Exercises, DFT cont.

All exercises this time is in O&S;

- Prob. 6 p.714
- Prob. 10 p.714
- Prob. 11, p.715
- Prob. 14, p.715
- Prob. 15, p.716

Problem 6, p. 714. (1)

Given $x[n] = \begin{cases} e^{j\omega_0 n} & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$

a) Fourier Transform of $x[n]$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{-j(\omega-\omega_0)n}$$

$$= \sum_{n=0}^{N-1} \left(e^{-j(\omega-\omega_0)}\right)^n \quad \text{Geometric series.}$$

$$= \frac{1 - e^{-j(\omega-\omega_0)N}}{1 - e^{-j(\omega-\omega_0)}}$$

$e^{-j(\omega-\omega_0)\frac{N}{2}} \cdot e^{j(\omega-\omega_0)\frac{N}{2}}$

$$= \frac{1 - e^{-j(\omega-\omega_0)\frac{N}{2}} \cdot e^{-j(\omega-\omega_0)\frac{N}{2}}}{1 - e^{-j(\omega-\omega_0)1/2} \cdot e^{-j(\omega-\omega_0)1/2}}$$

$e^{-j(\omega-\omega_0)\frac{1}{2}} \cdot e^{j(\omega-\omega_0)\frac{1}{2}}$

$$= \frac{e^{-j(\omega-\omega_0)\frac{N}{2}}\left(e^{j(\omega-\omega_0)\frac{N}{2}} - e^{-j(\omega-\omega_0)\frac{N}{2}}\right)}{e^{-j(\omega-\omega_0)\frac{1}{2}}\left(e^{j(\omega-\omega_0)\frac{1}{2}} - e^{-j(\omega-\omega_0)\frac{1}{2}}\right)} \quad (2)$$

$$= \frac{e^{-j(\omega-\omega_0)\frac{N}{2}} \cdot 2j\sin\left((\omega-\omega_0)\frac{N}{2}\right)}{e^{-j(\omega-\omega_0)\frac{1}{2}} \cdot 2j\sin\left((\omega-\omega_0)\frac{1}{2}\right)}$$

$$= \frac{e^{-j(\omega-\omega_0)\frac{N}{2}}}{e^{-j(\omega-\omega_0)\frac{1}{2}}} \cdot \frac{\sin\left((\omega-\omega_0)\frac{N}{2}\right)}{\sin\left((\omega-\omega_0)\frac{1}{2}\right)}$$

$$= e^{-j(\omega-\omega_0)\frac{N-1}{2}} \cdot \frac{\sin\left((\omega-\omega_0)\frac{N}{2}\right)}{\sin\left((\omega-\omega_0)\frac{1}{2}\right)}$$

b) The N-point DFT.

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot W_N^{kn} \qquad 0 \le k \le N-1$$

$$= \sum_{n=0}^{N-1} e^{j\omega_0 n} W_N^{kn}$$

$$= \sum_{n=0}^{N-1} e^{j\omega_0 n} \, e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \left(e^{-j\left(\frac{2\pi k}{N} - \omega_0\right)} \right)^n$$

$$= \frac{1 - e^{-j\left(\frac{2\pi k}{N} - \omega_0\right)N}}{1 - e^{-j\left(\frac{2\pi k}{N} - \omega_0\right)}}$$

Now, using the same set of arguments/calculations as before, we derive;

$$X[k] = e^{-j\left(\frac{2\pi k}{N} - \omega_0\right)\left(\frac{N-1}{2}\right)} \cdot \frac{\sin\left[\left(\frac{2\pi k}{N} - \omega_0\right)\frac{N}{2}\right]}{\sin\left[\left(\frac{2\pi k}{N} - \omega_0\right)\frac{1}{2}\right]}$$

from which we conclude;

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \qquad 0 \leq k \leq N-1$$

c) Find DFT of $x[n]$ for $\omega_0 = \frac{2\pi k_0}{N}$ where k_0 is an integer.

$$X[k] = \frac{1 - e^{-j\left(\frac{2\pi k}{N} - \omega_0\right)N}}{1 - e^{-j\left(\frac{2\pi k}{N} - \omega_0\right)}}$$

