The generation of gravitational waves by orbiting charges

Master's Thesis in Physics

Presented by

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Contents

1 Introduction

Einstein's theory of general relativity has been formulated more than a hundred years ago, and has not yielded since. Nevertheless, it does not explain all available evidence – misbehaved galaxy rotation curves [1] and inconsistencies in distance measurements [2] are prominent phenomena that do not seem to fit into the framework of Einstein's theory. Intriguingly, even the inverse square law of Newtonian mechanics, which is recovered from general relativity in the limit of weak fields, is becoming less sacrosanct in the light of new evidence gained from precision measurements [3]. But even though some of these problems have been well known for many years, so far no new theory of gravity managed to replace general relativity.

This lack of progress cannot be explained by a lack of attempts. Several modified theories of gravity were proposed since 1915, but none of them established itself in the mainstream. Often, these theories follow a common pattern: start with conventional general relativity, and modify it somehow to accommodate some yet unexplained phenomenon. Even though sometimes successful in dissolving the targeted dissonances, this approach often results in otherwise inconsistent theories. A typical problem is that hardly negotiable properties of general relativity, such as causality or locality, tend to break down when the theory is modified by tinkering with field equations on a phenomenological level.

One way to address this problem is to invert the method. Rather than modifying the theory at the phenomenological level first and checking consistency at the fundamental level only afterwards, one may instead raise the desired properties to a construction principle, and then *deduce* predictions for high-level phenomena. One particular implementation of such a bottom-up approach to the construction of gravity theories is the constructive gravity program [4], which takes the equations of motion for matter fields on a given background geometry as input and then furnishes this background geometry with gravitational dynamics.

The mechanism behind this surprising method to obtain gravitational dynamics from given matter dynamics is simple: it is plainly the systematic implementation of the non-negotiable requirement that the canonical evolution of the geometry proceed in lockstep with that of any stipulated matter fields that couple to the geometry. Indeed, the mechanism only provides dynamics to geometric degrees of freedom to which at least one of the postulated matter fields couples. On an intuitive level, this new "constructive" approach seems at least as likely to be successful as the old "adaptive" method, with the additional benefit that it has been tried less.

The work that I present in this thesis builds on the gravitational dynamics that have been derived as the gravitational closure of birefringent electrodynamics [5]. Birefringent electrodynamics is a matter model that allows for birefringence in vacuo [6], and features a quartic - instead of a quadratic - lightcone. While there has been no serendipitous observation of such vacuum birefringence so far [7], obtaining predictions for when and where related phenomena would occur is precisely the point of gravitationally closing such a matter theory.

This thesis is concerned with working out the purely gravitational effect of electromagnetic birefringence on gravitational wave generation, propagation and detection. More specifically, I present a double case study of a binary system that generates gravitational waves - once for the gravitational theory that follows by gravitational closure of standard Maxwell electrodynamics (which non-trivially but therefore immensely reassuringly turns out to be standard general relativity [8]), and once for the gravitational dynamics that are obtained for the refined background geometry that underlies the most general electrodynamics whose solutions still feature a linear superposition principle (and which has been worked out, in the here relevant perturbative limit, in [5]). I model the binary by two charged point masses, slowly orbiting each other due to mutual electromagnetic attraction. This system is the simplest system I found that can be treated consistently within the framework of linearized gravity theory while still yielding nontrivial results [1].

My exposition begins with a sketch of the gravitational closure algorithm in chapter 2. Aside from providing the gravitational laws I later use to investigate the generation of gravitational waves, the algorithm also determines the mathematical framework that I use for my analysis. In chapter 3, I then treat the model system with conventional electromagnetism à la Maxwell, in combination with general relativity. In this warm-up project, my analysis leads to a reproduction of the established results for gravitational wave production in binaries, whilst using the framework implied by the gravitational closure algorithm. In chapter 4, I repeat the analysis presented in chapter 3, with the sole but crucial difference of starting from birefringent electrodynamics. According to the gravitational closure algorithm, this starting point leads to a different theory of gravity, and it is of course this evolutionary compatible gravity theory that must then be employed. Chapter 4 is the heart of this thesis and contains my original contribution to the body of work on constructive gravity. There, I derive the result that vacuum birefringence implies the emission of additional, non-Einsteinian gravitational waves. I further investigate the effect of this partly novel radiation on test masses, which could be the basis of an experimental setup for their detection. A discussion of my results and their implications for area metric gravity will be provided in the final chapter 5. Throughout, I abide to notational conventions that are laid out in the appendix.

[1] Contrary to only slowly dying belief, it is not possible to treat a binary bound by gravitational interaction consistently in linearized gravity [9].

2 Gravitational closure of matter field equations

This section contains a summary of the constructive gravity program [4]. What does it mean for a matter field theory to be gravitationally closed? A matter field theory is gravitationally closed if its field equations are supplemented by additional equations that determine the coefficient functions of the field equations. In such a closed theory, there is no more field that needs to be described by hand or acts without being acted upon: All the partaking components mutually determine each other through coupled partial differential field equations.

As the simplest example, consider the coefficient functions g_{ab} of the Klein-Gordon equation

$$g^{ab}\partial_a\partial_b\phi - \frac{1}{2}g^{mn}g^{as}\left(\partial_m g_{sn} + \partial_n g_{sm} + \partial_s g_{mn}\right)\partial_a\phi - m^2\phi = 0 \tag{1}$$

for a scalar field ϕ. In order for this equation to be chart-independent, the coefficient functions must be the components functions of a metric tensor field g. The general theory of relativity claims that the metric, and therefore the coefficient functions of the Klein-Gordon equation, ought to be determined by Einstein's field equations

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}. \tag{2}$$

Einstein's equations thus provide one possible gravitational closure for the Klein-Gordon equation. Together, (1) and (2) fully determine the matter field as well as the gravitational field. The only ingredients that must still be supplemented are the two constants G and Λ. In principle, these constants could be measured in two different experiments. It is a result of the constructive gravity program that the two-parameter family (2) of field equations present the *only* gravitational dynamics that canonically evolve in lockstep with the matter field dynamics (1) and that employ a minimal number of gravitational degrees of freedom.

Although being dominant in contemporary physics, metric geometry is not the only geometry to appear in the coefficient functions of physically interesting matter field equations. A different geometric object, the area metric G^{abcd}, appears in the theory of light propagation in birefringent media - I will revisit this geometric object in chapter 4. Wherever different geometric objects appear in fundamental matter theories, an associated gravitational theory that determines the dynamics of those geometric objects is necessary to provide gravitational closure.

Remarkably, simple and physically virtually non-negotiable requirements severely constrain possible closing gravitational theories - so much so that typically only one gravitational theory with (again, typically) only a finite number of undetermined constants remains for any given matter field dynamics. This chapter outlines how the gravitational theory must be determined for given matter field equations according to the gravitational construction procedure.

Assume to be given a field theory for some tensorial matter field A on a 4-dimensional spacetime, mathematically represented by a smooth 4-dimensional manifold $\mathcal{M}$. Further assume that, apart from A, the equations of motion of that field theory also feature some undetermined tensorial field G. Enforcing gravitational closure for that matter field theory will ultimately result in field equations for the field G.

Without loss of generality, let us assume that the matter field theory is specified through an action functional $S_{\mathrm{M}}\,[A, G) = \int \mathrm{d}^4x\, \mathscr{L}_{\mathrm{M}}\;[A, G)$. One such action functional is specified in (33): the action of Maxwellian electromagnetism, which contains an inverse metric g^{ab} with Lorentzian signature in addition to the electromagnetic potential A_a.

The low-level requirements that ultimately determine the gravitational theory are most conveniently expressed in its canonical, i.e. Hamiltonian formulation. This formulation requires a technically essential, but conceptually ultimately harmless spacetime foliation [10], which I describe in section 2.1. Then, step-by-step instructions to set up the Hamiltonian formulation of the matter theory follow in section 2.2. Starting from the action functional defined in terms of spacetime quantities, we will finally reach a Hamiltonian density that generates the dynamics of a set of configuration variables on a hypersurface. Both formulations of the matter theory are completely equivalent.

All these steps require the spacetime foliation to be tailored to the matter field dynamics. More explicitly, spacetime must be foliated into initial value surfaces for the matter field equations. Smoothly deforming these surfaces typically turns one permissible foliation into another. I discuss such deformations in section 2.3, ultimately arriving at the so-called hypersurface deformation algebra. Finally, in section 2.4, I will supplement the Hamiltonian density of the matter theory with a Hamiltonian density for the geometric fields. This geometrical Hamiltonian is meant to generate the gravitational equations that close the matter field equations. To evolve the geometric fields consistently alongside the matter field, the geometrical Hamiltonian must respect the dynamical structure imposed by the matter theory, e.g. its parts must mimic the hypersurface deformation algebra.

Before starting the construction, it is necessary to briefly mention that there are some minimal technical prerequisites that must be fulfilled by the matter field equations in order to be closeable - for details, refer to [11]. In short, the matter field equations must be predictive, which means that they possess a well posed initial value formulation. They must also be canonically quantizable, which means that their solutions can be split into positive and negative frequency modes. These prerequisites are fulfilled by any respectable fundamental matter field theory of contemporary physics, and certainly by the theories that I will use later to investigate orbiting charges.

2.1 The foliation

The construction that enables the Hamiltonian formulation of the matter field theory starts with organising spacetime in a way tailored to the dynamic evolution of the matter field: we foliate spacetime into a collection of initial value surfaces of the matter field equations. Let Σ be a 3-dimensional manifold (with coordinates y^α), and let

$$X_t : \Sigma \hookrightarrow \mathcal{M} \tag{3}$$

be a family of embeddings labeled by $t \in \mathbb{R}$. Let the embedding maps X_t foliate $\mathcal{M}$ into surfaces $X_t(\Sigma)$ such that every one of these surfaces could serve as an initial value surface for the matter field equations. Since we may view any geometry on $\mathcal{M}$ as a 'movie' of geometries on Σ, we like to call Σ the screen manifold, and t the embedding time.

Whether or not a surface is an initial value surface for a set of field equations depends on the principal polynomial P_G of these field equations. More precisely, we consider a tensor field P_G which induces a homogeneous polynomial

$$P_G(x) : T_x^*\mathcal{M} \to \mathbb{R} \tag{4}$$

on each cotangent space, whose definition is extracted from the coefficients of the field equations and defined at every point x in spacetime. Since it is constructed from the coefficient functions, it must be build from the field G - hence the subscript. For P_G evaluated on a covector, write

$$P_G(x)(k) = P_G^{a_1 \dots a_{\deg P}}(x) k_{a_1} \dots k_{a_{\deg P_G}}. \tag{5}$$

Since P_G is a homogeneous polynomial, $P_G^{a_1 \dots a_{\deg P}}$ is symmetric under all possible index permutations. Now, a surface is an initial value surface of the field equations only if the conormals n of the surface lie within the hyperbolicity cones of the principal polynomial P_G everywhere on the hypersurface. Therefore, knowledge of P_G is necessary to classify a surface as initial value surface.

There is a coordinate system directly linked with the foliation: Any point $p \in \mathcal{M}$ can be uniquely addressed by specifying both t such that $p \in X_t(\Sigma)$, and the coordinates $y^\alpha(q)$ of the point $q = X_t^{-1}(p) \in \Sigma$. We will refer to the coordinate system $\{t, y^\alpha\}$ as the embedding coordinates.

The corresponding frame is partly consisting of the push forwards of the coordinate induced basis vectors to the surface $X_t(\Sigma)$,

$$e_\alpha(t) := (X_t)_* \, \partial/\partial y^\alpha. \tag{6}$$

Since the basis vectors e_α are induced by coordinates,

$$[e_\alpha, e_\beta] = 0 \tag{7}$$

holds for these tangential vectors. The vector $\partial/\partial t$, induced by the embedding time, completes the frame associated with the embedding coordinates, which is called the foliation frame.

Another very useful frame can be constructed from the foliation. It contains the same tangential vectors $\{e_\alpha\}$. Instead of $\partial/\partial t$ however, the basis vector that points away from the embedded surface is constructed from the conormal $n(t)$ of the hypersurface $X_t\,(\Sigma)$ and is defined by the conditions $n(e_\alpha(t)) = 0$, $P(n) = 1$ and the requirement that n(t) lie in the hyperbolicity cone. In order to turn $n(t)$ into a vector, we use the *Legendre map* L that is constructed from P_G:

$$L^a\,(k) := \frac{1}{\deg P}\frac{\mathrm{D}^a P\,(k)}{P\,(k)}. \tag{8}$$

The Legendre map maps from the hyperbolicity cone in cotangent space into tangent space, and provides $T(t) := L(n(t))$ to complete the frame $\{e_\alpha(t), T(t)\}$. This frame is called the orthogonal frame; its dual frame is denoted by $\{\epsilon^\alpha(t), n\}$.

The defining conditions for n can now be written as

$$P^{a_1 \ldots a_{\deg P}}(x)\epsilon^\alpha_{a_1} n_{a_2} \ldots n_{a_{\deg P}} = 0 \tag{9}$$
$$P^{a_1 \ldots a_{\deg P}}(x) n_{a_1} n_{a_2} \ldots n_{a_{\deg P}} = 1 \tag{10}$$

which I shall furthermore refer to as the frame conditions.

The orthogonal frame on a hypersurface is defined only by properties of that very surface, e.g. it depends on the embedding only locally. The embedding frame, in contrast, also depends on how the embedding changes as t is increased. We might use the orthogonal frame to quantify this change in the embedding: the vector field $\partial/\partial t$ has component functions

$$N := n(\partial/\partial t) \tag{11}$$
$$N^\alpha := \epsilon^\alpha(\partial/\partial t) \tag{12}$$

which are named lapse and shift. One can either think of the embedding determining lapse and shift, or of lapse and shift determining the embedding: Indeed, by specifying lapse and shift everywhere in spacetime, we can construct an embedding starting from a single initial value surface. Note that the embedding defines the embedding coordinates. Thus, lapse and shift indirectly define the coordinate system. Dirac [12] first noticed that invariance under coordinate transformations of a field theory also requires lapse and shift to fulfill a certain role in the associated Hamiltonian.

In summary, there are two relevant frames of the tangent space: the orthogonal frame $\{T, e_\alpha\}$ and the foliation frame $\{\dot{X}, e_\alpha\}$, with the corresponding cotangent space bases $\{n, \epsilon^\alpha\}$ and $\{\kappa, \tilde{\epsilon}^\alpha\}$ respectively. These two bases are related by

$$\dot{X} = NT + N^\alpha e_\alpha \tag{13}$$

and

$$\kappa = \frac{1}{N} n \tag{14}$$

$$\tilde{\epsilon}_\alpha = \epsilon_\alpha - \frac{1}{N} N^\alpha n. \tag{15}$$

2.2 The 3+1 split and canonical dynamics

In this section, we use the foliation described in section 2.1 to set up a Hamiltonian formalism for the matter theory: Instead of obtaining the equations of motion for the spacetime matter field as Euler-Lagrange equations from a spacetime action, we will be able to obtain Hamiltonian equations of motion for spatial fields on a hypersurface. Both ways to describe the dynamics of the matter field are completely equivalent.

Firstly, we project the matter field A from the spacetime manifold $\mathcal{M}$ onto the embedded surfaces $X_t(\Sigma)$. The components of the field with respect to the orthogonal frame are tensor fields on the hypersurface (they transform like tensors under change of the hypersurface coordinates), from which the original spacetime tensors can be reassembled. Those hypersurface tensors are then projected, by use of the orthonormal basis at each spacetime point, to fields $a^{\mathscr{A}}(t, y)$ on Σ that depend on the foliation parameter t. The index $\mathscr{A}$ represents any index structure that might arise for components of the spacetime tensors.

Secondly, the spacetime Lagragian density $\mathscr{L}_{\mathrm{M}}$ taken from the action functional $S[A, G) = \int \mathrm{d}^4x\, \mathscr{L}_{\mathrm{M}}(A, \partial A, \ldots, G)$ must be expressed in terms of the fields $a^{\mathscr{A}}(t, y)$, their derivatives with respect to the coordinates on Σ, and their derivatives with respect to the embedding time, $\dot{a}^{\mathscr{A}}(t, y)$. The derivative with respect to the embedding time is given as

$$\dot{a}^{\mathscr{A}}(t, y) := (\mathcal{L}_{\dot{X}} a)^{\mathscr{A}}(t, y)\,. \tag{16}$$

In summary, re-parametrize $\mathscr{L}_{\mathrm{M}}$:

$$\mathscr{L}_{\mathrm{M}}\left(A, \partial A, \ldots, G\right) = \mathscr{L}_{\mathrm{M}}\left(A\left(a\right), \partial A\left(\partial_{\alpha} a, \dot{a}\right) \ldots, G\right). \tag{17}$$

Thirdly, we perform a Legendre transform of the Lagrangian density to obtain the Hamiltonian density. If there are no constraints, as we will assume for simplicity (otherwise the Dirac procedure applies), this is done just as in standard mechanics: first, we calculate the canonical momenta

$$\pi_{\mathscr{A}} := \frac{\partial \mathscr{L}_{\mathrm{M}}}{\partial \dot{a}^{\mathscr{A}}}, \tag{18}$$

then we invert the relation to determine the time derivative of $a^{\mathscr{A}}$ as a function of the canonical momenta:

$$\dot{a}^{\mathscr{A}} = \dot{a}^{\mathscr{A}}\left(\pi_{\mathscr{A}}\right). \tag{19}$$

Finally, we determine the Hamiltonian density as the Legendre transform of the Lagrangian density, expressed fully in terms of the field and its canonical momenta:

$$\mathscr{H}_{\mathrm{M}} = \dot{a}^{\mathscr{A}}\left(\pi_{\mathscr{A}}\right)\pi_{\mathscr{A}} - \mathscr{L}_{\mathrm{M}}\left(a, \partial a, \dot{a}\left(\pi\right) \ldots, G\right). \tag{20}$$

This Hamiltonian generates the dynamics of the pulled back spatial tensor fields on Σ. Given an initial configuration of a and π at some time t, one might now calculate the evolution of these fields for all times. Then, one could use the embedding map and the orthogonal frame at the embedded surface to reassemble the spacetime quantity A at any time. If we had calculated the time evolution of A directly from the spacetime action, we would have arrived at the same result. The canonical formalism on the screen manifold is a equivalent way to represent the dynamics.

