



State Space Methods

Lecture 5: introducing reference signals, anti-windup, optimal control

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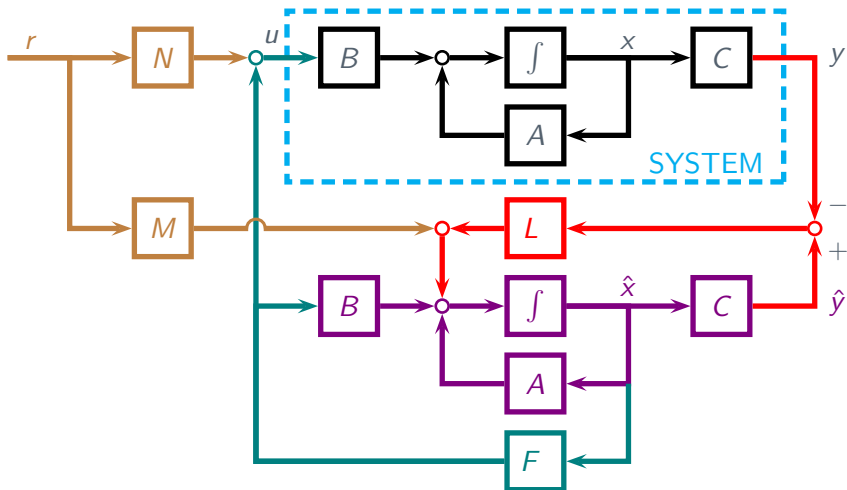
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Introducing reference signals



Introducing reference signals

System:

$$\begin{aligned}\dot{x} &= Ax + B(F\hat{x} + Nr) \\ y &= Cx\end{aligned}$$

Observer:

$$\dot{\hat{x}} = A\hat{x} + BF\hat{x} + L(C\hat{x} - y) + Mr$$

$$\begin{aligned}\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} &= \begin{pmatrix} A & BF \\ -LC & A + BF + LC \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{pmatrix} BN \\ M \end{pmatrix} r \\ y &= \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}\end{aligned}$$

Introducing reference signals

From these derivations, we can see that the closed loop system from reference to output has the following state space model:

$$\begin{aligned}\dot{x}_{cl} &= A_{cl}x_{cl} + B_{cl}r, \text{ where } x_{cl} = \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \\ y &= C_{cl}x_{cl}\end{aligned}$$

and

$$A_{cl} = \begin{pmatrix} A & BF \\ -LC & A + BF + LC \end{pmatrix}, \quad B_{cl} = \begin{pmatrix} BN \\ M \end{pmatrix}, \quad C_{cl} = (C \quad 0)$$

In the sequel, we will need to factor out the gain N , which yields:

$$\begin{aligned}\dot{x}_{cl} &= A_{cl}x_{cl} + \tilde{B}_{cl}Nr, \text{ where } \tilde{B}_{cl} = \begin{pmatrix} B \\ \tilde{M} \end{pmatrix}, \text{ and } \tilde{M} = MN^{-1} \\ y &= C_{cl}x_{cl}\end{aligned}$$

Zeros of systems

We have previously introduced this result:

Lemma

A square ($\#inputs=\#outputs$) system with a state space model of the form


$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

has a zero with value $z \in \mathbb{C}$ only if

$$\det \begin{pmatrix} A - zI & B \\ C & D \end{pmatrix} = 0$$

Zero assignment

$$\det \begin{pmatrix} A_{cl} - zI & B_{cl} \\ C_{cl} & D_{cl} \end{pmatrix} = 0$$

$$\det \begin{pmatrix} A - zI & B & B_N \\ -LC & A + BF + LC - zI & M \\ C & 0 & 0 \end{pmatrix} = 0$$