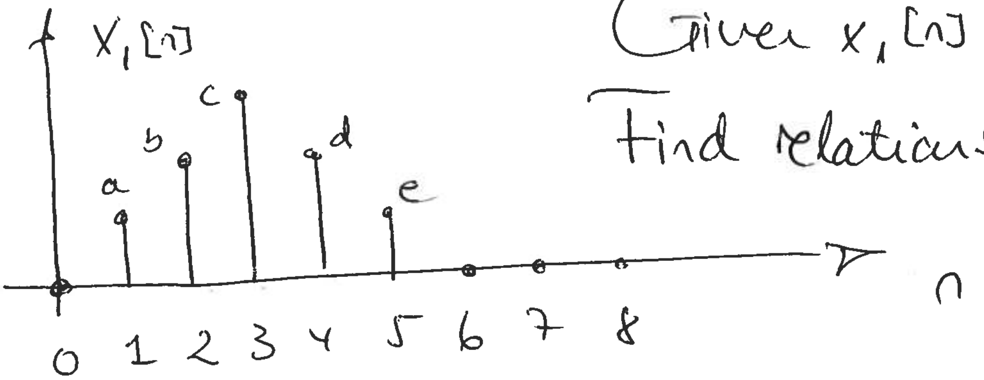
$$= \left. \frac{1 - e^{-j\left(\frac{2\pi k}{N} - \frac{2\pi k_0}{N}\right)N}}{1 - e^{-j\left(\frac{2\pi k}{N} - \frac{2\pi k_0}{N}\right)}} \right|_{\omega_0 = \frac{2\pi k_0}{N}}$$

$$= \frac{1 - e^{-j(k-k_0)2\pi}}{1 - e^{-j(k-k_0)\frac{2\pi}{N}}}$$

and once again using the same calculations as before;

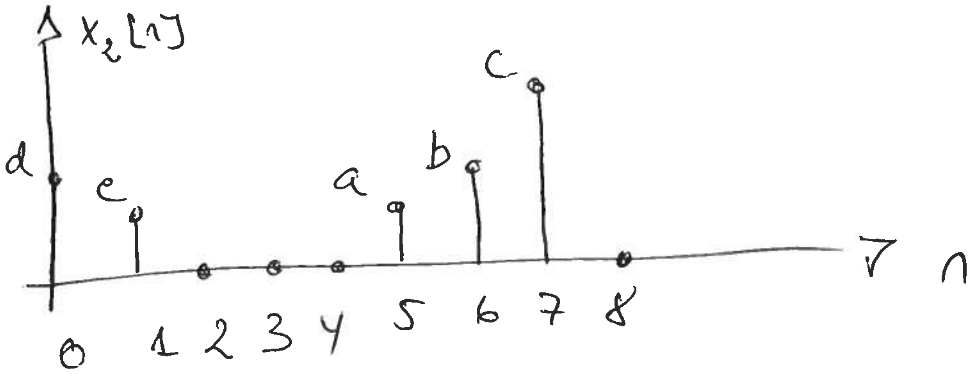
$$X[k] = e^{-j\frac{2\pi}{N}(k-k_0)\cdot\frac{N-1}{2}} \cdot \frac{\sin\left(\pi \cdot (k-k_0)\right)}{\sin\left(\pi (k-k_0)\frac{1}{N}\right)}$$

Prob. 10 p. 714.



Given $x_1[n]$ and $x_2[n]$.

Find relationship between $X_1[k]$ and $X_2[k]$.



Looking carefully at the two sequences, we realize that they are both 8 point sequences related through circular shift.;

$$x_2[n] = x_1[((n-4))_8]$$

We now use property 5 in table 2 p. 688

$$x[((n-m))_N] \xleftrightarrow{DFT} W_N^{kn} X[k]$$

Using this property, we can find an expression for $X_2[k]$

$$\mathrm{DFT}\{x_1[((n-4))_8]\} = W_8^{4k} X_1[k]$$

and thus

$$X_2[k] = W_8^{4k} \cdot X_1[k]$$

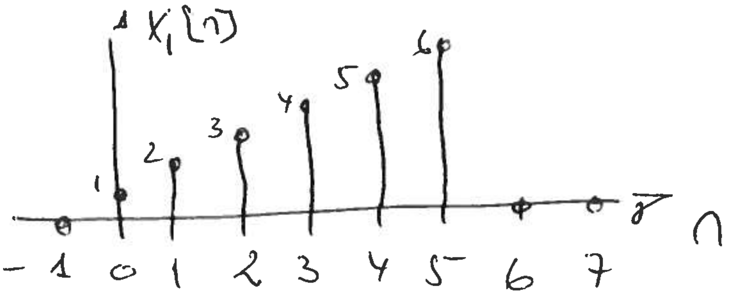
$$= e^{-j\frac{2\pi}{8}\cdot 4k} \cdot X_1[k]$$

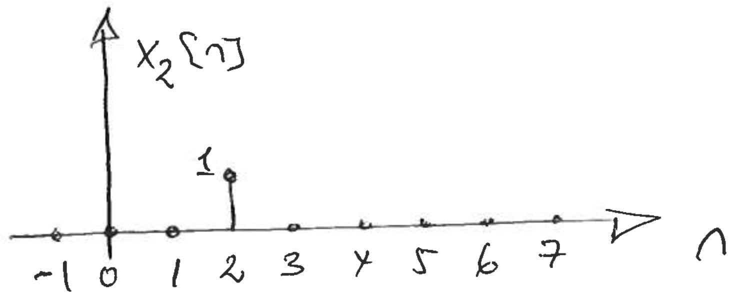
$$= e^{-j\pi k} \cdot X_1[k]$$

$$= (-1)^k \cdot X_1[k]$$

Prob. 11 p. 715.

