Digital Signal Processing
ESD-5 & IV-5 (elektro), E24

8. The Discrete Fourier Transform – The Discrete Fourier Series

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A few words on the mini project

- The topic is ”Studies and Application of the Frequency Sampling Method for FIR Filter Design”.
- The idea it to get more insigth into the important domain of FIR filter design.
- An alternative to the Window Method, known as the Frequency Sampling Method, should be studied, described, and applied on a given specification.
- The Mini Project accounts for 1 ECTS.
- Each group hand in an approx. 10 pages report on their findings – so ONE report per project group.
- Deadline is Monday December 2.
- At the oral exam, some time will be allocated for discussion on the theory and the findings reported in our project.

The Discrete Fourier Transform, DFT

In this lecture we will initiate our discussion on the Discrete Fourier Transform which is the fundamental transform used for all ”computerized” calculation of the spectral content of a discrete-time signal.

In four lectures we will address the following topics;

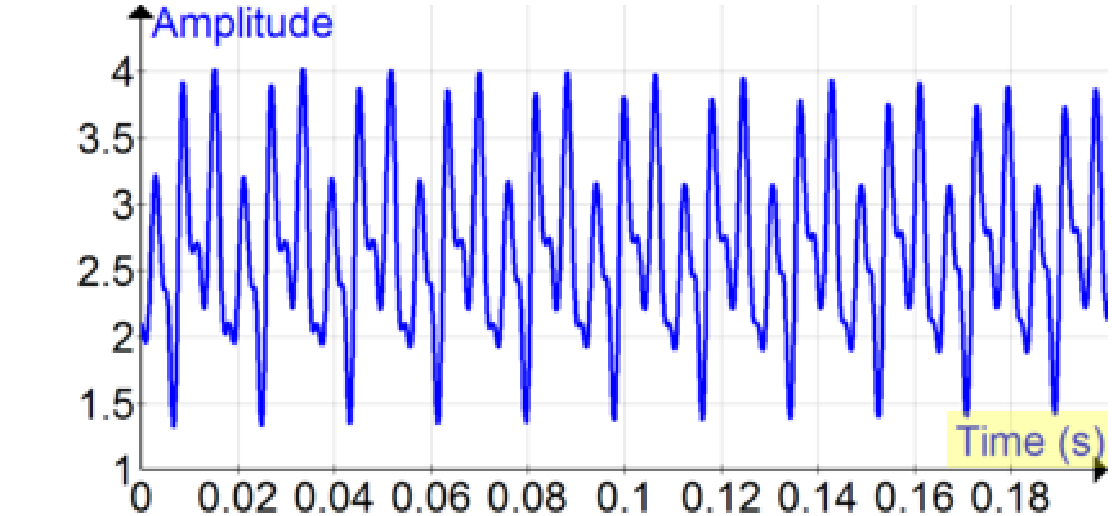
1. Introduction to the DFT, we will start out with the Discrete Fourier Series

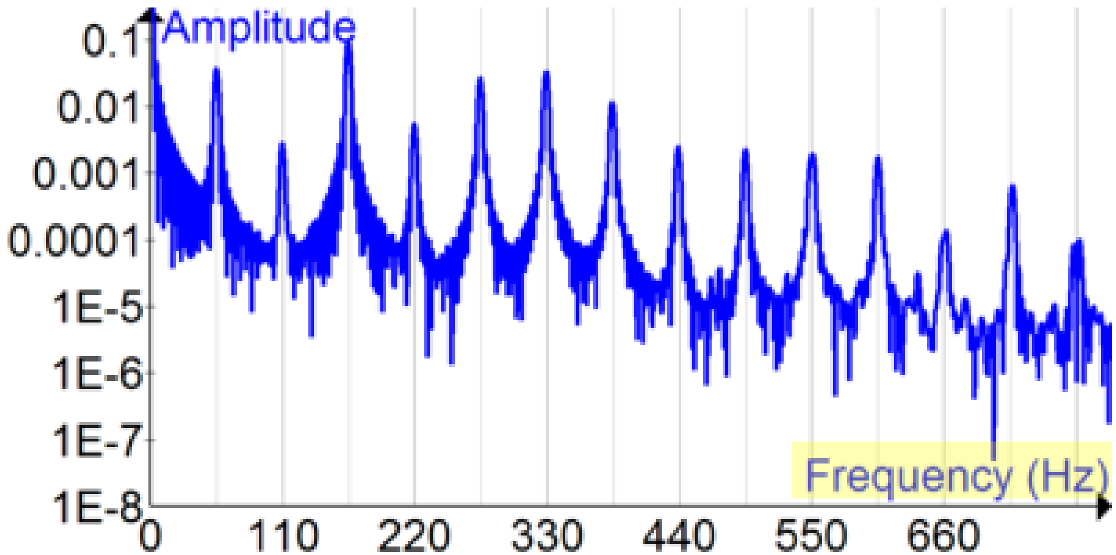
2. Properties of the DFT, e.g., various types of Convolution

3. Efficient computation of the DFT; The Fast Fourier Transform (FFT)

4. Practical implementation; The Short Time Fourier Transform (STFT)

Why concern about the Fourier Transform





Joseph Fourier

lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

The Fourier Transform

The Fourier transform exists for signals which are

- continuous in time
- discrete in time

For each of these two cases, the "time observation" can be divided into

- finite in duration
- infinite in duration

This results in four cases.

