I present a case study of moving matter that generates gravitational waves. The system I study consists of two charged point masses, slowly orbiting each other due to electromagnetic attraction.

1 The 3+1 Split

To study my cases, I use the screen manifold formalism. Say the causal structure of the matter theory is given by the principal polynomial P

There are two relevant bases of the tangent space: the orthogonal basis $\{T, e_\alpha\}$ and the foliation basis $\{\dot{X}, \tilde{e}_\alpha\}$, with the corresponding cotangent space bases $\{n, \epsilon^\alpha\}$ and $\{\kappa, \tilde{\epsilon}^\alpha\}$ respectively.

The defining property of the orthogonal basis is

$$P\left(n, \ldots, n\right) := 1 \tag{1}$$

$$P\left(\epsilon^\alpha, n, \ldots, n\right) := 0 \tag{2}$$

The the two bases are related by

$$\dot{X} = NT + N^\alpha e_\alpha \tag{3}$$

$$\tilde{e}_\alpha = e_\alpha \tag{4}$$

and

$$\kappa = \frac{1}{N} n \tag{5}$$

$$\tilde{\epsilon}_\alpha = \epsilon_\alpha - \frac{1}{N} N^\alpha n \tag{6}$$

Formulas to derive / explain / proove:

$$[e_\alpha, e_\beta] = 0 \tag{7}$$

$$\dot{T}^A = \left(\mathcal{L}_{\dot{X}} T\right)^A \tag{8}$$

This section should contain a recipe which explains how to switch from the spacetime formalism to the screen manifold formalism for a generic quantity T which comes with an action S_T:

1. Determine the projections of T on the screen manifold

2. Determine the screen manifold action for those projections by expressing the spacetime action in foliation coordinates, and in terms of the projections

3. From the action, calculate the Hamiltonian that generates the dynamics of the projections on the screen manifold with respect to the embedding parameter

2 Constructive Gravity

The appropriate gravitational theory for a given matter theory depends on the causal structure (i.e. the principal polynomial P) of that matter theory, and is obtained by insisting on consistent dynamics.

More precisely, a gravitational theory simply describes the dynamics of the causal structure of matter dynamics - it describes the dynamics of P.

P is the only relevant geometric object; the Legendre map L, which maps vectors to covectors, is obtained from P by

$$L^a\left(k\right) = \frac{1}{\deg P}\frac{\mathrm{D}^a P\left(k\right)}{P\left(k\right)} \tag{9}$$

The universal action of charged massive point particles in an electromagnetic field is

$$S = m\int \mathrm{d}\tau \left(P\left(L^{-1}\left(\frac{\mathrm{d}\gamma\left(\tau\right)}{\mathrm{d}\tau}\right)\right)^{\frac{1}{\deg P}} + eA_a\frac{\mathrm{d}\gamma^a\left(\tau\right)}{\mathrm{d}\tau}\right) \tag{10}$$

3 Orbiting Charges in Metric Spacetime

In this section, I firstly derive the Hamiltonians of the electromagnetic field and of N charged point particles. Then, I determine the motion of two charged point particles which interact via the electromagnetic field. Finally, I use the linearised equations of metric gravity to determine the gravitational waves generated by the matter system.

Metric specific formulae:

$$g^{00} = 1 \tag{11}$$
$$g^{\alpha 0} = 0 \tag{12}$$
$$g^{\alpha\beta} := g^{\alpha\beta}_{(3)} := -\gamma^{\alpha\beta} + \varphi^{\alpha\beta} \tag{13}$$
$$g_{\alpha\beta} = -\gamma^{\alpha\beta} - \varphi^{\alpha\beta} \tag{14}$$

3.1 The Electromagnetic field

This subsections contains the derivation of the Hamiltonian of the electromagnetic field. The laws of Maxwells electromagnetism are condensed in the action

$$S = \frac{1}{16\pi} \int \mathrm{d}^4 x \, (-\det g^{\cdot\cdot})^{-\frac{1}{2}} \, F_{ab} F_{cd} \; g^{ac} g^{bd} \tag{15}$$
$$F_{ab} := \partial_a A_b - \partial_b A_a \tag{16}$$

which can be found in the book of Landau and Lifshitz.
On the screen manifold, the spacetime object A_a decomposes into

$$\phi := T^a A_a \tag{17}$$
$$A_\alpha := e^a_\alpha A_a \tag{18}$$

The derivative with respect to the embedding parameter t is

$$\dot{A}_a := \left(\mathcal{L}_{\dot{X}} A\right)_a \tag{19}$$
$$\dot{\phi} := T^a \dot{A}_a \tag{20}$$
$$\dot{A}_\alpha := e^a_\alpha \dot{A}_a \tag{21}$$

Now, I express S in terms of fields ϕ and A_α on the screen manifold, their derivatives with respect to y_α and t, and lapse N and shift N^α. I find

$$F_{\alpha\beta} := e^a_\alpha e^b_\beta F_{ab} = F^{(3)}_{\alpha\beta} \tag{22}$$

where $F^{(3)}_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha$ and I used that $[e_\alpha, e_\beta] = 0$, and

$$F_{0\beta} := T^a e^b_\beta F_{ab} = \frac{1}{N}\left(\dot{A}_\beta - \partial_\beta\left(N\phi + N^\alpha A_\alpha\right) - N^\alpha F^{(3)}_{\alpha\beta}\right) \tag{23}$$

using $T = \frac{1}{N}\left(\dot{X} - N^\alpha e_\alpha\right)$. Kuchar and Stone grant that

$$\mathrm{d}^4 x \left(-\det g^{\cdot\cdot}\right)^{-\frac{1}{2}} = \mathrm{d}t\,\mathrm{d}^3 y N \left(-\det g^{\cdot\cdot}_{(3)}\right)^{-\frac{1}{2}} \tag{24}$$

