Digital Signal Processing
ESD5 & IV5-elektro, E24

4. Frequency Response, Difference Equation, and Structures for Digital Filters

Assoc. Prof. Peter Koch, AAU

In our discussion so far we have addressed the theory/method for designing a transfer function $H(z)$ for a digital filter with Infinite Impulse Response (IIR) using either the Impulse Invariant Method ot the Bilinear Transformation.

As we have seen, this is not (necessarily) an easy task – in particular there is a significant risk that something may go wrong when we are manipulating all the equations.

Besides, we are also much interested in figuring out whether the filter designed complies with the initial specifications used to design the analog proto-type filter…

Remember that: Design Specs $\rightarrow \; H(s) \; \rightarrow \; H(z)$

Normally these design specifications relate to

- LP/HP/BP/BS
- $3dB$ frequency f_c and sample frequency f_s
- Pass band and Stop band frequencies (and thus transition bandwidth)
- Pass band and Stop band attenuation and ripples
- Requirement of the phase
- Various numerical requirements such as SNR and/or ADC/DAC word-length
- …others

AALBORG UNIVERSITY
DENMARK

Many of these specifications are possible to check/evaluate once the transfer function $H(z)$ is available.

However, the transfer function itself is not really suited for performing such an evaluation.

Basically, we want to evaluate The Amplitude and The Phase as a function of the frequency.

In the discrete-time domain, the Unit Circle, i.e., $z = e^{j\omega}$, is the ”frequency axis”, and therefore we want to ”evaluate the transfer function $H(z)$ on the Unit Circle”.

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$$

This frequency-continuous function, $H(e^{j\omega})$, is what we denote the ”Frequency Response”. Due to its complex-valued nature, it is not, in itself, really useful…

But it holds all the important and interesting information that we want….

AALBORG UNIVERSITY
DENMARK

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$$

We now derive the modulus and the argument of the frequency response;

$|H(e^{j\omega})|$ The Amplitude Response

$\angle H(e^{j\omega})$ The Phase Response

Although $H(e^{j\omega})$ is often given in a closed form, a typical challenge is that it is quite complicated (high order of the numerator and/or the denominator polynomial) and thus not immediate applicable.

We therefore need to find a practicable expression for $H(e^{j\omega})$ before we can calculate $|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$.

Analysis of the Discrete-Time Systems $H(z)$

Assume that we have designed the filter $H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$

In order to analyse $H(z)$, we first need to find the roots in $B(z)$ and $A(z)$ – but for positive exponents of z.

Basically, this means that we want to find the ZEROs and the POLEs of $H(z)$.

There are three possible scenarios.

1) $M = N$. Extend $B(z)$ and $A(z)$ with z^M.

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{M-k}}{\sum_{k=0}^{M} a_k z^{M-k}}$$

2) $M > N$. Extend $B(z)$ and $A(z)$ with z^M.

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{M-k}}{z^{M-N} \sum_{k=0}^{N} a_k z^{M-k}}$$

In this scenario we get $M - N$ extra roots in $A(z)$, i.e., $M - N$ extra poles in $z = 0$.

3) $M < N$. Extend $B(z)$ and $A(z)$ with z^N.

$$H(z) = \frac{z^{N-M} \sum_{k=0}^{M} b_k z^{N-k}}{\sum_{k=0}^{N} a_k z^{N-k}}$$

In this scenario we get $N - M$ extra roots in $B(z)$, i.e., $N - M$ extra zeros in $z = 0$.

It is now possible to factorize $H(z)$ using positive exponents of z.

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{L}(z - c_k)}{\prod_{k=1}^{L}(z - d_k)} \qquad L = \max\{N, M\}$$

Using this expression, we can now write the frequency response, i.e., $z = e^{j\omega}$;

$$H\left(e^{j\omega}\right) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{L}(e^{j\omega} - c_k)}{\prod_{k=1}^{L}(e^{j\omega} - d_k)}$$

which is then used to derive the Amplitude- and the Phase Responses:

AMPLITUDE RESPONSE

$$|H\left(e^{j\omega}\right)| = |\frac{b_0}{a_0}| \cdot \frac{\prod_{k=1}^{L}|e^{j\omega} - c_k|}{\prod_{k=1}^{L}|e^{j\omega} - d_k|}$$

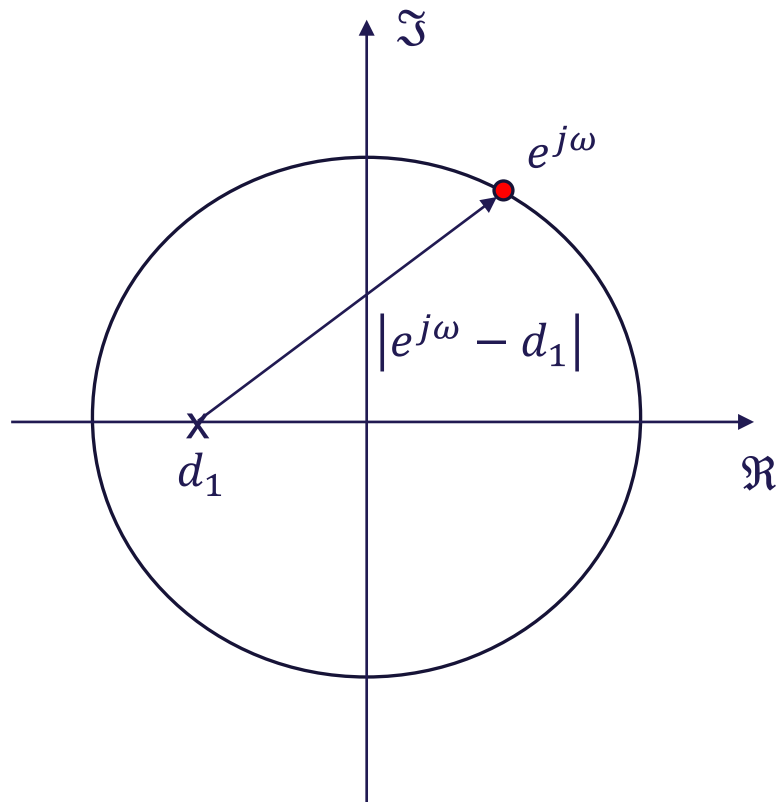
$|e^{j\omega} - c_k|$ and $|e^{j\omega} - d_k|$ represent the length of the vectors defined by the zeros and the poles of $H(z)$.

AALBORG UNIVERSITY
DENMARK

$$|H(e^{j\omega})| = |\frac{b_0}{a_0}| \cdot \frac{\prod_{k=1}^{L} |e^{j\omega} - c_k|}{\prod_{k=1}^{L} |e^{j\omega} - d_k|}$$

An example…

Let's assume there is a real pole at $z = -d_1$



AALBORG UNIVERSITY
DENMARK

$$H\left(e^{j\omega}\right) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{L}(e^{j\omega} - c_k)}{\prod_{k=1}^{L}(e^{j\omega} - d_k)}$$

PHASE RESPONSE

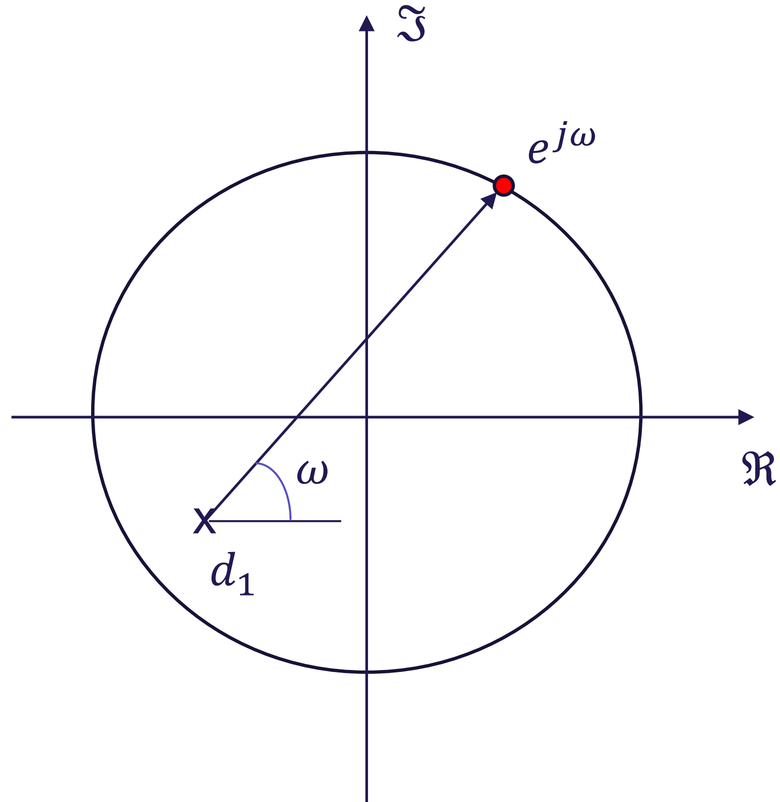
$$\angle H(e^{j\omega}) = \arg\{\frac{b_0}{a_0}\} + \sum_{k=1}^{L} \arg\{e^{j\omega} - c_k\} - \sum_{k=1}^{L} \arg\{e^{j\omega} - d_k\}$$

$\arg\{e^{j\omega} - c_k\}$ and $\arg\{e^{j\omega} - d_k\}$ represent the angles of the vectors defined by the zeros and the poles of $H(z)$.

$$\angle H(e^{j\omega}) = \arg\{\frac{b_0}{a_0}\} + \sum_{k=1}^{L} \arg\{e^{j\omega} - c_k\} - \sum_{k=1}^{L} \arg\{e^{j\omega} - d_k\}$$

An example…

Let's assume there is a complex pole at $z = -d_1$



AALBORG UNIVERSITY
DENMARK

Geometric interpretation of the Amplitude and Phase Responses - a complete example

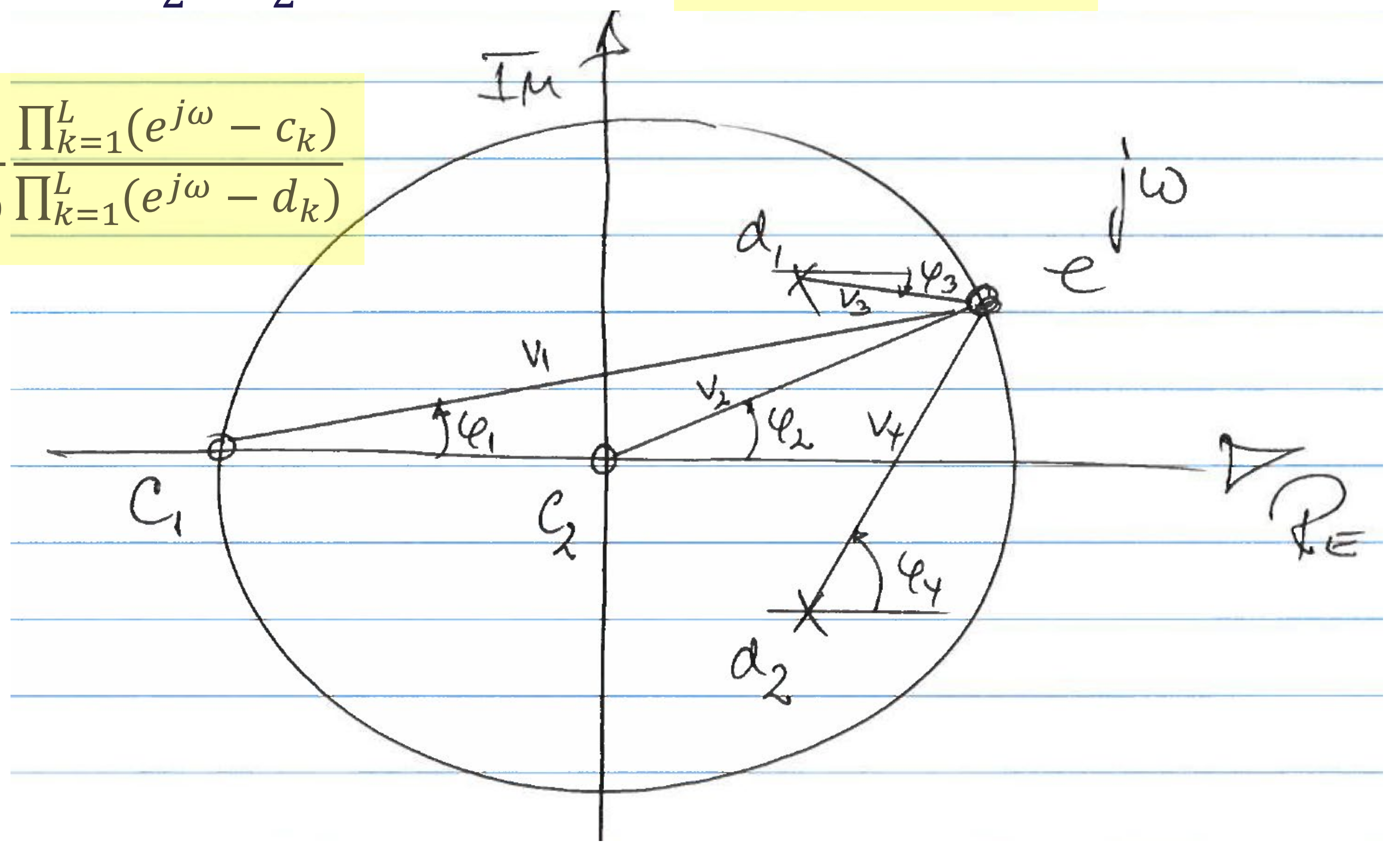
$$H(z) = \frac{1+z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}} = \frac{z^2+z}{z^2-z+\frac{1}{2}} = \frac{1}{1}\cdot\frac{(z-(-1))\cdot(z-0)}{\left(z-\left(\frac{1}{2}+j\frac{1}{2}\right)\right)\cdot\left(z-\left(\frac{1}{2}-j\frac{1}{2}\right)\right)}$$

Zeros: $z=0$ and $z=-1$

Poles: $z=\frac{1}{2}\pm j\frac{1}{2}$

Pole-Zero Plot…!!

$$H\left(e^{j\omega}\right) = \frac{b_0}{a_0}\frac{\prod_{k=1}^{L}\left(e^{j\omega}-c_k\right)}{\prod_{k=1}^{L}\left(e^{j\omega}-d_k\right)}$$



Amplitude Response

$$H\left(e^{j\omega}\right) = \frac{1}{1} \cdot \frac{(e^{j\omega} - (-1)) \cdot (e^{j\omega} - 0)}{(e^{j\omega} - \left(\frac{1}{2} + j\,\frac{1}{2}\right)) \cdot (e^{j\omega} - \left(\frac{1}{2} - j\,\frac{1}{2}\right))}$$

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{b_0}{a_0}\right| \cdot \frac{|V_1||V_2|}{|V_3||V_4|}$$

For each vector V_k we now express $|V_k|$ as a function of ω, e.g., $|V_1|$

$$|V_1| = \sqrt{(\cos(\omega) + 1)^2 + (\sin(\omega))^2}$$

Finally, evaluate $\left|H\left(e^{j\omega}\right)\right|$ for $\omega \in [0; \pi]$

