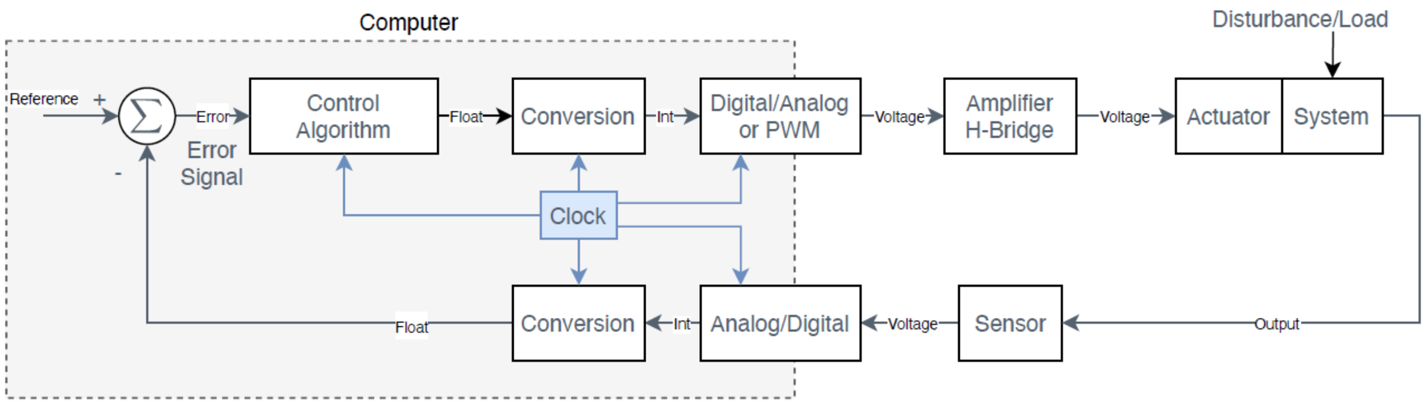
Time domain specifications steady state

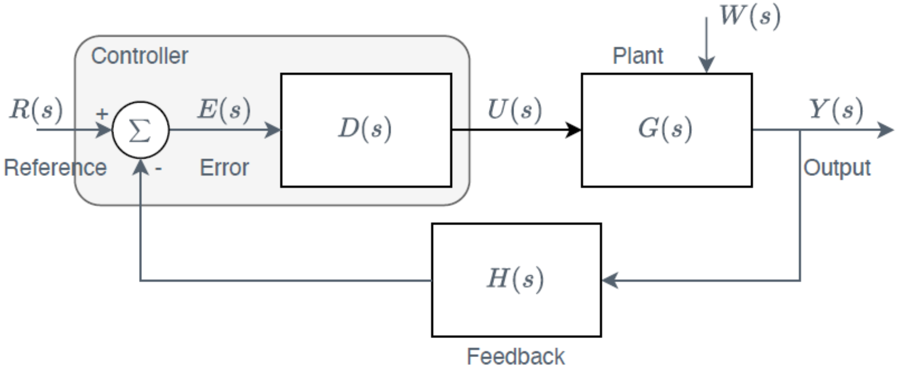
mm2

System setup

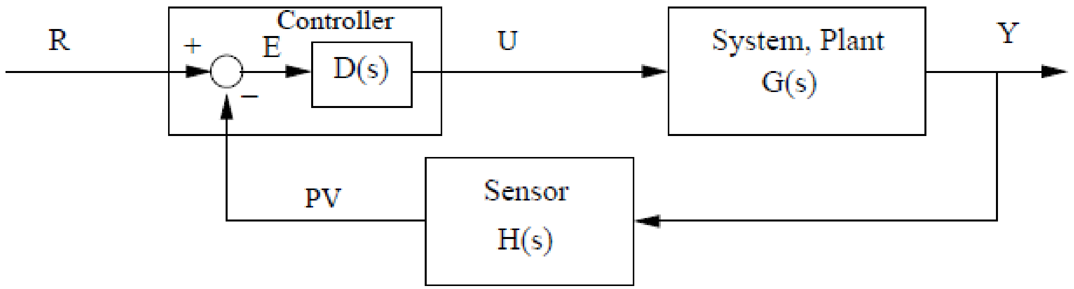
An example of a functional block diagram of a control system:



Same system on standard form for analysis:



Standard set-up



Goal: Control Y

The relation between R and Y is the closed loop transfer function

To fulfill the goal D(s) must be determined.

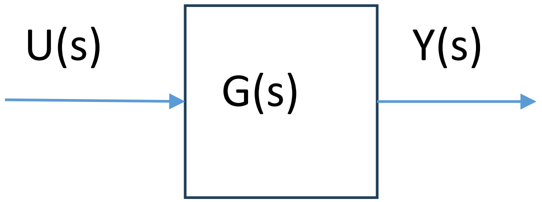
To determine D(s) the transfer functions G(s) and H(s) must be known, and we must have specifications for the total controlled system.

Ideal specifications
Time domain : $y(t) = 1 \cdot r(t) + 0 \cdot w(t)$
Frequency domain : $Y(s) = 1 \cdot R(s) + 0 \cdot W(s)$ In real life that is impossible

Time domain investigations

- Often based on step responses
  - Determination of model parameters
  - Model identification
  - Specifications for the controlled system
  - Poles and zeroes in the transfer function – their relation to dynamics

Transfer functions



The transfer function for a system can be written as

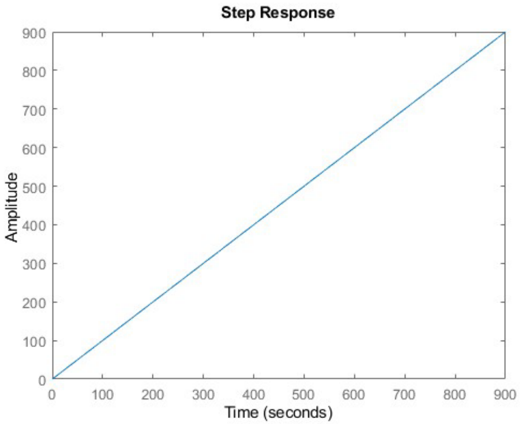
$$\frac{Y(s)}{U(s)} = G(s) = \mathrm{K}\frac{(s+a_m)(s+a_{m-1})\ldots(s+a_1)}{(s+b_n)(s+b_{n-1})\ldots.(s+b_1)}$$

$-a_m, -a_{m-1}, .., -a_1$ are zeroes, $-b_n, -b_{n-1}, .., -b_1$ are poles

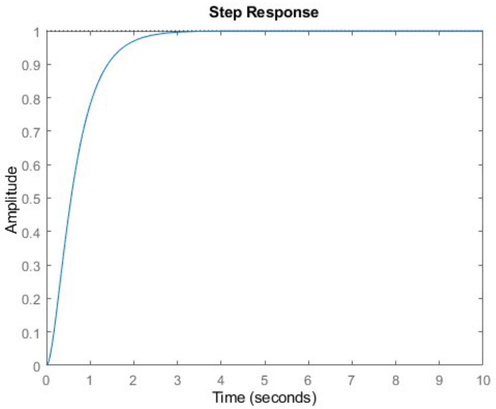
- Poles and zeros are important for the dynamic behavior for the system

