GLED Hamiltonian

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Hamiltonian of GLED

Start from action

$$\boxed{S = \frac{1}{32\pi} \int F_{ab} F_{cd} G^{abcd} \, \omega \, d^4x}$$

$G \sim 2gg$

$\rightsquigarrow$ factor $\frac{1}{2}$ in front of action when compared to $S_{Maxwell}$

$$\boxed{F_{ab} := \partial_a A_b - \partial_b A_a}$$

$$\boxed{T^a A_a =: \phi}, \quad \boxed{e^a_\alpha A_a =: A_\alpha}, \quad \boxed{(\mathcal{L}_{\dot{X}} A)_\alpha =: \dot{A}_\alpha}$$

Completeness relation for G:

$$\boxed{G^{abcd} = 4 G^{\beta\delta} T^{[a} e_\beta^{b]} T^{[c} e_\delta^{d]} + 2 G^{\beta\gamma\delta} T^{[a} e_\beta^{b]} e_\gamma^c e_\delta^d + 2 G^{\alpha\beta\delta} e^a_\alpha e^b_\beta T^{[c} e_\delta^{d]} + G^{\alpha\beta\gamma\delta} e^a_\alpha e^b_\beta e^c_\gamma e^d_\delta}$$

$$F_{0\beta} = T^a e^b_\beta F_{ab} = \frac{\dot{X}^a - N^\alpha e^a_\alpha}{N} e^b_\beta F_{ab}$$

$$= \frac{1}{N} \left[\dot{X}^a e^b_\beta F_{ab} - N^\alpha e^a_\alpha e^b_\beta F_{ab} \right]$$

$= \ldots$ (see particle_and_maxwell_hamiltonian p. 7, 8)

$$\boxed{F_{0\beta} = \frac{1}{N} \left(\dot{A}_\beta - \partial_\beta (N\phi + N^\alpha A_\alpha) \right) - \frac{N^\alpha}{N} F_{\alpha\beta}}$$

$F_{\alpha\beta} = e^a_\alpha e^b_\beta F_{ab} = \ldots$ (see same as above)

$$\boxed{F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha}$$

Instead of G^{abcd}, use

note that $\epsilon^{0123} = -1$

thus $\epsilon^{abcd}\epsilon_{abcd} = -\frac{1}{4!}$

$$\boxed{\tilde{G}^{abcd} = G^{abcd} + \epsilon^{abcd}\,\frac{1}{4!}\,G^{mnop}\,\epsilon_{mnop}}$$

Why? A total antisymmetric part of G does not contribute to the action, appart from surface terms:

$$\int \epsilon^{abcd} F_{ab} F_{cd} = \int 4\,\epsilon^{abcd}\,\partial_a A_b\,\partial_c A_d$$

$$= -\int 4\,\epsilon^{abcd}\,\partial_c \partial_a A_b A_d + \text{Surf.}$$

Schwarz.

Can thus reduce totally antisymmetric part to resemble metric Maxwell one.

$$\rightsquigarrow \quad S = \frac{1}{32\pi}\int F_{ab}F_{cd}\,\tilde{G}^{abcd}\,\omega\,d^4x$$

$$= \frac{1}{32\pi}\int\left[F_{ab}F_{cd}\,G^{abcd} + F_{ab}F_{cd}\,\epsilon^{abcd}\,\frac{1}{4!}\,G^{mnop}\epsilon_{mnop}\right]\omega\,d^4x$$

Now, choose

$$\boxed{\omega^{-1} =: \frac{1}{4!}\,G^{abcd}\,\epsilon_{abcd}}$$

(see Noldines thesis)

and get

$$S = \frac{1}{32\pi}\int\left[G^{abcd}F_{ab}F_{cd} + \frac{1}{\omega}\,\epsilon^{abcd}F_{ab}F_{cd}\right]\omega\,d^4x$$

now, let us start the 3+1-split!