This is done most easily by writing a program…

Phase Response

$$H\left(e^{j\omega}\right) = \frac{1}{1} \cdot \frac{(e^{j\omega} - (-1)) \cdot (e^{j\omega} - 0)}{(e^{j\omega} - \left(\frac{1}{2} + j\frac{1}{2}\right)) \cdot (e^{j\omega} - \left(\frac{1}{2} - j\frac{1}{2}\right))}$$

$$\arg\{H\left(e^{j\omega}\right)\} = \arg\{\frac{b_0}{a_0}\} + \sum_{k=1}^{L} \arg\{e^{j\omega} - c_k\} - \sum_{k=1}^{L} \arg\{e^{j\omega} - d_k\}$$

$$\arg\{H\left(e^{j\omega}\right)\} = \arg\{\frac{1}{1}\} + \sum_{k=1}^{2} \arg\{e^{j\omega} - c_k\} - \sum_{k=1}^{2} \arg\{e^{j\omega} - d_k\} = \varphi_1 + \varphi_2 - \varphi_3 - \varphi_4$$

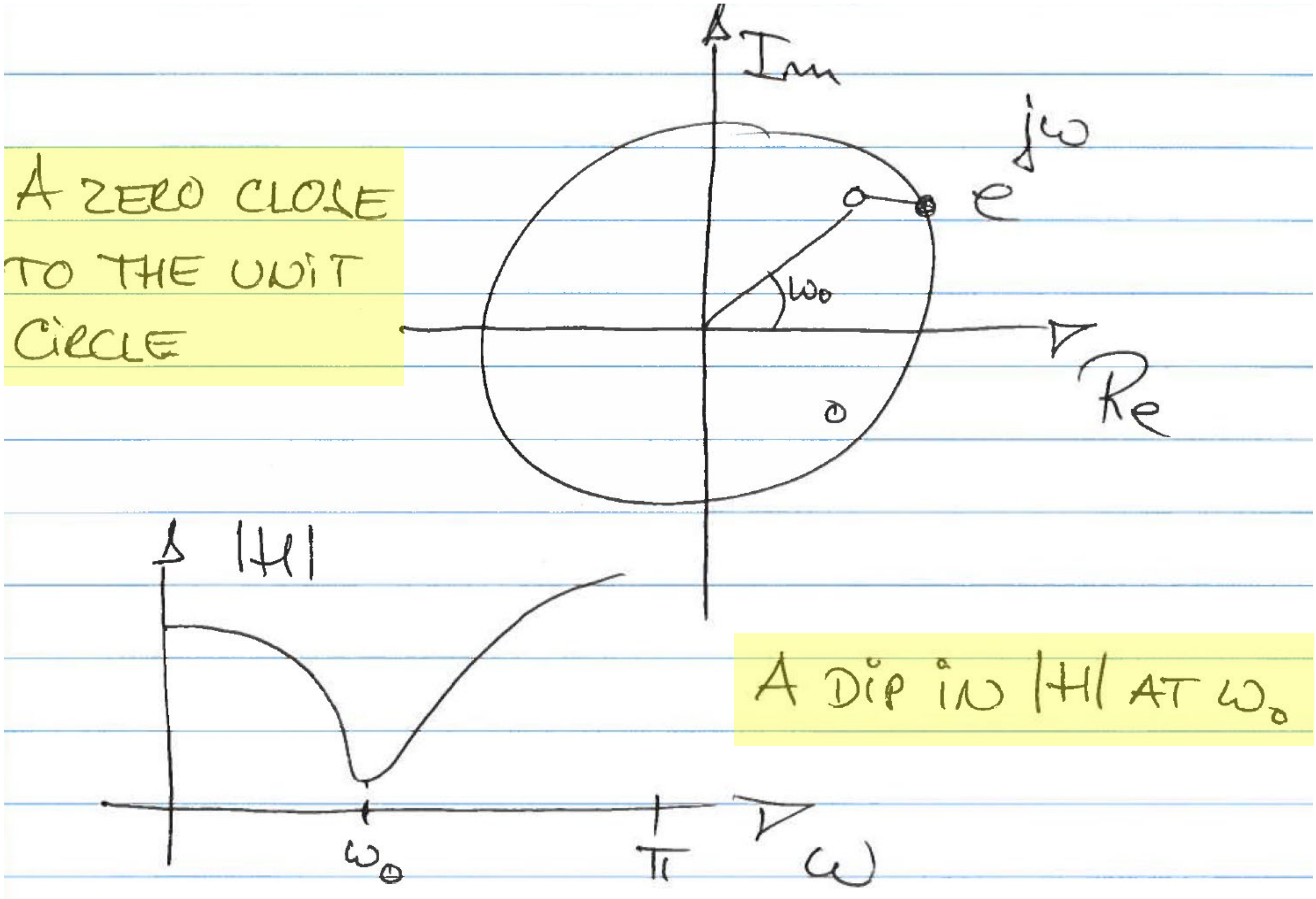
For each vector V_k we now express $\arg\{V_k\}$ as a function of ω, e.g., φ_1

$$\tan\varphi_1 = \frac{\sin(\omega)}{\cos(\omega) + 1} \Rightarrow \varphi_1 = \arctan\{\frac{\sin(\omega)}{\cos(\omega) + 1}\}$$

Finally, evaluate $\arg\{H\left(e^{j\omega}\right)\}$ for $\omega \in [0; \pi]$

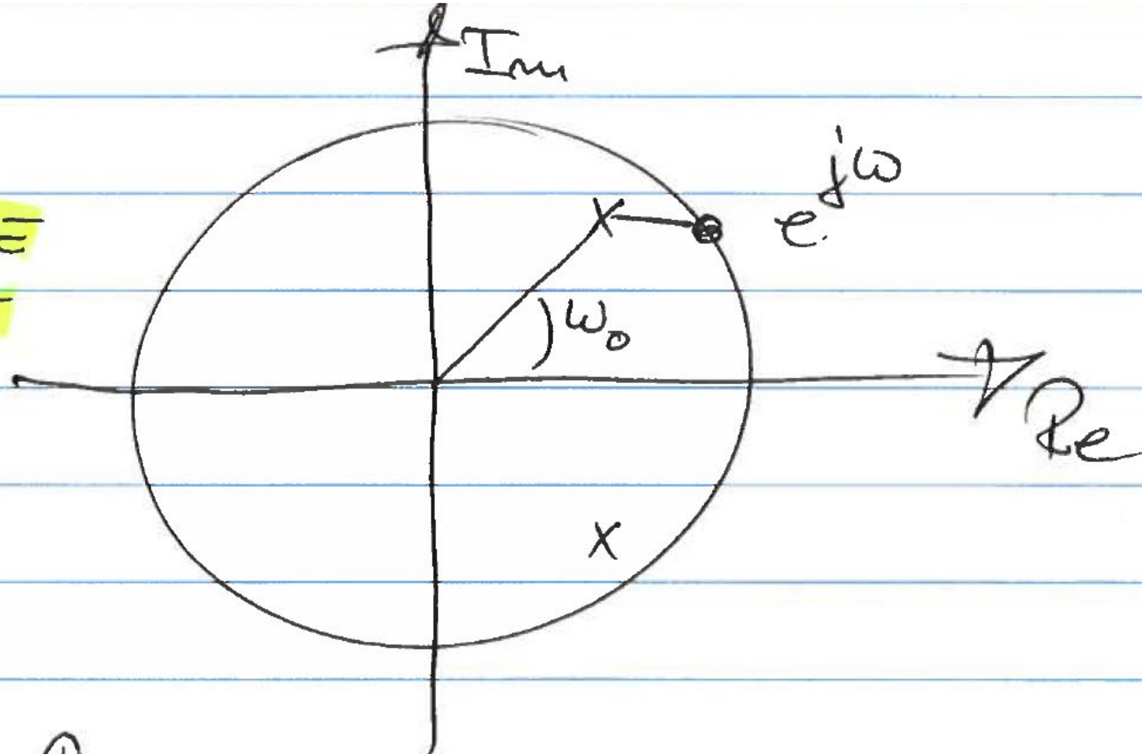
This is done most easily by writing a program…