Step responses
typical examples for different poles

Integration: $G(s) = \frac{1}{s}$

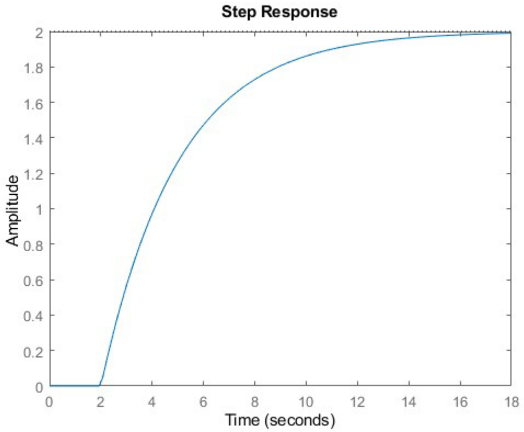


2. order system : $G(s) = \frac{10}{(s+2)(s+5)}$ real poles

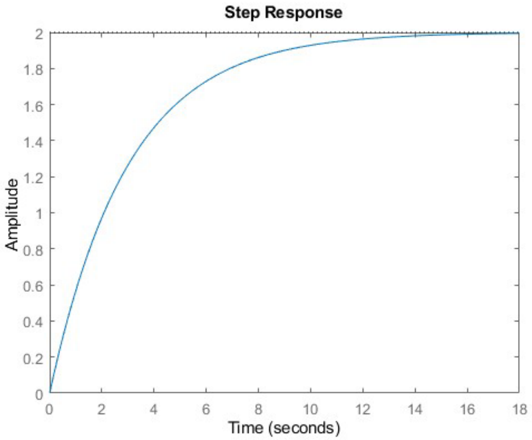


First order system with delay

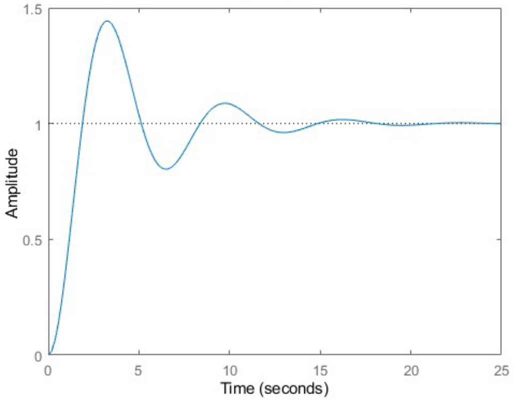
$$G(s) = \frac{2 * e^{-2s}}{3s+1}$$



First order system $G(s) = \frac{2}{3s+1}$



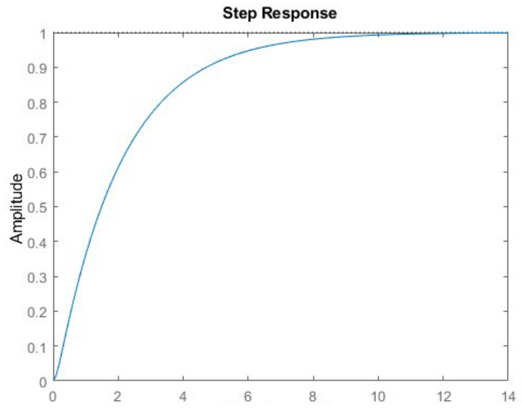
2. order system : $G(s) = \frac{1}{s^2+0.5s+1}$ complex poles



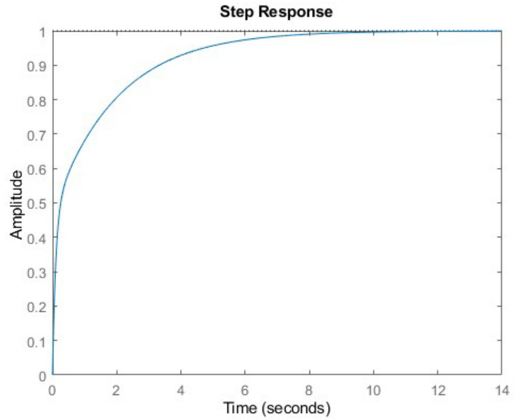
Effect of zeroes

- A LHP zero makes the step response faster.
  - increase the overshoot,
  - decrease the peak time,
  - decrease the rise time,
  - the settling time is not affected too much
- A RHP zero
  - the response can act in different ways
    - go in the opposite direction before rising to the final value. This phenomenon is called undershoot.

