

Complete Matter Hamiltonian:

$$P.P. + \epsilon H + L_{\text{back}}.$$

- From action to Hamiltonian
- Linearized to $O(q)$
- Ensures of motion
- Source terms
- Conservation of Energy and Momentum

From Matter action to matter Hamiltonian

$$S_m = S_{\text{p.p.}} + S_{\text{EM}} + S_I$$

point particles Electromagnetism Interaction

$$= \sum_i \int m_i \sqrt{g_{ab} \dot{\gamma}_i^a \dot{\gamma}_i^b} dx + e_i A_a \dot{\gamma}_i^a dx$$

$$+ \frac{1}{16\pi} \int d^4x g^{ax} g^{bd} F_{ab} F_{cd} - \frac{1}{2} \det g_{ab}$$

Linearise action! We'll need ... the metric:

$$g_{..} = \left[\begin{array}{c|c} N^2 + h_{\alpha\beta} N^\alpha N^\beta & h_{\alpha\beta} N^\beta \\ \hline h_{\alpha\beta} N^\alpha & h_{\alpha\beta} \end{array} \right] ..$$

from "Conservation of Energy & Momentum"

$$g^{-1}.. = \left[\begin{array}{c|c} \frac{1}{N^2} & -\frac{N^\beta}{N^2} \\ \hline -\frac{N^\alpha}{N^2} & R^{-1}\alpha\beta + \frac{N^\alpha N^\beta}{N^2} \end{array} \right] ..$$

In basis $\{\dot{X}, e_\alpha\}$.

$$\dot{X} = NT + N^\alpha e_\alpha$$

Basis dual to $\{\dot{X}, \tilde{e}_\alpha\}$

$$\tilde{e}_\alpha$$

The parametrised curves

$$\dot{\gamma}_i = \dot{X} + \omega^\alpha \tilde{e}_\alpha$$

given by

$$\kappa = \frac{1}{N} n$$

$$\tilde{e}^\alpha = e^\alpha - \frac{1}{N} N^\alpha n$$

First, linearise point-Matter action.

$$\sum_i \int m_i \sqrt{g_{ab} \dot{\gamma}_i^a \dot{\gamma}_i^b} + e_i A_a \dot{\gamma}_i^a dx$$

$$= \sum_i \int dt \sqrt{g_{++} + 2g_{+\alpha} \omega^\alpha + g_{\alpha\beta} \omega^\alpha \omega^\beta} m_i + e_i (N\phi + N^\alpha A_\alpha + \omega^\alpha A_\alpha)$$

$$\text{Since } A_\alpha T^\alpha = \phi \quad A_\alpha e_\alpha^\alpha = A_\alpha,$$

$$A = \phi n + \tilde{e}^\alpha A_\alpha = N\kappa\phi + (\tilde{e}^\alpha + N^\alpha\kappa) A_\alpha = \kappa(N\phi + N^\alpha A_\alpha) + \tilde{e}^\alpha A_\alpha$$

Now, to lowest order, get

$$\begin{aligned} N &= 1 + A \\ N^2 &= 0 + B^2 \end{aligned}$$

$$L^{-1} \alpha \beta = -\gamma^{\alpha \beta} + \varphi^{\alpha \beta}$$

Therefore

$$L \alpha \beta = -\gamma^{\alpha \beta} - \varphi^{\alpha \beta}$$

$$\varphi_{\alpha \beta} := \varphi^{\mu \nu} \gamma_{\mu \alpha} \gamma_{\nu \beta}$$

Insert to get:

$$\begin{aligned} S_{pp, I} &= \sum_i \int dt \sqrt{1 + 2A - 2\gamma^{\alpha \beta} B^{\beta} v^{\alpha} - (\gamma^{\alpha \beta} + \varphi^{\alpha \beta}) v^{\alpha} v^{\beta}} m_i \\ &\quad + e_i ((1+A)\phi + B^{\alpha} A_{\alpha} + v^{\alpha} A_{\alpha}) \\ &= \sum_i \int dt \sqrt{1 - \gamma^{\alpha \beta} v^{\alpha} v^{\beta} + 2A - 2\gamma^{\alpha \beta} B^{\beta} v^{\alpha} - \varphi^{\alpha \beta} v^{\alpha} v^{\beta}} m_i \\ &\quad + e_i (\phi + v^{\alpha} A_{\alpha} + A\phi + B^{\alpha} A_{\alpha}) \\ &= \sum_i \int dt m_i \sqrt{1 - v^2} \left(1 + \frac{A - \gamma^{\alpha \beta} B^{\beta} v^{\alpha} - \frac{1}{2} \varphi^{\alpha \beta} v^{\alpha} v^{\beta}}{1 - v^2} \right) \\ &\quad + e_i (\phi + v^{\alpha} A_{\alpha} + A\phi + B^{\alpha} A_{\alpha}) \\ &= \sum_i \int dt m_i \left[\sqrt{1 - v^2} + \frac{1}{\sqrt{1 - v^2}} \left(A - \gamma^{\alpha \beta} B^{\beta} v^{\alpha} - \frac{1}{2} \varphi^{\alpha \beta} v^{\alpha} v^{\beta} \right) \right] \\ &\quad + e_i [\phi + v^{\alpha} A_{\alpha} + A\phi + B^{\alpha} A_{\alpha}] // \end{aligned}$$

Lagrange \rightarrow Hamilton (do for single particle, then add up!)

