In-Class 5

Wednesday, November 27, 2019 3:05 PM

Problem 5: Given N numbers $x_1, ..., x_N \in \mathbb{R}$, construct a <u>minimization</u> problem such that its optimal value is $\max(x_1, \ldots, x_N)$ and derive the Lagrange dual problem.

MIN $\rightarrow$ $p^* = d^*$ MAX

(P) $\min_{b \in \mathbb{R}} b$ $\quad x_i - b \leq 0 \quad \forall i$

$\max_{w} \sum_{i=1}^{N} w_i x_i$ (D)

$\sum_{i=1}^{N} w_i = 1$

$w_i \geq 0$

Derive the dual for (P)

- $L(b, w) = b + \sum_{i=1}^{N} w_i (x_i - b) = \left(1 - \sum_{i=1}^{N} w_i\right) b + \sum_{i=1}^{N} w_i x_i$
- $g(w) = \min_{b} L(b, w) = \min_{b} \left(1 - \sum w_i\right) b + \sum w_i x_i$

$$g(w) = \begin{cases} -\infty, & \text{if } 1 - \sum_{i=1}^{N} w_i \neq 0 \\ \sum_{i=1}^{N} w_i x_i, & \text{if } \sum_{i=1}^{N} w_i = 1 \end{cases}$$

- $d^* = \max_{w \geq 0} g(w) = \max \sum_{i=1}^{N} w_i x_i$

$w \geq 0$

$\sum_{i=1}^{N} w_i = 1$ (D)

$$f(x) = 0 \quad \Longleftrightarrow \quad \begin{aligned} f(x) &\leq 0 \\ -f(x) &\leq 0 \end{aligned}$$

Problem 6: Given N numbers $x_1, ..., x_N \in \mathbb{R}$, construct a maximization problem such that its optimal value is the sum of the k largest values and derive the Lagrange dual problem.

$$\max_{w} \sum_{i=1}^{N} w_i x_i$$

s.t. $\sum_{i=1}^{N} w_i = k$

$-w_i \leq 0$ for $i = 1..N$

$w_i - 1 \leq 0$

$$(-) \min_{w} -\sum_{i=1}^{N} w_i x_i$$

b_+ $\quad \sum_{i=1}^{N} w_i - k \leq 0$

b_- $\quad -\sum_{i=1}^{N} w_i + k \leq 0$

$t \in \mathbb{R}^N$ $\quad -w_i \leq 0$ $\quad i = 1..N$

$s \in \mathbb{R}^N$ $\quad w_i - 1 \leq 0$

- $$L(\underbrace{w}_{"x"}, \underbrace{b_+, b_-, s, t}_{"\alpha"}) = -\sum_{i=1}^{N} w_i x_i + b_+ \left(\sum w_i - k\right) + b_- \left(-\sum w_i + k\right) + \sum s_i (w_i - 1) + \sum t_i (-w_i)$$

$$= \sum_{i=1}^{N} w_i \left(-x_i + b_+ - b_- + s_i - t_i\right) + (b_- - b_+) k - \sum s_i$$

- $g(b_+, b_-, t, s) = \min_w L(\_,\_) =$

$$= \begin{cases} (b_- - b_+)k - \sum s_i & \text{if } -x_i + b_+ - b_- + s_i - t_i = 0 \quad \forall i = 1..N \\ -\infty & \text{otherwise} \end{cases}$$

- $$\max_{b_+, b_-, t, s} (b_- - b_+)k - \sum s_i$$

$$b_+, b_-, t, s \geq 0$$

$$-x_i + b_+ - b_- + s_i - t_i = 0$$

$$\max_{b, t, s} -bk - \sum s_i$$

$$t, s \geq 0$$

$$-x_i + b + s_i = t_i$$

$$\max_{b, s} -bk - \sum s_i$$

$$s \geq 0$$

$$b \geq x_i - s_i$$

$$(-) \min_{b, s} bk + \sum s_i$$

$$s \geq 0$$

$$b \geq x_i - s_i$$

In-Class 7

Wednesday, November 27, 2019 3:10 PM

Problem 7: Given $\boldsymbol{c} \in \mathbb{R}^d$, $\boldsymbol{b} \in \mathbb{R}^M$ and $\boldsymbol{A} \in \mathbb{R}^{M\times d}$, derive the Lagrange dual problem of

$$(P) \quad \begin{aligned} &\text{minimize}_{\boldsymbol{x}} && \boldsymbol{c}^T\boldsymbol{x} \\ &\text{subject to} && \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}\,. \end{aligned}$$

$f(x) = c^T x$

$Ax - b \le 0$

$\mathcal{L} : \mathbb{R}^d \times \mathbb{R}^M \longrightarrow \mathbb{R}$

- $\mathcal{L}(x,\alpha) = c^T x + \underbrace{\alpha^T (Ax - b)}_{= \sum_{i=1}^{M} \alpha_i (a_i x - b_i)}$

$A = \begin{pmatrix} — & a_1 & — \\ & \vdots & \\ — & a_M & — \end{pmatrix} \in \mathbb{R}^{M\times d}$

- $g(\alpha) = \min_x \; c^T x + \alpha^T (Ax - b) =$

$= \min_x \; (\underset{d\times M}{A^T}\underset{M\times 1}{\alpha} + \underset{d\times 1}{c})^T \underset{d\times 1}{x} - b^T \alpha \qquad \sum_j \beta_j x_j$

$g(\alpha) = \begin{cases} -\infty, & \text{if } \underbrace{A^T\alpha + c}_{\beta} \neq 0 \\ -b^T\alpha, & \text{if } A^T\alpha + c = 0 \end{cases}$

- $\max_{\alpha \ge 0} g(\alpha) = \max \; -b^T\alpha$

$A^T\alpha + c = 0$

$\alpha \ge 0$