Area metric

Point particle Hamiltonian

$$\boxed{\text{Generic Action} \longrightarrow H(p;\ \bar{\varphi}, \bar{\bar{\varphi}}, \bar{\bar{\bar{\varphi}}}, A, N\alpha)}$$

Hamiltonian of the point particle

Action of massive p.p: $$\boxed{S = m\int P(L^{-1}(\dot{x}))^{\frac{1}{\deg P}} \, d\tau}$$

Area metric gravity: $$P(k) = P^{abcd} k_a k_b k_c k_d$$

and the Legendre map: $$L^m(k) = \frac{P^{mabc} k_a k_b k_c}{P(k)}$$

P is the principal polynomial, given by

$$\boxed{P^{abcd} = -\frac{1}{4!\,\psi^2} \varepsilon_{mnpq} \varepsilon_{rstu} G^{mnr(a} G^{b|ps|c} G^{d)qtu}}$$

where G^{abcd} is the area metric, and $\psi = \frac{1}{24}\varepsilon_{abcd} G^{abcd}$.

Area metric yields three hypersurface fields:

$$\boxed{\begin{aligned} \bar{g}^{\alpha\beta} &= -G^{0\alpha0\beta} \\ \bar{\bar{g}}_{\alpha\beta} &= \frac{1}{4\det\bar{g}} \varepsilon_{\alpha\mu\nu} \varepsilon_{\beta\tau\omega} G^{\mu\nu\tau\omega} \\ \bar{\bar{\bar{g}}}^{\alpha}{}_{\beta} &= \frac{1}{2\sqrt{\det\bar{g}}} \varepsilon_{\beta\mu\nu} G^{0\alpha\mu\nu} - \delta^{\alpha}{}_{\beta} \end{aligned}}$$

Indices relate to the dual basis $\{n, \varepsilon^\alpha\}$

These field must be parametrised with the degrees of freedom of the area metric. Here, use perturbative Ansatz:

$$\boxed{\begin{aligned} \bar{g}^{\alpha\beta} &= \gamma^{\alpha\beta} + \bar{\varphi}^{\alpha\beta} \\ \bar{\bar{g}}_{\alpha\beta} &= \gamma_{\alpha\beta} + \bar{\bar{\varphi}}_{\alpha\beta} \\ \bar{\bar{\bar{g}}}^{\alpha}{}_{\beta} &= \bar{\bar{\bar{\varphi}}}^{\alpha}{}_{\beta} + \mathcal{O}(\varphi^2) \end{aligned}}$$

where $\varphi \ll 1$

$$\bar{\varphi}^{\alpha\beta} := I^{\alpha\beta}{}_{\bar{A}} \varphi^{\bar{A}}$$

Using the results of HH et. al, now linearise the principal polynomial.

We still work in $\{ n, \epsilon^\alpha \}$ dual basis.

$$\boxed{\begin{aligned} p^{0000} &= 1 \\ p^{\alpha 000} &= 0 \end{aligned}}$$

Frame conditions. from those follows $\boxed{\overline{\overline{\overline{g}}}^{\mu}{}_{\mu} = 0}$

and $\boxed{\overline{\overline{\overline{g}}}^{\alpha}{}_{\mu} \, \overline{g}^{\mu\beta} - \overline{\overline{\overline{g}}}^{\beta}{}_{\mu} \, \overline{g}^{\mu\alpha} = 0}$

$$p^{\alpha\beta 00} = \frac{\det \overline{g}}{6 \psi^2} \Big[\overline{g}^{\alpha\lambda} \overline{g}^{\beta\kappa} \overline{\overline{g}}_{\lambda\kappa} - \overline{g}^{\alpha\beta} \overline{g}^{\lambda\kappa} \overline{\overline{g}}_{\lambda\kappa} + \mathcal{O}(\overline{\overline{\overline{g}}}^2) \Big]$$

