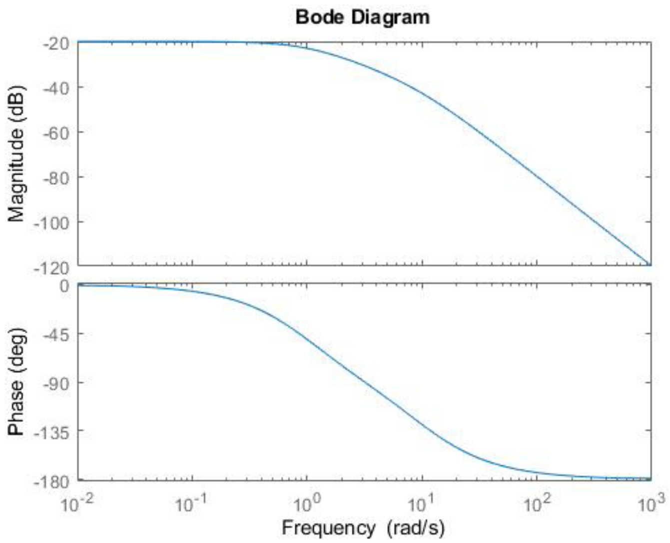
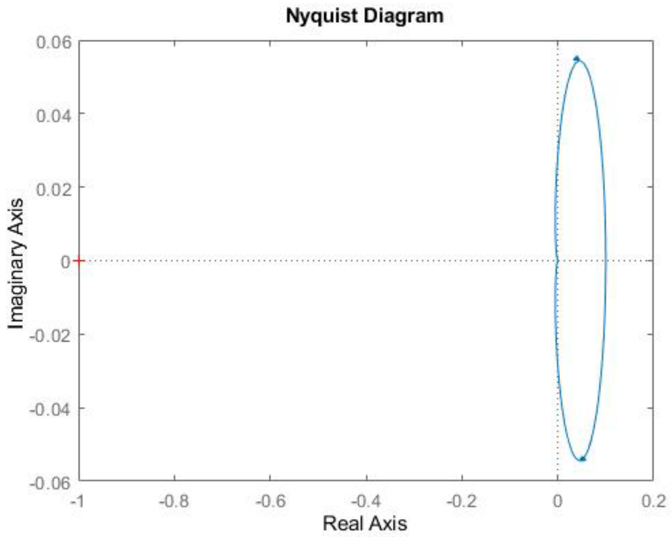
Nyquist stability criteion

Nyquist diagram

- Bode: separate plot for amplitude and phase
- Nyquist: Plotting the amplitude and phase in the same diagram
- $H(s) = \frac{1}{(s+1)(s+10)}$ $A(j\omega) = |H(j\omega)|$ and $\varphi = \arg(H(j\omega))$





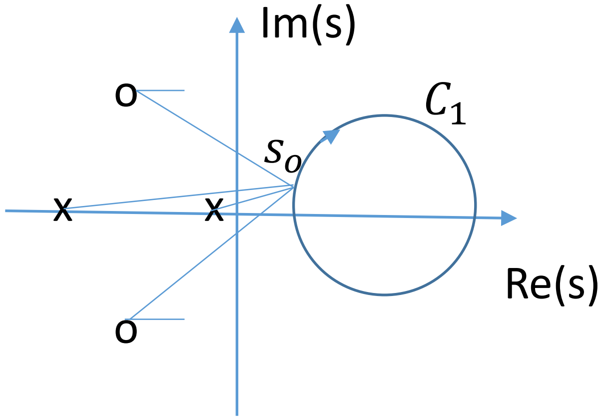
You can check stability for unstable poles and zeroes

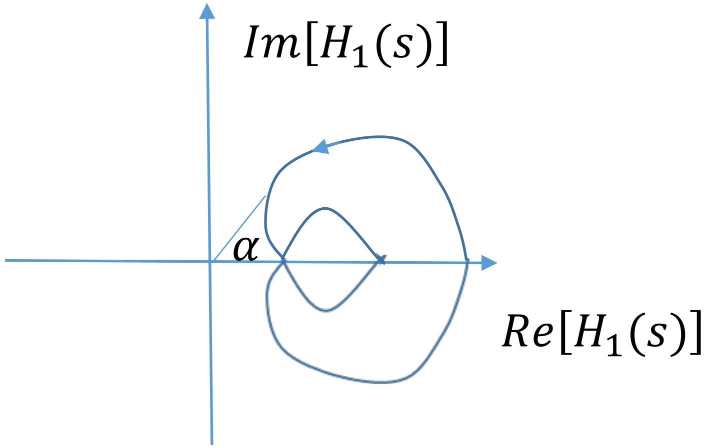
Contour evaluation

H(s) has 2 zeroes and 2 poles

Evaluate H_1 for values of s in the clockwise contour C_1

$H_1(s_o) = |\overline{v_1}|e^{j\alpha}$
$\alpha = \theta_1 + \theta_2 - (\varphi_1 + \varphi_2)$



