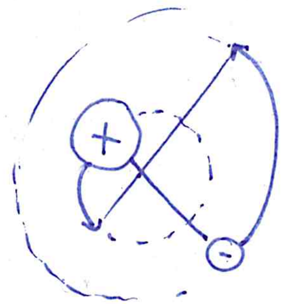
Binary system of
opposite charges
EM interaction



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Electro-magnetic field of a moving point charge

See Landau / Lifshitz:

The electric field

The electric field of a moving charge e @ position $\vec{x}_e$ with velocity $\vec{v}$ is given as

$$\vec{E}(\vec{x}) = e \frac{\vec{x} - \vec{x}_e}{|\vec{x} - \vec{x}_e|^3} \frac{1 - v^2}{\left(1 - v^2 \sin^2\theta\right)^{3/2}}$$

where $\langle \vec{x} - \vec{x}_e , \vec{v} \rangle = |\vec{v}| \, |\vec{x} - \vec{x}_e| \cos\theta$

The dominant contribution in the limit $v \searrow 0$ is

$$\boxed{\vec{E}(\vec{x}) = e \frac{\vec{x} - \vec{x}_e}{|\vec{x} - \vec{x}_e|^3} + \mathcal{O}(v^2)}$$

The magnetic field

The magnetic field of charge e @ position $\vec{x}_e$, velocity $\vec{v}$, is given as

$$\vec{H} = \vec{v} \times \vec{E}$$

The dominant contribution is

$$\boxed{\vec{H} = e \frac{\vec{v} \times (\vec{x} - \vec{x}_e)}{|\vec{x} - \vec{x}_e|^3} + \mathcal{O}(v^3)}$$

Next, we consider two charges in each others potential, moving on a circular orbit, and determine the frequency:

Eq. of motion of a particle of charge e in an external field:

$$\frac{d\vec{p}}{dt} = e\vec{E}_{ext} + e\,\vec{v} \times \vec{H}_{ext}$$

where $\vec{p} = \frac{m}{\sqrt{1-v^2}}\,\vec{v}$, For low velocities:

$$\boxed{m\,\frac{d\vec{v}}{dt} = e\left[\vec{E}_{ext} + \vec{v} \times \vec{H}_{ext}\right]}$$

Now, insert the fields produced by a charge e_2 @ $\vec{x}_2, \vec{v}_2 \ll 1$

$$m_1 \frac{d\vec{v}_1}{dt} = e_1\left[e_2 \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3} + e_2 \vec{v}_1 \times \left(\vec{v}_2 \times \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}\right)\right] + \mathcal{O}(v_2^2)$$

$$= e_1 e_2 \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3} + \mathcal{O}\left(\max\left(\vec{v}_1^{\,2}, \vec{v}_2^{\,2}, \vec{v}_1\vec{v}_2\right)\right)$$

For the particle 2, we obtain the same equation modulo $1 \leftrightarrow 2$ in the indices. Thus:

$$\frac{d}{dt}\left(\vec{v}_1 - \vec{v}_2\right) = e_1 e_2 \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

define $\vec{x}_1 - \vec{x}_2 = \vec{y}$, $\frac{m_1 m_2}{m_1 + m_2} =: \mu$, $e_1 e_2 =: \alpha$, get

$$\mu \frac{d^2}{dt^2}\vec{y} = \alpha \frac{\vec{y}}{|\vec{y}|^3}$$

we also find that $\frac{d}{dt}\left(m_1 \vec{v}_1 + m_2 \vec{v}_2\right) = 0$

And solve this with $\boxed{m_1 \vec{x}_1 + m_2 \vec{x}_2 = 0}$

The centre of mass of the charge-system shall not move!

Let's try the solution $\vec{y} = \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \\ 0 \end{pmatrix} d$, and obtain

$$-\mu\, \omega^2 \cancel{\vec{y}} = \alpha \frac{\cancel{\vec{y}}}{d^3} \iff \omega^2 = -\frac{\alpha}{d^3}\frac{1}{\mu}$$

Circular motion in case $\alpha < 0$ eg opposite charges!

Then, position of the charges:

$$\vec{x}_S = m_1 \vec{x}_1 + m_2 \vec{x}_2 \implies \vec{x}_2 = \frac{1}{m_2}\left(\vec{x}_S - m_1 \vec{x}_1\right)$$

$$\vec{y} = \vec{x}_1 - \vec{x}_2 \iff \vec{x}_1 = \vec{y} + \vec{x}_2$$

$$\Big\} \quad \vec{x}_1 = \vec{y} + \frac{\vec{x}_S}{m_2} - \frac{m_1}{m_2}\vec{x}_1$$

$$\implies \vec{x}_1 = \frac{1}{1 + \frac{m_1}{m_2}}\left(\vec{y} + \frac{\vec{x}_S}{m_2}\right) = \frac{m_2}{m_2 + m_1}\left(\vec{y} + \frac{\vec{x}_S}{m_2}\right)$$

Here: $\vec{x}_S = 0$, thus

$$\boxed{\begin{aligned} \vec{x}_1 &= \frac{m_2}{m_2 + m_1}\vec{y} \\ \vec{x}_2 &= -\frac{m_1}{m_2 + m_1}\vec{y} \end{aligned}}$$

All is checked out, results are gathered on final page

Binary system of opposite charges to order v^2

$$\vec{x}_1 = \frac{\mu}{m_1}\,\vec{y} \qquad \vec{x}_2 = -\frac{\mu}{m_2}\,\vec{y}$$

$$\vec{y} = \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \\ 0 \end{pmatrix} d \qquad \omega^2 = -\frac{e_1 e_2}{\mu d^3} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

fields:

$$\vec{E}(\vec{x}) = e_1 \frac{\vec{x}-\vec{x}_1}{|\vec{x}-\vec{x}_1|^3} + e_2 \frac{\vec{x}-\vec{x}_2}{|\vec{x}-\vec{x}_2|^3} + \mathcal{O}(v^2)$$

$$\vec{H}(\vec{x}) = \mathcal{O}(v)$$