Complete Matter Hamiltonian:

P.P. + EM + Interaction

- From action to Hamiltonian
- linearized to $\mathcal{O}(\varphi)$
- Equations of motion
- Source terms
- Conservation of Energy and Momentum

From Matter action to matter Hamiltonian

$$S_M = \underbrace{S_{p.p.}}_{\text{point particles}} + \underbrace{S_{EM}}_{\text{Electromagnetism}} + \underbrace{S_I}_{\text{Intraction}}$$

$$= \sum_i \int m_i \sqrt{g_{ab}\dot\gamma_i^a\dot\gamma_i^b}\, d\tau + e_i A_a \dot\gamma_i^a \qquad d\tau$$

$$+ \frac{1}{16\pi}\int d^4x \; g^{ac} g^{bd} F_{ab}F_{cd}\sqrt{|\det g_{..}|}$$

Linearise action! We'll need ... the metric:

from "Conservation of Energy & Momentum"

$$g_{..} = \left[\begin{array}{c|c} N^2 + h_{\alpha\beta}N^\alpha N^\beta & h_{\alpha\beta}N^\beta \\ \hline h_{\alpha\beta}N^\alpha & h_{\alpha\beta}\end{array}\right]_{..}$$

$$g^{-1\,..} = \left[\begin{array}{c|c} 1/N^2 & -\frac{N^\beta}{N^2} \\ \hline -\frac{N^\alpha}{N^2} & h^{-1\,\alpha\beta} + \frac{N^\alpha N^\beta}{N^2}\end{array}\right]^{..}$$

In basis $\{\dot x, e_\alpha\}$.

$$\dot x = NT + N^\alpha e_\alpha$$

Basis dual to $\{\dot x, \tilde e_\alpha\}$ is $\{\kappa, \tilde\varepsilon^\alpha\}$ given by $\kappa = \frac{1}{N} n$, $\tilde\varepsilon^\alpha = \varepsilon^\alpha - \frac{1}{N}N^\alpha n$

The parametrised curves $\dot\gamma_i = \dot x + v^\alpha \tilde e_\alpha$

First, linearise point-Matter action.

$$\sum_i \int m_i \sqrt{g_{ab}\dot\gamma_i^a\dot\gamma_i^b} + e_i A_a \dot\gamma_i^a \; d\tau$$

$$= \sum_i \int dt \sqrt{g_{tt} + 2g_{t\alpha}v^\alpha + g_{\alpha\beta}v^\alpha v^\beta}\; m_i + e_i\left(N\phi + N^\alpha A_\alpha + v^\alpha A_\alpha\right)$$

Since $A_a T^a =: \phi \quad A_a e_\alpha^a =: A_\alpha$,

$$A = \phi n + \varepsilon^\alpha A_\alpha = N\kappa\phi + (\tilde\varepsilon^\alpha + N^\alpha\kappa)A_\alpha = \kappa(N\phi + N^\alpha A_\alpha) + \tilde\varepsilon^\alpha A_\alpha$$

Now, to lowest order, get $\boxed{N = 1 + A}$ $\boxed{N^\alpha = 0 + B^\alpha}$ $\boxed{h^{-1\,\alpha\beta} = -\gamma^{\alpha\beta} + \varphi^{\alpha\beta}}$

Therefore

$$\boxed{h_{\alpha\beta} = -\gamma_{\alpha\beta} - \varphi_{\alpha\beta}} \longleftarrow \varphi_{\alpha\beta} := \varphi^{\mu\nu}\gamma_{\mu\alpha}\gamma_{\nu\beta}$$

Insert to get:

$$S_{pp,I} = \sum_i \int dt \sqrt{1 + 2A - 2\gamma_{\alpha\beta} B^\beta v^\alpha - (\gamma_{\alpha\beta} + \varphi_{\alpha\beta}) v^\alpha v^\beta}\; m_i$$
$$+ e_i \left((1+A)\phi + B^\alpha A_\alpha + v^\alpha A_\alpha \right)$$

$$= \sum_i \int dt \sqrt{1 - \gamma_{\alpha\beta} v^\alpha v^\beta + 2A - 2\gamma_{\alpha\beta} B^\beta v^\alpha - \varphi_{\alpha\beta} v^\alpha v^\beta}\; m_i$$
$$+ e_i \left(\phi + v^\alpha A_\alpha + A\phi + B^\alpha A_\alpha \right)$$

$$= \sum_i \int dt\; m_i \sqrt{1 - v^2} \left(1 + \frac{A - \gamma_{\alpha\beta} B^\beta v^\alpha - \frac{1}{2}\varphi_{\alpha\beta} v^\alpha v^\beta}{1 - v^2} \right)$$
$$+ e_i \left(\phi + v^\alpha A_\alpha + A\phi + B^\alpha A_\alpha \right)$$

$$= \sum_i \int dt\; m_i \left[\sqrt{1 - v^2} + \frac{1}{\sqrt{1 - v^2}} \left(A - \gamma_{\alpha\beta} B^\beta v^\alpha - \frac{1}{2}\varphi_{\alpha\beta} v^\alpha v^\beta \right) \right]$$
$$+ e_i \left[\phi + v^\alpha A_\alpha + A\phi + B^\alpha A_\alpha \right] \quad //$$

Lagrangian $\longrightarrow$ Hamiltonian (do for single particle, then add up!)