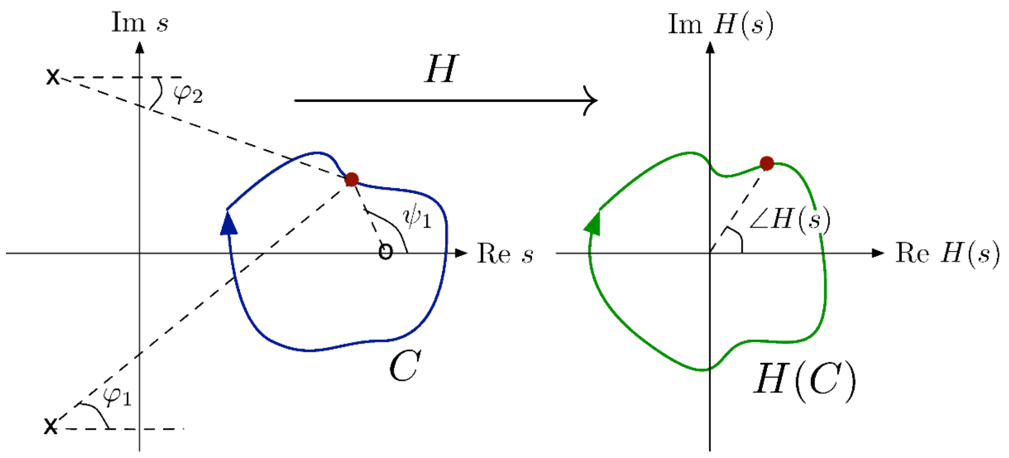


As s traverse C_1starting in s_o the angle α will change but it will not undergo a change of 360 degrees as long there is no poles or zeroes within C_1. All pole zero angles returns to their original value. =>
The plot will not encircle origin.

The plot will encircle the origo if there is a pole or a zero inside C_1

Countour evaluation - mapping

One zero inside C



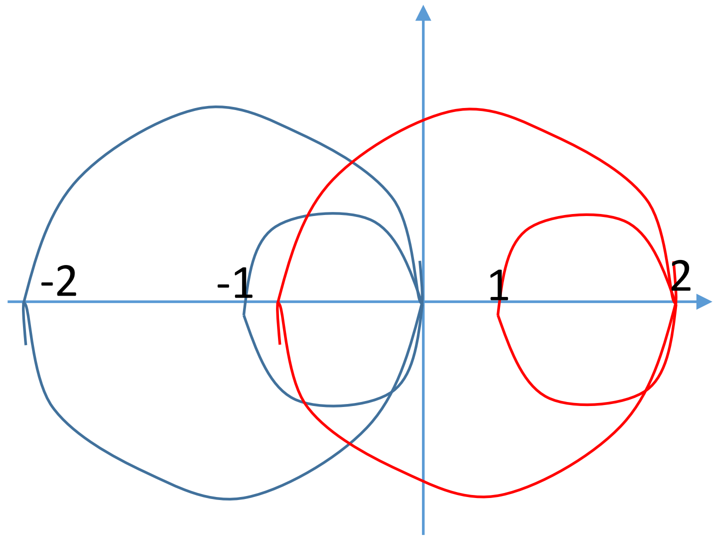
A countour map of a complex function will encircle the origin Z-P times where Z is the number of zeroes and P is the number of poles of the function inside the contour.

Countour mapping

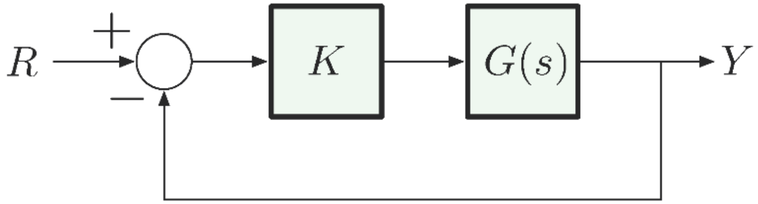
Countour plot for G(s)

countour plot for G(s)+2

Adding a constant will cause a parallel shift of the contour



Application to controller design



$$\frac{Y(s)}{R(s)} = T(s) = \frac{KG(s)}{1 + KG(s)}$$

Closed loop poles in the RHP will cause an unstable closed loop system.

Closed loop poles are the solution to $1 + KG(s) = 0$

If we evaluate the contour enclosing the entire RHP this function of s will encircle the origin N times. N is the number of poles and zeroes in the RHP.

$1 + KG(s) = 0 \Rightarrow KG(s) = -1$ therefore we can evaluate the open loop function around the RHP according to encirclements of (-1,0) to check the closed loop

Encirclements caused by zeroes or poles

$$1 + KG(s) = 1 + K\frac{b(s)}{a(s)} = \frac{a(s) + Kb(s)}{a(s)} \Rightarrow$$

Poles of 1+KG(s) are also poles of G(s)

A clockwise contour C_1 enclosing a zero of 1+KG(s) will result in KG(s) encircling (-1,0) in a clockwise direction

A clockwise contour C_1 enclosing a pole of 1+KG(s) will result in a KG(s) encircling (-1,0) in a counterclockwise direction

Determination of the number of RHP zeroes

- A countour map of $KG(s)$ using the clockwise direction will encircle the origin $N= Z-P$ times in the clockwise direction where Z is the number of zeroes and P is the number of poles of the function inside the contour.
- We know the number of RHP poles in the open loop system => we can calculate the number of RHP poles in the closed loop system.