3+1 :

$$G^{abcd} F_{ab} F_{cd} = 4\, G^{\beta\delta} F_{0\beta} F_{0\delta} + 2\left(G^{\beta\gamma\delta} + G^{\gamma\delta\beta}\right) F_{0\beta} F_{\gamma\delta} + G^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}$$

$$\epsilon^{abcd} F_{ab} F_{cd} = 4\, \epsilon^{0\alpha\beta\gamma} F_{0\alpha} F_{\beta\gamma}$$

to integrate over such stuff, must evaluate in right coordinates: $\{t, y^\alpha\}$

$$= -4\, \epsilon^{\alpha\beta\gamma} \underbrace{F_{t\alpha} F_{\beta\gamma}}$$

since $\boxed{\epsilon^{0123} := -1}$

$$\dot{X}^a e^b_\alpha F_{ab} F_{cd} e^c_\beta e^d_\gamma = N F_{0\alpha} F_{\beta\gamma} + N^\beta F_{\beta\alpha} F_{\beta\gamma}$$

$$\mathcal{L} = \frac{\omega}{32\pi}\Big[4 G^{\beta\delta} F_{0\beta} F_{0\delta} + \left\{ 2\left(G^{\beta\gamma\delta} + G^{\gamma\delta\beta}\right) - N\frac{4}{\omega}\, \epsilon^{\beta\gamma\delta} \right\} F_{0\beta} F_{\gamma\delta} + G^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} - \frac{4N^\mu}{\omega}\, \epsilon^{\alpha\beta\gamma} F_{\mu\alpha} F_{\beta\gamma}\Big]$$

$$\Pi^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{A}^\alpha} = \frac{\partial \mathcal{L}}{\partial F_{0\mu}} \frac{\partial F_{0\mu}}{\partial \dot{A}^\alpha}$$

$$= \frac{\omega}{32\pi}\left[8 G^{\mu\nu} F_{0\nu} + \left\{ 2\left(G^{\mu\alpha\beta} + G^{\alpha\beta\mu}\right) - \frac{4}{\omega}\, \epsilon^{\mu\alpha\beta} \right\} F_{\alpha\beta} \right] \frac{1}{N} \delta^\alpha_\mu$$

see later : in coordinates $\{t, y^\alpha\}$, $\omega = N \cdot \Omega$

So

$$\boxed{\Pi^\mu = \frac{\Omega}{4\pi}\left[G^{\mu\nu} F_{0\nu} + \left\{ \frac{1}{4}\left(G^{\mu\alpha\beta} + G^{\alpha\beta\mu}\right) - \frac{1}{2\Omega}\, \epsilon^{\mu\alpha\beta} \right\} F_{\alpha\beta} \right]}$$

and other way round :

$$F_{0\mu} = \left(G^{-1}\right)_{\mu\nu} \left[\frac{4\pi}{\Omega} \Pi^\mu - \left\{ \frac{1}{4}\left(G^{\mu\alpha\beta} + G^{\alpha\beta\mu}\right) - \frac{1}{2\Omega}\, \epsilon^{\mu\alpha\beta} \right\} F_{\alpha\beta} \right]$$

As for ϕ : $\Pi_\phi = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0$

Now, calculate the Hamiltonian:

$$\int H = \int \Pi^\alpha \dot{A}_\alpha + \pi\phi\,\dot{\phi} - \mathcal{L}$$

$$\dot{A}_\alpha = N F_{0\alpha} + \partial_\alpha (N\phi + N^\mu A_\mu) + N^\mu F_{\mu\alpha}$$

$$= \int N \frac{\Omega}{4\pi} \left[G^{\mu\nu} F_{0\nu} + \left\{ \frac{1}{4}\left(G^{\mu\alpha\beta} + G^{\alpha\beta\mu}\right) + \frac{1}{2\omega} \epsilon^{\mu\alpha\beta} \right\} F_{\alpha\beta} \right] F_{0\mu}$$

$$+ \Pi^\alpha \partial_\alpha (N\phi + N^\mu A_\mu) + \Pi^\alpha N^\mu F_{\mu\alpha}$$