2.3 The hypersurface deformation algebra

In this section, we investigate another question which will prove crucial: The pulled back projections a of the spacetime field A at a time t depend on how Σ is embedded in $\mathcal{M}$ by X_t. How would the projections change if the embedding is changed?

Changing the embedding means deforming the embedded surface. Roughly speaking, all possible deformations of an embedded surface form a group, and the infinitesimal deformations form an operator algebra. This group acts on hypersurfaces, but it has representations acting on hypersurface quantities. Changes in hypersurface quantities due to infinitesimal deformations may be seen as the result of applying deformation operators to these quantities.

It is eminently useful to distinguish two independent ways to deform the embedded surface: tangential and normal. Correspondingly, there are two types of deformation operators, given as

$$\mathbf{H}\left(A\right) := \int \mathrm{d}^3 y A\left(y\right) n_a \frac{\delta}{\delta X_t^a\left(y\right)} \tag{21}$$

and

$$\mathbf{D}(\vec{A}) := \int \mathrm{d}^3 y A^\alpha\left(y\right) \epsilon_a^\alpha \frac{\delta}{\delta X_t^a\left(y\right)}, \tag{22}$$

where $\mathbf{H}\left(A\right)$ corresponds to an infinitesimal deformation in normal direction parametrized by the function $A(y)$, while $\mathbf{D}(\vec{A})$ corresponds to tangential deformations parametrized by the vector field $A^\alpha\left(y\right)$.

If the surface is deformed twice, the order of those two deformations matters. Applying them in different order yields different final surfaces. This is illustrated by the algebra of the deformation operators :

$$\Big[\mathbf{H}(A), \mathbf{H}(B)\Big] = -\mathbf{D}((\deg P - 1) P^{\alpha\beta}(A\partial_\beta B - B\partial_\beta A)\partial_\alpha) \tag{23}$$

$$\Big[\mathbf{D}(\vec{A}), \mathbf{H}(B)\Big] = -\mathbf{H}(\mathcal{L}_{\vec{A}} B) \tag{24}$$

$$\Big[\mathbf{D}(\vec{A}), \mathbf{D}(\vec{B})\Big] = -\mathbf{D}(\mathcal{L}_{\vec{A}} \vec{B}). \tag{25}$$

Crucially, this algebra, which is called the deformation algebra, is not independent of the construction of the foliation - the principal polynomial is contained in the algebraic relations. The reason for this is that we needed the Legendre map, which is derived from principal polynomial, in order to construct the orthogonal frame. Without the principal polynomial as a geometrical structure, orthogonality is not defined.

Finally, note that the change of hypersurface quantities resulting from an infinitesimal increment in embedding time t is equivalent to the change resulting from the infinitesimal deformation with parameter functions N and $\vec{N}$:

$$\dot{a} = \mathbf{H}(N)a + \mathbf{D}(\vec{N})a. \tag{26}$$

The deformation algebra and the time evolution of hypersurface quantities are thus related in exactly this fashion.

2.4 Gravitational closure

In the final section of the chapter on gravitational closure, I will present the minimal requirements that gravitational field equations have to fulfill in order to close a given set of matter field equations.

Throughout this section, we shall adopt a canonical perspective: In section 2.2, the degrees of freedom in the spacetime matter field A were projected to a collection of one-parameter families of fields a on the screen manifold Σ. Likewise, one can project the geometric field G to a one-parameter families of tensor fields $g^{\mathscr{A}}$ on Σ.

Here, we encounter a subtlety: The frame conditions (9) constitute four possible nonlinear constraints on g. Further, algebraic symmetries of the spacetime geometry G often constrain the screen manifold fields g even more. Therefore, the evolution of g is not totally free, but must respect all these constraints to guarantee equivalence between the spacetime picture and the canonical picture.

Clearly, the component functions of the tensor fields g cannot all be independent degrees of freedom. Rather, we denote the true, independent and unconstrained degrees of freedom by φ^A. Those variables span the configuration space for the geometry, and parametrize the actual geometric fields:

$$g^{\mathscr{A}} = g^{\mathscr{A}}\left(\varphi^A\right). \tag{27}$$

Providing gravitational closure then amounts to specifying equations of motion for φ^A. We shall view those equations for the geometric degrees of freedom as Hamiltonian equations of motion, derived from the gravitational Hamiltonian $H_G = \int \mathrm{d}^3y\ \mathscr{H}_G(\varphi(y), p_\varphi(y))$.

Can we specify $\mathscr{H}_G$ freely, or do we need to consider a priori restrictions? First, the gravitational Hamiltonian should be invariant under diffeomorphisms. As discussed ealier, diffeomorphisms are parametrized by lapse and shift. Dirac showed [12] that a Hamiltonian that generates invariant evolution must be of the form

$$H_G = \mathcal{H}_G\left(N\right) + \mathcal{D}_G(\vec{N}). \tag{28}$$

Typically, $\mathcal{H}_G$ is called the super-hamiltonian and $\mathcal{D}_G$ is called the super-momentum.

Secondly, we require that that the evolution of the geometric degrees of freedom is consistent with the foliation. To see what that means, let us examine how the Hamiltonian generates the evolution:

$$\dot{\varphi}^A = \{\varphi, H_G\} = \{\varphi, \mathcal{H}_G\left(N\right)\} + \{\varphi, \mathcal{D}_G(\vec{N})\}. \tag{29}$$

Here, $\{A, B\}$ is the Poisson bracket of A and B. Now, this bears a strong resemblance with how the hypersurface deformation operators act on hypersurface fields, as elaborated in section 2.3. Indeed, $\mathcal{H}_G$ and $\mathcal{D}_G$ mimic those spacetime operators. In order for the canonical dynamics to be consistent, the algebra of $\mathcal{H}_G$ and $\mathcal{D}_G$ must mirror the hypersurface deformation algebra (23) - (25):

$$\{\mathcal{H}_G\left(N\right), \mathcal{H}_G\left(M\right)\} = \mathcal{D}_G((\mathrm{deg}P - 1)P^{\alpha\beta}(N\partial_\beta M - M\partial_\beta N)\vec{e}_\alpha) \tag{30}$$

$$\{\mathcal{D}_G(\vec{N}), \mathcal{H}_G(M)\} = \mathcal{H}_G(\mathcal{L}_{\vec{N}}M) \tag{31}$$

$$\{\mathcal{D}_G(\vec{N}), \mathcal{D}_G(\vec{M})\} = \mathcal{D}_G(\mathcal{L}_{\vec{N}}\vec{M}). \tag{32}$$

These relations are functional equations for $\mathcal{H}_G$ and $\mathcal{D}_G$. Solving these equations means determining the most general form of consistent closing gravitational dynamics. Rewriting the nonlinear algebra relations in terms of a linear system of partial differential equations yields the so-called gravitational closure equations [4]. The closure equations have solutions that are typically labeled by a finite number of constants. In the metric case, only two such constants arise: the gravitational constant and the cosmological constant.

Often, the closure equations are very hard to solve. However, if one only deals with comparatively weak gravitational fields, the closure equations can be solved pertubatively, then yielding the zeroth, first and second orders of $\mathcal{H}_G$ and $\mathcal{D}_G$. From the respective pertubative Hamiltonian, one then derives the linearized equations of motion. It is these linearized field equations for the gravitational field that I will use later in chapter 3 and chapter 4 to determine the gravitational waves emitted by the charged binary system.

3 Orbiting charges in metric spacetime

In this chapter, I analyze the generation of gravitational waves by two slowly orbiting point charges which adhere to the laws of Maxwellian electrodynamics, and hence to general relativity via gravitational closure.

Maxwell condensed a variety of experimental insights into a set of equations that describe the behavior of electric and magnetic fields. These laws of Maxwellian electrodynamics, in turn, are compactly encoded in the action

$$S = \frac{1}{16\pi} \int \mathrm{d}^4 x \, (-\det g^{\cdot\cdot})^{-\frac{1}{2}} \, F_{ab} F_{cd} \; g^{ac} g^{bd} \tag{33}$$

$$F_{ab} := \partial_a A_b - \partial_b A_a \tag{34}$$

which marks my starting point and only dynamical postulate. The coefficient functions of the equations of motion derived from that action arise from a single Lorenzian metric g_{ab}, just as those of the Klein Gordon equation (1). The principal polynomial of the Maxwellian equations of motion is slightly more difficult to calculate, due to the gauge invariance in the electromagnetic field, but can be calculated carefully by an argument due to Itin [13], and ultimately is exactly as before for the Klein-Gordon field, namely

$$P(k) = g^{ab} k_a k_b. \tag{35}$$

Thus, the causal structure of Maxwellian electrodynamics is determined by the inverse metric g^{ab}. The procedure of gravitational closure outlined in chapter 2 yields equations that determine g^{ab}. Just as in the case of the Klein Gordon equation, the closing field equations turn out to be the field equations of general relativity.

In section 3.1, I begin with deriving the Hamiltonian of the electromagnetic field, and commence with that of a charged point particle in section 3.2. These Hamiltonian formulations are needed in order to prescribe the energy-momentum of matter in the version of constructive gravity employed in this thesis (see section 3.4). Then, using the Hamiltonian formalism, I determine the motion of two charged point particles which interact via the electromagnetic field in section 3.3. Finally, I use the linearized gravitational field equations for the geometry that underlies Maxwellian electrodynamics, as previously obtained via the gravitational closure procedure, with the ultimate aim to determine the gravitational waves generated by that matter system in section 3.4.

3.1 The electromagnetic field

This section is dedicated to the derivation of the Hamiltonian of the electromagnetic field. First, I shall perform a 3+1 split of the action (33). The spacetime object A_a gives rise to two hypersurface projections:

$$\phi := T^a A_a \tag{36}$$

$$A_\alpha := e^a_\alpha A_a. \tag{37}$$

To obtain the derivative of the hypersurface projections with respect to the embedding time, I project the Lie derivative with respect to the embedding flow $\dot{X}$ of the respective spacetime quantity onto the hypersurface:

$$\dot{\phi} := T^a \left(\mathcal{L}_{\dot{X}} A\right)_a \tag{38}$$

$$\dot{A}_\alpha := e^a_\alpha \left(\mathcal{L}_{\dot{X}} A\right)_a . \tag{39}$$

Next, I shall express the action (33) in terms of the screen manifold fields ϕ and A_α, their derivatives with respect to y_α and t, and the auxiliary embedding fields called lapse N and shift N^α. To that end, I need some identities. Firstly,

$$F_{\alpha\beta} := e^a_\alpha e^b_\beta F_{ab} = F^{(3)}_{\alpha\beta}, \tag{40}$$

where $F^{(3)}_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha$. Here I used that $[e_\alpha, e_\beta] = 0$, which holds since e_α and e_β belong to a coordinate induced basis. Further,

$$F_{0\beta} := T^a e^b_\beta F_{ab} = \frac{1}{N}\left(\dot{A}_\beta - \partial_\beta\left(N\phi + N^\alpha A_\alpha\right) - N^\alpha F^{(3)}_{\alpha\beta}\right) \tag{41}$$

using $T = \frac{1}{N}\left(\dot{X} - N^\alpha e_\alpha\right)$. Finally, Kuchař and Stone [14] grant that

$$\mathrm{d}^4x \left(-\det g^{\cdot\cdot}\right)^{-\frac{1}{2}} = \mathrm{d}t\,\mathrm{d}^3y N \left(-\det g^{\cdot\cdot}_{(3)}\right)^{-\frac{1}{2}} . \tag{42}$$

By substituting the above expressions into the action (33), I obtain an action for the hypersurface projection fields $\phi(t, y)$ and $A_\alpha(t, y)$:

$$S = \frac{1}{16\pi}\int \mathrm{d}t \int \mathrm{d}^3y\ N \left(-\det g^{\cdot\cdot}_{(3)}\right)^{-\frac{1}{2}} \tag{43}$$

$$\times \left(2g^{\alpha\beta}_{(3)} F_{\alpha 0} F_{\beta 0} + g^{\alpha\beta}_{(3)} g^{\gamma\delta}_{(3)} F^{(3)}_{\alpha\gamma} F^{(3)}_{\beta\delta}\right) . \tag{44}$$

The embedding parameter t controls the dynamics on the screen manifold, and serves as the time parameter that is needed to cast those dynamics in the canonical (i.e. Hamiltonian) form. The canonical treatment starts with the computation of the canonical momenta conjugate to $\partial \dot{A}_\alpha$,

$$\Pi^\alpha := \frac{\partial \mathscr{L}}{\partial \dot{A}_\alpha} = \frac{1}{4\pi}\left(-\det g_{\ddot{(3)}}\right)^{-\frac{1}{2}} g^{\alpha\beta}_{(3)} F_{0\beta}. \tag{45}$$

The derivative $\dot{\phi}$ does not feature in the Lagrangian density; ϕ thus automatically enters the action as a Lagrange multiplier [14]. Now, I perform the Legendre transformation to arrive at the Hamiltonian density

$$\mathscr{H} := \Pi^\alpha \dot{A}_\alpha - \mathscr{L} = N\mathcal{H} + N^\alpha \mathcal{D}_\alpha \tag{46}$$

$$\tag{47}$$

with

$$\mathcal{H} := 2\pi\left(-\det g_{\ddot{(3)}}\right)^{\frac{1}{2}} g^{(3)}_{\alpha\beta}\Pi^\alpha\Pi^\beta - \frac{1}{16\pi}\left(-\det g_{\ddot{(3)}}\right)^{-\frac{1}{2}} g^{\alpha\beta}_{(3)} g^{\gamma\delta}_{(3)} F^{(3)}_{\alpha\gamma} F^{(3)}_{\beta\delta} - \phi\partial_\alpha\Pi^\alpha \tag{48}$$

and

$$\mathcal{D}_\alpha := \Pi^\beta F^{(3)}_{\alpha\beta} - A_\alpha \partial_\beta \Pi^\beta. \tag{49}$$

We will only look at scenarios that include weak gravitational fields. As we shall discuss later, the involved geometric fields can then be written as

$$g^{\alpha\beta}_{(3)} := -\gamma^{\alpha\beta} + \varphi^{\alpha\beta} \tag{50}$$

$$N := 1 + A, \tag{51}$$

where $\varphi^{\alpha\beta}$ and A are much smaller than one. Also, I take N^α to be much smaller than one. I obtain an approximation to the electromagnetic Hamiltonian by substituting these redefined fields, and by then expanding to first order in A, N^α and $\varphi^{\alpha\beta}$. I arrive at

$$\begin{aligned} H := &-(1+A)\left[2\pi\gamma_{\alpha\beta}\Pi^\alpha\Pi^\beta + \frac{1}{8\pi}\gamma_{\alpha\beta}H^\alpha H^\beta + \phi\partial_\alpha\Pi^\alpha\right] \\ &- \varphi_{\mu\nu}\left(\delta^\mu_\alpha\delta^\nu_\beta - \frac{1}{2}\gamma^{\mu\nu}\gamma_{\alpha\beta}\right)\left[2\pi\Pi^\alpha\Pi^\beta + \frac{1}{8\pi}H^\alpha H^\beta\right] \\ &+ N^\mu\left[\epsilon_{\mu\alpha\beta}\Pi^\alpha H^\beta - A_\mu\partial_\alpha\Pi^\alpha\right] + \mathscr{O}(2), \end{aligned} \tag{52}$$

where

$$H^{\alpha} := \frac{1}{2}\epsilon^{\alpha\beta\gamma}F^{(3)}_{\alpha\beta}. \tag{53}$$

This approximate Hamiltonian provides the dynamics of the electromagnetic fields. To investigate a system of point charges, it must be supplemented by the Hamiltonian of massive point masses and a coupling term between both. That indeed is the topic of the following section.

3.2 Point particles

In this section, I derive the Hamiltonian of the charged point mass. Generally, the point to start from is a point particle action that is compatible with the geometric-optical limit of the (possibly massless) corresponding matter field theory that populates spacetime and from which the closure procedure starts. In this case, the corresponding matter field theory is Maxwellian electrodynamics. As has been shown [11], the general free point particle action contains the Legendre map

$$S = m\int \mathrm{d}\tau \left[P\left(L^{-1}\left(\frac{\mathrm{d}\gamma\left(\tau\right)}{\mathrm{d}\tau}\right)\right)^{-\frac{1}{\deg P}}\right]. \tag{54}$$

If the point particle is charged, a minimal coupling term to the electromagnetic potential must be added to the reparametrization-invariant action (54) [15], such that

$$S = m\int \mathrm{d}\tau \left[P\left(L^{-1}\left(\frac{\mathrm{d}\gamma\left(\tau\right)}{\mathrm{d}\tau}\right)\right)^{-\frac{1}{\deg P}} + eA_a\frac{\mathrm{d}\gamma^{a}\left(\tau\right)}{\mathrm{d}\tau}\right]. \tag{55}$$

In relativistic theory, a point particle corresponds to a curve γ in spacetime. To construct a Hamiltonian theory for the particle moving on the screen manifold Σ - where it is represented by a position $\lambda(t)$ which changes with the foliation parameter t - I parametrize the curve γ with the foliation parameter t and decompose the particle's velocity:

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} =: \dot{X} + e_{\alpha}v^{\alpha}. \tag{56}$$

This decomposition is defined such that $v = \frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} := \dot{\lambda}(t)$. That definition makes sure that $v^{\alpha} = 0$ holds exactly when the particle remains at the same spot on the screen manifold, which means that it moves with the embedding in the spacetime picture.