Note that $\psi = \sqrt{\det \overline{g}}$ and $\overline{\overline{\overline{g}}} \sim \mathcal{O}(1)$, thus

$$\begin{aligned} p^{\alpha\beta 00} &= \frac{1}{6} \Big[(\gamma^{\alpha\lambda} + \overline{\varphi}^{\alpha\lambda})(\gamma^{\beta\kappa} + \overline{\varphi}^{\beta\kappa})(\gamma_{\lambda\kappa} + \overline{\overline{\varphi}}_{\lambda\kappa}) \\ &\quad - (\gamma^{\alpha\beta} + \overline{\varphi}^{\alpha\beta})(\gamma^{\lambda\kappa} + \overline{\varphi}^{\lambda\kappa})(\gamma_{\lambda\kappa} + \overline{\overline{\varphi}}_{\lambda\kappa}) \Big] + \mathcal{O}(2) \\ &= \frac{1}{6} \Big[\gamma^{\alpha\beta} + 2\overline{\varphi}^{\alpha\beta} + \overline{\overline{\varphi}}^{\alpha\beta} - 3\gamma^{\alpha\beta} - 3\overline{\varphi}^{\alpha\beta} \\ &\quad - \gamma^{\alpha\beta} \overline{\varphi} - \gamma^{\alpha\beta} \overline{\overline{\varphi}} \Big] + \mathcal{O}(2) \\ &= \frac{1}{6} \Big[-2\gamma^{\alpha\beta} - \overline{\varphi}^{\alpha\beta} + \overline{\overline{\varphi}}^{\alpha\beta} - \gamma^{\alpha\beta} \overline{\varphi} - \gamma^{\alpha\beta} \overline{\overline{\varphi}} \Big] + \mathcal{O}(2) \end{aligned}$$

$$\boxed{p^{\alpha\beta 00} = -\frac{1}{3} \gamma^{\alpha\beta} + \frac{1}{6} \Big[-\overline{\varphi}^{\alpha\beta} + \overline{\overline{\varphi}}^{\alpha\beta} - \gamma^{\alpha\beta} \overline{\varphi} - \gamma^{\alpha\beta} \overline{\overline{\varphi}} \Big] + \mathcal{O}(2)}$$

$$p^{\alpha\beta\gamma 0} = \frac{(\det \overline{g})^{3/2}}{12 \psi^2} \Big[\epsilon^{\alpha\mu\nu} \Big(2 \overline{g}^{\beta\gamma} \overline{\overline{g}}_{\mu\tau} \overline{\overline{\overline{g}}}^{\tau}{}_{\nu} + \overline{g}^{\beta\tau} \overline{\overline{g}}_{\nu\tau} \overline{\overline{\overline{g}}}^{\gamma}{}_{\mu} + \overline{g}^{\gamma\tau} \overline{\overline{g}}_{\nu\tau} \overline{\overline{\overline{g}}}^{\beta}{}_{\mu} \Big) + \text{cyclic per. of } \alpha\beta\gamma + \mathcal{O}(\overline{\overline{\overline{g}}}_2) \Big]$$

Note:

$$\det \bar{g} = \det(\gamma + \bar{\varphi}) = 1 + \bar{\varphi} + \mathcal{O}(2)$$

therefore:

$$\frac{(\det \bar{g})^{3/2}}{12\ \psi^2} = \frac{(\det \bar{g})^{3/2}}{12 \det \bar{g}} = \frac{1}{12} + \mathcal{O}(1)$$

We get

1st. order of 2nd frame condition, together with $\varepsilon^{\alpha\mu\nu}$

$$\rho^{\alpha\beta\gamma o} = \frac{1}{12}\Big[\varepsilon^{\alpha\mu\nu}\Big(\cancel{2\gamma^{\beta\gamma}\gamma_{\mu\tau}\bar{\bar{\bar{\varphi}}}^{\tau}{}_{\nu}} + \gamma^{\beta\tau}\gamma_{\nu\tau}\bar{\bar{\bar{\varphi}}}^{\gamma}{}_{\mu} + \gamma^{\gamma\tau}\gamma_{\nu\tau}\bar{\bar{\bar{\varphi}}}^{\beta}{}_{\mu}\Big) + \text{cyclic per. of } \alpha\beta\gamma + \mathcal{O}(2)\Big]$$