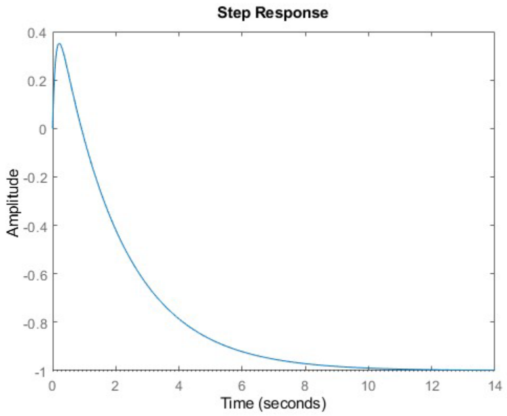
$$G(s) = \frac{5}{(s+0.5)(s+10)}$$



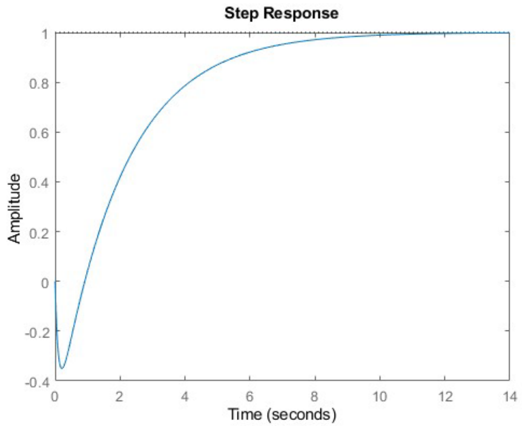
$$G(s) = \frac{5(s+1)}{(s+0.5)(s+10)}$$



$$G(s) = \frac{5(s-1)}{(s+0.5)(s+10)}$$

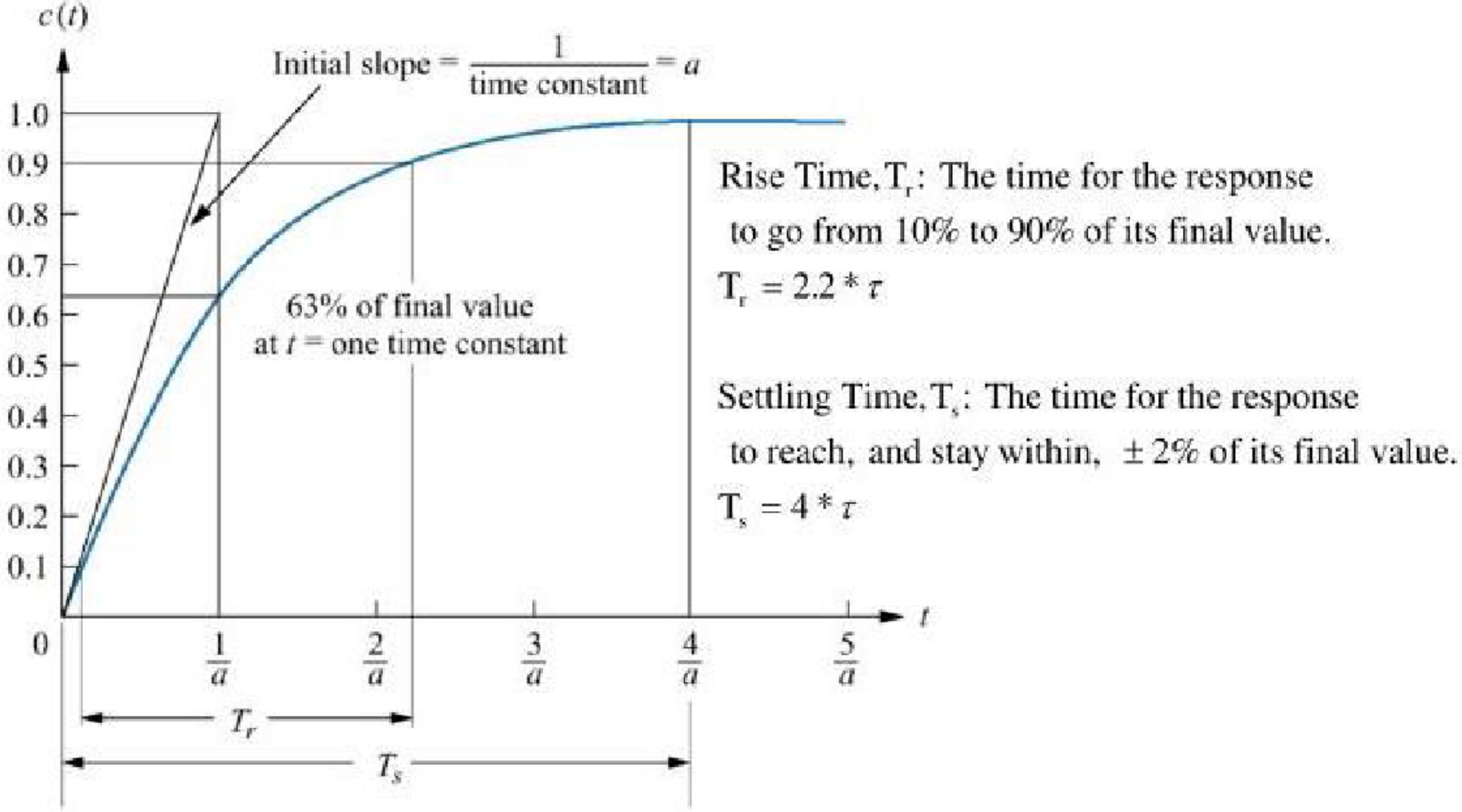


$$G(s) = \frac{-5(s-1)}{(s+0.5)(s+10)}$$

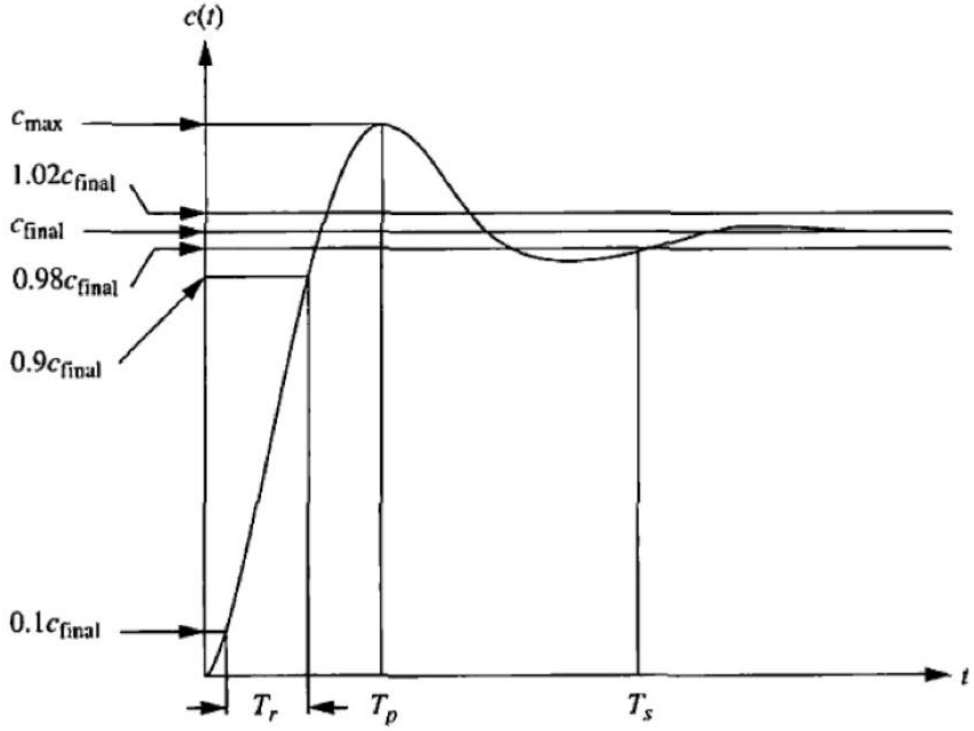


Time domain specifications 1. order system
step response

$$\frac{Y(s)}{U(s)} = \frac{1}{\tau s + 1} = \frac{a}{s+a}, \qquad \tau = \frac{1}{a}$$



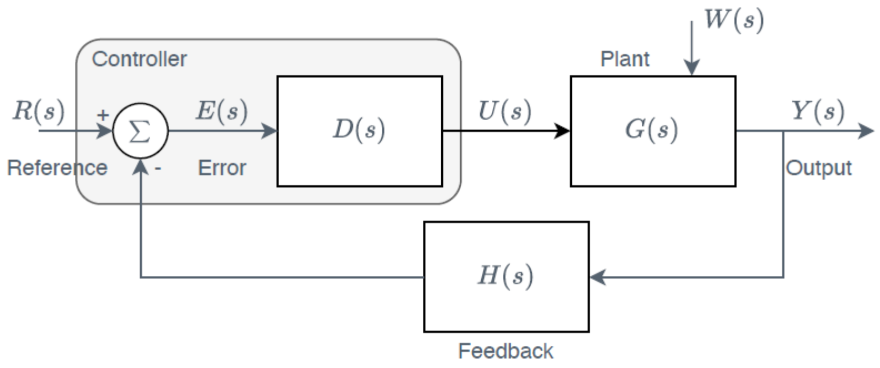
Time domain specifications - 2. order systems
step response



$$\frac{Y(s)}{U(s)} = \frac{b}{s^2 + as + b}$$

Where T_r is the rise time, T_s is the settling time, T_p is the peak time, and c_{max} (M_p) is the overshoot.