$$- N \frac{\Omega}{4\pi} \Big[\frac{1}{2} G^{\mu\nu} F_{0\mu} F_{0\nu} + \left\{ \frac{1}{4}\left(G^{\beta\mu\nu} + G^{\mu\nu\beta}\right) + \frac{1}{2\omega}\epsilon^{\beta\mu\nu} \right\} \times F_{0\beta} F_{\mu\nu}$$

$$+ \frac{1}{8} G^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} \Big] + \frac{1}{8\pi} N^\mu \epsilon^{\alpha\beta\gamma} F_{\mu\alpha} F_{\beta\gamma}$$

$$= \int N \frac{\Omega}{4\pi} \left[\frac{1}{2} G^{\mu\nu} F_{0\nu} F_{0\mu} - \frac{1}{8} G^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \right] + \Pi^\alpha \left\{ \partial_\alpha (N\phi + N^\mu A_\mu) + N^\mu F_{\mu\alpha} \right\}$$

$$+ \frac{1}{8\pi} N^\mu \epsilon^{\alpha\beta\gamma} F_{\mu\alpha} F_{\beta\gamma}$$

$$= \int N \left[\frac{\Omega}{8\pi} \left(G^{\mu\nu} F_{0\nu} F_{0\mu} - \frac{1}{4} G^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \right) - \phi\, \partial_\alpha \Pi^\alpha \right]$$

$$+ N^\mu \left[F_{\mu\alpha} \Pi^\alpha - A_\mu \partial_\alpha \Pi^\alpha + \frac{1}{8\pi} \epsilon^{\alpha\beta\gamma} F_{\mu\alpha} F_{\beta\gamma} \right]$$

use

$$\boxed{p^\alpha := G^{\alpha\beta} F_{0\beta} \frac{\Omega}{4\pi} = \Pi^\alpha - \frac{\Omega}{4\pi} \left\{ \frac{1}{4}\left(G^{\mu\alpha\beta} + G^{\alpha\beta\mu}\right) + \frac{1}{2\omega}\epsilon^{\mu\alpha\beta} \right\} F_{\mu\beta}}$$

to get

$$\boxed{\begin{aligned} H = N\Big[& \frac{2\pi}{\Omega} (G^{-1})_{\alpha\beta}\, p^\alpha p^\beta - \frac{\Omega}{32\pi} G^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} - \phi\, \partial_\alpha \Pi^\alpha \Big] \\ & + N^\mu \left[\frac{1}{8\pi} \epsilon^{\alpha\beta\gamma} F_{\mu\alpha} F_{\beta\gamma} + F_{\mu\alpha} \Pi^\alpha - A_\mu \partial_\alpha \Pi^\alpha \right] \end{aligned}}$$

$$\boxed{\bar{g}^{\alpha\beta} =: -G^{\alpha\beta} = -G^{abcd} n_a \epsilon_b^\alpha n_c \epsilon_d^\beta}$$ $\bar{g}^{\alpha\beta} = \bar{g}^{\beta\alpha}$

$\bar{\bar{\bar{g}}}^\alpha{}_\alpha = 0$

$$\boxed{\bar{\bar{\bar{g}}}^\alpha{}_\beta =: \frac{1}{2} (\omega_G)_{\beta\gamma\delta} G^{abcd} n_a \epsilon_b^\alpha \epsilon_c^\gamma \epsilon_d^\delta - \delta^\alpha_\beta}$$

$\bar{\bar{\bar{g}}}^\alpha{}_\mu \bar{g}^{\mu\beta} = \bar{\bar{\bar{g}}}^\beta{}_\mu \bar{g}^{\mu\alpha}$

$$\boxed{\bar{\bar{g}}_{\alpha\beta} := \frac{1}{4} (\omega_G)_{\alpha\mu\nu} (\omega_G)_{\beta\rho\sigma} G^{abcd} \epsilon_a^\mu \epsilon_b^\nu \epsilon_c^\rho \epsilon_d^\sigma}$$ $\bar{\bar{g}}_{\alpha\beta} = \bar{\bar{g}}_{\beta\alpha}$