By substituting the above expressions in S, I obtain an action for the fields $\phi(t,y)$ and $A_\alpha(t,y)$ on the screen manifold.

$$S = \frac{1}{16\pi}\int \mathrm{d}t \int \mathrm{d}^3 y\; N\left(-\det g^{\cdot\cdot}_{(3)}\right)^{-\frac{1}{2}} \tag{25}$$

$$\times\left(2 g^{\alpha\beta}_{(3)} F_{\alpha 0} F_{\beta 0} + g^{\alpha\beta}_{(3)} g^{\gamma\delta}_{(3)} F^{(3)}_{\alpha\gamma} F^{(3)}_{\beta\delta}\right) \tag{26}$$

The embedding parameter t controls the dynamics on the screen manifold and serves as the time parameter for the Hamiltonian treatment; the canonical momenta are

$$\Pi^\alpha := \frac{\partial L}{\partial \dot{A}_\alpha} = \frac{1}{4\pi}\left(-\det g^{\cdot\cdot}_{(3)}\right)^{-\frac{1}{2}} g^{\alpha\beta}_{(3)} F_{0\beta} \tag{27}$$

and the Hamiltonian is

$$H := \Pi^\alpha \dot{A}_\alpha - L = N\mathcal{H} + N^\alpha \mathcal{D}_\alpha \tag{28}$$

$$\tag{29}$$

with

$$\mathcal{H} := 2\pi\left(-\det g^{\cdot\cdot}_{(3)}\right)^{\frac{1}{2}} g^{(3)}_{\alpha\beta}\Pi^\alpha\Pi^\beta - \frac{1}{16\pi}\left(-\det g^{\cdot\cdot}_{(3)}\right)^{-\frac{1}{2}} g^{\alpha\beta}_{(3)} g^{\gamma\delta}_{(3)} F^{(3)}_{\alpha\gamma} F^{(3)}_{\beta\delta} - \phi\partial_\alpha\Pi^\alpha \tag{30}$$

$$\mathcal{D}_\alpha := \Pi^\beta F^{(3)}_{\alpha\beta} - A_\alpha \partial_\beta \Pi^\beta \tag{31}$$

For scenarios including only weak gravitational fields, I obtain a decent approximation to this Hamiltonian by substituting

$$g_{(3)}^{\alpha\beta} := -\gamma^{\alpha\beta} + \varphi^{\alpha\beta} \tag{32}$$

$$N := 1 + A \tag{33}$$

and then expanding the Hamiltonian to first order. I arrive at

$$\begin{aligned} H := &-(1+A)\left[2\pi\gamma_{\alpha\beta}\Pi^\alpha\Pi^\beta + \frac{1}{8\pi}\gamma_{\alpha\beta}H^\alpha H^\beta + \phi\partial_\alpha\Pi^\alpha\right] \\ &- \varphi_{\mu\nu}\left(\delta^\mu_\alpha\delta^\nu_\beta - \frac{1}{2}\gamma^{\mu\nu}\gamma_{\alpha\beta}\right)\left[2\pi\Pi^\alpha\Pi^\beta + \frac{1}{8\pi}H^\alpha H^\beta\right] \\ &+ N^\mu\left[\epsilon_{\mu\alpha\beta}\Pi^\alpha H^\beta - A_\mu\partial_\alpha\Pi^\alpha\right] \end{aligned} \tag{34}$$

where

$$H^\alpha := \frac{1}{2}\epsilon^{\alpha\beta\gamma}F^{(3)}_{\alpha\beta} \tag{35}$$

3.2 Point Particles

In relativistic theory, a point particle corresponds to a curve γ in spacetime. To switch to the screen manifold formalism - where the particle is represented by a position $\lambda(t)$ which changes with the foliation parameter t - I parametrise γ with the foliation parameter t and decompose the particles velocity (i.e. the vector tangent to the parametrised curve) according to

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} =: \dot{X} + e_\alpha v^\alpha \tag{36}$$

This decomposition is defined such that $v = \frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} := \dot{\lambda}(t)$, so if $v^\alpha = 0$ the particle moves with the embedding e.g. remains at the same spot on the screen manifold.

The next step towards the autonomous screen manifold formalism requires to express the action that governs the dynamics of the particle solely in screen manifold quantities. For metric geometry, the universal action for a point mass in the presence of the electromagnetic field reads

$$S = \int \mathrm{d}\tau \left[m\left(g_{ab}(\gamma)\frac{\mathrm{d}\gamma^a(\tau)}{\mathrm{d}\tau}\frac{\mathrm{d}\gamma^b(\tau)}{\mathrm{d}\tau}\right)^{\frac{1}{2}} + eA_a(\gamma)\frac{\mathrm{d}\gamma^a(\tau)}{\mathrm{d}\tau}\right] \tag{37}$$

Now parametrise γ with t and use the foliation frame to obtain the screen manifold action

$$\begin{aligned} S = \int \mathrm{d}\tau m \left(N^2 + g^{(3)}_{\alpha\beta} N^\alpha N^\beta + 2 g^{(3)}_{\alpha\beta} v^\alpha N^\beta + g^{(3)}_{\alpha\beta} v^\alpha v^\beta\right)^{\frac{1}{2}} \\ + e \left(N\phi + N^\alpha A_\alpha + v^\alpha A_\alpha\right) \end{aligned} \tag{38}$$

To calculate the Hamiltonian of the point mass, I proceed slightly differently than above: this time, I firstly linearise the action and then execute the Legendre transform with the benefit that the inversion in favour of the velocity must not be done exactly, but pertubatively.