- The number of RHP zeroes in the characteristic equation can be calculated as $Z=N+P$

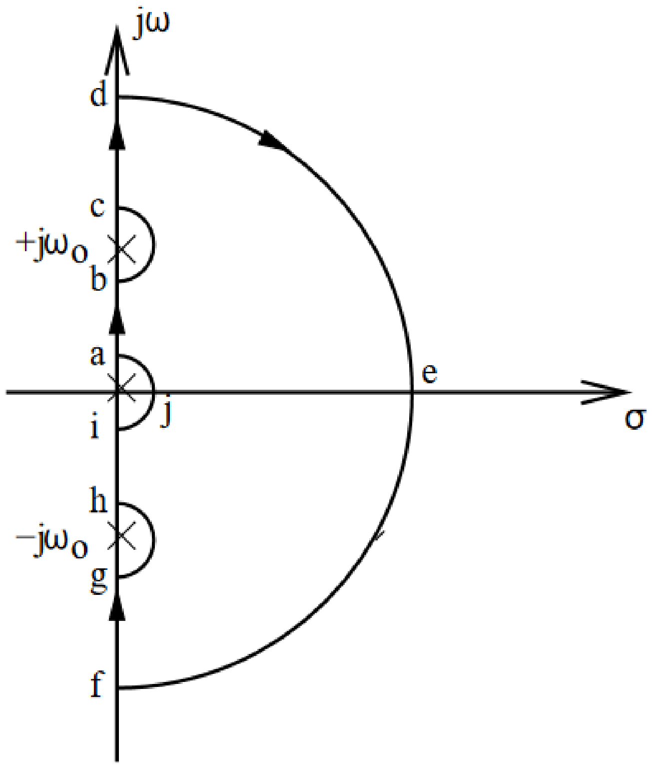
Nyquist stability

The sequence in the proof:

1. Poles in the closed loop in RHP gives an unstable closed loop system.
2. Poles in the closed loop are identical with zeroes in $1 + D(s)G(s)H(s)$ meaning that the zeroes in the RHP in $1 + D(s)G(s)H(s)$ gives an unstable close loop.
3. Poles in $D(s)G(s)H(s)$ are identical with poles in $1 + D(s)G(s)H(s)$.
4. The number of poles in $D(s)G(s)H(s)$ in RHP is counted and called P.
5. The number of zeroes in $1 + D(s)G(s)H(s)$ is found by making a Nyquist plot of $1 + D(s)G(s)H(s)$ and count the number of clockwise encircles in the MAP, this is called N.
6. Instead of using $1 + D(s)G(s)H(s)$ the open-loop transfer function $D(s)G(s)H(s)$ can be Nyquist plotted and N is now the number of clockwise encircles around -1 in the MAP.
7. The closed loop system is stable if Z=N+P is zero. Note that P is zero or a positive integer, N is negative or positive or zero.

The Nyquist contour

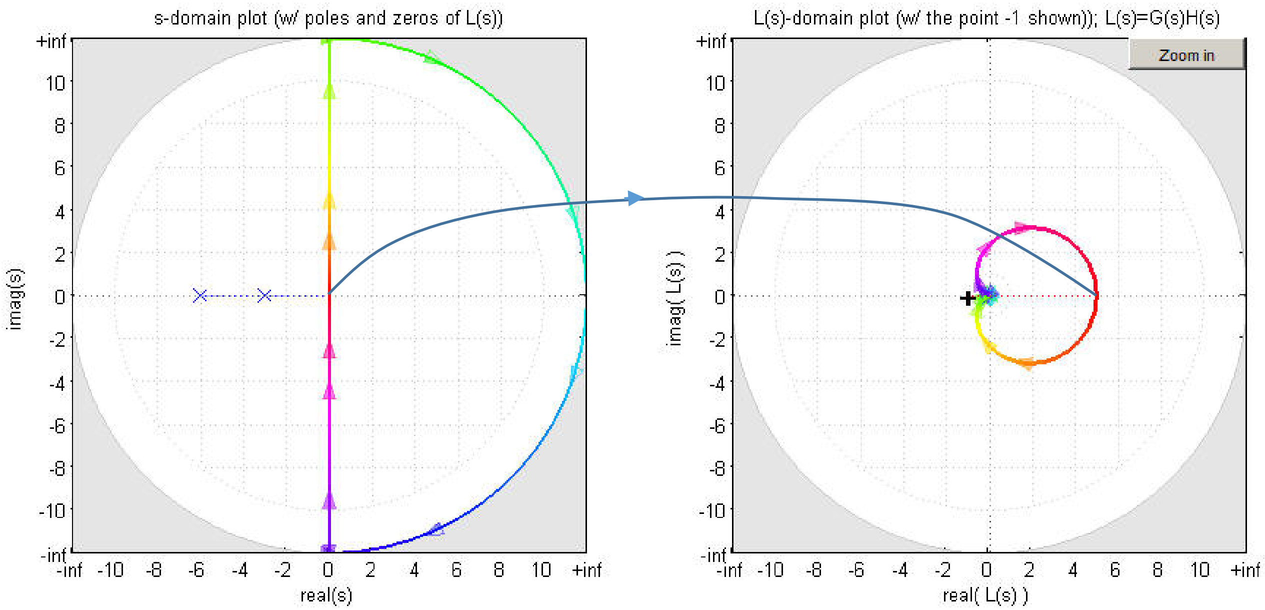
For poles on the imaginary axis make a small contour around the pole

