Control Theory

Root locus

Lecture 1

Outline

The Root Locus Design Method

- Introduction
  - Idea, general aspects. A graphical picture of how changes of one system parameter will change the closed loop poles.
- Sketching a root locus
  - Definitions
  - 6 rules for sketching root locus/root locus characteristics

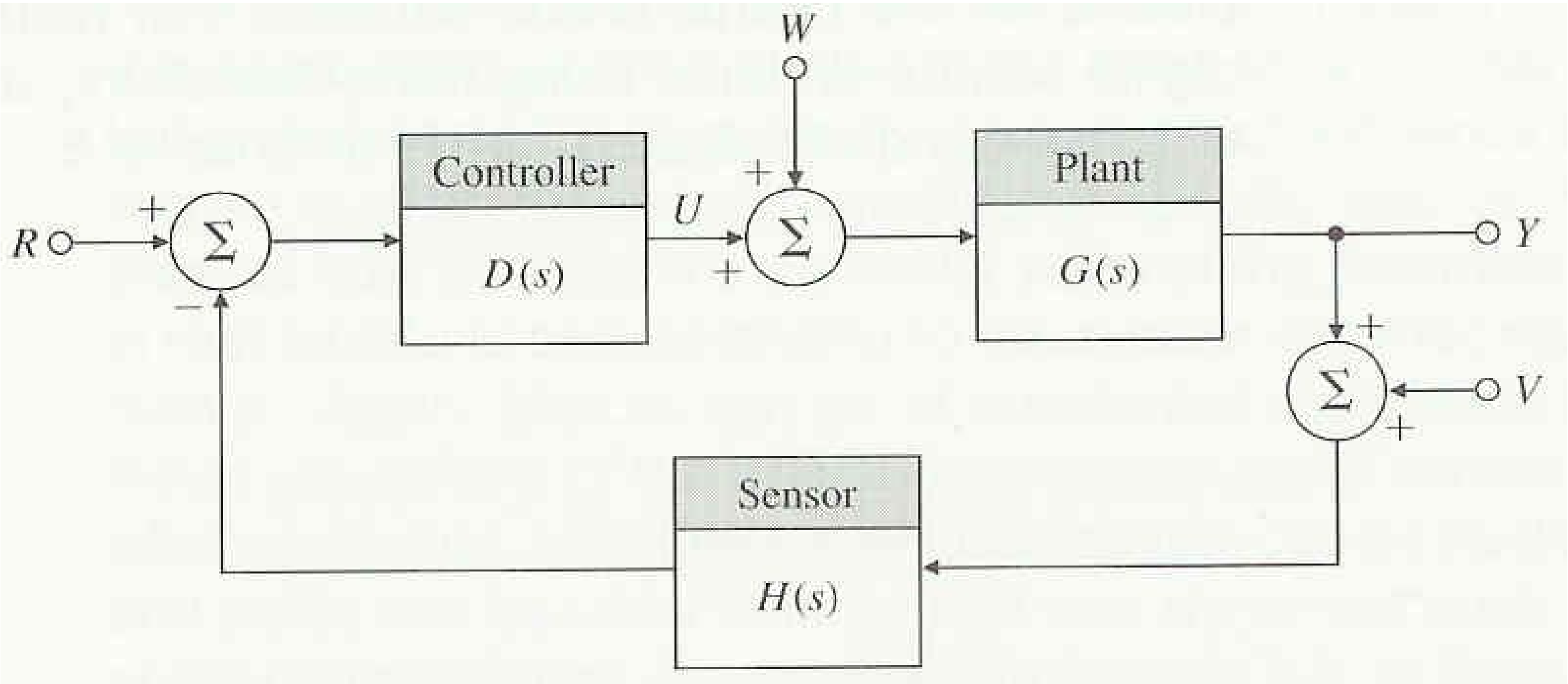
Introduction

Closed loop transfer function

$$\frac{Y(s)}{R(s)} = T(s) = \frac{D(s)G(s)}{1 + D(s)G(s)H(s)}$$

Characteristic equation, roots are poles in *T(s)*

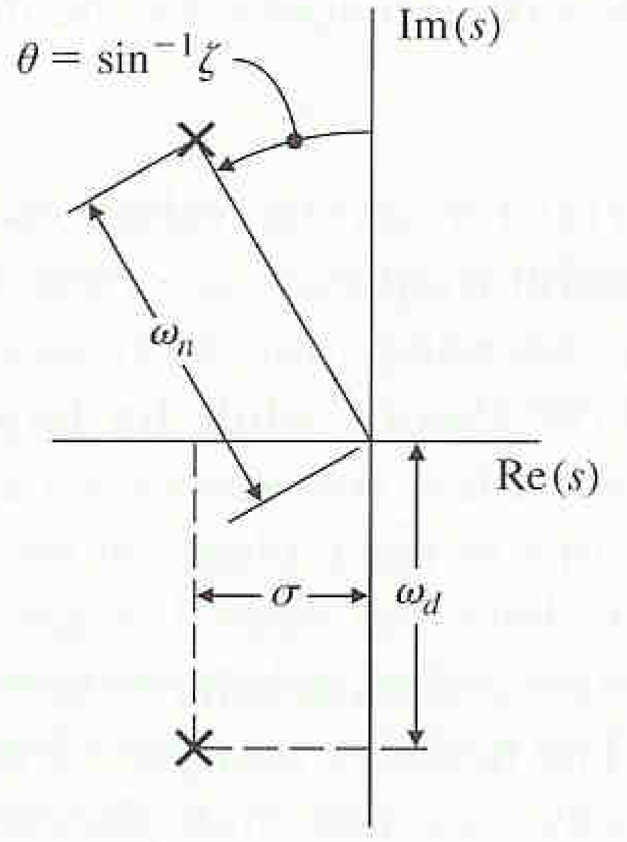
$$1 + D(s)G(s)H(s) = 0$$



Introduction

Dynamic features depend on the pole locations.

For example, time constant τ, rise time t_r, and overshoot M_p



$$H_1(s) = \frac{1}{\tau s + 1}$$ (1st order)

$$H_2(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \sigma + j\omega_d)(s + \sigma - j\omega_d)}$$

$$t_r \approx 1.8/\omega_n \quad , \quad M_p(\zeta)$$ (2nd order)

Introduction

Root locus

- Determination of the closed loop pole locations under varying *K*.
- For example, *K* could be the control gain.