Given two finite-length sequences $x_1[n]$ and $x_2[n]$;

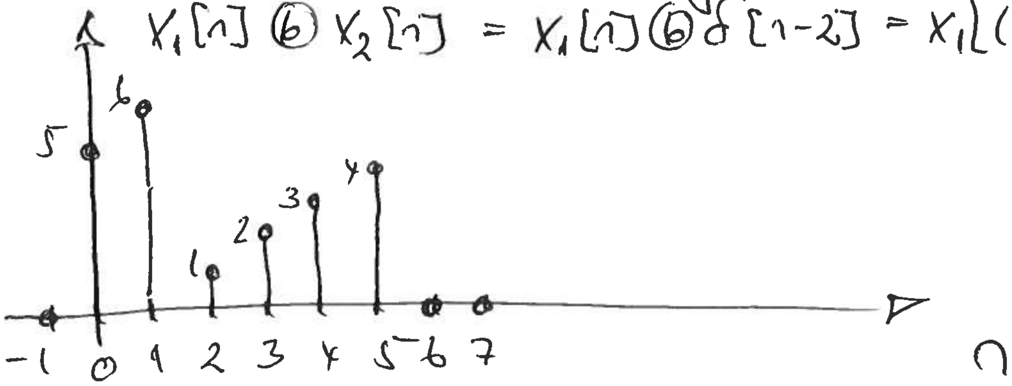




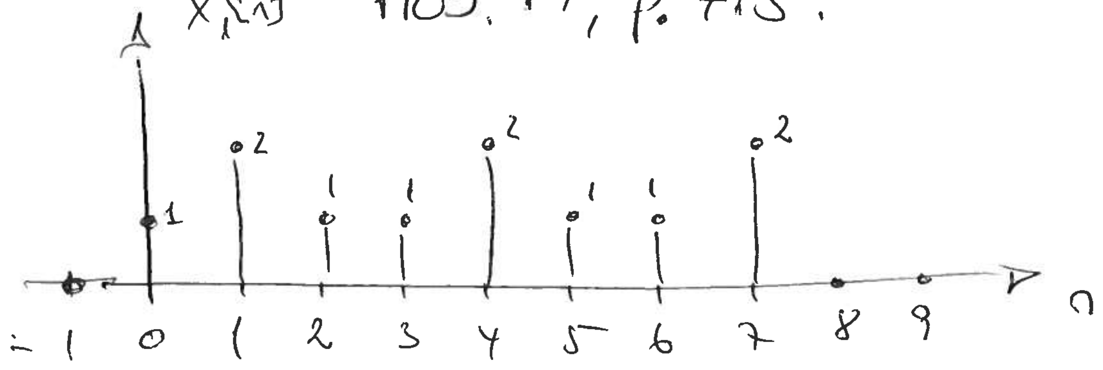
Draft their 6-point circular convolution.

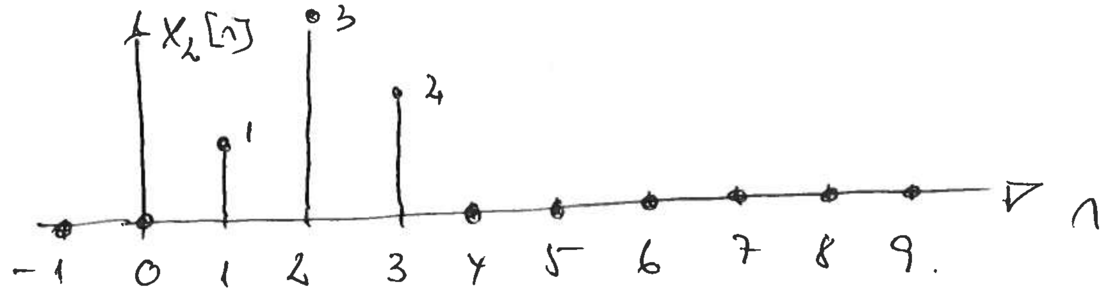
We see that $x_2[n]$ is just a shifted impulse, and thus the circular convolution is simply a circular shift of $x_1[n]$ by two points:

$$x_1[n] \circledast_6 x_2[n] = x_1[n] \circledast_6 \delta[n-2] = x_1[((n-2))_6]$$



Prob. 14, p. 715.