Time domain specifications for the controlled system
loop equation



Loop Equations:

$$\textit{Direct term}: \quad D(s)G(s) \tag{3}$$

$$\textit{Open loop}: \quad L(s) = D(s)G(s)H(s) \tag{4}$$

$$\textit{Closed loop}: \quad T(s) = \frac{Y(s)}{R(s)} = \frac{D(s)G(s)}{1 + D(s)G(s)H(s)} \tag{5}$$

Time domain specifications always refers to closed loop $T(s)$!

Time domain specifications
special case: 2nd order system

In the special case where the closed loop is a second order syster

$$T(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

where K is the gain, ζ is the damping factor, and ω_n is the natural frequency.

We have the following design rules:

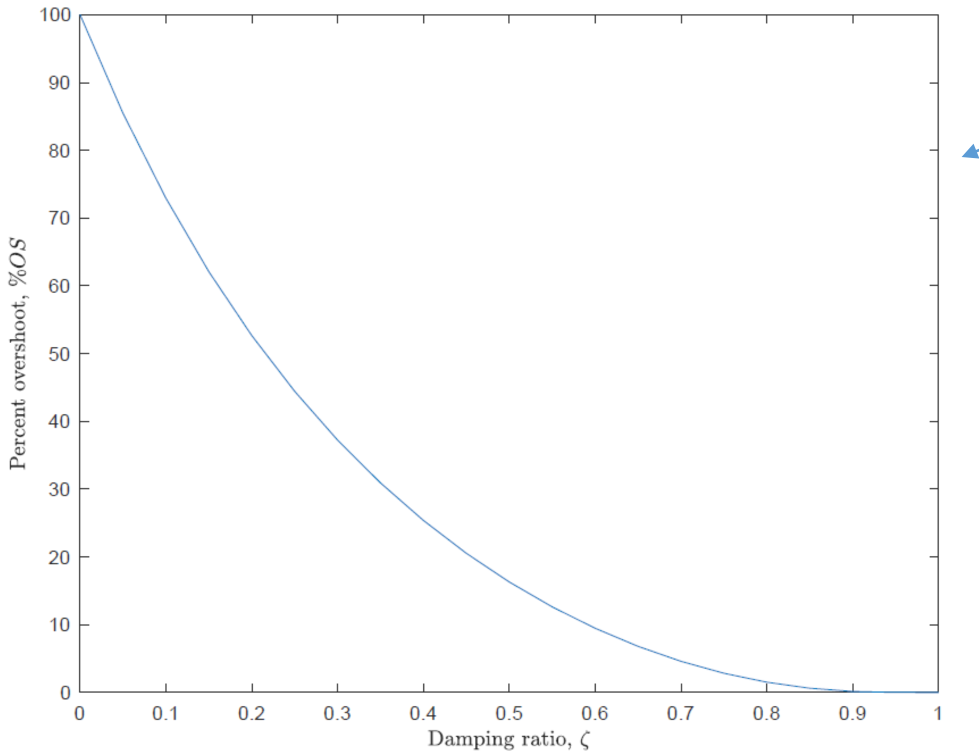
- $T_r \approx \frac{1.8}{\omega_n}$
- $T_s(x) = \frac{-\ln(x)}{\zeta\omega_n} \Rightarrow T_s(0.01) = \frac{4.6}{\zeta\omega_n}$
- $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

Time domain specifications
overshoot versus damping

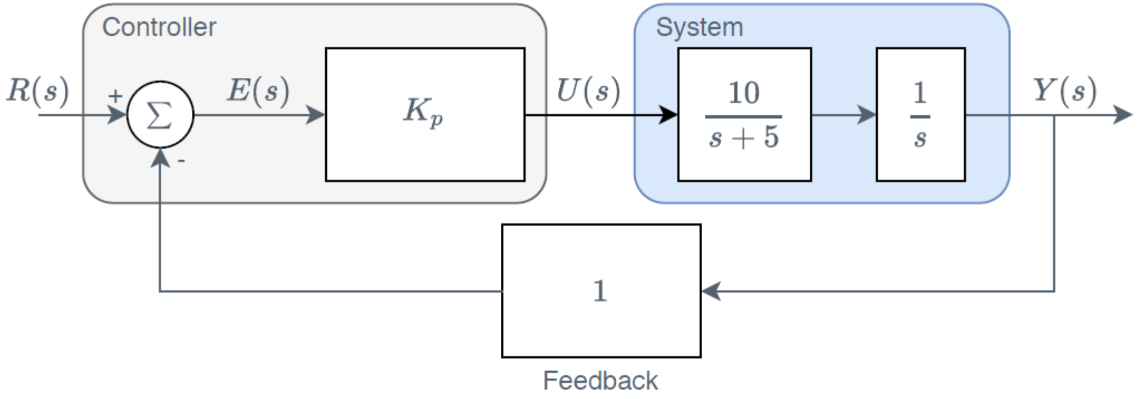
The overshoot percentage can be calculated as:

$$\%OS = 100M_p = 100e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

You can use the graph



Time domain specifications
design example



Closed loop system:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\frac{10K_p}{(s+5)s}}{1 + \frac{10K_p}{(s+5)s}}$$

$$= \frac{10K_p}{(s+5)s + 10K_p} = \frac{10K_p}{s^2 + 5s + 10K_p}$$

Time domain specifications

Design example

$$T(s) = \frac{Y(s)}{R(s)} = \frac{10K_p}{s^2 + 5s + 10K_p} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} \tag{10}$$