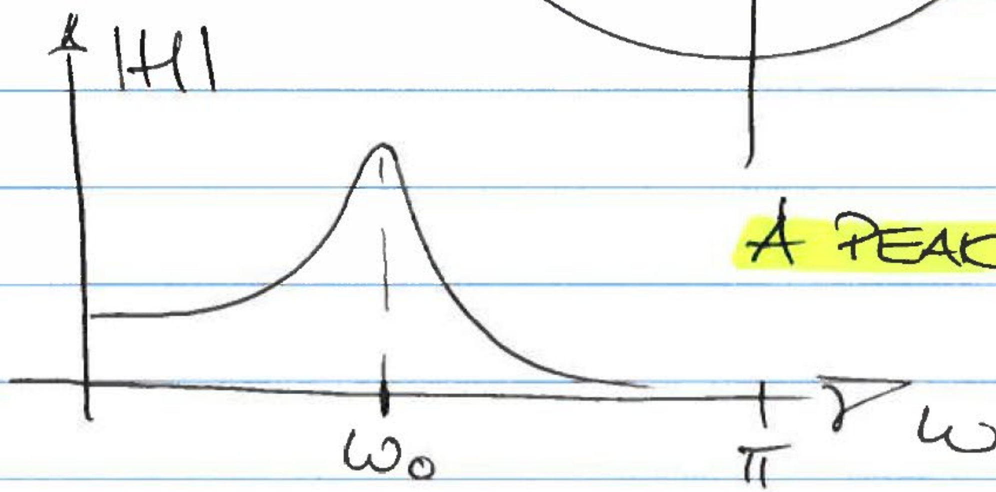
Interpretation of Zero Locations



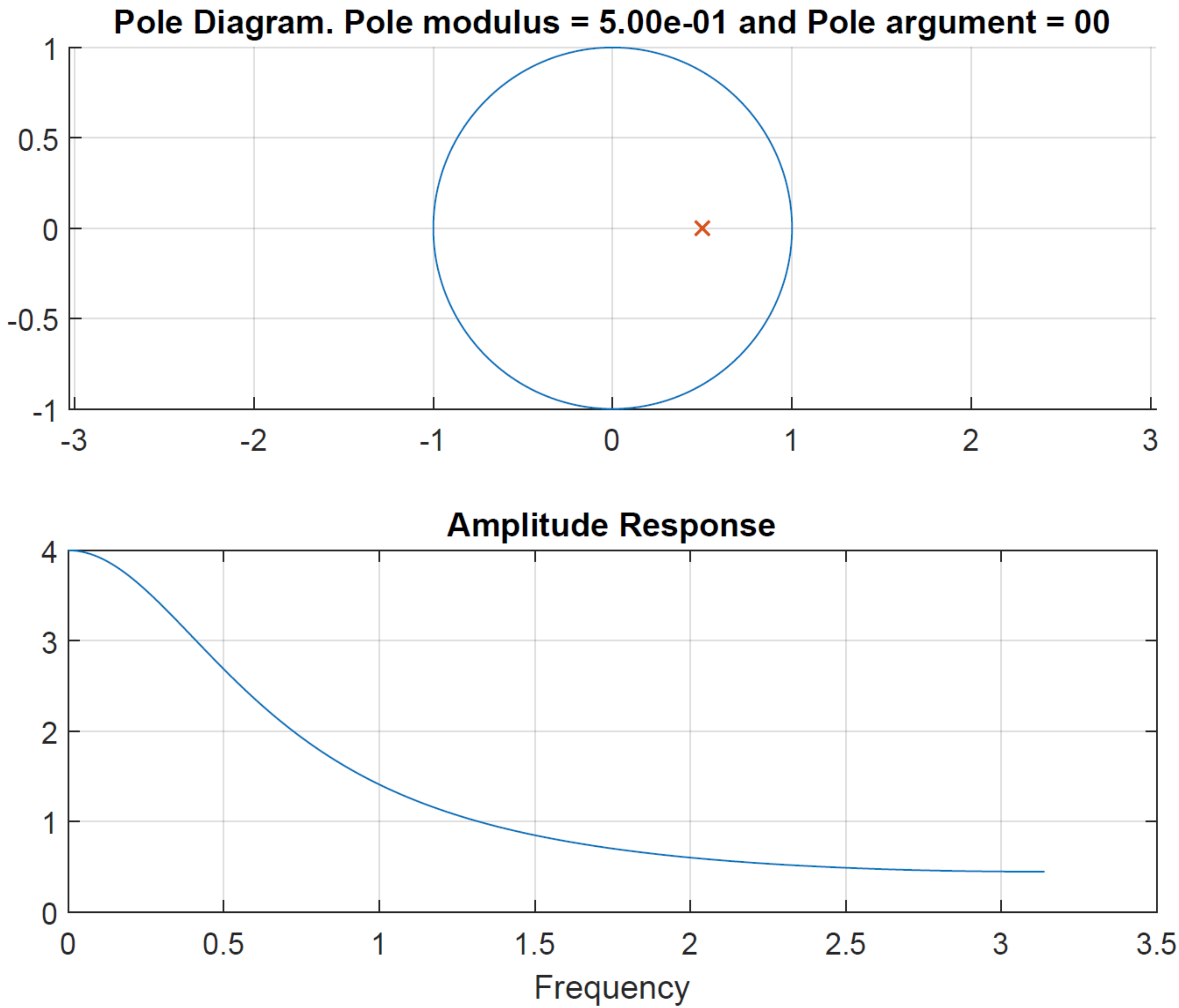
Interpretation of Pole Locations

A POLE CLOSE TO THE UNIT CIRCLE

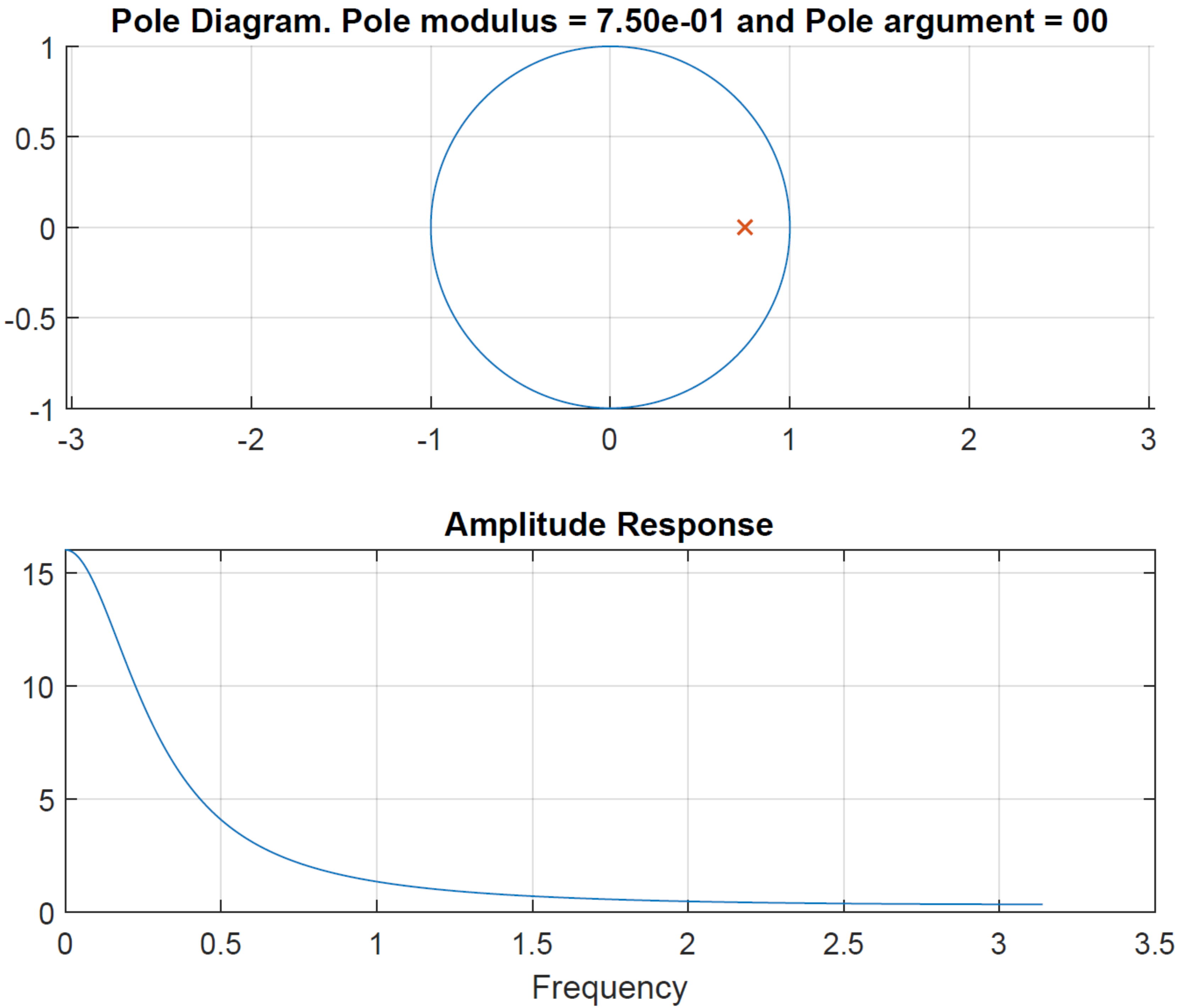




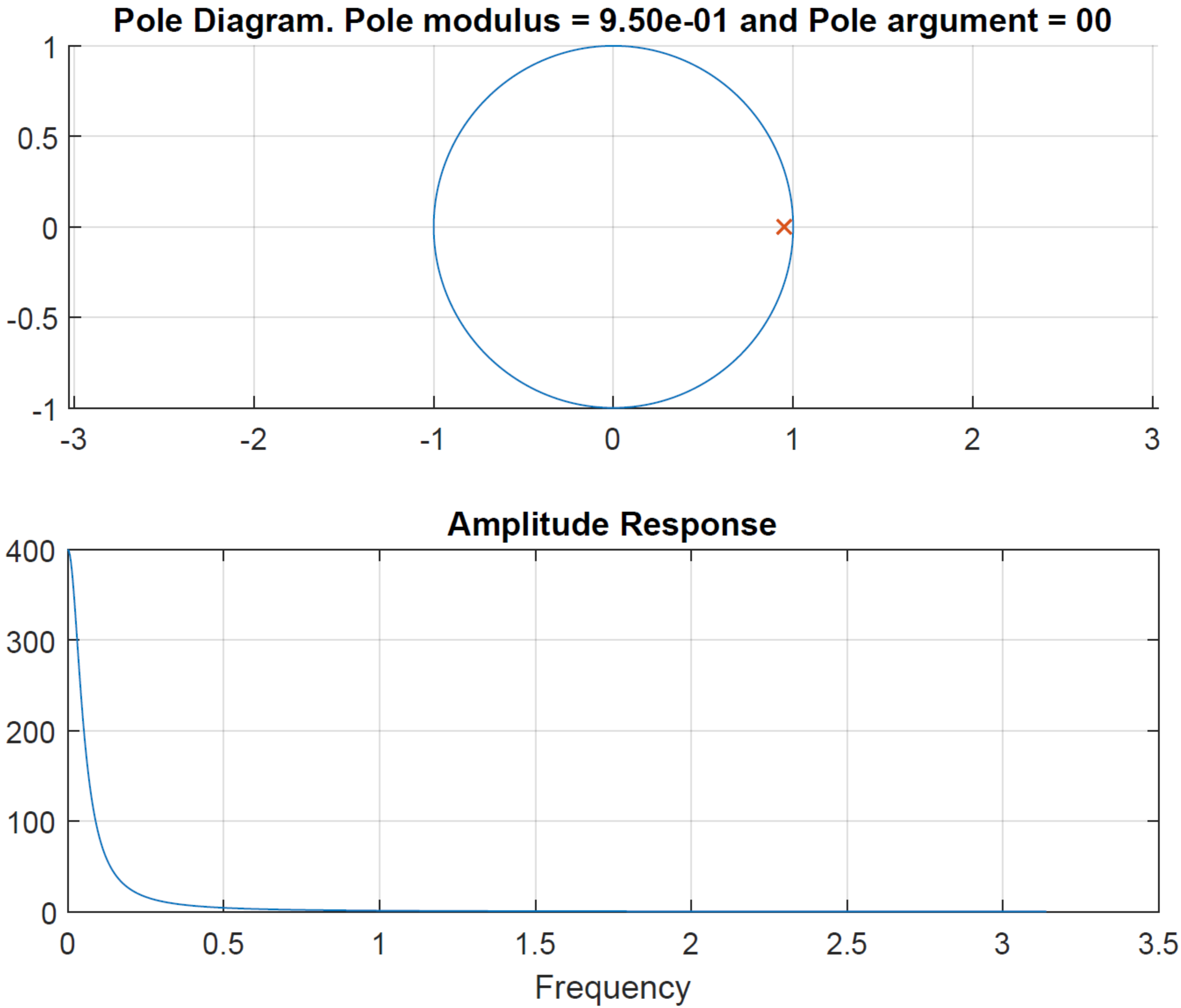
A PEAK IN $|H|$ AT ω_0



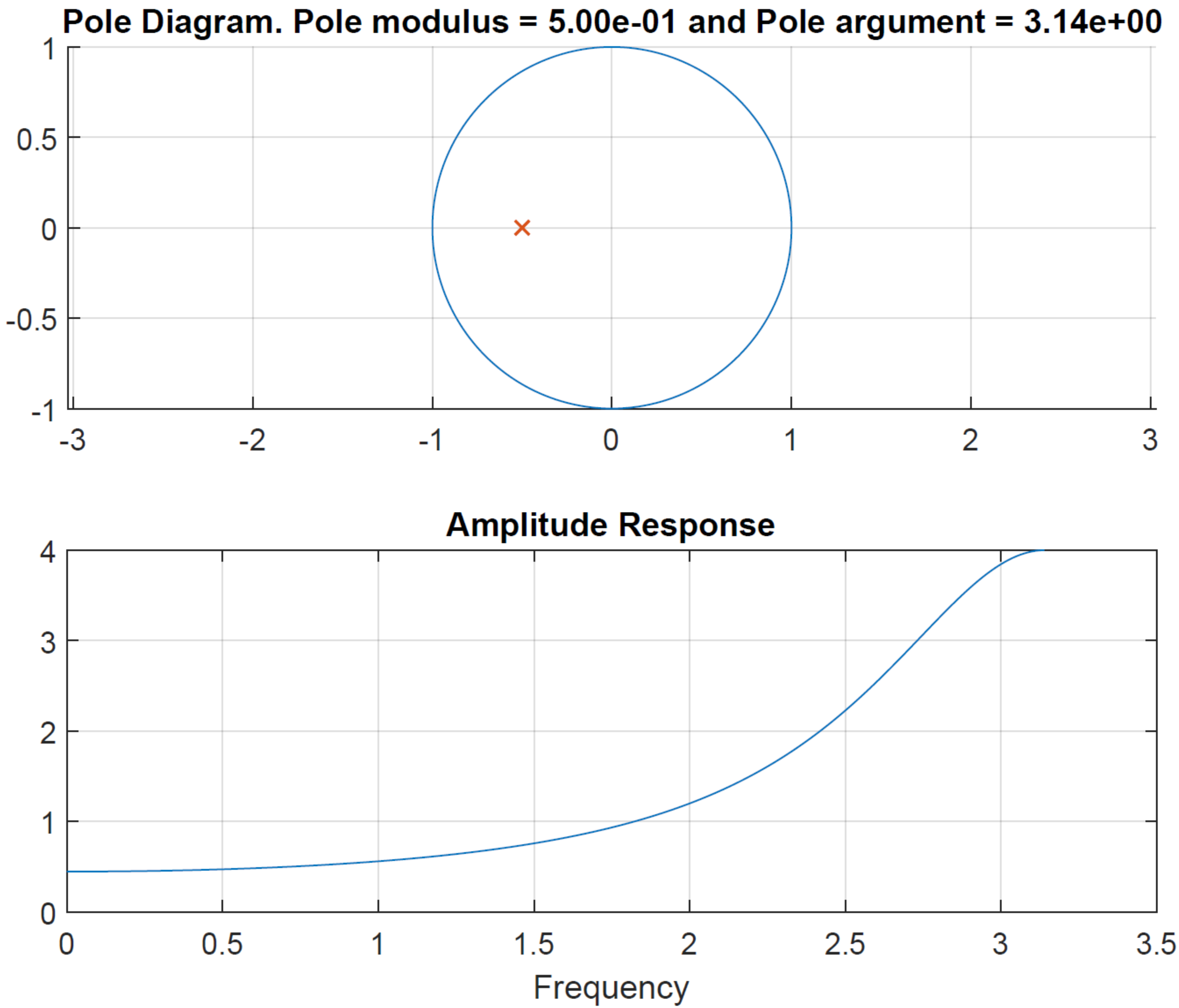
Pole Diagram. Pole modulus = 5.00e-01 and Pole argument = 00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
4
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

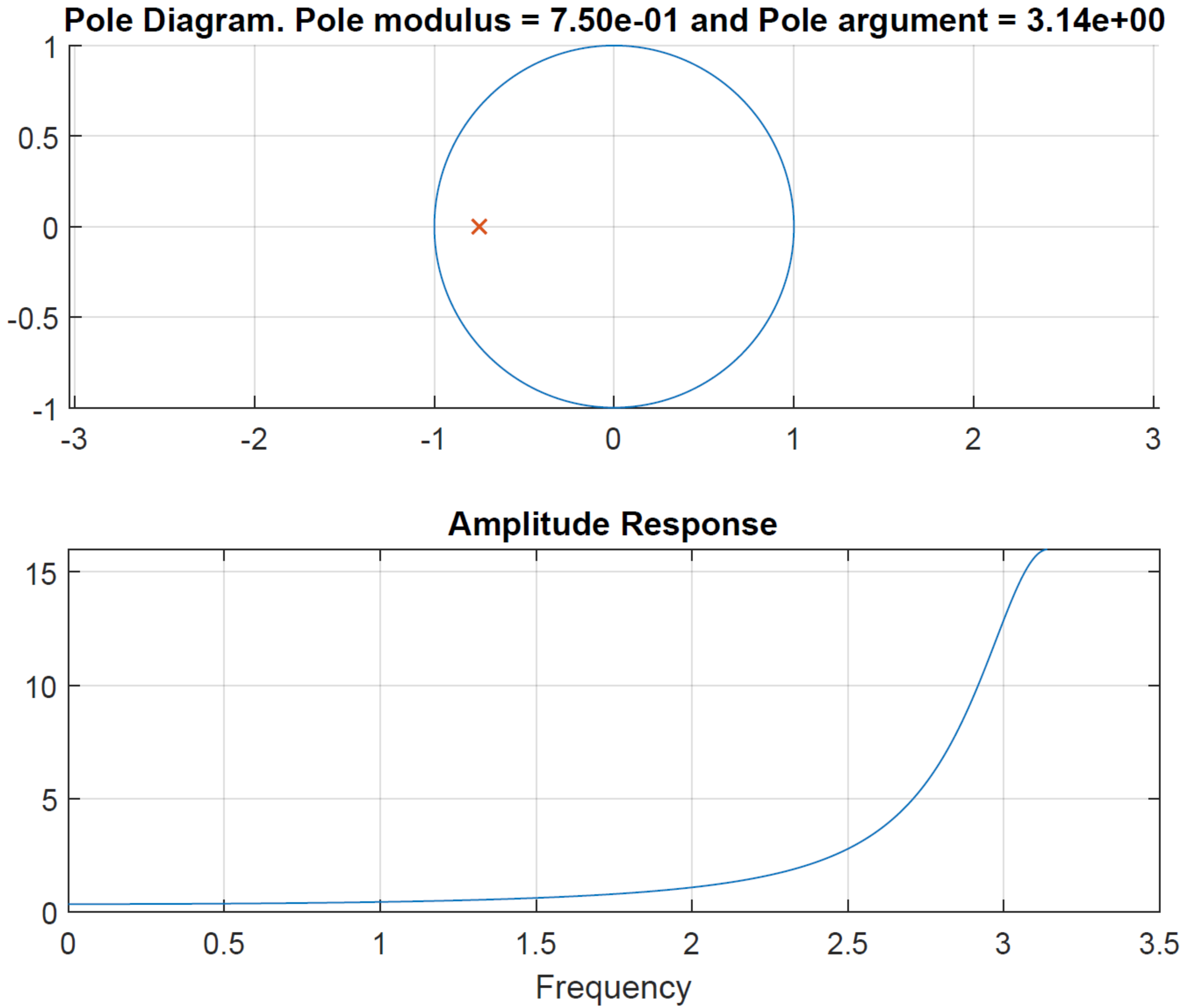


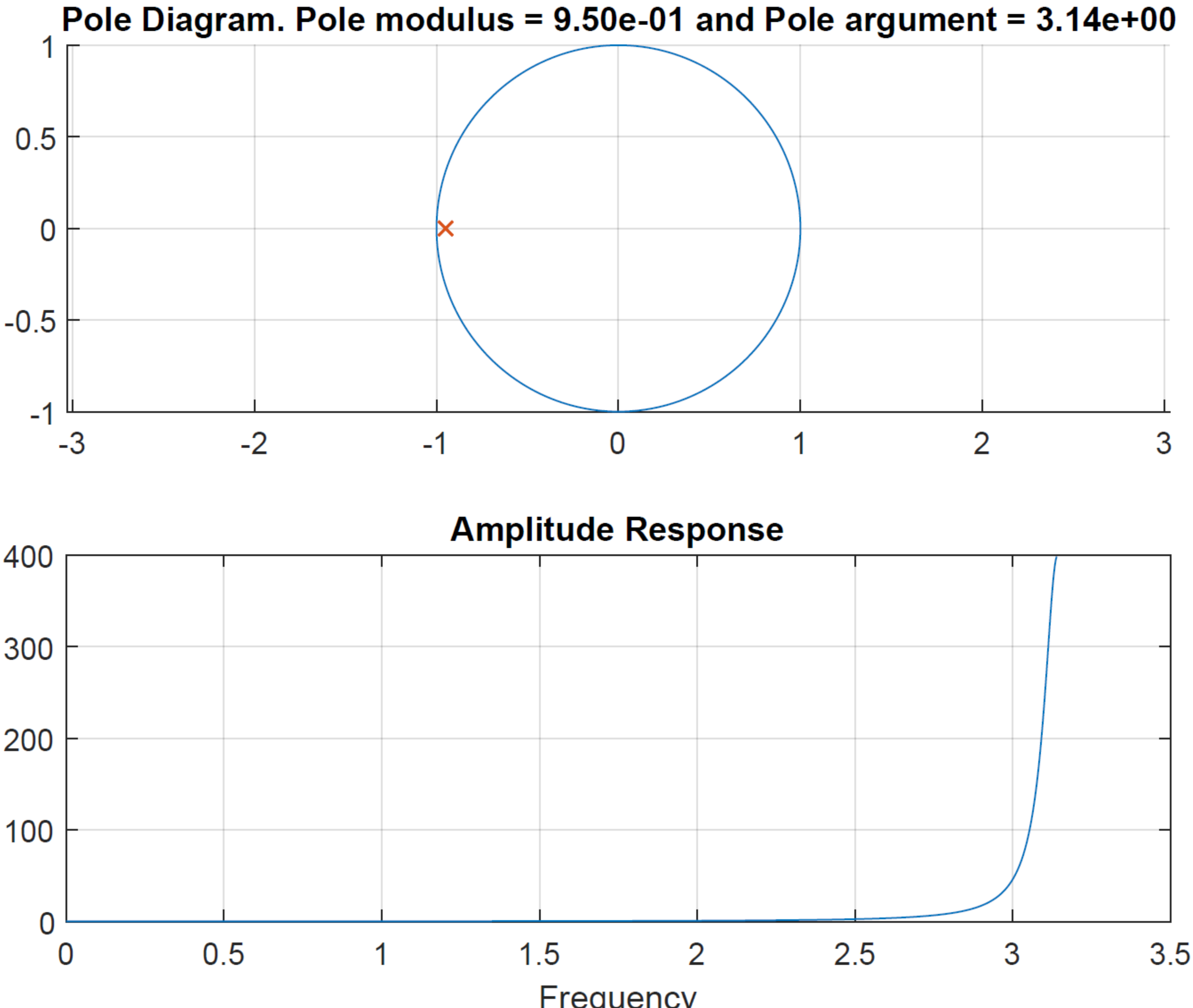
Pole Diagram. Pole modulus = 7.50e-01 and Pole argument = 00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
15
10
5
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