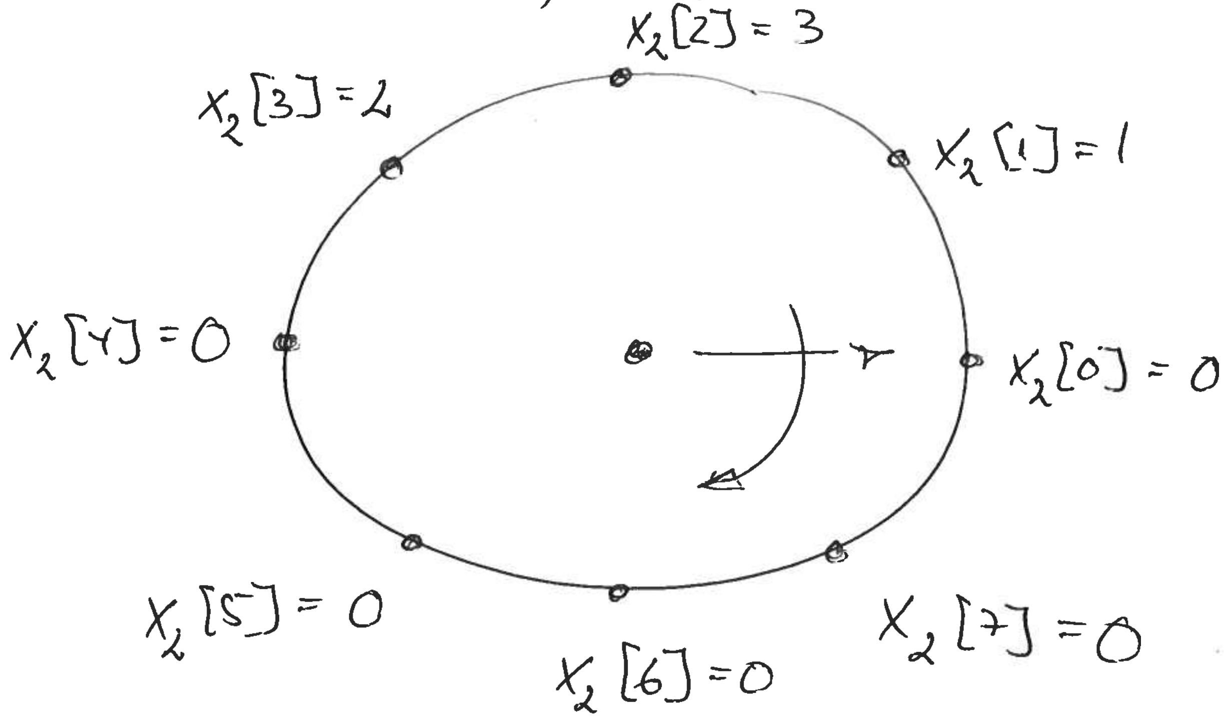




x_3 is the 8-point circular convolution of x_1 and x_2.

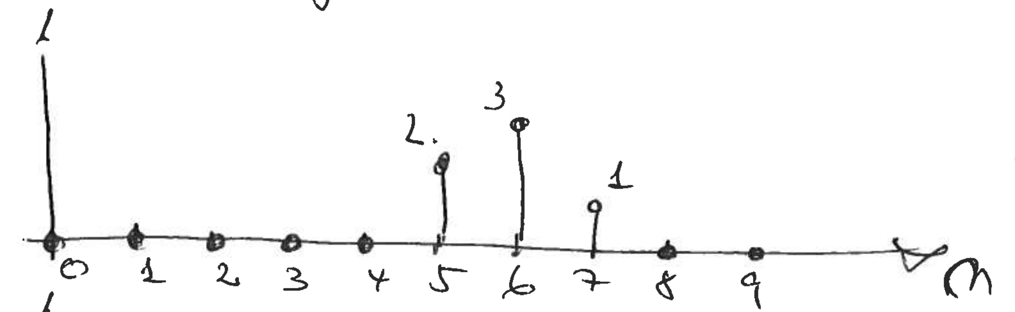
$x_3[n] = x_1[n] \circledast_8 x_2[n]$

One possible way to calculate $x_3[n]$ (and thus find $x_3[2]$) is to first draw $x_2[n]$ on a circle;