$$= \frac{1}{12}\Big[\varepsilon^{\alpha\mu\beta}\bar{\bar{\bar{\varphi}}}^{\gamma}{}_{\mu} + \varepsilon^{\alpha\mu\gamma}\bar{\bar{\bar{\varphi}}}^{\beta}{}_{\mu} + \varepsilon^{\beta\mu\gamma}\bar{\bar{\bar{\varphi}}}^{\alpha}{}_{\mu} + \varepsilon^{\beta\mu\alpha}\bar{\bar{\bar{\varphi}}}^{\gamma}{}_{\mu} + \varepsilon^{\gamma\mu\alpha}\bar{\bar{\bar{\varphi}}}^{\beta}{}_{\mu} + \varepsilon^{\gamma\mu\beta}\bar{\bar{\bar{\varphi}}}^{\alpha}{}_{\mu} + \mathcal{O}(2)\Big]$$

$$= \frac{1}{12}\Big[\cancel{\varepsilon^{\alpha\mu\beta}\bar{\bar{\bar{\varphi}}}^{\gamma}{}_{\mu}} + \cancel{\varepsilon^{\alpha\mu\gamma}\bar{\bar{\bar{\varphi}}}^{\beta}{}_{\mu}} + \cancel{\varepsilon^{\beta\mu\gamma}\bar{\bar{\bar{\varphi}}}^{\alpha}{}_{\mu}} + \cancel{\varepsilon^{\alpha\beta\mu}\bar{\bar{\bar{\varphi}}}^{\gamma}{}_{\mu}} + \cancel{\varepsilon^{\alpha\gamma\mu}\bar{\bar{\bar{\varphi}}}^{\beta}{}_{\mu}} + \cancel{\varepsilon^{\beta\gamma\mu}\bar{\bar{\bar{\varphi}}}^{\alpha}{}_{\mu}} + \mathcal{O}(2)\Big] + \mathcal{O}(2)$$

$$= \mathcal{O}(2)$$

$$\boxed{\rho^{\alpha\beta\gamma o} = \mathcal{O}(2)}$$

$$\rho^{\alpha\beta\gamma\delta} = \frac{(\det \bar{g})^2}{12\ \psi^2}\Big[\varepsilon^{\alpha\mu\nu}\varepsilon^{\beta\omega\tau}\bar{\bar{g}}_{\mu\omega}\bar{g}^{\gamma\delta}\bar{\bar{g}}_{\nu\tau} + 5 \text{ per. of } \alpha\beta\gamma\delta + \mathcal{O}(\bar{\bar{g}}^2)\Big]$$

$$= \frac{1}{12}\det\bar{g}\Big[\varepsilon^{\alpha\mu\nu}\varepsilon^{\beta\omega\tau}(\gamma_{\mu\omega} + \bar{\bar{\varphi}}_{\mu\omega})(\gamma^{\gamma\delta} + \bar{\varphi}^{\gamma\delta})(\gamma_{\nu\tau} + \bar{\bar{\varphi}}_{\nu\tau})$$
$$+ 5 \text{ per. of } \alpha\beta\gamma\delta + \mathcal{O}(2)\Big]$$

$$= \frac{1}{12}(1+\bar{\varphi})\Big[\varepsilon^{\alpha\mu\nu}\varepsilon^{\beta\omega\tau}\big(\gamma_{\mu\omega}\gamma_{\nu\tau}\gamma^{\gamma\delta} + 2\bar{\bar{\varphi}}_{\mu\omega}\gamma^{\gamma\delta}\gamma_{\nu\tau} + \gamma_{\mu\omega}\bar{\varphi}^{\gamma\delta}\gamma_{\nu\tau} + \text{per.} + \mathcal{O}(2)\Big]$$