$$\text{Solt } m \left[\frac{1}{1-v^2} + \frac{1}{\sqrt{1-v^2}} \left(A - \gamma_{\alpha\beta} N^\beta v^\alpha - \frac{1}{2} \varphi_{\alpha\beta} v^\alpha v^\beta \right) \right] \\ + e \left[\phi + v^\alpha A_\alpha + A \phi + N^\alpha A_\alpha \right] + O(\varphi^2)$$

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial v_\alpha} = m \left[- \frac{v_\alpha}{\sqrt{1-v^2}} \right]$$

$$+ (-) \frac{v_\alpha}{\sqrt{1-v^2}} \left(A - \gamma_{\alpha\beta} N^\beta v^\alpha - \frac{1}{2} \varphi_{\alpha\beta} v^\alpha v^\beta \right)$$

$$+ \frac{1}{\sqrt{1-v^2}} \left(- N_\alpha - \varphi_{\alpha\beta} v^\beta \right) + e A_\alpha$$

Assume $v_\alpha = v_\alpha^0 + v_\alpha^1$, solve for v_α^0 .

$$k_\alpha := p_\alpha - e A_\alpha = -m \frac{v_\alpha^0}{\sqrt{1-v_0^2}}$$

$$-v_\alpha^0 = -\frac{k_\alpha}{\sqrt{m^2 + k^2}} = -\frac{k_\alpha}{E_K}$$

$$\boxed{1 - (v_\alpha^0 + v_\alpha^1)^2 = \sqrt{1 - v_0^{02} + 2v_\alpha^1 v_\alpha^{00}}} = \sqrt{1 - v_0^{02}} \sqrt{1 - \frac{2v_\alpha^1 v_\alpha^{00}}{1 - v_0^{02}}} \\ = \sqrt{1 - v_0^{02}} \left(1 - \frac{v_\alpha^1 v_\alpha^{00}}{1 - v_0^{02}} \right) = \frac{m}{E_K} \left(1 + \frac{E_K}{m^2} K_\alpha^0 v_\alpha^1 \right)$$

$$\boxed{1 - v_0^{02} = 1 - \frac{k^2}{m^2 + k^2} = \frac{m^2 + k^2 - k^2}{m^2 + k^2} = \frac{m^2}{E_K^2}}$$

$$= K^\alpha \left(1 + \frac{K^2}{m^2 - E_K^2} - \frac{E_K^2 k^2}{m^2(m^2 - E_K^2)} (A + \dots) - \frac{E_K^2}{m^2} (A + \dots) \right)$$

$$+ \frac{E_K}{m^2 - E_K^2} \left(-N \cdot K + \frac{\varphi(k \cdot K)}{E_K} \right)$$

ken

Bock mehr!

gehts auch einfacher?

Inserat in H!

$$H = p_\alpha v^\alpha - \mathcal{L}$$

$$= p_\alpha (v_0^\alpha + v_1^\alpha) - m \left[\frac{m}{E_K} \left(1 + \frac{E_K}{m^2} K^\alpha v_2^\alpha \right) \right]$$

$$+ \frac{E_K}{m} \left(A - \gamma_{\alpha\beta} N^\beta v_0^\alpha - \frac{1}{2} \varphi_{\alpha\beta} v_0^\alpha v_0^\beta \right] - e\phi$$

$$+ v_0^\alpha A_\alpha + v_1^\alpha A_\alpha + A\phi + N^\alpha A_\alpha \Big] + O(\varphi^2)$$

$$= -p_\alpha \frac{K^\alpha}{E_K} + v_0^\alpha \underbrace{(p_\alpha - e A_\alpha)}_{k^\alpha} - \frac{m^2}{E_K} - K^\alpha \cancel{v_2^\alpha}$$

$$- E_K \left(A - \gamma_{\alpha\beta} N^\beta \left(-\frac{K^\alpha}{E_K} \right) - \frac{1}{2} \varphi_{\alpha\beta} \frac{K^\alpha K^\beta}{E_K^2} \right)$$

$$- e\phi + e \frac{K^\alpha}{E_K} A_\alpha - e A\phi - e N^\alpha A_\alpha + O(\varphi^2)$$

$$\begin{aligned} &= -\frac{k^2}{E_K} - \frac{m^2}{E_K} - E_K \left(A + \gamma_{\alpha\beta} N^\beta K^\alpha \frac{1}{E_K} - \frac{1}{2} \varphi_{\alpha\beta} \frac{K^\alpha K^\beta}{E_K^2} \right) \\ &\quad - e\phi - e A\phi - e N^\alpha A_\alpha \end{aligned}$$

$$= -E_K - E_{KA} - N^\alpha k_\alpha - N^\alpha e A_\alpha - e A \phi - e \phi + \frac{1}{2e_K} \varphi_{\alpha\beta} k^\alpha k^\beta$$

$$H = -(\epsilon_K + e\phi) - A(\epsilon_K + e\phi) - N^\alpha p_\alpha + \frac{1}{2e_K} \varphi_{\alpha\beta} k^\alpha k^\beta$$

$$\epsilon_K = \sqrt{m^2 + (p - eA)^2}$$

$$k^\alpha = p^\alpha - e A^\alpha$$

For multiple particles then:

$$H = \sum_i \left[-(1+A) (\epsilon_{K^i} + e\phi(x_i)) - N^\alpha p_\alpha^i + \frac{1}{2e_{K^i}} \varphi_{\alpha\beta}(x_i) k_i^\alpha k_i^\beta \right]$$

Next: Split Maxwell action!