The characteristic equation can be written in various ways

$$1 + D(s)G(s)H(s) = 0$$

$$\Downarrow$$

$$1 + KL(s) = 0$$

$$1 + K\frac{b(s)}{a(s)} = 0$$

$$a(s) + Kb(s) = 0$$

$$L(s) = -\frac{1}{K}$$

$$\frac{b(s)}{a(s)} = -\frac{1}{K}$$

Introduction

Polynomials *b(s)* and *a(s)*

$$L(s) = \frac{b(s)}{a(s)} \quad , \quad a(s) \text{ and } b(s) \text{ are monic}$$

$$\text{(coefficient to the highest power} = 1\text{)}$$

$$b(s) = s^m + b_1 s^{m-1} + \ldots + b_m$$

$$= (s - z_1)(s - z_2)\cdots(s - z_m) = \prod_{i=1}^{m}(s - z_i)$$

$$a(s) = s^n + a_1 s^{n-1} + \ldots + a_m$$

$$= (s - p_1)(s - p_2)\cdots(s - p_m) = \prod_{i=1}^{n}(s - p_i)$$

Definition of Root Locus

Definition 1

- The root locus is the values of *s* for which *1+KL(s)=0* is satisfied as *K* varies from 0 to infinity (pos.).

Definition 2

- The root locus is the points in the s-plane where the phase of *L(s)* is 180°.
  - Def: The angle to the test point from zero number *i* is ψ_i.
  - Def: The angle to the test point from pole number *i* is ϕ_i.
  - Therefore, $\sum \psi_i - \sum \phi_i = 180° + 360° (l-1), \quad l \quad \text{is an integer}$

In def. 2, notice, $L(s) = -1/K \,, \quad \angle(-1/K) = 180°$

for K real and positive

Root locus of a motor position control (example)

DC - motor position $\dfrac{\Theta_m(s)}{V_a(s)} = \dfrac{Y(s)}{U(s)} = G(s) = \dfrac{A}{s(s+1)}$

We have,

$$K = A\,, \quad L(s) = \frac{1}{s(s+1)}\,,$$

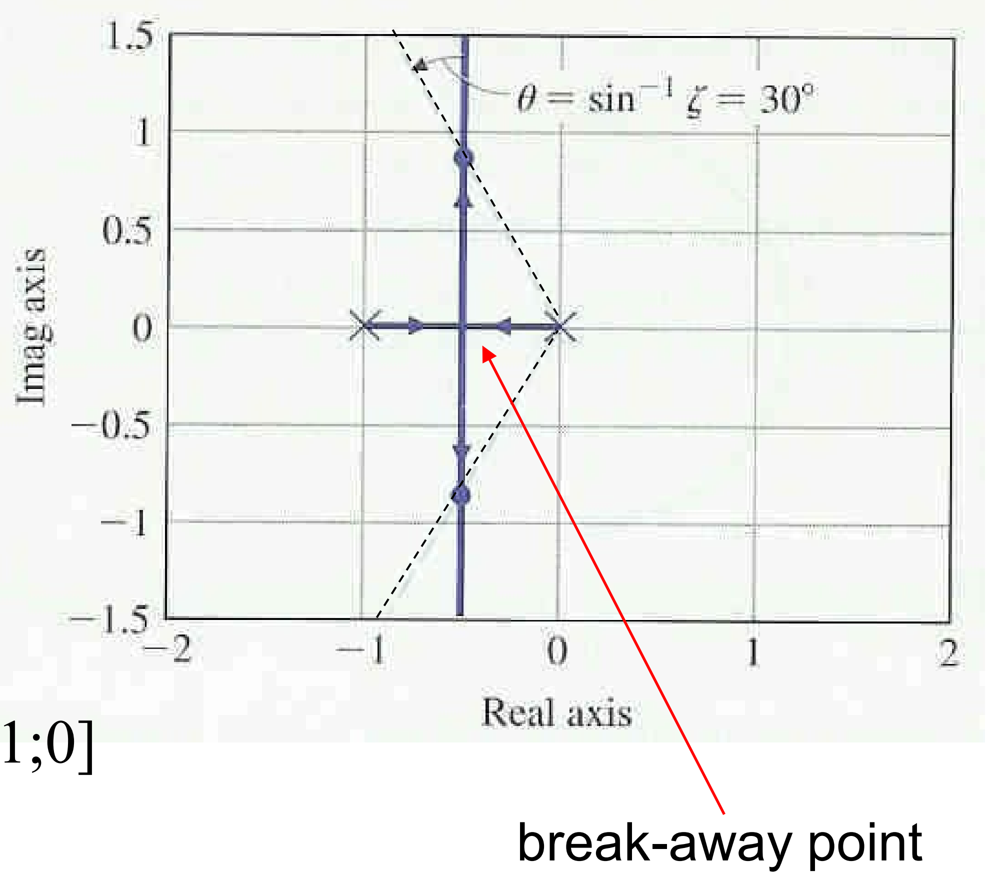
$$b(s) = 1\,, \quad a(s) = s^2 + s$$

Root locus is a graph of roots of

$$a(s) + Kb(s) = s^2 + s + K = 0$$

$$r_1\,, r_2 = -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2}$$

Root locus of a motor position control (example)



$$(r_1, r_2) = -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2}$$

For $K = 0$, (open-loop)

$$(r_1, r_2) = -\frac{1}{2} \pm \frac{1}{2} = \begin{cases} 0 \\ -1 \end{cases}$$

For $0 \le K \le 1/4$,

(r_1, r_2): real in the interval [-1;0]