$$= \frac{1}{12}(1+\bar{\varphi})\Big[2\gamma^{\alpha\beta}(\gamma^{\gamma\delta} + \bar{\varphi}^{\gamma\delta}) + 2(\gamma^{\alpha\beta}\gamma^{\mu\omega} - \gamma^{\alpha\omega}\gamma^{\beta\mu})\bar{\bar{\varphi}}_{\mu\omega}\gamma^{\gamma\delta} + \text{per.} + \mathcal{O}(2)\Big]$$

$$= \frac{1}{12}(1+\bar{\varphi})\Big[2\gamma^{\alpha\beta}\gamma^{\gamma\delta} + 2\gamma^{\alpha\beta}\bar{\varphi}^{\gamma\delta} + 2\gamma^{\alpha\beta}\gamma^{\gamma\delta}\bar{\bar{\varphi}} - 2\bar{\bar{\varphi}}^{\alpha\beta}\gamma^{\gamma\delta} + \text{per.} + \mathcal{O}(2)\Big]$$

$$= \frac{1}{6}\Big[\gamma^{\alpha\beta}\gamma^{\gamma\delta} + \bar{\varphi}\gamma^{\alpha\beta}\gamma^{\gamma\delta} + \gamma^{\alpha\beta}\bar{\varphi}^{\gamma\delta} + \gamma^{\alpha\beta}\gamma^{\gamma\delta}\bar{\bar{\varphi}} - \bar{\bar{\varphi}}^{\alpha\beta}\gamma^{\gamma\delta} + \text{per.} + \mathcal{O}(2)\Big]$$

$$\boxed{P^{\alpha\beta\gamma\delta} = \gamma^{(\alpha\beta}\gamma^{\gamma\delta)} + \bar{\varphi}\gamma^{(\alpha\beta}\gamma^{\gamma\delta)} + \gamma^{(\alpha\beta}\bar{\varphi}^{\gamma\delta)} + \gamma^{(\alpha\beta}\gamma^{\gamma\delta)}\bar{\bar{\varphi}} - \bar{\bar{\varphi}}^{(\alpha\beta}\gamma^{\gamma\delta)} + \mathcal{O}(2)}$$

Next, must evaluate P in the general coframe $\{K, \tilde{\varepsilon}^\alpha\}$ which is dual to $\{\dot{X}, e_\alpha\}$.

In terms of $\{n, \varepsilon^\alpha\}$, we have $\boxed{K = \frac{1}{N} n}$, $\boxed{\tilde{\varepsilon}^\alpha = -\frac{1}{N} N^\alpha n + \varepsilon^\alpha}$
(this is to be found in Nadiles thesis, p. 83)

Further, set $\boxed{N = 1 + A}$, $A, N^\alpha \sim \mathcal{O}(1)$

To further distinguish to coefficients of P evaluated in the frame $\{n, \varepsilon^\alpha\}$ and in the frame $\{\kappa, \tilde{\varepsilon}^\alpha\}$, we write $\tilde{P}$ for the latter.

$$\tilde{P}^{0000} = P(\kappa, \kappa, \kappa, \kappa) = P(n, n, n, n)\frac{1}{N^4} = \frac{1}{(1+A)^4} P^{0000}$$

$$= 1 - 4A + \mathcal{O}(2)$$

$$\tilde{P}^{\alpha 000} = P(\tilde{\varepsilon}^\alpha, \kappa, \kappa, \kappa) = P(n, n, n, n)\left(-\frac{N^\alpha}{N^4}\right) + \frac{1}{N^3} P(\varepsilon^\alpha, n, n, n)$$

$$= -N^\alpha \frac{1}{N^4} P^{0000} + \cancel{\frac{1}{N^3} P^{\alpha 000}}$$

$$= -N^\alpha + \mathcal{O}(2)$$

$$\tilde{P}^{\alpha\beta 00} = P(\tilde{\varepsilon}^\alpha, \tilde{\varepsilon}^\beta, \kappa, \kappa) = P\left(-\frac{N^\alpha}{N} n + \varepsilon^\alpha, -\frac{N^\beta}{N} n + \varepsilon^\beta, \frac{n}{N}, \frac{n}{N}\right)$$