where $$\boxed{(\omega_G)_{\alpha\beta\gamma} := \epsilon_{\alpha\beta\gamma} \left(\det \bar{g}^{\cdot\cdot}\right)^{-\frac{1}{2}}}$$

now, in basis $\{\dot{X}, e^\alpha\}$, dual to $\left\{ K = \frac{1}{N} n, \; \tilde{\epsilon}^\alpha = \epsilon^\alpha - \frac{N^\alpha}{N} n \right\}$

$$\omega^{-1} = \frac{1}{4!} G^{abcd} \epsilon_{abcd}$$

$$= \frac{1}{3!} \epsilon_{\alpha\beta\gamma} G^{abcd} K_a \tilde{\epsilon}_b^\alpha \tilde{\epsilon}_c^\beta \tilde{\epsilon}_d^\gamma$$

$$= \frac{1}{3!} \epsilon_{\beta\gamma\delta} G^{abcd} \frac{1}{N} n_a \left(\epsilon_b^\beta - \frac{N^\beta}{N} n_b\right)\left(\epsilon_c^\gamma - \frac{N^\gamma}{N} n_c\right)\left(\epsilon_d^\delta - \frac{N^\delta}{N} n_d\right)$$

$$= \frac{1}{3!}\frac{1}{N} \epsilon_{\beta\gamma\delta} G^{0bcd} \Big[\epsilon_b^\beta \epsilon_c^\gamma \epsilon_d^\delta - \frac{N^\beta}{N} n_b \epsilon_c^\gamma \epsilon_d^\delta + \frac{N^\beta}{N}\frac{N^\gamma}{N} n_b n_c \epsilon_d^\delta$$
$$- \epsilon_b^\beta \frac{N^\gamma}{N} n_c \epsilon_d^\delta + \frac{N^\beta}{N}\frac{N^\delta}{N} n_b \epsilon_c^\gamma n_d$$
$$- \frac{N^\beta N^\gamma N^\delta}{N^3} n_b n_c n_d - \epsilon_b^\beta \epsilon_c^\gamma \frac{N^\delta}{N} n_d + \frac{N^\gamma}{N}\frac{N^\delta}{N} \epsilon_b^\beta n_c n_d$$

$G^{abcd} = G^{[ab][cd]}$

$$= \frac{1}{N\,3!} \epsilon_{\beta\gamma\delta} \Big[G^{0\beta\gamma\delta} - \cancel{\frac{N^\beta}{N} G^{00\gamma\delta}} - \frac{N^\gamma}{N} G^{0\beta0\delta} - \frac{N^\delta}{N} G^{0\beta\gamma0} - \cancel{\frac{N^\beta N^\gamma N^\delta}{N^3} G^{0000}}$$
$$+ \cancel{\frac{N^\beta}{N}\frac{N^\gamma}{N} G^{000\delta}} + \cancel{\frac{N^\beta}{N}\frac{N^\delta}{N} G^{00\gamma0}} + \cancel{\frac{N^\gamma}{N}\frac{N^\delta}{N} G^{0\beta00}} \Big]$$

$$= \frac{1}{3!N} \varepsilon_{\beta\gamma\delta} \left[G^{0\beta\gamma\delta} - \cancel{\frac{2N^\gamma}{N} G^{0\beta0\delta}} \right]$$

$$G^{(AB)} = G^{AB}$$

$A = ab \quad B = cd$

$$= \frac{1}{3!N} \varepsilon_{\beta\gamma\delta} G^{0\beta\gamma\delta}$$

$$= \frac{1}{N} \frac{1}{3!} \delta^\beta_\alpha \varepsilon_{\beta\gamma\delta} G^{0\alpha\gamma\delta} = \frac{1}{N} \frac{1}{3!} \delta^\beta_\alpha \left(2 \left(\det \bar{g}^{\cdot\cdot} \right)^{\frac{1}{2}} \left(\bar{\bar{g}}^\alpha{}_\beta + \delta^\alpha{}_\beta \right) \right)$$

$$= \frac{1}{N} \frac{1}{3!} \left(2 \cdot 3 \left(\det \bar{g}^{\cdot\cdot} \right)^{\frac{1}{2}} \right) = \frac{\left(\det \bar{g}^{\cdot\cdot} \right)^{\frac{1}{2}}}{N}$$

get : $\boxed{\omega = N \left(\det \bar{g}^{\cdot\cdot} \right)^{-\frac{1}{2}}} \rightsquigarrow \boxed{\Omega = \left(\det \bar{g}^{\cdot\cdot} \right)^{-\frac{1}{2}}}$