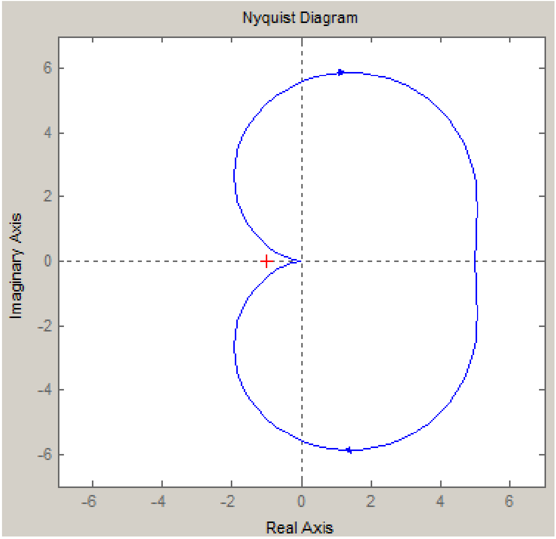


Path	*Equation*	*Valid for*
ab	$s = j\omega$	$0 < \omega < \omega_o$
bc	$s = \lim_{r\to 0}(j\omega_o + re^{j\Theta})$	$-90^o \leq \Theta \leq +90^o$
cd	$s = j\omega$	$\omega_o < \omega < \infty$
def	$s = \lim_{R\to\infty} Re^{j\Theta}$	$+90^o \geq \Theta \geq -90^o$
fg	$s = j\omega$	$-\infty < \omega < -\omega_o$
gh	$s = \lim_{r\to 0}(-j\omega_o + re^{j\Theta})$	$-90^o \leq \Theta \leq +90^o$
hi	$s = j\omega$	$-\omega_o < \omega < 0$
ija	$s = \lim_{r\to 0} re^{j\Theta}$	$-90^o \leq \Theta \leq +90^o$

Eksample 1

$$G(s) = \frac{90}{s^2 + 9s + 18} = \frac{90}{(s+3)(s+6)}$$

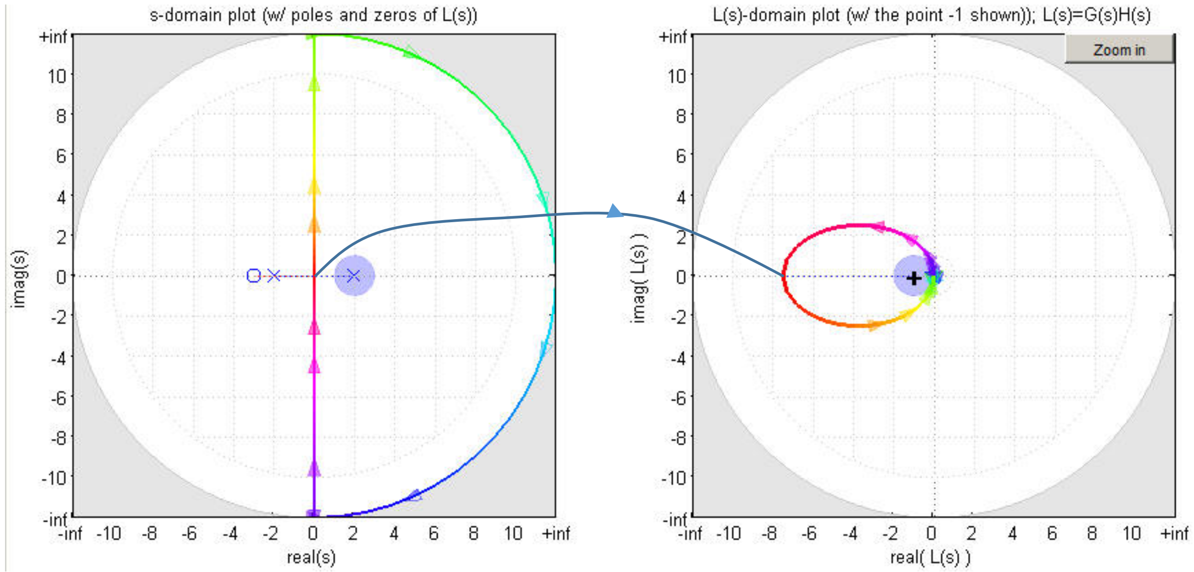


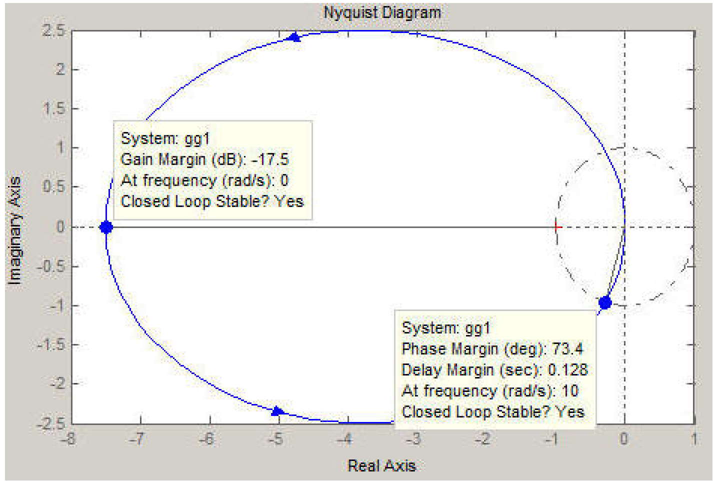


The -1+j0 point is not encircled so N=0. There are no poles of G(s) in the right half plane so P=0. Since N=Z-P, Z=0. This means that the characteristic equation of the closed loop transfer function has no zeros in the right half plane (the closed loop transfer function has no poles there). The system is stable.