The linearised action reads

$$\begin{aligned} S = \int \mathrm{d}\tau\ m \left(\left(1 - v^2\right)^{\frac{1}{2}} + \left(1 - v^2\right)^{-\frac{1}{2}} \left(A - \gamma_{\alpha\beta} N^\beta v^\alpha - \frac{1}{2} \varphi_{\alpha\beta} v^\alpha v^\beta\right)\right) \\ + e \left(\phi + v^\alpha A_\alpha + A\phi + N^\alpha A_\alpha\right) + \mathcal{O}\left(\varphi^2\right) \end{aligned} \tag{39}$$

The canonical momentum of the point mass is

$$p_\alpha := \frac{\partial L}{\partial v^\alpha} = -m \frac{v^\alpha}{\sqrt{1 - v^2}} + e A_\alpha + \mathcal{O}\left(\varphi\right) \tag{40}$$

I define $k_\alpha = p_\alpha - e A_\alpha$, expand $v^\alpha = v_0^\alpha + v_1^\alpha + \ldots$ in orders of φ and, by virtue of

$$k_\alpha = -m \frac{v^\alpha}{\sqrt{1 - v^2}} \qquad \Leftrightarrow v_\alpha = -\frac{k_\alpha}{\sqrt{k^2 + m^2}} \tag{41}$$

I obtain

$$v_\alpha = -\frac{k_\alpha}{E_k} + \mathcal{O}\left(\varphi\right) \tag{42}$$

where $E_k := \sqrt{k^2 + m^2}$. Accommodatingly, knowledge of the zeroth order of v^α suffices to obtain the linearised Hamiltonian throught the Legendre transform:

$$H := p_\alpha v^\alpha - L = -\left(1 + A\right)\left(E_k + e\phi\right) - N^\alpha p_\alpha + \frac{1}{2E_k} \varphi_{\alpha\beta} k^\alpha k^\beta + \mathcal{O}\left(\varphi^2\right) \tag{43}$$

where all occurring fields are evaluated at the position λ of the particle.

3.3 The Motion of Two Charged Point Masses

The Hamiltonian of the complete system containing both charged point masses and electromagnetic fields is the sum of the respective Hamiltonians:

$$H_{\text{Matter}} = \sum_i H^{(i)}_{\text{Point particle}} + H_{\text{Electromagnetism}} \tag{44}$$

This Hamiltonian is a screen manifold object, which generates the dynamics on the screen manifold with respect to the embedding parameter t.

The Hamiltonian contains the gravitational fields to linear order, so the resulting equations of motion describe the dynamics of electromagnetic fields and point masses in presence of weak gravitational fields. However - as I shall elaborate on later - to consistently analyse gravitational waves in linearised theory, the dynamics of the matter that generates those waves must happen on a flat background. Therefore, for the purpose of determining the equations of motion for the present matter and henceforth the dynamics of the source system, I shall neglect any gravitational fields.

The Hamiltonian of one charged point mass and the electromagnetic field in a flat background is

$$\begin{aligned} H =& -\left(E_k + e\phi\left(\lambda\right)\right) \\ & - \int \mathrm{d}^3y \left(2\pi\gamma_{\alpha\beta}\Pi^\alpha\Pi^\beta + \frac{1}{8\pi}\gamma_{\alpha\beta}H^\alpha H^\beta + \phi\partial_\alpha\Pi^\alpha\right) \end{aligned} \tag{45}$$

Using the Hamiltonian field equations $\dot{A}_\alpha = \delta H/\delta\Pi^\alpha$ and $\dot{\Pi}^\alpha = -\delta H/\delta A_\alpha$, the constraint equation $0 = \delta H/\delta\phi$ and the defining equation of H^α, I obtain Maxwells equations

$$\dot{E}_\alpha - \left(\partial \times H\right)_\alpha = 4\pi e\dot{\lambda}_\alpha\delta_\lambda \tag{46}$$

$$\dot{H}_\alpha + \left(\partial \times E\right)_\alpha = 0 \tag{47}$$

$$\partial_\alpha E^\alpha = -4\pi e\delta_\lambda \tag{48}$$

$$\partial_\alpha H^\alpha = 0 \tag{49}$$

for the electromagnetic fields on the screen manifold, where $E^\alpha := 4\pi\Pi^\alpha = \partial_\alpha\phi - \dot{A}_\alpha$. Further, combining the Hamiltonian equations $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$ for the point mass, I obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mv_\alpha}{\sqrt{1-v^2}}\right) = -e\left[(v\times H)_\alpha + E_\alpha\right] \tag{50}$$

The force on the right hand side is the familiar Lorentz Force.

With the equations of motion at hand, I shall next present a particular solution, namely the slowly orbiting binary system. Slowly moving point masses move with velocities much less than the speed of light, e.g. $v \ll 1$ in units where $c = 1$. For such masses, I shall therefore neglect all but the leading order contribution in v.

A point charge e_1 at the position λ_1, moving with a velocity v_1, generates an electric field

$$E^\alpha(y) = e_1\frac{y^\alpha - \lambda_1^\alpha}{|y-\lambda_1|^3} + \mathcal{O}\left(v_1^2\right) \tag{51}$$

and a magnetic field

$$H^\alpha = \mathcal{O}\left(v_1\right) \tag{52}$$

according to Landau and Lifshitz. A second particle at position λ_2 with charge e_2, mass m_2 and velocity v_2 exposed to the field of the first particle then obeys

$$m_2\dot{v_2}^\alpha = e_1e_2\frac{(\lambda_1-\lambda_2)^\alpha}{|\lambda_1-\lambda_2|^3} + \mathcal{O}\left(v^2\right) \tag{53}$$

Of course, the situation of the second particle is totally similar; I obtain an equation for its velocity simply by exchanging the labels 1 and 2 in eq. 53.

