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2) a) $\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$ (1) $\frac{dM}{dr} = 4\pi r^2 \rho(r)$ (2)

We can set these equations to each other by rearranging (1) and taking the derivative with respect to r to set it to (2).

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP}{dr}\right) = -G\frac{dM(r)}{dr} = -4\pi G r^2 \rho(r)$$

$$\rightarrow \boxed{-4\pi G\rho = \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right)}$$

b) Let's now assume $P = K\rho^\gamma$ and insert that into $\frac{d}{dr}P(r)$

$$\rightarrow -4\pi G\rho = \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}\left[K\rho^\gamma\right]\right)$$

$$\rightarrow -\rho = \frac{K}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho^{-1}\frac{d\rho^\gamma}{dr}\right) \qquad \rho^{-1}\frac{d\rho^\gamma}{dr} = \rho^{-1}\gamma\rho^{\gamma-1}\frac{d\rho}{dr} = \gamma\rho^{\gamma-2}\frac{d\rho}{dr}$$

IDK if this is valid :) just assuming it is

$$\rightarrow \boxed{-\rho = \frac{K\gamma}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right)}$$

c) Let's make the substitution $x = \frac{r}{\alpha}$ and $\theta = \left(\frac{\rho}{\rho_c}\right)^{\gamma-1}$ and $n = \frac{1}{\gamma-1}$ into the equation discovered in (b); we must also acknowledge that $r = x\alpha \rightarrow dr = \alpha\, dx$, α is const.

$$-\rho = \frac{K\gamma}{4\pi G}\frac{1}{(x\alpha)^2}\frac{1}{\alpha}\frac{d}{dx}\left((x\alpha)^2\rho^{\gamma-2}\frac{1}{\alpha}\frac{d\rho}{dx}\right) \quad \text{making x-sub}$$

$$\rightarrow -\rho = \frac{K\gamma}{4\pi G}\frac{1}{\alpha^2}\frac{1}{x^2}\frac{d}{dx}\left(x^2\rho^{\gamma-2}\frac{d\rho}{dx}\right)$$

Now let's use $P = K\rho^\gamma$ to get $P = K\rho^{1+\gamma-1} = K\rho_c^{1+\gamma-1}\theta^{1+\frac{1}{\gamma-1}}$ and $\rho = \rho_c\theta^n$ and regress a bit to the equation in (a).

$$\rightarrow -\rho_c\theta^n = \frac{K}{4\pi G}\frac{1}{\alpha^2}\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{1}{\rho_c\theta^n}\frac{d}{dx}\left(\rho_c^{1+\gamma-1}\theta^{n+1}\right)\right) \quad \text{let } \alpha = \sqrt{\frac{K}{4\pi G}\rho_c^{\gamma-2}(n+1)}$$

and apply $\theta^{-n}\frac{d}{dx}\theta^{n+1} = (n+1)\frac{d\theta}{dx}$

$$\rightarrow \boxed{-\theta^n = \frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\theta}{dx}\right)} \Longleftrightarrow \boxed{-\int x^2\theta^n dx = x^2\frac{d\theta}{dx}}$$

since $\int \frac{d}{dx}f(x)dx = f(x)$

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2) d) Let's consider what happens at $x=0$ for $\theta, \dot{\theta}$. Consider what $x=0$ means $\Rightarrow x = \frac{1}{\alpha} r \Rightarrow r=0$, What is r in this? It is the radial depth of the star in question. Let's look at the equation for θ now:

$$\theta = \left(\frac{\rho}{\rho_c}\right)^{\gamma-1} = \left(\frac{\rho(x)}{\rho_c}\right)^{\gamma-1} \propto \left(\frac{\rho(r)}{\rho_c}\right)^{\gamma-1}$$

What is ρ_c? It is the core density, which is given to us by considering $\lim_{r\to 0} \frac{M(r)}{\frac{4}{3}\pi r^3} = \rho_c$. Therefore, it follows that:

$$\theta(0) = \lim_{r\to 0}\left(\frac{\rho(r)}{\rho_c}\right)^{\gamma-1} = \left[\frac{1}{\rho_c}\lim_{r\to 0}\rho(r)\right]^{\gamma-1} = \left[\frac{1}{\rho_c}\rho_c\right]^{\gamma-1} = 1$$

which clearly implies that $\dot{\theta}(0) = \frac{d}{dx}\left[\theta(0)\right] = \frac{d}{dx}(1) = 0$.

$\boxed{\text{Therefore, the boundary conditions at } x=0 \text{ are } \theta(0)=1,\ \dot{\theta}(0)=0}$

e) If we take the derivative of the integral form of the Lane-Emden equation we get the following:

$$\rightarrow -x^2\theta^n = \frac{d}{dx}\left(x^2\frac{d\theta}{dx}\right) = 2x\frac{d\theta}{dx} + x^2\frac{d^2\theta}{dx} \rightarrow -\theta^n - \frac{2}{x}\frac{d\theta}{dx} = \frac{d^2\theta}{dx^2}$$

$\rightarrow$ if we solve $\frac{d^2\theta}{dx^2}$ for first step, we can solve $\frac{d\theta}{dx}$ & then θ :o

f) $\frac{dM}{dr} = 4\pi r^2\rho$ let $r = x\alpha$, $dr = \alpha dx$, $\rho = \theta^n\rho_c$

$$\rightarrow \int dM = 4\pi\alpha^3\rho_c\int x^2\theta^n dx = 4\pi\alpha^3\rho_c\left(-x^2\frac{d\theta}{dx}\right)$$

$$\rightarrow \boxed{M_r = -4\pi\alpha^3\rho_c x^2\frac{d\theta}{dx}}$$