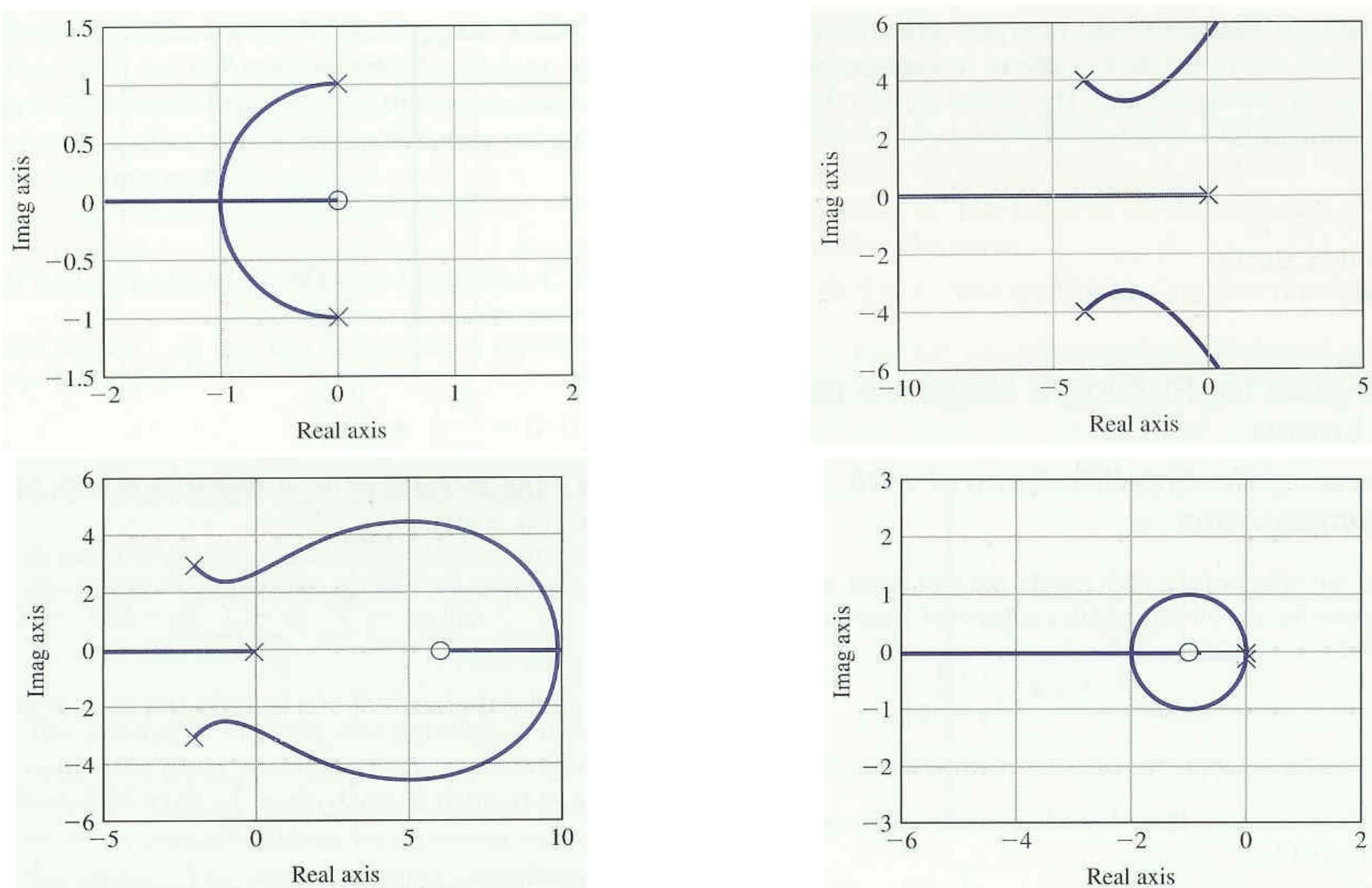
For $K > 1/4$,

(r_1, r_2): complex conjugated, real part $-1/2$

K can be calculated for some ζ (here, $\zeta = 0.5$)

Introduction

Some root loci examples (K from zero to infinity)

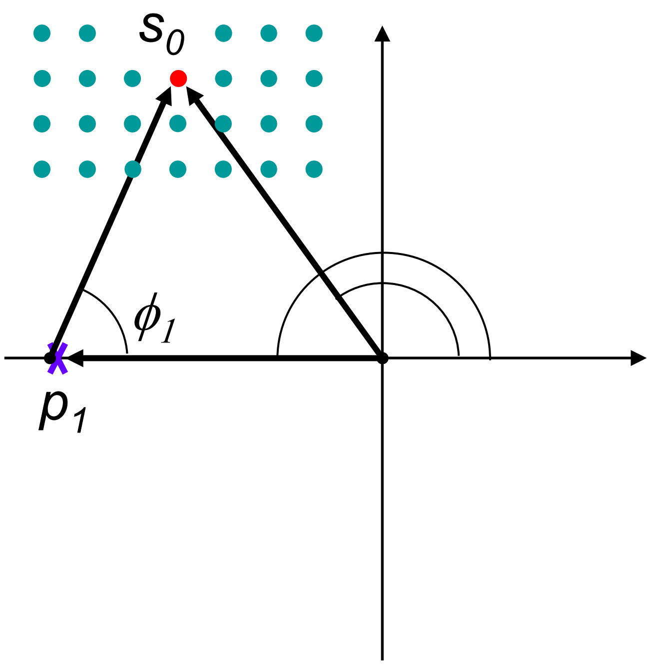


Sketching a Root Locus

How do we find the the root locus = closed loop pole locations for varying gain K

$$1+D(s)G(s)H(s) = 1+K\,L(s) = 0$$

- We could try a lot of test points and investigate the angle criteria (a lot of work)
- Sketch by hand
- Matlab