Eksempel 2:   pole in RHP, and is stable

$$G(s) = 10\frac{(s+3)}{(s^2-4)} = 10\frac{(s+3)}{(s+2)(s-2)}$$

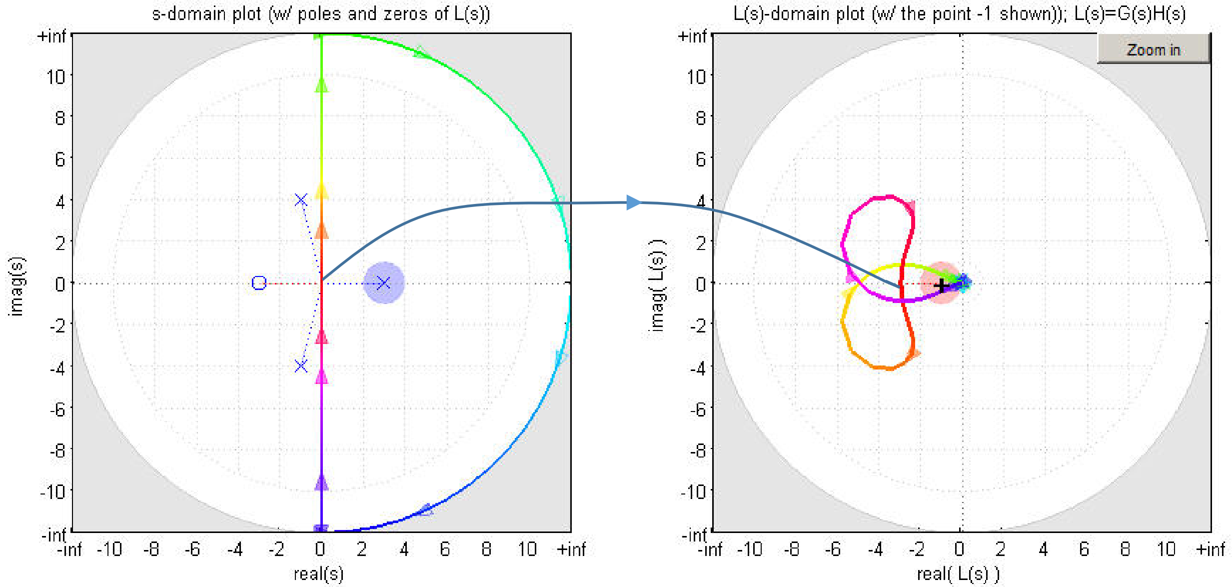


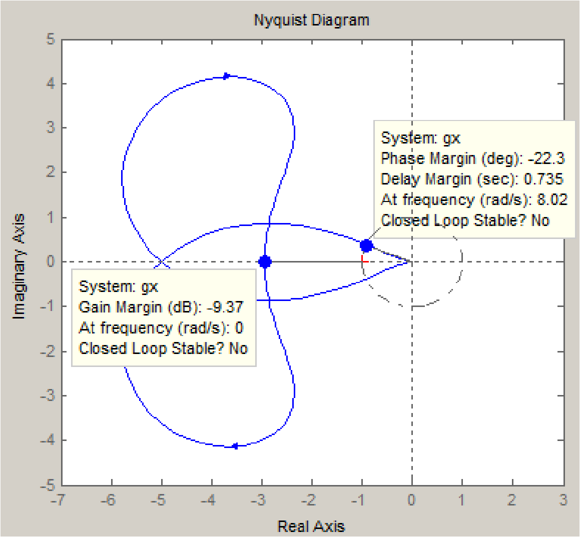


The -1+j0 point is encircled one time in the counterclockwise direction so N=-1. There is one pole of G(s) in the right half plane so P=-1. Since N=Z-P, Z=0. This means that the characteristic equation of the closed loop transfer function has no zeros in the right half plane (the closed loop transfer function has no poles there). The system is stable.