3+1 Split of Maxwell Action

$$S = \frac{1}{16\pi} \int F_{ab} F^{cd} g^{ac} g^{bd} \sqrt{|\det g_{..}|} d^4x \quad \left(\text{dow landau} \right. \\ \left. - \text{Lichitz} \right)$$

$F_{ab} = \partial_a A_b - \partial_b A_a$

For 3+1 split: — Cast metric in ADM form.

— Write F_{ab} in 3-quantities

$$T^a A_a =: \phi \quad A_a e_\alpha^a =: A_\alpha$$

$$F_H = 0$$

$$\begin{aligned} F_{+\beta} &= \dot{x}^\alpha e_\beta^b F_{ab} = \dot{x}^\alpha e_\beta^b (\partial_a A_b - \partial_b A_a) \\ &= \dot{x}^\alpha \partial_a (A_\beta) - (\dot{x}^\alpha \partial_a e_\beta^b) A_b - e_\beta^b \partial_b (\dot{x}^\alpha A_a) + e_\beta^b (\partial_b \dot{x}^\alpha) A_a \\ &= \dot{x}^\alpha \partial_a (A_\beta) - \partial_\beta ((N T^\alpha + N^\alpha e_\alpha^a) A_a) - (\cancel{\mathcal{L}_{\dot{x}} e_\beta})^b A_b \\ &= \dot{x}^\alpha \partial_a (A_\beta) - \partial_\beta (N \phi + N^\alpha A_\alpha) \end{aligned}$$

coordinate basis

What else?

$$\begin{aligned} \dot{A}_\alpha &= e_\alpha^a (\mathcal{L}_{\dot{x}} A)_a = e_\alpha^a (\dot{x}^b \partial_b A_a + \partial_a \dot{x}^b A_b) \\ &= \dot{x}^b \partial_b (A_\alpha) - \dot{x}^b (\partial_b e_\alpha^a) A_a + \partial_\alpha \dot{x}^b A_b \end{aligned}$$

Can also rather look @

$$F_{\alpha\beta} = T^\alpha e_\beta^b F_{ab} = \dot{x}^b \partial_b (A_\alpha) - (\cancel{\mathcal{L}_{\dot{x}} e_\alpha})^b A_b$$

What's the idea? Eliminate $\dot{x}^\alpha \partial_a$ for $\mathcal{L}_{\dot{x}}$.

Again :

$$\begin{aligned} F_{+\beta} &= \dot{x}^a e_\beta^b F_{ab} \\ &= \dot{x}^a e_\beta^b (\partial_a A_b - \partial_b A_a) \\ &= (\dot{x}^a \partial_a A_b + \partial_b \dot{x}^a A_a - \partial_b \dot{x}^a A_a) e_\beta^b - \dot{x}^a \partial_\beta A_a \\ &= (\mathcal{L}_{\dot{x}} A)_\beta e_\beta^b - (\partial_\beta \dot{x}^a) A_a + \dot{x}^a (\partial_\beta A_a) \\ &= \dot{A}_\beta - \partial_\beta (N^a A_a + N^\alpha e_\alpha^a A_a) \end{aligned}$$

$F_{+\beta} = \dot{A}_\beta - \partial_\beta (N^a + N^\alpha A_\alpha)$

$$\begin{aligned} F_{\alpha\beta} &= e_\alpha^a e_\beta^b (\partial_a A_b - \partial_b A_a) \\ &= (\partial_\alpha A_\beta) - (\partial_\alpha e_\beta^b) A_b - (\partial_\beta A_\alpha) + (\partial_\beta e_\alpha^a) A_a \\ &= {}^{(3)}F_{\alpha\beta} - (\mathcal{L}_{e_\alpha} e_\beta^b) A_b \quad \text{coordinate basis} \end{aligned}$$

$F_{\alpha\beta} = {}^{(3)}F_{\alpha\beta}$

$$\begin{aligned} F_{ab} F_{cd} g^{ac} g^{bd} &= 2 g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} + 4 g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} \\ &\quad + g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} \end{aligned}$$

Make life easier:

$$F_{\alpha\beta} = T^a e_\beta^b F_{ab} = \frac{\dot{x}^a - N^\alpha e_\alpha^a}{N} e_\beta^b F_{ab}$$

$$= \frac{1}{N} \dot{A}_\alpha^a e_\beta^b F_{ab} - \frac{N^\alpha}{N} e_\alpha^a e_\beta^b F_{ab}$$

$$F_{\alpha\beta} = \frac{1}{N} F_{+\beta} - \frac{N^\alpha}{N} F_{\alpha\beta} = \frac{1}{N} (\dot{A}_\beta - \partial_\beta (N\phi + N^\alpha A_\alpha)) - \frac{N^\alpha}{N} F_{\alpha\beta}$$

Then: $F_{ab} F_{cd} g^{ac} g^{bd}$

$$= 2 g^{00} g^{\alpha\beta} F_{\alpha 0} F_{\beta 0} + g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta}$$

$$= 2 \eta^{\alpha\beta} F_{\alpha 0} F_{\beta 0} + \eta^{\alpha\beta} \eta^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta}$$

Also

note

$$d^4x \sqrt{|\det g_{..}|} = N \sqrt{|\det \eta_{..}|} dt d^3x$$

(Klein, Stone 3.8)

$$S = \int dt \int d^3x \underbrace{\frac{N \sqrt{|\det \eta_{..}|}}{16\pi}}_{\propto} \left[2 \eta^{\alpha\beta} F_{\alpha 0} F_{\beta 0} + \eta^{\alpha\beta} \eta^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} \right]$$