Sketching a Root Locus

What is meant by a test point? For example

$$G(s) = \frac{1}{(s+a)} = \frac{1}{(s-p_1)}$$

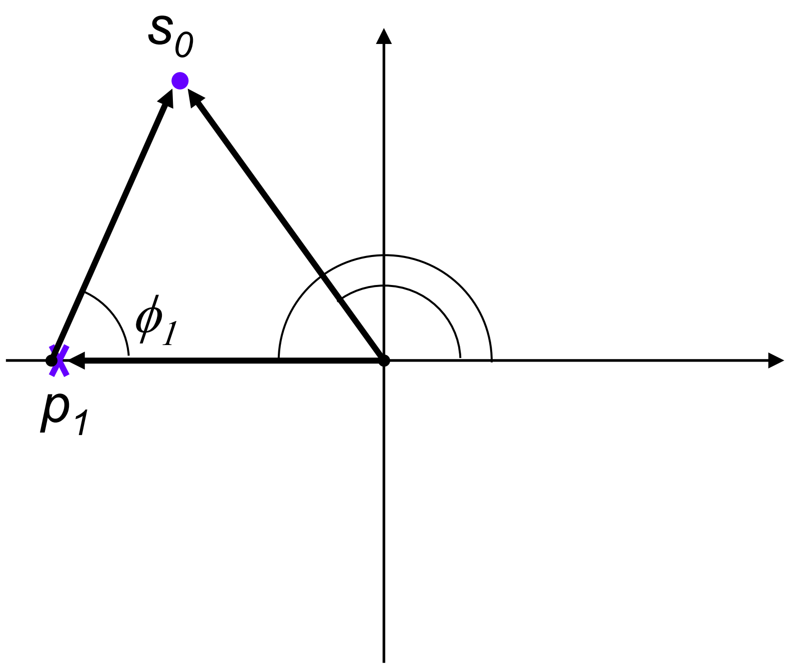
Test point s_0

$$G(s_0) = \frac{1}{(s_0-p_1)} = \frac{1}{R\,e^{j\phi_1}}$$

where

$$|s_0 - p_1| = R$$

$$\angle(s_0 - p_1) = \phi_1$$



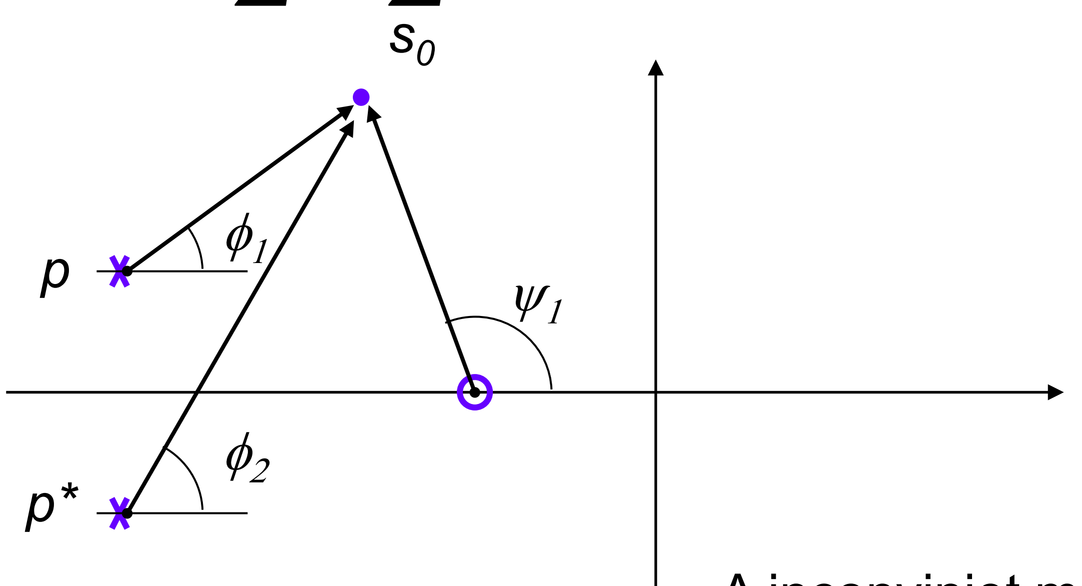
$\varphi_1 \neq 180^o \Rightarrow s_0$ is not on the root locus

Sketching a Root Locus

In general, $$G(s_0)=\frac{(s_0-z_1)\cdots(s_0-z_m)}{(s_0-p_1)\cdots(s_0-p_n)}=\frac{r_{b1}e^{j(\psi_1)}\cdots r_{bm}e^{j(\psi_m)}}{r_{a1}e^{j(\phi_1)}\cdots r_{an}e^{j(\phi_n)}}$$

$$=\frac{r_{b1}\cdots r_{b2}}{r_{a1}\cdots r_{a2}}\exp\left(j(\psi_1+\ldots+\psi_m-\phi_1-\ldots-\phi_n)\right)$$

Root locus definition : $\sum\psi_i-\sum\phi_i=180°+360°(l-1)$, l is an integer



A inconviniet method !

Root Locus characteristics

The *n* branches of the locus start at the poles of *L(s)* and *m* of these branches end on the zeros of *L(s)*, n-m branches ends in ∞.

Notice,

$$a(s) \text{ is of order } n, \text{ and } b(s) \text{ is of order } m, \quad n > m$$

$$a(s) + Kb(s) = 0 \quad \Leftrightarrow \quad \frac{b(s)}{a(s)} = -\frac{1}{K}$$

$$\text{if } \ K = 0 \ \ \text{(start)}, \quad a(s) = 0 \, \text{(poles)}$$

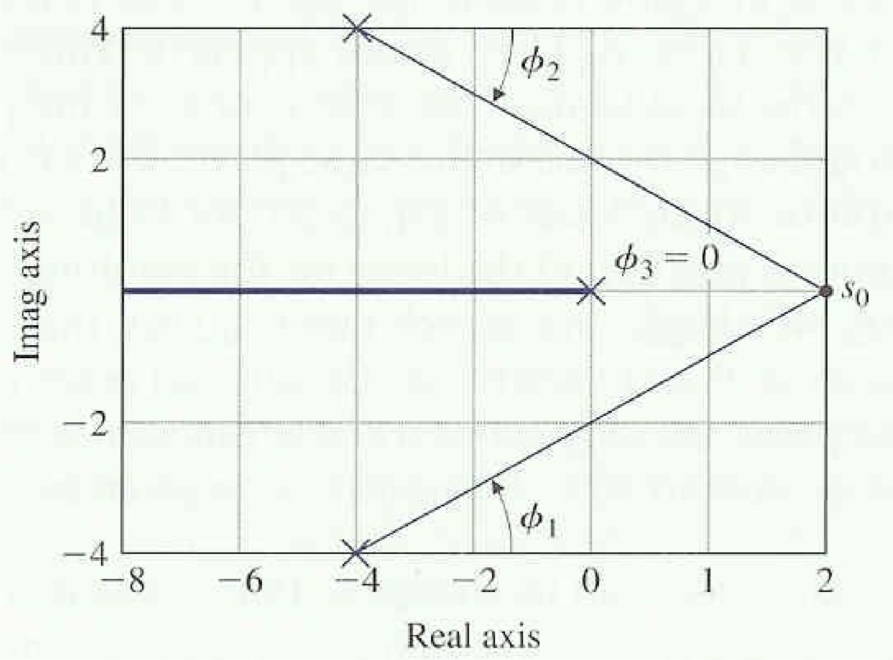
$$\text{if } \ K \to \infty \ \ \text{(end)}, \quad \begin{cases} b(s) = 0 & (zeros)(\text{Rule } 1) \\ a(s) \to \infty & (\text{Rule } 3) \end{cases}$$

Root Locus characteristics

The loci on the real axis (real-axis part) are to the left of an odd number of poles plus zeros.

Notice, if we take a test point s_0 on the real axis :

- The angle of complex poles cancel each other.
- Angles from real poles or zeros are 0° if s_0 are to the right.
- Angles from real poles or zeros are 180° if s_0 are to the left.
- Total angle = $180° + 360° \, l$



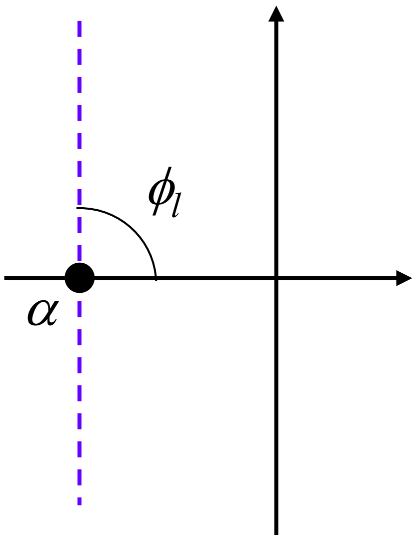
Root Locus characteristics

- For large s and K, n-m branches of the loci are asymptotic to lines at angles ϕ_l radiating out from a point $s = \alpha$ on the real axis.

$$\phi_l = \frac{180° + 360°(l-1)}{n-m}, \quad l = 1, 2, \ldots, n-m$$

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m}$$

For example



Root Locus characteristics

We have, $\dfrac{b(s)}{a(s)} = L(s) = -1/K \quad , \quad n > m$

If $K \to \infty, \quad L(s) = 0 \quad \Rightarrow \begin{cases} b(s) = 0 & \text{(Rule 1)} \\ a(s) \to \infty & \text{(Rule 3)} \end{cases}$