Eksample 3: pole in the RHP and is unstable

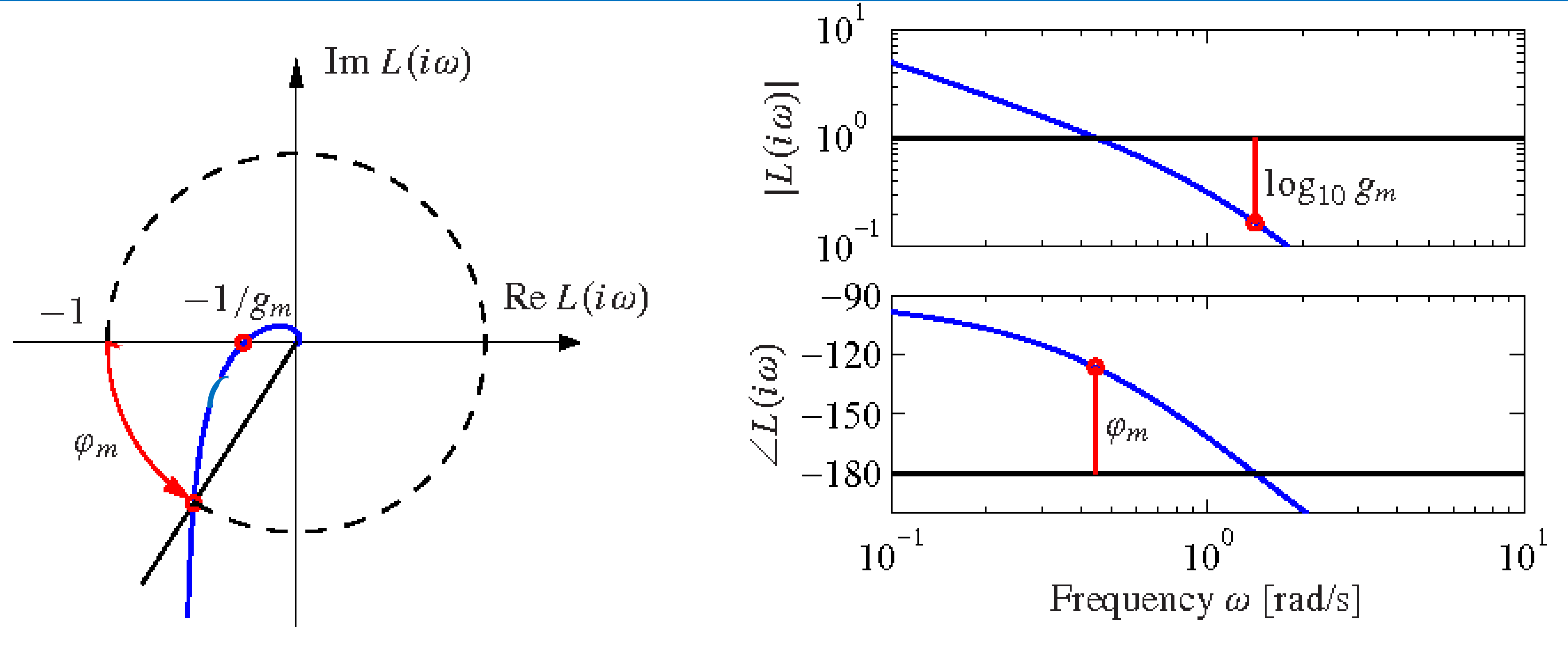
$$G(s) = 50\frac{s+3}{s^3 - s^2 + 11s - 51} = 50\frac{s+3}{(s+1+4j)(s+1-4j)(s-3)}$$





The -1+j0 point is encircled one time in the clockwise direction so N=1. There is one pole of G(s) in the right half plane so P=-1. Since N=Z-P, Z=2. This means that the characteristic equation of the closed loop transfer function has two zeros in the right half plane (the closed loop transfer function has two poles there). The system is unstable.

Gain and Phase margin



(a) Nyquist plot (b) Bode plot

Stability for a range of K

- We scale KG(s) by K and examine G(s)

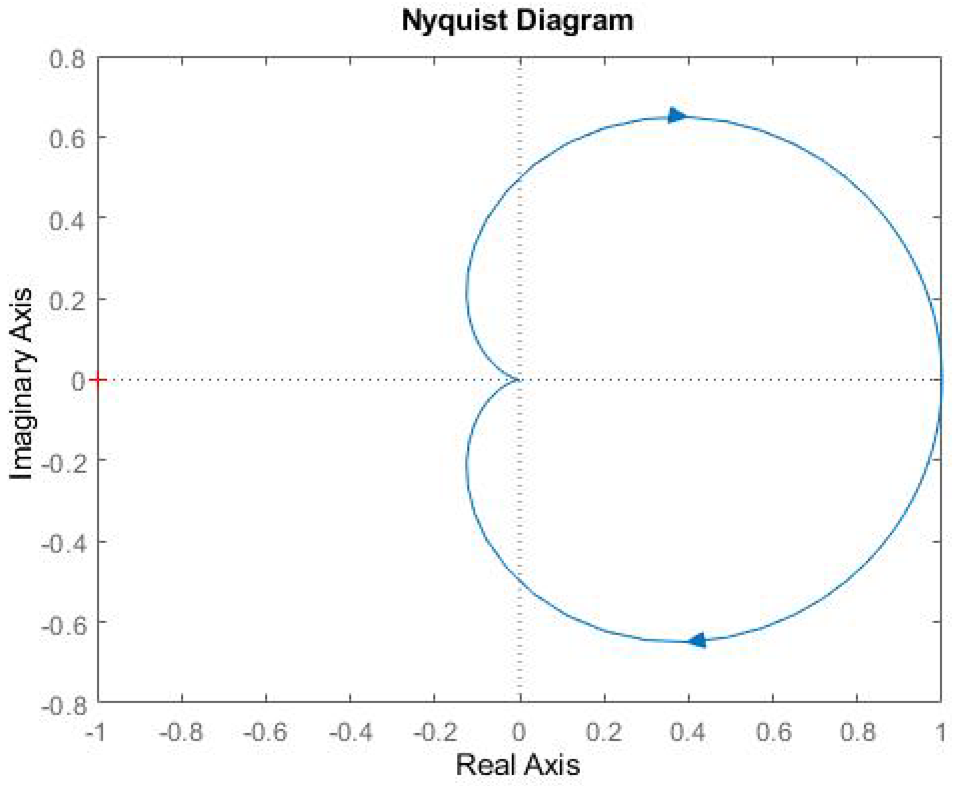
- The encirclements of -1 by KG(s) is the same as the encirclements of -1/K by G(s)