Then:

$$\Pi^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{A}_\alpha}$$

$$= \frac{\partial \mathcal{L}}{\partial F_{\alpha 0}} \frac{\partial F_{\alpha 0}}{\partial \dot{A}_\alpha} = \frac{\partial \mathcal{L}}{\partial F_{\alpha 0}} \frac{1}{N} S_p^\alpha = \frac{\partial \mathcal{L}}{\partial F_{\alpha 0}} \frac{1}{N}$$

$$= \frac{1}{N} \underbrace{\frac{2N \sqrt{|\det \eta_{..}|}}{16\pi}}_{\propto} 2 \eta^{\alpha\beta} F_{\beta 0} = \frac{1}{4\pi} \sqrt{|\det \eta_{..}|} \eta^{\alpha\beta} F_{\beta 0}$$

$$\Pi^\alpha = \frac{1}{4\pi} \sqrt{|\det \eta_{..}|} \eta^{\alpha\beta} F_{\beta 0}$$

$$\Leftrightarrow \boxed{\frac{4\pi}{|\det \eta_{..}|} \Pi^\beta \eta_{\beta\alpha} = F_{\alpha 0}}$$

Next, we need Hamiltonian:

$$H = \Pi^\alpha A_\alpha - \mathcal{L}$$

Note that $N F_{\alpha\beta} = A_\alpha - \partial_\alpha (N\phi + N^h A_h) - N^{(3)} F_{\mu\alpha}$

$$\Rightarrow A_\alpha = N F_{\alpha\beta} + \partial_\alpha (N\phi + N^h A_h) + N^{(3)} F_{\mu\alpha}$$

get

$$\begin{aligned}
 H &= \Pi^\alpha \left(N \frac{4\pi}{|\det g_{\alpha\beta}|} h_{\alpha\beta} \Pi^\beta + \partial_\alpha (N\phi + N^h A_h) + N^{(3)} F_{\mu\alpha} \right) \\
 &\quad - N \frac{\sqrt{|\det g_{\alpha\beta}|}}{16\pi} \left[2g^{\mu\nu} \left(\frac{4\pi}{|\det g_{\alpha\beta}|} \right)^2 \Pi^\alpha \partial_\mu \Pi^\beta \partial_\nu \right. \\
 &\quad \left. + g^{\alpha\beta} h^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} \right] \\
 &= N \left[\frac{4\pi}{|\det g_{\alpha\beta}|} h_{\alpha\beta} \Pi^\alpha \Pi^\beta - \frac{2\pi}{\sqrt{|\det g_{\alpha\beta}|}} g_{\alpha\beta} \Pi^\alpha \Pi^\beta \right. \\
 &\quad \left. - \frac{\sqrt{|\det g_{\alpha\beta}|}}{16\pi} g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma}^{(3)} F_{\beta\delta} \right] + \Pi^\alpha \partial_\alpha (N\phi + N^h A_h) \\
 &\quad + N^{(3)} F_{\mu\alpha} \Pi^\alpha \\
 &= N \left[\frac{2\pi}{\sqrt{|\det g_{\alpha\beta}|}} g_{\alpha\beta} \Pi^\alpha \Pi^\beta - \frac{\sqrt{|\det g_{\alpha\beta}|}}{16\pi} g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma}^{(3)} F_{\beta\delta} \right] \\
 &\quad + \Pi^\alpha \partial_\alpha (N\phi + N^h A_h) + N^{(3)} F_{\mu\alpha} \Pi^\alpha
 \end{aligned}$$

Now, assuming that partial integration is allowed since 70

$$S = \int \pi^\alpha \dot{A}_\alpha - H d^3x dt$$

get

$$H = N \left[\frac{2\pi}{\sqrt{|\det h_{..}|}} h_{\alpha\beta} \pi^\alpha \pi^\beta - \frac{\sqrt{|\det h_{..}|}}{16\pi} \gamma^{\alpha\beta} h^{\gamma\delta} \overset{(2)}{F}_{\alpha\gamma} \overset{(3)}{F}_{\beta\delta} \right. \\ \left. - \phi \partial_\alpha \pi^\alpha \right] + N^n \left[\pi^\alpha \overset{(3)}{F}_{\mu\alpha} - A_\mu \partial_\alpha \pi^\alpha \right]$$

which coincides with Stone & Kuchar (86) up to minus signs (from signature) and factors of 4π (from normalisation).

Next step is: linearise!

$$N = 1 + A, \quad h_{\alpha\beta} = -\gamma_{\alpha\beta} - \varphi_{\alpha\beta}$$

$$\det h_{..} = -\det(\gamma_{..} + \varphi_{..}) = - (1 + \gamma^{\alpha\beta} \varphi_{\alpha\beta} + \mathcal{O}(\varphi^2))$$

$$\sqrt{|\det h_{..}|} = \left(1 + \gamma^{\alpha\beta} \varphi_{\alpha\beta} + \mathcal{O}(\varphi^2) \right)^{\frac{1}{2}} = 1 + \frac{1}{2} \gamma^{\alpha\beta} \varphi_{\alpha\beta}$$

$$H = (1 + A) \left[2\pi \left(1 - \frac{1}{2} \gamma^{\alpha\beta} \varphi_{\alpha\beta} \right) (-\gamma_{\alpha\beta} - \varphi_{\alpha\beta}) \pi^\alpha \pi^\beta \right.$$