$$\int dt \quad m\left[\sqrt{1-v^2} + \frac{1}{\sqrt{1-v^2}}\left(A - \gamma_{\alpha\beta}N^\beta v^\alpha - \frac{1}{2}\varphi_{\alpha\beta}v^\alpha v^\beta\right)\right]$$
$$+ e\left[\phi + v^\alpha A_\alpha + A\phi + N^\alpha A_\alpha\right] + \mathcal{O}(\varphi^2)$$

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial v^\alpha} = m\left[- \frac{v_\alpha}{\sqrt{1-v^2}}\right.$$
$$+ \frac{v_\alpha}{\sqrt{1-v^2}^3}\left(A - \gamma_{\alpha\beta}N^\beta v^\alpha - \frac{1}{2}\varphi_{\alpha\beta}v^\alpha v^\beta\right)$$
$$\left.+ \frac{1}{\sqrt{1-v^2}}\left(- N_\alpha - \varphi_{\alpha\beta} v^\beta\right)\right] + eA_\alpha$$

Assume $v_\alpha = v_\alpha^0 + v_\alpha^1$, solve for v_α^0:

$\boxed{\text{0th order:}}$

$$k_\alpha := p_\alpha - eA_\alpha = -m\frac{v_\alpha^0}{\sqrt{1-v_0^2}}$$

$$v_\alpha^0 = -\frac{k_\alpha}{\sqrt{m^2+k^2}} =: -\frac{k_\alpha}{E_k}$$

$$\sqrt{1-(v_\alpha^0+v_\alpha^1)^2} = \sqrt{1-v^{0\,2}-2v_\alpha^1 v^{\alpha 0}} = \sqrt{1-v^{02}}\sqrt{1-\frac{2v_\alpha^1 v^{\alpha 0}}{1-v^{02}}}$$
$$= \sqrt{1-v^{02}}\left(1-\frac{v_\alpha^1 v^{\alpha 0}}{1-v^{02}}\right) = \frac{m}{E_k}\left(1+\frac{E_k}{m^2}k^\alpha v_\alpha^1\right)$$

$$1-v^{02} = 1-\frac{k^2}{m^2+k^2} = \frac{m^2+k^2-k^2}{m^2+k^2} = \frac{m^2}{E_k^2}$$

$$= k^\alpha \Big(1 + \frac{k^2}{m^2-k^2} - \frac{E_k^2 k^2}{m^2(m^2-k^2)} (A + \dots) - \frac{E_k^2}{m^2}(A + \dots)$$

$$+ \frac{E_k}{m^2-k^2}\left(-N\cdot k + \frac{\varphi(k,k)}{E_k}\right)$$

$$-$$

kein Bock mehr! gehts auch einfach?

Insert in H!

$$H = p_\alpha v^\alpha - \mathcal{L}$$

$$= p_\alpha (v_0^\alpha + v_1^\alpha) - m\left[\frac{m}{E_k}\left(1 + \frac{E_k}{m^2} k^\alpha v_\alpha^1\right) + \frac{E_k}{m}\left(A - \gamma_{\alpha\beta} N^\beta v_0^\alpha - \frac{1}{2}\varphi_{\alpha\beta} v_0^\alpha v_0^\beta\right)\right] - e\Big[\phi + v_0^\alpha A_\alpha + v_1^\alpha A_\alpha + A\phi + N^\alpha A_\alpha\Big] + O(\varphi^2)$$

$$= -p_\alpha \frac{k^\alpha}{E_k} + v_1^\alpha \overbrace{\cancel{(p_\alpha - e A_\alpha)}}^{k_\alpha} - \frac{m^2}{E_k} - \cancel{k^\alpha v_\alpha^1}$$

$$- E_k\left(A - \gamma_{\alpha\beta} N^\beta\left(-\frac{k^\alpha}{E_k}\right) - \frac{1}{2}\varphi_{\alpha\beta}\frac{k^\alpha k^\beta}{E_k^2}\right)$$

$$- e\phi + e\frac{k^\alpha}{E_k} A_\alpha - eA\phi - eN^\alpha A_\alpha + O(\varphi^2)$$

$$= -\frac{k^2}{E_k} - \frac{m^2}{E_k} - E_k\left(A + \gamma_{\alpha\beta} N^\beta k^\alpha \frac{1}{E_k} - \frac{1}{2}\varphi_{\alpha\beta}\frac{k^\alpha k^\beta}{E_k^2}\right)$$

$$- e\phi - eA\phi - eN^\alpha A_\alpha$$

$$= - E_k - E_k A - N^\alpha k_\alpha - N^\alpha e A_\alpha - e A \phi - e \phi + \frac{1}{2E_k} \varphi_{\alpha\beta} k^\alpha k^\beta$$

$$\boxed{H = -(E_k + e\phi) - A(E_k + e\phi) - N^\alpha p_\alpha + \frac{1}{2E_k} \varphi_{\alpha\beta} k^\alpha k^\beta}$$

$$\boxed{E_k = \sqrt{m^2 + (p - eA)^2}} \qquad \boxed{k^\alpha = p^\alpha - eA^\alpha}$$

For multiple particles then:

$$\boxed{H = \sum_i \left[-(1+A)\left(E_k^i + e\,\phi(x_i)\right) - N^\alpha p_\alpha^i + \frac{1}{2E_k^i} \varphi_{\alpha\beta}(x^i)\, k_i^\alpha k_i^\beta \right]}$$

Next: split Maxwell action!