Therefore, the frequency domain also has four cases; continuous and discrete frequency, and finite and infinite bandwidth.

When time is continuous, the frequency axis is infinite, and vice versa.

A overview of the four categories of the Fourier transform

After a brief recap of the DTFT, today we will start with the FS and then end up with the DFT – we are aiming for discrete frequency.

Time Duration		
Finite	Infinite	
Discrete FT (DFT) $$X(k)=\sum_{n=0}^{N-1} x(n)e^{-j\omega_k n}$$ $k=0,1,\ldots,N-1$	Discrete Time FT (DTFT) $$X(\omega)=\sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$$ $\omega\in[-\pi,+\pi)$	discr. time n
Fourier Series (FS) $$X(k)=\frac{1}{P}\int_0^{P} x(t)e^{-j\omega_k t}dt$$ $k=-\infty,\ldots,+\infty$	Fourier Transform (FT) $$X(\omega)=\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$ $\omega\in(-\infty,+\infty)$	cont. time t
discrete freq. k	continuous freq. ω	

P denotes one period of the signal.

What we already have used many times…
The Discrete Time Fourier Transform (DTFT)

The transformation from time to frequency, the DTFT

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

…and here the transformation from frequency to time, the IDTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

Note that $X\left(e^{j\omega}\right)$ is a function in continuous frequency…!

The DTFT is a complex number

$$X\left(e^{j\omega}\right) = X_R\left(e^{j\omega}\right) + jX_I\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right| e^{j\measuredangle X\left(e^{j\omega}\right)}$$

Rectangular form Polar form

As we have seen many times, this complex number has a modulus and an argument, also known as the Amplitude Response $\left|X\left(e^{j\omega}\right)\right|$ and the Phase Response $\angle X\left(e^{j\omega}\right)$.

DTFT pair

Sequence	Fourier Transform	
1. $\delta[n]$	1	Impulse <-> Flat spectrum
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$	Time shifted impulse
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$	DC
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$	Exponential descreasing
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$	
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$	
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2 e^{-j2\omega}}$	
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$	Lowpass filter
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$	Rect. window
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\omega} 2\pi\delta(\omega-\omega_0+2\pi k)$	Complex exponential
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$	

To get started with the DFT, let's first have a look at the Fourier Series, i.e., initially the Continuous-time domain

What is a Fourier Series…??

A Fourier Series is a representation of a CONTINUOUS-TIME PERIODIC signal using sinusoids…

- This is extremely usefull because sinusoids have simple interaction with LTI systems; If we know how the system responds to a sinusoid, and our signal is represented as a sum of sinusoids, then using the principle of super-position we can find the response to the signal – how..??

- Fourier Series are also used as an Analysis Tool for finding amplitude- and phase information of the periodic continuous-time signal.

Fourier Series – the basic approach

A continuous-time periodic signal can be represented as a sum of continuous-time complex sinusoids.

The (observable) signal $x(t)$ is periodic with period P.

The sinusoids should also have period P, i.e., they are hamonically related with individual frequencies;

$$\Omega_k = k \cdot 2\pi f = k\frac{2\pi}{P} = k\Omega_f$$

where $\Omega_f = \frac{2\pi}{P}$ is the fundamental (angular) frequency (dansk: grund-frekvensen)

So, the signal $x(t)$ can be expressed as

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_f t}}$$

where X_k are weight factors (complex amplitudes), and the exponentials are complex sinusoids with frequencies $k\Omega_f$.

Main point:
For periodic signals, all spectral lines have frequencies that are *integer* multiples of the fundamental frequency

Are these spectral lines located for only positive frequencies, or are they located also for negative frequencies…???

Although it may be difficult to imagine negative frequencies, it is nevertheless what we have, when we study the signal from a mathematical point of view.

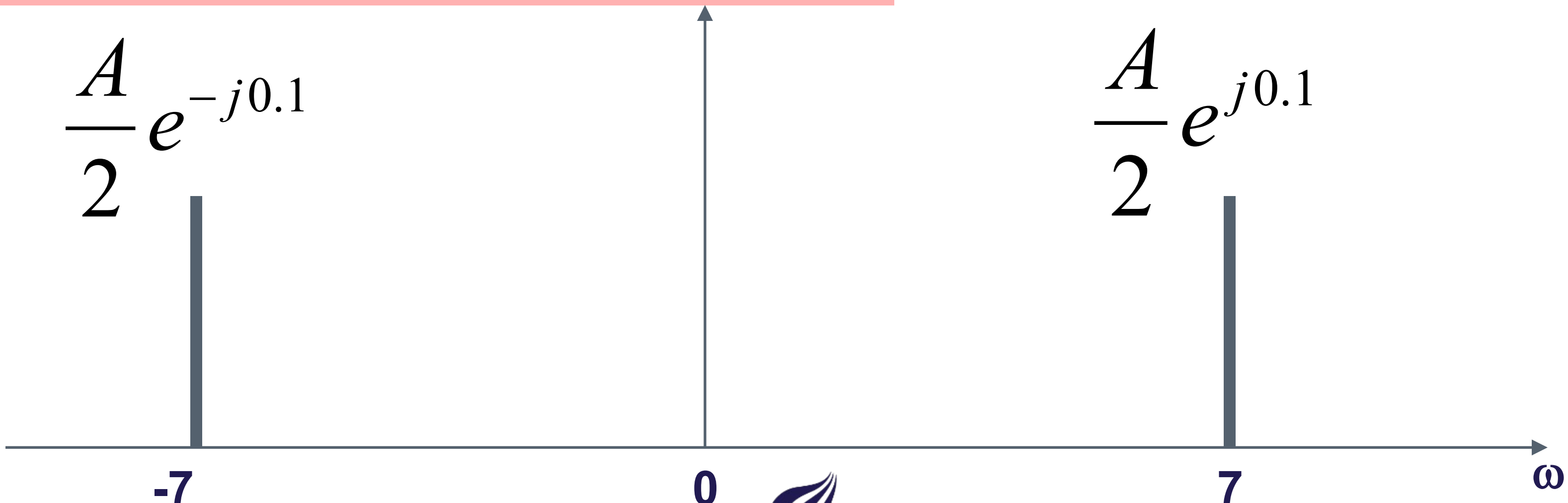
Example; $x(t) = A\cos(7t) = \frac{A}{2}e^{j7t} + \frac{A}{2}e^{-j7t}$