$$- \frac{1}{16\pi} \left(1 + \frac{1}{2} \gamma^{\alpha\beta} \varphi_{\alpha\beta} \right) (-\gamma^{\alpha\beta} + \varphi^{\alpha\beta}) (-\gamma^{\gamma\delta} + \varphi^{\gamma\delta}) F_{\alpha\gamma} F_{\beta\delta}$$

$$\left. - \phi \partial_\alpha \pi^\alpha \right] + N^n \left[\pi^\alpha \overset{(3)}{F}_{\mu\alpha} - A_\mu \partial_\alpha \pi^\alpha \right]$$

$$= (1+A) \left[2\pi (-\gamma_{\alpha\beta} - \gamma_{\alpha\beta} + \frac{1}{2} \gamma^{\mu\nu} \gamma_{\mu\nu} \gamma^{\alpha\beta}) \Pi^\alpha \Pi^\beta \right. \\ \left. - \frac{1}{16\pi} (\gamma^{\alpha\beta} - 2\gamma^{\alpha\beta} + \frac{1}{2} \gamma^{\mu\nu} \gamma_{\mu\nu} \gamma^{\alpha\beta}) \gamma^{\sigma\delta} F_{\alpha\gamma} F_{\beta\delta} \right. \\ \left. - \phi \partial_\alpha \Pi^\alpha \right] + N^\mu \left[\Pi^\alpha F_{\mu\alpha} - A_\mu \partial_\alpha \Pi^\alpha \right]$$

$$= (1+A) \left[-2\pi \gamma_{\alpha\beta} \Pi^\alpha \Pi^\beta - \frac{1}{16\pi} \gamma^{\alpha\beta} \gamma^{\sigma\delta} F_{\alpha\gamma} F_{\beta\delta} - \phi \partial_\alpha \Pi^\alpha \right] \\ + \left[2\pi (-\delta_\alpha^\mu \delta_\beta^\nu + \frac{1}{2} \gamma^{\mu\nu} \gamma_{\alpha\beta}) \varphi_{\mu\nu} \Pi^\alpha \Pi^\beta \right. \\ \left. - \frac{1}{16\pi} (-2\gamma^{\mu\alpha} \gamma^{\nu\beta} + \frac{1}{2} \gamma^{\mu\nu} \gamma^{\alpha\beta}) \varphi_{\mu\nu} \gamma^{\sigma\delta} F_{\alpha\gamma} F_{\beta\delta} \right] \\ + N^\mu \underbrace{\left[\Pi^\alpha F_{\mu\alpha} - A_\mu \partial_\alpha \Pi^\alpha \right]}_{D_\mu}$$

finally write $F_{\mu\nu} = \epsilon_{\mu\nu\sigma} H^\sigma$

$$\text{get } \gamma^{\sigma\delta} F_{\alpha\gamma} F_{\beta\delta} = \gamma^{\sigma\delta} \epsilon_{\alpha\gamma\sigma} \epsilon_{\beta\delta\tau} H^\sigma H^\tau \\ = (\gamma_{\alpha\beta} \gamma_{\sigma\tau} - \gamma_{\alpha\tau} \gamma_{\sigma\beta}) H^\sigma H^\tau$$

$$\gamma^{\alpha\beta} \gamma^{\sigma\delta} F_{\alpha\gamma} F_{\beta\delta} = 2 \gamma_{\sigma\tau} H^\sigma H^\tau$$

$$= (1+A) \left[-2\pi \gamma_{\alpha\beta} \Pi^\alpha \Pi^\beta - \frac{1}{8\pi} \gamma_{\sigma\tau} H^\sigma H^\tau - \phi \partial_\alpha \Pi^\alpha \right] \\ + \left[-2\pi (\delta_\alpha^\mu \delta_\beta^\nu - \frac{1}{2} \gamma^{\mu\nu} \gamma_{\alpha\beta}) \varphi_{\mu\nu} \Pi^\alpha \Pi^\beta \right. \\ \left. - \frac{1}{16\pi} (-2\gamma^{\mu\alpha} \gamma^{\nu\beta} + \frac{1}{2} \gamma^{\mu\nu} \gamma^{\alpha\beta}) (\gamma_{\alpha\beta} \gamma_{\sigma\tau} - \gamma_{\alpha\tau} \gamma_{\sigma\beta}) H^\sigma H^\tau \varphi_{\mu\nu} \right] + N^\mu D_\mu$$

$$\begin{aligned}
H = & - (1+A) \left[-2\pi \gamma_{\alpha\beta} \Pi^\alpha \Pi^\beta + \frac{1}{8\pi} \gamma_{\tau\zeta} H^\tau H^\zeta + \phi \partial_\alpha \Pi^\alpha \right] \\
& - \delta_{\mu\nu} \left[-2\pi (\delta_\alpha^\mu \delta_\beta^\nu - \frac{1}{2} \delta^{\mu\nu} \gamma_{\alpha\beta}) \Pi^\alpha \Pi^\beta \right. \\
& \quad \left. + \frac{1}{8\pi} (\delta_\tau^\mu \delta_\sigma^\nu - \frac{1}{2} \delta^{\mu\nu} \gamma_{\tau\sigma}) H^\tau H^\zeta \right] \\
& + N^\mu \left[\epsilon_{\mu\nu\beta} \Pi^\alpha H^\beta - A_\mu \partial_\alpha \Pi^\alpha \right]
\end{aligned}$$

The linearised Hamiltonian of Maxwell theory !