Define : $\boxed{F_{\alpha\beta} =: \varepsilon_{\alpha\beta\gamma} H^\gamma} \rightsquigarrow \boxed{\frac{1}{2} \varepsilon^{\alpha\beta\gamma} F_{\alpha\beta} = H^\gamma}$

and invert relations:

$$\boxed{G^{\alpha\beta} = - \bar{g}^{\alpha\beta}}$$

$$\boxed{G^{0\beta\gamma\delta} = \left(\det \bar{g}^{\cdot\cdot} \right)^{\frac{1}{2}} \varepsilon^{\alpha\gamma\delta} \left(\bar{\bar{g}}^\beta{}_\alpha + \delta^\beta_\alpha \right)}$$

$$\boxed{G^{\alpha\beta\gamma\delta} = \varepsilon^{\alpha\beta\mu} \varepsilon^{\gamma\delta\nu} \left(\det \bar{g}^{\cdot\cdot} \right) \bar{\bar{g}}_{\mu\nu}}$$

Next: insert this into the Hamiltonian!

$$H = N\Big[-2\pi\,(\det \bar g^{\cdot\cdot})^{\frac12}\,(\bar g^{-1}_{\alpha\beta})\,p^\alpha p^\beta$$

$$-\frac{1}{8\pi}\,(\det \bar g^{\cdot\cdot})^{\frac12}\,\bar{\bar g}_{\alpha\beta}H^\alpha H^\beta - \phi\,\partial_\alpha \Pi^\alpha\Big]$$

$$+ N^\mu\Big[F_{\mu\alpha}\Pi^\alpha - A_\mu \partial_\alpha \Pi^\alpha + \cancel{\frac{1}{4\pi}F_{\mu\alpha}H^\alpha}\Big]$$

note: $H^\alpha F_{\mu\alpha} = H^\alpha \varepsilon_{\mu\alpha\tau} H^\tau = 0$

and

$$p^\alpha = \Pi^\alpha - \frac{1}{4\pi}(\det \bar g^{\cdot\cdot})^{-\frac12}\Big\{\frac12(\det \bar g^{\cdot\cdot})^{\frac12}\,(\bar{\bar{\bar g}}{}^\alpha{}_\beta + \delta^\alpha_\beta)\,\epsilon^{\beta\mu\nu}$$

$$-\frac12(\det \bar g^{\cdot\cdot})^{\frac12}\,\epsilon^{\alpha\mu\nu}\Big\}F_{\mu\nu}$$

Obtain

$$\boxed{p^\alpha = \Pi^\alpha - \frac{1}{4\pi}\,\bar{\bar{\bar g}}{}^\alpha{}_\beta H^\beta}$$

So :

$$\boxed{\begin{aligned} H = N\Big[&-2\pi\,(\det \bar g^{\cdot\cdot})^{\frac12}\,(\bar g^{-1})_{\alpha\beta}\Big(\Pi^\alpha - \frac{1}{4\pi}\bar{\bar{\bar g}}{}^\alpha{}_\nu H^\nu\Big)\Big(\Pi^\beta - \frac{1}{4\pi}\bar{\bar{\bar g}}{}^\beta{}_\mu H^\mu\Big) \\ &-\frac{1}{8\pi}(\det \bar g^{\cdot\cdot})^{\frac12}\,\bar{\bar g}_{\alpha\beta}H^\alpha H^\beta - \phi\,\partial_\alpha\Pi^\alpha\Big] \\ &+ N^\mu\Big[-\varepsilon_{\mu\alpha\beta}H^\alpha \Pi^\beta - A_\mu\,\partial_\alpha \Pi^\alpha\Big]\end{aligned}}$$

Next step: linearise!