Zero assignment

$$\det \begin{pmatrix} A_{cl} - zI & B_{cl} \\ C_{cl} & D_{cl} \end{pmatrix} = 0$$

$$\det \begin{pmatrix} A - zI & BF & BN \\ -LC & A + BF + LC - zI & M \\ C & 0 & 0 \end{pmatrix} = 0$$

$$\det \begin{pmatrix} A - zI & BF - BNN^{-1}F & BN \\ -LC & A + BF + LC - zI - MN^{-1}F & M \\ C & 0 & 0 \end{pmatrix} = 0$$

$$\det \begin{pmatrix} A - zI & 0 & BN \\ -LC & A + BF + LC - MN^{-1}F - zI & M \\ C & 0 & 0 \end{pmatrix} = 0$$

$$\det \begin{pmatrix} A - zI & 0 & B \\ -LC & A + BF + LC - \tilde{M}F - zI & \tilde{M} \\ C & 0 & 0 \end{pmatrix} = 0$$

Zero assignment

$$\det \begin{pmatrix} A-zI & 0 & B \\ -LC & A+BF+LC-\tilde{M}F-zI & \tilde{M} \\ C & 0 & 0 \end{pmatrix} = 0$$

$$\begin{cases} \det \begin{pmatrix} A-zI & B \\ C & 0 \end{pmatrix} = 0 & \text{or} \\ \det (A+BF+LC-\tilde{M}F-zI) = 0 \end{cases}$$

$$\begin{cases} z \text{ is a zero of } \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} & \text{or} \\ z \text{ is an eigenvalue of } A+BF+LC-\tilde{M}F \end{cases}$$



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Zero assignment

Lemma

If \tilde{M} is an 'observer gain' such that the characteristic polynomial of the matrix $A_{za} + \tilde{M}C_{za}$ has the characteristic polynomial

$$\det \left(sI - \left(A_{za} + \tilde{M}C_{za} \right) \right) = (s - z_1) \cdots (s - z_n)$$

with $A_{za} = A + BF + LC$ and $C_{za} = -F$, then the numbers z_1, \dots, z_n are all zeros of the closed loop transfer function from r to y .

Algorithm for zero assignment

1. Design \tilde{M} assigning zeros close to the cut-off frequency of the Bode plot, such that the 'horizontal' part is extended.
2. Compute N such that the DC-value of the transfer function from r to y is unity:

$$N = - \left(C_{cl} A_{cl}^{-1} \tilde{B}_{cl} \right)^{-1}$$

where

$$A_{cl} = \begin{pmatrix} A & BF \\ -LC & A + BF + LC \end{pmatrix}, \quad \tilde{B}_{cl} = \begin{pmatrix} B \\ \tilde{M} \end{pmatrix}$$

$$C_{cl} = (C \quad 0)$$

3. Compute $M = MN^{-1}N = \tilde{M}N$.



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Example: zero assignment

We consider again the system

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y &= \begin{pmatrix} -3 & 2 \end{pmatrix} x\end{aligned}$$

A state feedback F that assign poles in $\{-3, -4\}$ and an observer gain L that assigns poles in $\{-9, -12\}$ are given by:

$$F = (22 \quad -16), \quad L = \begin{pmatrix} -122 \\ -192 \end{pmatrix}$$

We would like to assign zeros from r to y in $\{-3, -4\}$ to cancel the poles from F .

Example: zero assignment

With these values of F and L we obtain:

$$A_{za} = A + BF + LC = \begin{pmatrix} 412 & -279 \\ 646 & -437 \end{pmatrix}$$

$$C_{za} = -F = \begin{pmatrix} -22 & 16 \end{pmatrix}$$

An 'observer gain' that assigns poles in $\{-3, -4\}$ for $A_{za} + \tilde{M}C_{za}$ is