Now, I first insert the principal polynomial of Maxwellian electrodynamics (35) into the action and obtain

$$S = \int \mathrm{d}\tau \left[m \left(g_{ab}\left(\gamma\right) \frac{\mathrm{d}\gamma^a\left(\tau\right)}{\mathrm{d}\tau} \frac{\mathrm{d}\gamma^b\left(\tau\right)}{\mathrm{d}\tau} \right)^{\frac{1}{2}} + eA_a\left(\gamma\right) \frac{\mathrm{d}\gamma^a\left(\tau\right)}{\mathrm{d}\tau} \right]. \tag{57}$$

Next, I express the action in terms of the decomposed spacetime velocity, yielding

$$\begin{aligned} S = \int \mathrm{d}t \; m & \left(N^2 + g^{(3)}_{\alpha\beta} N^\alpha N^\beta + 2 g^{(3)}_{\alpha\beta} v^\alpha N^\beta + g^{(3)}_{\alpha\beta} v^\alpha v^\beta \right)^{\frac{1}{2}} \\ & + e\left(N\phi + N^\alpha A_\alpha + v^\alpha A_\alpha \right). \end{aligned} \tag{58}$$

To calculate the Hamiltonian of the charged point mass, I proceed slightly differently than for the electromagnetic field theory in section 3.1, where I first evaluated the Legendre transform and then expanded the resulting Hamiltonian to linear order in the geometric fields. This time, I first expand the action to linear order in the geometric fields, and then execute the Legendre transform. This strategy is equivalent to the strategy used in section 3.1, and has the benefit that the inversion needed to find the velocities in terms of the momenta must not be carried out exactly, but only pertubatively. The action up to linear order in φ reads

$$\begin{aligned} S = \int \mathrm{d}\tau \; m & \left(\left(1 - v^2\right)^{\frac{1}{2}} + \left(1 - v^2\right)^{-\frac{1}{2}} \left(A - \gamma_{\alpha\beta} N^\beta v^\alpha - \frac{1}{2} \varphi_{\alpha\beta} v^\alpha v^\beta \right) \right) \\ & + e\left(\phi + v^\alpha A_\alpha + A\phi + N^\alpha A_\alpha \right) + \mathscr{O}\left(\varphi^2\right). \end{aligned} \tag{59}$$

The canonical momentum of the point mass is

$$p_\alpha := \frac{\partial L}{\partial v^\alpha} = -m \frac{v^\alpha}{\sqrt{1 - v^2}} + eA_\alpha + \mathscr{O}\left(\varphi\right). \tag{60}$$

I define $k_\alpha = p_\alpha - eA_\alpha$, expand $v^\alpha = v^\alpha_0 + v^\alpha_1 + \dots$ in orders of φ, and by inverting

$$k_\alpha = -m \frac{v^\alpha}{\sqrt{1 - v^2}} \tag{61}$$

I obtain

$$v_\alpha = -\frac{k_\alpha}{E_k} + \mathscr{O}\left(\varphi\right), \tag{62}$$

where $E_k := \sqrt{k^2 + m^2}$. Conveniently, knowledge of the zeroth order of v^α suffices to obtain the linearized Hamiltonian through the Legendre transform:

$$H := p_\alpha v^\alpha - L = -\left(1 + A\right)\left(E_k + e\phi\right) - N^\alpha p_\alpha + \frac{1}{2E_k} \varphi_{\alpha\beta} k^\alpha k^\beta + \mathscr{O}\left(\varphi^2\right), \tag{63}$$

where all occurring fields are evaluated at the position $\lambda(t)$ of the particle on the screen manifold Σ.

3.3 The motion of two charged point masses

The Hamiltonian of the complete system containing both charged point masses (labeled by $i = 1, 2$) and electromagnetic fields is the following sum of the respective Hamiltonians that I derived in sections 3.1 and 3.2:

$$H_{\text{Matter}} = \sum_i H^{(i)}_{\text{Point particle}} + H_{\text{Electromagnetic}}. \tag{64}$$

This Hamiltonian is a screen manifold object, which generates the dynamics on the screen manifold with respect to the embedding parameter t. It contains the gravitational fields $\varphi^{\alpha\beta}$ and the electromagnetic potentials A and ϕ to linear order, so the resulting equations of motion describe the dynamics of electromagnetic fields and point masses in presence of weak gravitational fields. However - as I shall elaborate on later - to consistently analyze gravitational waves in linearized theory, the dynamics of the matter that generates those waves must occur on a flat background. For the purpose of determining the equations of motion for the present matter and henceforth the dynamics of the source system, I use that one may neglect any gravitational fields due to the chosen approximation.

The Hamiltonian of one charged point mass and the electromagnetic field in a flat background is

$$\begin{aligned} H = &- \left(E_k + e\phi\left(\lambda\right)\right) \\ &- \int \mathrm{d}^3y \left(2\pi\gamma_{\alpha\beta}\Pi^\alpha\Pi^\beta + \frac{1}{8\pi}\gamma_{\alpha\beta}H^\alpha H^\beta + \phi\partial_\alpha\Pi^\alpha\right). \end{aligned} \tag{65}$$

Using the Hamiltonian field equations $\dot{A}_\alpha = \delta H/\delta\Pi^\alpha$ and $\dot{\Pi}^\alpha = -\delta H/\delta A_\alpha$, the constraint equation $0 = \delta H/\delta\phi$, and the defining equation of H^α, I obtain Maxwell's equations

$$\dot{E}_\alpha - \left(\partial \times H\right)_\alpha = 4\pi e\dot{\lambda}_\alpha\delta_\lambda \tag{66}$$

$$\dot{H}_\alpha + \left(\partial \times E\right)_\alpha = 0 \tag{67}$$

$$\partial_\alpha E^\alpha = -4\pi e\delta_\lambda \tag{68}$$

$$\partial_\alpha H^\alpha = 0 \tag{69}$$

for the electromagnetic fields on the screen manifold, where $E_\alpha := 4\pi\Pi_\alpha = \partial_\alpha\phi - \dot{A}_\alpha$. Further, combining the Hamiltonian equations $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$ for the point mass, I obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mv_\alpha}{\sqrt{1-v^2}}\right) = -e\left[(v\times H)_\alpha + E_\alpha\right]. \tag{70}$$

The force on the right hand side is the familiar Lorentz Force.

With the equations of motion at hand, the next step is to find a particular solution, namely the slowly orbiting binary system. Slowly moving point masses move with velocities much less than the speed of light, e.g. $v \ll 1$. For such masses, I can therefore neglect all but the leading order contribution in v, which also includes loss of energy due to electromagnetic radiation. According to Landau and Lifshitz [15], a point charge e_1 at the position λ_1, moving with a velocity v_1, generates an electric field

$$E^\alpha(y) = e_1 \frac{y^\alpha - \lambda_1^\alpha}{|y-\lambda_1|^3} + \mathscr{O}\left(v_1^2\right) \tag{71}$$

and a magnetic field

$$H^\alpha = \mathscr{O}\left(v_1\right). \tag{72}$$

A second particle at position λ_2 with charge e_2, mass m_2 and velocity v_2 exposed to the field of the first particle then obeys

$$m_2 \dot{v_2}^\alpha = e_1 e_2 \frac{(\lambda_1-\lambda_2)^\alpha}{|\lambda_1-\lambda_2|^3} + \mathscr{O}\left(v^2\right). \tag{73}$$

Of course, the situation of the second particle is totally similar; I obtain an equation for its velocity simply by exchanging the labels 1 and 2 in (73). Defining $\lambda_{\mathrm{rel}} = \lambda_1 - \lambda_2$ and $\mu = m_1 m_2/\left(m_1+m_2\right)$, I rewrite the equations of motion for the system as

$$\mu \frac{\mathrm{d}^2}{\mathrm{d}t^2}\lambda_{\mathrm{rel}}^\alpha - e_1 e_2 \frac{\lambda_{\mathrm{rel}}^\alpha}{|\lambda_{\mathrm{rel}}|^3} = 0 \tag{74}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(m_1\lambda_1 + m_2\lambda_2\right)^\alpha = 0. \tag{75}$$

Equation (75) allows the solution $m_1\lambda_1 + m_2\lambda_2 = 0$ which corresponds to a resting centre of mass. The ansatz for circular motion

$$\lambda_{\mathrm{rel}} = d\begin{pmatrix}\cos\omega t\\ \sin\omega t\\ 0\end{pmatrix} \tag{76}$$

solves (74) for frequencies

$$\omega = \pm\sqrt{-e_1 e_2/\mu d^3}. \tag{77}$$

I denote the positive frequency by ω_{bin}.

In summary, a particular approximate solution (realistic as long as the velocities are very small) of the flat space equations of motion for two charged point masses in a Maxwellian universe is given by

$$\lambda_1 = \frac{\mu}{m_1} d \begin{pmatrix} \cos\sqrt{-e_1 e_2/\mu d^3}t \\ \sin\sqrt{-e_1 e_2/\mu d^3}t \\ 0 \end{pmatrix} \tag{78}$$

$$\lambda_2 = -\frac{\mu}{m_2} d \begin{pmatrix} \cos\sqrt{-e_1 e_2/\mu d^3}t \\ \sin\sqrt{-e_1 e_2/\mu d^3}t \\ 0 \end{pmatrix} \tag{79}$$

$$E^\alpha(y) = e_1 \frac{y^\alpha - \lambda_1^\alpha}{|y-\lambda_1|^3} + e_2 \frac{y^\alpha - \lambda_2^\alpha}{|y-\lambda_2|^3} \tag{80}$$

$$H^\alpha = 0. \tag{81}$$

The rest of the chapter provides an analysis of that system's ability to generate waves in the gravitational fields.

3.4 The generation of gravitational waves in metric spacetime

In this section, I combine my results from sections 3.2 and 3.1 with previous work [16] to determine the gravitational waves that the system of two point charges described in section 3.3 emits in Maxwellian spacetime.

To investigate the behaviour of the gravitational fields, I need the respective field equations. Those must not be postulated, but can be derived through the gravitational closure algorithm. To obtain the field equations, one first identifies the inverse metric g^{ab} as geometric field, and then parametrize its hypersurface projections. We may take for granted that $g(n,n) = 1$ and $g(n,\epsilon^\alpha) = 0$ due to the frame conditions (9) and (10). For the purely tangential part of the metric, one can choose the parametrization

$$g(\epsilon^\alpha, \epsilon^\beta) =: -\gamma^{\alpha\beta} - I_A^{\alpha\beta}\varphi^A, \tag{82}$$

anticipating that one could solve the gravitational closure equations perturbatively: if gravitational forces are weak, $g \approx \eta$ and so the components $\varphi^{\alpha\beta} := I_A^{\alpha\beta}\varphi^A$, are small in a suitable frame. Higher than linear orders in $\varphi^{\alpha\beta}$ can then be neglected while yielding still excellent physical results. The gravitational field equations become linear.

I was provided with the linearized equations that provide gravitational closure for a metric geometry [16] [2], where also $N =: 1 + A$:

$$
\begin{aligned}
-\frac{\delta H_M}{\delta \varphi^{\alpha\beta}} &= 2g\left(\gamma_{\alpha\beta}\gamma_{\gamma\delta} - \gamma_{\alpha\gamma}\gamma_{\beta\delta}\right)\left[\ddot{\varphi}^{\gamma\delta} - \partial^\gamma \dot{N}^\delta - \partial^\delta \dot{N}^\gamma\right] \\
&\quad + 4g\left(\gamma^{\sigma\tau}\gamma_{\alpha\beta} - \delta^\sigma_\alpha \delta^\tau_\beta\right)\partial_\sigma\partial_\tau A \\
&\quad + g\left(-2\gamma^{\gamma\delta}\gamma_{\alpha\beta}\gamma_{\mu\nu} + 2\gamma^{\gamma\delta}\gamma_{\alpha\mu}\gamma_{\beta\nu} + 2\delta^\gamma_\alpha\gamma^\delta_\beta\gamma_{\mu\nu}\right. \\
&\quad \left. + 2\delta^\gamma_\mu\delta^\delta_\nu\gamma_{\alpha\beta} - 2\delta^\gamma_\mu\delta^\delta_\beta\gamma_{\alpha\nu} - 2\delta^\delta_\mu\delta^\gamma_\beta\gamma_{\alpha\nu}\right)\partial_\gamma\partial_\delta\varphi^{\mu\nu},
\end{aligned}
\tag{83}
$$

$$
-\frac{\delta H_M}{\delta N} = 4g\left(\gamma^{\alpha\beta}\gamma_{\sigma\tau} - \delta^\alpha_\sigma\delta^\beta_\tau\right)\partial_\alpha\partial_\beta\varphi^{\sigma\tau} \tag{84}
$$

$$
\frac{\delta H_M}{\delta N^\alpha} = 4g\left(\gamma_{\mu\alpha}\gamma_{\nu\tau} - \gamma_{\mu\nu}\gamma_{\alpha\tau}\right)\partial^\mu\left(\dot{\varphi}^{\nu\tau} - \partial^\tau N^\nu - \partial^\nu N^\tau\right). \tag{85}
$$

To disentangle these equations, I use a technique that is often applied in perturbative cosmology [17]. This technique consists of decomposing spatial tensor fields, e.g. tensor fields on a three-dimensional. Euclidean space, uniquely into a sum of independent components. Now, any linear equation holds for a tensor if and only if it holds for all those components individually. Let us therefore be reductionist, and split the above system of complicated equations into several simpler equations for said components. The linearity of the equations ensures that different components do not get mixed up.

Two types of spatial tensor fields will occur: vector fields v^α and symmetric (0,2) tensor fields $w_{\alpha\beta}$. To decompose those, I follow Mukhanov [17] in using the Helmholtz theorem and its generalizations.

According to the Helmholtz theorem, every vector field v^α might be decomposed uniquely into a gradient and a divergence-free component,

$$
v^\alpha = \partial^\alpha V + V^\alpha, \tag{86}
$$

where V^α is *solenoidal*, which means $\partial_\alpha V^\alpha = 0$.

A generalization of the Helmholtz theorem states that every spatial symmetric (0,2) tensor field $w_{\alpha\beta}$ is a sum of a collection of components:

$$
w_{\alpha\beta} = \tilde{W}\gamma_{\alpha\beta} + \Delta_{\alpha\beta}W + \partial_\alpha W_\beta + \partial_\beta W_\alpha + W_{\alpha\beta}. \tag{87}
$$

[2]The equations are to be found on p. 48 in [16] , corrected for a mistake in the ansatz on p. 39, which really should have been $\hat{g}^{\alpha\beta} =: -\gamma^{\alpha\beta} - \varphi^{\alpha\beta}$.

The components are: a trace $\tilde{W}$, a tracefree Hessian

$$\Delta_{\alpha\beta}W = (\partial_\alpha\partial_\beta - 1/3\Delta\gamma_{\alpha\beta})\, W, \tag{88}$$

a solenoidal vector W_α and a transverse and tracefree tensor $W_{\alpha\beta}$. For $W_{\alpha\beta}$ to be transverse and tracefree means that $\gamma^{\alpha\beta}W_{\alpha\beta} = \partial^\alpha W_{\alpha\beta} = \partial^\beta W_{\alpha\beta} = 0$.

Now, let us decompose the gravitational fields $\varphi^{\alpha\beta}$, A and N^α accordingly:

$$\varphi^{\alpha\beta} =: \tilde{V}\gamma_{\alpha\beta} + \Delta_{\alpha\beta}V + 2\partial_{(\alpha}V_{\beta)} + V_{\alpha\beta} \tag{89}$$

$$N_\alpha =: \partial_\alpha B + B_\alpha. \tag{90}$$

Using these definitions, I can now decompose both sides of the equations (83), (84) and (85). Since these equations are linear, only one type of component will appear in each of them - scalars, solenoidal vectors and transverse, tracefree tensors do not mix in linear theory. I then read off equations for each component.

Before I display the resulting equations, I need to account for an additional complication: the gravitational theory that I use is by construction invariant under diffeomorphisms, which creates ambiguity: If we find a solution, every other solution related to the original one through some diffeomorphism will also constitute a solution which is in no way distinguishable from the original.

To account for this ambiguity, I shall first show how a general infinitesimal transformation of coordinates affects the Helmholtz components of a perturbation. Then, I can specify some conditions on the components that can only be met by a unique coordinate system. Thereby, the ambiguity of choosing coordinates is broken.

As per assumption, the metric is a Minkowski metric η, with a small perturbation φ added:

$$g^{ab} = \eta^{ab} + \varphi^{ab}. \tag{91}$$

According to e.g. Carroll [18], if φ is a small quantity, an infinitesimal coordinate transformation $\hat{x}^a = x^a + \xi^a$ will have the effect

$$\hat{\varphi}^{ab} = \varphi^{ab} + \mathcal{L}_\xi\eta^{ab} \tag{92}$$

on the components of φ [3].