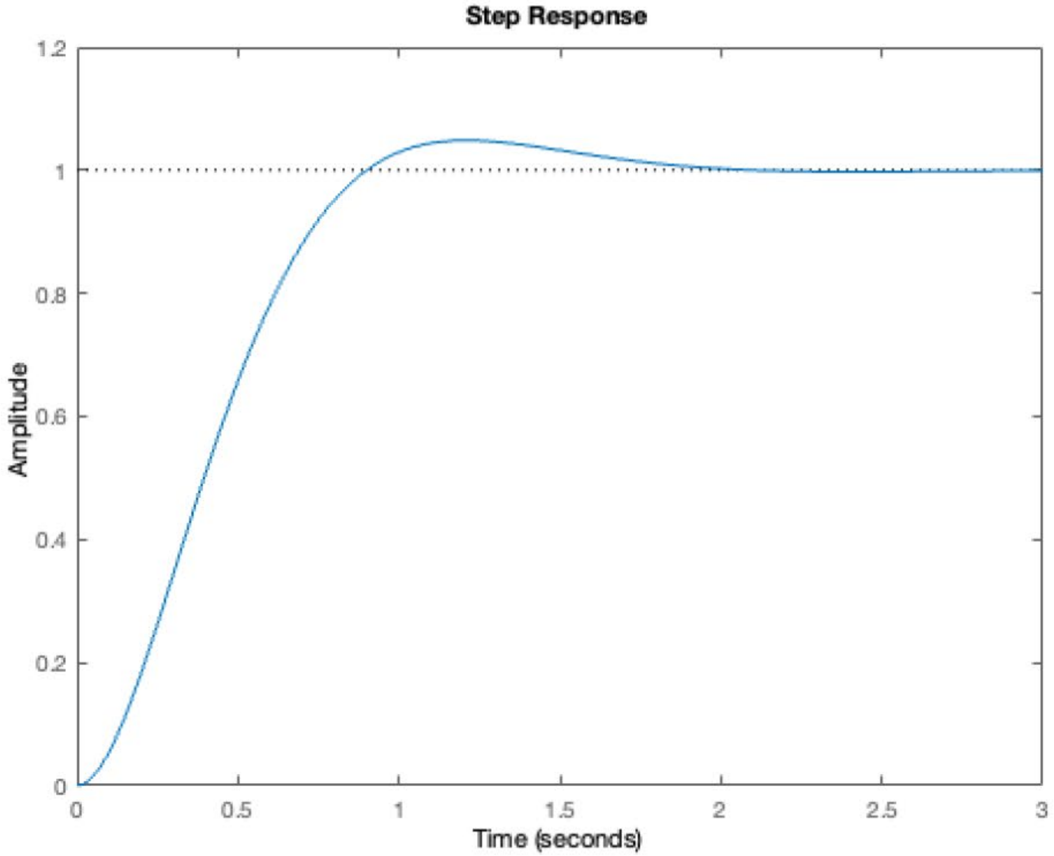
Having: $2\zeta\omega_n = 5$ and $\omega_n^2 = 10K_p$.
Design Specification: Rise time, $T_r = 0.5$ sec.

Procedure:

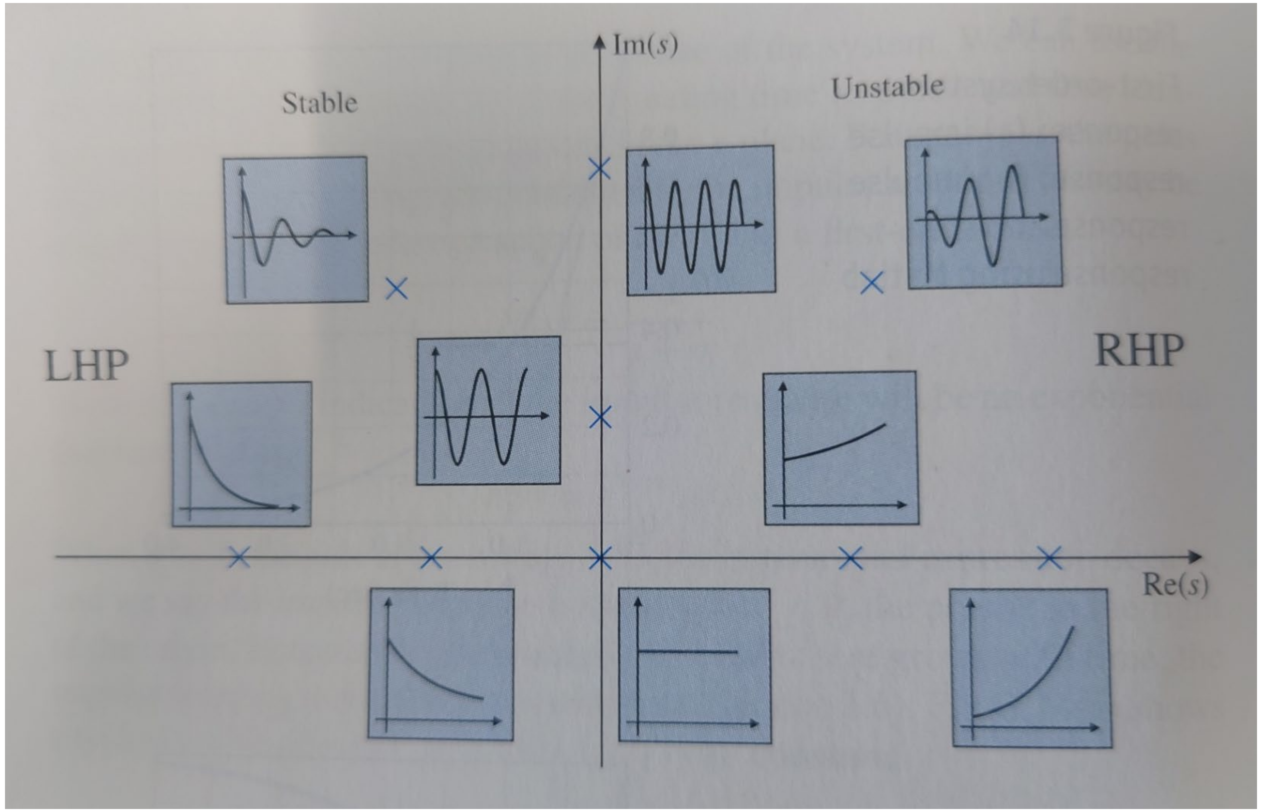
- Find K_p and calculate the overshoot
- $T_r \approx \frac{1.8}{\omega_n} = 0.5 \Rightarrow \omega_n = 3.6$
- $\omega_n^2 = 10K_p \Rightarrow 3.6^2 = 10K_p \Rightarrow \underline{K_p = 1.3}$
- $2\zeta\omega_n = 5 \Rightarrow \zeta = 0.7$
- Resulting in an overshoot of 5%