$$\tilde{M} = \begin{pmatrix} 7.0460 \\ 10.8133 \end{pmatrix}$$

Example: zero assignment

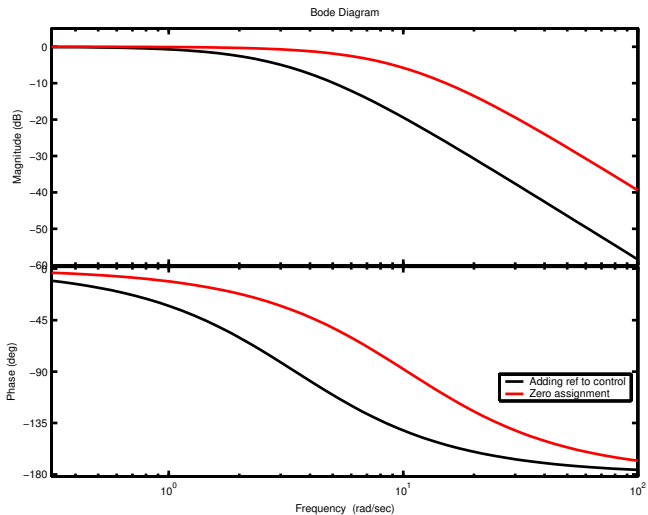
N can be computed as:

$$\begin{aligned}
 N &= \\
 &= \left((C \quad 0) \begin{pmatrix} A & BF \\ -LC & A + BF + LC \end{pmatrix}^{-1} \begin{pmatrix} B \\ \tilde{M} \end{pmatrix} \right)^{-1} \\
 &= 108
 \end{aligned}$$

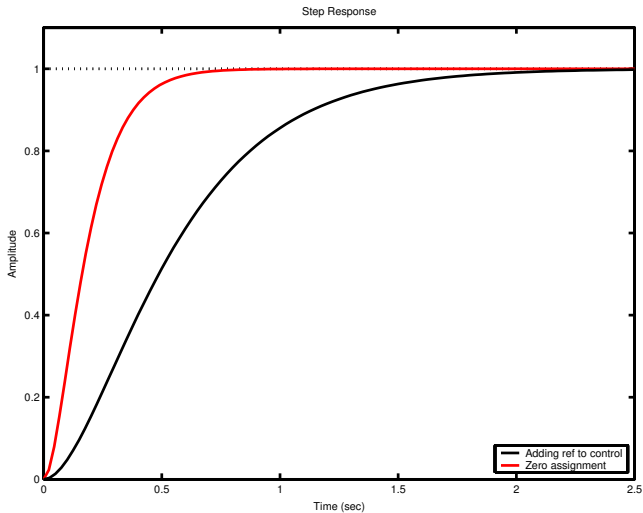
M is obtained from:

$$M = \tilde{M}N = \begin{pmatrix} 7.0460 \\ 10.8133 \end{pmatrix} \cdot 108 = \begin{pmatrix} 760.97 \\ 1167.84 \end{pmatrix}$$