3+1 Split of Maxwell Action

$$S = \frac{1}{16\pi} \int F_{ab} F_{cd}\, g^{ac} g^{bd} \sqrt{|\det g_{..}|}\, d^4x$$

(from Landau - Livshitz)

$$F_{ab} = \partial_a A_b - \partial_b A_a$$

For 3+1 split: – Cast metric in ADM form.
– Write F_{ab} in 3-quantities

$$T^a A_a =: \phi \qquad A_a e^a_\alpha =: A_\alpha$$

$$F_{tt} = 0$$

$$F_{t\beta} = \dot{X}^a e^b_\beta F_{ab} = \dot{X}^a e^b_\beta (\partial_a A_b - \partial_b A_a)$$

$$= \dot{X}^a \partial_a (A_\beta) - (\dot{X}^a \partial_a e^b_\beta) A_b - e^b_\beta \partial_b (\dot{X}^a A_a) + e^b_\beta (\partial_b \dot{X}^a) A_a$$

$$= \dot{X}^a \partial_a (A_\beta) - \partial_\beta ((N T^a + N^\alpha e^a_\alpha) A_a) - \cancel{(\mathcal{L}_{\dot{X}} e_\beta)^b} A_b$$

$$= \dot{X}^a \partial_a (A_\beta) - \partial_\beta (N \phi + N^\alpha A_\alpha)$$

coordinate basis

What else?

$$\dot{A}_\alpha =: e^a_\alpha (\mathcal{L}_{\dot{X}} A)_a = e^a_\alpha \left(\dot{X}^b \partial_b A_a + \partial_a \dot{X}^b A_b \right)$$

$$= \dot{X}^b \partial_b (A_\alpha) - \dot{X}^b (\partial_b e^a_\alpha) A_a + \partial_\alpha \dot{X}^b A_b$$

$$= \dot{X}^b \partial_b (A_\alpha) - \cancel{(\mathcal{L}_{\dot{X}} e_\alpha)^a} A_a$$

coordinate basis

Can also rather look @

$$F_{0\beta} = T^a e^b_\beta F_{ab}$$

What's the idea? Eliminate $\dot{X}^a \partial_a$ for $\mathcal{L}_{\dot{X}}$.

Again:

$$F_{t\beta} = \dot{X}^a e^b_\beta F_{ab}$$

$$= \dot{X}^a e^b_\beta \left(\partial_a A_b - \partial_b A_a\right)$$

$$= \left(\dot{X}^a \partial_a A_b + \partial_b \dot{X}^a A_a - \partial_b \dot{X}^a A_a\right) e^b_\beta - \dot{X}^a \partial_\beta A_a$$

$$= \left(\mathcal{L}_{\dot{X}} A\right)_b e^b_\beta - \left(\left(\partial_\beta \dot{X}^a\right) A_a + \dot{X}^a \left(\partial_\beta A_a\right)\right)$$

$$= \dot{A}_\beta - \partial_\beta\left(N T^a A_a + N^\alpha e^a_\alpha A_a\right)$$

$$\boxed{F_{t\beta} = \dot{A}_\beta - \partial_\beta\left(N\phi + N^\alpha A_\alpha\right)}$$

$$F_{\alpha\beta} = e^a_\alpha e^b_\beta \left(\partial_a A_b - \partial_b A_a\right)$$

$$= \left(\partial_\alpha A_\beta\right) - \left(\partial_\alpha e^b_\beta\right) A_b - \left(\partial_\beta A_\alpha\right) + \left(\partial_\beta e^a_\alpha\right) A_a$$

$$= {}^{(3)}F_{\alpha\beta} - \cancel{\left(\mathcal{L}_{e_\alpha} e_\beta\right)^b A_b}$$

coordinate basis

$$\boxed{F_{\alpha\beta} = {}^{(3)}F_{\alpha\beta}}$$

$$F_{ab} F_{cd} g^{ac} g^{bd} = 2 g^{\alpha\beta} g^{tt} F_{\alpha t} F_{\beta t} + 4 g^{\alpha\beta} g^{\gamma t} F_{\alpha\gamma} F_{\beta t} + g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta}$$

Make life easier:

$$F_{0\beta} = T^a e^b_\beta F_{ab} = \frac{\dot{X}^a - N^\alpha e^a_\alpha}{N} e^b_\beta F_{ab}$$

ℓ

$$= \frac{1}{N} \dot{x}^a e^b_\beta F_{ab} - \frac{N^\alpha}{N} e^a_\alpha e^b_\beta F_{ab}$$

$$\boxed{F_{0\beta} = \frac{1}{N} F_{t\beta} - \frac{N^\alpha}{N} F_{\alpha\beta} = \frac{1}{N}\left(\dot{A}_\beta - \partial_\beta\left(N\phi + N^\alpha A_\alpha\right)\right) - \frac{N^\alpha}{N} {}^3F_{\alpha\beta}}$$

Then : $F_{ab} F_{cd} g^{ac} g^{bd}$

$$= 2 g^{00} g^{\alpha\beta} F_{\alpha 0} F_{\beta 0} + g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta}$$

$$= 2 h^{\alpha\beta} F_{\alpha 0} F_{\beta 0} + h^{\alpha\beta} h^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta}$$

Also note $\boxed{d^4x \sqrt{|\det g_{..}|} = N \sqrt{|\det h_{..}|}\, dt\, d^3x}$ (Kuchar, Stone 3.8)

$$S = \int dt \int d^3x \underbrace{\frac{N\sqrt{|\det h_{..}|}}{16\pi}\left[2 h^{\alpha\beta} F_{\alpha 0} F_{\beta 0} + h^{\alpha\beta} h^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta}\right]}_{\mathcal{L}}$$