In this example, the angular frequency is $\omega = \pm 7\ rad/s$.

Making it a bit more realistic, let's also introduce a phase in $x(t)$;

$$x(t) = A\cos(7t + 0.1) = \frac{A}{2}e^{j0.1}e^{j7t} + \frac{A}{2}e^{-j0.1}e^{-j7t}$$

The frequency domain representation of $x(t)$ then looks like this;



SPECTRAL REPRESENTATION

Fourier Series – the basic approach

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_f t}$$

We are now interested in finding the weight factors X_k, the Fourier coefficients.

X_k can be found by integrating the signal $x(t)$ against a complex sinusoid over one period of the signal $x(t)$;

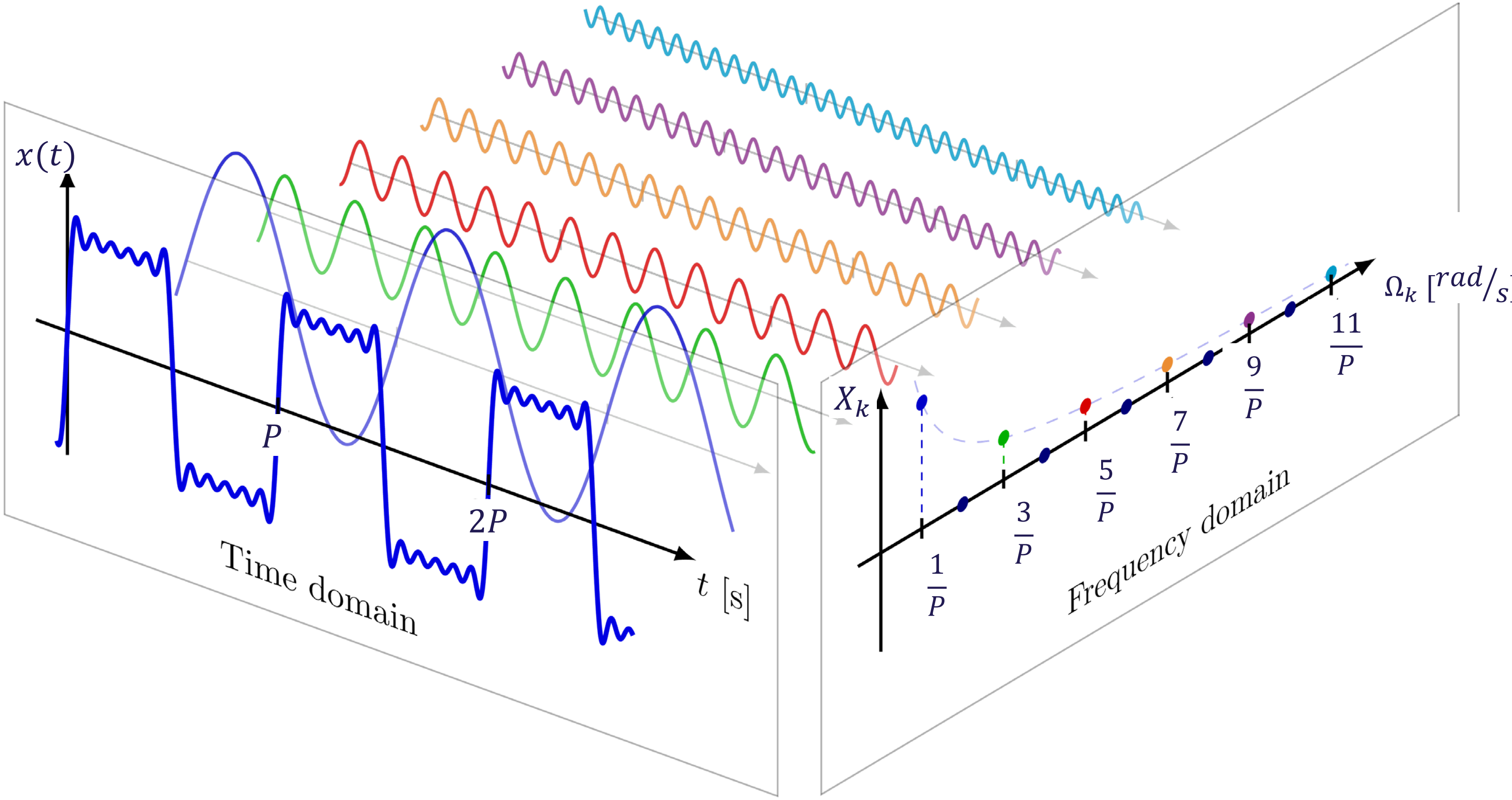
$$X_k = \frac{1}{P}\int_0^P x(t)e^{-jk\Omega_f t}\,dt = \frac{\Omega_f}{2\pi}\int_0^P x(t)e^{-jk\Omega_f t}\,dt$$

