State Space Methods

Lecture 5: introducing reference signals, anti-windup, optimal control

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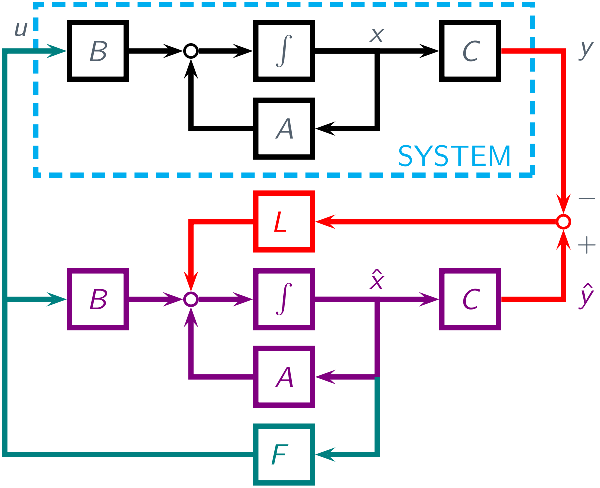
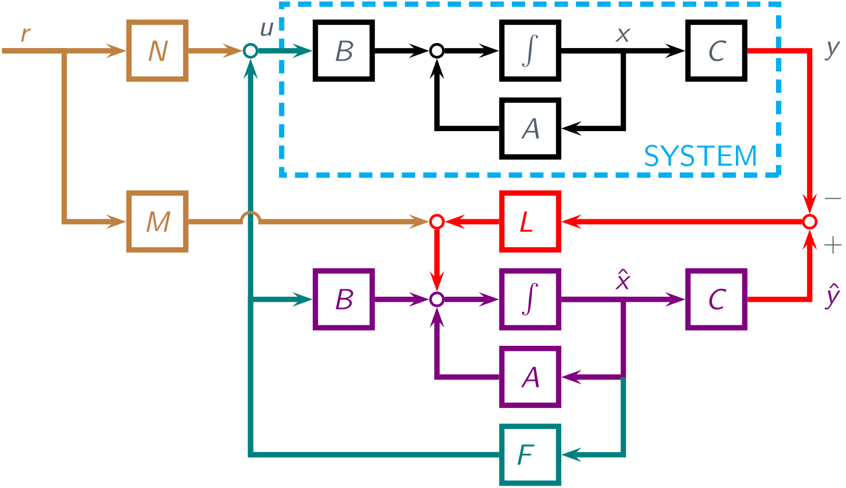


Introducing reference signals



Introducing reference signals

System:

$$\begin{aligned} \dot{x} &= Ax + B(F\hat{x} + Nr) \\ y &= Cx \end{aligned}$$

Observer:

$$\dot{\hat{x}} = A\hat{x} + BF\hat{x} + L(C\hat{x} - y) + Mr$$

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Observer:

$$\dot{\hat{x}} = A\hat{x} + BF\hat{x} + L(C\hat{x} - y) + Mr$$

$$\begin{pmatrix}\dot{x}\\ \dot{\hat{x}}\end{pmatrix} = \begin{pmatrix}A & BF\\ -LC & A+BF+LC\end{pmatrix}\begin{pmatrix}x\\ \hat{x}\end{pmatrix} + \begin{pmatrix}BN\\ M\end{pmatrix} r$$

$$y = \begin{pmatrix}C & 0\end{pmatrix}\begin{pmatrix}x\\ \hat{x}\end{pmatrix}$$

Introducing reference signals

From these derivations, we can see that the closed loop system from reference to output has the following state space model:

$$\begin{array}{rcl} \dot{x}_{\mathrm{cl}} & = & A_{\mathrm{cl}} x_{\mathrm{cl}} + B_{\mathrm{cl}} r \\ y & = & C_{\mathrm{cl}} x_{\mathrm{cl}} \end{array}, \quad \text{where } x_{\mathrm{cl}} = \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$$

and

$$A_{\mathrm{cl}} = \begin{pmatrix} A & BF \\ -LC & A + BF + LC \end{pmatrix}, \quad B_{\mathrm{cl}} = \begin{pmatrix} BN \\ M \end{pmatrix}, \quad C_{\mathrm{cl}} = \begin{pmatrix} C & 0 \end{pmatrix}$$