Example: Bode plot



Example: step response



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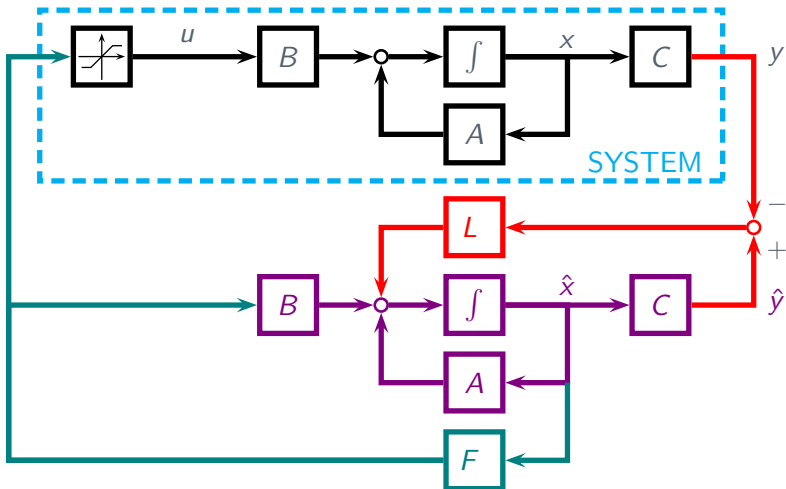
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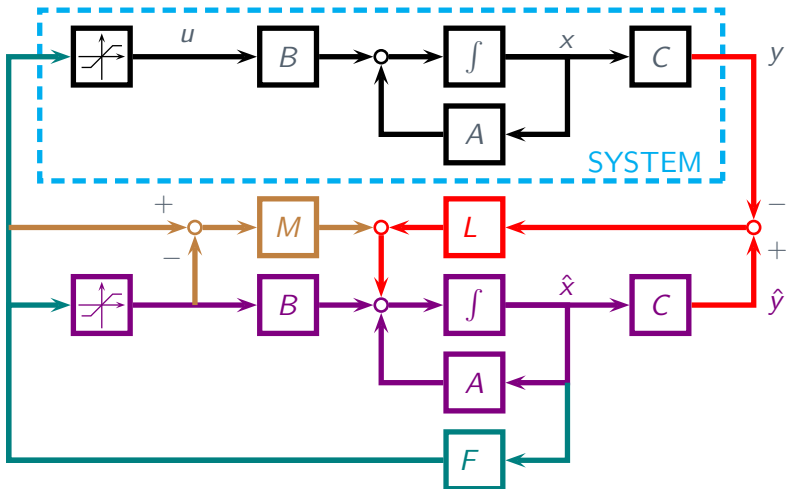
Example: optimal control

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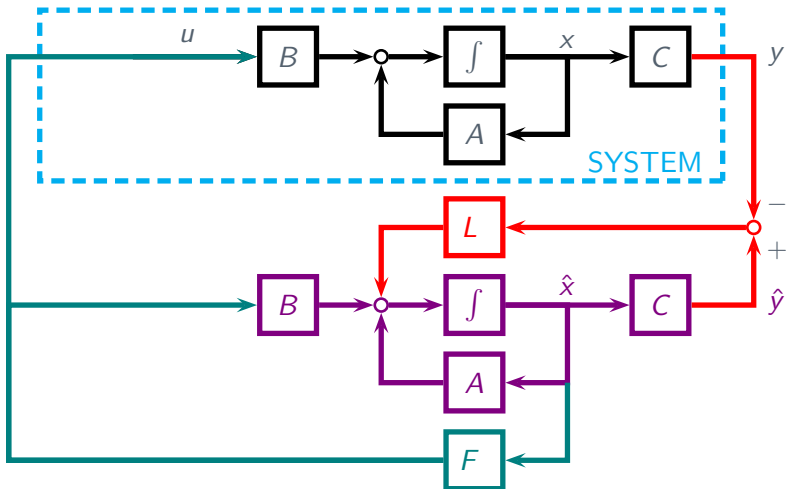
Anti-windup architecture