$$= \cancelto{\mathcal{O}(2)}{\frac{N^\alpha N^\beta}{N^4}} P(n, n, n, n) - \frac{N^\beta}{N^3} P(\cancelto{P^{\alpha 000} = 0}{\varepsilon^\alpha, n, n, n})$$

$$- \frac{N^\alpha}{N^3} P(\cancelto{P^{0\beta 00} = 0}{n, \varepsilon^\beta, n, n}) + P(\varepsilon^\alpha, \varepsilon^\beta, n, n)\frac{1}{N^2}$$

$$= (1 - 2A)\, P^{\alpha\beta 00}$$

$$= -\frac{1}{3}\gamma^{\alpha\beta} + \frac{2}{3} A \gamma^{\alpha\beta} + \frac{1}{6}\left[-\bar{\varphi}^{\alpha\beta} + \bar{\bar{\varphi}}^{\alpha\beta} - \gamma^{\alpha\beta}\bar{\varphi} - \gamma^{\alpha\beta}\bar{\bar{\bar{\varphi}}}\right]$$

$$+ \mathcal{O}(2)$$

$$\tilde{P}^{\alpha\beta\gamma 0} = P(\tilde{\varepsilon}^\alpha, \tilde{\varepsilon}^\beta, \tilde{\varepsilon}^\gamma, \kappa) = P\left(-\frac{N^\alpha}{N} n + \varepsilon^\alpha, -\frac{N^\beta}{N} n + \varepsilon^\beta, -\frac{N^\gamma}{N} n + \varepsilon^\gamma, \frac{n}{N}\right)$$

$$= -\frac{N^\alpha N^\beta N^\gamma}{N^4} \cancelto{\mathcal{O}(3)}{P(n, n, n, n)} + \cancelto{\mathcal{O}(2)}{\mathcal{O}(N^{\alpha 2})}$$

$$- 3 \frac{N^{(\alpha}}{N^2} P(n, \varepsilon^\beta, \varepsilon^{\gamma)}, n) + \frac{1}{N} \cancelto{P^{\alpha\beta\gamma 0} = 0 + \mathcal{O}(2)}{P(\varepsilon^\alpha, \varepsilon^\beta, \varepsilon^\gamma, n)}$$

$$= -3N^{(\alpha}\left(-\frac{1}{3}\gamma^{\beta\gamma)}\right) = N^{(\alpha}\gamma^{\beta\gamma)}$$

$$
\begin{aligned}
\tilde p^{\alpha\beta\gamma\delta} &= p(\tilde\varepsilon^{\alpha}, \tilde\varepsilon^{\beta}, \tilde\varepsilon^{\gamma}, \tilde\varepsilon^{\delta}) \\
&= p\left(-\frac{N^{\alpha}}{N}n + \varepsilon^{\alpha}, -\frac{N^{\beta}}{N}n + \varepsilon^{\beta}, -\frac{N^{\gamma}}{N}n + \varepsilon^{\gamma}, -\frac{N^{\delta}}{N}n + \varepsilon^{\delta}\right) \\
&= \cancel{-4\frac{N^{(\alpha}}{N}p^{\beta\gamma\delta)0}} + p^{\alpha\beta\gamma\delta} \\
&= \gamma^{(\alpha\beta}\gamma^{\gamma\delta)} + \bar\varphi\,\gamma^{(\alpha\beta}\gamma^{\gamma\delta)} + \gamma^{(\alpha\beta}\,\bar\varphi\,\gamma^{\gamma\delta)} \\
&\quad + \gamma^{(\alpha\beta}\gamma^{\gamma\delta)}\,\bar{\bar\varphi} - \bar{\bar\varphi}^{(\alpha\beta}\gamma^{\gamma\delta)} + \mathcal{O}(2)
\end{aligned}
$$