So, knowing $x(t)$, we can find X_k, and vice versa…

$$x(t) \overset{FS,\Omega_f}{\longleftrightarrow} X_k$$

If $x(t)$ is a real signal, it can be express as a sum of sines and cosines (and not as a sum of complex exponentials, as we have just seen for the general case…) - check your notes from 2nd semester ROD course.

The famous square-wave example…



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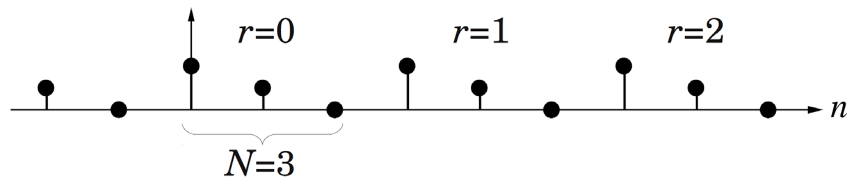
Now, let's turn our attention to the Fourier Series of a periodic sequence - The Discrete Fourier Series

Tilde denotes "periodicity"

$$\tilde{x}[n] = \tilde{x}[n+rN]$$



N is the period and r represents the period number.

Similarly to the continuous-time case, in the general case, the periodic sequence can be constructed as a sum of complex exponentials…

Also periodic with period N

$$e_k[n] = e^{j\underbrace{(2\pi/N)}kn} = e_k[n+rN]$$

The fundamental frequency. Similar to the continuous-time, $k\Omega_f = k\frac{2\pi}{P}$

k denotes the frequency index which is multiplied onto the "fundamental frequency".

The periodic sequence is given as a weighted sum of complex exponentials, normally also denoted complex sinusoids;

$$\tilde{x}[n] = \frac{1}{N} \sum_{k} \tilde{X}[k] e^{j(2\pi/N)kn}$$

Note the immediate similarity with the continuous-time case on p.12...

Question now is; how many FS coefficients $\tilde{X}[k]$ are needed…?

Since $e_{k+lN}[n] = e_k[n]$ only N complex exponentials are needed to describe $\tilde{x}[n]$.

Remember that frequency is 2π periodic in the discrete-time domain, i.e., $N \sim 2\pi$

Thus we now have the Fourier Series representation of a periodic sequence $\tilde{x}[n]$;

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn}$$

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We can now obtain the FS coefficients (see Equations 4-9 on p. 652-653)

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn} \Leftrightarrow$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)kn}$$

which is periodic in N, i.e.

So, again we conclude that discrete time leads to periodic frequency...!

$$\tilde{X}[0] = \tilde{X}[N], \tilde{X}[1] = \tilde{X}[N+1] \text{ etc.}$$

In conclusion; the Fourier Series is a periodic sequence for all k.

Basically, this means that now the frequency representation is given by a set of discrete and ”evenly spaced” complex amplitude values.

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DFS of a Periodic Sequence $\tilde{x}[n]$

DFS analysis: $$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

DFS synthesis: $$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

where

$W_N = e^{-j(2\pi/N)}$ ("Twiddle-factor")

Example 8.1 Discrete Fourier Series of a Periodic Impulse Train, p. 654 in O&S

DFS analysis

We consider the periodic impulse train

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \begin{cases} 1, & n = rN, \quad r \text{ any integer}, \\ 0, & \text{otherwise}. \end{cases} \tag{8.14}$$

Since $\tilde{x}[n] = \delta[n]$ for $0 \le n \le N - 1$, the DFS coefficients are found, using Eq. (8.11), to be

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \delta[n] W_N^{kn} = W_N^0 = 1. \tag{8.15}$$

In this case, $\tilde{X}[k]$ is the same for all k. Thus, substituting Eq. (8.15) into Eq. (8.12) leads to the representation

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} e^{j(2\pi/N)kn}. \tag{8.16}$$

DFS synthesis

So, what we see here is that the impulse (in time) gives a flat spectrum (in freq.), and that the impulse train can be expressed as a sum of N complex exponentials of equal amplitude

Duality in the Discrete Fourier Series

Here we let the discrete Fourier series coefficients be the periodic impulse train

$$\tilde{Y}[k] = \sum_{r=-\infty}^{\infty} N\delta[k - rN].$$

Substituting $\tilde{Y}[k]$ into Eq. (8.12) gives

DFS synthesis

$$\tilde{y}[n] = \frac{1}{N}\sum_{k=0}^{N-1} N\delta[k]W_N^{-kn} = W_N^{-0} = 1.$$

Note that we need to look over one period only and thus $\delta[k - rN]$ reduces to $\delta[k]$, i.e., $r = 0$.