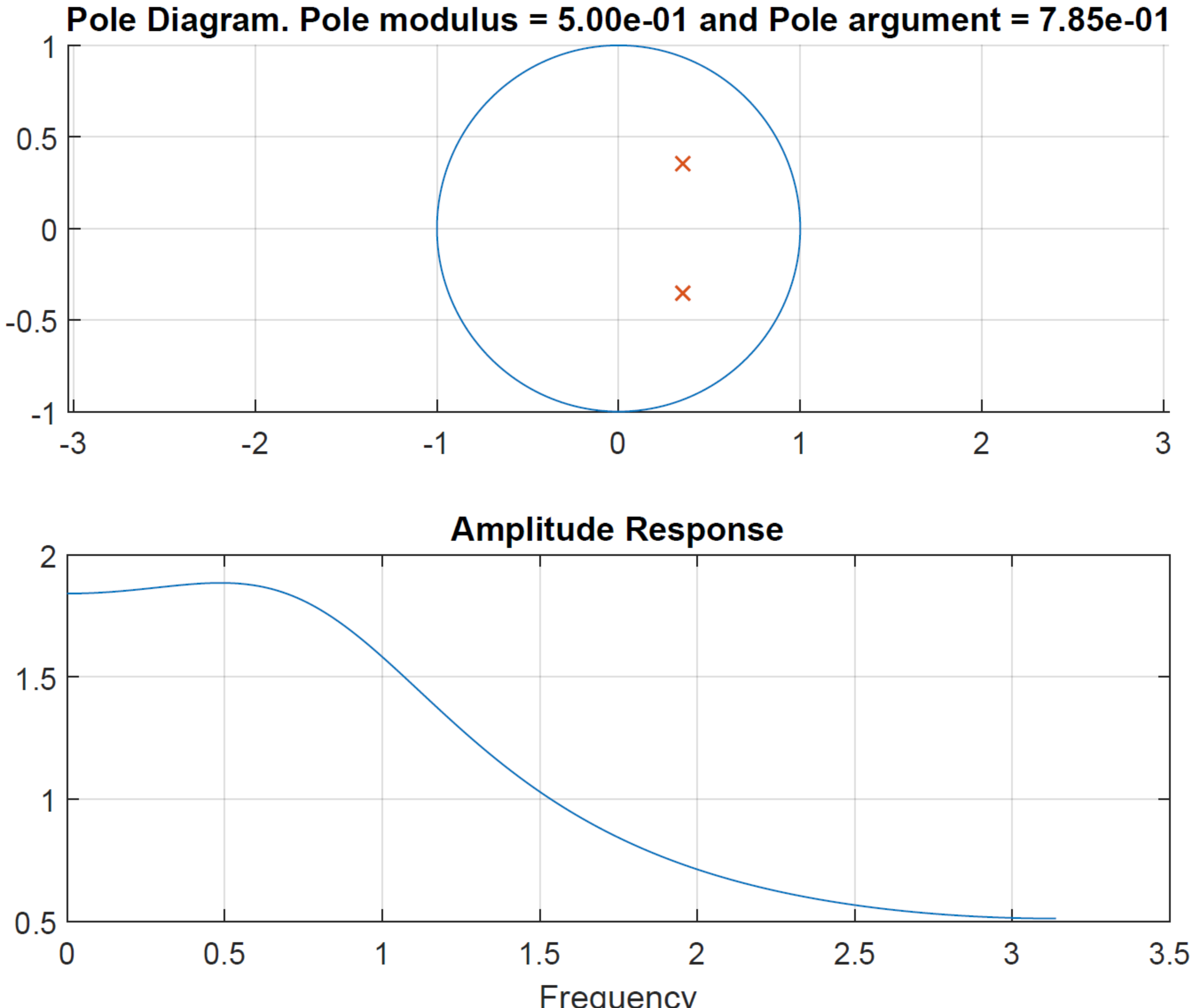


Pole Diagram. Pole modulus = 9.50e-01 and Pole argument = 00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
400
300
200
100
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

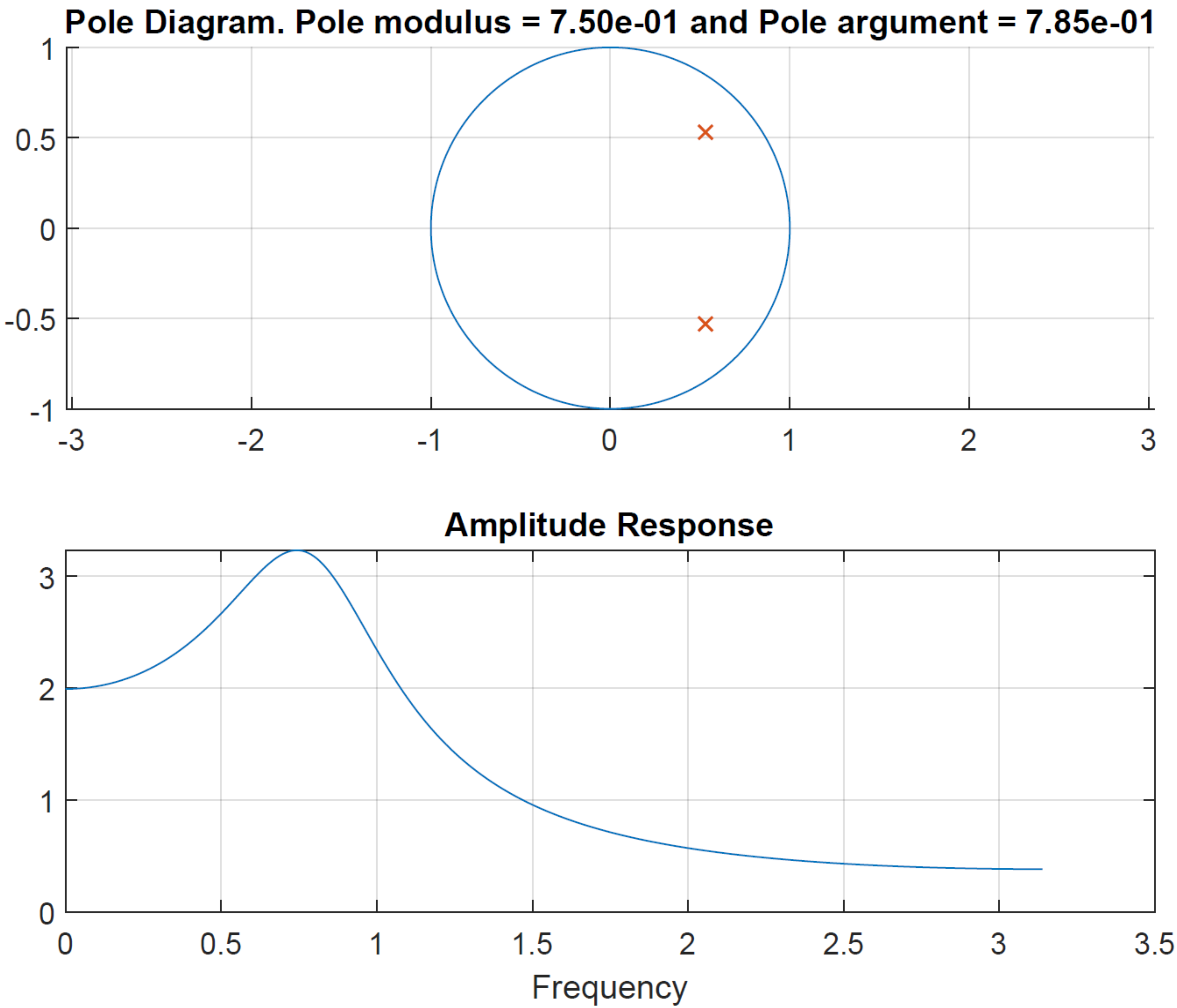

Pole Diagram. Pole modulus = 5.00e-01 and Pole argument = 3.14e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
4
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

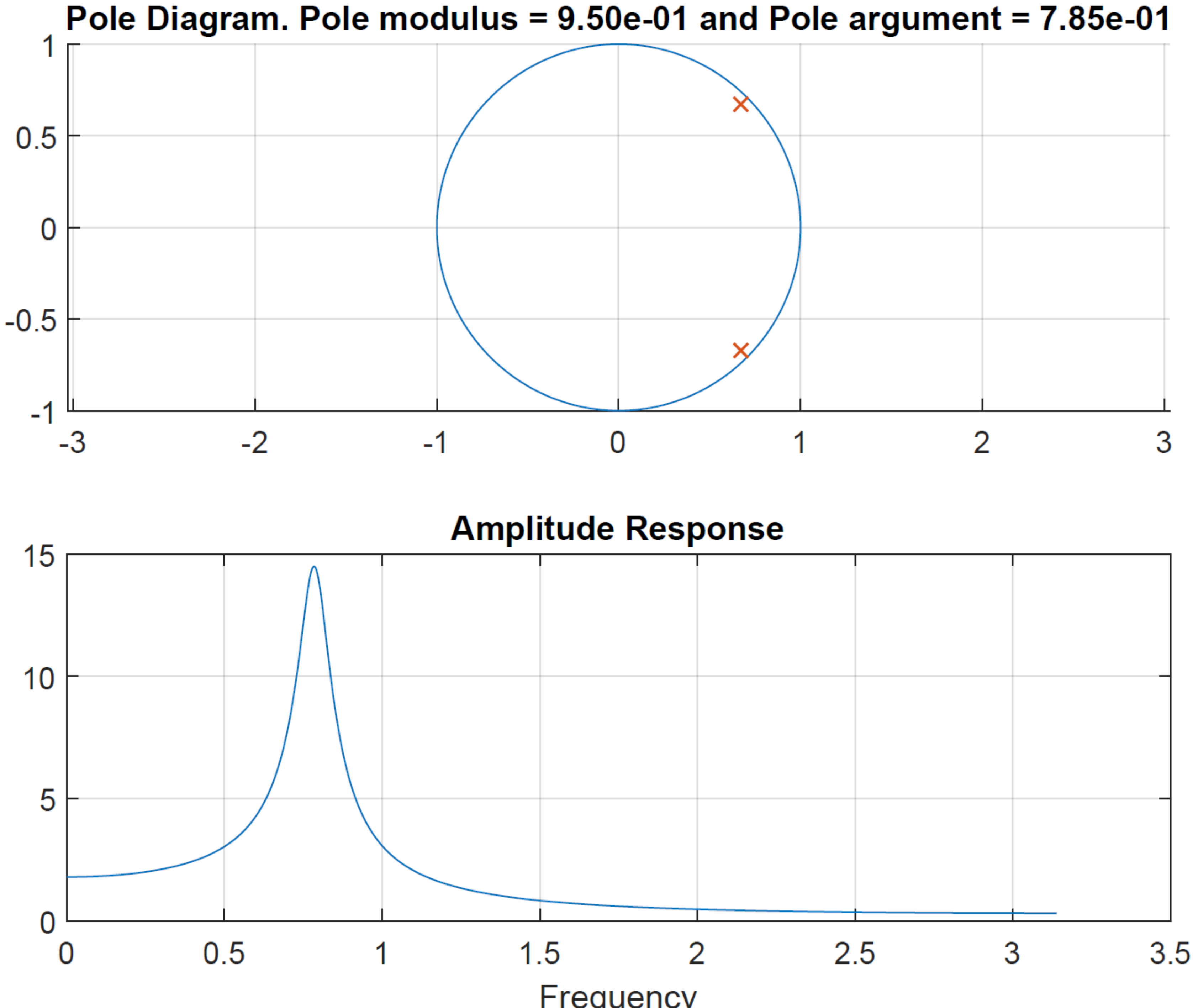

Pole Diagram. Pole modulus = 7.50e-01 and Pole argument = 3.14e+00
Amplitude Response
Frequency


Pole Diagram. Pole modulus = 9.50e-01 and Pole argument = 3.14e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
400
300
200
100
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