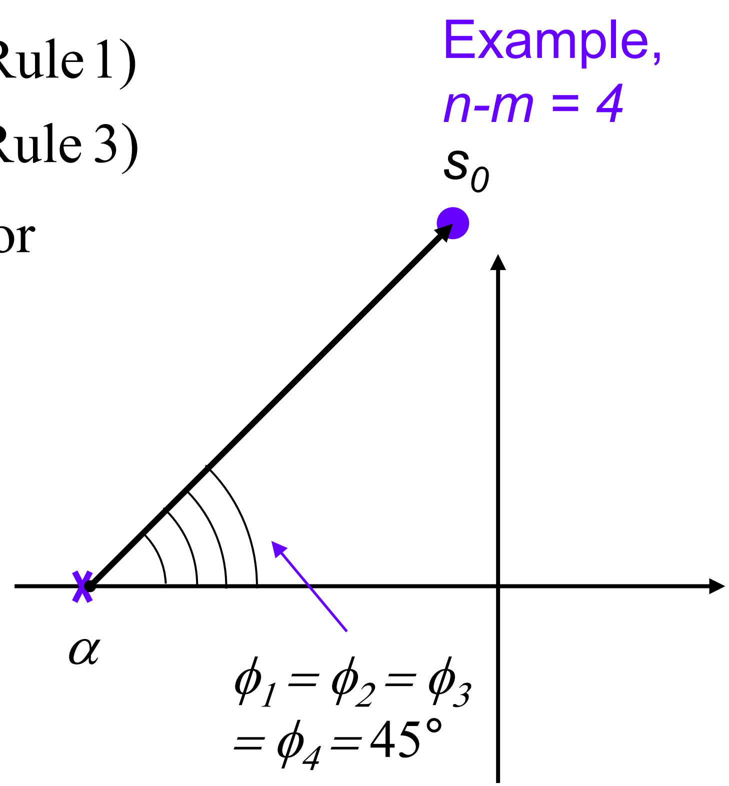
For large s we can derive an approximation for

$$1 + K\frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = 0$$

Approximation

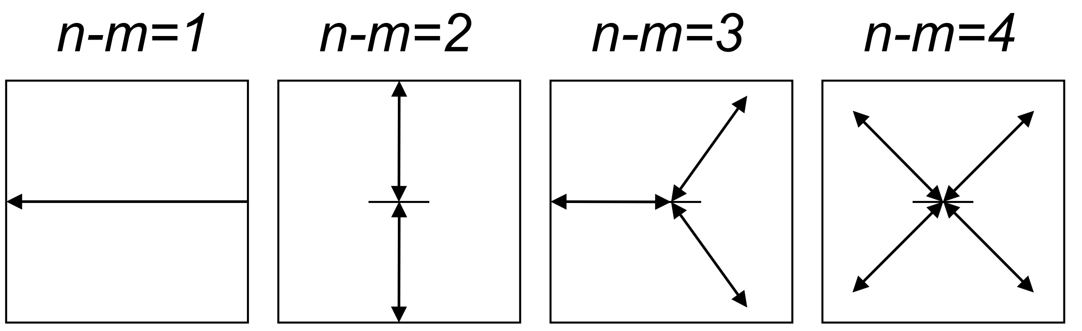
$$1 + K\frac{1}{(s-\alpha)^{n-m}} = 0\,, \quad \alpha \in R$$

Notice, we must have $\displaystyle\sum_{i=1}^{n\text{-}m} \angle(s-\alpha)_i = 180°$



Thus, $\phi_l = 45°$

Root Locus characteristics

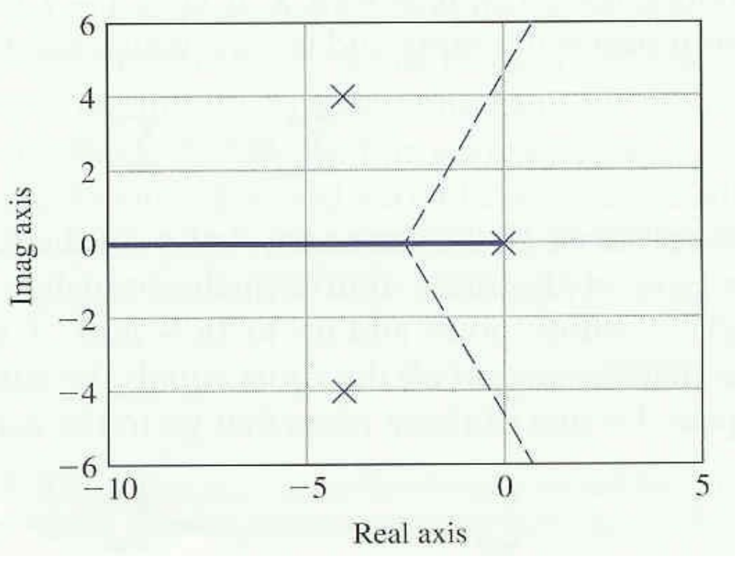


Rule 3

$$\phi_l = \frac{180° + 360°(l-1)}{n-m},$$

$$l = 1, 2, \ldots, n-m$$

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m}$$



For ex., *n-m=3*

Root Locus characteristics

- The angle of departure of a branch of the locus from a pole of multiplicity *q* is given by (1)
- The angle of departure of a branch of the locus from a zero of multiplicity *q* is given by (2)

$$\text{Pole of mult. } q, \quad \frac{b(s)}{a(s)} = \frac{1}{(s+p)^q}\frac{b(s)}{a_0(s)}$$

$$(1) \quad q\phi_{l,dep} = \sum \psi_i - \sum_{i \neq l} \phi_i - 180° - 360°(l-1)$$

$$\text{Zero of mult. } q, \quad \frac{b(s)}{a(s)} = (s+z)^q \frac{b_0(s)}{a(s)}$$

$$(2) \quad q\psi_{l,dep} = \sum \phi_i - \sum_{i \neq l} \psi_i + 180° + 360°(l-1)$$

Notice, the situation is similar to the approximation in Rule 3

Root Locus characteristics

In general, we must find a s_0 such that *angle(L(s))=180°*

Because we have

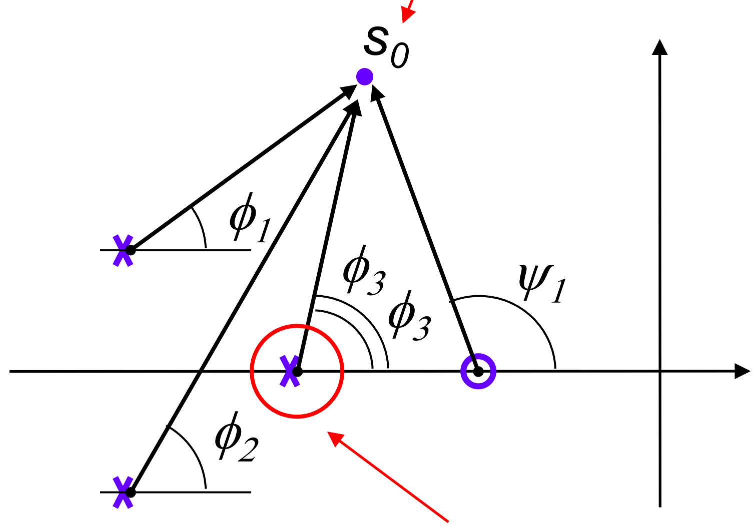
$$\sum \psi_i - \sum \phi_i = 180° + 360°(l-1)$$

Notice,

$$\sum \phi_i = q\phi_{l,dep} + \sum_{i \neq l} \phi_i$$

$$\sum \psi_i = q\psi_{l,arr} + \sum_{i \neq l} \psi_i$$

So, we can find $q\phi_{l,dep}$ and $q\psi_{l,arr}$



Vary s_0 until *angle(L(s))=180°*

Departure and arrival?