Starting from this formula, I shall determine how the Helmholtz components change due

[3]The notation here is as follows: I denote the components of a tensor φ in the frame induced by the coordinates $\{x^a\}$ by φ^{ab}. The components of the same tensorin the frame induced by the coordinates $\{\hat{x}^a\}$ are denoted by $\hat{\varphi}^{ab}$

to an infinitesimal coordinate transformation. First, I define the Helmholtz components of ξ:

$$\xi^0 =: T \tag{93}$$

$$\xi^\alpha =: \partial^\alpha L + L^\alpha. \tag{94}$$

Now, I take the original coordinates x^α to be the embedding coordinates - it is from those coordinates that N, N^α and φ emerge, so it is those coordinates that we must distort to learn about the distortion of gravitational fields. The components of the inverse metric tensor in the embedding frame are

$$g^{00} = N^2 + g_{\alpha\beta}N^\alpha N^\beta \tag{95}$$

$$g^{0\alpha} = -\gamma_{\alpha\beta}N^\beta + \varphi_{\alpha\beta}N^\beta \tag{96}$$

$$g_{\alpha\beta} = -\gamma_{\alpha\beta} + \varphi_{\alpha\beta}. \tag{97}$$

The deviation from flat space is small: $\varphi_{\alpha\beta} \ll 1$. Further, we decide to look only at embeddings such that $A := N - 1 = \mathscr{O}(\varphi)$ and $N^\alpha = \mathscr{O}(\varphi)$. Then, the linear order of the components of the metric is given as

$$g^{00} = 1 + 2A + \mathscr{O}(2) \tag{98}$$

$$g^{0\alpha} = -\gamma_{\alpha\beta}N^\beta + \mathscr{O}(2). \tag{99}$$

Using the transformation law (92), we can determine how all different fields are affected at linear order by a change of coordinates. Let us look at an example:

$$\hat{g}^{00} = g^{00} + 2\eta_{0c}\partial_0\xi^c \tag{100}$$

$$= 1 + 2A + 2\dot{T} + \mathscr{O}(2). \tag{101}$$

$$\tag{102}$$

Comparing $\hat{g}^{00}$ and g^{00}, we can read off that the transformation law for A is $\hat{A} = A + \dot{T}$.

One finds the transformation behavior of the all other Helmholtz components analogously:

$$\hat{A} = A + \dot{T} \tag{103}$$

$$\hat{B} = B - X + \dot{L} \tag{104}$$

$$\hat{B}^\alpha = B^\alpha + \dot{L}^\alpha \tag{105}$$

$$\hat{\tilde{W}} = \tilde{W} - \frac{2}{3}\Delta L \tag{106}$$

$$\hat{W} = W - 2\Delta L \tag{107}$$

$$\hat{W}_\alpha = W_\alpha - L_\alpha \tag{108}$$

$$\hat{W}_{\alpha\beta} = W_{\alpha\beta}. \tag{109}$$

Note that we can think of this change of coordinates also as a change of embedding, by reinterpreting the new coordinates as resulting from a different embedding. Embeddings are specified by N and N^α, and so is the coordinate change. Say we change the embedding according to $N \leftarrow \tilde{N}$ and $N^\alpha \leftarrow \tilde{N}^\alpha$. Then simply set

$$\dot{T} = \tilde{N} - N \tag{110}$$

$$\dot{L} = \Delta^{-1}\partial_\alpha(\tilde{N}^\alpha - N^\alpha) + X \tag{111}$$

$$\dot{L}^\alpha = (\delta^\alpha_\beta - \Delta^{-1}\partial^\alpha\partial_\beta)(\tilde{N}^\beta - N^\beta) \tag{112}$$

and insert that in the transformation laws. We see that it is equivalent to change coordinates or change the embedding which defines a coordinate system. In both cases, there are four independent parameters that determine the change.

Now, let us break the ambiguity of coordinates (which can also be though of an ambiguity in embedding): I will impose conditions on the Helmholtz components of the geometric fields, and then, starting from some generic embedding, transform coordinates such that these conditions are fulfilled. If there is only one embedding in which they are fulfilled, the conditions fix the ambiguity. Choosing these conditions is often called choosing a gauge. I shall choose the gauge

$$N^\alpha = V = 0. \tag{113}$$

After implementing the gauge, I finally combine the remaining decomposed equations, and obtain the gravitational field equations in a simple form:

$$-\frac{\delta H_M}{\delta N} = 8g\Delta\tilde{V} \tag{114}$$

$$3\Delta^{\alpha\beta}\frac{\delta H_M}{\delta\varphi^{\alpha\beta}} - \frac{1}{2}\Delta\frac{\delta H_M}{\delta N} = 8g\Delta^2 A \tag{115}$$

$$-\left[\frac{\delta H_M}{\delta N^\alpha}\right]^V = 4g\Delta\dot{V}_\alpha \tag{116}$$

$$\left[\frac{\delta H_M}{\delta\varphi^{\alpha\beta}}\right]^{TT} = 2g\Box V_{\alpha\beta}. \tag{117}$$

A divergence free three-dimensional vector field has two degrees of freedom, so equation (114), (115) and (116) determine four degrees of freedom. These equations are constraint equations, which means that they determine the respective gravitational fields at each time directly and fully - there is no freedom for these fields to propagate. Together with the gauge fixing equations (113), which determine four degrees of freedom, eight of the ten degrees of freedom inherent to the symmetric, 4-dimensional metric tensor are determined by constraint equations, and thus do not propagate.

Equation (117), however, is a sourced wave equation. A transverse traceless symmetric (3,3) tensor has two degrees of freedom - the remaining two - and these are propagating. The only gravitational waves in a Maxwellian universe are thus TT waves, and they are generated according to equation (117).

To determine the gravitational waves emitted by slowly orbiting point charges, I need to solve equation (117). Again, I restrict myself to solving the equation approximately - here is how it is done: Consider the problem of solving the differential equation

$$\left(\Box + M^2\right) X_A\left(x,t\right) = \rho_A\left(x,t\right) \tag{118}$$

which I henceforth refer to as the massive Klein-Gordon equation. The A in X_A stands for any index structure; (118) is an equation for the components of a tensor. The mass term, which is taken to be positive ($M^2 \geq 0$) makes the problem more general (and thus more difficult) than necessary at this point; the wave equation that I want to solve right now is massless. However, at a later point in this work, the general solution will prove useful.

After Fourier transforming the time dependence of both X_A and ρ using the conventions (242) and (243) defined in the appendix, (118) becomes

$$\left(-\Delta - \omega^2 + M^2\right) X_A\left(x,\omega\right) = \rho_A\left(x,\omega\right). \tag{119}$$

This is the so called screened Poisson equation, which I tackle with Green's functions. The defining equation for the relevant Green function is

$$(c + \Delta) G_x(y) = \delta_x(y) . \tag{120}$$

The solutions - which I obtain using Fourier transforms and contour integrals, but which are very established results in mathematical physics - read

$$G_x(y) = -\frac{1}{4\pi |x-y|} e^{-\sqrt{|c|}|x-y|} \tag{121}$$

for $c < 0$ and

$$G_x^{\pm}(y) = -\frac{1}{4\pi |x-y|} e^{\pm i\sqrt{|c|}|x-y|}. \tag{122}$$

for $c > 0$. Here, $|x| := \gamma_{\alpha\beta} x^\alpha x^\beta$. The two different solutions for $c > 0$ correspond to absorption and radiation respectively, as is discussed in any standard textbook on field theory. Picking the Green's function labeled by the sign $-\mathrm{sgn}(\omega)$ corresponds to modeling a radiation scenario.

According to the method of Green's functions, the solution of (119) for a radiation scenario is

$$\begin{aligned} X_A(x,\omega) &= -\int \mathrm{d}y^3 G_x(y) \rho_A(y,\omega) \\ &= \int \mathrm{d}y^3 \frac{\rho_A(y,\omega)}{4\pi |x-y|} e^{-i\omega\sqrt{1-M^2/\omega^2}|x-y|} \end{aligned} \tag{123}$$

if $\omega^2 > M^2$.

So far, the solution is exact, but plugging in the source term to calculate the amplitude of the waves is still not feasible. I shall therefore present a few steps of approximation. The resulting approximate solution is much easier to handle, but less accurate. Explicitly, the approximation I am about to present is accurate only if

1. the source is confined to a compact region in the universe,
2. the source moves slowly, i.e. typical velocities of the matter within the source are much smaller than the speed of light, and
3. the gravitational wave is observed far away from the source.

Without loosing generality, I assume that the source is located at the origin of the coordinate system $\{y^\alpha\}$ of the screen manifold, and, as it is compact, is contained in a sphere of diameter d around the origin. Further, I evaluate the field X_A at a position x such that $|x| \gg d$.

The integrand in (123) contains ρ_A, which is only supported within the sphere of diameter d around the origin. The integrand thus vanishes unless $|y| < d$ in which case $|y| / |x| \ll 1$ holds. Defining $\epsilon := |y| / |x|$ and expanding (123) in ϵ, keeping only the dominant contribution, yields

$$X_A(x,\omega) = \frac{e^{-i\omega\sqrt{1-M^2/\omega^2}r}}{4\pi r} \int \mathrm{d}y^3 \rho_A(y,\omega) \exp\left(-i\omega\sqrt{1-M^2/\omega^2}\hat{x}^\alpha y_\alpha\right) \times (1 + \mathscr{O}(\epsilon)) \quad (124)$$

where $r := |x|$ and $\hat{x} := x/r$.

Finally, note that the order of magnitude of a typical velocity within the source is $v \sim \omega d$. I assume a slowly moving source, which corresponds to $\omega d \ll 1$ and allows a second expansion. I arrive at

$$X_A(x,\omega) = \frac{e^{-i\omega r\sqrt{1-M^2/\omega^2}}}{4\pi r} \sum_{n=0}^{\infty} \frac{1}{n!} \left(i\omega\sqrt{1-M^2/\omega^2}\right)^n \hat{x}^{\alpha_1} \dots \hat{x}^{\alpha_n} \int \mathrm{d}y^3\, y_{\alpha_1} \dots y_{\alpha_n}\ \rho_A(y,\omega) \times (1 + \mathscr{O}(\epsilon)) . \quad (125)$$

Inspecting this expression, one notices that the source is now organised into a multipole expansion, where higher moments are suppressed by higher powers of v. To obtain a consistent result, it is necessary to keep in mind that also ρ_A contains v, and thus must be expanded, too. Then, all contributions up to some specified order of v must be collected.

Now, I shall use this very general solution to solve the wave equation (117). The solution (125) is easily customized:

1. I set M^2 to zero,
2. I substitute $X_A = V_{\alpha\beta}$,
3. I identify the source ρ_A as $\frac{1}{2g}\left[\frac{\delta H_M}{\delta\varphi^{\alpha\beta}}\right]^{TT}$.

The main remaining work of this chapter consists in evaluating the source term for the system of charged point masses. The Hamiltonian of that matter system is

$$H_M = H_{\text{Maxwell Electromagnetism}} + H_{\text{Maxwell Point Mass},1} + + H_{\text{Maxwell Point Mass},2} . \quad (126)$$

Plugging in the approximate expressions (63) and (52) for the respective Hamiltonians, I get

$$
\begin{aligned}
H_M :=& \int \Bigg(-(1+A)\left[2\pi\gamma_{\alpha\beta}\Pi^\alpha\Pi^\beta + \frac{1}{8\pi}\gamma_{\alpha\beta}H^\alpha H^\beta + \phi\partial_\alpha\Pi^\alpha\right] \\
& - \varphi_{\mu\nu}\left(\delta^\mu_\alpha\delta^\nu_\beta - \frac{1}{2}\gamma^{\mu\nu}\gamma_{\alpha\beta}\right)\left[2\pi\Pi^\alpha\Pi^\beta + \frac{1}{8\pi}H^\alpha H^\beta\right] \\
& + N^\mu\left[\epsilon_{\mu\alpha\beta}\Pi^\alpha H^\beta - A_\mu\partial_\alpha\Pi^\alpha\right]\Bigg) \\
& - (1 + A(\lambda_1))\left(E(k_1) + e_1\phi(\lambda_1)\right) \\
& - N^\alpha(\lambda_1)(p_1)_\alpha + \frac{1}{2E(k_1)}\varphi_{\alpha\beta}(\lambda_1)k_1^\alpha k_1^\beta \\
& - (1 + A(\lambda_2))\left(E(k_2) + e\phi(\lambda_2)\right) \\
& - N^\alpha(\lambda_2)(p_2)_\alpha + \frac{1}{2E(k_2)}\varphi_{\alpha\beta}(\lambda_2)k_2^\alpha k_2^\beta + \mathscr{O}\left(\varphi^2\right)
\end{aligned}
\tag{127}
$$

from which I obtain

$$
\begin{aligned}
\frac{\delta H_M}{\delta\varphi^{\mu\nu}} =& -\frac{1}{8\pi}\left(\gamma_{\alpha\mu}\gamma_{\beta\nu} - \frac{1}{2}\gamma_{\mu\nu}\gamma_{\alpha\beta}\right)\left(E^\alpha E^\beta + H^\alpha H^\beta\right) \\
& + \frac{1}{2E(k_1)}\delta_{\lambda_1}(k_1)_\mu(k_1)_\nu + \frac{1}{2E(k_2)}\delta_{\lambda_2}(k_2)_\mu(k_2)_\nu + \mathscr{O}(\varphi)\,,
\end{aligned}
\tag{128}
$$

where $E^\alpha := 4\pi\Pi^\alpha$. To obtain the source term, I need the transverse traceless part of that expression. I will, however take a slightly different route: Instead of annihilating the longitudinal components and the trace of the source term before plugging it into the general solution (125), I will only annihilate the trace of (128) before plugging it into (125). Then, I will annihilate the longitudinal components of the resulting wave as a final step. This is the standard approach presented in GR textbooks.

Now, let us determine the dominating contribution to the general solution (125). As mentioned earlier, the general solution is organized into a multipole expansion, where higher moments are suppressed by higher orders in v. Thus, the monopole (if non-vanishing) dominates the solution at low source velocities. I now calculate this monopole moment of the source to lowest non-vanishing order in v, evaluating all expressions on the explicit solution (78) of the charged binary system.

I start from (128) and implement the result (81) of section 3.3 before expanding in v, and yield

$$\int \mathrm{d}y^3 \left[\frac{\delta H_M}{\delta \varphi^{\mu\nu}}\right]^{\mathrm{TF}} (y)$$
$$= \left[-\frac{1}{8\pi}\int \mathrm{d}^3y \, \left(E_\mu E_\nu - \frac{1}{2}\gamma_{\mu\nu}E^\sigma E_\sigma\right) + \frac{1}{2}m_1 \left(v_1\right)_\mu \left(v_1\right)_\nu + \frac{1}{2}m_2 \left(v_2\right)_\mu \left(v_2\right)_\nu + \mathscr{O}\left(v^3\right) \right]^{\mathrm{TF}} . \quad (129)$$

Next, I examine the electromagnetic terms:

$$-\int \mathrm{d}^3y \, \frac{1}{8\pi}\left(E_\mu E_\nu - \frac{1}{2}\gamma_{\mu\nu}E^\sigma E_\sigma\right)$$
$$= -\frac{1}{8\pi}\int \mathrm{d}^3y \, \partial^\tau y_\mu \left(E_\tau E_\nu - \frac{1}{2}\gamma_{\tau\nu}E^\sigma E_\sigma\right)$$
$$= \frac{1}{8\pi}\int \mathrm{d}^3y \, y_\mu \left(E_\nu \partial_\tau E^\tau + E^\tau \left[\partial_\tau E_\nu - \partial_\nu E_\tau\right]\right) . \quad (130)$$

To further process this expression, I use (66) and (70) alongside the fact that $H_\alpha = 0 + \mathscr{O}\left(v^2\right)$ in the system under investigation, and obtain

$$\frac{1}{8\pi}\int \mathrm{d}^3y \, y_\mu \left(E_\nu \partial_\tau E^\tau + E^\tau \left[\partial_\tau E_\nu - \partial_\nu E_\tau\right]\right)$$
$$= -\frac{1}{2}\int \mathrm{d}^3y \, y_\mu E_\nu \left(\delta_{\lambda_1} e_1 + \delta_{\lambda_2} e_2\right) + \mathscr{O}\left(v^3\right)$$
$$= \frac{1}{2}\left(m_1 \left(\lambda_1\right)_\mu \left(\dot{v}_1\right)_\nu + m_1 \left(\lambda_1\right)_\mu \left(\dot{v}_1\right)_\nu\right) + \mathscr{O}\left(v^3\right) . \quad (131)$$

This can be made more explicit by substituting the particle trajectories into the expression; I arrive at

$$-\frac{1}{8\pi}\int \mathrm{d}^3y \, \left(E_\mu E_\nu - \frac{1}{2}\gamma_{\mu\nu}E^\sigma E_\sigma\right)$$
$$= -\frac{\mu d^2 \omega_{\mathrm{bin}}^2}{2} \begin{pmatrix} \cos \omega_{\mathrm{bin}} t \\ \sin \omega_{\mathrm{bin}} t \\ \\ \end{pmatrix}_\nu \begin{pmatrix} \cos \omega_{\mathrm{bin}} t \\ \sin \omega_{\mathrm{bin}} t \\ \\ \end{pmatrix}_\mu + \mathscr{O}\left(v^3\right) . \quad (132)$$

Now, I put (132) and the particle trajectories into (129), and apply a Fourier transform to the time dependence. I end up with

$$\int \mathrm{d}y^3 \left[\frac{\delta H_M}{\delta \varphi^{\alpha\beta}}\right]^{TF} (y,\omega)$$
$$= \frac{\mu\omega_{\text{bin}}^2 d^2}{4}\left[2\pi\delta\left(\omega - 2\omega_{\text{bin}}\right)\begin{pmatrix} -1 & i & \\ i & 1 & \\ & & \end{pmatrix}_{\alpha\beta} + 2\pi\delta\left(\omega + 2\omega_{\text{bin}}\right)\begin{pmatrix} -1 & -i & \\ -i & 1 & \\ & & \end{pmatrix}_{\alpha\beta}\right]$$
$$+ \text{const.} + \mathscr{O}\left(v^3\right). \quad (133)$$

In this result, I do not show the constant contribution ($\omega = 0$), for it does not contribute to the wave.