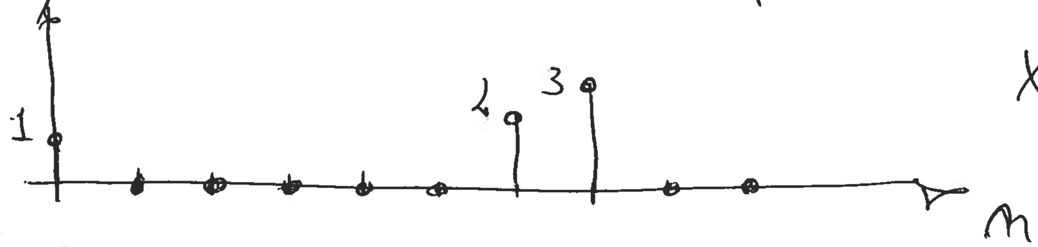


8 points of x_2 from $n=0$ to $n=7$.

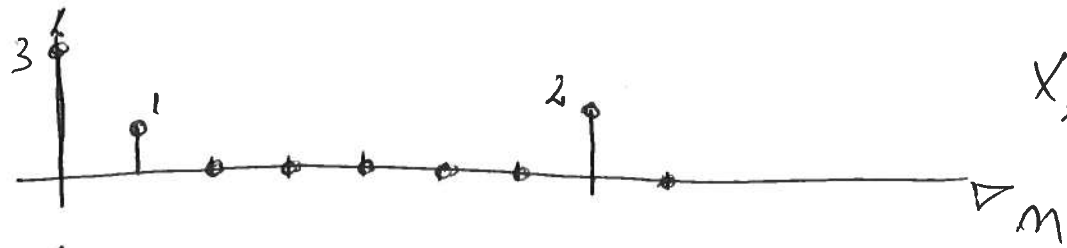
From this figure we can obtain the circularly time reversed sequence $x_2[((n-m))_8]$