For K increasing from zero to infinity poles go towards zeros or infinity. Thus,

- A branch corresponding to a pole departs at some angle.
- A branch corresponding to a zero arrives at some angle.

Rouths stability criterion

Given a characteristic equation of an n-th order system

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

The system is stable if all elemets af the firs column of the Routh Array are positive

The root locus crosses the jω axis when an element in the first row changes sign

$$\begin{array}{cc|c|cccc}
\mathrm{Row}(s^{n}) & n & 1 & a_2 & a_4 & \cdots \\
\mathrm{Row}(s^{n-1}) & n-1 & a_1 & a_3 & a_5 & \cdots \\
\mathrm{Row}(s^{n-2}) & n-2 & b_1 & b_2 & b_3 & \cdots \\
\mathrm{Row}(s^{n-3}) & n-3 & c_1 & c_2 & c_3 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \\
\mathrm{Row}(s^{0}) & 0 & * & * & * & *
\end{array}$$

$$b_1 = \frac{-1}{a_1}\det\begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}$$

$$b_2 = \frac{-1}{a_1}\det\begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}$$

$$c_1 = \frac{-1}{b_1}\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}$$

Routh stability -example

Characteristic equation

$$1 + K\frac{(s+1)}{s(s-1)(s+6)} = 0$$

$$s^3 + 5s^2 + (K-6)s + K = 0$$

Routh array

$$\begin{array}{lcc} s^3 & 1 & k-6 \\ s^2 & 5 & K \\ s & (4K-30)/5 & 0 \\ s^0 & K & \end{array}$$

K >0.

The system is stable if 4K-30>0=> K>7.5

Root Locus characteristics

- The locus will have multiplicative roots of *q* at points on the locus where (1) applies.
- The branches will approach a point of *q* roots at angles separated by (2) and will depart at angles with the same separation.

(1) $$\left(b\frac{da}{ds} - a\frac{db}{ds}\right) = 0$$

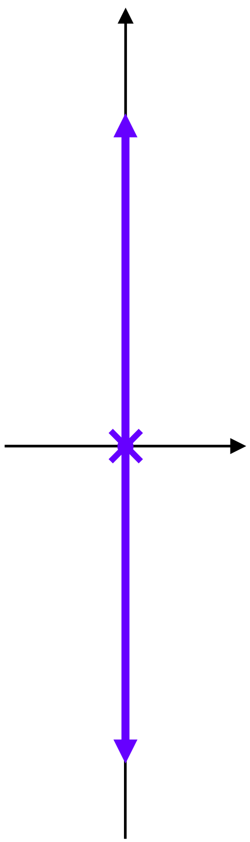
(2) $$\frac{180° + 360°(l-1)}{q}$$

Root Locus sketching

Example

Root locus for double integrator with P-control (Satellite control)

$$G(s) = \frac{1}{s^2} \quad , \quad D(s) = k_p \quad \Rightarrow \quad 1 + k_p \frac{1}{s^2} = 0 \quad \text{(Char. Eq.)}$$



The locus has two poles ≈ two branches that starts in *s=0*. There are no zeros. Thus, the branches do not end at zeros but goes towards infinity.

Two branches have asymptotes for *s* going towards infinity.

$$\phi_l = \frac{180° + 360°(l-1)}{n-m} = \frac{180° + 360°(l-1)}{2} = \begin{cases} 90° \\ 270° \end{cases}$$

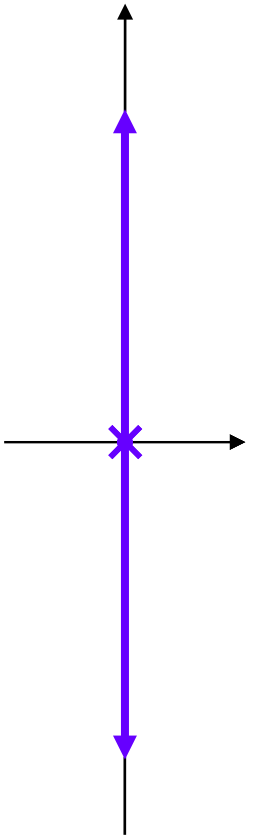
$$\alpha = \frac{\sum p_i - \sum z_i}{n-m} = \frac{0}{2} = 0$$

Marginal stability for all K

Sketching a Root Locus

$$G(s) = \frac{1}{s^2} \quad , \quad D(s) = k_p \quad \Rightarrow \quad 1 + k_p \frac{1}{s^2} = 0 \quad \text{(Char. Eq.)}$$

No branches at the real axis.



The loci remain on the imaginary axis. Thus, no crossings of the jω-axis.

Easy to see, no further multiple poles. Verification:

$$a(s) = s^2 \quad , \quad b(s) = 1$$

$$0 = \left(b\frac{da}{ds} - a\frac{db}{ds} \right) = \left(1\frac{d(s^2)}{ds} - s^2 \frac{d1}{ds} \right) = 2s - 0 \quad \Rightarrow \quad s = 0$$

Sketching a Root Locus

Example

Root locus for satellite attitude control with PD-control

$$G(s) = \frac{1}{s^2} \quad , \quad D(s) = k_p + k_D s \quad \Rightarrow$$

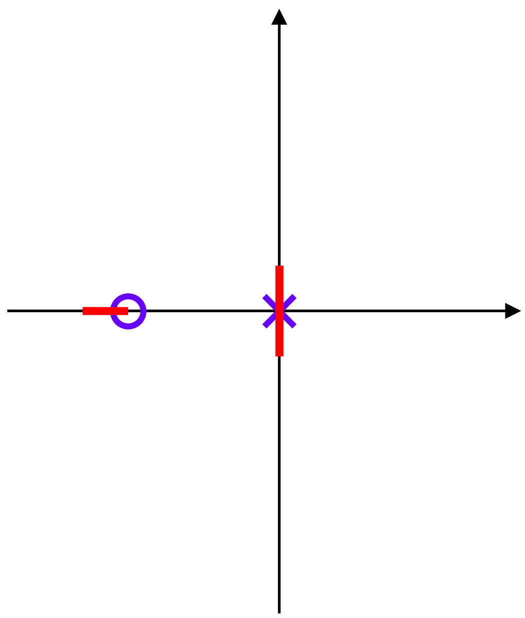
$$1 + (k_p + k_D s)\frac{1}{s^2} = 0 \quad \text{(Char. Eq.)}$$

$$K = k_D \quad \text{and} \quad \frac{k_p}{k_D} = 1 \quad \Rightarrow$$

$$\frac{1}{K} + (1+s)\frac{1}{s^2} = 0 \quad \Leftrightarrow \quad 1 + K\frac{(1+s)}{s^2} = 0 \quad \text{(Char. Eq.)}$$

Sketching a Root Locus

$$1 + K\frac{(1+s)}{s^2} = 0 \quad \Leftrightarrow \quad \frac{(1+s)}{s^2} = -\frac{1}{K}$$



- There are two branches that start at $s = 0$.