Example I

$D(s) = K \qquad G(s) = \frac{1}{(s+1)^2}$

No poles of G(s) in RHP => P=0
For stability we want Z=0
Therefore we want N=0 =
The nyquistplot cannot encircle (-1/K,0)

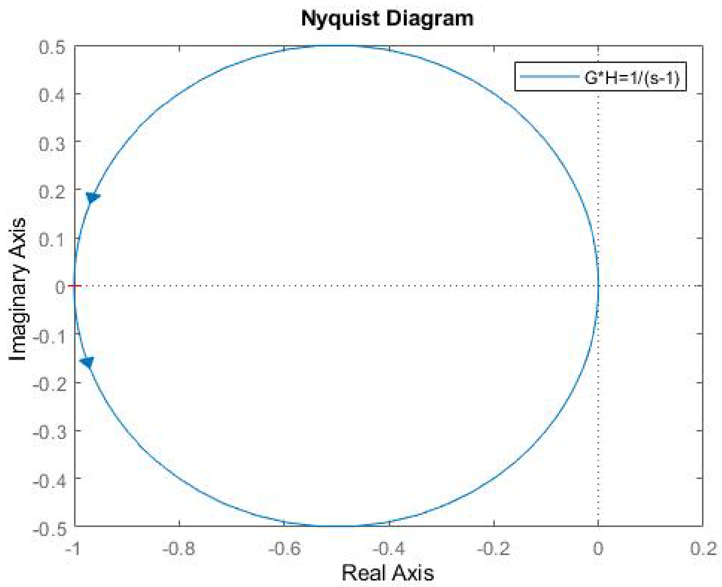
This is true when $-\frac{1}{K} < 0 \;\; or \;\; -\frac{1}{K} > 0$



Example II

The open loop $G(s)H(s) = \frac{K}{s-1}$
It has one right half plane pole => P=1
For stability we want Z= P+N=0 => N=-1

The number of encirclements of (-1/K,0) must be -1 in the clockwise direction = 1 encirclement in the counterclockwise direction.
Encirclements for -1/K <1 ⇔ K>1 (1/K is inside the contour)

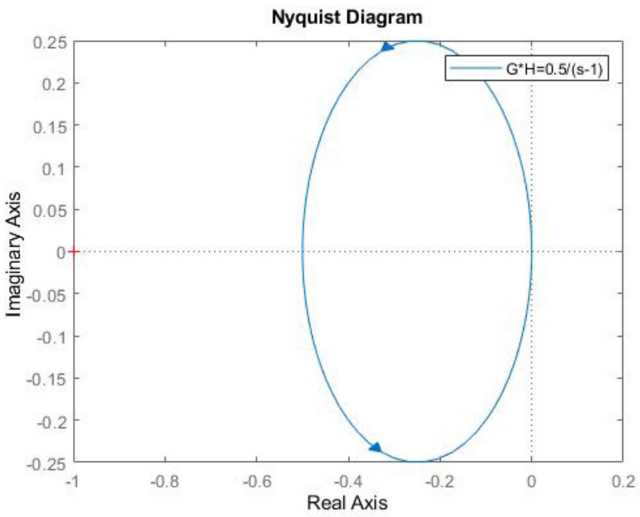


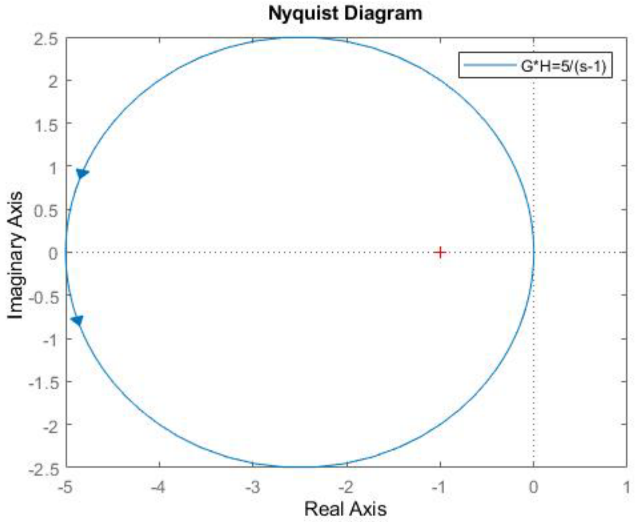
K=0.5 => N=0 => Z=1+0=1
K=5 => N=-1 (counter clockwise) => Z=1+(-1)=0