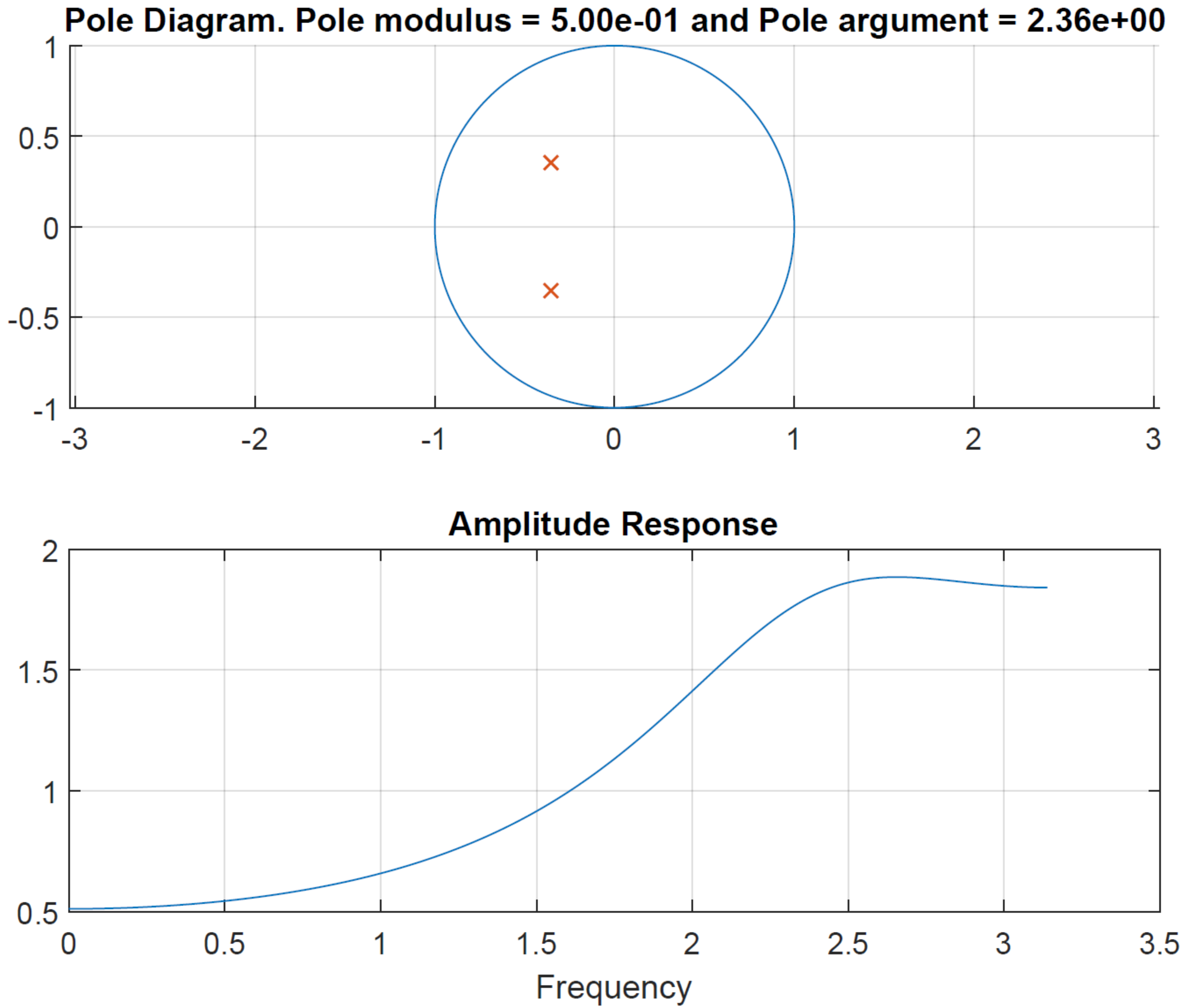


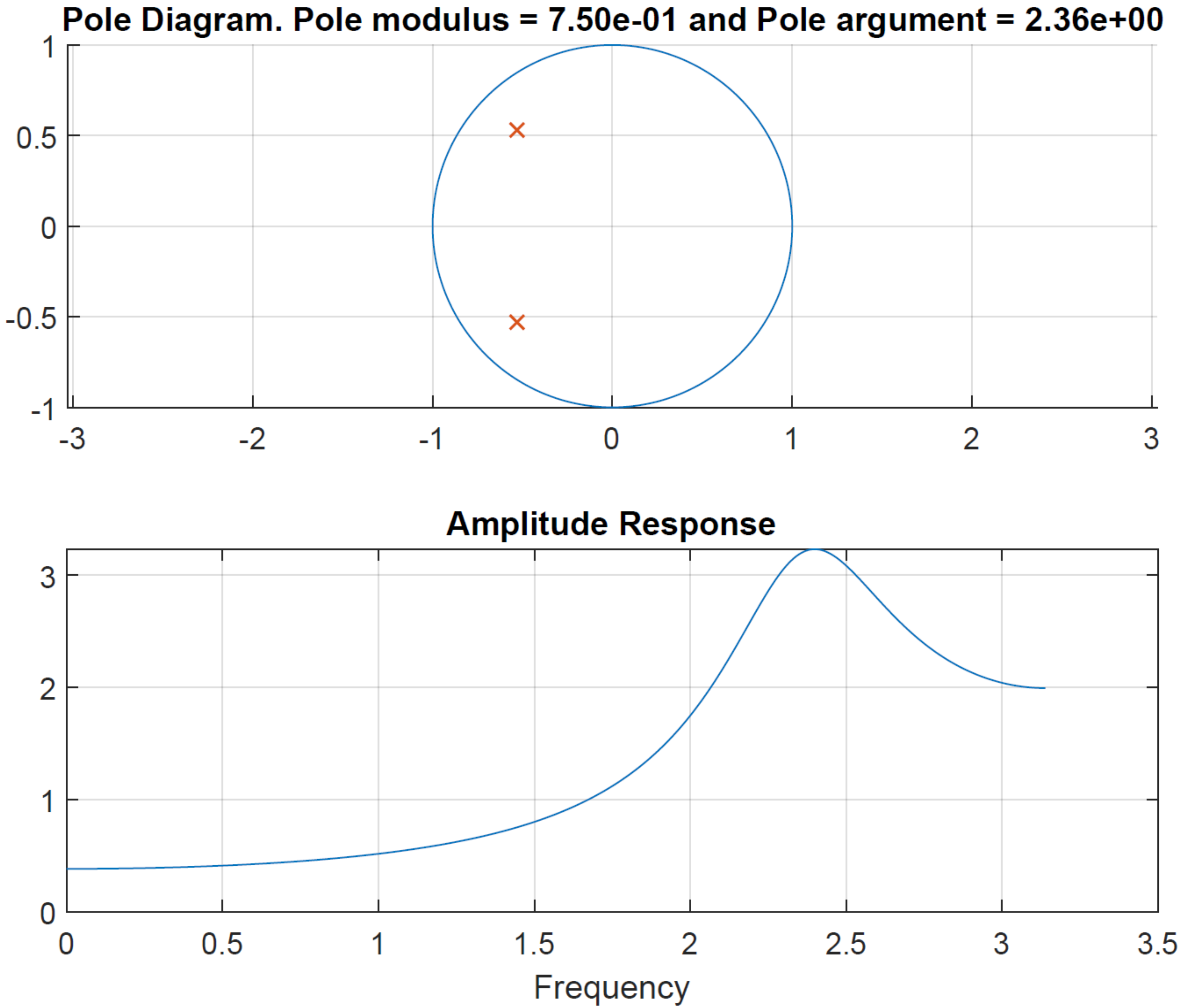
Pole Diagram. Pole modulus = 5.00e-01 and Pole argument = 7.85e-01
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
2
1.5
1
0.5
0
0.5
1
1.5
2
2.5
3
3.5
Frequency


Pole Diagram. Pole modulus = 7.50e-01 and Pole argument = 7.85e-01
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

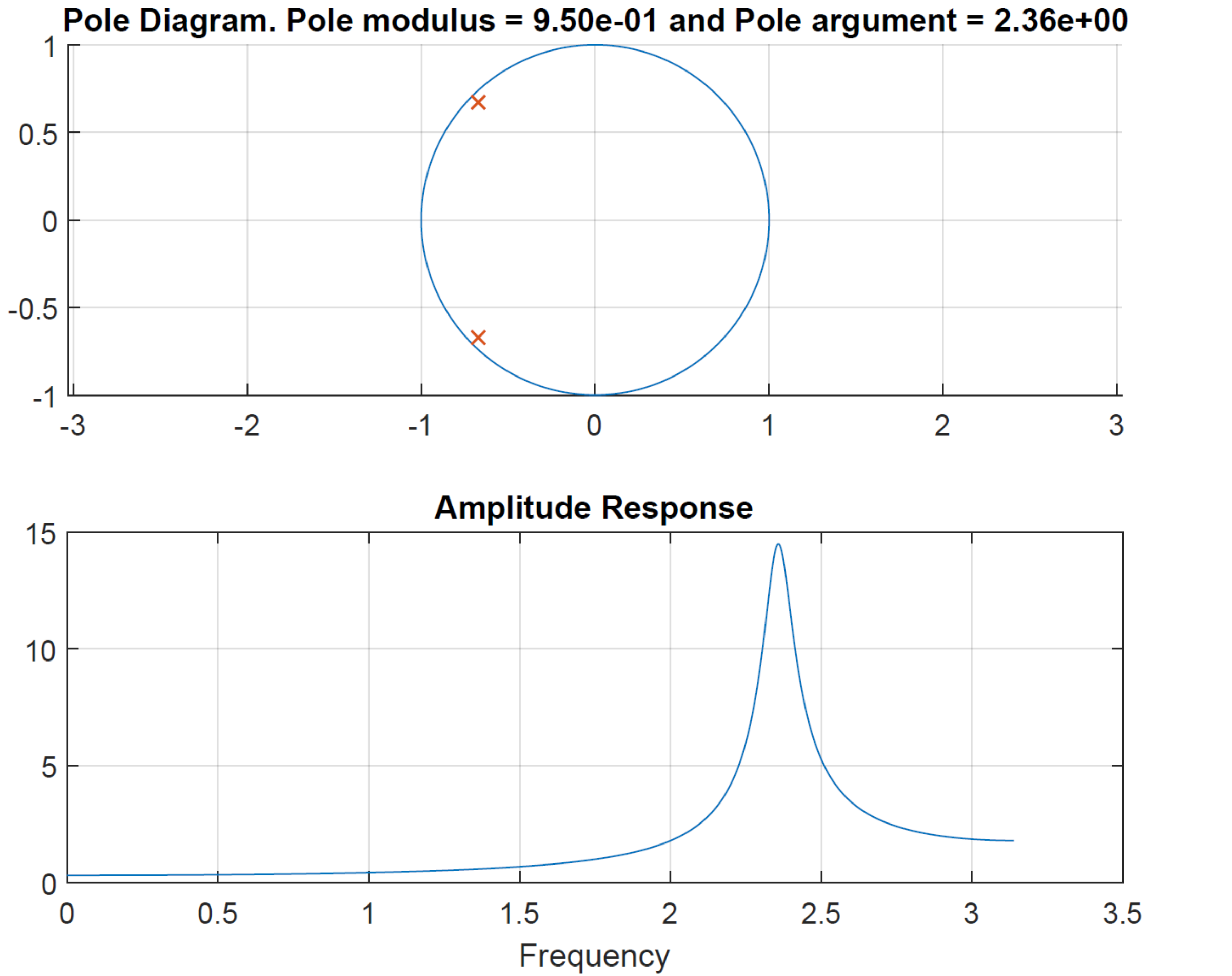


Pole Diagram. Pole modulus = 9.50e-01 and Pole argument = 7.85e-01
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
15
10
5
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

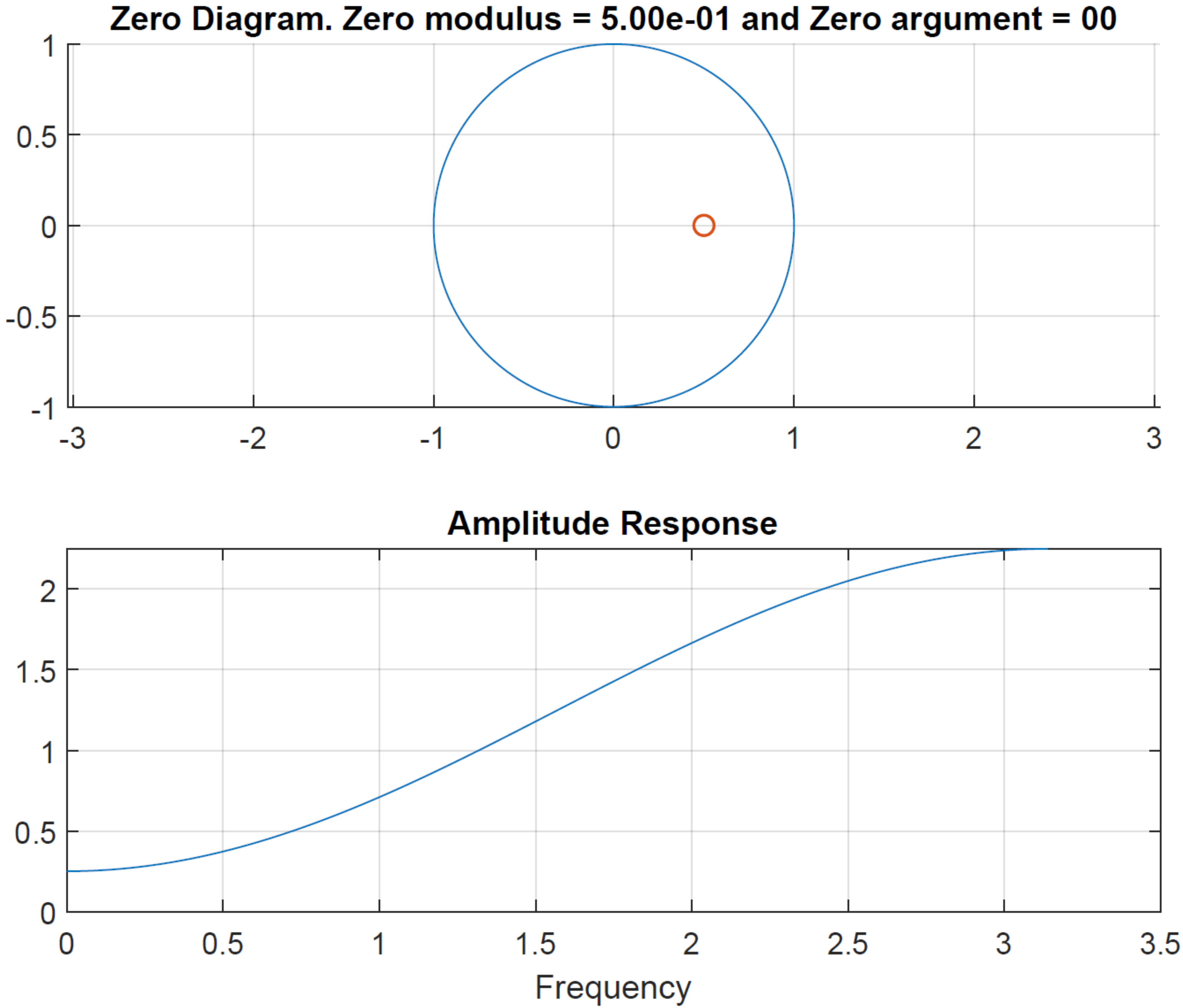

Pole Diagram. Pole modulus = 5.00e-01 and Pole argument = 2.36e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
2
1.5
1
0.5
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