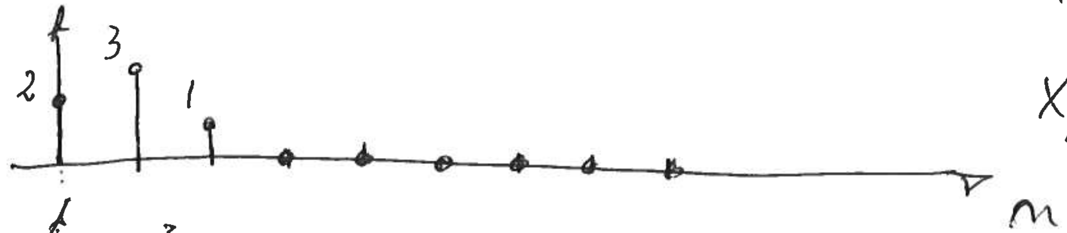
$x_2[((-m))_8]$



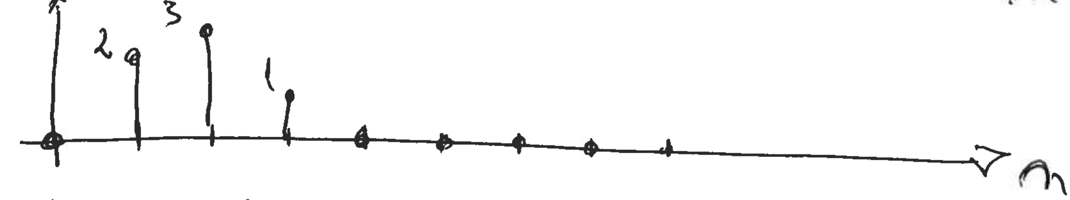
$x_2[((1-m))_8]$



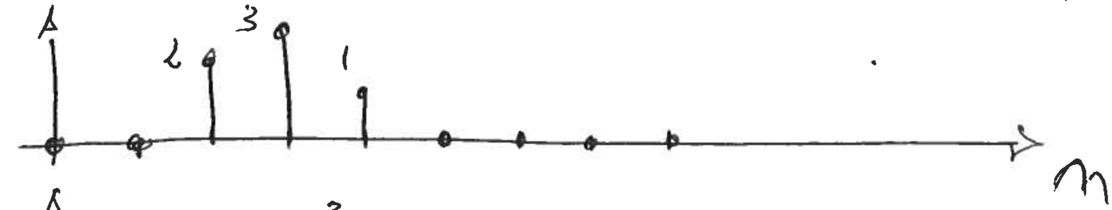
$x_2[((2-m))_8]$



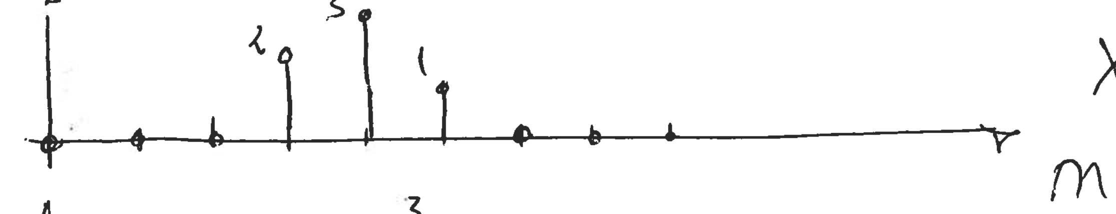
$x_2[((3-m))_8]$



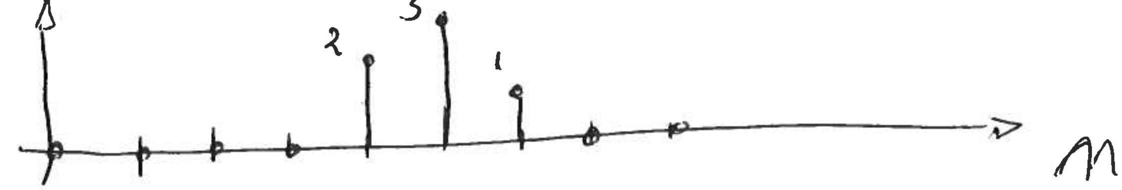
$x_2[((4-m))_8]$



$x_2[((5-m))_8]$



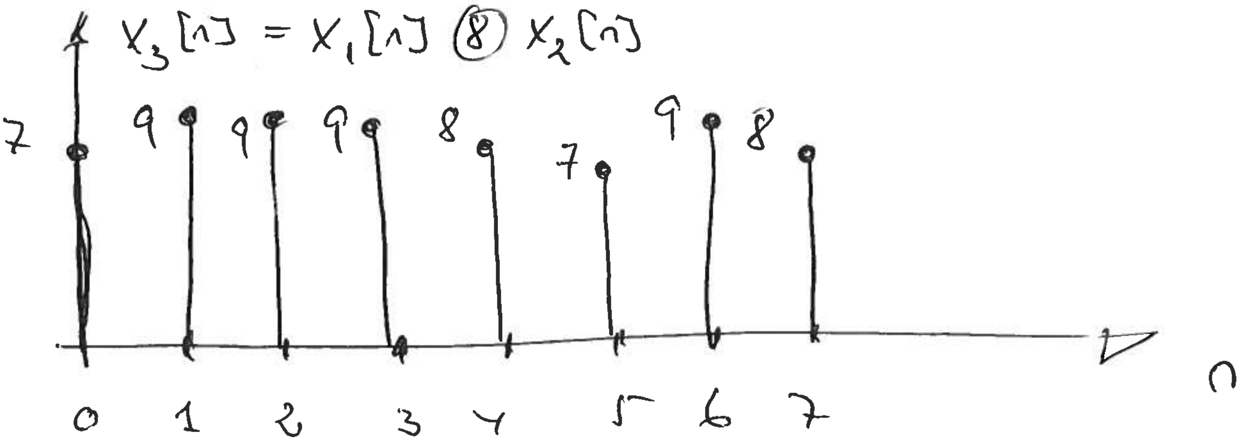