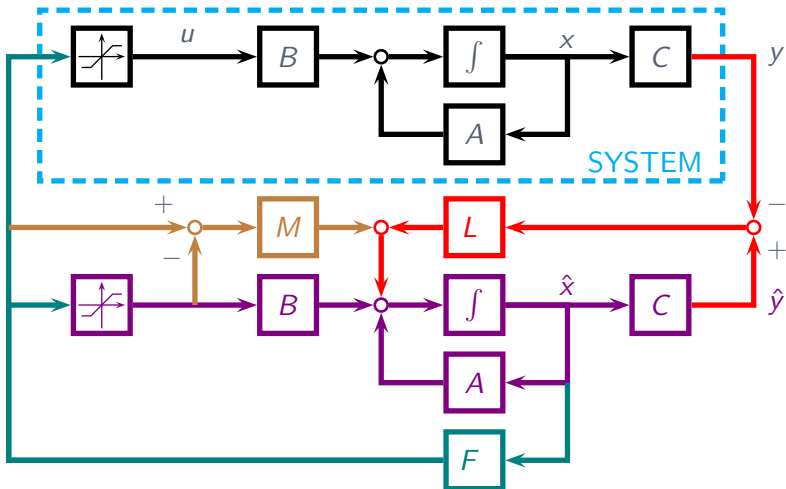
Anti-windup architecture



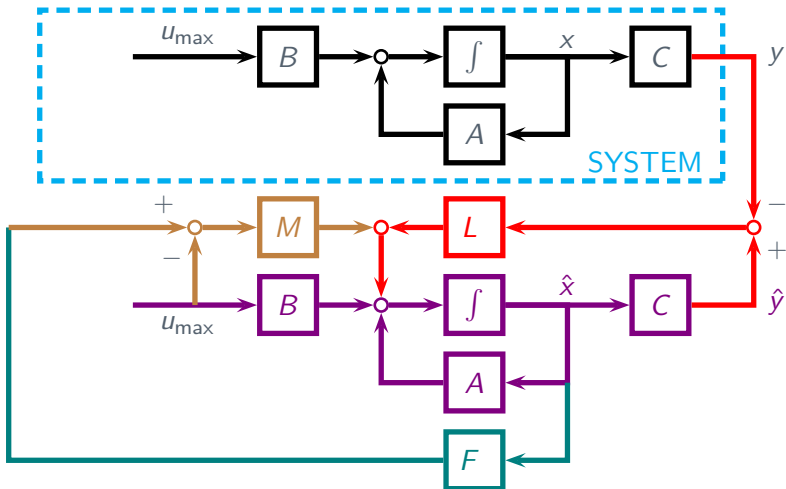
Anti-windup architecture, nominal



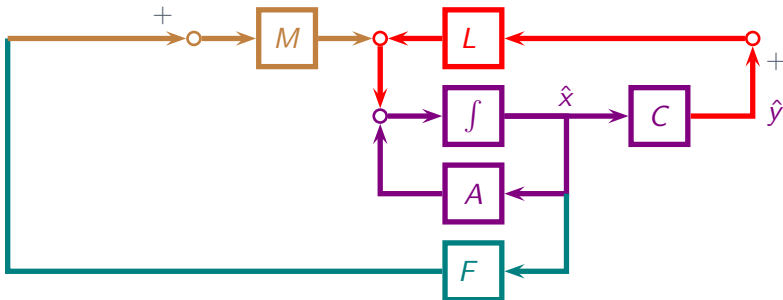
Anti-windup architecture, saturated



Anti-windup architecture, saturated



Anti-windup architecture, saturated



Designing saturation gain

Dynamics of controller during saturation:

$$\dot{\hat{x}} = A\hat{x} + LC\hat{x} + MF\hat{x}$$

or

$$\dot{\hat{x}} = (A + LC + MF)\hat{x}$$

Determining M can be recognized as an observer gain design problem:

$$\dot{\hat{x}} = (\tilde{A} + \tilde{L}\tilde{C})\hat{x}$$

with $\tilde{A} = A + LC$, $\tilde{L} = M$, and $\tilde{C} = F$, from which the unknown $\tilde{L} = M$ can be chosen to assign any desired poles to the saturated controller.

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Optimal control

We consider a linear control system of the form:

$$\begin{aligned}\dot{x} &= Ax + Bu, & x(0) &= x_0 \\ y &= Cx\end{aligned}$$

A control law for such a system is said to be *optimal*, if it minimizes the cost functional:

$$\mathcal{J} = \int_0^\infty x^T Q x + u^T R u \, dt$$

where $Q = Q^T$ is a positive semi-definite matrix and $R = R^T$ is a positive definite matrix.

The algebraic Riccati equation

An *Algebraic Riccati Equation* is a second order matrix equation in an indeterminate $P = P^T \in \mathbb{R}^{n \times n}$ of the form:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ are matrices, $R = R^T \in \mathbb{R}^{m \times m}$ is a positive definite matrix, and $Q = Q^T \in \mathbb{R}^{n \times n}$ is a positive semidefinite matrix.