Then : $\pi^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{A}_\alpha}$

$$= \frac{\partial \mathcal{L}}{\partial F_{\mu 0}} \frac{\partial F_{\mu 0}}{\partial \dot{A}_\alpha} = \frac{\partial \mathcal{L}}{\partial F_{\mu 0}} \frac{1}{N} \delta^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial F_{\alpha 0}} \frac{1}{N}$$

$$= \frac{1}{N} \frac{2N\sqrt{|\det h_{..}|}}{16\pi} 2 h^{\alpha\beta} F_{\beta 0} = \frac{1}{4\pi} \sqrt{|\det h_{..}|}\, h^{\alpha\beta} F_{\beta 0}$$

$$\boxed{\pi^\alpha = \frac{1}{4\pi} \sqrt{|\det h_{..}|}\, h^{\alpha\beta} F_{\beta 0}}$$

$$\Leftrightarrow \boxed{\frac{4\pi}{\sqrt{|\det h_{..}|}} \pi^\beta h_{\beta\alpha} = F_{\alpha 0}}$$

Next I need Hamiltonian:

$$H = \Pi^\alpha \dot{A}_\alpha - \mathcal{L}$$

Note that $\quad N F_{0\alpha} = \dot{A}_\alpha - \partial_\alpha (N\phi + N^\mu A_\mu) - N^\mu\, {}^{(3)}F_{\mu\alpha}$

$$\Rightarrow \quad \dot{A}_\alpha = N F_{0\alpha} + \partial_\alpha (N\phi + N^\mu A_\mu) + N^\mu\, {}^{(3)}F_{\mu\alpha}$$

get

$$H = \Pi^\alpha \left(N \frac{4\pi}{\sqrt{|\det h_{..}|}} h_{\alpha\beta} \Pi^\beta + \partial_\alpha (N\phi + N^\mu A_\mu) + N^\mu\, {}^{(3)}F_{\mu\alpha} \right)$$

$$- N \frac{\sqrt{|\det h_{..}|}}{16\pi} \left[2 h^{\mu\nu} \left(\frac{4\pi}{\sqrt{|\det h_{..}|}} \right)^2 \Pi^\sigma h_{\sigma\mu} \Pi^\tau h_{\tau\nu} \right.$$

$$\left. + h^{\alpha\beta} h^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} \right]$$

$$= N \left[\frac{4\pi}{\sqrt{|\det h_{..}|}} h_{\alpha\beta} \Pi^\alpha \Pi^\beta - \frac{2\pi}{\sqrt{|\det h_{..}|}} h_{\alpha\beta} \Pi^\alpha \Pi^\beta \right.$$

$$\left. - \frac{\sqrt{|\det h_{..}|}}{16\pi} h^{\alpha\beta} h^{\gamma\delta}\, {}^{(3)}F_{\alpha\gamma}\, {}^{(3)}F_{\beta\delta} \right] + \Pi^\alpha \partial_\alpha (N\phi + N^\mu A_\mu)$$

$$+ N^\mu\, {}^{(3)}F_{\mu\alpha} \Pi^\alpha$$

$$= N \left[\frac{2\pi}{\sqrt{|\det h_{..}|}} h_{\alpha\beta} \Pi^\alpha \Pi^\beta - \frac{\sqrt{|\det h_{..}|}}{16\pi} h^{\alpha\beta} h^{\gamma\delta}\, {}^{(3)}F_{\alpha\gamma}\, {}^{(3)}F_{\beta\delta} \right]$$

$$+ \Pi^\alpha \partial_\alpha (N\phi + N^\mu A_\mu) + N^\mu\, {}^{(3)}F_{\mu\alpha} \Pi^\alpha$$

Now, assuming that partial integration is allowed since

$$S = \int \Pi^\alpha \dot{A}_\alpha - H \, d^3x \, dt$$

get

$$\boxed{\begin{aligned} H = \; & N \left[\frac{2\pi}{\sqrt{|\det h_{..}|}} h_{\alpha\beta} \Pi^\alpha \Pi^\beta - \frac{\sqrt{|\det h_{..}|}}{16\pi} h^{\alpha\beta} h^{\gamma\delta} \, {}^{(3)}F_{\alpha\gamma} \, {}^{(3)}F_{\beta\delta} \right. \\ & \left. - \phi \, \partial_\alpha \Pi^\alpha \right] + N^\mu \left[\Pi^\alpha \, {}^{(3)}F_{\mu\alpha} - A_\mu \partial_\alpha \Pi^\alpha \right] \end{aligned}}$$

which coincides with Stone & Kuchař (86) up to minus signs (from signature) and factors of 4π (from normalisation).

Next step is: linearise! $\quad N = 1 + A \quad , \quad h_{\alpha\beta} = -\gamma_{\alpha\beta} - \varphi_{\alpha\beta}$

$$\det h_{..} = -\det(\gamma_{..} + \varphi_{..}) = -\left(1 + \gamma^{\alpha\beta}\varphi_{\alpha\beta} + \mathcal{O}(\varphi^2)\right)$$

$$\sqrt{|\det h_{..}|} = \left(1 + \gamma^{\alpha\beta}\varphi_{\alpha\beta} + \mathcal{O}(\varphi^2)\right)^{\frac{1}{2}} = 1 + \tfrac{1}{2}\gamma^{\alpha\beta}\varphi_{\alpha\beta}$$