At this point, the hard labor is over. All that is left to do to obtain the dominant contribution to the gravitational wave is plugging the monopole expression (133) into the general solution (125), and then annihilating the longitudinal components. After transforming the amplitude into the time domain, I obtain

$$V_{\alpha\beta}^{\text{TF}}(x,t) = -\frac{\mu d^2 \omega_{\text{bin}}^2}{16 g\pi r}\begin{pmatrix} \cos 2\omega_{\text{bin}}\tilde{t} & \sin 2\omega_{\text{bin}}\tilde{t} & \\ \sin 2\omega_{\text{bin}}\tilde{t} & -\cos 2\omega_{\text{bin}}\tilde{t} & \\ & & \end{pmatrix}_{\alpha\beta} + \mathscr{O}\left(v^3, d/r\right), \quad (134)$$

where $\tilde{t} := t - r$. The last step consists in applying the transverse projector

$$\mathscr{P}_{\mu}^{\alpha} := \delta_{\mu}^{\alpha} - \left(\Delta^{-1}\right)\partial^{\alpha}\partial_{\mu} \quad (135)$$

to both indices of $V_{\alpha\beta}^{\text{TF}}$. Note that if used on expression (134), the partial derivatives can be substituted, using that

$$\partial^{\alpha}\partial_{\beta} = -4\omega_{\text{bin}}^2 \frac{x^{\alpha}x_{\beta}}{r^2} + \mathscr{O}\left(\frac{d}{r}\right). \quad (136)$$

Therefore, the projector becomes

$$\mathscr{P}_{\mu}^{\alpha} = \delta_{\mu}^{\alpha} + \frac{x^{\alpha}x_{\mu}}{r^2} + \mathscr{O}\left(\frac{d}{r}\right) = \delta_{\mu}^{\alpha} + \hat{x}^{\alpha}\hat{x}_{\mu} + \mathscr{O}\left(\frac{d}{r}\right) \quad (137)$$

and I find the final result

$$V\left(x,t\right)_{\mu\nu}=\frac{\mu d^2\omega_{\rm bin}^2}{16g\pi r}\times \left(\delta_\mu^\alpha+\hat{x}^\alpha\hat{x}_\mu\right)\left(\delta_\nu^\beta+\hat{x}^\beta\hat{x}_\nu\right)\begin{pmatrix}-\cos 2\omega_{\rm bin}\tilde{t} & -\sin 2\omega_{\rm bin}\tilde{t} & \\ -\sin 2\omega_{\rm bin}\tilde{t} & \cos 2\omega_{\rm bin}\tilde{t} & \\ & & \end{pmatrix}_{\alpha\beta} + \mathscr{O}\left(v^3, d/r\right), \quad (138)$$

which constitutes the dominant contribution to the gravitational radiation emitted by two orbiting, massive point charges.

Is this result comparable with the corresponding result for a gravitationally bound system? The issue with treating a gravitationally bound system is that the gravitational interaction itself contributes to the generation of waves. Those contributions appear in the form of quadratic source terms in the gravitational equations of motion, and can therefore not be captured in the above presented straightforward calculation at linear order. In contrast, the electromagnetic field strength quadratic terms survived the expansion to linear order in the gravitational fields, and contributed as sources in the above presented derivation.

The slowly dying belief that a gravitationally bound system might be treated in linearized gravity originates from a clever trick: general relativity provides a shortcut based on conservation laws, which allows one to arrive at the correct result even at linear order (see, for instance, [18]). This trick does not typically carry through to more complex theories of gravity, thus making the investigation of gravitational waves emitted by a gravitationally bound system impossible within linearized theories.

However, in both the gravitationally and the electromagnetically bound system, the dominating part of the gravitational radiation is generated by two masses that orbit each other due to mutual attraction via a inverse square force. It is not surprising that the final results are equivalent: comparing equation (138) to the literature (see [18], p. 307) suggests that I must only set $G = 1/\left(64\pi g\right)$ in my result for the electromagnetically bound binary to recover the established result for the gravitational bound binary.

I conclude that the route taken to calculate wave amplitudes presented in this chapter yields reliable results, which additionally provide an idea of - but not necessarily an accurate prediction for - the gravitational waves emitted by a gravitationally bound system. I shall use the same route in the next chapter - only that time, the gravitational theory will be novel. The strategy is set up to step into the unknown.

4 Orbiting charges in weakly birefringent spacetime

In this chapter, I develop an answer to this question: Assuming that the propagation of light in vacuum depends on the polarization (i.e. assuming the vacuum is birefringent), what kind gravitational radiation does a slowly orbiting binary system emit? On the first glance, the premise in this question - about matter dynamics - seems entirely insufficient to allow a conclusion about gravitational dynamics in general, and thus gravitational waves in particular. However, as we saw, this intuition is false! The birefringent theory of light in vacuum inevitably determines a theory of gravitational interaction via the algorithm of gravitational closure. The resulting theory of gravitational interactions, henceforth called area metric gravity for brevity, then predicts the different kinds of radiation from the binary system. It is that last part of the logical chain that I focus on in this chapter.

Just as in chapter 3, I start by deriving and simplifying the relevant Hamiltonians. First, this is done for the electromagnetic field in section 4.1, then for the point mass in section 4.2. Finally, in section 4.3, I investigate how the system I presented in section 3.3 generates waves in the area metric gravitational fields. The last section 4.4 is devoted to explore the effect of one type of novel radiation on test masses.

4.1 The electromagnetic field

Birefringent electrodynamics is a generalization of Maxwell's electrodynamics [6]: just as Maxwell's theory, birefringent electrodynamics features a linear equation of motion and thus shares the superposition principal, but in contrast to Maxwell theory, it allows for birefringence in vacuo. In birefringent electrodynamics, the dynamics of the electromagnetic potential A_a are controlled by the action

$$S = \frac{1}{32\pi} \int \mathrm{d}^4x \omega F_{ab} F_{cd}\, G^{abcd}, \tag{139}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ as usual and the scalar density of weight one

$$\omega = \left(\frac{1}{4!} G^{abcd} \epsilon_{abcd} \right)^{-1}. \tag{140}$$

The fourth-rank tensor field G^{abcd} is called the area metric and may be assumed to feature the symmetries

$$G^{abcd} = G^{cdab} = -G^{bacd}. \tag{141}$$

Implementing these symmetries in the expansion of G in the orthogonal frame yields

$$\begin{aligned} G^{abcd} &= 4G^{\beta\delta}T^{[a}e_\beta^{b]}T^{[c}e_\beta^{b]} \\ &+ 2G^{\beta\gamma\delta}T^{[a}e_\beta^{b]}e_\gamma^c e_\delta^d + 2G^{\alpha\beta\delta}e_\alpha^a e_\beta^b T^{[c}e_\delta^{d]} \\ &+ G^{\alpha\beta\gamma\delta}e_\alpha^a e_\beta^{b]}e_\gamma^c e_\delta^d, \end{aligned} \tag{142}$$

where

$$G^{\beta\delta} := G\left(n, \epsilon^\beta, n, \epsilon^\delta\right), \tag{143}$$

$$G^{\beta\gamma\delta} := G\left(n, \epsilon^\beta, \epsilon^\gamma, \epsilon^\delta\right), \tag{144}$$

$$G^{\alpha\beta\gamma\delta} := G\left(\epsilon^\alpha, \epsilon^\beta, \epsilon^\gamma, \epsilon^\delta\right). \tag{145}$$

In the following, I undertake the construction of a screen manifold Hamiltonian that generates the same dynamics as the spacetime action (139). The definition of the screen manifold fields ϕ and A_α, as well as the expressions of $F_{0\beta}$ and $F_{\alpha\beta}$ in terms of $N, N^\alpha, \dot{A}_\alpha$ and $F^{(3)}_{\alpha\beta}$, are the same as in section 3.1. For later convenience, I shall at this point explicitly reduce the action (139) by a zero contribution. Since

$$\int \mathrm{d}^4x\epsilon^{abcd}F_{ab}F_{cd} = \int \mathrm{d}^4x\epsilon^{abcd}\partial_a A_b\partial_c A_d = -\int \mathrm{d}^4x\epsilon^{abcd}\partial_c\partial_a A_b A_d = 0 \tag{146}$$

due to integration by parts and the assumption of asymptotically vanishing electromagnetic field strength, the contribution of the totally antisymmetric part of G to the action vanishes and might as well be explicitly subtracted. The action then becomes

$$S = \frac{1}{32\pi}\int \mathrm{d}^4x\omega F_{ab}F_{cd}\ \left[G^{abcd} + \frac{1}{4!}\epsilon^{abcd}G^{mnop}\epsilon_{mnop}\right]. \tag{147}$$

The weird + sign arises due to convention (240). Expressed only in screen manifold fields and coordinates, the action becomes

$$\begin{aligned} S = \frac{1}{32\pi}\int \mathrm{d}^4x\ \omega[&4G^{\beta\delta}F_{0\beta}F_{0\delta} \\ &+ \left(2\left(G^{\beta\gamma\delta} + G^{\gamma\delta\beta}\right) - \frac{4N}{\omega}\epsilon^{\beta\gamma\delta}\right)F_{0\beta}F_{\gamma\delta} \\ &+ G^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} - \frac{4N^\mu}{\omega}\epsilon^{\alpha\beta\gamma}F_{\mu\alpha}F_{\beta\gamma}]. \end{aligned} \tag{148}$$

The canonical momentum is

$$\Pi^\mu = \frac{\omega}{4\pi N}\left[G^{\mu\nu}F_{0\nu} + \left(\frac{1}{2}G^{\mu\alpha\beta} - \frac{N}{2\omega}\epsilon_{\mu\alpha\beta}\right)F_{\alpha\beta}\right], \tag{149}$$

so that by defining

$$P^\alpha := \frac{\omega}{4\pi N}G^{\alpha\beta}F_{0\beta} = \Pi^\alpha - \frac{\omega}{4\pi N}\left(\frac{1}{2}G^{\mu\alpha\beta} - \frac{N}{2\omega}\epsilon^{\mu\alpha\beta}\right)F_{\alpha\beta}, \tag{150}$$

I obtain the Hamiltonian

$$\begin{aligned} H = N&\left[\frac{2\pi N}{\omega}G^{-1}_{\alpha\beta}P^\alpha P^\beta - \frac{\omega}{32\pi N}G^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} - \phi\partial_\alpha\Pi^\alpha\right] \\ + N^\mu&\left[\frac{1}{8\pi}\epsilon^{\alpha\beta\gamma}F_{\mu\alpha}F_{\beta\gamma} + F_{\mu\alpha}\Pi^\alpha - A_\mu\partial_\alpha\Pi^\alpha\right] \end{aligned} \tag{151}$$

by way of the usual Legendre transform.

Following [5], we do not work with the orthogonal frame components of G directly, but rather with the equivalent set of fields

$$\bar{g}^{\alpha\beta} := -G^{\alpha\beta}, \tag{152}$$

$$\bar{\bar{g}}_{\alpha\beta} := \frac{1}{4}\left(\det \bar{g}^{\cdot\cdot}\right)\epsilon_{\alpha\mu\nu}\epsilon_{\beta\rho\sigma}G^{\mu\nu\rho\sigma}, \tag{153}$$

$$\bar{\bar{\bar{g}}}^\alpha{}_\beta := \frac{1}{2}\left(\det \bar{g}^{\cdot\cdot}\right)^{-\frac{1}{2}}\epsilon_{\beta\mu\nu}G^{\alpha\mu\nu} - \delta^\alpha_\beta. \tag{154}$$

The frame conditions (9) and (10) translate into

$$\bar{\bar{\bar{g}}}^\alpha{}_\alpha = 0, \tag{155}$$

$$\bar{\bar{\bar{g}}}^\alpha{}_\mu \bar{g}^{\mu\beta} = \bar{\bar{\bar{g}}}^\beta{}_\mu \bar{g}^{\mu\alpha}. \tag{156}$$

Of course, the components of G might be reassembled from the fields $\bar{g}$, $\bar{\bar{g}}$ and $\bar{\bar{\bar{g}}}$ via

$$G^{\alpha\beta} = -\bar{g}^{\alpha\beta}, \tag{157}$$

$$G^{\beta\gamma\delta} = \left(\det \bar{g}^{\cdot\cdot}\right)^{\frac{1}{2}}\epsilon^{\alpha\gamma\delta}\left(\bar{\bar{\bar{g}}}^\beta{}_\alpha + \delta^\beta_\alpha\right), \tag{158}$$

$$G^{\alpha\beta\gamma\delta} = \epsilon^{\alpha\beta\mu}\epsilon^{\gamma\delta\nu}\left(\det \bar{g}^{\cdot\cdot}\right)\bar{\bar{g}}_{\mu\nu}. \tag{159}$$

In terms of $\bar{g}$, $\bar{\bar{g}}$ and $\bar{\bar{\bar{g}}}$, the density ω reads

$$\omega = N\left(\det \bar{g}^{\cdot\cdot}\right)^{-\frac{1}{2}} \tag{160}$$

and the Hamiltonian becomes

$$
\begin{aligned}
H = &-N[2\pi \left(\det \bar{g}^{\cdot\cdot}\right)^{\frac{1}{2}} \bar{g}^{-1}_{\alpha\beta} \left(\Pi^\alpha - \frac{1}{4\pi}\bar{\bar{\bar{g}}}^{\alpha}{}_{\mu} H^\mu\right)\left(\Pi^\beta - \frac{1}{4\pi}\bar{\bar{\bar{g}}}^{\beta}{}_{\nu} H^\nu\right) \\
&- \frac{1}{8\pi} \left(\det \bar{g}^{\cdot\cdot}\right)^{\frac{1}{2}} \bar{\bar{g}}_{\alpha\beta} H^\alpha H^\beta - \phi \partial_\alpha \Pi^\alpha] \\
&- N^\mu \left[\epsilon_{\mu\alpha\beta} H^\alpha \Pi^\beta + A_\mu \partial_\alpha \Pi^\alpha\right],
\end{aligned}
\tag{161}
$$

where H^α is defined in (53).

If the gravitational fields are very weak, neglecting all but linear terms in these fields yields a good approximation. The parametrization used for this perturbative approach is

$$
\bar{g}^{\alpha\beta} := \gamma^{\alpha\beta} + \bar{\varphi}^{\alpha\beta}, \tag{162}
$$

$$
\bar{\bar{g}}_{\alpha\beta} := \gamma_{\alpha\beta} + \bar{\bar{\varphi}}_{\alpha\beta}, \tag{163}
$$

$$
\bar{\bar{\bar{g}}}^{\alpha}{}_{\beta} := \bar{\bar{\bar{\varphi}}}^{\alpha}{}_{\beta} + \mathscr{O}\left(\bar{\bar{\bar{\varphi}}}^2\right). \tag{164}
$$

Substituting this into the Hamiltonian and neglecting all but the leading, linear order in φ, yields an approximate Hamiltonian which is valid as long as the gravitational fields are small enough, eg. $\varphi \ll 1$.

Using Jacobi's formula, I obtain

$$
\det \bar{g}^{\cdot\cdot} = 1 + \gamma_{\alpha\beta}\bar{\varphi}^{\alpha\beta} + \mathscr{O}\left(\varphi^2\right) \tag{165}
$$

and arrive at

$$
\begin{aligned}
H = &-(1+A)\left[\gamma_{\alpha\beta}\left(2\pi\Pi^\alpha\Pi^\beta + \frac{1}{8\pi}H^\alpha H^\beta\right) + \phi\partial_\alpha\Pi^\alpha\right] \\
&- \frac{1}{2}\left(\bar{\bar{\varphi}}^{\alpha\beta} - \bar{\varphi}^{\alpha\beta}\right)\left(\delta^\alpha_\sigma\delta^\beta_\tau - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\sigma\tau}\right) \\
&\times \left[2\pi\Pi^\alpha\Pi^\beta + \frac{1}{8\pi}H^\alpha H^\beta\right] \\
&+ \frac{1}{2}\left(\bar{\bar{\varphi}}^{\alpha\beta} + \bar{\varphi}^{\alpha\beta}\right) \\
&\times \left[2\pi\left(\delta^\alpha_\sigma\delta^\beta_\tau - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\sigma\tau}\right)\Pi^\alpha\Pi^\beta + \frac{1}{8\pi}\left(-\delta^\alpha_\sigma\delta^\beta_\tau - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\sigma\tau}\right)H^\alpha H^\beta\right] \\
&+ \bar{\bar{\bar{\varphi}}}^{\beta}{}_{\mu}\left(\delta^\mu_\nu\delta^\alpha_\beta - \frac{1}{3}\delta^\mu_\beta\delta^\alpha_\nu\right)H^\nu\Pi_\alpha \\
&- N^\mu\left[\epsilon_{\mu\alpha\beta}H^\alpha\Pi^\beta + A_\mu\partial_\alpha\Pi^\alpha\right].
\end{aligned}
\tag{166}
$$

Confidence about this result arises from comparing it with the metric case: Choosing

$$G^{abcd} = g^{ac}g^{bd} - g^{ad}g^{bc} - \left(-\det g^{\cdot\cdot}\right)^{\frac{1}{2}} \epsilon^{abcd} \tag{167}$$

turns birefringent electrodynamics into electrodynamics à la Maxwell. The resulting area metric G thus becomes a metrically induced area metric. Using this equation, I determine the value of the area metric gravitational fields in the case where G is metrically induced, and obtain

$$\frac{1}{2}\left(\bar{\bar{\varphi}}^{\alpha\beta} - \bar{\varphi}^{\alpha\beta}\right) = \varphi^{\alpha\beta}, \tag{168}$$

$$\frac{1}{2}\left(\bar{\bar{\varphi}}^{\alpha\beta} + \bar{\varphi}^{\alpha\beta}\right) = 0, \tag{169}$$

$$\bar{\bar{\bar{\varphi}}}^{\alpha}{}_{\beta} = 0, \tag{170}$$

where φ is defined in (50). Substituting this into the Hamiltonian (166) yields the Hamiltonian of Maxwell electrodynamics (52), proving at least partial correctness of the result.