P is called a *stabilizing solution* to the ARE, if it satisfies the equation, and further satisfies that the eigenvalues of $A - BR^{-1}B^T P$ are in the open left half plane.

Optimal state feedback control

Theorem

Consider a linear system of the form:

$$\begin{aligned}\dot{x} &= Ax + Bu, \quad x(0) = x_0 \\ y &= Cx\end{aligned}$$

Let P be a stabilizing solution to the ARE:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

Then the optimal state feedback law is given by:

$$u = Fx \quad \text{where} \quad F = -R^{-1}B^T P$$

Optimal state estimation

Given the system

$$\begin{aligned}\dot{x} &= Ax + Bu + Gw \\ y &= Cx + Du + v\end{aligned}$$

with unbiased process noise w and measurement noise v with covariances

$$\mathcal{E}\{ww^T\} = Q, \quad \mathcal{E}\{vv^T\} = R$$

Then an optimal state estimator is given by:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y)$$

Optimal state estimation

with unbiased process noise w and measurement noise v with covariances

$$\mathcal{E}\{ww^T\} = Q, \quad \mathcal{E}\{vv^T\} = R$$

Then an optimal state estimator is given by:

$$\dot{\hat{x}} = A\hat{x} + Bu + \textcolor{red}{L}(C\hat{x} - y)$$

where

$$\textcolor{red}{L} = -PC^TR^{-1}$$

P is a stabilizing solution to the ARE:

$$AP + PA^T - PC^TR^{-1}CP + Q = 0$$

Output variance minimization

Introducing $y = Cx$ into a cost functional of the type

$$\mathcal{J} = \int_0^{\infty} \rho y^T y + u^T u \, dt, \quad \rho \in \mathbb{R}$$

this can be written as an optimal control problem

$$\begin{aligned} \mathcal{J} &= \int_0^{\infty} \rho y^T y + u^T u \, dt \\ &= \int_0^{\infty} \rho x^T C^T C x + u^T u \, dt \\ &= \int_0^{\infty} x^T Q x + u^T R u \, dt, \quad Q = \rho C^T C, R = I \end{aligned}$$

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Computing an optimal state feedback for the cost functional:

$$\mathcal{J} = \int_0^{\infty} 2.25 y^T y + u^T u dt$$

can be done with the MATLABTM command

$$F_{opt} = -\text{lqr}(A, B, 2.25 * C' * C, 1)$$

Example: optimal control

This yields the result:

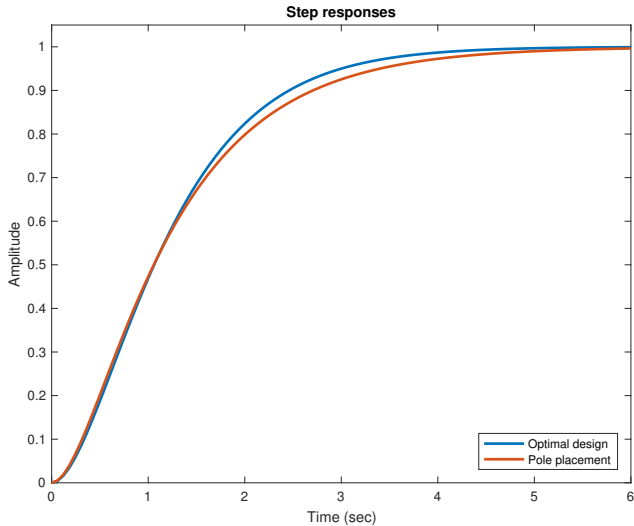
$$F_{\text{opt}} = (1.1754 \quad -0.8377)$$

In comparison, a pole assignment with the poles $\{-1, -3\}$ leads to the gain:

$$F_{\text{pa}} = (1 \quad -1)$$

It can be seen that the two feedbacks have comparable gains. The optimal controller, however, gives a slightly better rise time.

Example: optimal control



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