Parametrisation:

$$\boxed{\begin{array}{l} \bar{g}^{\alpha\beta} = \gamma^{\alpha\beta} + \bar{\varphi}^{\alpha\beta} \\ \bar{\bar{g}}_{\alpha\beta} = \gamma_{\alpha\beta} + \bar{\bar{\varphi}}_{\alpha\beta} \\ \bar{\bar{\bar{g}}}^{\alpha}{}_{\beta} = \bar{\bar{\bar{\varphi}}}^{\alpha}{}_{\beta} + \mathcal{O}(2) \end{array}}$$

$N = 1 + A$

$$\det \bar{g}^{\cdot\cdot} = \det\left(\gamma^{\cdot\cdot} + \bar{\varphi}^{\cdot\cdot}\right) = 1 + \gamma_{\alpha\beta}\bar{\varphi}^{\alpha\beta} + \mathcal{O}(2)$$

So:

$$H = (1+A)\Big[-2\pi\left(1 + \tfrac{1}{2}\gamma_{\sigma\tau}\bar{\varphi}^{\sigma\tau}\right)\left(\gamma_{\alpha\beta} - \bar{\varphi}_{\alpha\beta}\right)\left(\Pi^{\alpha} - \tfrac{1}{4\pi}\bar{\bar{\bar{\varphi}}}^{\alpha}{}_{\nu}H^{\nu}\right)\left(\Pi^{\beta} - \tfrac{1}{4\pi}\bar{\bar{\bar{\varphi}}}^{\beta}{}_{\mu}H^{\mu}\right)$$
$$- \tfrac{1}{8\pi}\left(1 + \tfrac{1}{2}\gamma_{\sigma\tau}\bar{\varphi}^{\sigma\tau}\right)\left(\gamma_{\alpha\beta} + \bar{\bar{\varphi}}_{\alpha\beta}\right)H^{\alpha}H^{\beta} - \phi\,\partial_{\alpha}\Pi^{\alpha}\Big]$$
$$+ N^{\mu}\mathcal{D}_{\mu}$$

$$= -(1+A)\left[2\pi\,\gamma_{\alpha\beta}\Pi^{\alpha}\Pi^{\beta} + \tfrac{1}{8\pi}\gamma_{\alpha\beta}H^{\alpha}H^{\beta} + \phi\,\partial_{\alpha}\Pi^{\alpha}\right]$$
$$+ N^{\mu}\mathcal{D}_{\mu}$$
$$+ \Big[2\pi\,\bar{\varphi}^{\sigma\tau}\left(\delta^{\alpha}_{\sigma}\delta^{\beta}_{\tau} - \tfrac{1}{2}\gamma_{\sigma\tau}\gamma^{\alpha\beta}\right)\Pi^{\alpha}\Pi^{\beta}$$
$$+ \gamma_{\alpha\beta}\bar{\bar{\bar{\varphi}}}^{\alpha}{}_{\nu}H^{\nu}\Pi^{\beta}$$
$$- \tfrac{1}{8\pi}\left(\bar{\bar{\varphi}}_{\alpha\beta} + \tfrac{1}{2}\bar{\varphi}^{\sigma\tau}\gamma_{\sigma\tau}\gamma_{\alpha\beta}\right)H^{\alpha}H^{\beta}\Big]$$

Connect with metric theory : Metric induced Area - metric

$$G^{abcd} = g^{ac}g^{bd} - g^{ad}g^{bc} - \sqrt{-\det g^{\cdot\cdot}}\,\epsilon^{abcd}$$

get $-\bar{g}^{\alpha\beta} = G^{0\alpha0\beta} = g^{00}g^{\alpha\beta} - g^{0\alpha}g^{0\beta} - \cancel{\sqrt{-\det g^{\cdot\cdot}}}\,\epsilon^{0\alpha0\beta}$

note: $\boxed{g^{00} = 1}$, $\boxed{g^{0\alpha} = 0}$, $\boxed{g^{\alpha\beta} = -\gamma^{\alpha\beta} + h^{\alpha\beta}}$