$x_2[((6-m))_8]$



$x_2[((7-m))_8]$

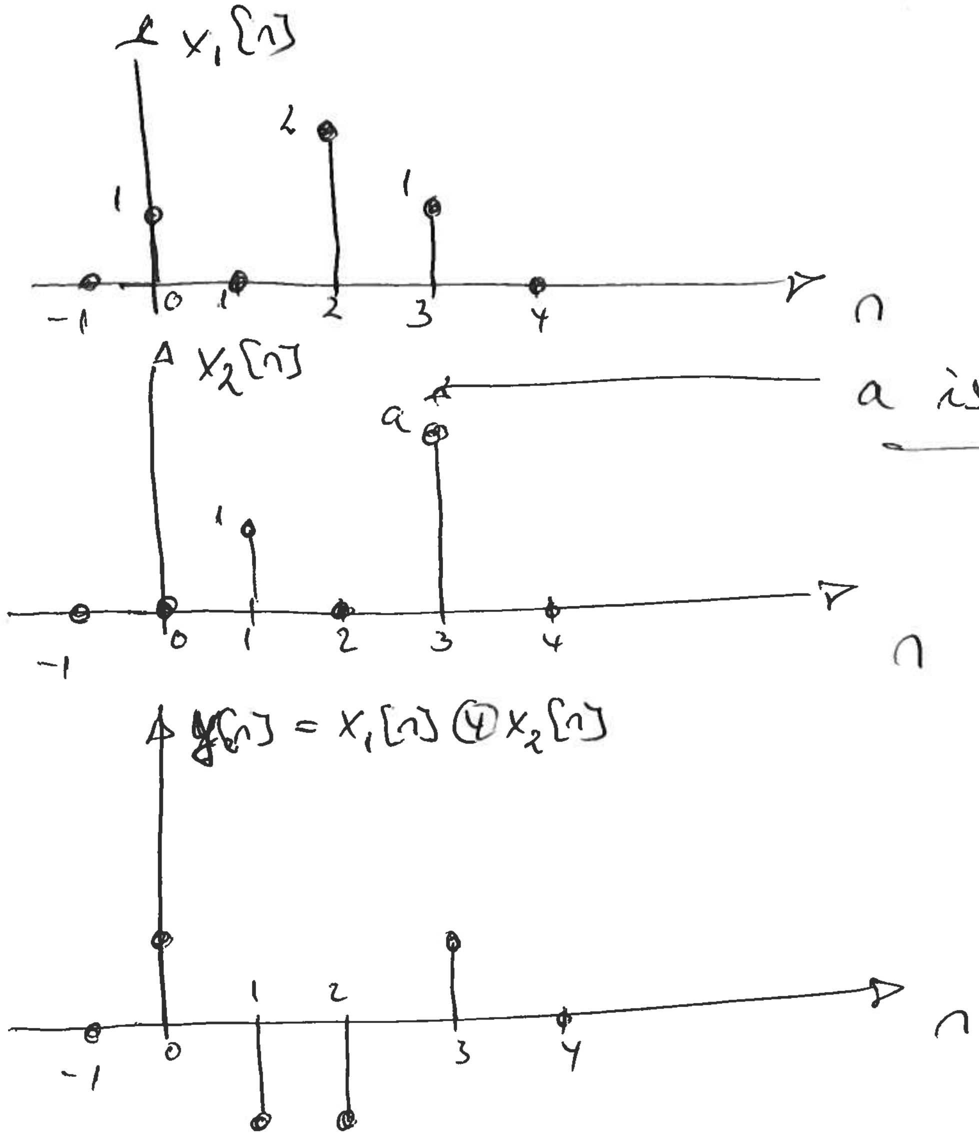
We now use these circularly time reversed sequences to calculate x_3 by multiplying/adding with the sequence x_1;

$x_3[n] = x_1[n] \; (8) \; x_2[n]$



So, the answer is $x_3[2] = 9$

Prob. 15, p. 716.



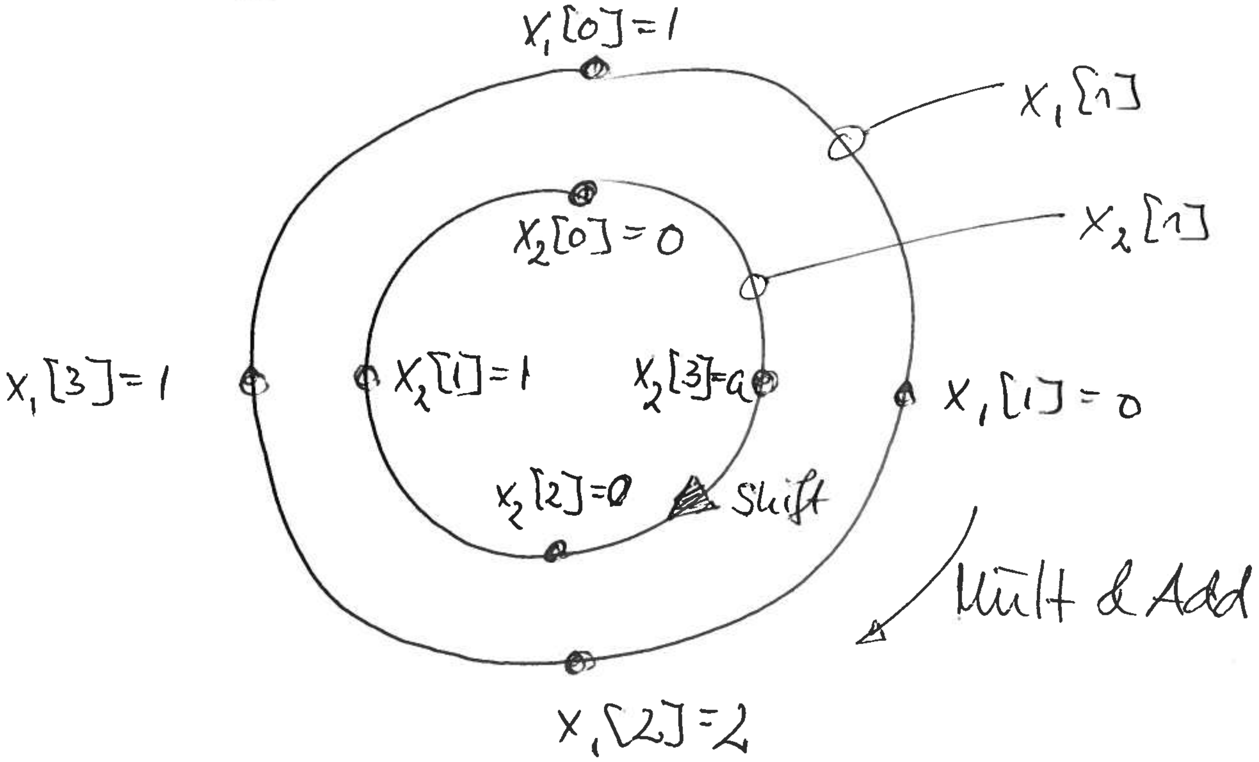
a is unknown.