Can now provide the coefficients of p in the general frame $\{n, \tilde\varepsilon^{\alpha}\}$ to linear order:

$$
\boxed{
\begin{aligned}
\tilde p^{0000} &= 1 - 4A + \mathcal{O}(2) \\
\tilde p^{\alpha 000} &= -N^{\alpha} + \mathcal{O}(2) \\
\tilde p^{\alpha\beta 00} &= -\frac{1}{3}\gamma^{\alpha\beta} + \frac{2}{3}A\gamma^{\alpha\beta} + \frac{1}{6}\left[-\bar\varphi^{\alpha\beta} + \bar{\bar\varphi}^{\alpha\beta} - \gamma^{\alpha\beta}\bar\varphi - \gamma^{\alpha\beta}\bar{\bar\varphi}\right] + \mathcal{O}(2) \\
\tilde p^{\alpha\beta\gamma 0} &= N^{(\alpha}\gamma^{\beta\gamma)} + \mathcal{O}(2) \\
\tilde p^{\alpha\beta\gamma\delta} &= \gamma^{(\alpha\beta}\gamma^{\gamma\delta)} + \bar\varphi\,\gamma^{(\alpha\beta}\gamma^{\gamma\delta)} + \gamma^{(\alpha\beta}\,\bar\varphi\,\gamma^{\gamma\delta)} + \gamma^{(\alpha\beta}\gamma^{\gamma\delta)}\bar{\bar\varphi} \\
&\quad - \bar{\bar\varphi}^{(\alpha\beta}\gamma^{\gamma\delta)} + \mathcal{O}(2)
\end{aligned}
}
$$

We see that

$$
\left.
\begin{aligned}
\tilde p^{0000} &= 1 + \mathcal{O}(1) &&= \eta^{(00}\eta^{00)} \\
\tilde p^{\alpha 000} &= 0 + \mathcal{O}(1) &&= \eta^{(\alpha 0}\eta^{00)} \\
\tilde p^{\alpha\beta 00} &= -\frac{1}{3}\gamma^{\alpha\beta} + \mathcal{O}(1) &&\overset{*}{=} \eta^{(\alpha\beta}\eta^{00)} \\
\tilde p^{\alpha\beta\gamma 0} &= 0 + \mathcal{O}(1) &&= \eta^{(\alpha\beta}\eta^{\gamma 0)} \\
\tilde p^{\alpha\beta\gamma\delta} &= \gamma^{(\alpha\beta}\gamma^{\gamma\delta)} + \mathcal{O}(1) &&= \eta^{(\alpha\beta}\eta^{\gamma\delta)}
\end{aligned}
\right\}
\quad
\boxed{\tilde p^{abcd} = \eta^{(ab}\eta^{cd)} + \mathcal{O}(1)}
$$

*: Since

$$\eta^{(\alpha\beta}\eta^{00)} = \frac{1}{4\cdot 3}\Big[\eta^{\alpha\beta}\eta^{00} + \cancel{\eta^{\alpha 0}\eta^{\beta 0}} + \cancel{\eta^{\alpha 0}\eta^{0\beta}} + \cancel{\eta^{0\alpha}\eta^{\beta 0}} + \cancel{\eta^{0\alpha}\eta^{0\beta}} + \eta^{00}\eta^{\alpha\beta} + \alpha \leftrightarrow \beta\Big]$$

$$= -\frac{1}{3}\gamma^{\alpha\beta}$$

Using this information, the Lagrangian can now be determined.

define $$\boxed{\tilde{P}^{abcd} = \eta^{(ab}\eta^{cd)} + \Sigma^{abcd}} \quad , \quad \Sigma \sim O(1)$$

and insert that into S via P and C^{-1}. Arrive at

These indices are pulled with η!

$$S = \int \mathcal{L}\, d\tau \quad , \quad \boxed{\mathcal{L} = m\left[\sqrt{\eta_{ab}\dot{x}^a\dot{x}^b} - \frac{1}{4}\frac{\Sigma_{abcd}\,\dot{x}^a\dot{x}^b\dot{x}^c\dot{x}^d}{\sqrt{\eta_{ab}\dot{x}^a\dot{x}^b}^{\,3}}\right]}$$