Defining $\lambda_{\mathrm{rel}} = \lambda_1 - \lambda_2$ and $\mu = m_1m_2/\left(m_1+m_2\right)$, I rewrite the equations of motion for the system as

$$\mu\frac{\mathrm{d}^2}{\mathrm{d}t^2}\lambda_{\mathrm{rel}}^\alpha = e_1e_2\frac{\lambda_{\mathrm{rel}}^\alpha}{|\lambda_{\mathrm{rel}}|^3} \tag{54}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(m_1\lambda_1 + m_2\lambda_2\right)^\alpha = 0 \tag{55}$$

Eq. 55 allows the solution $m_1\lambda_1 + m_2\lambda_2 = 0$ which corresponds to a resting centre of mass.

The ansatz for circular motion

$$\lambda_{\mathrm{rel}} = d \begin{pmatrix} \cos\omega t \\ \sin\omega t \\ 0 \end{pmatrix} \tag{56}$$

solves eq. 54 for a frequencies $\omega = \pm\sqrt{-e_1 e_2/\mu d^3}$. Therefore, an particular approximate solution (realistic as long as the velocities are very small) of the flat space equations of motion is given by

$$\lambda_1 = \frac{\mu}{m_1} d \begin{pmatrix} \cos\sqrt{-e_1 e_2/\mu d^3} t \\ \sin\sqrt{-e_1 e_2/\mu d^3} t \\ 0 \end{pmatrix} \tag{57}$$

$$\lambda_2 = \frac{\mu}{m_2} d \begin{pmatrix} \cos\sqrt{-e_1 e_2/\mu d^3} t \\ \sin\sqrt{-e_1 e_2/\mu d^3} t \\ 0 \end{pmatrix} \tag{58}$$

$$E^\alpha (y) = e_1 \frac{y^\alpha - \lambda_1^\alpha}{|y - \lambda_1|^3} + e_1 \frac{y^\alpha - \lambda_1^\alpha}{|y - \lambda_1|^3} \tag{59}$$

$$H^\alpha = 0 \tag{60}$$

which shall henceforth be analysed with respect it's ability to generate waves in the gravitational fields.

3.4 The Generation of Gravitational Waves in Metric Spacetime

4 Orbiting Charges in Weakly Birefringent Spacetime

How would gravity work if the electromagnetic field obeyed the equations of motion of birefringent electrodynamics? These equations feature a different causal structure than the familiar light cone of Maxwell electrodynamics, namely a double light cone. The appropriate gravitational theory, i.e. the dynamics of this double cone, has been determined as part of the constructive gravity program.

In this section, I shall firstly implement the screen manifold formalism for birefringent electrodynamics. Then, I shall investigate the affiliated point particle theory, which also features the double cone as its causal structure. Finally, I shall investigate how the system I presented in sec. 3.3 generates waves in the gravitational fields that make up the causal structure.

4.1 The Electromagnetic Field

Birefringent electrodynamics is a generalisation of Maxwells electrodynamics: just as Maxwells theory, birefringent electrodynamics features a linear equation of motion and thus shares the superposition principle, but in contrast to Maxwell theory, it allows for birefringence in vacua.

In birefringent electrodynamics, the dynamics of the electromagnetic potential A_a are controlled by the action

$$S = \frac{1}{32\pi} \int \mathrm{d}^4 x \omega F_{ab} F_{cd} \; G^{abcd} \tag{61}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ as usual, and

$$\omega = \left(\frac{1}{4!} G^{abcd} \epsilon_{abcd} \right)^{-1} \tag{62}$$

G^{abcd} is called the area metric, and features the symmetries

$$G^{abcd} = G^{cdab} = -G^{bacd} \tag{63}$$

Implementing these symmetries in the expansion of G in the orthogonal basis yields

$$\begin{aligned} G^{abcd} &= 4G^{\beta\delta} T^{[a} e_\beta^{b]} T^{[c} e_\beta^{b]} \\ &+ 2G^{\beta\gamma\delta} T^{[a} e_\beta^{b]} e_\gamma^c e_\delta^d + 2G^{\alpha\beta\delta} e_\alpha^a e_\beta^b T^{[c} e_\delta^{d]} \\ &+ G^{\alpha\beta\gamma\delta} e_\alpha^a e_\beta^{b]} e_\gamma^c e_\delta^d \end{aligned} \tag{64}$$

where

$$G^{\beta\delta} := G\left(n, \epsilon^\beta, n, \epsilon^\delta\right) \tag{65}$$

$$G^{\beta\gamma\delta} := G\left(n, \epsilon^\beta, \epsilon^\gamma, \epsilon^\delta\right) \tag{66}$$

$$G^{\alpha\beta\gamma\delta} := G\left(\epsilon^\alpha, \epsilon^\beta, \epsilon^\gamma, \epsilon^\delta\right) \tag{67}$$

Now, I again undertake the construction of a screen manifold Hamiltonian that generates the same dynamics as the spacetime action 61. The definition of the screen manifold fields ϕ and A_α, as well as the expressions of $F_{0\beta}$ and $F_{\alpha\beta}$ in terms of $N, N^\alpha, \dot{A}_\alpha$ and $F^{(3)}_{\alpha\beta}$, are the same as in sec. 3.1

For later convenience, I shall at this point explicitly reduce the action 61 by a zero contribution. Since

$$\int \mathrm{d}^4x \epsilon^{abcd} F_{ab} F_{cd} = \int \mathrm{d}^4x \epsilon^{abcd} \partial_a A_b \partial_c A_d = -\int \mathrm{d}^4x \epsilon^{abcd} \partial_c \partial_a A_b A_d = 0 \tag{68}$$

using integration by parts, the contribution of the totally antisymmetric part of G to the action vanishes and might as well be explicitly subtracted. The action then becomes

$$S = \frac{1}{32\pi} \int \mathrm{d}^4x \omega F_{ab} F_{cd} \left[G^{abcd} + \frac{1}{4!} \epsilon^{abcd} G^{mnop} \epsilon_{mnop} \right] \tag{69}$$