In the sequel, we will need to factor out the gain N, which yields:

$$\begin{array}{rcl} \dot{x}_{\mathrm{cl}} & = & A_{\mathrm{cl}} x_{\mathrm{cl}} + \tilde{B}_{\mathrm{cl}} N r \\ y & = & C_{\mathrm{cl}} x_{\mathrm{cl}} \end{array}, \text{where } \tilde{B}_{\mathrm{cl}} = \begin{pmatrix} B \\ \tilde{M} \end{pmatrix}, \text{and } \tilde{M} = M N^{-1}$$

Zeros of systems

We have previously introduced this result:

Lemma

A square (#inputs=#outputs) system with a state space model of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

has a zero with value $z \in \mathbb{C}$ only if

$$\det \begin{pmatrix} A - zI & B \\ C & D \end{pmatrix} = 0$$

Zero assignment

$$\det \begin{pmatrix} A_{\mathrm{cl}} - zI & B_{\mathrm{cl}} \\ C_{\mathrm{cl}} & D_{\mathrm{cl}} \end{pmatrix} = 0$$

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$$\det \begin{pmatrix} A - zI & BF - BNN^{-1}F & BN \\ -LC & A + BF + LC - zI - MN^{-1}F & M \\ C & 0 & 0 \end{pmatrix} = 0$$

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$$\det \begin{pmatrix} A - zI & 0 & B \\ -LC & A + BF + LC - \tilde{M}F - zI & \tilde{M} \\ C & 0 & 0 \end{pmatrix} = 0$$

Zero assignment

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Zero assignment

$$\det \begin{pmatrix} A - zI & 0 & B \\ -LC & A + BF + LC - \tilde{M}F - zI & \tilde{M} \\ C & 0 & 0 \end{pmatrix} = 0$$

$$\begin{cases} \det \begin{pmatrix} A - zI & B \\ C & 0 \end{pmatrix} = 0 \qquad \text{or} \\ \det \left(A + BF + LC - \tilde{M}F - zI \right) = 0 \end{cases}$$

Zero assignment

$$\det \begin{pmatrix} A - zI & 0 & B \\ -LC & A + BF + LC - \tilde{M}F - zI & \tilde{M} \\ C & 0 & 0 \end{pmatrix} = 0$$

$$\left\{ \begin{array}{ll} \det \begin{pmatrix} A - zI & B \\ C & 0 \end{pmatrix} = 0 & \text{or} \\ \det \left(A + BF + LC - \tilde{M}F - zI \right) = 0 & \end{array} \right.$$

$$\left\{ \begin{array}{llll} z & \text{is a zero of} & \left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx \end{array} \right\} & \text{or} \\ z & \text{is an eigenvalue of} & A + BF + LC - \tilde{M}F & \end{array} \right.$$

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Zero assignment

Lemma

If $\tilde{M}$ is an 'observer gain' such that the characteristic polynomial of the matrix $A_{za} + \tilde{M}C_{za}$ has the characteristic polynomial

$$\det\left(sI - \left(A_{za} + \tilde{M}C_{za}\right)\right) = (s - z_1)\cdots(s - z_n)$$

with $A_{za} = A+BF+LC$ and $C_{za} = -F$, then the numbers $z_1, \ldots, z_n$ are all zeros of the closed loop transfer function from r to y.

Algorithm for zero assignment

1. Design $\tilde{M}$ assigning zeros close to the cut-off frequency of the Bode plot, such that the 'horisontal' part is extended.

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2. Compute N such that the DC-value of the transfer function from r to y is unity:

$$N = -\left(C_{\mathrm{cl}} A_{\mathrm{cl}}^{-1} \tilde{B}_{\mathrm{cl}}\right)^{-1}$$

where

$$A_{\mathrm{cl}} = \begin{pmatrix} A & BF \\ -LC & A + BF + LC \end{pmatrix}, \quad \tilde{B}_{\mathrm{cl}} = \begin{pmatrix} B \\ \tilde{M} \end{pmatrix}$$

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3. Compute $M = MN^{-1}N = \tilde{M}N$.