So, what we see here is that an impulse located in $k = 0$ (in frequency) is a ”flat” sequence (in time), i.e., a DC signal.

This is ”opposite” to the DFS of an impulse train – thus called a DUALITY…

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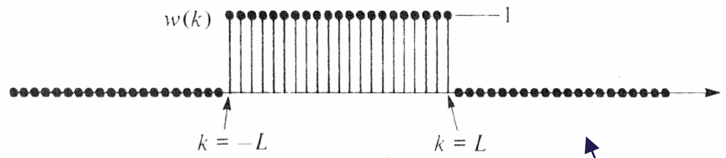
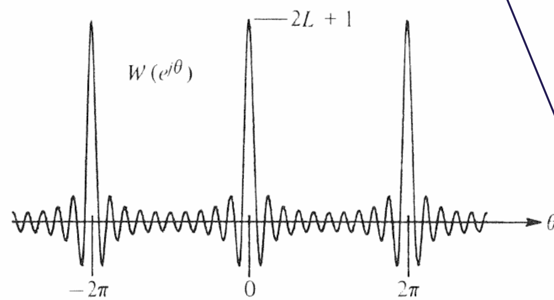
Duality – we have seen something quite similar previously (FIR filter design)

Rectangular Window in time

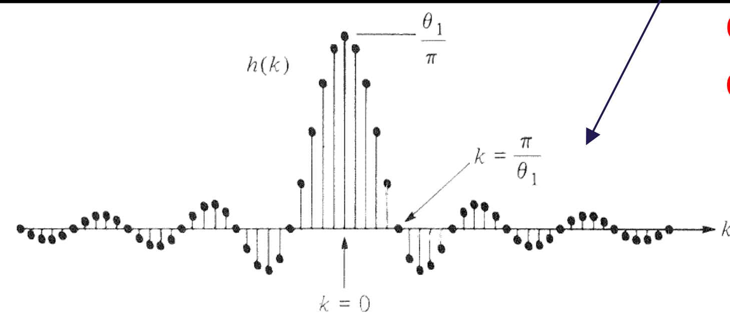


Rectangular Window in freq., Dirichlet function

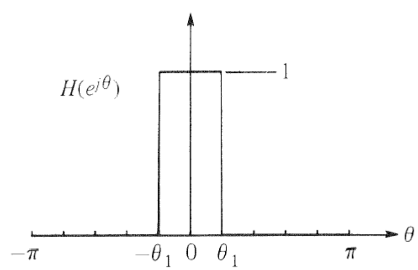


Note however, that these are NOT periodic sequences and thus the freq.-responses are continuous functions of Θ

Impulse response of ideal LP, SINC function



Freq./ Amplitude response of LP



Duality is a consequense of the similarity between DFS analysis and synthesis – let's see it once more…

DFS analysis: $$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

DFS synthesis: $$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

Other important properties of the DFS

Linearity:

$$\left.\begin{array}{l} \tilde{x}_1[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_1[k] \\ \\ \tilde{x}_2[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_2[k] \end{array}\right\} \text{yields}$$

$$a\,\tilde{x}_1[n] + b\,\tilde{x}_2[n] \overset{DFS}{\longleftrightarrow} a\,\tilde{X}_1[k] + b\,\tilde{X}_2[k]$$

Time-shift:

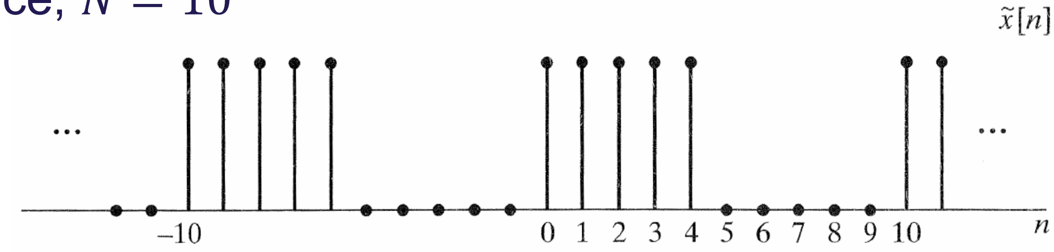
$$\tilde{x}[n-m] \overset{DFS}{\longleftrightarrow} W_N^{km}\,\tilde{X}[k]$$

Frequency-shift:

$$\tilde{X}[k-l] \overset{DFS}{\longleftrightarrow} W_N^{-nl}\,\tilde{x}[n]$$

Example – find the DFS

Periodic sequence, $N = 10$

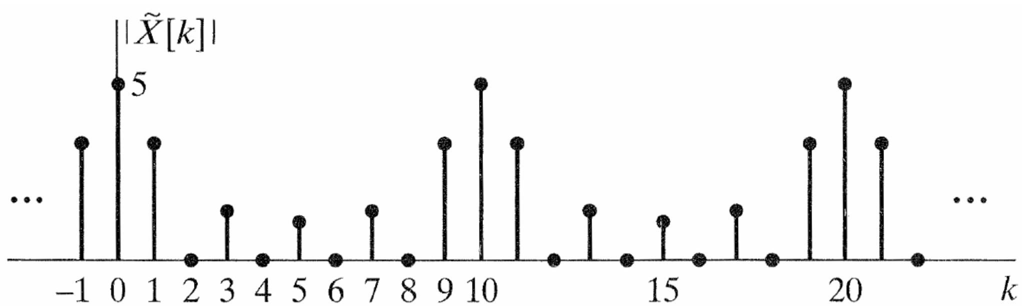


$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

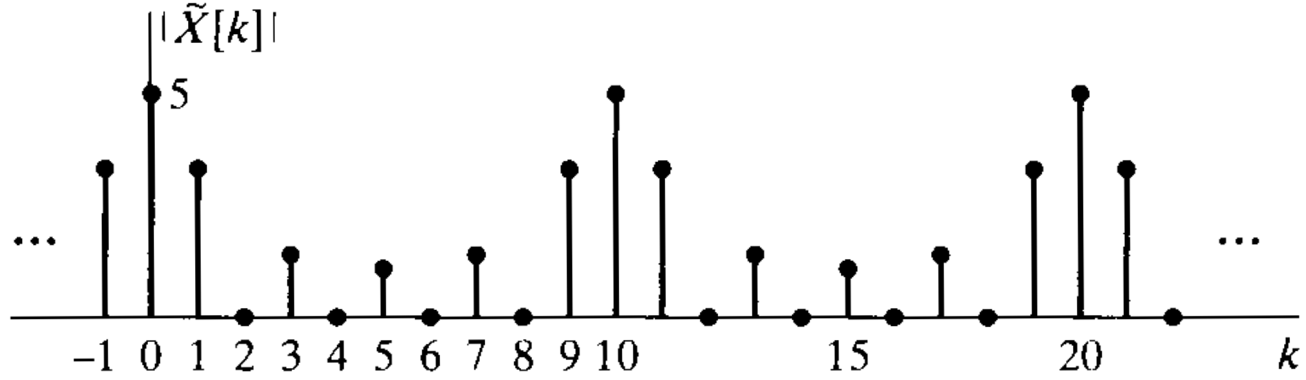
Geometric series where $q = W_{10}^{k}$

$$\tilde{X}[k] = \sum_{n=0}^{4} W_{10}^{kn} = \sum_{n=0}^{4} e^{-j(2\pi/10)kn} \Rightarrow$$

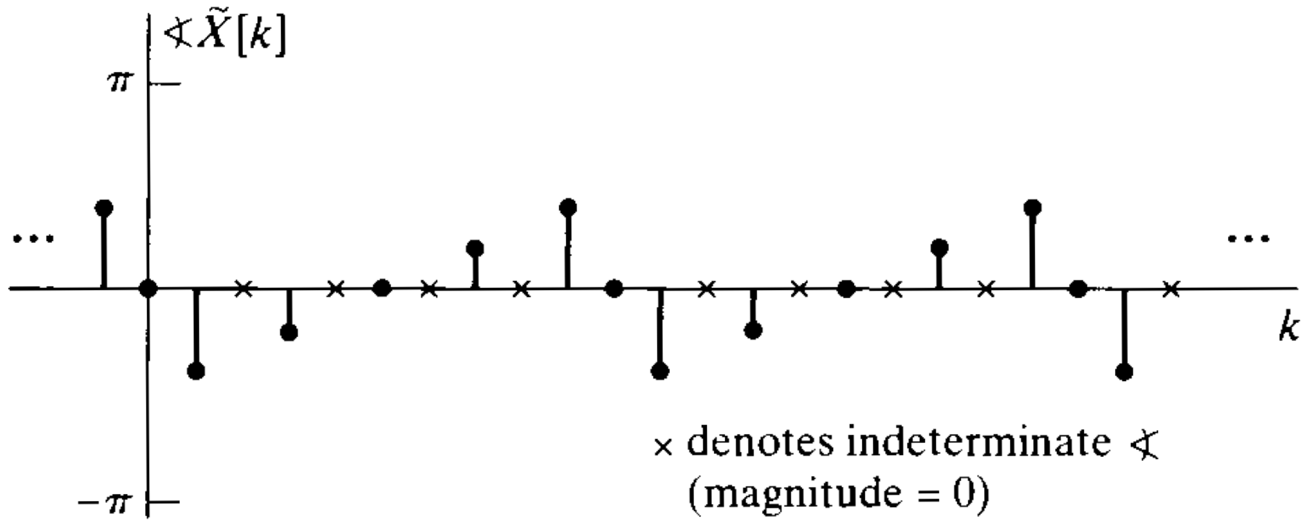
$$\tilde{X}[k] = e^{-j(4\pi k/10)} \frac{\sin(\pi k/2)}{\sin(\pi k/10)}$$



…an interesting observation related to the phase response



(a)



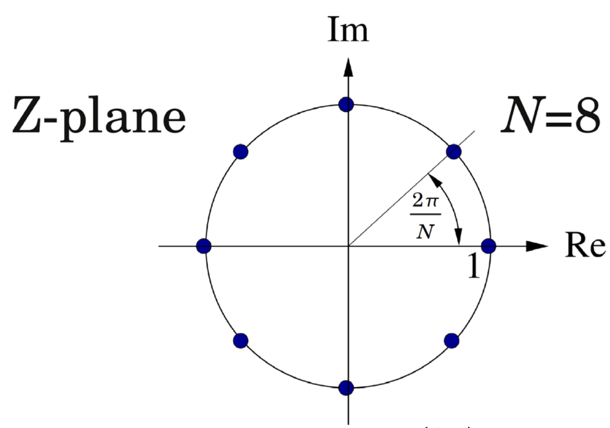
(b)

Relation between periodic and aperiodic sequences

A signal $x[n]$ is not necessarily a periodic sequence – typically, it is aperiodic..! So, how do we handle this situation…?