Can now write down complete matter Hamiltonian!

The Hamiltonian for multiple point particles in the electromagnetic field is

$$\begin{aligned}
 H = & \sum_i \left[- (1 + A(x_i)) (E_k^i + e\phi(x_i)) - N^\alpha(x_i) p_\alpha^i + \frac{1}{2e_k^i} \varphi_{\alpha\beta}(x_i) k_i^\alpha k_i^\beta \right] \\
 = & (1 + A) \left[2\pi \gamma_{\alpha\beta} \Pi^\alpha \Pi^\beta + \frac{1}{8\pi} \gamma_{\alpha\beta} H^\alpha H^\beta + \phi \partial_\alpha \Pi^\alpha \right] \\
 = & \varphi_{\mu\nu} \left[2\pi (\delta_\alpha^\mu \delta_\beta^\nu - \frac{1}{2} \gamma^{\mu\nu} \gamma_{\alpha\beta}) \Pi^\alpha \Pi^\beta + \frac{1}{8\pi} (\delta_\alpha^\mu \delta_\beta^\nu - \frac{1}{2} \gamma^{\mu\nu} \gamma_{\alpha\beta}) H^\alpha H^\beta \right] \\
 = & N^\alpha \left[E_{\alpha\mu} H^\mu \Pi^\nu + A_\alpha \partial_\mu \Pi^\mu \right]
 \end{aligned}$$

Now various things are of interest:

1. The $\mathcal{O}(0)$ Hamiltonian provides the eq. of motion.
Do they coincide with the ones solved?

For simplicity: one particle:

$$\begin{aligned}
 H^0 = & - [E_k + e\phi(x(+))] \\
 = & - \int d^3x \left\{ 2\pi \gamma_{\alpha\beta} \Pi^\alpha \Pi^\beta + \frac{1}{8\pi} \gamma_{\alpha\beta} H^\alpha H^\beta + \phi \partial_\alpha \Pi^\alpha \right\}
 \end{aligned}$$

Now, eq. of motion for particle:

$$\begin{aligned}
 \dot{x}^\alpha = \frac{\partial H}{\partial p_\alpha} &= - \frac{\partial E_k}{\partial p_\alpha} = - \frac{\partial}{\partial p_\alpha} \left(\sqrt{m^2 + (p - eA)^2} \right) \\
 \boxed{\dot{x}^\alpha} &= - \frac{(p - eA)^\alpha}{\sqrt{m^2 + (p - eA)^2}}
 \end{aligned}$$

$$\dot{p}_\alpha = - \frac{\partial H}{\partial \dot{x}^\alpha} = - \partial_\alpha [E_K + e \phi(\dot{x}(t))]$$

$$= -e \frac{(p - eA)_n \partial_\alpha A^m}{\gamma m^2 + (p - eA)^2} + e (\partial_\alpha \phi)(\dot{x}(t))$$

$$= e \left[\dot{x}^n \partial_\alpha A_n + \partial_\alpha \phi \right]$$

using eqn 1

$$\text{Now } (\dot{x} \times H)_\mu = \epsilon_{\mu \nu \rho} \dot{x}^\nu H^\rho = \dot{x}^\alpha F_{\mu \alpha}$$

$$= \dot{x}^\alpha (\partial_\mu A_\alpha - \partial_\alpha A_\mu)$$

so

$$\begin{aligned} \frac{d}{dt} (p_\alpha - eA_\alpha) &= e \left[\dot{x}^n \partial_\alpha A_n + \partial_\alpha \phi \right] - e \partial_\mu A_\alpha \dot{x}^n - e \frac{\partial}{\partial t} A_\alpha \\ &= e \left[(\dot{x} \times H)_\alpha + \partial_\alpha \phi - \frac{\partial}{\partial t} A_\alpha \right] \end{aligned}$$

Now, from eq. 1: $\dot{x}^\alpha = - \frac{k^\alpha}{\sqrt{m^2 + k^2}} \quad i^{K^*} = p - eA$

$$\Leftrightarrow k^\alpha = -m \frac{\dot{x}^\alpha}{\sqrt{1 - \dot{x}^2}}$$

$$\text{So } \boxed{\frac{d}{dt} \left(\frac{m \dot{x}^\alpha}{\sqrt{1 - \dot{x}^2}} \right) = -e \left[(\dot{x} \times H)_\alpha + \partial_\alpha \phi - \frac{\partial}{\partial t} A_\alpha \right]}$$

$$F_{0\alpha} = \partial_0 A_\alpha - \partial_\alpha A_0 = \frac{\partial}{\partial t} A - \partial_\alpha \phi \quad , \text{ all seems fine...}$$

Next, need equations of motion for ϕ field!

