2^2, 3 = 2^3, ..., n = 2^n$.

In $3d$ Legendre's three squares theorem gives answers, similar to the $2d$ case. Can be further extended for any n.

Groups, Rings and Modules Theory applications due to combinatorics. Factorial arrangement of coordinates if all are distinct for any N dimension case.

Prime factorisation and number theory applications showing symmetries and even cryptological applications, cyber-security, rotor machine designing.

Theoretical geometry applications, Hyper-spheres, n spheres, vectors, Physics applications, space and string theory.