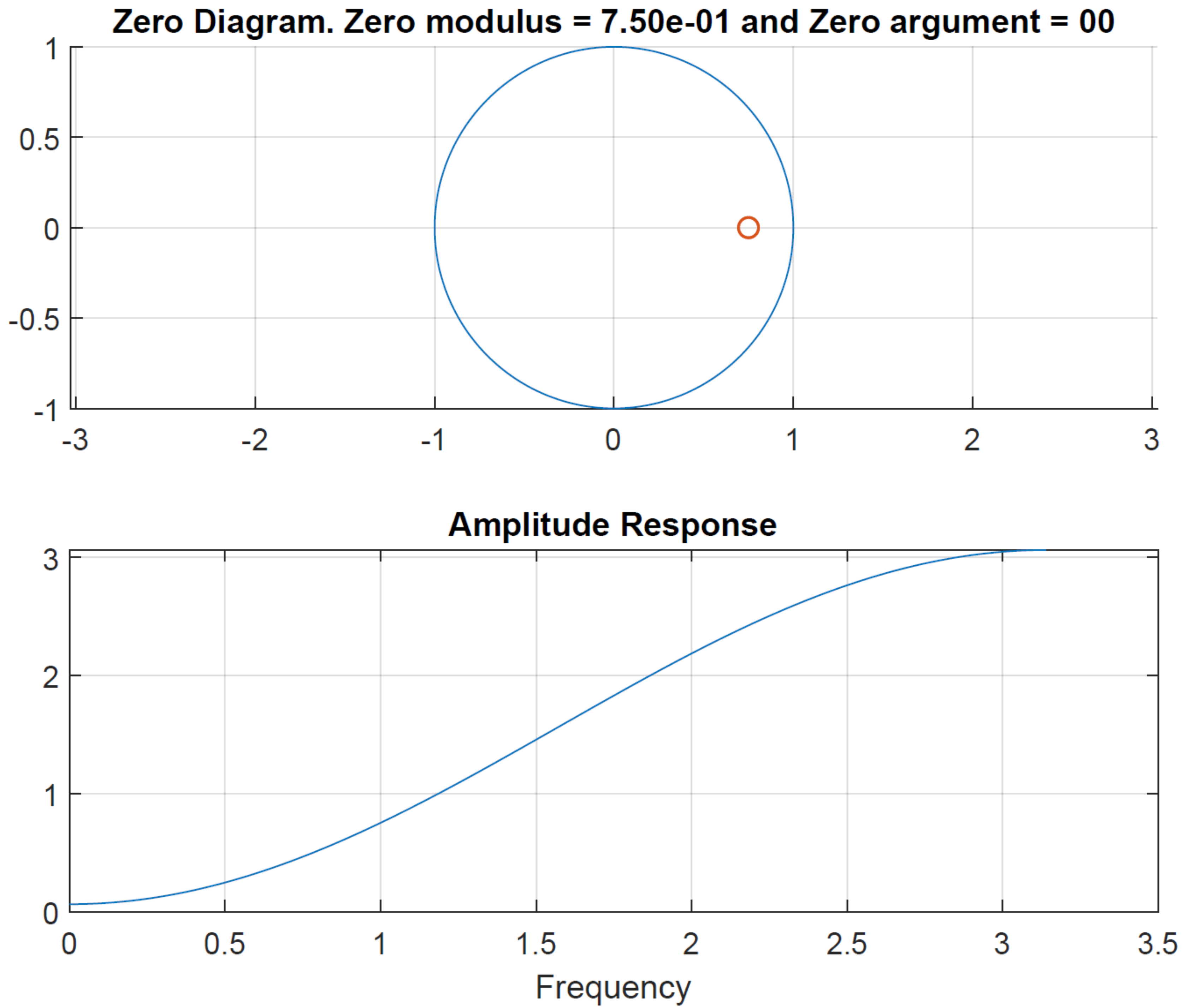


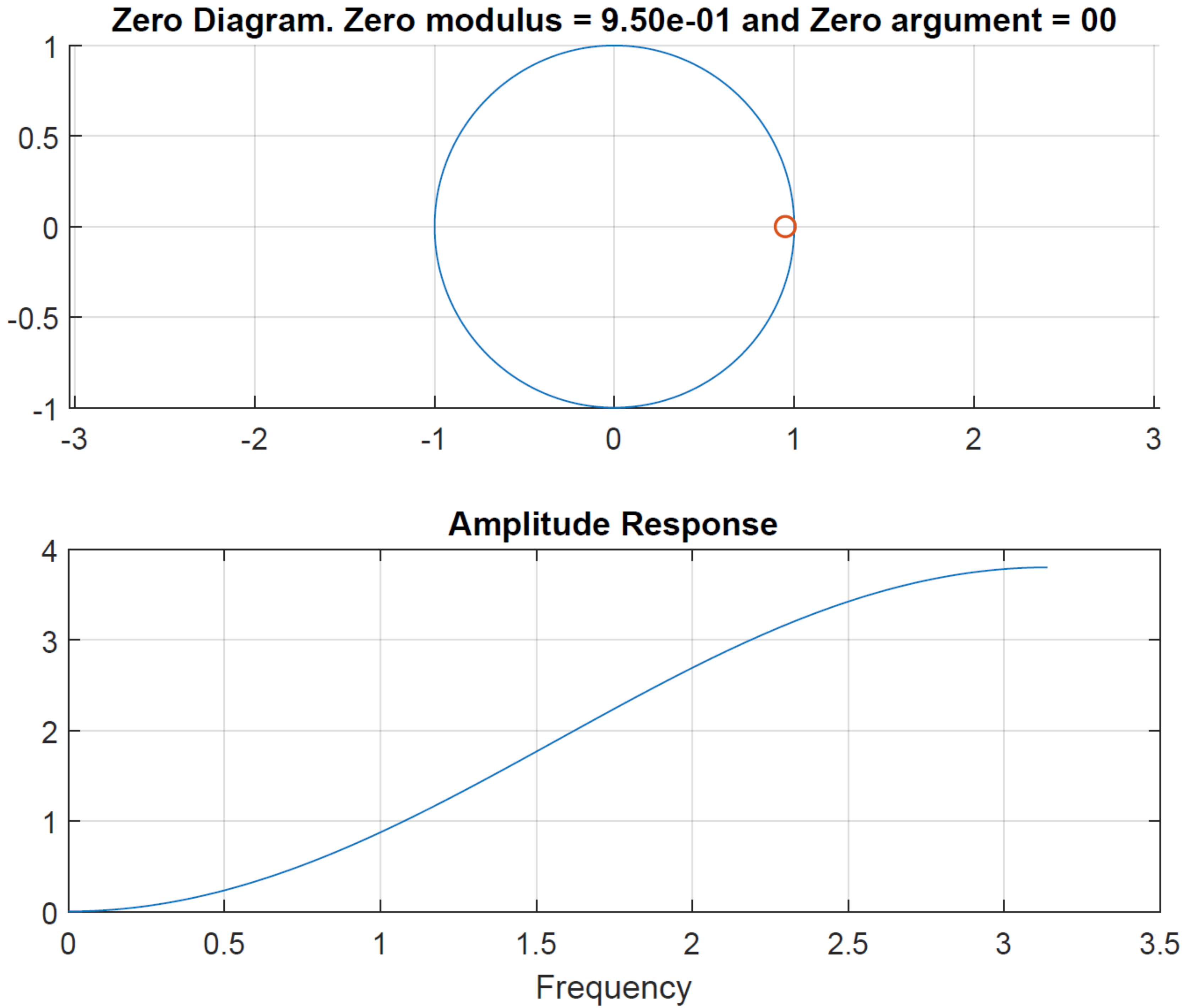
Pole Diagram. Pole modulus = 7.50e-01 and Pole argument = 2.36e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency



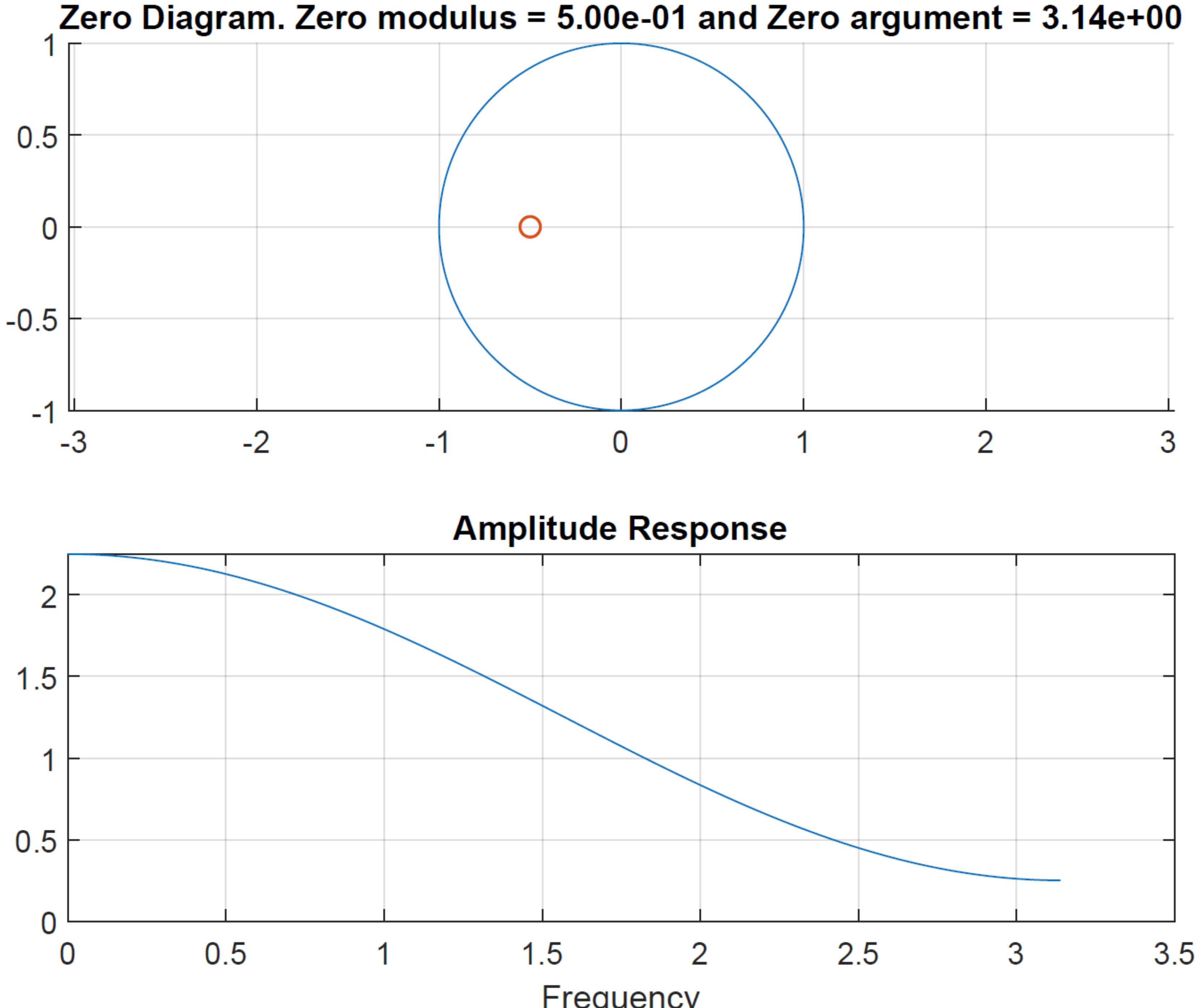
Pole Diagram. Pole modulus = 9.50e-01 and Pole argument = 2.36e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
15
10
5
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

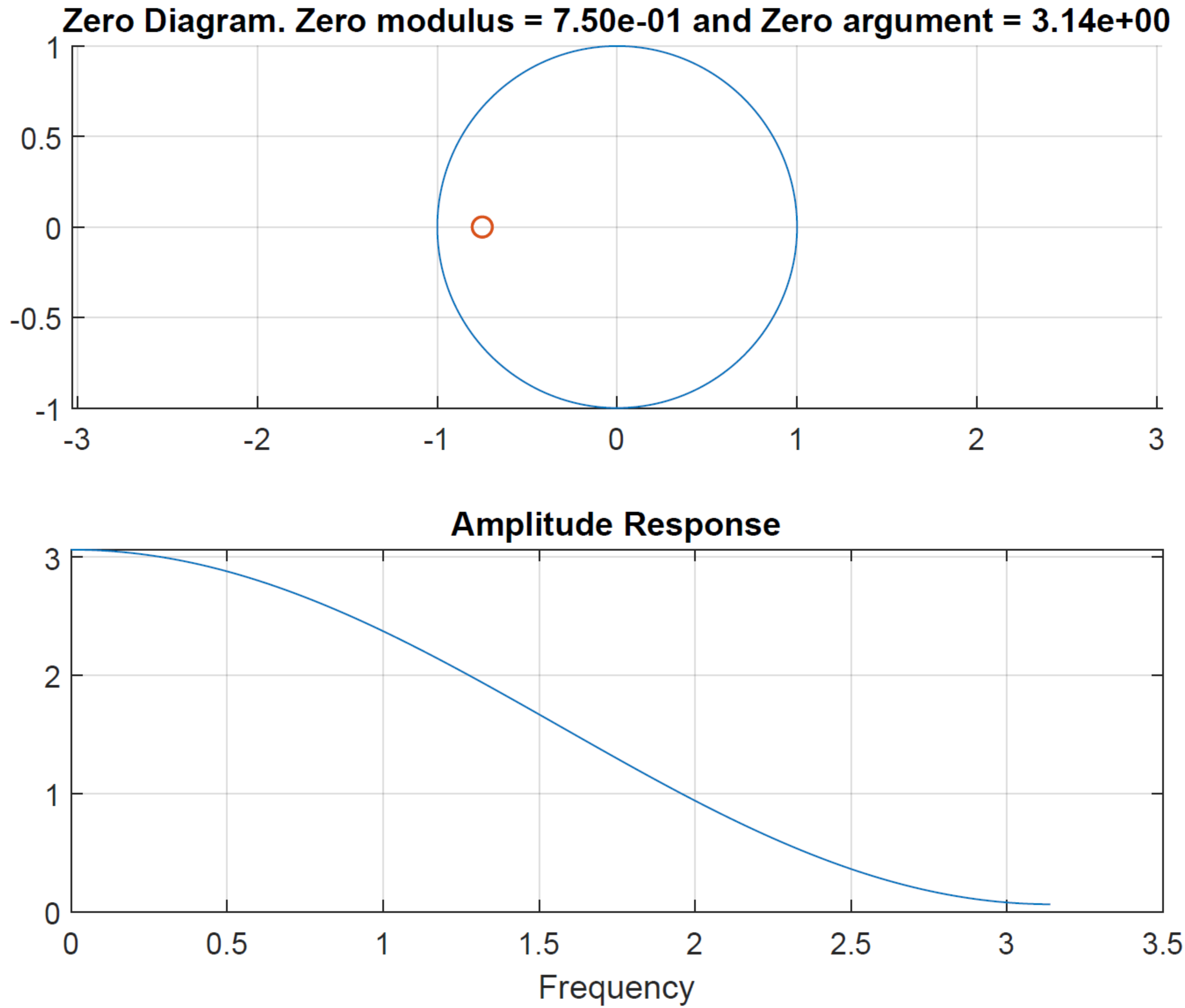

Zero Diagram. Zero modulus = 5.00e-01 and Zero argument = 00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