The weird + sign arises due to convention 118. Expressed only in screen manifold fields and coordinates, the action becomes

$$\begin{aligned} S = \frac{1}{32\pi} \int \mathrm{d}^4x \ \omega [& 4 G^{\beta\delta} F_{0\beta} F_{0\delta} \\ & + \left(2 \left(G^{\beta\gamma\delta} + G^{\gamma\delta\beta} \right) - \frac{4N}{\omega} \epsilon^{\beta\gamma\delta} \right) F_{0\beta} F_{\gamma\delta} \\ & + G^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} - \frac{4N^\mu}{\omega} \epsilon^{\alpha\beta\gamma} F_{\mu\alpha} F_{\beta\gamma}] \end{aligned} \tag{70}$$

The canonical momentum is

$$\Pi^\mu = \frac{\omega}{4\pi N} \left[G^{\mu\nu} F_{0\nu} + \left(\frac{1}{2} G^{\mu\alpha\beta} - \frac{N}{2\omega} \epsilon_{\mu\alpha\beta} \right) F_{\alpha\beta} \right] \tag{71}$$

I define

$$P^\alpha := \frac{\omega}{4\pi N} G^{\alpha\beta} F_{0\beta} = \Pi^\alpha - \frac{\omega}{4\pi N} \left(\frac{1}{2} G^{\mu\alpha\beta} - \frac{N}{2\omega} \epsilon^{\mu\alpha\beta} \right) F_{\alpha\beta} \tag{72}$$

and, through the usual Legendre transform, obtain the Hamiltonian

$$\begin{aligned} H = N & \left[\frac{2\pi N}{\omega} G^{-1}_{\alpha\beta} P^\alpha P^\beta - \frac{\omega}{32\pi N} G^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} - \phi \partial_\alpha \Pi^\alpha \right] \\ & + N^\mu \left[\frac{1}{8\pi} \epsilon^{\alpha\beta\gamma} F_{\mu\alpha} F_{\beta\gamma} + F_{\mu\alpha} \Pi^\alpha - A_\mu \partial_\alpha \Pi^\alpha \right] \end{aligned} \tag{73}$$

Within the community of constructive gravity, it is common not to work with the orthogonal frame components of G directly, but rather with the equivalent set of fields

$$
\begin{aligned}
\bar{g}^{\alpha\beta} &:= -G^{\alpha\beta} && (74)\\
\bar{\bar{g}}_{\alpha\beta} &:= \frac{1}{4}\left(\det \bar{g}^{\cdot\cdot}\right) \epsilon_{\alpha\mu\nu}\epsilon_{\beta\rho\sigma}G^{\mu\nu\rho\sigma} && (75)\\
\bar{\bar{\bar{g}}}^{\alpha}{}_{\beta} &:= \frac{1}{2}\left(\det \bar{g}^{\cdot\cdot}\right)^{-\frac{1}{2}} \epsilon_{\beta\mu\nu}G^{\alpha\mu\nu} - \delta^{\alpha}_{\beta} && (76)
\end{aligned}
$$

The frame conditions 1 and 2 translate into

$$
\begin{aligned}
\bar{\bar{\bar{g}}}^{\alpha}{}_{\alpha} &= 0 && (77)\\
\bar{\bar{\bar{g}}}^{\alpha}{}_{\mu}\bar{g}^{\mu\beta} &= \bar{\bar{\bar{g}}}^{\beta}{}_{\mu}\bar{g}^{\mu\alpha} && (78)
\end{aligned}
$$

Of course, the components of G might be reassembled from the fields $\bar{g}$, $\bar{\bar{g}}$ and $\bar{\bar{\bar{g}}}$ via

$$
\begin{aligned}
G^{\alpha\beta} &= -\bar{g}^{\alpha\beta} && (79)\\
G^{\beta\gamma\delta} &= \left(\det \bar{g}^{\cdot\cdot}\right)^{\frac{1}{2}} \epsilon^{\alpha\gamma\delta}\left(\bar{\bar{\bar{g}}}^{\beta}{}_{\alpha} + \delta^{\beta}_{\alpha}\right) && (80)\\
G^{\alpha\beta\gamma\delta} &= \epsilon^{\alpha\beta\mu}\epsilon^{\gamma\delta\nu}\left(\det \bar{g}^{\cdot\cdot}\right)\bar{\bar{g}}_{\mu\nu} && (81)
\end{aligned}
$$

In terms of $\bar{g}$, $\bar{\bar{g}}$ and $\bar{\bar{\bar{g}}}$, the density ω reads

$$
\omega = N\left(\det \bar{g}^{\cdot\cdot}\right)^{-\frac{1}{2}} \tag{82}
$$

and the Hamiltonian becomes

$$
\begin{aligned}
H = -N[2\pi\left(\det \bar{g}^{\cdot\cdot}\right)^{\frac{1}{2}} \bar{g}^{-1}_{\alpha\beta}&\left(\Pi^{\alpha} - \frac{1}{4\pi}\bar{\bar{\bar{g}}}^{\alpha}{}_{\mu}H^{\mu}\right)\left(\Pi^{\beta} - \frac{1}{4\pi}\bar{\bar{\bar{g}}}^{\beta}{}_{\nu}H^{\nu}\right)\\
&- \frac{1}{8\pi}\left(\det \bar{g}^{\cdot\cdot}\right)^{\frac{1}{2}}\bar{\bar{g}}_{\alpha\beta}H^{\alpha}H^{\beta} - \phi\partial_{\alpha}\Pi^{\alpha}]\\
&- N^{\mu}\left[\epsilon_{\mu\alpha\beta}H^{\alpha}\Pi^{\beta} + A_{\mu}\partial_{\alpha}\Pi^{\alpha}\right]
\end{aligned} \tag{83}
$$

where H^{α} is defined in eq. 35.