Is it possible to specify a uniquely?

And if „yes" – what is the value of a?

(Basically what we do is that we calculate $y[n] = x_1[n] ④ x_2[n]$.

Let's try the „Concentric Circle" Method



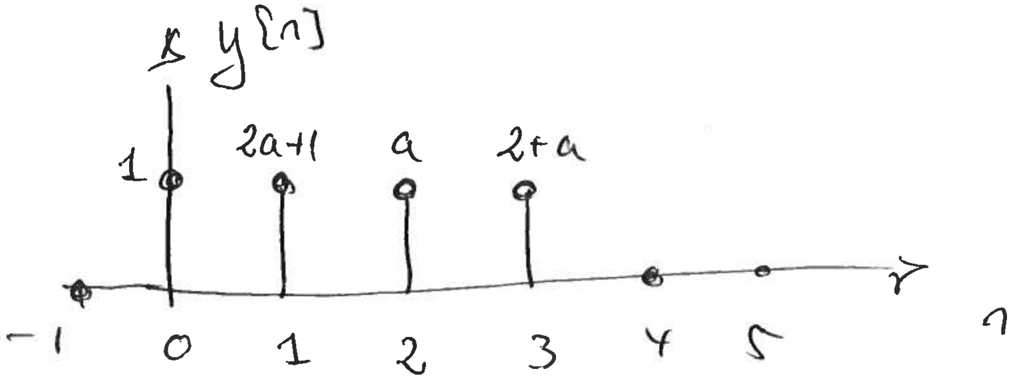
For the four possible „locations" of the outer circle, we spin the inner circle, and do the multiplications and additions.

$$y[0] = 0\times1 + a\times0 + 0\times2 + 1\times1 = 1$$

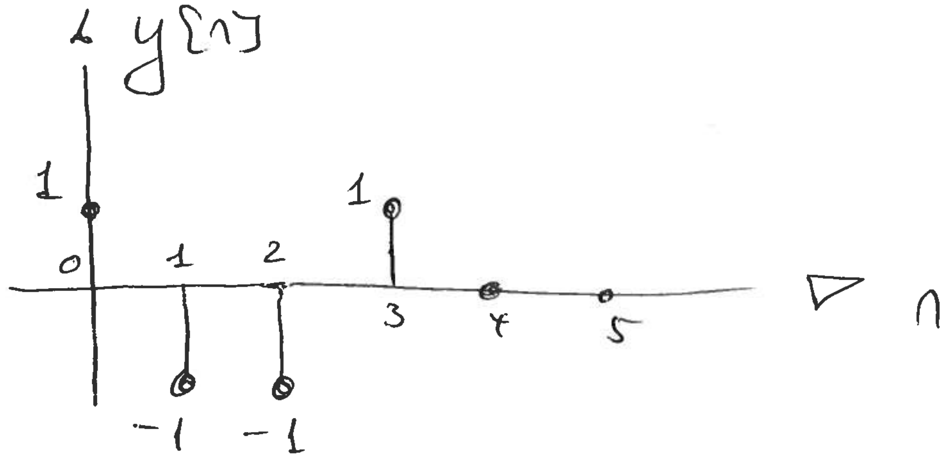
$$y[1] = 1\times1 + 0\times0 + a\times2 + 0\times1 = 2a+1$$

$$y[2] = 0\times1 + 1\times0 + 0\times2 + a\times1 = a$$

$$y[3] = a\times1 + 0\times0 + 1\times2 + 0\times1 = a+2$$



Which we now compare to the sequence given in the problem formulation;



Comparing these two sequences, we conclude that it is possible to determine a uniquely; $a=-1$