4.2 Point particles

The point particle theory must match birefringent electrodynamics, i.e., it must share the same causality (and, for massless point particles, constitute the geometric optical limit of birefringent electrodynamics). As in section 3.2, such a theory is simply formulated by inserting the causal structure P of birefringent electrodynamics into the universal point particle action (54).

In terms of the area metric G that first appeared in the action (139), the principal polynomial P which encodes the causal structure reads [16]

$$\begin{aligned} P\left(k\right) &= -\frac{4!}{\left(\epsilon_{efgh}G^{efgh}\right)^2}\epsilon_{mnpq}\epsilon_{rstu}G^{mnr(a}G^{b|ps|c}G^{d)qtu}k_a k_b k_c k_d \\ &=: P^{abcd}k_a k_b k_c k_d. \end{aligned} \tag{171}$$

To substitute this into the universal point particle action, I need the components of P in the foliation frame. These have been determined by H. M. Rieser and others [19]. If, to match the accuracy of the Hamiltonian (166), one discards all but the linear oder in the gravitational fields, one obtains

$$P(\kappa,\kappa,\kappa,\kappa) = 1 - 4A, \tag{172}$$

$$P(\tilde{\epsilon}^\alpha,\kappa,\kappa,\kappa) = -N^\alpha, \tag{173}$$

$$P\left(\tilde{\epsilon}^\alpha,\tilde{\epsilon}^\beta,\kappa,\kappa\right) = -\frac{1}{3}\gamma^{\alpha\beta} + \frac{2}{3}A\gamma^{\alpha\beta} \tag{174}$$

$$+ \frac{1}{6}\left[-\left(\delta^\alpha_\mu\delta^\beta_\nu + \gamma^{\alpha\beta}\gamma_{\mu\nu}\right)\bar{\varphi}^{\mu\nu} + \left(\delta^\alpha_\mu\delta^\beta_\nu - \gamma^{\alpha\beta}\gamma_{\mu\nu}\right)\bar{\bar{\varphi}}^{\mu\nu}\right], \tag{175}$$

$$P\left(\tilde{\epsilon}^\alpha,\tilde{\epsilon}^\beta,\tilde{\epsilon}^\gamma,\kappa\right) = N^{(\alpha}\gamma^{\beta\gamma)}, \tag{176}$$

$$P\left(\tilde{\epsilon}^\alpha,\tilde{\epsilon}^\beta,\tilde{\epsilon}^\gamma,\tilde{\epsilon}^\delta\right) = \gamma^{(\alpha\beta}\gamma^{\gamma\delta)} + \gamma^{(\alpha\beta}\left(\delta^\gamma_\mu\delta^{\delta)}_\nu + \gamma^{\gamma\delta)}\gamma_{\mu\nu}\right)\bar{\varphi}^{\mu\nu} \tag{177}$$

$$- \gamma^{(\alpha\beta}\left(\delta^\gamma_\mu\delta^{\delta)}_\nu - \gamma^{\gamma\delta)}\gamma_{\mu\nu}\right)\bar{\bar{\varphi}}^{\mu\nu}. \tag{178}$$

I read off the zeroth order, and collect all first order contributions into a new variable Σ^{abcd} such that

$$P^{abcd} = \eta^{(ab}\eta^{cd)} + \Sigma^{abcd} + \mathscr{O}\left(\varphi^2\right). \tag{179}$$

Using this parametrisation, and starting from the universal point particle action, one obtains

$$S = m\int \mathrm{d}\tau\left[\left(\eta_{ab}\gamma'^a\gamma'^b\right)^{\frac{1}{2}} - \frac{1}{4}\frac{\Sigma_{abcd}\gamma'^a\gamma'^b\gamma'^c\gamma'^d}{\left(\eta_{ab}\gamma'^a\gamma'^b\right)^{\frac{3}{2}}} + \mathscr{O}\left(\varphi^2\right) + e\gamma'^a A_a\right], \tag{180}$$

where $\gamma' := \frac{\mathrm{d}\gamma(\tau)}{\mathrm{d}\tau}$. Just as in section 3.2, I proceed by parameterizing γ with the foliation parameter t and expressing the action in the foliation frame and obtain

$$S = m\int \mathrm{d}t\Big[\left(1-v^2\right)^{\frac{1}{2}} - \frac{1}{4}\frac{\sum_{n=0}^4\binom{4}{n}\Sigma_{\alpha_1\ldots\alpha_n 0\ldots 0}v^{\alpha_1}\ldots v^{\alpha_n}}{\left(1-v^2\right)^{\frac{3}{2}}} \tag{181}$$
$$+ e\left(\phi + A\phi + A_\alpha\left(N^\alpha + v^\alpha\right)\right)\Big] + \mathscr{O}\left(\varphi^2\right),$$

where again $v = \frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} =: \dot{\lambda}(t)$ and $v^2 := \gamma_{\alpha\beta}v^\alpha v^\beta$.

I proceed just as in section 3.2, and find the same results concerning the momenta and velocities to order $\mathscr{O}(1)$ as given in (60) - (62). Again, this knowledge suffices to derive the Hamiltonian to order $\mathscr{O}(\varphi)$ through the Legendre transform. After inserting the expressions (172) - (178), the approximate Hamiltonian for a charged point mass in an electromagnetic field that obeys the causal structure of birefringent electrodynamics reads

$$
\begin{aligned}
H = & -(1+A)\left(E_k + e\phi\right) - N^\alpha p_\alpha \\
& - \frac{1}{2E_k}\left[\frac{1}{2}\left(\bar{\varphi}^{\alpha\beta} - \bar{\bar{\varphi}}^{\alpha\beta}\right) k_\alpha k_\beta + \frac{1}{2}\left(\bar{\varphi}^{\alpha\beta} + \bar{\bar{\varphi}}^{\alpha\beta}\right) \gamma_{\alpha\beta}\gamma^{\mu\nu} k_\mu k_\nu\right],
\end{aligned}
\tag{182}
$$

with the same notational conventions as in (63).

4.3 Gravitational waves

Collecting the results from sections 4.1 and 4.2, I am now in possession of the complete matter Hamiltonian that describes the two electromagnetically bound particles, including the electromagnetic field:

$$
H_{\mathrm{M}} = \sum_i H^{(i)}_{\text{Point particle}} + H_{\text{Electromagnetism}}. \tag{183}
$$

To determine the gravitational radiation emitted from a specific system, these source terms must be evaluated on a specific solution of the matter equations of motion. I will evaluate them on the solution presented in section 3.3.

Now, I first present the linearized equations of motion for the gravitational fields that share the causal structure of birefringent electrodynamics, as obtained by the members of the constructive gravity group [5]. In this work, I restrict myself to area metric gravitational dynamics that only contain short-range modifications compared to standard metric general relativity, see [5].

The linearized equations of motion are formulated in terms of for fields which stem from a decomposition of the fields $\bar{\varphi}$, $\bar{\bar{\varphi}}$ and $\bar{\bar{\bar{\varphi}}}$ into scalar fields, divergence free vector fields and traceless, divergence free symmetric tensor fields :

$$
\bar{\varphi}_{\alpha\beta} =: \tilde{F}\gamma_{\alpha\beta} + \Delta_{\alpha\beta}F + 2\partial_{(\alpha}F_{\beta)} + F_{\alpha\beta}, \tag{184}
$$

$$
\bar{\bar{\varphi}}_{\alpha\beta} =: \tilde{E}\gamma_{\alpha\beta} + \Delta_{\alpha\beta}E + 2\partial_{(\alpha}E_{\beta)} + E_{\alpha\beta}, \tag{185}
$$

$$
\bar{\bar{\bar{\varphi}}}_{\alpha\beta} =: \Delta_{\alpha\beta}C + 2\partial_{(\alpha}C_{\beta)} + C_{\alpha\beta}, \tag{186}
$$

see section 3.4 for a detailed explanation of this decomposition.

I shall now give my version of the equations of motion for these fields. It differs from the version presented in [5] in seven aspects:

1. As already stated, I focus on the case of short-range modifications compared to general relativity. This becomes manifest in some of the κ-constants (see [5]) being set to zero.

2. For some historic reason, the authors of [5] chose the gauge-fixing $E^\alpha = F = B = 0$. That gauge fixing, however, is rather impractical for the study of gravitational wave generation, and I choose $B^\alpha = B = \tilde{F} - \tilde{E} = 0$ instead.

3. It appears to me that for both practical and conceptual reasons, instead of working with the fields F_A, E_A and C_A (where the A stands for any index-type - I am talking about scalars, vectors and tensors here) one should rather work with the fields $V_A := F_A - E_A$, $U_A = E_A + F_A$ and $I_A = 2C_A$. The practical reasons for this redefinition of variables reveal themselves to everyone who can be bothered to follow my calculations, whereas the conceptual reason is that V_A captures exactly the metric degrees of freedom, and is therefore the only field which would survive the metric limit described at the end of section 4.1.

4. Since it reduces redundancy in notation, I partially reverse the decomposition presented in (184) - (186), and work with the tracefree tensor fields

$$\bar{U}_{\alpha\beta} = \Delta_{\alpha\beta} U + 2\partial_{(\alpha} U_{\beta)} + U_{\alpha\beta}, \tag{187}$$

$$\bar{I}_{\alpha\beta} = \Delta_{\alpha\beta} I + 2\partial_{(\alpha} I_{\beta)} + I_{\alpha\beta} \tag{188}$$

   instead of with the parts from which they are assembled.

5. The analysis of the linearized equations of motion in vacuo and their solutions in [5] reveiled that certain combinations of constants must vanish in order to make flat Minkowski spacetime a stable solution of the theory. Since this criterion is non-negotiable, I impose these stability conditions on the respective constants. The conditions are given in detail in the appendix, section 6.

6. In order to clear up the notation, the involved combinations of the mutually independent κ-constants in [5] are replaced by constants labeled with m, s, v and t, which however are no longer entirely independent of each other. In section 6, I provide the equations to convert the former into the latter.

7. I have combined the equations in a way as to clearly separate truly dynamical equations, i.e. wave equations, from the constraint equations. This allows to systematically solve them in few steps.

Wave equations:

$$-m_4\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\varphi}^{\alpha\beta}}+\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\varphi}}^{\alpha\beta}}\right]^{\mathrm{TF}}+m_2\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\bar{\varphi}}}^{\alpha\beta}}\right]^{\mathrm{TF}}=\left(m_3m_4-m_1m_2\right)\Box\bar{I}_{\alpha\beta}+\left(m_4^2-m_2^2\right)\bar{I}_{\alpha\beta} \tag{189}$$

$$-m_4\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\bar{\varphi}}}^{\alpha\beta}}\right]^{\mathrm{TF}}+m_2\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\varphi}^{\alpha\beta}}+\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\varphi}}^{\alpha\beta}}\right]^{\mathrm{TF}}=\left(m_3m_4-m_1m_2\right)\Box\bar{U}_{\alpha\beta}+\left(m_4^2-m_2^2\right)\bar{U}_{\alpha\beta} \tag{190}$$

$$-\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\varphi}^{\alpha\beta}}-\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\varphi}}^{\alpha\beta}}\right]^{\mathrm{TT}}=\left(t_1-t_3\right)\Box V_{\alpha\beta} \tag{191}$$

$$\begin{aligned}\left(3\left(s_1+s_2\right)+2s_8\right)\gamma^{\alpha\beta}\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\varphi}^{\alpha\beta}}-\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\varphi}}^{\alpha\beta}}\right]-3\left(s_1+s_2\right)\gamma^{\alpha\beta}\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\varphi}^{\alpha\beta}}+\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\varphi}}^{\alpha\beta}}\right]+\left(3\left(s_1+s_2\right)-s_8\right)\frac{\delta H_{\mathrm{M}}}{\delta N}\\=-\left(108\left(s_1+s_2\right)^2+72\left(s_1+s_2\right)s_8-18\left(s_1+s_2\right)\right)s_{28}+12s_8^2\right)\Box\tilde{U}+18\left(s_1+s_2\right)s_{32}\tilde{U}\end{aligned} \tag{192}$$

Constraint equtions:

$$\frac{\delta H_{\mathrm{M}}}{\delta N}+\left(6\left(s_1+s_2\right)+4s_8\right)\Delta\tilde{U}=2\left(s_1+s_2\right)\Delta^2V \tag{193}$$

$$-\frac{\delta H_{\mathrm{M}}}{\delta N}-2\gamma^{\alpha\beta}\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\varphi}^{\alpha\beta}}-\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\varphi}}^{\alpha\beta}}\right]-\left(18\left(s_1+s_2\right)+12s_8\right)\ddot{\tilde{U}}-\left(12\left(s_1+s_2\right)-4s_8\right)\Delta\tilde{U}=24\left(s_1+s_2\right)\Delta A \tag{194}$$

$$-\left[\frac{\delta H_{\mathrm{M}}}{\delta N^{\alpha}}\right]^{\mathrm{V}}=2\left(v_1+v_2\right)\Delta\dot{V}_{\alpha} \tag{195}$$

A tracefree symmetric (2,0) tensor field on a three-dimensional manifold yields 5 degrees of freedom, a divergence free vector yields two degrees of freedom, as does a transverse (i.e. divergence free) tracefree symmetric (2,0) tensor field. A scalar field yields one degree of freedom.

The constraint equations (193) - (195) determine 4 degrees of freedom, and further 4 degreed of freedom are determined by the gauge fixing $N^\alpha = \tilde{V} = 0$. The wave equations (189) - (192) finally determine the remaining 13 degrees of freedom, the propagating degrees of freedom. It is precisely those wave equations that I will solve in the remainder of this chapter. They are all of the form (118), so it will be possible to solve them using the general approximate solution (123) to an accuracy of $\mathscr{O}\left(v^2\right)$, taking the source matter to be the system composed of two orbiting charges and an electric field, which I analyzed in section 3.3.

I start with equation (189): The source term for the field denoted by $\bar{I}_{\alpha\beta}$ is

$$(\rho_{\bar{I}})_{\alpha\beta} = -\sigma_1 \mathscr{P}^{\mu\nu}_{\alpha\beta}\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\varphi}^{\mu\nu}} + \frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\varphi}}^{\mu\nu}}\right] + \sigma_2 \mathscr{P}^{\mu\nu}_{\alpha\beta}\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\bar{\varphi}}}^{\mu\nu}}\right], \tag{196}$$

where $\mathscr{P}^{\mu\nu}_{\alpha\beta} = \delta^\mu_\alpha\delta^\nu_\beta - \frac{1}{3}\gamma^{\mu\nu}_{\alpha\beta}$ and the constants

$$\sigma_1 = \frac{m_4}{m_4 m_3 - m_1 m_2} \qquad\qquad \sigma_2 = \frac{m_2}{m_4 m_3 - m_1 m_2}. \tag{197}$$

I evaluate the source term for the orbiting charges to order v^2 and obtain

$$\begin{aligned}
\int \mathrm{d}^3 y\, (\rho_{\bar{I}})_{\alpha\beta}(y,\omega) &= -\int \mathrm{d}t\, e^{-i\omega t}\int \mathrm{d}^3 y\, \frac{\sigma_1}{8\pi}\mathscr{P}^{\mu\nu}_{\alpha\beta}\left(E_\mu E_\nu - \frac{1}{2}\gamma_{\mu\nu}E^\sigma E_\sigma\right) \\
&= -\frac{\sigma_1}{8\pi}\mathscr{P}^{\mu\nu}_{\alpha\beta}\int \mathrm{d}t\, e^{-i\omega t}\int \mathrm{d}^3 y\, \partial^\tau y_\mu\left(E_\tau E_\nu - \frac{1}{2}\gamma_{\mu\nu}E^\sigma E_\sigma\right) \\
&= \frac{\sigma_1}{8\pi}\mathscr{P}^{\mu\nu}_{\alpha\beta}\int \mathrm{d}t\, e^{-i\omega t}\int \mathrm{d}^3 y\, y_\mu\left(E_\nu\partial_\tau E^\tau + E^\tau\left[\partial_\tau E_\nu - \partial_\nu E_\tau\right]\right),
\end{aligned} \tag{198}$$

using integration by parts. Equation (67) combined with the result (81) grants that $\partial_\tau E_\nu -$

$\partial_\nu E_\tau = 0$, while, via (68) and the result (80) , I arrive at

$$\int \mathrm{d}^3 y \left(\rho_{\bar{I}}\right)_{\alpha\beta}(y,\omega) = -\frac{\pi\mu\sigma_1 d^2\omega_{\text{bin}}^2}{2}\times$$
$$\left[\frac{\delta(\omega)}{3}\begin{pmatrix}1 & & \\ & 1 & \\ & & -2\end{pmatrix}_{\mu\nu} + \frac{\delta(\omega-2\omega_{\text{bin}})}{2}\begin{pmatrix}1 & -i & \\ -i & -1 & \\ & & \end{pmatrix}_{\mu\nu} + \frac{\delta(\omega+2\omega_{\text{bin}})}{2}\begin{pmatrix}1 & i & \\ i & -1 & \\ & & \end{pmatrix}_{\mu\nu}\right]$$
$$+ \text{infinite contributions}, \quad (199)$$

where I used (132). The infinite contributions originate from the ultimately unphysical assumption of point particles. The integral technically demands evaluation of the electric fields sourced by the point charges also at their very position, where those fields exhibit a singularity. It is conventional to assume that point charges do not see their own potential. I therefore but neglect the infinite contributions which arise from corresponding terms.