If the gravitational fields are very weak, neglecting all but linear terms in these fields yields a good approximation. The parametrisation used for this perturbative approach is

$$
\begin{aligned}
\bar{g}^{\alpha\beta} &:= \gamma^{\alpha\beta} + \bar{\varphi}^{\alpha\beta} && (84)\\
\bar{\bar{g}}_{\alpha\beta} &:= \gamma_{\alpha\beta} + \bar{\bar{\varphi}}_{\alpha\beta} && (85)\\
\bar{\bar{\bar{g}}}^{\alpha}{}_{\beta} &:= \bar{\bar{\bar{\varphi}}}^{\alpha}{}_{\beta} + \mathcal{O}\left(\bar{\bar{\bar{\varphi}}}^{2}\right) && (86)
\end{aligned}
$$

Substituting this into the Hamiltonian, and neglecting all but the leading, linear order in φ, yields an approximate Hamiltonian which is valid as long as the gravitational fields are small enough, eg. $\varphi \ll 1$.

Using Jacobi's formula, I obtain

$$
\det \bar{g}^{\cdot\cdot} = 1 + \gamma_{\alpha\beta}\bar{\varphi}^{\alpha\beta} + \mathcal{O}\left(\varphi^{2}\right) \tag{87}
$$

and arrive at

$$
\begin{aligned}
H = -\left(1+A\right) \Bigg[& \gamma_{\alpha\beta}\left(2\pi\Pi^{\alpha}\Pi^{\beta} + \frac{1}{8\pi}H^{\alpha}H^{\beta}\right) + \phi\partial_{\alpha}\Pi^{\alpha} \Bigg] \\
& -\frac{1}{2}\left(\bar{\bar{\varphi}}^{\alpha\beta} - \bar{\varphi}^{\alpha\beta}\right)\left(\delta^{\alpha}_{\sigma}\delta^{\beta}_{\tau} - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\sigma\tau}\right) \\
& \times \left[2\pi\Pi^{\alpha}\Pi^{\beta} + \frac{1}{8\pi}H^{\alpha}H^{\beta}\right] \\
& +\frac{1}{2}\left(\bar{\bar{\varphi}}^{\alpha\beta} + \bar{\varphi}^{\alpha\beta}\right) \\
\times \Bigg[2\pi & \left(\delta^{\alpha}_{\sigma}\delta^{\beta}_{\tau} - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\sigma\tau}\right)\Pi^{\alpha}\Pi^{\beta} + \frac{1}{8\pi}\left(-\delta^{\alpha}_{\sigma}\delta^{\beta}_{\tau} - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\sigma\tau}\right)H^{\alpha}H^{\beta}\Bigg] \\
& +\bar{\bar{\bar{\varphi}}}^{\beta}{}_{\mu}\left(\delta^{\mu}_{\nu}\delta^{\alpha}_{\beta} - \delta^{\mu}_{\beta}\delta^{\alpha}_{\nu}\right)H^{\nu}\Pi_{\alpha} \\
& -N^{\mu}\left[\epsilon_{\mu\alpha\beta}H^{\alpha}\Pi^{\beta} + A_{\mu}\partial_{\alpha}\Pi^{\alpha}\right]
\end{aligned} \tag{88}
$$

Confidence about this result arises in comparison with the metric case: Choosing

$$
G^{abcd} = g^{ac}g^{bd} - g^{ad}g^{bc} - \left(-\det g^{\cdot\cdot}\right)^{\frac{1}{2}}\epsilon^{abcd} \tag{89}
$$

turns birefringent electrodynamics into electrodynamics la Maxwell - G is then called a metric induced area metric. Using this equation, I determine the value of the area metric gravitational fields in the case where G is metric induced, and obtain

$$\frac{1}{2}\left(\bar{\bar{\varphi}}^{\alpha\beta} - \bar{\varphi}^{\alpha\beta}\right) = \varphi^{\alpha\beta} \tag{90}$$

$$\frac{1}{2}\left(\bar{\bar{\varphi}}^{\alpha\beta} + \bar{\varphi}^{\alpha\beta}\right) = 0 \tag{91}$$

$$\bar{\bar{\bar{\varphi}}}^{\alpha}{}_{\beta} = 0 \tag{92}$$

where φ is defined in eq. 32. Substituting this into the Hamiltonian 88 yields the Hamiltonian of Maxwell electrodynamics 34, prooving at least partial correctness of the result.

4.2 Point Particles

The point particle theory that matches birefringent electrodynamics, i.e. that shares the same causality (and, for massless point particles, constitutes the geometric optical limit of birefringent electrodynamics) is again, as in section 3.2 formulated as soon as the causal structure P of birefringent electrodynamics is inserted into the universal point particle action 10.

In terms of the area metric G that first appeared in the action 61, the principle polynomial P which encodes the causal structure reads

$$\begin{aligned} P(k) &= -\frac{4!}{\left(\epsilon_{efgh}G^{efgh}\right)^2}\epsilon_{mnpq}\epsilon_{rstu}G^{mnr(a}G^{b|ps|c}G^{d)qtu}k_a k_b k_c k_d \\ &=: P^{abcd}k_a k_b k_c k_d \end{aligned} \tag{93}$$

To substitute this into the universal point particle action, I need the components of P in the foliation frame. These have been determined by H. M. Rieser and others, whom I trust to provide correct expressions.

