Lagrange Multipliers

- This method can be applied, when we want to optimize a function *f(x,y,z)*, subject to the constraint *g(x,y,z) = k*.

- To do so we define the Lagrangian of the system:

$$\mathcal{L} = f(x, y, z) + \lambda \left[g(x, y, z) - k \right]$$

- Calculating the derivatives
$$\frac{\partial \mathcal{L}}{\partial x}, \frac{\partial \mathcal{L}}{\partial y}, \frac{\partial \mathcal{L}}{\partial z}, \frac{\partial \mathcal{L}}{\partial \lambda}$$

  and setting them equal to 0 results in a system of equations, we can now solve.

- Note: You can add more constraints with λ_1, λ_2, ...

Lagrange multipliers

Example $f(x, y) = 5x - 3y$ $x^2 + y^2 = 136$

Lagrangian:

$$\mathcal{L} = 5x - 3y + \lambda \left(x^2 + y^2 - 136 \right)$$

Derivatives:

$$\frac{\partial \mathcal{L}}{\partial x} = 5 + 2\lambda x$$

$$\frac{\partial \mathcal{L}}{\partial y} = -3 + 2\lambda y$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - 136$$

Lagrange multipliers

At an optimum the derivatives have to be 0. Thus:

$$-5 = 2\lambda x$$

$$3 = 2\lambda y$$

$$x^2 + y^2 = 136$$

The first two equations can be solved for *x* and *y*:

$$-\frac{5}{2\lambda} = x \qquad \frac{3}{2\lambda} = y$$

Setting this into the 3rd equation yields:

$$\frac{25}{4\lambda^2} + \frac{9}{4\lambda^2} = 136$$

Lagrange multipliers

$$\frac{25}{4\lambda^2} + \frac{9}{4\lambda^2} = 136$$

$$\frac{34}{4\lambda^2} = 136$$

$$\frac{34}{136} = 4\lambda^2$$

$$\frac{1}{4} = 4\lambda^2$$

$$\frac{1}{16} = \lambda^2$$

$$\lambda = \pm\frac{1}{4}$$

Lagrange multipliers

If $\lambda = +\dfrac{1}{4}$

$$x = -\frac{5}{2\lambda} = -10$$

$$y = \frac{3}{2\lambda} = 6$$

If $\lambda = -\dfrac{1}{4}$

$$x = -\frac{5}{2\lambda} = 10$$

$$y = \frac{3}{2\lambda} = -6$$