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Example: zero assignment

We consider again the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y &= \begin{pmatrix} -3 & 2 \end{pmatrix} x \end{aligned}$$

A state feedback F that assign poles in $\{-3, -4\}$ and an observer gain L that assigns poles in $\{-9, -12\}$ are given by:

$$F = \begin{pmatrix} 22 & -16 \end{pmatrix}, \quad L = \begin{pmatrix} -122 \\ -192 \end{pmatrix}$$

We would like to assign zeros from r to y in $\{-3, -4\}$ to cancel the poles from F.

Example: zero assignment

With these values of F and L we obtain:

$$A_{\text{za}} = A + BF + LC = \begin{pmatrix} 412 & -279 \\ 646 & -437 \end{pmatrix}$$

$$C_{\text{za}} = -F = \begin{pmatrix} -22 & 16 \end{pmatrix}$$

An 'observer gain' that assigns poles in $\{-3, -4\}$ for $A_{\text{za}} + \tilde{M} C_{\text{za}}$ is

$$\tilde{M} = \begin{pmatrix} 7.0460 \\ 10.8133 \end{pmatrix}$$

Example: zero assignment

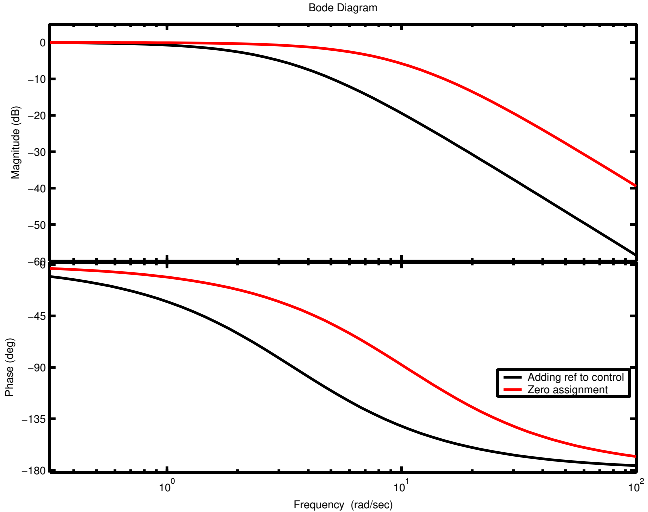
N can be computed as:

$$N = -\left(\begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} A & BF \\ -LC & A+BF+LC \end{pmatrix}^{-1} \begin{pmatrix} B \\ \tilde{M} \end{pmatrix}\right)^{-1} = 108$$

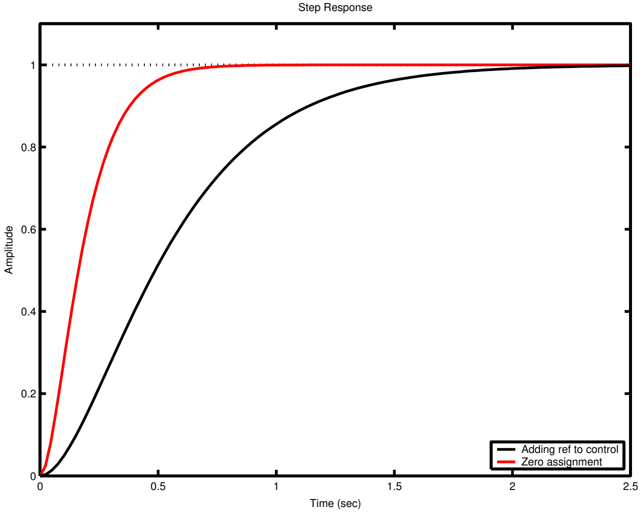
M is obtained from:

$$M = \tilde{M}N = \begin{pmatrix} 7.0460 \\ 10.8133 \end{pmatrix} \cdot 108 = \begin{pmatrix} 760.97 \\ 1167.84 \end{pmatrix}$$