$$\dot{A}_\alpha = \frac{\delta H}{\delta \Pi^\alpha} = -4\pi \gamma_{\alpha\beta} \Pi^\beta + \partial_\alpha \phi$$

$$4\pi \Pi^\beta \gamma_{\alpha\beta} = \partial_\alpha \phi - \partial_\beta A_\alpha =: E_\alpha$$

$$\dot{\Pi}^\alpha = -\frac{\delta H}{\delta A_\alpha} = \frac{\delta E_K}{\delta A_\alpha} + \frac{1}{4\pi} \gamma_{\mu\nu} H^\nu \frac{\delta H^\nu}{\delta A_\alpha}$$

$$\begin{aligned} \frac{\delta E_K}{\delta A_\alpha} &= \frac{S}{\delta A_\alpha} \sqrt{m^2 + (p - eA)^2} \\ &= \frac{(p - eA)^\mu}{E_K} (-e S_\mu^\alpha S_2) = -e \frac{(p - eA)^\alpha}{E_K} S_2 \\ &= e \dot{x}^\alpha S_2 \end{aligned}$$

$$\epsilon^{\mu\nu\sigma} F_{\mu\nu} = \epsilon_{\mu\nu\sigma} H^\sigma \quad \epsilon^{\mu\nu\sigma} = 2 S_\sigma^\nu H^\sigma = 2 H^\nu$$

$$\begin{aligned} \frac{\delta H^\nu}{\delta A_\alpha} &= \frac{\delta}{\delta A_\alpha} \frac{1}{2} \epsilon^{\sigma\tau\nu} F_{\sigma\tau} = \frac{\delta}{\delta A_\alpha} \epsilon^{\sigma\tau\nu} \partial_\sigma A_\tau \\ &= \epsilon^{\sigma\alpha\tau} \partial_\tau S \end{aligned}$$

get

$$\dot{\Pi}^\alpha = e \dot{x}^\alpha S_2 - \frac{1}{4\pi} \epsilon^{\sigma\alpha\tau} \partial_\sigma H_\tau$$

$$\Leftrightarrow 4\pi \dot{\Pi}^\alpha \gamma_{\alpha\beta} = e 4\pi \dot{x}_\beta S_2 + \gamma_{\alpha\beta} \epsilon^{\mu\nu\alpha} \partial_\mu H_\nu$$

$$\vec{E}_\beta = 4\pi e \dot{x}_\beta S_2 + (\vec{B} \times \vec{H})_\beta$$

Finally, must use constraint stuff:

$$0 = \frac{\delta H}{\delta \phi} = -e \delta_2 - \partial_\alpha \Pi^\alpha$$

$$\Leftrightarrow \partial_\alpha E_\beta, \gamma^{\alpha\beta} = -e \delta_2 4\pi$$

$$\partial_\alpha E^\alpha = -4\pi e \delta(z)$$

Equations are gathered:

$$\boxed{\dot{E}_\beta - (\partial \times H)_\beta = 4\pi e \dot{x}_\beta \delta(z)}$$

$$\boxed{\partial_\alpha E^\alpha = -4\pi e \delta(z)}$$

Further, know that $\frac{1}{2} F_{\mu\nu} \epsilon^{\mu\nu\zeta} = H^\zeta$

and $\partial_\zeta H^\zeta = \frac{1}{2} \epsilon^{\mu\nu\zeta} \partial_\zeta F_{\mu\nu} = \epsilon^{\mu\nu\zeta} \partial_\zeta \partial_\mu A_\nu = 0$

$$\boxed{\partial_\zeta H^\zeta = 0}$$

and

$$E_\alpha = \partial_\alpha \phi - \partial_\alpha A_\alpha$$

so

$$(\partial \times E)_\alpha = \epsilon_{\mu\nu\alpha} \partial^\mu E^\nu = \epsilon_{\mu\nu\alpha} \partial^\mu (\partial^\nu \phi - \partial_\nu A^\nu)$$

$$= -\partial_\mu \epsilon_{\mu\nu\alpha} \partial^\mu A^\nu = -\partial_\alpha H_\alpha$$

so

$$\boxed{(\partial \times E)_\alpha + \partial_\alpha H_\alpha = 0}$$

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Obtained eq. of motion for field & charges:

$$\frac{d}{dt} \left[\frac{m \dot{z}^\alpha}{\sqrt{1 - \dot{z}^2}} \right] = -e (\dot{z} \times H)^\alpha - e E^\alpha$$

$$\partial_\alpha E^\alpha = -4\pi e \delta(z(+))$$

$$\partial_\alpha H^\alpha = 0$$

$$(\partial \times E)_\alpha = -\overset{\circ}{H}_\alpha$$

$$(\partial \times H)_\alpha = \overset{\circ}{E}_\beta - 4\pi e \dot{z}_\beta \delta(z(+))$$

which are exactly the correct equations given in Landau/Lifshitz,
if $e \rightarrow -e$ convention.

Next: Obtain the source-terms for gravity!

From Hamiltonian, get

$$\frac{\delta H_{\text{pp}}}{\delta \varphi^{\alpha\beta}} = \sum_i \left\{ \frac{1}{2E_k} k_\alpha^i k_\beta^i \delta_x \right\}$$

$$\frac{\delta H_{EH}}{\delta \varphi^{\alpha\beta}} = - \frac{1}{8\pi} \left\{ E_\alpha E_\beta - \frac{1}{2} \gamma_{\alpha\beta} E^2 + H_\alpha H_\beta - \frac{1}{2} \gamma_{\alpha\beta} H^2 \right\}$$

↑
Factor $\frac{1}{2}$ difference from Landau-Lifshitz!

Further

$$\frac{\delta H_{\text{pp}}}{\delta N} = - (E_k + e\phi) \delta_x$$

$$\frac{\delta H_{EH}}{\delta N} = - \frac{1}{8\pi} [E^2 + H^2] - \frac{1}{4\pi} \partial_\alpha E^\alpha \phi$$

$$\frac{\delta H_{\text{pp}}}{\delta N^\alpha} = - p_\alpha$$

$$\frac{\delta H_{EH}}{\delta N^\alpha} = \frac{1}{4\pi} \left[\epsilon_{\alpha\mu\nu} E^\mu H^\nu - A_\mu \partial_\alpha E^\mu \right]$$

Furthermore, can use the connection of GT Tensor to relate and check p.p. calculation!