$$\begin{aligned} H = \; & (1 + A) \left[2\pi \left(1 - \tfrac{1}{2}\gamma^{\alpha\beta}\varphi_{\alpha\beta}\right)\left(-\gamma_{\alpha\beta} - \varphi_{\alpha\beta}\right) \Pi^\alpha \Pi^\beta \right. \\ & - \frac{1}{16\pi}\left(1 + \tfrac{1}{2}\gamma^{\alpha\beta}\varphi_{\alpha\beta}\right)\left(-\gamma^{\alpha\beta} + \varphi^{\alpha\beta}\right)\left(-\gamma^{\gamma\delta} + \varphi^{\gamma\delta}\right) F_{\alpha\gamma} F_{\beta\delta} \\ & \left. - \phi \, \partial_\alpha \Pi^\alpha \right] + N^\mu \left[\Pi^\alpha \, {}^{(3)}F_{\mu\alpha} - A_\mu \partial_\alpha \Pi^\alpha \right] \end{aligned}$$

$$= (1+A)\Big[2\pi \left(-\gamma_{\alpha\beta} - \varphi_{\alpha\beta} + \tfrac{1}{2}\gamma^{\mu\nu}\varphi_{\mu\nu}\gamma_{\alpha\beta}\right)\Pi^\alpha\Pi^\beta$$

$$- \frac{1}{16\pi}\left(\gamma^{\alpha\beta} - 2\varphi^{\alpha\beta} + \tfrac{1}{2}\gamma^{\mu\nu}\varphi_{\mu\nu}\gamma^{\alpha\beta}\right)\gamma^{\gamma\delta} F_{\alpha\gamma}F_{\beta\delta}$$

$$- \phi\, \partial_\alpha \Pi^\alpha \Big] + N^\mu\left[\Pi^\alpha F_{\mu\alpha} - A_\mu \partial_\alpha \Pi^\alpha\right]$$

$$= (1+A)\Big[-2\pi\gamma_{\alpha\beta}\Pi^\alpha\Pi^\beta - \frac{1}{16\pi}\gamma^{\alpha\beta}\gamma^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta} - \phi\,\partial_\alpha\Pi^\alpha\Big]$$

$$+ \Big[2\pi\left(-\delta^\mu_\alpha\delta^\nu_\beta + \tfrac{1}{2}\gamma^{\mu\nu}\gamma_{\alpha\beta}\right)\varphi_{\mu\nu}\Pi^\alpha\Pi^\beta$$

$$- \frac{1}{16\pi}\left(-2\gamma^{\mu\alpha}\gamma^{\nu\beta} + \tfrac{1}{2}\gamma^{\mu\nu}\gamma^{\alpha\beta}\right)\varphi_{\mu\nu}\gamma^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta}\Big]$$

$$+ N^\mu \underbrace{\left[\Pi^\alpha F_{\mu\alpha} - A_\mu\partial_\alpha\Pi^\alpha\right]}_{D_\mu}$$

finally write $\quad F_{\mu\nu} = \varepsilon_{\mu\nu\sigma} H^\sigma$

get $\quad \gamma^{\gamma\delta} F_{\alpha\gamma}F_{\beta\delta} = \gamma^{\gamma\delta}\varepsilon_{\alpha\gamma\sigma}\varepsilon_{\beta\delta\tau}H^\sigma H^\tau$

$$= \left(\gamma_{\alpha\beta}\gamma_{\sigma\tau} - \gamma_{\alpha\tau}\gamma_{\sigma\beta}\right)H^\sigma H^\tau$$

$$\gamma^{\alpha\beta}\gamma^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta} = 2\gamma_{\sigma\tau}H^\sigma H^\tau$$

$$= (1+A)\Big[-2\pi\gamma_{\alpha\beta}\Pi^\alpha\Pi^\beta - \frac{1}{8\pi}\gamma_{\sigma\tau}H^\sigma H^\tau - \phi\,\partial_\alpha\Pi^\alpha\Big]$$

$$+ \Big[-2\pi\left(\delta^\mu_\alpha\delta^\nu_\beta - \tfrac{1}{2}\gamma^{\mu\nu}\gamma_{\alpha\beta}\right)\varphi_{\mu\nu}\Pi^\alpha\Pi^\beta$$

$$- \frac{1}{16\pi}\left(-2\gamma^{\mu\alpha}\gamma^{\nu\beta} + \tfrac{1}{2}\gamma^{\mu\nu}\gamma^{\alpha\beta}\right)\left(\gamma_{\alpha\beta}\gamma_{\sigma\tau} - \gamma_{\alpha\tau}\gamma_{\sigma\beta}\right)H^\sigma H^\tau\varphi_{\mu\nu}\Big] + N^\mu D_\mu$$

$$\boxed{\begin{aligned} H = &-(1+A)\left[2\pi\,\gamma_{\alpha\beta}\Pi^\alpha\Pi^\beta + \frac{1}{8\pi}\gamma_{\sigma\tau}H^\sigma H^\tau + \phi\,\partial_\alpha\Pi^\alpha\right] \\ &- \varphi_{\mu\nu}\Big[2\pi\left(\delta^\mu_\alpha\delta^\nu_\beta - \frac{1}{2}\gamma^{\mu\nu}\gamma_{\alpha\beta}\right)\Pi^\alpha\Pi^\beta \\ &\qquad + \frac{1}{8\pi}\left(\delta^\mu_\tau\delta^\nu_\sigma - \frac{1}{2}\gamma^{\mu\nu}\gamma_{\sigma\tau}\right)H^\sigma H^\tau\Big] \\ &+ N^\mu\left[\varepsilon_{\mu\alpha\beta}\Pi^\alpha H^\beta - A_\mu\partial_\alpha\Pi^\alpha\right] \end{aligned}}$$

The linearised Hamiltonian of Maxwell theory!