I insert that into the general solution (125) and transform the resulting amplitude into the time domain to obtain the final expression for the gravitational wave:

$$\bar{I}_{\alpha\beta}(x,t) = -\frac{\sigma_1}{4\pi r}\frac{\mu d^2\omega_{\text{bin}}^2}{4}\begin{pmatrix}\cos 2\omega_{\text{bin}}\tilde{t} & \sin 2\omega_{\text{bin}}\tilde{t} & \\ \sin 2\omega_{\text{bin}}\tilde{t} & -\cos 2\omega_{\text{bin}}\tilde{t} & \\ & & \end{pmatrix}_{\mu\nu} + \text{ const.} + \mathscr{O}\left(\omega^3\right), \quad (200)$$

where $\tilde{t} = t - r\sqrt{1 - M_{\hat{I}}^2/\left(2\omega_{\text{bin}}\right)^2}$ and $M_{\hat{I}}^2 = \left(m_4^2 - m_2^2\right)/\left(m_3 m_4 - m_1 m_2\right)$. After a short inspection of the similarities between (189) and (190), the solution of (190) can be easily obtained from (200):

$$\bar{U}_{\alpha\beta} = -\frac{\sigma_2}{\sigma_1}\bar{I}_{\alpha\beta}. \quad (201)$$

Next, I turn to the solution of equation (192). The relevant contributions to the source term, evaluated on the particular solution presented in section 3.3, are

$$\gamma^{\alpha\beta}\left[\frac{\delta H_{\text{M}}}{\delta\bar{\varphi}^{\alpha\beta}} - \frac{\delta H_{\text{M}}}{\delta\bar{\bar{\varphi}}^{\alpha\beta}}\right] = -\frac{1}{2}\frac{k_1^2}{E_{k_1}}\delta_{\lambda_1} - \frac{1}{2}\frac{k_2^2}{E_{k_2}}\delta_{\lambda_2} - \frac{1}{16\pi}E^2, \quad (202)$$

$$\gamma^{\alpha\beta}\left[\frac{\delta H_{\text{M}}}{\delta\bar{\varphi}^{\alpha\beta}} + \frac{\delta H_{\text{M}}}{\delta\bar{\bar{\varphi}}^{\alpha\beta}}\right] = -\frac{3}{2}\frac{k_1^2}{E_{k_1}}\delta_{\lambda_1} - \frac{3}{2}\frac{k_2^2}{E_{k_2}}\delta_{\lambda_2} - \frac{1}{16\pi}E^2, \quad (203)$$

$$\frac{\delta H}{\delta N} = -E_{k_1}\delta_{\lambda_1} - E_{k_2}\delta_{\lambda_2} - \frac{1}{8\pi}E^2. \quad (204)$$

Combined, I obtain the source term

$$\rho_{\tilde{U}} = -\frac{3\left(s_1+s_2\right)-s_8}{\kappa}\left[\frac{m_1^2}{E_{k_1}}\delta_{\lambda_1}+\frac{m_2^2}{E_{k_2}}\delta_{\lambda_2}\right]-\frac{3\left(s_1+s_2\right)}{\kappa}\frac{E^2}{8\pi} \tag{205}$$

where, for brevity, I defined

$$\kappa = 108\left(s_1+s_2\right)^2+72\left(s_1+s_2\right)s_8-18\left(s_1+s_2\right))\,s_{28}+12s_8^2. \tag{206}$$

Next, I must analyse the different moments of that source. The zeroth moment, or monopole, does not vanish, but is constant:

$$\int \mathrm{d}y^3 \rho_{\tilde{U}}\left(y,\omega\right) = \delta\left(\omega\right). \tag{207}$$

The monopole does therefore contribute only to a static configuration, but not to gravitational radiation. The first moment, or dipole, vanishes to leading order due to the symmetry of the source:

$$\int \mathrm{d}y^3 \rho_{\tilde{U}}\left(y,\omega\right)y^{\alpha} = 0+\mathscr{O}\left(v^2\right). \tag{208}$$

Since the dipole moment enters the amplitude of the wave with an additional factor ω (see (125)) , it does not contribute at order v^2.

The second moment, or quadrupole, picks up a factor of ω^2 when entering the amplitude of the wave, and therefore contributes to overall order v^2 only through its zeroth order. I use

$$\rho_{\tilde{U}} = -\frac{3\left(s_1+s_2\right)-s_8}{\kappa}\left[\frac{m_1^2}{E_{k_1}}\delta_{\lambda_1}+\frac{m_2^2}{E_{k_2}}\delta_{\lambda_2}\right]-\frac{3\left(s_1+s_2\right)}{\kappa}\frac{E^2}{8\pi} \tag{209}$$

$$= -\tau\left[m_1\delta_{\lambda_1}+m_2\delta_{\lambda_2}\right]+\mathscr{O}\left(v^2\right), \tag{210}$$

where $\tau := \left(3\left(s_1+s_2\right)-s_8\right)/\kappa$, and obtian

$$\int \mathrm{d}y^3 \rho_{\tilde{U}}\left(y,\omega\right) y_\alpha y_\beta = -\pi\mu\tau d^2 \times \\ \left[\delta\left(\omega\right)\begin{pmatrix}1 & & \\ & 1 & \\ & & \end{pmatrix}_{\alpha\beta} + \frac{\delta\left(\omega-2\omega_{\mathrm{bin}}\right)}{2}\begin{pmatrix}1 & -i & \\ -i & -1 & \\ & & \end{pmatrix}_{\alpha\beta} \right. \\ \left. + \frac{\delta\left(\omega+2\omega_{\mathrm{bin}}\right)}{2}\begin{pmatrix}1 & i & \\ i & -1 & \\ & & \end{pmatrix}_{\alpha\beta}\right]. \quad (211)$$

Again, inserting this into (125) and transforming back into the time domain yields the final wave

$$\tilde{U}\left(x,t\right) = \frac{\tau\mu d^2\omega_{\mathrm{bin}}^2}{4\pi r}\left(1-\frac{M_{\tilde{U}}^2}{\left(2\omega_{\mathrm{bin}}\right)^2}\right)\hat{x}^\alpha\hat{x}^\beta\begin{pmatrix}\cos 2\omega_{\mathrm{bin}}\tilde{t} & \sin 2\omega_{\mathrm{bin}}\tilde{t} & \\ \sin 2\omega_{\mathrm{bin}}\tilde{t} & -\cos 2\omega_{\mathrm{bin}}\tilde{t} & \\ & & \end{pmatrix}_{\alpha\beta} \\ + \mathrm{const.} + \mathscr{O}\left(\omega^3\right), \quad (212)$$

where $\tilde{t}$ is defined as above and

$$M_{\tilde{U}}^2 := -\frac{18\left(s_1+s_2\right)s_{32}}{\kappa}. \quad (213)$$

The last equation to solve is (191). This equation is a massless wave equation. Both the field that fulfills the equation and the source term are traceless and transverse. This bears a great resemblance with the the problem discussed in section 3.4 and indeed, the steps to solve it are similar. I first set M^2 to zero in (125). Secondly, I solve the equation for the traceless part of the source. Finally, I project out the transverse part of that solution.

$$\begin{aligned}\left(\rho_V\right)_{\alpha\beta} &= -\frac{1}{t_1-t_3}\mathscr{P}_{\alpha\beta}^{\mu\nu}\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\varphi}^{\mu\nu}}-\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\varphi}}^{\mu\nu}}\right] \\ &= \vartheta\mathscr{P}_{\alpha\beta}^{\mu\nu}\left[\frac{1}{2}\frac{k_{1\alpha}k_{1\beta}}{E_{k_1}}\delta_{\lambda_1}+\frac{1}{2}\frac{k_{2\alpha}k_{2\beta}}{E_{k_2}}\delta_{\lambda_2}-\frac{1}{8\pi}\left(\delta_\alpha^\sigma\delta_\beta^\tau-\frac{1}{2}\gamma_{\alpha\beta}\gamma^{\sigma\tau}\right)E_\sigma E_\tau\right],\end{aligned} \quad (214)$$

where $\vartheta := 1/\left(t_1-t_3\right)$. Now using the results from section 3.3 and (132), I determine the leading order contribution to the amplitude to be

$$\int \mathrm{d}y^3 \left(\rho_V\right)_{\alpha\beta}(y,\omega) = -\frac{\vartheta\mu\omega_{\mathrm{bin}}^2 d^2\pi}{2}\times \left[\frac{\delta\left(\omega-2\omega_{\mathrm{bin}}\right)}{2}\begin{pmatrix}1 & -i & \\ -i & -1 & \\ & & \end{pmatrix}_{\alpha\beta}+\frac{\delta\left(\omega+2\omega_{\mathrm{bin}}\right)}{2}\begin{pmatrix}1 & i & \\ i & -1 & \\ & & \end{pmatrix}_{\alpha\beta}\right]. \tag{215}$$

and thus arrive at the tracefree solution

$$V^{\mathrm{TF}}(x,t)_{\alpha\beta} = \frac{\varphi\mu d^2\omega_{\mathrm{bin}}^2}{8\pi r}\begin{pmatrix}\cos 2\omega_{\mathrm{bin}}\tilde{\tilde{t}} & \sin 2\omega_{\mathrm{bin}}\tilde{\tilde{t}} & \\ \sin 2\omega_{\mathrm{bin}}\tilde{\tilde{t}} & -\cos 2\omega_{\mathrm{bin}}\tilde{\tilde{t}} & \\ & & \end{pmatrix}_{\alpha\beta} + \mathscr{O}\left(\omega^3\right) \tag{216}$$

where $\tilde{\tilde{t}} := t - r$. The last step consists in applying the transverse projector

$$\mathscr{P}^{\alpha}_{\mu} := \delta^{\alpha}_{\mu} - \left(\Delta^{-1}\right)\partial^{\alpha}\partial_{\mu} \tag{217}$$

to both indices of $V^{\mathrm{TF}}_{\alpha\beta}$. Note that if used on expression (134), a partial derivative can be substituted by

$$\partial^{\alpha}\partial_{\beta} = -4\omega_{\mathrm{bin}}^2\frac{x^{\alpha}x_{\beta}}{r^2} + \mathscr{O}\left(\frac{1}{r}\right). \tag{218}$$

Therefore, the projector becomes

$$\mathscr{P}^{\alpha}_{\mu} = \delta^{\alpha}_{\mu} + \frac{x^{\alpha}x_{\mu}}{r^2} + \mathscr{O}\left(\frac{1}{r}\right) = \delta^{\alpha}_{\mu} + \hat{x}^{\alpha}\hat{x}_{\mu} + \mathscr{O}\left(\frac{1}{r}\right) \tag{219}$$

and I find the final result up to and including second order terms in ω:

$$V(x,t)_{\mu\nu} = \frac{\vartheta\mu d^2\omega_{\mathrm{bin}}^2}{8\pi r}\left(\delta^{\alpha}_{\mu} + \hat{x}^{\alpha}\hat{x}_{\mu}\right)\left(\delta^{\beta}_{\nu} + \hat{x}^{\beta}\hat{x}_{\nu}\right)\begin{pmatrix}\cos 2\omega_{\mathrm{bin}}\tilde{\tilde{t}} & \sin 2\omega_{\mathrm{bin}}\tilde{\tilde{t}} & \\ \sin 2\omega_{\mathrm{bin}}\tilde{\tilde{t}} & -\cos 2\omega_{\mathrm{bin}}\tilde{\tilde{t}} & \\ & & \end{pmatrix}_{\alpha\beta}. \tag{220}$$

4.4 The effect of scalar waves on test particles

In this last section, I shortly illustrate the true novelty of the phenomenology associated with area metric gravitational waves. To this end, I investigate the effect of the scalar wave in the gravitational potential $\tilde{U}$ on test particles.

The dynamics of a point particle in birefringent spacetime are generated by the Hamiltonian (182). The effect that I want to investigate does only require an uncharged, non-relativistic point mass as a test particle. Simplifying the the Hamiltonian (182) accordingly entails setting $e = 0$ and an expansion to $\mathscr{O}(p^2)$, which yields

$$H = -(1 + A)\left(m + \frac{p^2}{2m}\right) - \frac{1}{2m}\left(\frac{1}{2}\bar{V}^{\alpha\beta}p_\alpha p_\beta + \frac{3}{2}\tilde{U}p^2\right). \tag{221}$$

This expression already contains the gauge fixing $N^\alpha = \tilde{V} = 0$. Now, imagine the neutral, slow point mass far away from the radiating, charged binary system I investigated in this thesis. The gravitational potentials are weak, rendering the already small p^2-terms negligible. The leading order contribution to the particles dynamics is then generated by the Hamiltonian

$$H = -(1 + A)\, m - \frac{p^2}{2m} \tag{222}$$

where, as usual, A is evaluated at the particle's position q. As I showed above, A is not produced directly by the binary system. However, it is sourced by the field $\tilde{U}$ through (194). In the next paragraphs, I will firstly demonstrate how to determine A from $\tilde{U}$, and then show how a point mass reacts to a passing scalar gravitational wave.

Far away from the binary, $\tilde{U}$ essentially resembles a massive plane wave:

$$\tilde{U} \approx \tilde{U}_0 e^{i\left(\omega t - k^\alpha x^\beta \gamma_{\alpha\beta}\right)}, \tag{223}$$

with $\omega^2 = M_{\tilde{U}}^2 + k^2$. The amplitude $\tilde{U}_0$ of the wave decays as the distance to the source grows. I switched to a complex representation to simplify algebraic manipulations. The true value of the field $\tilde{U}$ is of course real, and can be regained at each point simply by taking the real part of the complex wave. I propose

$$A = A_0 e^{i\left(\omega t - k^\alpha x^\beta \gamma_{\alpha\beta}\right)} \tag{224}$$

as an ansatz and use (194) to solve for A_0. I obtain

$$A_0 = \left(a\frac{1}{1 - M_{\tilde{U}}^2/\omega^2} + b\right)\tilde{U}_0, \tag{225}$$

where $a = -\left(3\left(s_1 + s_2\right) + 2s_8\right)/4(s_1 + s_2)$ and $b = -\left(3\left(s_1 + s_2\right) - s_8\right)/6(s_1 + s_2)$.

Note that A_0 seems to diverge as $\omega \to M_{\tilde{U}}$. However, (212) asserts that $\tilde{U}_0 \sim 1 - M_{\tilde{U}}^2/\omega^2$. Therefore, the divergence cancels out, and as we shall see every observable quantity stays finite.

Next, I plug the gravitational potential A into the Hamiltonian (222), compute the Hamiltonian equations of motion, and finally insert them into each other. Using (224), I arrive at

$$\ddot{q}^{\alpha} = -A_0 \sin\left(\omega t - k^{\mu} q^{\nu} \gamma_{\nu\mu}\right) k^{\alpha}. \tag{226}$$

This equation means that the wave accelerates the point mass along its direction of propagation, and not transversal to it. This is in sharp contrast to transverse traceless tensor waves, which accelerate point masses only transversal to their direction of propagation.

To solve (226), I shall point towards a simplifying approximation made earlier: $A_0 \sim \tilde{U}_0 \sim 1/r$ if $\tilde{U}$ stems from a binary system, which I assumed to be far away from the observer. Accordingly, I shall expand q^{α} in powers of A_0 and solve only to first order. The first two orders are determined by the equations

$$\ddot{q}_0^{\alpha} = 0, \tag{227}$$

$$\ddot{q}_1^{\alpha} = -A_0 \sin\left(\omega t - k^{\mu} q_0^{\nu} \gamma_{\mu\nu}\right) k^{\alpha}, \tag{228}$$

which are easily solved by

$$q_0^{\alpha} = \text{const.}, \tag{229}$$

$$q_1^{\alpha} = \frac{A_0}{\omega^2} \sin\left(\omega t - k^{\mu} q_0^{\nu} \gamma_{\mu\nu}\right) k^{\alpha}. \tag{230}$$

This results into

$$q^{\alpha} = q_0^{\alpha} + \frac{A_0}{\omega^2} \sin\left(\omega t - k^{\mu} q_0^{\nu} \gamma_{\mu\nu}\right) k^{\alpha} + \mathscr{O}\left(1/r^2\right). \tag{231}$$

The particle harmonically oscillates about q_0. Note that the coordinate functions describing the position of a single particle are not a meaningful, observable quantity. However, the relative distance between two particles is! To predict an observable effect, I shall add another particle.

