Area - metric (17/8/2017)

Hamiltonian for point particle

in EM field

Next task: Hamiltonian of point matter in presence of EM field!

Step 0: Postulate action

$$S^{P.P} = \int \left[m\, P\left(L^{-1}(\dot{x})\right)^{-\frac{1}{\deg P}} + e A_a \dot{x}^a \right] d\tau$$

Step 1: 3+1 split, linearisation (see area-metric-point-particle-hamiltonian)

$$P(L^{-}(\dot{x}))^{-\frac{1}{\deg P}} = \left(\eta_{ab}\dot{x}^a\dot{x}^b\right)^{\frac{1}{2}} - \frac{1}{4}\frac{\Sigma_{abcd}\,\dot{x}^a\dot{x}^b\dot{x}^c\dot{x}^d}{\left(\eta_{ab}\dot{x}^a\dot{x}^b\right)^{3/2}}$$

In $\{t, y^\alpha\}$ coordinates/basis:

$$A = \kappa\,(N\phi + N^\alpha A_\alpha) + \tilde{\varepsilon}^\alpha A_\alpha$$

so

$$\dot{x}^a A_a = (\dot{X}^a + e^a_\alpha \dot{\lambda}^\alpha)\, A_a$$

$$= N\phi + N^\alpha A_\alpha + \dot{\lambda}^\alpha A_\alpha$$

So:

$$\boxed{\mathcal{L} = m\left((1-\dot{\lambda}^2)^{\frac{1}{2}} - \frac{1}{4}\frac{\sum_{n=0}^{4}\binom{4}{n}\Sigma_{\alpha_1\ldots\alpha_n 0\ldots 0}\,\dot{\lambda}^{\alpha_1}\ldots\dot{\lambda}^{\alpha_n}}{(1-\dot{\lambda}^2)^{3/2}}\right) + e\left(\phi + A\phi + A_\alpha(N^\alpha + \dot{\lambda}^\alpha)\right) + \mathcal{O}(2)}$$

Step 2: Canonical momentum

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = -\frac{m \dot{x}_\alpha}{\sqrt{1-\dot{x}^2}} + e A_\alpha + \mathcal{O}(1)$$

Again, expand $\dot{x}_\alpha = \dot{x}^0_\alpha + \mathcal{O}(1)$

Solve for $\dot{x}^0_\alpha$:

$$k_\alpha := p_\alpha - e A_\alpha = -\frac{m \dot{x}^0_\alpha}{\sqrt{1-\dot{x}_0^2}}$$

$$\Longleftrightarrow \quad \dot{x}^0_\alpha = -\frac{k_\alpha}{\sqrt{m^2 + k^2}} =: -\frac{k_\alpha}{E_k}$$

see
non-relativistic-point-particle-hamiltonian

Now, insert into H:

First note
$$\sqrt{1-\dot{x}^2} = \sqrt{1-\dot{x}_0^2}\left(1 - \frac{\dot{x}^\alpha_0}{1-\dot{x}^2_0}\dot{x}^1_\alpha\right) + \mathcal{O}(2)$$

$$= \frac{m}{E_k}\left(1 - \frac{1}{m\sqrt{1-\dot{x}^2}}\frac{m\dot{x}^\alpha_0}{\sqrt{1-\dot{x}^2_0}}\dot{x}^1_\alpha\right) + \mathcal{O}(2)$$

$$= \frac{m}{E_k}\left(1 + \frac{E_k}{m^2} k_\alpha \dot{x}^{\alpha 1}\right) + \mathcal{O}(2)$$

$$= \frac{m}{E_k} + \frac{k_\alpha}{m}\dot{x}^\alpha_1 + \mathcal{O}(2)$$

$$H = p_\alpha \dot{x}^\alpha - \mathcal{L}$$

$$= p_\alpha \dot{x}^\alpha_0 + p_\alpha \dot{x}^\alpha_1$$

$$- m \left[(1-\dot{x}^2)^{1/2} - \frac{1}{4} \frac{\sum_{n=0}^{4} \binom{4}{n} \varepsilon_{\alpha_1 \dots \alpha_n 0 \dots 0} \dot{x}^{\alpha_1} \dots \dot{x}^{\alpha_n}}{(1-\dot{x}^2)^{3/2}} \right]$$

$$- e \left[\phi + A\phi + A_\alpha N^\alpha + A_\alpha \dot{x}^\alpha_0 + A_\alpha \dot{x}^\alpha_1 \right] + \mathcal{O}(3)$$

$$= k_\alpha \dot{x}^\alpha_0 + \cancel{k_\alpha \dot{x}^\alpha_1}$$

$$- \frac{m^2}{E_k} - \cancel{k_\alpha \dot{x}^\alpha_1} + \frac{m}{4} \frac{\sum_{n=0}^{4} \dots}{(1-\dot{x}^2)^{3/2}}$$

$$- e\phi - eA\phi - eA_\alpha N^\alpha$$

$$= - E_k + \frac{E_k^3}{4} \frac{\sum_{n=0}^{4} \binom{4}{n} \varepsilon_{\alpha_1 \dots \alpha_n 0 \dots 0} \left(-\frac{k^{\alpha_1}}{E_k}\right) \dots \left(-\frac{k^{\alpha_n}}{E_k}\right)}{m^2}$$

$$- e\phi - eA\phi - eA_\alpha N^\alpha$$

$$\boxed{H = H_{free}(k) - e\phi - eA\phi - eA_\alpha N^\alpha}$$

Take H_{free} from the ~~Additi~~ free non-metric theory Hamiltonian:

$$\boxed{H_{p.p} = -(1+A)(E_k + e\phi) - N^\alpha p_\alpha - \frac{1}{2} E_k \left[\frac{1}{2} (\bar{\varphi}^{\alpha\beta} - \bar{\bar{\varphi}}^{\alpha\beta}) \frac{k_\alpha k_\beta}{E_k^2} + \frac{1}{2} (\bar{\varphi} + \bar{\bar{\varphi}}) \frac{k^2}{E_k^2} \right]}$$