Can now write down complete matter Hamiltonian!

The Hamiltonian for multiple point particles in the electromagnetic field is

$$H = \sum_i \Big[-(1+A(x_i))\big(E_k^i + e\,\phi(x_i)\big) - N^\alpha(x_i)\,p_\alpha^i + \frac{1}{2E_k^i}\varphi_{\alpha\beta}(x_i)\,k_i^\alpha k_i^\beta \Big]$$

$$-(1+A)\Big[2\pi\,\gamma_{\alpha\beta}\Pi^\alpha\Pi^\beta + \frac{1}{8\pi}\gamma_{\alpha\beta}H^\alpha H^\beta + \phi\,\partial_\alpha\Pi^\alpha\Big]$$

$$-\varphi_{\mu\nu}\Big[2\pi\big(\delta^\mu_\alpha\delta^\nu_\beta - \tfrac{1}{2}\gamma^{\mu\nu}\gamma_{\alpha\beta}\big)\Pi^\alpha\Pi^\beta + \frac{1}{8\pi}\big(\delta^\mu_\alpha\delta^\nu_\beta - \tfrac{1}{2}\gamma^{\mu\nu}\gamma_{\alpha\beta}\big)H^\alpha H^\beta$$

$$-N^\alpha\Big[\epsilon_{\alpha\mu\nu}H^\mu\Pi^\nu + A_\alpha\,\partial_\mu\Pi^\mu\Big]$$

Now various things are of interest:

1. The $\mathcal{O}(0)$ Hamiltonian provides the eq. of motion.
   Do they coincide with the ones solved?
   For simplicity: one particle:

$$H^0 = -\big[E_k + e\,\phi(x(t))\big]$$

$$-\int d^3x\,\Big\{2\pi\,\gamma_{\alpha\beta}\Pi^\alpha\Pi^\beta + \frac{1}{8\pi}\gamma_{\alpha\beta}H^\alpha H^\beta + \phi\,\partial_\alpha\Pi^\alpha\Big\}$$

Now, eq. of motion for particle:

$$\dot{x}^\alpha = \frac{\partial H}{\partial p_\alpha} = -\frac{\partial E_k}{\partial p_\alpha} = -\frac{\partial}{\partial p_\alpha}\Big(\sqrt{m^2 + (p-eA)^2}\Big)$$

$$\boxed{\dot{x}^\alpha = -\frac{(p-eA)^\alpha}{\sqrt{m^2+(p-eA)^2}}}$$

$$\dot{p}_\alpha = -\frac{\partial H}{\partial x^\alpha} = \partial_\alpha \left[E_K + e\,\phi(x(t)) \right]$$

$$= -e\frac{(p-eA)_\mu\, \partial_\alpha A^\mu}{\sqrt{m^2+(p-eA)^2}} + e\,(\partial_\alpha \phi)(x(t))$$

$$\underset{\uparrow}{=} e\left[\dot{x}^\mu\, \partial_\alpha A_\mu + \partial_\alpha \phi \right]$$

using Eqn 1

Now $(\dot{x} \times H)_\mu = \varepsilon_{\mu\alpha\beta}\, \dot{x}^\alpha H^\beta = \dot{x}^\alpha F_{\mu\alpha}$

$$= \dot{x}^\alpha (\partial_\mu A_\alpha - \partial_\alpha A_\mu)$$

So

$$\frac{d}{dt}(p_\alpha - eA_\alpha) = e\left[\dot{x}^\mu \partial_\alpha A_\mu + \partial_\alpha \phi \right] - e\,\partial_\mu A_\alpha\, \dot{x}^\mu - e\frac{\partial}{\partial t} A_\alpha$$

$$= e\left[(\dot{x} \times H)_\alpha + \partial_\alpha \phi - \frac{\partial}{\partial t} A_\alpha \right]$$

Now, in eq. 1: $\dot{x}^\alpha = -\dfrac{K^\alpha}{\sqrt{m^2+K^2}}$; $K = p - eA$

$$\Leftrightarrow \boxed{K^\alpha = -m\frac{\dot{x}^\alpha}{\sqrt{1-\dot{x}^2}}}$$

So

$$\boxed{\frac{d}{dt}\left(\frac{m\dot{x}^\alpha}{\sqrt{1-\dot{x}^2}} \right) = -e\left[(\dot{x} \times H)_\alpha + \partial_\alpha \phi - \frac{\partial}{\partial t} A_\alpha \right]}$$

$F_{0\alpha} = \partial_0 A_\alpha - \partial_\alpha A_0 = \frac{\partial}{\partial t} A - \partial_\alpha \phi$, all seems fine...

Next, need equations of motion for em field!

$$\dot{A}_\alpha = \frac{\delta H}{\delta \Pi^\alpha} = - 4\pi \gamma_{\alpha\beta} \Pi^\beta + \partial_\alpha \phi$$

$$\boxed{4\pi \Pi^\beta \gamma_{\alpha\beta} = \partial_\alpha \phi - \partial_t A_\alpha =: E_\alpha}$$

$$\dot{\Pi}^\alpha = - \frac{\delta H}{\delta A_\alpha} = \frac{\delta E_K}{\delta A_\alpha} + \frac{1}{4\pi} \gamma_{\mu\nu} H^\mu \frac{\delta H^\nu}{\delta A_\alpha}$$

$$\frac{\delta E_K}{\delta A_\alpha} = \frac{\delta}{\delta A_\alpha} \sqrt{m^2 + (p - eA)^2}$$