Let these two particles be aligned along the direction of propagation of the gravitational wave:

$$\left(q_0^1\right)^{\alpha} - \left(q_0^2\right)^{\alpha} = c k^{\alpha}. \tag{232}$$

The relative distance between the particles is given as

$$d = \sqrt{\left[(q_0^1)^{\alpha} - (q_0^2)^{\alpha}\right]\left[(q_0^1)^{\beta} - (q_0^2)^{\beta}\right] \gamma_{\alpha\beta}}. \tag{233}$$

To evaluate that expression, I use the approximate solution (231). Further, I shall define $d_0 := c\left|\vec{k}\right|$ and set $\vec{q}_0^{\,1} = 0$ for convenience. I also acknowledge that the wavelength of the gravitational wave is given by $\lambda = 2\pi / \left|\vec{k}\right|$, and that $\omega^2 = M_{\tilde{U}}^2 + k^2$. Taking all this into account, I finally arrive at

$$d = d_0 - \frac{2A_0}{\omega}\sqrt{1 - M_{\tilde{U}}^2/\omega^2}\,\sin\left(\frac{d_0}{\lambda}\pi\right)\sin\left(\omega t + \frac{d_0}{\lambda}\pi\right) + \mathscr{O}\left(1/r^2\right). \tag{234}$$

Two particles which are aligned with the direction of propagation of the scalar gravitational wave thus oscillate harmonically.

The results of section 4.3 suggest that a wave in the field $\tilde{U}$ is generated by the binary system at sufficiently high frequencies, which might arise in coalescence scenarios. If so, these waves could be directly observed through the calculated change of distance, with a detector similar to the LIGO/VIRGO setup.

5 Conclusions

In this thesis, I studied the generation of gravitational waves by two electromagnetically bound point charges orbiting each other. I did this in the context of two different theories of gravity, using a weak field approximation in both cases.

Most general relativity textbooks focus on gravitationally instead of electromagnetically bound binaries to discuss the generation of gravitational waves. Often, these discussions involve a weak field approximation in combination with certain conservation laws (see for instance [18]). This seemingly elegant derivation, however, is not conceptually rigorous enough to be generalized - as a matter of fact, it is impossible to investigate the generation of gravitational waves consistently in the framework of linearized theory [9]. For this reason, I investigate electromagnetically bound masses. The mutual electromagnetic attraction mimics Newtonian gravitational interaction well, and can be tackled with linearized theory in a straightforward manner. Doing so, I obtain nontrivial, qualitatively interesting results, which might well be directly transferable to the case of gravitationally bound masses.

First, I considered Einsteins theory of general relativity, which arises as the gravitational closure for Maxwell's theory of electrodynamics. Application of the theory of general relativity to the chosen model system of our two orbiting charges recovered a well established result: the slow binary emits transverse tracefree tensor waves with two propagating degrees of freedom. Although lacking any novelty as far as the final result is concerned, this part of the thesis gives confidence in the employed formalism and the route I took with my calculations. I chose both formalism and route because they fully generalize to the other gravity theories, as opposed to the textbook method mentioned above, which is tailored specifically to general relativity.

Secondly, I turned to area metric gravity, which provides the gravitational closure for birefringent electrodynamics. In contrast to general relativity, area metric gravity predicts that the binary system emits scalar, vector and tensor waves. However, all these waves with the exception of one transverse tracefree tensor wave are massive. Any such massive wave propagates more slowly than the speed of light. Moreover, in order to produce such massive waves, the system frequency must surpass a certain threshold. A similar — though of course technically and conceptually entirely different — phenomenon also appears in quantum field theory [20]: a photon can only decay into an electron and a positron if its energy - which is proportional to its frequency - surpasses the rest masses of the electron and the positron.

Birefringent electrodynamics is a generalization of Maxwellian electrodynamics, which means that the latter is still contained in the former. Consequently, general relativity is also still contained in areametric gravity. Hence not surprisingly, the exceptional massless transverse tracefree tensor wave of area metric gravity is exactly the Einsteinian remainder that is still embedded in the post-Einsteinian theory. This Einsteinian wave is massless, and thus produced at all orbit frequencies.

For area metric phenomenology, the distinction between the massive and massless waves is critical. Sufficiently high masses effectively suppress the emission of non-metric waves. Slowly orbiting binaries, like the famous binary pulsars [21], will then only produce gravitational waves below the non-metric regime. In that case, all radiation would be of Einsteinian type, possibly leading to a timing formula prediction essentially resembling the one of general relativity. Only at high frequencies would the area metric prediction deviate significantly from the Einsteinian one.

In 2015, the LIGO/VIRGO Collaboration succeeded in measuring gravitational radiation directly [22], for which they received the Nobel price in physics 2017. The measured gravitational radiation had been emitted during an extreme event, probably during the coalescence of two black holes. The detection mechanism exploited change in the distance between mirrors caused by a passing burst of gravitational radiation. This change in distance was characteristic for a transverse traceless tensor wave: it appears differentially in the two spatial directions orthogonal to the direction of propagation.

Above, I demonstrated that area metric gravity predicts the additional emission of a scalar wave during coalescences that exhibit sufficiently high frequencies. In section 4.4, I showed that the resulting scalar wave would have an effect on point masses that is quite distinct from the effect that TT-waves exert. Those scalar waves could be measured by a system similar to the one employed in LIGO/VIRGO. Importantly, the experimentalists would be able to clearly distinguish between the conventional TT-waves and the novel scalar waves.

There are several limitations of my study. First, it must again be stressed that my results apply directly only to electromagnetically bound binaries. In reality, binary systems are bound by gravitational potentials, and might thus generate a different set of waves. However, I expect that at least certain qualitative predictions like the emission of scalar radiation with longitudinal effects on test masses should also be valid for gravitationally bound binaries. The full, nonlinear theory of area metric gravity is needed to confirm this. That theory could eventually also provide the waveform signatures of extreme events like the coalescence of black holes. Such waveforms–which are of crucial importance for the direct detection of gravitational waves–can not be computed using the weak-field theory employed in this thesis, since that approximation breaks down in extreme scenarios. Another limitation is due to my restriction to circular orbits. Most observable binaries do in fact feature elliptic orbits, which require a more involved analysis. Even though beyond the scope of this thesis, such systems might in principle be investigated using the approach presented in this work.

The results that I obtained may be developed further also in other directions: Pulsar timing formulas provide an important testbed for new theories of gravity. These formulas describe how the frequencies of binary pulsars are affected by the loss of energy due to the gravitational radiation [23]. My results in combination with conservation equations for gravitational energy, obtained from the Lagrangian density through the Noether procedure, will allow to find such formulas for area metric gravity. Another line of work might start from the source terms I derived for point mass and electromagnetic fields. I used them to study the generation of gravitational waves, but they might well be used to investigate any other scenario that involves weak fields sourced by matter. Especially, one may determine what kind of matter sources vacuum birefringence, or calculate the area metric corrections to the Newtonian inverse square law.

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6 Appendix

6.1 Stability conditions

$$(t_1 + t_3)\, t_1 3 = 2 t_5 t_1 1 \tag{235}$$

$$-v_2 v_9 = v_8 v_3 \tag{236}$$

$$(s_1 - 3s_2)\, s_7 = 4 s_6 s_3 \tag{237}$$

6.2 Conversion of constants

$s_1 = \kappa_1$ $\quad$ $s_2 = \kappa_2$

$s_3 = \kappa_3$ $\quad$ $s_4 = -\kappa_3$

$s_5 = \kappa_4$ $\quad$ $s_6 = \kappa_5$

$s_7 = \kappa_6$ $\quad$ $s_8 = \kappa_7$

$s_9 = \kappa_8$ $\quad$ $s_{10} = 2\kappa_7 - 6\kappa_1 - 6\kappa_2 - 2\kappa_8$

$s_{11} = \kappa_1 - \kappa_7 + \kappa_8$ $\quad$ $s_{12} = (1/3)\kappa_7 - \kappa_1 - (1/3)\kappa_8$

$s_{13} = \kappa_3$ $\quad$ $s_{14} = -\kappa_3$

$s_{15} = \kappa_4$ $\quad$ $s_{16} = \kappa_5$

$s_{17} = \kappa_6$ $\quad$ $s_{18} = 3\kappa_7 - 6\kappa_1 - 6\kappa_2 - 4\kappa_8$

$s_{19} = 2\kappa_7 - 3\kappa_1 - 3\kappa_2 - 3\kappa_8$ $\quad$ $s_{20} = 2\kappa_7 - 2\kappa_8$

$s_{21} = \kappa_3$ $\quad$ $s_{22} = -\kappa_3$

$s_{23} = \kappa_9$ $\quad$ $s_{24} = -\kappa_9$

$s_{25} = -\kappa_4$ $\quad$ $s_{26} = \kappa_6$

$s_{27} = -4\kappa_5$ $\quad$ $s_{28} = \kappa_{10}$

$s_{29} = 16\kappa_1 + 16\kappa_2 - (8/3)\kappa_7 + (16/3)\kappa_8 - \kappa_{10}$

$s_{30} = 2\kappa_7 - 12\kappa_1 - 12\kappa_2 - 4\kappa_8 + \kappa_{10}$ $\quad$ $s_{31} = 20\kappa_1 + 20\kappa_2 - 4\kappa_7 + 8\kappa_8 - \kappa_{10}$

$s_{32} = \kappa_1 1$ $\quad$ $s_{33} = \kappa_1 1$

$s_{34} = 8\kappa_1 + 8\kappa_2 - (4/3)\kappa_7 + (8/3)\kappa_8$ $\quad$ $s_{35} = -(4/3)\kappa_1 - (4/3)\kappa_2 + 2/3\kappa_7 - 8/9\kappa_8$

$s_{36} = 2\kappa_7 - 12\kappa_1 - 12\kappa_2 - 4\kappa_8 + \kappa_{10}$ $\quad$ $s_{37} = 20\kappa_1 + 20\kappa_2 - 4\kappa_7 + 8\kappa_8 - \kappa_{10}$

$s_{38} = 4\kappa_7 - 18\kappa_1 - 18\kappa_2 - 8\kappa_8 + \kappa_{10}$ $\quad$ $s_{39} = 22\kappa_1 + 22\kappa_2 - (16/3)\kappa_7 + (32/3)\kappa_8 - \kappa_{10}$

$s_{40} = \kappa_{11}$ $\quad$ $s_{41} = \kappa_{11}$

$s_{42} = 4\kappa_1 + 4\kappa_2 - (4/3)\kappa_7 + (8/3)\kappa_8$ $\quad s_{43} = -(2/3)\kappa_1 - (2/3)\kappa_2 + (4/9)\kappa_7 - (2/3)\kappa_8$

$s_{44} = 24\kappa_1 + 24\kappa_2 - 4\kappa_7 + 8\kappa_8$ $\quad s_{45} = 12\kappa_1 + 12\kappa_2 - 4\kappa_7 + 8\kappa_8$

$s_{46} = (4/3)\kappa_7 - (4/3)\kappa_8$ $\quad t_1 = -2\kappa_1 - 3\kappa_2 + \kappa_7 - \kappa_8$

$t_2 = -3\kappa_2 + \kappa_7 - \kappa_8 + \kappa_9$ $\quad t_3 = \kappa_1$

$t_4 = -3\kappa_2 + \kappa_9$ $\quad t_5 = \kappa_3$

$t_6 = -\kappa_3$ $\quad t_7 = 2\kappa_1 + 6\kappa_2 - 2\kappa_7 + 2\kappa_8 - \kappa_9$

$t_8 = \kappa_4$ $\quad t_9 = 2\kappa_4$

$t_{10} = 2\kappa_4$ $\quad t_{11} = \kappa_5$

$t_{12} = \kappa_5$ $\quad t_{13} = \kappa_6$

$t_{14} = \kappa_1$ $\quad t_{15} = -3\kappa_2 + \kappa_9$

$t_{16} = \kappa_1 - \kappa_7 + \kappa_8$ $\quad t_{17} = 3\kappa_1 - \kappa_7 + \kappa_8 + \kappa_9$

$t_{18} = \kappa_3$ $\quad t_{19} = -\kappa_3$

$t_{20} = -4\kappa_1 + 2\kappa_7 - 2\kappa_8 - \kappa_9$ $\quad t_{21} = \kappa_4$

$t_{22} = 2\kappa_4$ $\quad t_{23} = 2\kappa_4$

$t_{24} = \kappa_5$ $\quad t_{25} = \kappa_5$

$t_{26} = \kappa_6$ $\quad t_{27} = \kappa_3$

$t_{28} = -\kappa_3$ $\quad t_{29} = \kappa_3$

$t_{30} = -\kappa_3$ $\quad t_{31} = \kappa_9$

$t_{32} = 4\kappa_1 - 12\kappa_2 + 3\kappa_9$ $\quad t_{33} = -2\kappa_1 - 6\kappa_2 + 2\kappa_7 - 2\kappa_8 + \kappa_9$

$t_{34} = 4\kappa_1 - 2\kappa_7 + 2\kappa_8 + \kappa_9$ $\quad t_{35} = -\kappa_4$

$t_{36} = -\kappa_4$ $\quad t_{37} = 8\kappa_4$

$t_{38} = \kappa_6$ $\quad t_{39} = \kappa_6$

$t_{40} = -4\kappa_5$ $\quad v_1 = -4\kappa_1 - 6\kappa_2 + 2\kappa_7 - 2\kappa_8$

$v_2 = \kappa_9/2$ $\quad v_3 = 2\kappa_3$

$v_4 = -2\kappa_3$ $\quad v_5 = 2\kappa_1 + 6\kappa_2 - 2\kappa_7 + 2\kappa_8 - \kappa_9$

$v_6 = 2\kappa_4$ $\quad v_7 = 2\kappa_4$

$v_8 = 2\kappa_5$ $\quad v_9 = 2\kappa_6$

$v_{10} = -6\kappa_1 - 6\kappa_2 + 2\kappa_7 - 2\kappa_8$ $\quad v_{11} = 2\kappa_1$

$v_{12} = \kappa_9/2$ $\quad v_{13} = 2\kappa_3$

$v_{14} = -2\kappa_3$ $\quad v_{15} = -4\kappa_1 + 2\kappa_7 - 2\kappa_8 - \kappa_9$

$v_{16} = 2\kappa_4$ $\quad v_{17} = 2\kappa_4$

$v_{18} = 2\kappa_5$ $\quad v_{19} = 2\kappa_6$

$v_{20} = 2\kappa_7 - 2\kappa_8$ $\quad v_{21} = 2\kappa_3$

$v_{22} = -2\kappa_3$ $\quad v_{23} = 2\kappa_9$

$v_{24} = 2\kappa_1 - 6\kappa_2$ $\quad v_{25} = -2\kappa_1 - 6\kappa_2 + 2\kappa_7 - 2\kappa_8 + \kappa_9$

$v_{26} = -2\kappa_4$ $\quad v_{27} = 8\kappa_4$

$v_{28} = 2\kappa_6$ $\quad v_{29} = -8\kappa_5$

$v_{30} = -6\kappa_1 - 6\kappa_2 + 4\kappa_7 - 4\kappa_8$ $\quad v_{31} = -6\kappa_1 - 6\kappa_2 + 2\kappa_7 - 2\kappa_8$

$v_{32} = -6\kappa_1 - 6\kappa_2$ $\quad v_{33} = 6\kappa_1 + 6\kappa_2 - 4\kappa_7 + 4\kappa_8$

6.3 Conventions

- I work in units where $c = 1$.
- Latin indices are 4-dimensional spacetime indices, i.e. $a, b, c, d, \ldots$ take the values $0, 1, 2$ and 3.
- Greek indices are 3-dimensional screen manifold indices, i.e. $\alpha, \beta, \mu, \nu, \ldots$ take the values $1, 2$ and 3.
- The flat metric in four dimensions is denoted by η. There exists a special chart in which the components of η read

$$\eta_{ab} = \operatorname{diag}(1, -1, -1, -1)_{ab} \tag{238}$$

  at each point in spacetime.

- The euclidean metric in three dimensions is denoted by γ. There exists a special chart in which the components of γ read

$$\gamma_{\alpha\beta} = \operatorname{diag}(1, 1, 1)_{\alpha\beta} \tag{239}$$

  at each point in space.

- The overall sign of the totally antisymmetric tensor density ϵ^{abcd} is fixed by

$$\epsilon^{0123} = -1. \tag{240}$$

In addition, I set $\epsilon_{0123} = 1$. This enables me to raise and lower the indices of ϵ consistently with the flat metric η.

- The d'Alembert operator is defined as

$$\Box = (\partial/\partial t)^2 - \Delta \tag{241}$$

where Δ is the Laplace operator.

- Fourier transforms in one dimension are defined as

$$f(\omega) := \int \mathrm{d}t \, e^{-i\omega t} f(t) \tag{242}$$

$$f(t) = \int \frac{\mathrm{d}\omega}{2\pi} e^{i\omega t} f(\omega) \tag{243}$$

which implies that one has the integral representation

$$\delta(\omega_1 - \omega_2) = \int \frac{\mathrm{d}t}{2\pi} e^{i(\omega_1 - \omega_2)t}. \tag{244}$$

of the Dirac distribution.

Selbstständigkeitserklärung

Ich versichere, dass ich meine Masterarbeit ohne Hilfe Dritter und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel angefertigt habe und die aus benutzten Quellen wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe. Diese Arbeit hat in gleicher oder ähnlicher Form noch keiner Prüfungsbehörde vorgelegen.

Erlangen, den 25. Juni 2018
Moritz Möller