Test

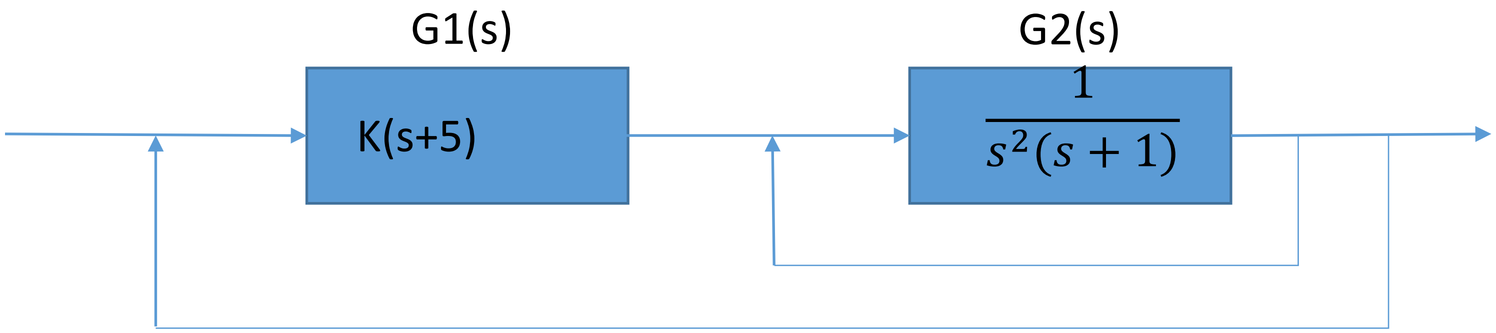
The closed loop $\frac{\frac{K}{s-1}}{1+\frac{K}{s-1}} = \frac{K}{s-1+K} = \frac{K}{s+(-1+K)}$

Stable if $-1 + K > 0 \Rightarrow K > 1$





Example II



G(s) = G1(s)G2(s)

$$G1 = K(s+5)$$

$$G2(s) = \frac{1}{s^3 + s^2 + 1}$$

The Routh array for G2(s) is

s^3	1	0
s^2	1	1
s^1	-1	0
s^0	1	0

There is two sign changes in first column => two RHP poles =>P=2

We want to find the K values that ensure stability Z=N+P=0

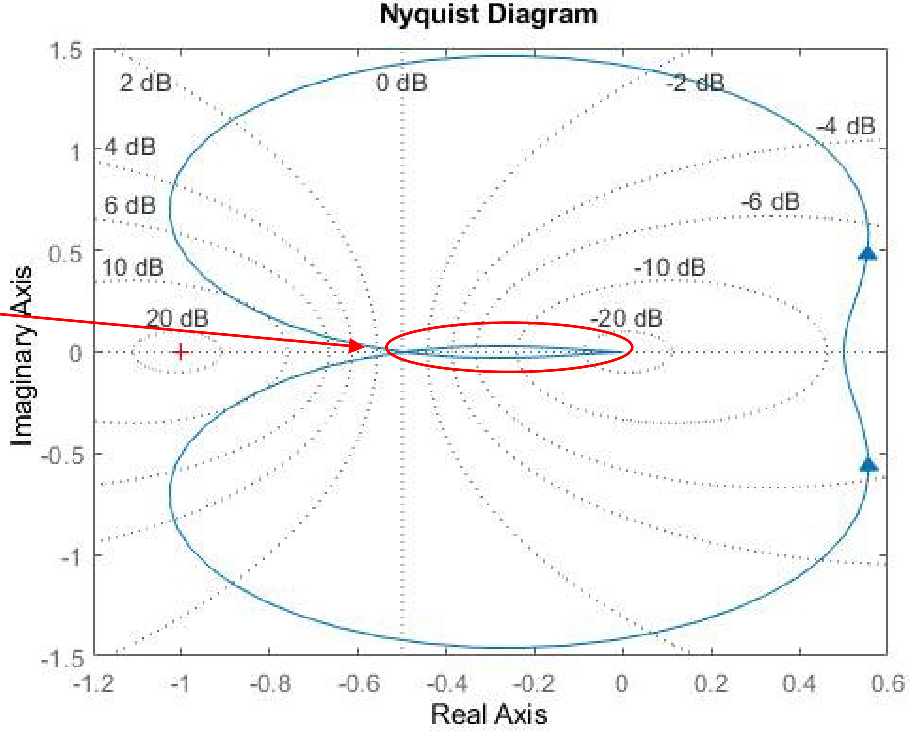
$G(s) = G1(s)G2(s) = \frac{K(s+0.5)}{s^3+s^2+1}$ P =2 =>

N=-2 = two encirclements 0f (-1,0) in the counter clockwise direction.

Example II

$$\frac{1}{K}G(s) = \frac{s + 0.5}{s^3 + s^2 + 1}$$

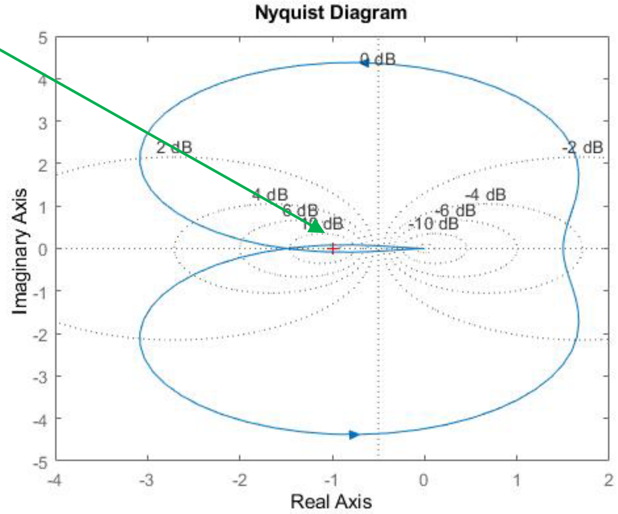
$$G(s) = \frac{K(s + 0.5)}{s^3 + s^2 + 1}$$



In the figure there are two encirclements of points between 0 and -0.5.
We can scale using K
We require

$$-0.5K < -1 \quad \Rightarrow$$
$$2 > K$$

For stability



K=3