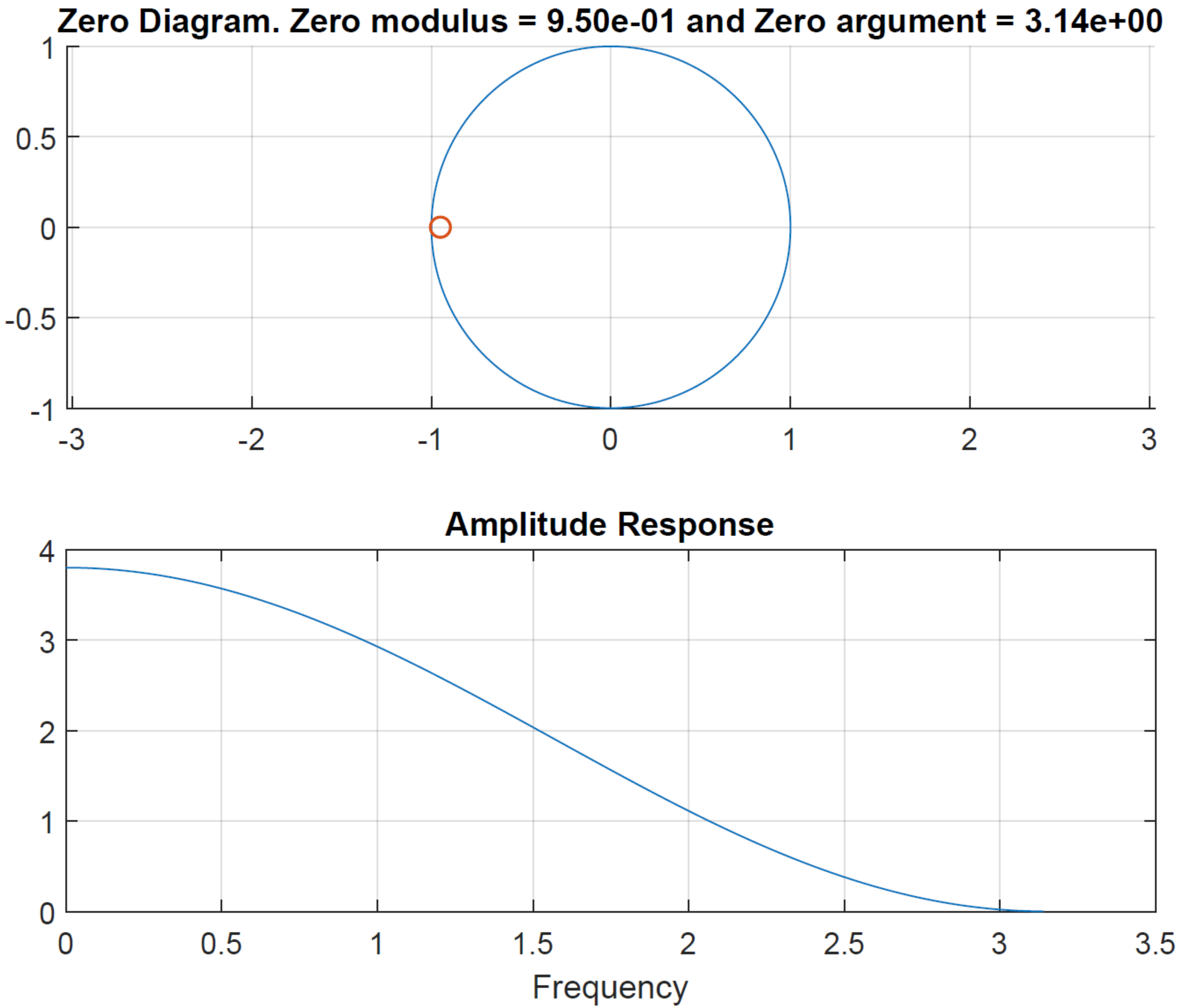

Zero Diagram. Zero modulus = 7.50e-01 and Zero argument = 00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

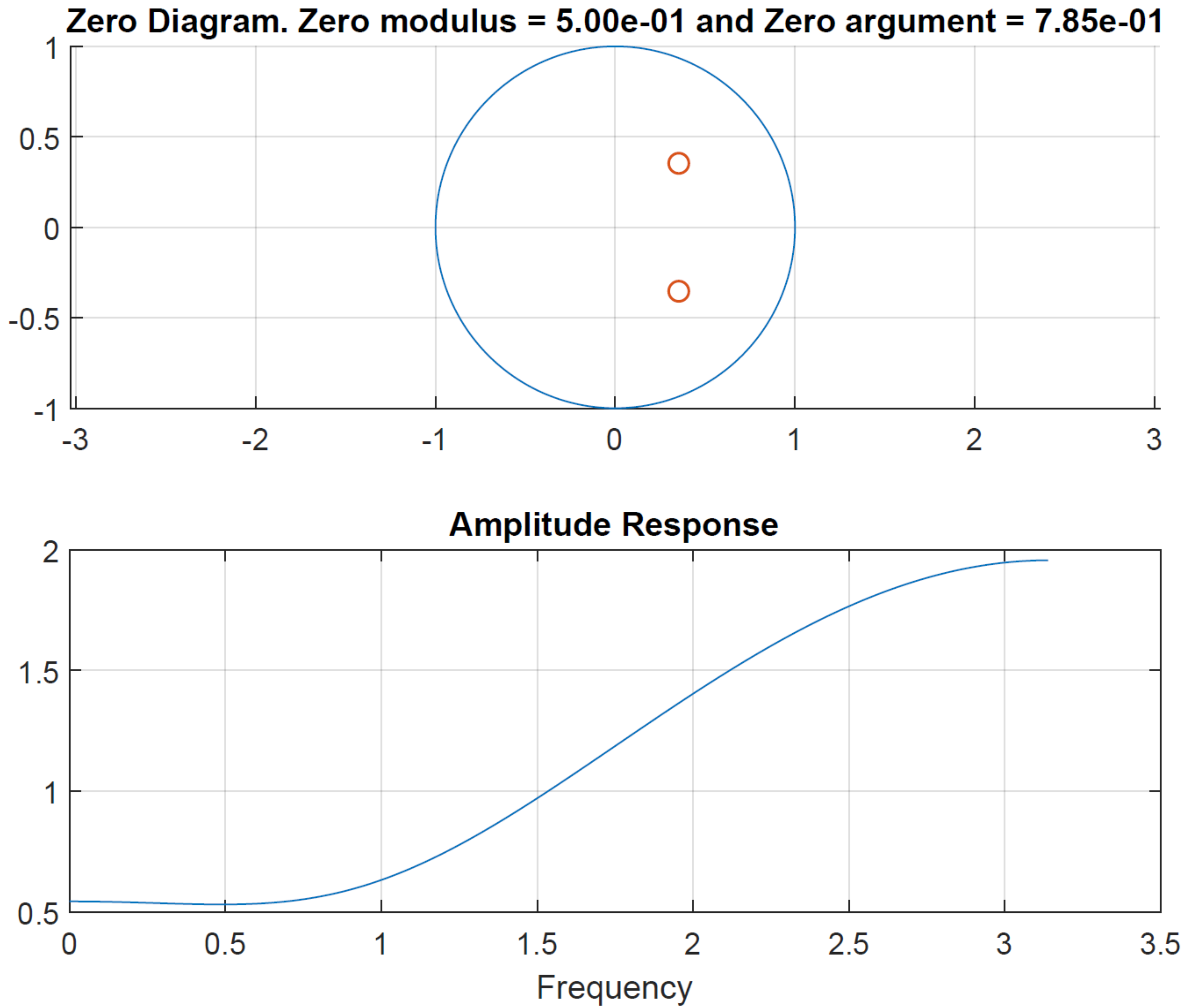


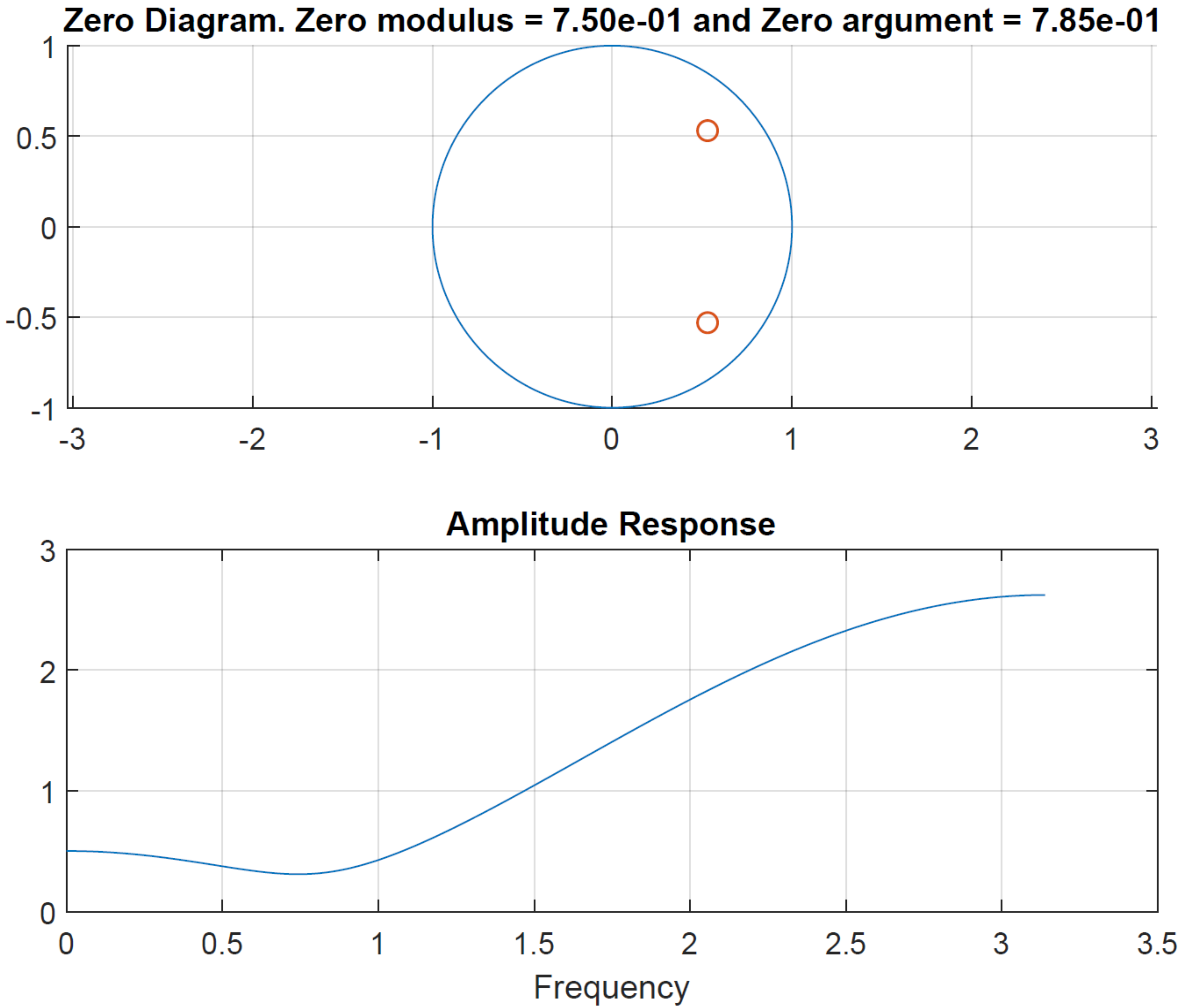
Zero Diagram. Zero modulus = 9.50e-01 and Zero argument = 00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
4
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency


Zero Diagram. Zero modulus = 5.00e-01 and Zero argument = 3.14e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency


Zero Diagram. Zero modulus = 7.50e-01 and Zero argument = 3.14e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

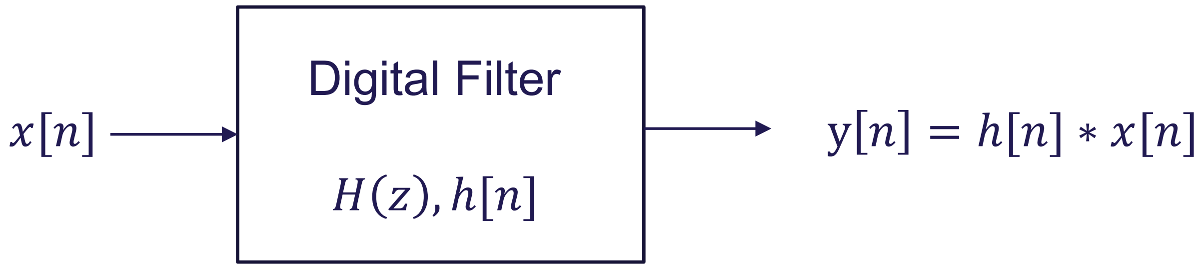

Zero Diagram. Zero modulus = 9.50e-01 and Zero argument = 3.14e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
4
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency


Zero Diagram. Zero modulus = 5.00e-01 and Zero argument = 7.85e-01
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
2
1.5
1
0.5
0
0.5
1
1.5
2
2.5
3
3.5
Frequency


Zero Diagram. Zero modulus = 7.50e-01 and Zero argument = 7.85e-01
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

Difference Equations

The transfer function $H(z)$ is a frequency-domain representaion of the I/O-relation for the filter, and thus it cannot be used for real-time implementation purposes where a time-domain representation of the I/O-relation is needed.



Previously we have discussed that the convolution sum is not (necessarily) is viable solution for those situations where $h[n]$ is an infinite-length sequence…

We therefore need another approach which, in the time-domain, relates input to output, i.e., $y[n] = f(x[n], var_1[n], var_2[n], \ldots, var_p[n])$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{l=1}^{N} a_l z^{-l}}$$

$$Y(z) \cdot \left(1 - \sum_{l=1}^{N} a_l z^{-l}\right) = X(z) \cdot \left(\sum_{k=0}^{M} b_k z^{-k}\right)$$

$$Y(z) = X(z) \cdot \left(\sum_{k=0}^{M} b_k z^{-k}\right) + Y(z)\left(\sum_{l=1}^{N} a_l z^{-l}\right)$$

$$Y(z) = \left(\sum_{k=0}^{M} b_k X(z) z^{-k}\right) + \left(\sum_{l=1}^{N} a_l Y(z) z^{-l}\right)$$

Applying now the Inverse z transform on this expression yields the DIFFERENCE EQUATION

$$y[n] = \left(\sum_{k=0}^{M} b_k x[n-k]\right) + \left(\sum_{l=1}^{N} a_l y[n-l]\right)$$

AALBORG UNIVERSITY
DENMARK

Graphical Representation of the Difference Equation - in terms of a Data Flow Graph

From an engineer's perspective it would be convenient if we could somehow transform the difference equation into a graphical description…

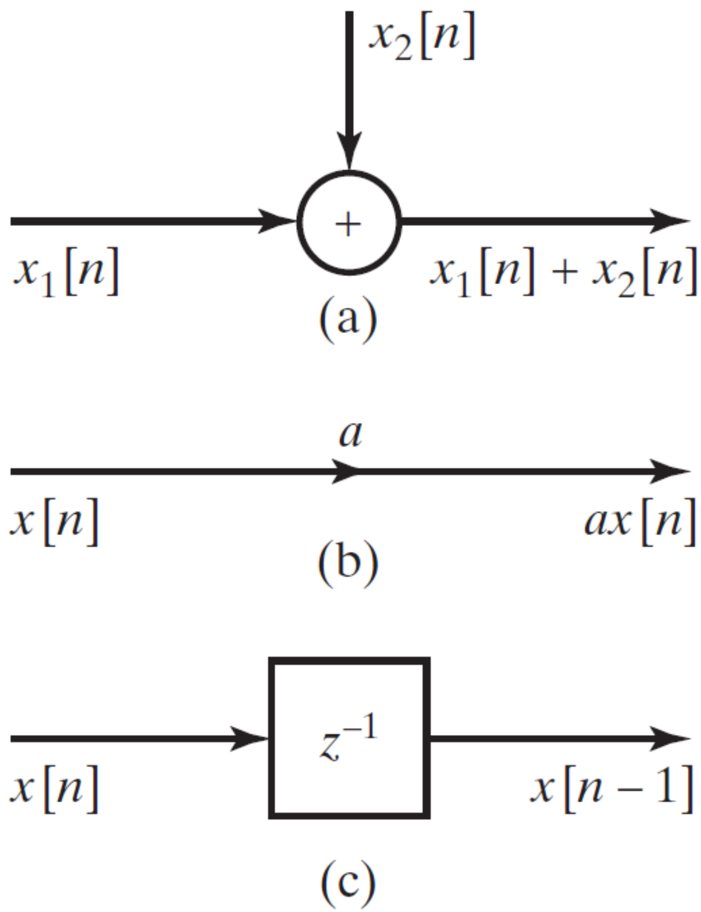
$$y[n] = \left(\sum_{k=0}^{M} b_k x[n-k]\right) + \left(\sum_{l=1}^{N} a_l y[n-l]\right)$$