so $\boxed{-\bar{g}^{\alpha\beta} = g^{\alpha\beta}}$ $\rightsquigarrow$ linearized $\boxed{-\bar{\varphi}^{\alpha\beta} = h^{\alpha\beta}}$

$$\bar{\bar{g}}_{\mu\nu} = \frac{1}{4}\left(\det \bar{g}^{\cdot\cdot}\right)^{-1} \varepsilon_{\alpha\mu\nu}\,\varepsilon_{\beta\rho\sigma}\, G^{\mu\nu\rho\sigma}$$

$$= -\frac{1}{4}\left(\det g^{\cdot\cdot}\right)^{-1} \varepsilon_{\alpha\mu\nu}\,\varepsilon_{\beta\rho\sigma}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho} - \cancel{\sqrt{\cdot\cdot}\;\epsilon^{\mu\nu\rho\sigma}}\right)$$

$$\boxed{\bar{\bar{g}}_{\mu\nu} = -\frac{1}{2}\left(\det g^{\cdot\cdot}\right)^{-1} \varepsilon_{\alpha\mu\nu}\,\varepsilon_{\beta\rho\sigma}\, g^{\mu\rho}g^{\nu\sigma}}$$

linearized

$$\gamma_{\mu\nu} + \bar{\bar{\varphi}}_{\mu\nu} = \frac{1}{2}\left(\det\left(\gamma^{\cdot\cdot} + h^{\cdot\cdot}\right)\right)^{-1} \varepsilon_{\alpha\mu\nu}\,\varepsilon_{\beta\rho\sigma}\left(-\gamma^{\mu\rho} + h^{\mu\rho}\right) \times \left(-\gamma^{\nu\sigma} + h^{\nu\sigma}\right)$$

$$= \frac{1}{2}\left(1 + \gamma_{\alpha\beta}h^{\alpha\beta}\right) \varepsilon_{\alpha\mu\nu}\,\varepsilon_{\beta\rho\sigma}\left(\gamma^{\mu\rho}\gamma^{\nu\sigma} - 2\gamma^{\nu\sigma}h^{\mu\rho}\right)$$

$$= \left(1 + \gamma_{\alpha\beta}h^{\alpha\beta}\right)\left(\gamma_{\alpha\beta} - \left(\gamma_{\alpha\beta}\gamma_{\mu\rho} - \gamma_{\alpha\rho}\gamma_{\mu\beta}\right)h^{\mu\rho}\right)$$

$$= \left(1 + h\right)\left(\gamma_{\alpha\beta} - \gamma_{\alpha\beta}h + h_{\alpha\beta}\right) = \gamma_{\alpha\beta} + h_{\alpha\beta} \quad \boxed{\bar{\bar{\varphi}}_{\mu\nu} = h_{\alpha\beta}}$$

$$\left(\delta^\alpha{}_\beta + \overline{\overline{\overline{g}}}{}^\alpha{}_\beta\right) = \frac{1}{2}\left(\det \overline{g}_{\cdot\cdot}\right)^{-\frac{1}{2}} \varepsilon_{\beta\gamma\delta}\, G^{\circ\alpha\gamma\delta}$$

$$= \frac{1}{2}\cancel{\left(-\det g_{\cdot\cdot}\right)^{-\frac{1}{2}}} \varepsilon_{\beta\gamma\delta} \cdot \cancel{\left(-\det g_{\cdot\cdot}\right)^{\frac{1}{2}}} \cdot \varepsilon^{\alpha\gamma\delta}$$

$$= \delta^\alpha_\beta \quad \rightsquigarrow \quad \boxed{\overline{\overline{\overline{g}}}{}^\alpha{}_\beta = 0}$$