- One terminates on the zero at $s = -1$.

  the other approaches infinity.

- the real axis to the left of $s = -1$ is on the locus

- because $n - m = 1$, there is one asymptote along the negative real axis

- the angles of departure from the double pole $s = 0$ at are $\pm 90°$

Sketching a Root Locus

$$1 + K\frac{(1+s)}{s^2} = 0 \qquad \text{(Char. Eq.)}$$

- Applying Routh's criterion, we find the array below.
  thus, the locus does not cross the imaginary axis.

$$\begin{array}{ll} 1 & K \\ K & \\ K & \end{array}$$

- The points of multiple roots are found from

$$b(s) = s+1 \quad , \quad a(s) = s^2 \quad , \quad \frac{db}{ds} = 1 \quad , \quad \frac{da}{ds} = 2s$$

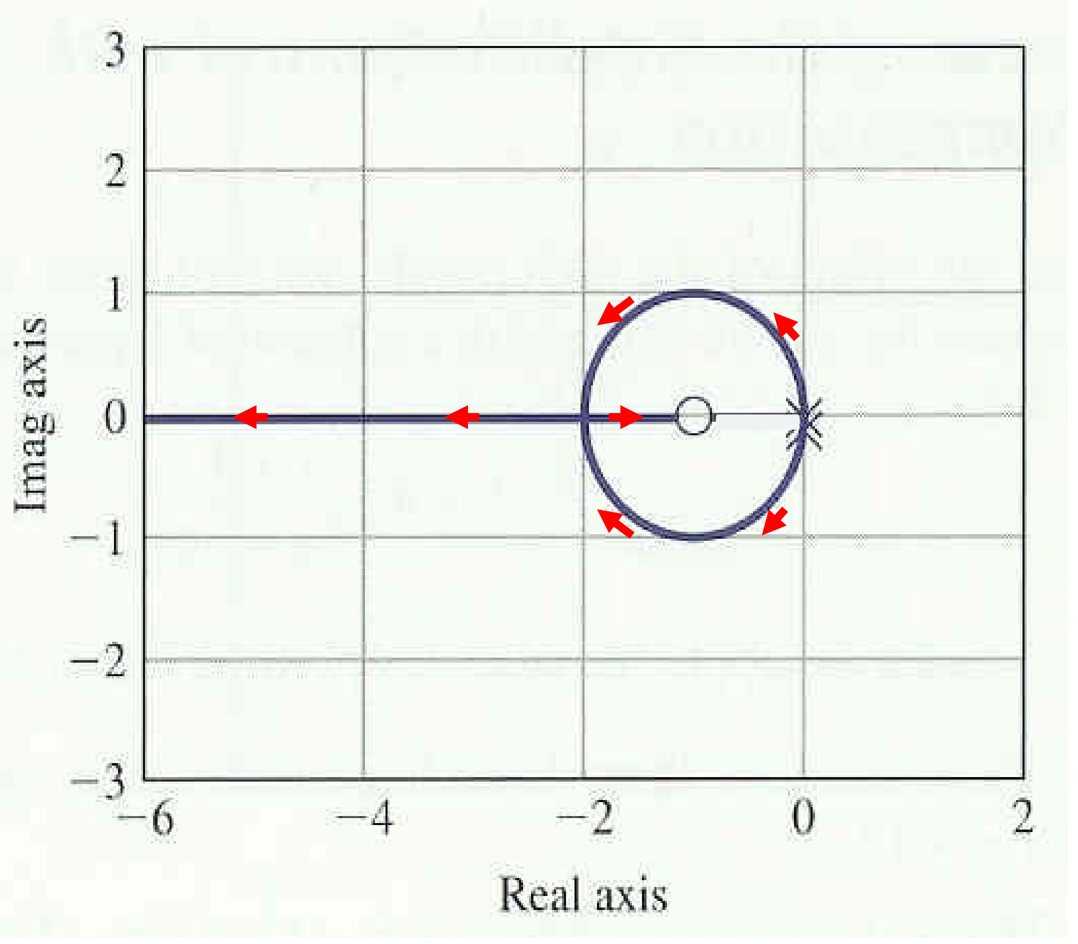
$$\Rightarrow \qquad b\frac{da}{ds} - a\frac{db}{ds} = (s+1)2s - s^2 = 0 \quad \Rightarrow \quad s_i = \{0, -2\}$$

Sketching a Root Locus

The system will be stable for all values of K >0

Real poles for large values of K

Complex poles for small values of K



Selecting the Parameter Value

The (positive) root locus

- A plot of all possible locations of roots to the equation $1+KL(s)=0$ for some real positive value of K.
- The purpose of design is to select a particular value of *K* that will meet the specifications for static and dynamic characteristics.
- For a given root locus we have (1). Thus, for some desired pole locations it is possible to find *K.*