Time domain specifications design example Matlab

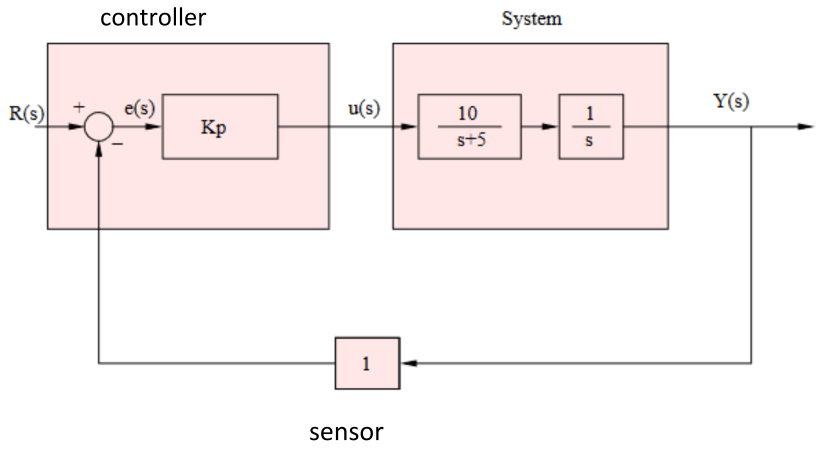
```
s=tf('s');
G=10/((s+5)*s);
H=1;
Kp=1.3;
T=feedback(Kp*G,H);
step(T)
stepinfo(T)
----------------------
Step info:
        RiseTime:  0.5844
    SettlingTime:  1.6615
     SettlingMin:  0.9098
     SettlingMax:  1.0486
       Overshoot:  4.8642
      Undershoot:  0
            Peak:  1.0486
        PeakTime:  1.2158
```



Pole placement and time domain behaviour



Design Example



Closed loop:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\frac{10K_p}{(s+5)s}}{1 + \frac{10K_p}{(s+5)s}}$$

$$= \frac{10K_p}{(s+5)s + 10K_p} = \frac{10K_p}{s^2 + 5s + 10K_p}$$

Design Example

$$\begin{aligned} T(s) &= \frac{10K_p}{s^2 + 5s + 10K_p} \\ &= \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned}$$

Giving $2\zeta\omega_n = 5$ og $\omega_n^2 = 10K_p$.
Specification: Rise time = 0.5 sek.
Find K_p and calculate the overshoot

The design rule gives $t_r \cong \frac{1.8}{\omega_n} = 0.5 \Rightarrow \omega_n = 3.6$. The controller gain K_p is

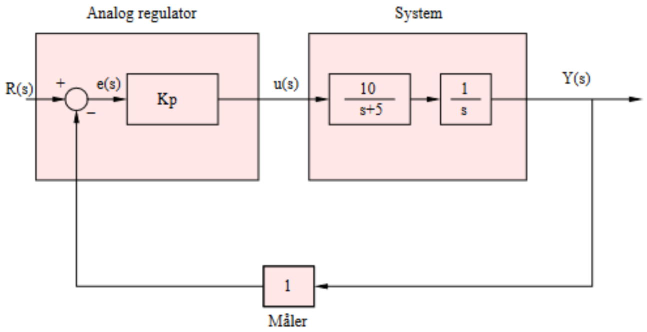
$\omega_n^2 = 10K_p \Rightarrow 3.6^2 = 10K_p \Rightarrow \underline{K_p = 1.3}$. $\omega_n = 3.6$ and $2\zeta\omega_n = 5$ gives $\zeta = 0.7$

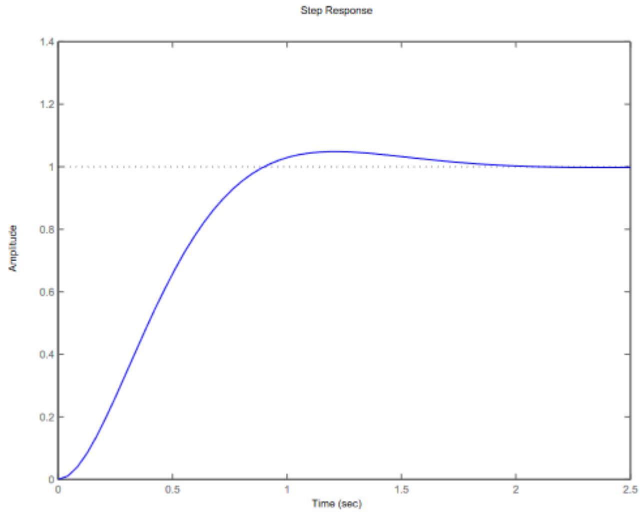
resulting in an overshoot of 5 %.

Design example

```
s=tf('s');
G=10/((s+5)*s); H=1; D=1.3;
T=feedback(D*G,H)
step(T)
stepinfo(T)

Transfer function:
      13
--------------
s^2 + 5 s + 13
ans =
        RiseTime: 0.5851
    SettlingTime: 1.6617
     SettlingMin: 0.9284
     SettlingMax: 1.0485
       Overshoot: 4.8533
      Undershoot: 0
            Peak: 1.0485
        PeakTime: 1.2292
```





BREAK

Example on blackboard

A system is given as $G(s) = \frac{1}{s+5} = \frac{\frac{1}{5}}{\frac{1}{5}s+1}$

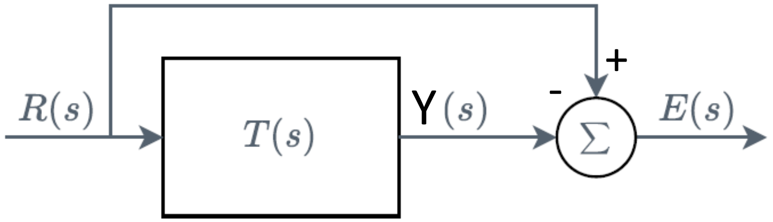
Design a proportional controller K to give a closed loop time constant twice as fast as the open loop time constant.

Steady State error
Definition

The steady state error is defined as the difference between the reference signal, $r(t)$, and the output of the system y(t) as time goes towards infinity

$$e(\infty) = \lim_{t \to \infty} e(t) = \lim_{t \to \infty} (r(t) - y(t)),$$

for a given reference signal. Typically, we would like the error to converge to zero.
In block diagram form, this can be illustrated as:

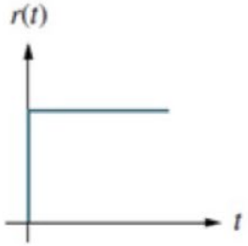
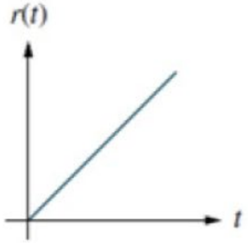
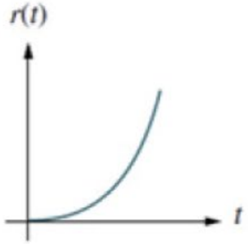