We find:

$$\boxed{e^{\alpha\beta} = \frac{1}{2}\left(\overline{\overline{\varphi}}{}^{\alpha\beta} - \overline{\varphi}{}^{\alpha\beta}\right) =: \varphi^{\alpha\beta}}$$

$$\boxed{0 = \frac{1}{2}\left(\overline{\overline{\varphi}}{}^{\alpha\beta} + \overline{\varphi}{}^{\alpha\beta}\right) =: \psi^{\alpha\beta}}$$

$$\boxed{0 = \overline{\overline{\overline{\varphi}}}{}^\alpha{}_\beta}$$

So

$$\boxed{\begin{aligned} \varphi^{\alpha\beta} + \psi^{\alpha\beta} &= \overline{\overline{\varphi}}{}^{\alpha\beta} \\ \psi^{\alpha\beta} - \varphi^{\alpha\beta} &= \overline{\varphi}{}^{\alpha\beta} \end{aligned}}$$

Insert in relevant 1st order part:

$$2\pi\left(\psi^{\sigma\tau} - \varphi^{\sigma\tau}\right)\left(\delta^\alpha_\sigma \delta^\beta_\tau - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\sigma\tau}\right)\Pi_\alpha \Pi_\beta$$

$$-\frac{1}{8\pi}\left(\varphi^{\alpha\beta} + \psi^{\alpha\beta} + \frac{1}{2}\left(\psi^{\alpha\beta} - \varphi^{\alpha\beta}\right)\gamma_{\sigma\tau}\gamma_{\alpha\beta}\right) H^\alpha H^\beta$$

$$= -\varphi^{\sigma\tau}\left[2\pi\left(\delta^\alpha_\sigma \delta^\beta_\tau - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\sigma\tau}\right)\Pi_\alpha\Pi_\beta + \frac{1}{8\pi}\left(\delta^\alpha_\sigma \delta^\beta_\tau - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\sigma\tau}\right) H_\alpha H_\beta\right]$$

$$+ \psi^{\sigma\tau}\left[2\pi\left(\delta^\alpha_\sigma \delta^\beta_\tau - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\sigma\tau}\right)\Pi_\alpha\Pi_\beta + \frac{1}{8\pi}\left(-\delta^\alpha_\sigma \delta^\beta_\tau - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\sigma\tau}\right) H_\alpha H_\beta\right]$$

Thus get:

$$
\boxed{
\begin{aligned}
H = & -(1+A)\left[2\pi\,\gamma_{\alpha\beta}\Pi^{\alpha}\Pi^{\beta} + \frac{1}{8\pi}\gamma_{\alpha\beta}H^{\alpha}H^{\beta} + \phi\,\partial_{\alpha}\Pi^{\alpha}\right] \\
& - \frac{1}{2}\left(\bar{\bar{\varphi}}^{\alpha\beta} - \bar{\varphi}^{\alpha\beta}\right)\Big[2\pi\left(\delta^{\alpha}_{\sigma}\delta^{\beta}_{\tau} - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\sigma\tau}\right)\Pi_{\alpha}\Pi_{\beta} \\
& \qquad\qquad + \frac{1}{8\pi}\left(\delta^{\alpha}_{\sigma}\delta^{\beta}_{\tau} - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\sigma\tau}\right)H_{\alpha}H_{\beta} \\
& + \frac{1}{2}\left(\bar{\bar{\varphi}}^{\alpha\beta} + \bar{\varphi}^{\alpha\beta}\right)\Big[2\pi\left(\delta^{\alpha}_{\sigma}\delta^{\beta}_{\tau} - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\sigma\tau}\right)\Pi_{\alpha}\Pi_{\beta} \\
& \qquad\qquad + \frac{1}{8\pi}\left(-\delta^{\alpha}_{\sigma}\delta^{\beta}_{\tau} - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\sigma\tau}\right)H_{\alpha}H_{\beta}\Big] \\
& + \bar{\bar{\bar{\varphi}}}^{\alpha}{}_{\nu}\left[H^{\nu}\Pi_{\alpha}\right] \\
& - N^{\mu}\left[\varepsilon_{\mu\alpha\beta}H^{\alpha}\Pi^{\beta} - A_{\mu}\partial_{\alpha}\Pi^{\alpha}\right]
\end{aligned}
}
$$