If, to match the accuracy of the Hamiltonian 88, one discards all but the linear oder in the gravitational fields, one obtains

$$P\left(\kappa,\kappa,\kappa,\kappa\right) = 1 - 4A \tag{94}$$

$$P\left(\tilde{\epsilon}^{\alpha},\kappa,\kappa,\kappa\right) = -N^{\alpha} \tag{95}$$

$$P\left(\tilde{\epsilon}^{\alpha},\tilde{\epsilon}^{\beta},\kappa,\kappa\right) = -\frac{1}{3}\gamma^{\alpha\beta} + \frac{2}{3}A\gamma^{\alpha\beta} \tag{96}$$

$$+\frac{1}{6}\left[-\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + \gamma^{\alpha\beta}\gamma_{\mu\nu}\right)\bar{\varphi}^{\mu\nu} + \left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \gamma^{\alpha\beta}\gamma_{\mu\nu}\right)\bar{\bar{\varphi}}^{\mu\nu}\right] \tag{97}$$

$$P\left(\tilde{\epsilon}^{\alpha},\tilde{\epsilon}^{\beta},\tilde{\epsilon}^{\gamma},\kappa\right) = N^{(\alpha}\gamma^{\beta\gamma)} \tag{98}$$

$$P\left(\tilde{\epsilon}^{\alpha},\tilde{\epsilon}^{\beta},\tilde{\epsilon}^{\gamma},\tilde{\epsilon}^{\delta}\right) = \gamma^{(\alpha\beta}\gamma^{\gamma\delta)} + \gamma^{(\alpha\beta}\left(\delta^{\gamma}_{\mu}\delta^{\delta)}_{\nu} + \gamma^{\gamma\delta)}\gamma_{\mu\nu}\right)\bar{\varphi}^{\mu\nu} \tag{99}$$

$$-\gamma^{(\alpha\beta}\left(\delta^{\gamma}_{\mu}\delta^{\delta)}_{\nu} - \gamma^{\gamma\delta)}\gamma_{\mu\nu}\right)\bar{\bar{\varphi}}^{\mu\nu} \tag{100}$$

I read off the zeroth order

$$P^{abcd} = \eta^{(ab}\eta^{cd)} + \Sigma^{abcd} \tag{101}$$

where $\Sigma \sim \mathcal{O}\left(\varphi\right)$.

Using this parametrisation and starting from the universal point particle action, F. P. Schuller obtained

$$S = m\int \mathrm{d}\tau\left[\left(\eta_{ab}\gamma'^{a}\gamma'^{b}\right)^{\frac{1}{2}} - \frac{1}{4}\frac{\Sigma_{abcd}\gamma'^{a}\gamma'^{b}\gamma'^{c}\gamma'^{d}}{\left(\eta_{ab}\gamma'^{a}\gamma'^{b}\right)^{\frac{3}{2}}} + \mathcal{O}\left(\varphi^{2}\right) + e\gamma'^{a}A_{a}\right] \tag{102}$$

where $\gamma' := \frac{\mathrm{d}\gamma(\tau)}{\mathrm{d}\tau}$. Just as in section 3.2, I proceed by parametrising γ with the foliation parameter t and expressing the action in the foliation frame. I yield

$$S = m\int \mathrm{d}t[\left(1-v^{2}\right)^{\frac{1}{2}} - \frac{1}{4}\frac{\sum_{n=0}^{4}\binom{4}{n}\Sigma_{\alpha_{1}\ldots\alpha_{n}0\ldots0}v^{\alpha_{1}}\ldots v^{\alpha_{n}}}{\left(1-v^{2}\right)^{\frac{3}{2}}} \tag{103}$$

$$+ e\left(\phi + A\phi + A_{\alpha}\left(N^{\alpha} + v^{\alpha}\right)\right)] + \mathcal{O}\left(\varphi^{2}\right)$$

where again $v = \frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} =: \dot{\lambda}(t)$ and $v^{2} := \gamma_{\alpha\beta}v^{\alpha}v^{\beta}$.

I proceed just as in section 3.2, and find that I obtain the same results concerning the momenta and velocities to order $\mathcal{O}\left(1\right)$ as given in eq. 40 -

42. Again, this knowledge suffices to derive the Hamiltonian to order $\mathcal{O}(\varphi)$ through the Legendre transform.

After inserting the expressions 94 - 100, the approximate Hamiltonian for a charged point mass in an electromagnetic field that obeys the causal structure of birefringent electrodynamics reads

$$\begin{aligned} H = &- (1+A)\,(E_k + e\phi) - N^{\alpha} p_{\alpha} \\ &- \frac{1}{2E_k}\left[\frac{1}{2}\left(\bar{\varphi}^{\alpha\beta} - \bar{\bar{\varphi}}^{\alpha\beta}\right) k_{\alpha} k_{\beta} + \frac{1}{2}\left(\bar{\varphi}^{\alpha\beta} + \bar{\bar{\varphi}}^{\alpha\beta}\right)\gamma_{\alpha\beta}\gamma^{\mu\nu} k_{\mu} k_{\nu}\right] \end{aligned} \tag{104}$$

with the same notational conventions as in eq. 43

4.3 Gravitational Waves

Collecting the results from sec. 4.1 and 4.2, I am now in possession of complete matter Hamiltonian. This is necessary if one wishes to calculate the gravitational waves generated by a matter system, because derivatives of the matter Hamiltonian appear as imhomogenities or source terms in the equations of motion of the gravitational fields. To determine the gravitational radiation emitted from a specific system, these source terms must be evaluated on a specific solution of the matter equations of motion - I will evaluate them on the solution presented in sec. 3.3.

Now, firstly, I present the linearised equations of motion for the gravitational fields that form the causal structure of birefringent electrodynamics, as obtained by the members of the constructive gravity group. These equations are published in an article called "Gravitational closure of weakly birefringent electrodynamics". In this work, I restrict myself to the special case called the $\xi = 0$ case. In this special case, some parts of the full theory are switched of (by setting the respective constants of nature to zero).