T(s) is the closed loop transfer function

$$E_{steady\ state}(s) = R(s)(1 - Y(s))_{s \to 0}$$

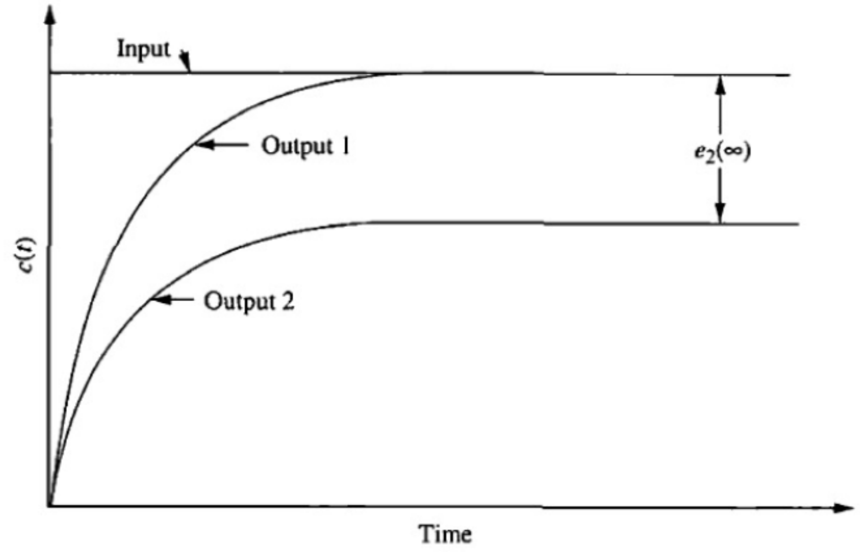
Steady State error

test signals

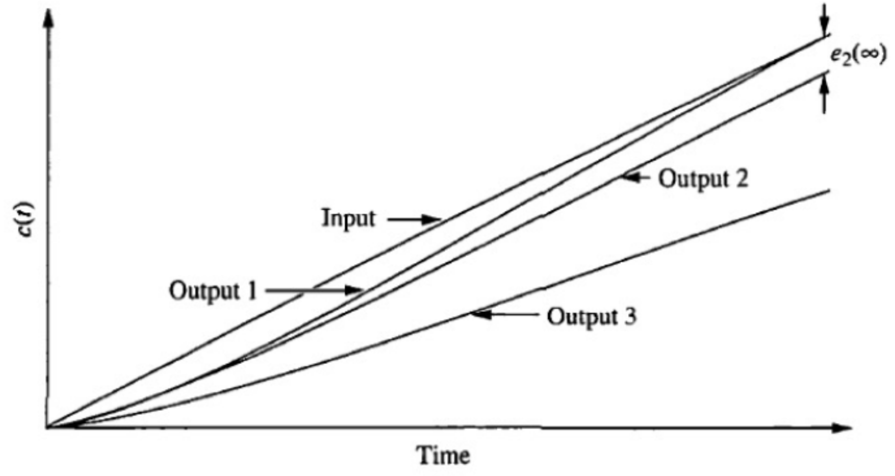
Waveform	Name	Time function	Laplace transform
	Step	1	$\frac{1}{s}$
	Ramp	t	$\frac{1}{s^2}$
	Parabola	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Steady State error

step and ramp input



(*a*)

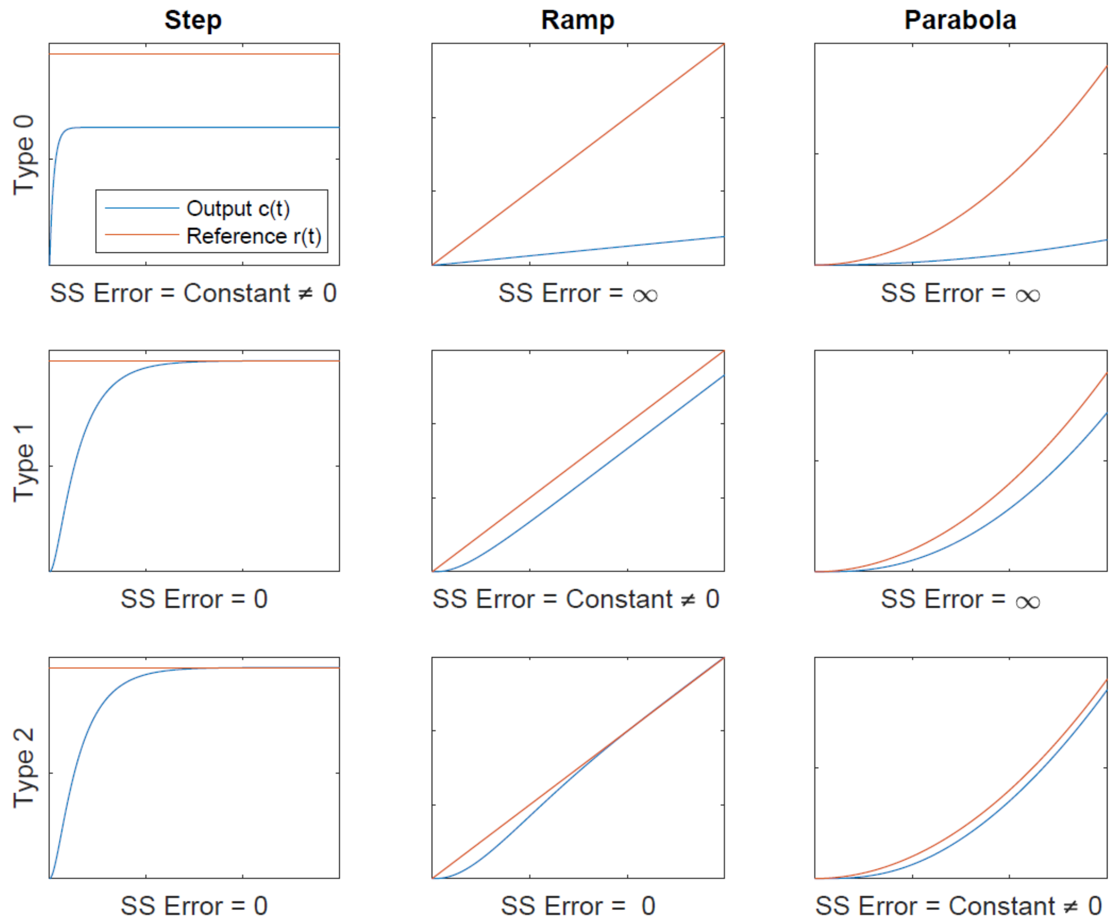


(*b*)

Steady State error

system type and input response

The system type korresponds to the number of integrations



Steady State Error

system type when unity feedback (fx ideal sensor)

System type table

Input		Steady-state error Formula	Type 0		Type 1		Type 2	
			Static error constant	Error	Static error constant	Error	Static error constant	Error
Step,	$u(t)$	$\frac{1}{1+K_p}$	$K_p = \text{constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp,	$tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola,	$\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{constant}$	$\frac{1}{K_a}$

K_p is the position-error constant, K_v is the velocity-error constant, and K_a is the acceleration-error constant.