Energy-momentum-conservation:

In flat space, have $\dot{H} = -\partial_\alpha \partial^\alpha$

$$\dot{\partial}_\mu = -2\partial_\alpha \left(\frac{\delta S}{\delta \varphi_{\alpha\mu}} \delta \varphi^\mu \right)$$

$$\begin{aligned}\dot{H} &= -\partial_+ \frac{\delta H}{\delta N} = -\partial_+ \left[(\epsilon_k + e\phi) \delta(z) + \frac{1}{8\pi} [\epsilon^2 + H^2] + \frac{1}{4\pi} \partial_\alpha \epsilon^\alpha \phi \right] \\ &= -\partial_+ \left[\epsilon_k \delta(z) + \frac{1}{8\pi} [\epsilon^2 + H^2] + \underbrace{\frac{\phi}{4\pi} (\partial_\alpha \epsilon^\alpha + e \delta(z) \frac{1}{4\pi})}_{=0} \right] \\ &= -\partial_+ \left[\sqrt{m^2 + (p - eA)^2} \delta(z) + \frac{1}{8\pi} (\epsilon^2 + H^2) \right] \xrightarrow{\text{supp. of eom!}} \\ \boxed{?} &= -\partial_+ [m \delta(z)] + \mathcal{O}(p^2)\end{aligned}$$

For two particles:

$$\dot{H} = -m_1 \partial_\alpha \delta(z_1) \dot{z}_1^\alpha - m_2 \partial_\alpha \delta(z_2) \dot{z}_2^\alpha$$

$$\partial_\alpha \partial^\alpha = \partial_\alpha \frac{\delta H}{\delta N^\alpha} = \partial_\alpha \left[-p_\alpha \delta_z - \frac{1}{4\pi} (\epsilon \times H)_\alpha - \frac{1}{4\pi} A_\mu \partial_\alpha \epsilon^\mu \right]$$

In our solution: $A = 0$, thus, for two particles

$$\partial_\alpha \partial^\alpha = -p_1^\alpha \partial^\alpha \delta(z_1) - p_2^\alpha \partial^\alpha \delta(z_2)$$

which yields

$$\boxed{m_i \dot{z}_i^\alpha = -p_i^\alpha}$$

correct result!

further:

$$\ddot{D}_\mu = \sum_i \partial_\alpha (-p_\mu^i s_{x^i}) = - \sum_i \left[\dot{p}_\mu^i s_{x^i} + p_\mu \partial_\alpha s_{x^i} x^{i\alpha} \right]$$

$$- 2 \partial_\alpha \left(\frac{eS}{8\pi G} \gamma_{\alpha\mu} \right)$$

$$= 2 \partial^\alpha \frac{eH}{8\pi G} = 2 \partial^\alpha \left[\sum_i \frac{1}{2m_i} k_\alpha^i k_\mu^i s_{x^i} - \frac{1}{8\pi} \left\{ E_\alpha E_\mu - \frac{1}{2} \gamma_{\alpha\mu} E^2 + H_\alpha H_\mu - \frac{1}{2} \gamma_{\alpha\mu} H^2 \right\} \right]$$

Newtonian limit: $E_\alpha^i = m_i$, $A = 0$, $H = 0$

$$= 2 \partial^\alpha \left[\sum_i \frac{1}{2m_i} p_\alpha^i p_\mu^i s(x^i) - \frac{1}{8\pi} \left\{ E_\alpha E_\mu - \frac{1}{2} \gamma_{\alpha\mu} E^2 \right\} \right]$$

$$= \sum_i -\frac{1}{m_i} p_\alpha^i p_\mu^i \partial^\alpha(x^i) - \frac{1}{4\pi} \left\{ \partial^\alpha(E_\alpha E_\mu) - \frac{1}{2} \partial_\mu E^2 \right\}$$

$$= - \sum_i p_\mu^i \partial^\alpha(x^i) \dot{x}_\alpha + \frac{1}{4\pi} \left\{ \partial^\alpha E_\alpha E_\mu + E^\alpha (\partial_\alpha E_\mu - \underbrace{\partial_\mu E_\alpha}_{0 \text{ since } H=0}) \right\}$$

$$= - \sum_i p_\mu^i \partial^\alpha(x^i) \dot{x}_\alpha + \frac{1}{4\pi} \partial^\alpha E_\alpha E_\mu$$

Thus: $\sum_i \dot{p}_\mu^i s(x^i) = \sum_i \frac{1}{4\pi} \partial^\alpha E_\alpha E_\mu$

$$\Leftrightarrow \begin{cases} \dot{s}(x^i) = -\frac{1}{4\pi e} \partial^\alpha E_\alpha \\ \dot{p}_\mu^i = -e E_\mu \end{cases}$$

which is correct!

$$\begin{aligned} \epsilon^{\mu\nu\alpha} \dot{H}_\mu &= - \epsilon^{\mu\nu\alpha} \epsilon_{\alpha\beta\gamma} \partial^\beta E^\gamma \\ &= - (\delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu) \partial^\beta E^\gamma \\ &= - (\partial^\mu E^\nu - \partial^\nu E^\mu) \end{aligned}$$