See Frederics calculation: lagrangian-point-particle-AM

Next, insert $\dot{x}^a$ of the point particle in the general frame.
In coordinates $\{t, y^\alpha\}$, and parametrised by the embedding parameter t, the velocity of the particle is $\dot{x}^a = \begin{pmatrix}1\\ \dot{\lambda}^\alpha\end{pmatrix}$, where $\lambda^\alpha(t)$ is the path of the particle in the embedded space.
Inserting $\dot{x}^a = \begin{pmatrix}1\\ \dot{\lambda}^\alpha\end{pmatrix}$ into $\mathcal{L}$ yields

$$\mathcal{L} = m\left[\sqrt{1-\dot{\lambda}^2} - \frac{1}{4}\frac{\sum_{n=0}^{4}\binom{4}{n}\Sigma_{\alpha_1\ldots\alpha_n 0\ldots 0}\,\dot{\lambda}^{\alpha_1}\ldots\dot{\lambda}^{\alpha_n}}{\sqrt{1-\dot{\lambda}^2}^{\,3}}\right]$$

where $\dot{\lambda}^2 = \dot{\lambda}^\alpha\dot{\lambda}^\beta\gamma_{\alpha\beta}$

Now, do a Legendre-trafo to determine the Hamiltonian.

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}$$

$$\boxed{p_\alpha = m\left[-\frac{\dot{x}_\alpha}{\sqrt{1-\dot{x}^2}} + \mathcal{O}(1)\right]}$$

Expand $\dot{x}_\alpha$: $\boxed{\dot{x}_\alpha = \dot{x}_\alpha^0 + \dot{x}_\alpha^1}$ where $\dot{x}^1_\alpha \sim \mathcal{O}(1)$

Find $\dot{x}^0_\alpha(p)$. For that, assume that p_α is $\mathcal{O}(0)$.

$$p_\alpha = -m\frac{\dot{x}^0_\alpha}{\sqrt{1-\dot{x}_0^2}} \iff \boxed{\dot{x}^0_\alpha = -\frac{p_\alpha}{\sqrt{m^2+p^2}} =: -\frac{p_\alpha}{E}}$$

Proof: simply plug in $\dot{x}^0_\alpha$ into $p_\alpha(\dot{x}^0_\alpha)$ $\square$

Now, insert ansatz into H:

$$H = p_\alpha \dot{x}^\alpha - \mathcal{L}$$

$$= p_\alpha \dot{x}^\alpha_0 + p_\alpha \dot{x}^\alpha_1 - m\left[\sqrt{1-\dot{x}^2} - \frac{1}{4}\frac{\sum_{m=0}^{4}\binom{4}{m}\varepsilon_{\alpha_1\ldots\alpha_m 0\ldots 0}\,\dot{x}^{\alpha_1}\ldots\dot{x}^{\alpha_m}}{\sqrt{1-\dot{x}^2}^{\,3}}\right.$$

now, expand $\sqrt{1-\dot{x}^2} = \sqrt{1-\dot{x}_0^2 - 2\dot{x}^\alpha_0\dot{x}^1_\alpha} + \mathcal{O}(2)$

$$= \sqrt{1-\dot{x}_0^2}\left(1 - \frac{\dot{x}^\alpha_0}{1-\dot{x}_0^2}\dot{x}^1_\alpha\right) + \mathcal{O}(2)$$

$$= \frac{m}{E}\left(1 + \frac{E}{m^2}p^\alpha\dot{x}^1_\alpha\right) + \mathcal{O}(2)$$

$$= \frac{m}{E} + \frac{p^\alpha}{m}\dot{x}^1_\alpha + \mathcal{O}(2)$$

Insert this into H:

$$H = -\frac{p^2}{E} + p_\alpha \dot{\lambda}^\alpha_1 - \left[\frac{m^2}{E} + p^\alpha \dot{\lambda}^1_\alpha - \frac{1}{4} \frac{\sum_{n=0}^{4} \binom{4}{n} \Sigma_{\alpha_1 \ldots \alpha_n 0 \ldots 0} \dot{\lambda}^{\alpha_1} \ldots \dot{\lambda}^{\alpha_n}}{\sqrt{1-\dot{\lambda}^2}^{\,3}} \right]$$

$$= -E + \frac{1}{4} \frac{E^3}{m^2} \sum_{n=0}^{4} \binom{4}{n} \Sigma_{\alpha_1 \ldots \alpha_n 0 \ldots 0} \left(-\frac{p^{\alpha_1}}{E}\right) \ldots \left(-\frac{p^{\alpha_n}}{E}\right)$$