$$= \frac{(p-eA)^\mu}{E_K} (-e \delta^\alpha_\mu \delta_\lambda) = -e \frac{(p-eA)^\alpha}{E_K} \delta_\lambda$$

$$= e \dot{\lambda}^\alpha \delta_\lambda$$

$$\varepsilon^{\mu\nu\tau} F_{\mu\nu} = \varepsilon_{\mu\nu\sigma} H^\sigma \varepsilon^{\mu\nu\tau} = 2 \delta^\tau_\sigma H^\sigma = 2 H^\tau$$

$$\leadsto \quad \frac{\delta H^\nu}{\delta A_\alpha} = \frac{\delta}{\delta A_\alpha} \frac{1}{2} \varepsilon^{\sigma\tau\nu} F_{\sigma\tau} = \frac{\delta}{\delta A_\alpha} \varepsilon^{\sigma\tau\nu} \partial_\sigma A_\tau$$

$$= \varepsilon^{\sigma\alpha\nu} \partial_\sigma \delta$$

get $$\dot{\Pi}^\alpha = e \dot{\lambda}^\alpha \delta_\lambda - \frac{1}{4\pi} \varepsilon^{\sigma\alpha\nu} \partial_\sigma H_\nu$$

$$\Leftrightarrow \quad 4\pi \dot{\Pi}^\alpha \gamma_{\alpha\beta} = e 4\pi \dot{\lambda}_\beta \delta_\lambda + \gamma_{\alpha\beta} \varepsilon^{\mu\nu\alpha} \partial_\mu H_\nu$$

$$\dot{E}_\beta = 4\pi e \dot{\lambda}_\beta \delta_\lambda + (\partial \times H)_\beta$$

Finally, must use contraint stuff:

$$0 = \frac{\delta H}{\delta \phi} = -e\,\delta_x - \partial_\alpha \Pi^\alpha$$

$$\Leftrightarrow \quad \partial_\alpha E_\beta \, \gamma^{\alpha\beta} = -e\,\delta_x\, 4\pi$$

$$\partial_\alpha E^\alpha = -4\pi e\,\delta(x)$$

Equations are gathered:

$$\boxed{\dot{E}_\beta - (\partial \times H)_\beta = 4\pi e\,\dot{x}_\beta\,\delta(x)}$$

$$\boxed{\partial_\alpha E^\alpha = -4\pi e\,\delta(x)}$$

Further, know that $\frac{1}{2} F_{\mu\nu}\, \varepsilon^{\mu\nu\tau} = H^\tau$

and $\partial_\tau H^\tau = \frac{1}{2} \varepsilon^{\mu\nu\tau}\, \partial_\tau F_{\mu\nu} = \varepsilon^{\mu\nu\tau}\, \partial_\tau \partial_\mu A_\nu = 0$

$$\boxed{\partial_\tau H^\tau = 0}$$

and

$$E_\alpha = \partial_\alpha \phi - \partial_t A_\alpha$$

so

$$(\partial \times E)_\alpha = \varepsilon_{\mu\nu\alpha}\, \partial^\mu E^\nu = \varepsilon_{\mu\nu\alpha}\, \partial^\mu (\partial^\nu \phi - \partial_t A^\nu)$$

$$= -\partial_t\, \varepsilon_{\mu\nu\alpha}\, \partial^\mu A^\nu = -\partial_t H_\alpha$$

So $\boxed{(\partial \times E)_\alpha + \partial_t H_\alpha = 0}$

Obtained eq. of motion for field & charges:

$$\boxed{\begin{aligned}
&\frac{d}{dt}\left[\frac{m\dot{x}^\alpha}{\sqrt{1-\dot{x}^2}}\right] = -e\,(\dot{x}\times H)^\alpha - eE_\alpha \\
&\partial_\alpha E^\alpha = -4\pi e\,\delta(x(t)) \\
&\partial_\alpha H^\alpha = 0 \\
&(\partial\times E)_\alpha = -\dot{H}_\alpha \\
&(\partial\times H)_\alpha = \dot{E}_\beta - 4\pi e\,\dot{x}_\beta\,\delta(x(t))
\end{aligned}}$$

which are exactly the correct equations given in Landau/Lifshitz, if $e \to -e$ convention.

Next: Obtain the source-terms for gravity!

From Hamiltonian, get

$$\frac{\delta H_{p.p}}{\delta \varphi^{\alpha\beta}} = \sum_i \left\{ \frac{1}{2E_k^i} k_\alpha^i k_\beta^i \delta_x \right\}$$

$$\frac{\delta H_{EM}}{\delta \varphi^{\alpha\beta}} = - \frac{1}{8\pi} \left\{ E_\alpha E_\beta - \frac{1}{2}\gamma_{\alpha\beta} E^2 + H_\alpha H_\beta - \frac{1}{2}\gamma_{\alpha\beta} H^2 \right\}$$

Factor $\frac{1}{2}$ difference from Landau-Lifshitz!

Further

$$\frac{\delta H_{p.p.}}{\delta N} = - \left(E_k + e\phi \right) \delta_x$$

$$\frac{\delta H_{EM}}{\delta N} = - \frac{1}{8\pi} \left[E^2 + H^2 \right] - \frac{1}{4\pi} \partial_\alpha E^\alpha \phi$$

$$\frac{\delta H_{p.p.}}{\delta N^\alpha} = - p_\alpha$$

$$\frac{\delta H_{EM}}{\delta N^\alpha} = \frac{1}{4\pi} \left[\varepsilon_{\alpha\mu\nu} E^\mu H^\nu - A_\mu \partial_\alpha E^\alpha \right]$$

Furthermore, can use the conservation of GT Tensor to relate and check p.p. calculation!