Remember that the DTFT of a sequence $x[n]$, i.e., $X(e^{j\omega})$, is identically equal to the z-transform $X(z)$ on the unit circle; $z = e^{j\omega}$.

We now consider N equally spaces point on the unit circle (e.g., $N = 8$);



We next sample $X(z)$ in the points $z = e^{j\left(\frac{2\pi}{N}\right)k}$ which leads to a sampled frequency response;

$$X(z)\big|_{z=e^{j\left(\frac{2\pi}{N}\right)k}} = X\left(e^{j\left(\frac{2\pi}{N}\right)k}\right) = X\left(e^{j\omega_k}\right) = \tilde{X}[k] \qquad 0 \le k \le N-1$$

Equ. 50, p. 666

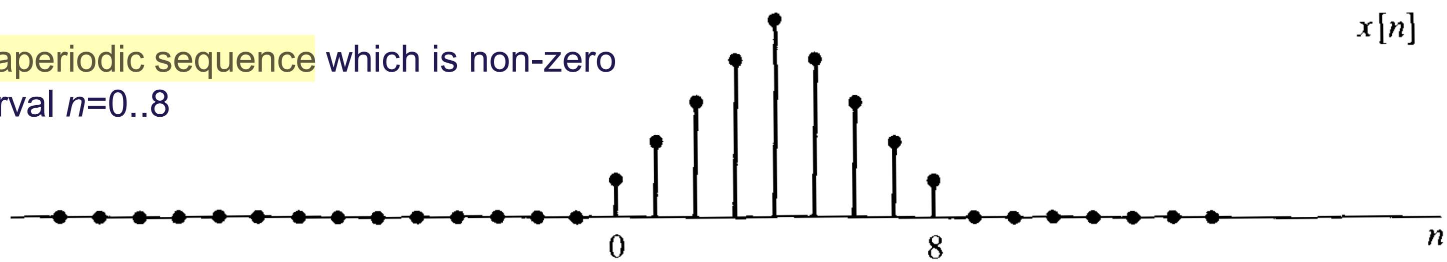
Tilde denotes "periodicity" in k with period N.

$$\tilde{X}[k] = X(z)\Big|_{z=e^{j\left(\frac{2\pi}{N}\right)k}} = X\left(e^{j\left(\frac{2\pi}{N}\right)k}\right) = X(e^{j\omega_k}) \qquad 0 \le k \le N-1$$

This expression represents an N-periodic sequence of samples which could be the sequence of Discrete Fourier Series coefficients of a sequence $\tilde{x}[n]$.

On p. 667 you'll find the math leading to the conclusion that $\tilde{x}[n]$, which corresponds to $\tilde{X}[k]$ obtained by sampling $X(z)$, is formed from $x[n]$ by adding together an infinite number of shifted replicas of $x[n]$. The shifts are all positive and negative integer multiples of *N.*

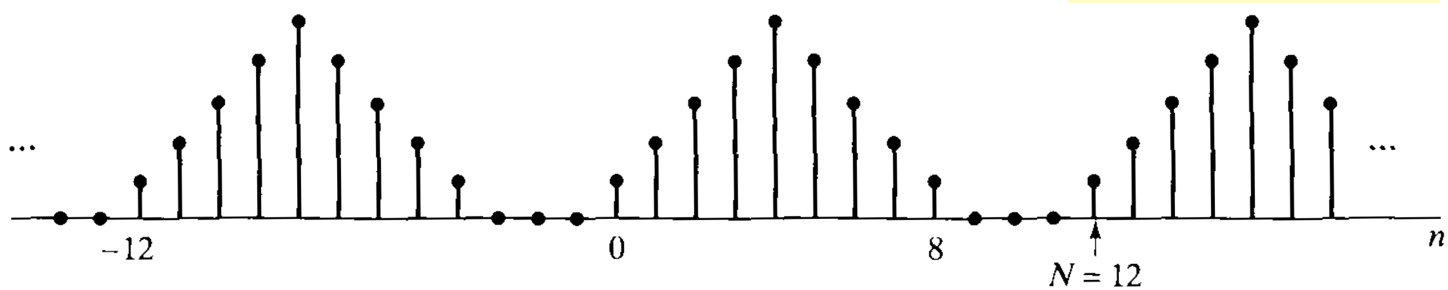
x[*n*] is an aperiodic sequence which is non-zero in the interval *n*=0..8



(a)