The linearised equations of motion approximate the full nonlinear ones, solutions of the former accurately resemble solutions of the later as long as the field strengths are very small, i.e. as $\varphi \ll 1$.

The fields that obey the equations of motion stem from a decomposition of the fields $\bar{\varphi}$, $\bar{\bar{\varphi}}$ and $\bar{\bar{\bar{\varphi}}}$ into scalar fields, divergence free vector fields and traceless, divergence free symmetric tensor fields :

$$\bar{\varphi}_{\alpha\beta} =: \tilde{F}\gamma_{\alpha\beta} + \Delta_{\alpha\beta} F + 2\partial_{(\alpha} F_{\beta)} + F_{\alpha\beta} \tag{105}$$

$$\bar{\bar{\varphi}}_{\alpha\beta} =: \tilde{E}\gamma_{\alpha\beta} + \Delta_{\alpha\beta} E + 2\partial_{(\alpha} E_{\beta)} + E_{\alpha\beta} \tag{106}$$

$$\bar{\bar{\bar{\varphi}}}_{\alpha\beta} =: \Delta_{\alpha\beta} C + 2\partial_{(\alpha} C_{\beta)} + C_{\alpha\beta} \tag{107}$$

I shall now give my version of the equations of motion for these fields. It differs from the version presented in the above quoted article in six aspects:

1. As already stated, I focus on the $\xi = 0$ case. This becomes manifest in some of the κ-constants being set to zero.

2. For some reason, the authors of the above quoted article chose the gauge-fixing $E^\alpha = F = B = 0$. I personally find that gauge fixing rather impractical, and prefer to use $B^\alpha = B = F - E = 0$ instead.

3. It appears to me that for both practical and conceptual reasons, instead of working with the fields F_A, E_A and C_A (where the A stands for any index-type, I am talking about scalars, vectors and tensors here) one should rather work with the fields $V_A := F_A - E_A$, $U_A = E_A + F_A$ and $I_A = 2C_A$. The practical reasons for this redefinition of variables reveal themselves to everyone who can be bothered to follow my calculations, whereas the conceptual reason is that V_A captures exactly the metric degrees of freedom, and is therefore the only field which would survive the metric limit described at the end of sec. 4.1.

4. Since it reduces redundancy in notation, I partially reverse the decomposition presented in eq. 105 - 107, and work with the trace free tensor fields

$$\bar{U}_{\alpha\beta} = \Delta_{\alpha\beta} U + 2\partial_{(\alpha} U_{\beta)} + U_{\alpha\beta} \tag{108}$$

$$\bar{I}_{\alpha\beta} = \Delta_{\alpha\beta} I + 2\partial_{(\alpha} I_{\beta)} + I_{\alpha\beta} \tag{109}$$

   instead of with the parts from which they are assembled.

5. In his analysis of the linearised equations of motion in vacua and their solutions, N. Alex found that certain combinations of constants must vanish in order to make flat Minkowski spacetime a stable solution of the theory. Since this criterion is non-negotiable, I impose these stability conditions on the respective constants.

6. I have combined the equation in a way as to clearly separate truly dynamical equations, i.e. wave equations, from the constraint equations. The way the equations are written down allows to systematically solve them in few steps.

My formulation of the equations of motion for the gravitational fields of birefringent electrodynamics:

Wave equations

$$-m_4\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\varphi}^{\alpha\beta}}+\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\varphi}}^{\alpha\beta}}\right]^{\mathrm{TF}}+m_2\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\bar{\varphi}}}^{\alpha\beta}}\right]^{\mathrm{TF}}=\left(m_3m_4-m_1m_2\right)\Box\bar{I}_{\alpha\beta} \tag{110}$$

$$+\left(m_4^2-m_2^2\right)\bar{I}_{\alpha\beta} \tag{111}$$

$$-m_4\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\bar{\varphi}}}^{\alpha\beta}}\right]^{\mathrm{TF}}+m_2\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\varphi}^{\alpha\beta}}+\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\varphi}}^{\alpha\beta}}\right]^{\mathrm{TF}}=\left(m_3m_4-m_1m_2\right)\Box\bar{U}_{\alpha\beta} \tag{112}$$

$$+\left(m_4^2-m_2^2\right)\bar{U}_{\alpha\beta} \tag{113}$$

$$-\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\varphi}^{\alpha\beta}}-\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\varphi}}^{\alpha\beta}}\right]^{\mathrm{TT}}=\left(t_1-t_3\right)\Box\bar{V}_{\alpha\beta} \tag{114}$$

$$\left(3\left(s_1+s_2\right)+2s_8\right)\gamma^{\alpha\beta}\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\varphi}^{\alpha\beta}}-\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\varphi}}^{\alpha\beta}}\right] \tag{115}$$

$$-3\left(s_1+s_2\right)\gamma^{\alpha\beta}\left[\frac{\delta H_{\mathrm{M}}}{\delta\bar{\varphi}^{\alpha\beta}}+\frac{\delta H_{\mathrm{M}}}{\delta\bar{\bar{\varphi}}^{\alpha\beta}}\right]+\left(3\left(s_1+s_2\right)-s_8\right)\frac{\delta H_{\mathrm{M}}}{\delta N} \tag{116}$$

$$= \tag{117}$$

$$-\Big(108\left(s_1+s_2\right)^2+72\left(s_1+s_2\right)s_8-18\left(s_1+s_2\right)\right)s_28+12s_8^2\Box\tilde{U}$$

5 Conventions

- latin indices are spacetime indices
- greek indices are screen manifold indices

$$\epsilon^{0123}=-1 \tag{118}$$