Energy - momentum - conservation:

In flat space, have
$$\dot{\mathcal{H}} = -\partial_\alpha \mathcal{D}^\alpha$$
$$\dot{\mathcal{D}}_\mu = -2\partial_\alpha\left(\frac{\delta S}{\delta \varphi_{\alpha\gamma}}\,\gamma_{\gamma\mu}\right)$$

$$\dot{\mathcal{H}} = \partial_t \frac{\delta H}{\delta N} = -\partial_t\left[\left(E_k + e\phi\right)\delta(x) + \frac{1}{8\pi}\left[E^2 + H^2\right] + \frac{1}{4\pi}\partial_\alpha E^\alpha \phi\right]$$
$$= -\partial_t\left[E_k\,\delta(x) + \frac{1}{8\pi}\left[E^2 + H^2\right] + \underbrace{\frac{\phi}{4\pi}\left(\partial_\alpha E^\alpha + e\,\delta(x)\,4\pi\right)}_{=0}\right]$$
$$= -\partial_t\left[\sqrt{m^2 + (p - eA)^2}\,\delta_x + \frac{1}{8\pi}\left(E^2 + H^2\right)\right] \text{ on supp. of eom!}$$
$$\boxed{?}\; = -\partial_t\left[m\,\delta_x\right] + \mathcal{O}(p^2)$$

For two particles:
$$\boxed{\dot{\mathcal{H}} = -m_1\,\partial_\alpha\delta(x_1)\,\dot{x}_1^\alpha - m_2\,\partial_\alpha\delta(x_2)\,\dot{x}_2^\alpha}$$

$$\partial_\alpha \mathcal{D}^\alpha = \partial_\alpha \frac{\delta H}{\delta N_\alpha} = \partial_\alpha\left[-p_\alpha\delta_x\;\frac{1}{4\pi}\left(E\times H\right)_\alpha - \frac{1}{4\pi}A_\mu\partial_\alpha E^\alpha\right]$$

In our solution: $A = 0$, thus, for two particles
$$\partial_\alpha \mathcal{D}^\alpha = -p_\alpha^1\,\partial^\alpha\delta(x_1) - p_\alpha^2\,\partial^\alpha\delta(x_2)$$

which yields
$$\boxed{m_i\,\dot{x}_i^\alpha = -p^{i\alpha}}$$
correct result!

further:

$$\mathcal{D}^{\circ}_{\mu} = \sum_i \partial_t\left(-p^i_\mu \, \delta_{x^i}\right) = -\sum_i \left[\dot p^i_\mu \, \delta(x^i) + p_\mu \, \partial_\alpha \delta(x^i) \, \dot x^{i\alpha}\right]$$

$$-2\,\partial_\alpha\left(\frac{\delta S}{\delta \varphi_{\alpha\gamma}}\,\gamma_{\gamma\mu}\right)$$

$$= 2\,\partial^\alpha \frac{\delta H}{\delta \varphi^{\alpha\mu}} = 2\,\partial^\alpha\Big[\sum_i \frac{1}{2E^i_k}\,k^i_\alpha\,k^i_\mu\,\delta_{x^i}$$

$$- \frac{1}{8\pi}\left\{E_\alpha E_\mu - \frac12 \gamma_{\alpha\mu} E^2 + H_\alpha H_\mu - \frac12 \gamma_{\alpha\mu} H^2\right\}\Big]$$

Newtonian limit: $E^i_k = m^i$, $A = 0$, $H = 0$

$$= 2\,\partial^\alpha\left[\sum_i \frac{1}{2m^i}\,p^i_\alpha p^i_\mu\,\delta(x^i) - \frac{1}{8\pi}\left\{E_\alpha E_\mu - \frac12 \gamma_{\alpha\mu} E^2\right\}\right]$$

$$= \sum_i \frac{1}{m^i}\,p^i_\alpha p^i_\mu\,\partial^\alpha(x^i) - \frac{1}{4\pi}\left\{\partial^\alpha(E_\alpha E_\mu) - \frac12 \partial_\mu E^2\right\}$$

$$= -\sum_i p^i_\mu\,\partial^\alpha(x^i)\,\dot x_\alpha + \frac{1}{4\pi}\Big\{\partial^\alpha E_\alpha E_\mu + \underbrace{E^\alpha(\partial_\alpha E_\mu - \partial_\mu E_\alpha)}_{0 \text{ since } \dot H = 0}$$

$$= -\sum_i p^i_\mu\,\partial^\alpha(x^i)\,\dot x_\alpha + \frac{1}{4\pi}\,\partial^\alpha E_\alpha E_\mu$$

$$\dot H_\sigma = -(\partial \times E)_\sigma$$

$$\varepsilon^{\mu\nu\sigma}\dot H_\sigma = -\varepsilon^{\mu\nu\sigma}\varepsilon_{\sigma\alpha\beta}\,\partial^\alpha E^\beta = -(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha)\,\partial^\alpha E^\beta = -(\partial^\mu E^\nu - \partial^\nu E^\mu)$$

Thus: $\sum_i \dot p^i_\mu\,\delta(x^i) = \sum_i \frac{1}{4\pi}\,\partial^\alpha E_\alpha E_\mu$

$$\Longleftrightarrow \boxed{\begin{aligned} \delta(x^i) &= -\frac{1}{4\pi e}\,\partial^\alpha E_\alpha \\ \dot p^i_\mu &= -e E_\mu \end{aligned}}$$

which is correct!