Example: Bode plot



Example: step response



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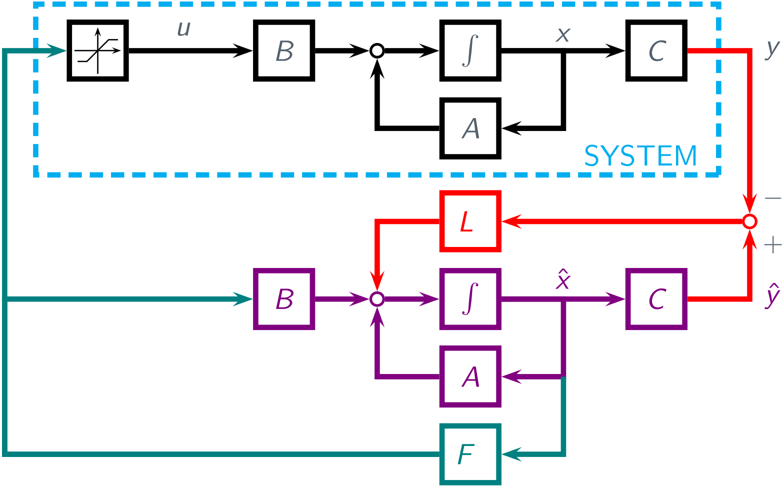
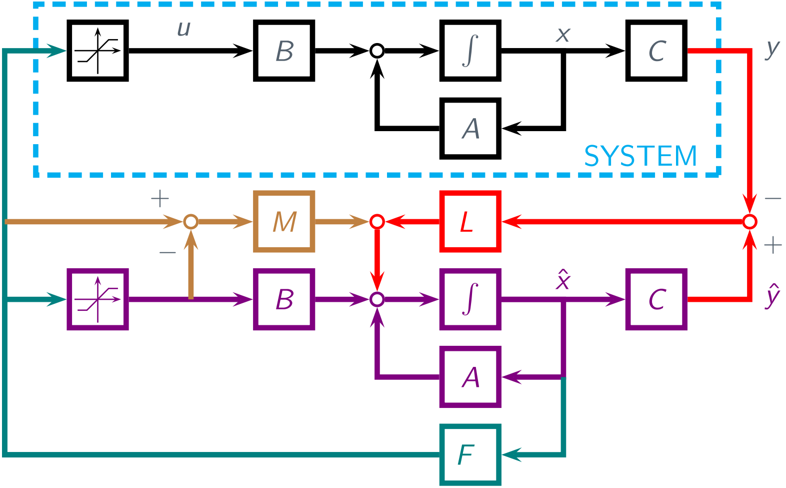
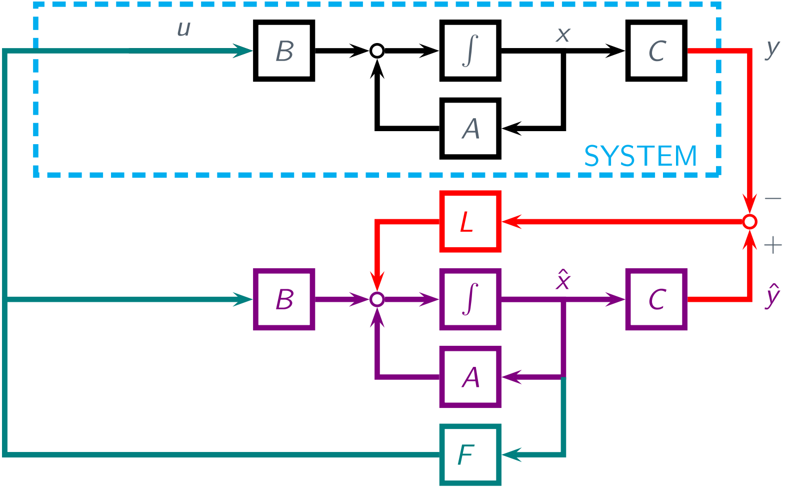
Anti-windup architecture