Looking at the difference equation, we find three mathematical operation;

- Multiplication
- Addition
- Time Delay

AALBORG UNIVERSITY
DENMARK

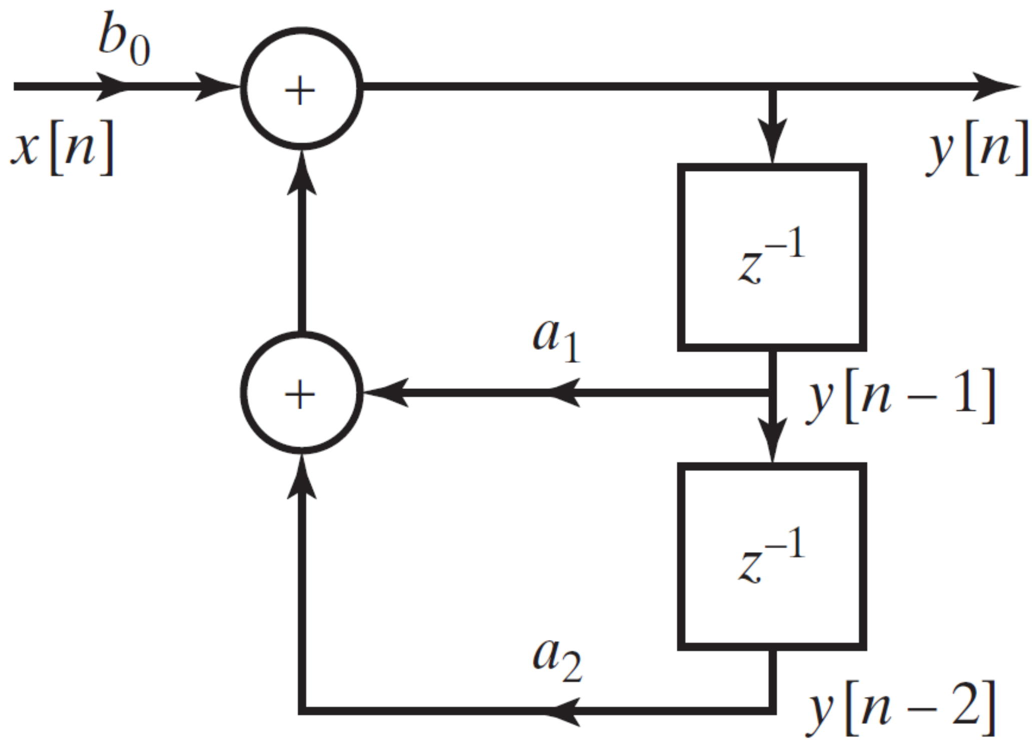
Multiplication, Addition and Time Delay



An Example…

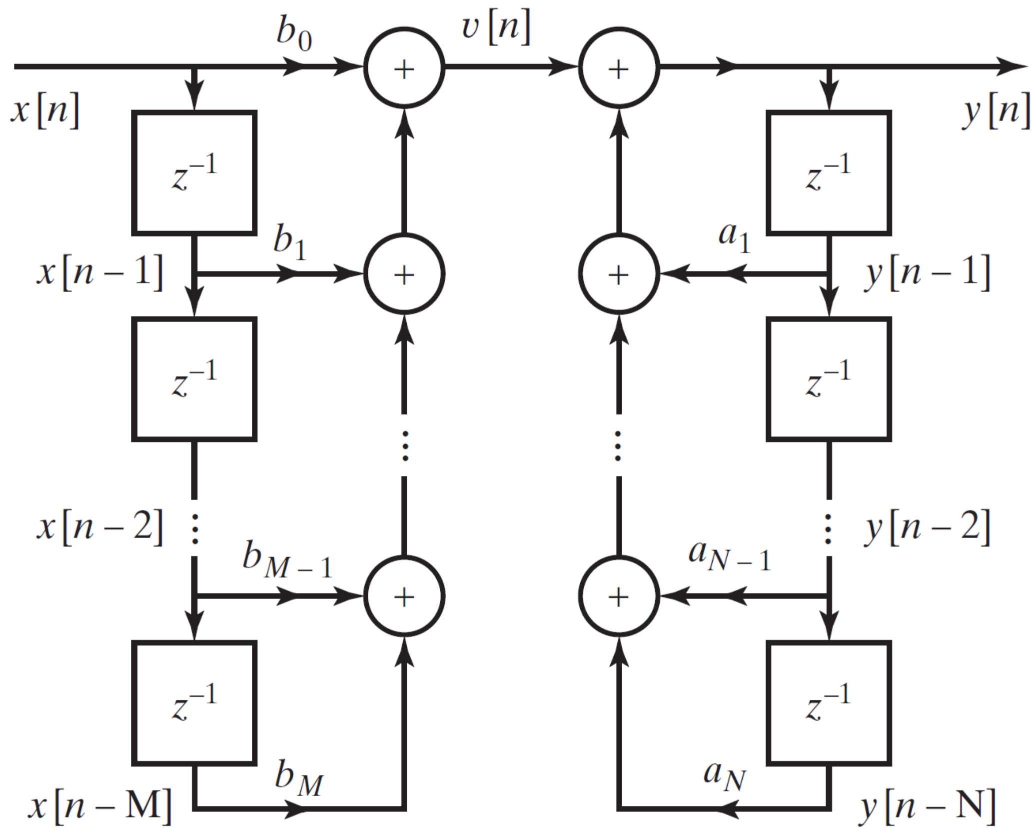
$$H(z) = \frac{b_0}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{Y(z)}{X(z)}$$

$$y[n] = b_0 x[n] + a_1 y[n-1] + a_2 y[n-2]$$



...and now the general case – The Direct Form I

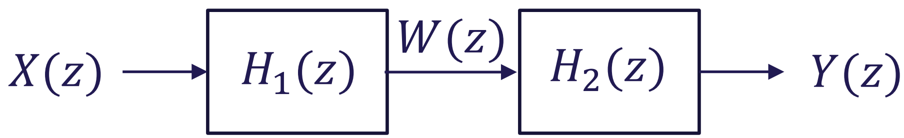
$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$



The zeros The poles

The DF-I structure can now be re-arranged

We now part $H(z) = \frac{Y(z)}{X(z)}$ into two transfer functions – the ”pole function” $H_1(z)$ and the ”zero function” $H_2(z)$.



$$H(z) = H_1(z) \cdot H_2(z) = \left\{\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}}\right\} \cdot \left\{\sum_{k=0}^{M} b_k z^{-k}\right\} = \frac{W(z)}{X(z)} \cdot \frac{Y(z)}{W(z)}$$

$$W(z) \cdot \left\{1 - \sum_{k=1}^{N} a_k z^{-k}\right\} = X(z)$$

$$W(z) \cdot \left\{\sum_{k=0}^{M} b_k z^{-k}\right\} = Y(z)$$

…and now inverse z transform

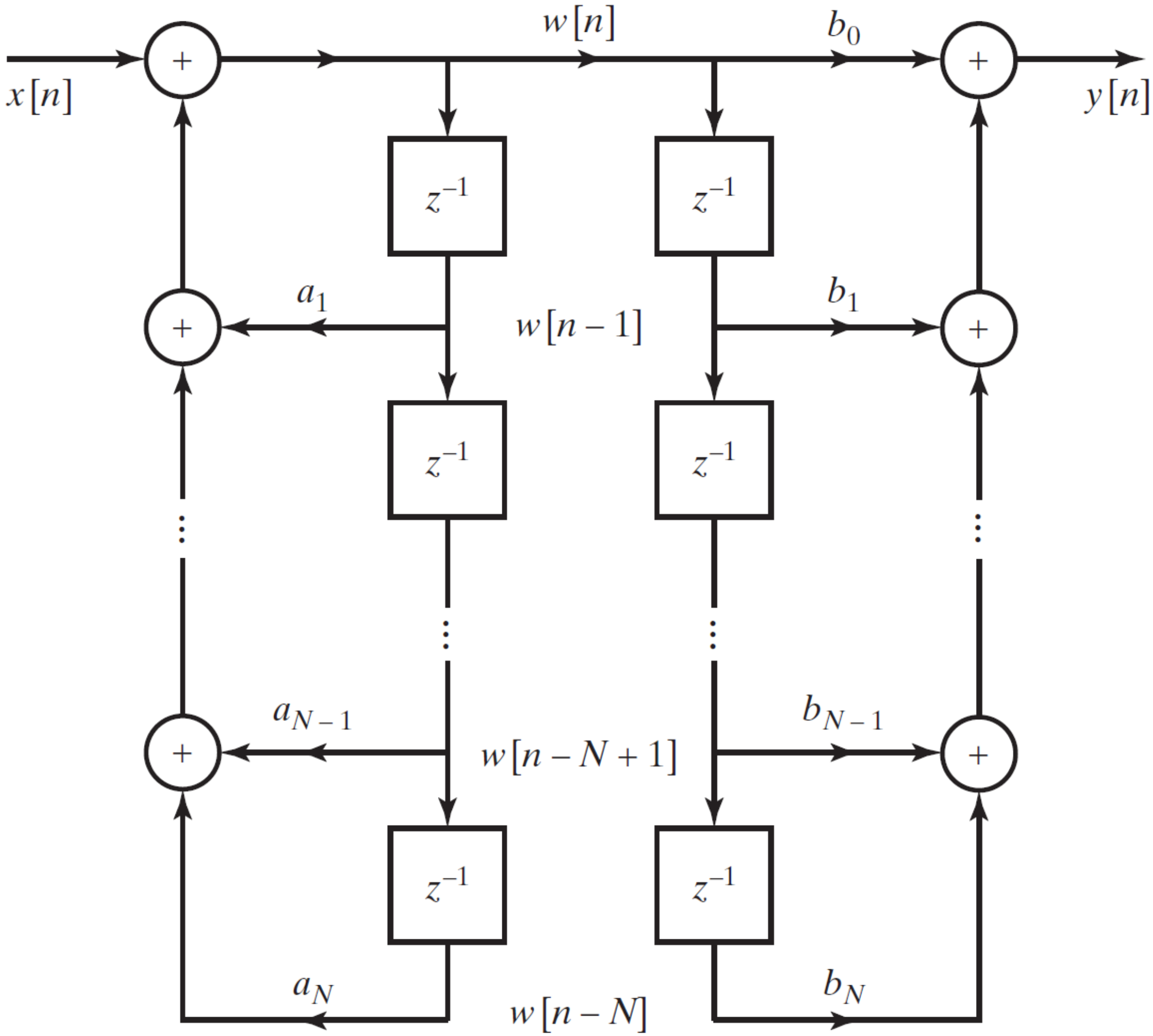
$$w[n] = x[n] + \sum_{k=1}^{N} a_k w[n-k]$$

$$y[n] = \sum_{k=0}^{M} b_k w[n-k]$$

So, what we see here is that the intermediate variable $w[n]$ is derived based on the input signal and N delayed versions of itself, and the output $y[n]$ is generated based on $w[n]$ and M delayed versions of $w[n]$.

AALBORG UNIVERSITY
DENMARK

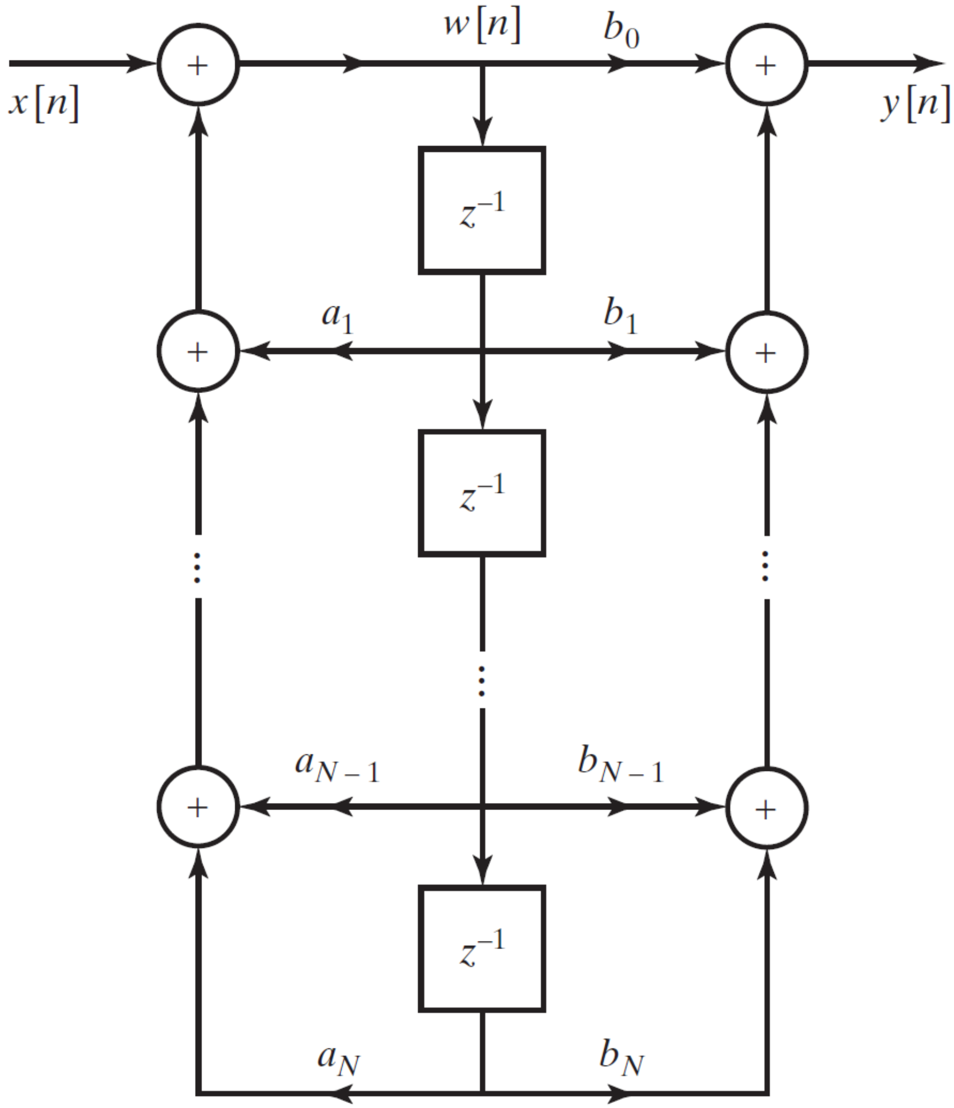
The Direct Form II Structure



The poles

The zeros

The Direct Form II Structure



AALBORG UNIVERSITY
DENMARK

An alternative to the Data Flow Graph - The Signal Flow Graph