Hamiltonian:

$$\rightsquigarrow \boxed{H = E \left[-1 + \frac{1}{4} \frac{E^2}{m^2} \sum_{n=0}^{4} \binom{4}{n} \Sigma_{\alpha_1 \ldots \alpha_n 0 \ldots 0} \left(-\frac{p^{\alpha_1}}{E}\right) \ldots \left(-\frac{p^{\alpha_n}}{E}\right) \right]}$$

next thing to do is inserting the $\Sigma_{\alpha_1 \ldots \alpha_n 0 \ldots 0}$ in the general frame. We get

Note! Indices are pulled with $\eta_{\alpha\beta} = -\gamma_{\alpha\beta}$

$$H = -E + \frac{1}{4} \frac{E^3}{m^2} \Big[-4A - 4 \frac{N^\alpha p_\alpha}{E}$$

$$+ 6 \frac{2}{3} A \gamma^{\alpha\beta} p_\alpha p_\beta \frac{1}{E^2}$$

$$+ \left[-\bar{\varphi}^{\alpha\beta} + \bar{\bar{\varphi}}^{\alpha\beta} - \gamma^{\alpha\beta} \bar{\varphi} - \gamma^{\alpha\beta} \bar{\bar{\varphi}} \right] \frac{p_\alpha}{E} \frac{p_\beta}{E}$$

$$+ 4 N^\alpha p_\alpha p^2 \frac{1}{E^3}$$

$$+ \left[\bar{\varphi} p^2 + \bar{\varphi}^{\alpha\beta} p_\alpha p_\beta + p^2 \bar{\bar{\varphi}} - \bar{\bar{\varphi}}^{\alpha\beta} p_\alpha p_\beta \right] \frac{p^2}{E^4} \Big]$$

$$= -E + \frac{1}{4}\frac{E^3}{m^2}\Bigg[+ 4A\left(-1+\frac{p^2}{E^2}\right) - 4N^\alpha p_\alpha\left(\frac{1}{E} - \frac{p^2}{E^3}\right)$$

$$+ \left(\bar{\varphi}^{\alpha\beta} - \bar{\bar{\varphi}}^{\alpha\beta}\right)\left[-\frac{p_\alpha}{E}\frac{p_\beta}{E} + \frac{p_\alpha\, p_\beta\, p^2}{E\;E\;E^2}\right]$$

$$+ \left(\bar{\varphi} + \bar{\bar{\varphi}}\right)\left[-\frac{p^2}{E^4} + \frac{p^4}{E^4}\right]\Bigg]$$

$$= -E + \frac{1}{4}\frac{E^3}{m^2}\Bigg[- 4A\frac{m^2}{E^2} - 4N^\alpha p_\alpha \frac{m^2}{E^3}$$

$$- \left(\bar{\varphi}^{\alpha\beta} - \bar{\bar{\varphi}}^{\alpha\beta}\right)\frac{p_\alpha p_\beta}{E^2}\frac{m^2}{E^2}$$

$$- \left(\bar{\varphi} + \bar{\bar{\varphi}}\right)\frac{p^2 m^2}{E^4}\Bigg]$$

$$\boxed{H = -E - EA - N^\alpha p_\alpha - \frac{1}{4}E\left[\left(\bar{\varphi}^{\alpha\beta} - \bar{\bar{\varphi}}^{\alpha\beta}\right)\frac{p_\alpha p_\beta}{E^2} + \left(\bar{\varphi} + \bar{\bar{\varphi}}\right)\frac{p^2}{E^2}\right]}$$

This is exactly the Hamiltonian that I had been calculating bevor!