In the Laplace domain the static error constants are defined as:

$$K_p = \lim_{s\to 0} G(s), \quad K_v = \lim_{s\to 0} sG(s), \quad K_a = \lim_{s\to 0} s^2G(s) \qquad (12)$$

Steady State Error
system type when unity feedback

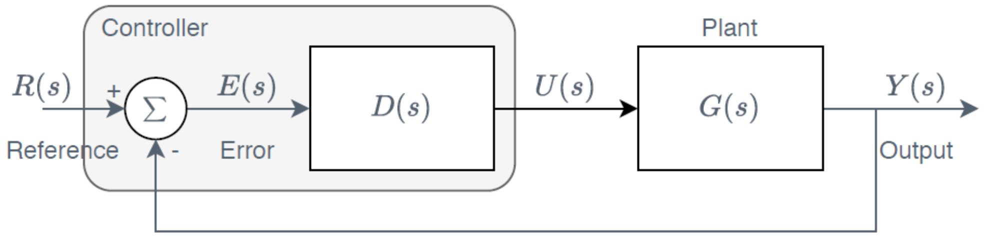
Steady State Procedure

1. Determine the system type by finding n (the number of *poles* in zero).
2. Use the table from the previous slide.
3. Calculate K_p, K_v, and K_a.

You now know the steady state error for a step, ramp, and parabola input for the given system.

Steady State Error

Example: unity feedback, H(s)=1



System type is given by the number of poles in 0 in $D(s)G(s)$.

Examples:

- If $D(s)G(s) = \frac{10}{s(s+5)}$, there is a pole in 0 and one in -5. Thus, system type is 1.
- If $D(s)G(s) = \frac{10}{(s+10)(s+5)}$, there is a pole in -10 and one in -5. Thus, system type is 0.

Notice: $D(s)G(s) = L(s)$, the open loop equation.

Steady State Error

Matlab example

Plotting step, ramp, and parabola response of system

$$T(s) = \frac{s+3}{s^2+3s+5}$$

```
% Matlab program
s=tf('s');
T=(s+3)/(s^2+3*s+5);
figure(1); step(T,3)
figure(2); step(T/s,3)
figure(3); step(T/s^2,3)
```

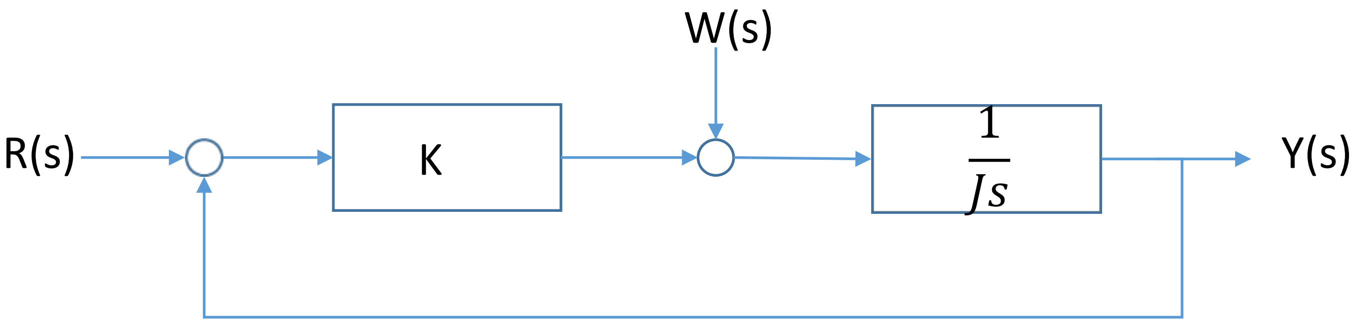
Step response : $T(s)\frac{1}{s}$

Ramp response : $T(s)\frac{1}{s}\frac{1}{s}$

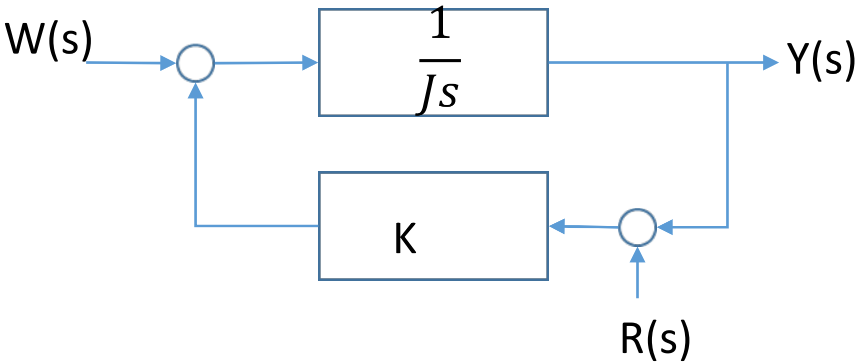
Parabola response : $T(s)\frac{1}{s}\frac{1}{s}\frac{1}{s}$

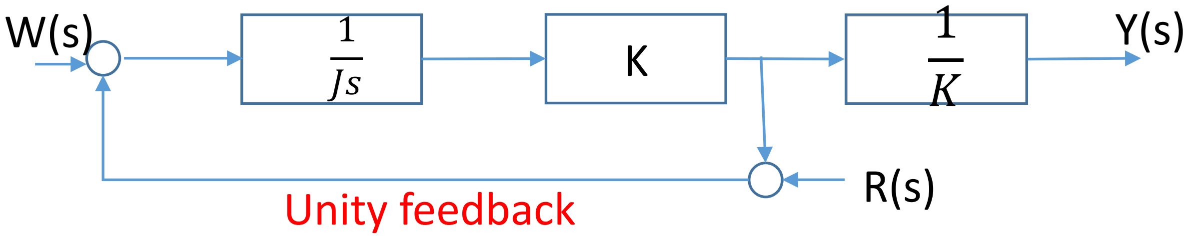
General idea: put $\frac{1}{s}$ and $\frac{1}{s^2}$ in the system and calculate the step response.

Steady state error from reference and disturbance



$\frac{Y(s)}{R(s)} = \frac{K\frac{1}{Js}}{1+K\frac{1}{Js}} = \frac{K}{Js+K} \Rightarrow y(\infty) \Rightarrow \lim_{s\to 0} s * \frac{K}{Js+K} * \frac{1}{s} = \frac{K}{K} = 1$ for step in the reference





$\frac{Y(s)}{W(s)} = \frac{K\frac{1}{Js}}{1+K\frac{1}{Js}}\frac{1}{K} = \frac{1}{Js+K} \Rightarrow y(\infty) = \lim_{s\to 0} s * \frac{1}{Js+K} * \frac{1}{s} = \frac{1}{K}$ for step in the disturbance

We can solve that in the controller design later in the course

Steady State Error

Final remarks

How can a ramp and a parabola be considered as steady state, they are not constants as time goes to infinity?

It is an old story from mechanics where we have position, velocity and acceleration. If we have a system described using the velocity then for a constant (steady state) velocity, the position is a ramp. If we have a system described using the parameter acceleration, then for a steady state acceleration the position is a parabola.

So in the future if we say steady state we consider that our variables have found constant values. Otherwise we talk about ramp and parabola steady state errors.