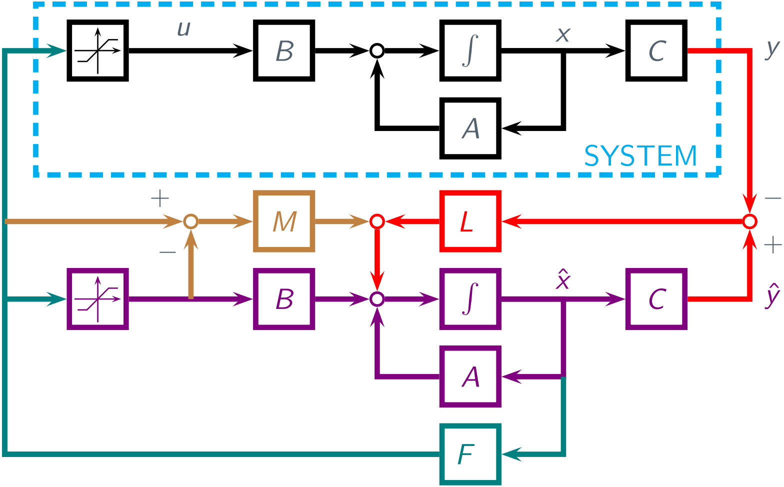
Anti-windup architecture



Anti-windup architecture, nominal



Anti-windup architecture, saturated



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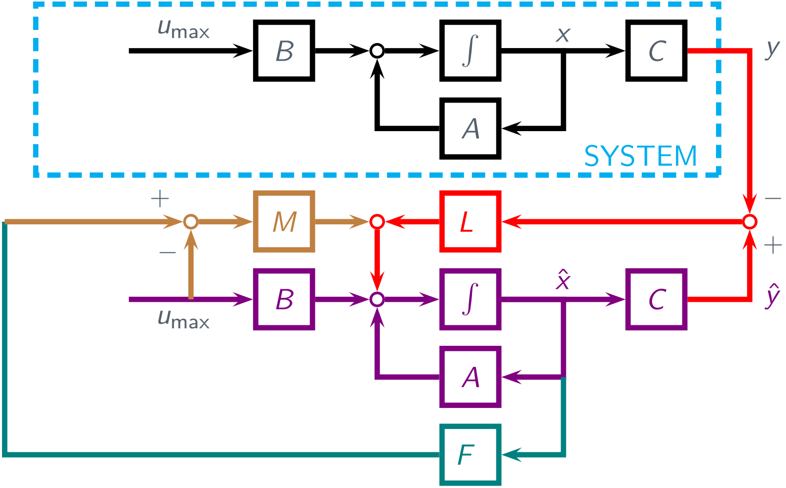

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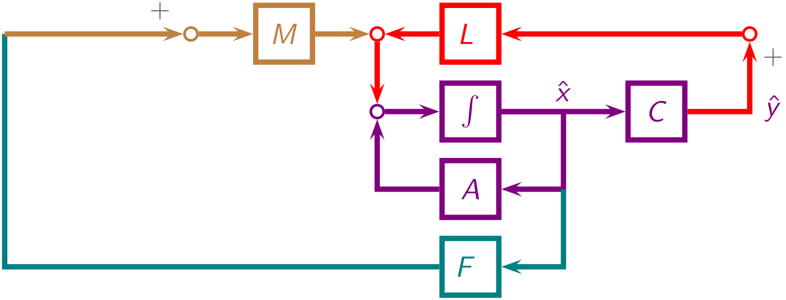

Designing saturation gain

Dynamics of controller during saturation:

$$\dot{\hat{x}} = A\hat{x} + LC\hat{x} + MF\hat{x}$$

or

$$\dot{\hat{x}} = (A + LC + MF)\,\hat{x}$$

Designing saturation gain

Dynamics of controller during saturation:

$$\dot{\hat{x}} = A\hat{x} + LC\hat{x} + MF\hat{x}$$

or

$$\dot{\hat{x}} = (A + LC + MF)\,\hat{x}$$

Determining M can be recognized as an observer gain design problem:

$$\dot{\hat{x}} = \left(\tilde{A} + \tilde{L}\tilde{C}\right)\hat{x}$$

with $\tilde{A} = A + LC$, $\tilde{L} = M$, and $\tilde{C} = F$, from which the unknown $\tilde{L} = M$ can be chosen to assign any desired poles to the saturated controller.

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Optimal control

We consider a linear control system of the form:

$$\begin{aligned} \dot{x} &= Ax + Bu\,, \quad x(0) = x_0 \\ y &= Cx \end{aligned}$$

A control law for such a system is said to be *optimal*, if it minimizes the cost functional:

$$\mathcal{J} = \int_0^\infty x^T Q x + u^T R u \; dt$$

where $Q = Q^T$ is a positive semi-definite matrix and $R = R^T$ is a positive definite matrix.