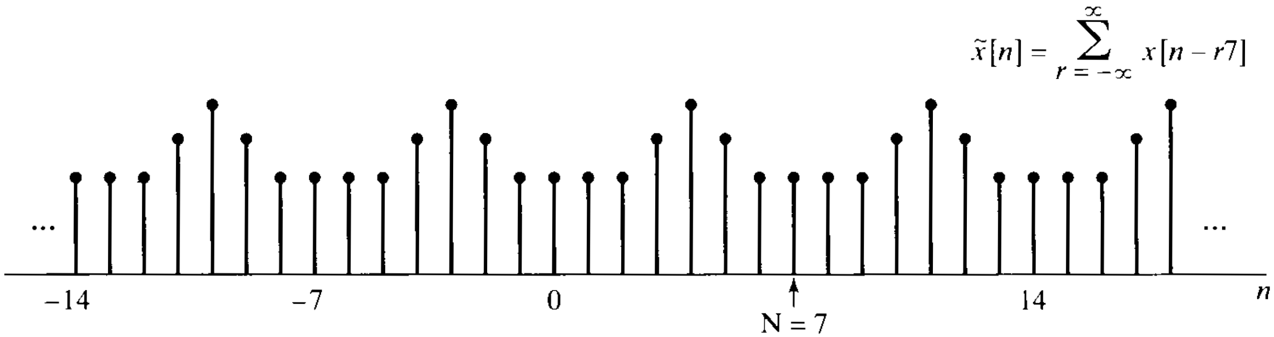
Here we see the periodic sequence derived from *x*[*n*]. The period *N*=12.

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - r12]$$

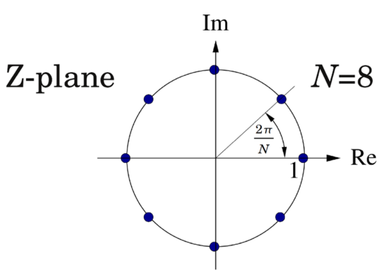


What happens if the period is less than 9 (in the example)..?

Here we have the same sequence $x[n]$, but now the period $N = 7$.



Basically what we see here is "an overlap in the time domain" which can be considered as "aliasing" – N is too small, i.e., we have chosen too few samples in the frequency domain when we sample $X(z)$ on the unit circle.



Consequently, time domain aliasing can be avoided only if $x[n]$ has finite length, just as frequency domain aliasing can be avoided only for signals that have bandlimited Fourier transforms.

OBS…!!

Now we can state the definition of the Discrete Fourier Transform, DFT

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \leqslant k \leqslant N-1 \\ 0, & \text{otherwise} \end{cases}$$

As a consequence, the Discrete Fourier Transform, DFT, can be written as

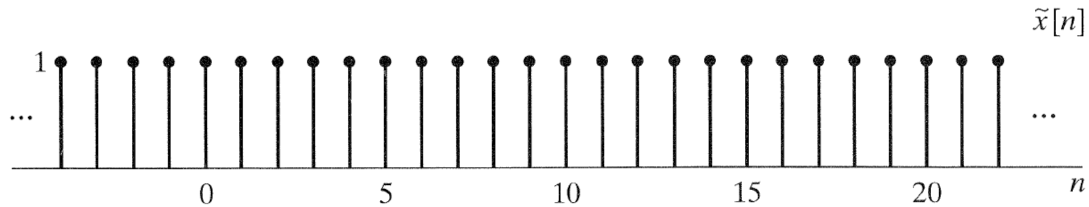
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad \text{and}$$ Equ. 67, p. 672

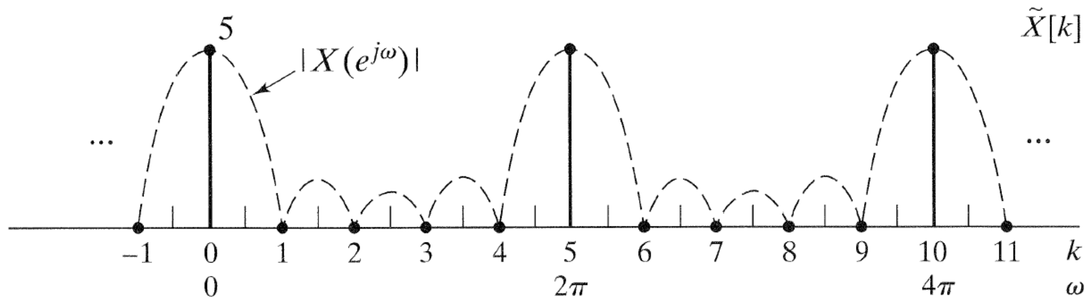
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$ Equ. 68, p. 672

However, we must require that both $X[k]$ and $x[n]$ are identially equal to zero (0) for (k, n) outside $[0; N-1]$, which however is not always stated…

DFT of a pulse…

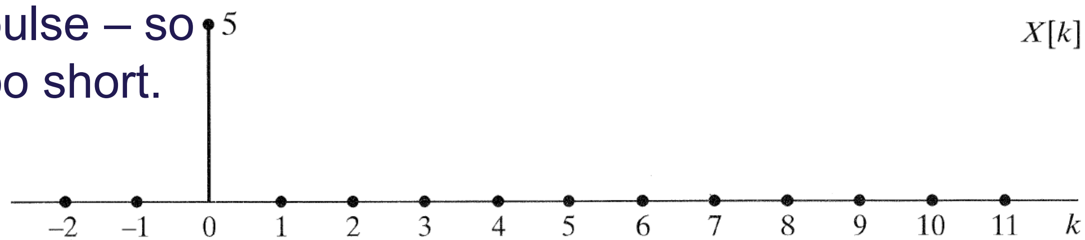
Equal length of the sequence and the period





$X[k]$ must be 0 outside $k = 0..N-1$

We expect to see a Direchlet function but rather we see an impulse – so the period obviously is too short.



DFT of a pulse…

The period is now longer than the sequence length

Yes, now it looks much better – and the larger we choose N, the more pronounced the Dirichlet function will appear…