$$(1) \qquad K = -\frac{1}{L(s)} \quad \Rightarrow \quad K = \frac{1}{|L|}$$

Selecting the Parameter Value

Example

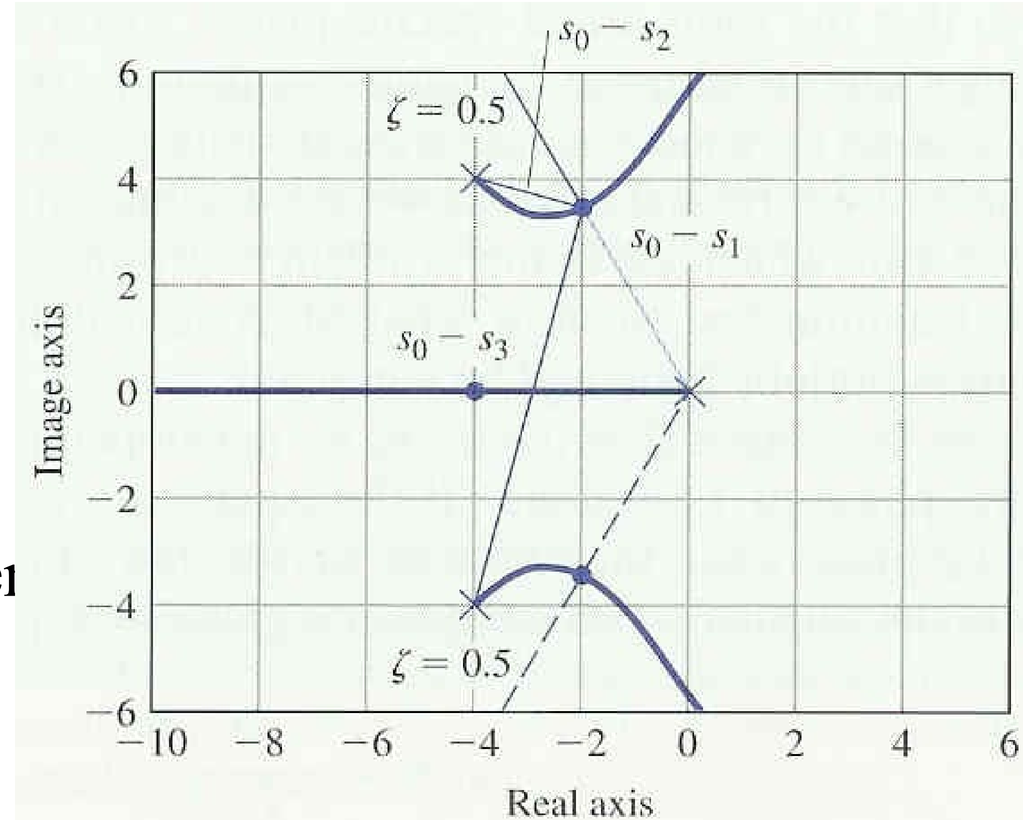
$$L(s) = \frac{1}{s((s+4)^2+16)}$$

$$= \frac{1}{s(s+4+j4)(s+4-j4)}$$

Poles at s_1, s_2, s_3

Let us say we want $\zeta = 0.5$, ther

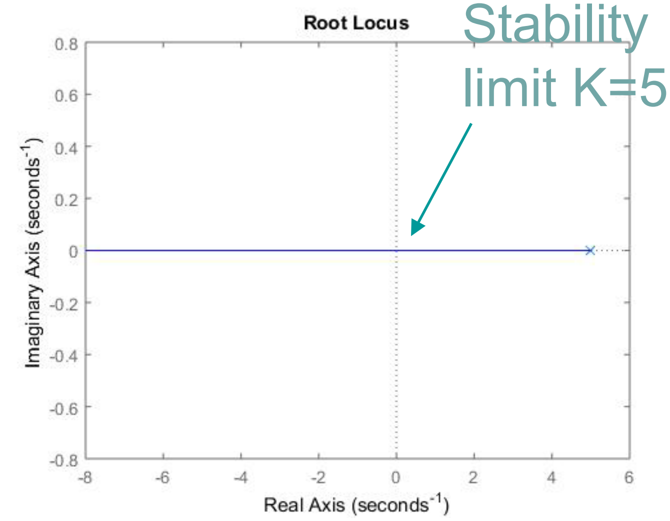
$$L(s_0) = \frac{1}{s_0(s_0-s_2)(s-s_3)} \quad \Rightarrow$$



$$K = \frac{1}{|L(s_0)|} = |s_0||(s_0-s_2)||(s_0-s_3)| \approx 4*2.1*7.7 \approx 65$$

Unstable systems

Systems with positive poles can not be handled by Bode plots. They can be handled by root locuses



Given the unstable system

$$L(s) = \frac{1}{s-5}$$

The characteristic equation is

$$1 + K\frac{1}{s-5} = 0 \Rightarrow s - 5 + K = 0$$

Using the Routh array to find the limits for stability

$$\begin{array}{cc} 1 & 0 \\ -5+K & 0 \\ 0 & 0 \end{array}$$

the limit for positive array values in 1. column is $-5 + K \geq 0 \; or \; K \geq 5$

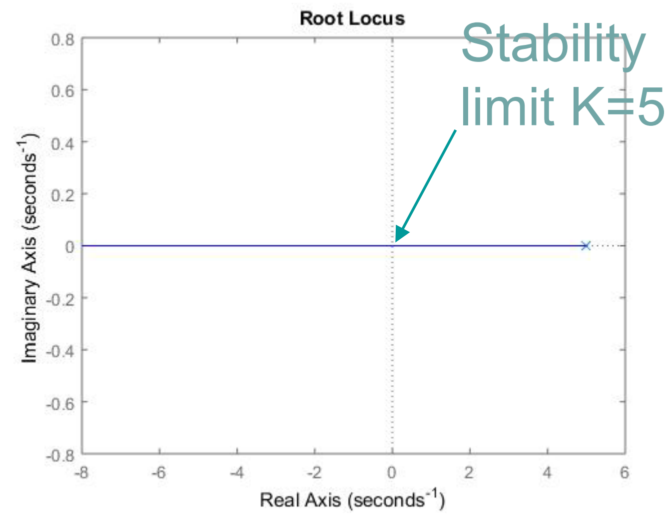
Unstable systems

$$L(s) = \frac{1}{s-5}$$

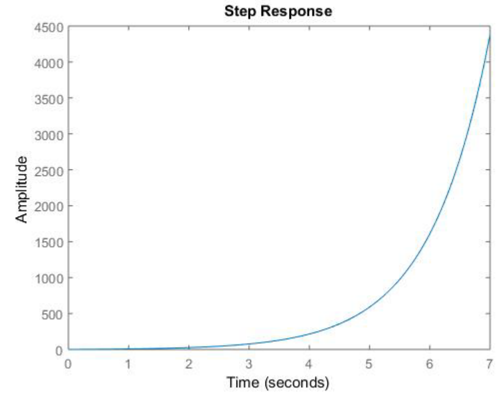
The closed loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{K\dfrac{1}{s-5}}{1+K\dfrac{1}{s-5}} = \frac{K}{s-5+K}$$

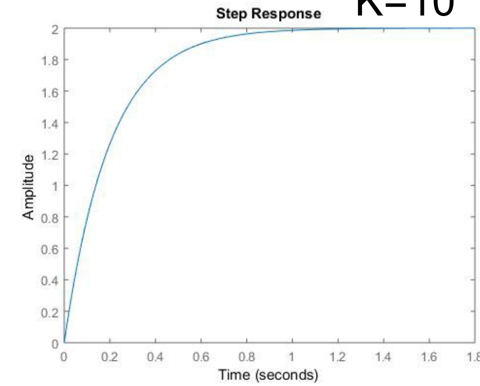
Pole in 0 for K=5
Negative pole for K>5



K=4



K=10



Selecting the Parameter Value

Matlab

A root locus can be plotted using Matlab

- rlocus(sysL)

Selection of K

- [K,p]=rlocfind(sysL)