The algebraic Riccati equation

An *Algebraic Riccati Equation* is a second order matrix equation in an indeterminate $P = P^T \in \mathbb{R}^{n \times n}$ of the form:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ are matrices, $R = R^T \in \mathbb{R}^{m \times m}$ is a positive definite matrix, and $Q = Q^T \in \mathbb{R}^{n \times n}$ is a positive semidefinite matrix.

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P is called a *stabilizing solution* to the ARE, if it satisfies the equation, and further satisfies that the eigenvalues of $A - BR^{-1}B^T P$ are in the open left half plane.

Optimal state feedback control

Theorem

Consider a linear system of the form:

$$\begin{aligned} \dot{x} &= Ax + Bu\,, \quad x(0) = x_0 \\ y &= Cx \end{aligned}$$

Let P be a stabilizing solution to the ARE:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

Then the optimal state feedback law is given by:

$$u = Fx \quad \text{where} \quad F = -R^{-1}B^T P$$

Optimal state estimation

Given the system

$$\begin{aligned} \dot{x} &= Ax + Bu + Gw \\ y &= Cx + Du + v \end{aligned}$$

with unbiased process noise w and measurement noise v with covariances

$$\mathcal{E}\{ww^T\} = Q\,, \quad \mathcal{E}\{vv^T\} = R$$

Then an optimal state estimator is given by:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y)$$

Optimal state estimation

with unbiased process noise w and measurement noise v with covariances

$$\mathcal{E}\{ww^T\} = Q\,, \quad \mathcal{E}\{vv^T\} = R$$

Then an optimal state estimator is given by:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y)$$

where

$$L = -PC^T R^{-1}$$

P is a stabilizing solution to the ARE:

$$AP + PA^T - PC^T R^{-1} CP + Q = 0$$

Output variance minimization

Introducing $y = Cx$ into a cost functional of the type

$$\mathcal{J} = \int_0^\infty \rho y^T y + u^T u \, dt \,, \quad \rho \in \mathbb{R}$$

this can be written as an optimal control problem

$$\begin{aligned} \mathcal{J} &= \int_0^\infty \rho y^T y + u^T u \, dt \\ &= \int_0^\infty \rho x^T C^T C x + u^T u \, dt \\ &= \int_0^\infty x^T Q x + u^T R u \, dt \,, \quad Q = \rho C^T C \,, R = I \end{aligned}$$

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Example: optimal control

We consider once again the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y &= \begin{pmatrix} -3 & 2 \end{pmatrix} x \end{aligned}$$

Computing an optimal state feedback for the cost functional:

$$\mathcal{J} = \int_0^\infty 2.25\, y^T y + u^T u \, dt$$

can be done with the MATLAB™ command

```
Fopt = -lqr(A,B,2.25*C'*C,1)
```

Example: optimal control

This yields the result:

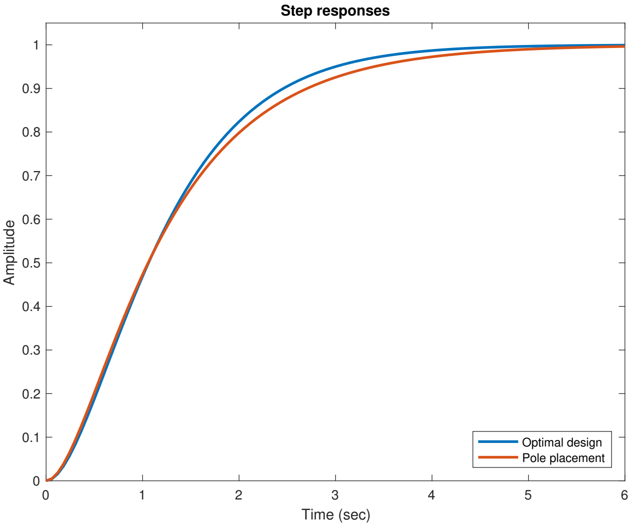
$$F_{\text{opt}} = \begin{pmatrix} 1.1754 & -0.8377 \end{pmatrix}$$

In comparison, a pole assignment with the poles $\{-1, -3\}$ leads to the gain:

$$F_{\text{pa}} = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

It can be seen that the two feedbacks have comparable gains. The optimal controller, however, gives a slightly better rise time.

Example: optimal control



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