



# State Space Methods

## Lecture 4: reduced order observers, integral control

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The reduced order observer

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# The reduced order observer

Possibly following a state space transformation, a state space model can be partitioned as:

$$\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u$$
$$y = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Writing out the equations, we obtain:

$$\begin{aligned} \dot{\mathbf{x}}_1 &= A_{11}\mathbf{x}_1 + A_{12}x_2 + B_1u \\ \dot{x}_2 &= A_{21}\mathbf{x}_1 + A_{22}x_2 + B_2u \\ y &= \mathbf{x}_1 \end{aligned}$$

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$$y = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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# The reduced order observer

By rearranging the equation for  $\dot{x}_1 = \dot{y}$  :

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$$\underbrace{\dot{y} - A_{11}y - B_1u}_{\text{known}} = \underbrace{A_{12}x_2}_{\text{unknown}}$$

it can be seen as a 'measurement equation'.

The corresponding state estimation equation is formed from the equation for  $\dot{x}_2$ :

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The corresponding state estimation equation is formed from the equation for  $\dot{x}_2$ :

$$\dot{\hat{x}}_2 = A_{21}y + A_{22}\hat{x}_2 + B_2u + L(A_{12}\hat{x}_2 - A_{12}x_2)$$

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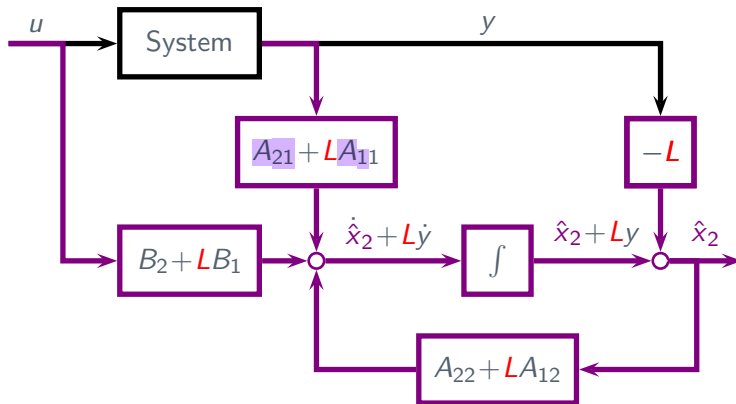
The corresponding state estimation equation is formed from the equation for  $\dot{x}_2$ :

$$\begin{aligned}\dot{\hat{x}}_2 &= A_{21}y + A_{22}\hat{x}_2 + B_2u + L(A_{12}\hat{x}_2 - A_{12}x_2) \\ &= A_{21}y + A_{22}\hat{x}_2 + B_2u + L(A_{12}\hat{x}_2 - (\dot{y} - A_{11}y - B_1u))\end{aligned}$$

Rearranging the terms, we obtain:

$$\dot{\hat{x}}_2 + L\dot{y} = (A_{22} + LA_{12})\hat{x}_2 + (A_{21} + LA_{11})y + (B_2 + LB_1)u$$

# The reduced order observer



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# The reduced order observer

System equation for  $x_2$ :

$$\dot{x}_2 = A_{21}y + A_{22}x_2 + B_2u$$

Observer equation:

$$\dot{\hat{x}}_2 = A_{21}y + A_{22}\hat{x}_2 + B_2u + L(A_{12}\hat{x}_2 - A_{12}x_2)$$

Estimation error:  $e = \hat{x}_2 - x_2$ .

$$\begin{aligned}\dot{e} &= \dot{\hat{x}}_2 - \dot{x}_2 \\ &= A_{21}y + A_{22}\hat{x}_2 + B_2u + L(A_{12}\hat{x}_2 - A_{12}x_2) \\ &\quad - (A_{21}y + A_{22}x_2 + B_2u) \\ &= (A_{22} + LA_{12})(\hat{x}_2 - x_2) = (A_{22} + LA_{12})e\end{aligned}$$

# The reduced order observer

## Theorem

*Assume that the auxiliary system*

$$\dot{x}_2 = A_{22}x_2, \quad y = A_{12}x_2$$

*is observable. Then there exists an observer gain  $L$  such that  $A_{22} + LA_{12}$  is stable.*

*With this observer gain, the observer*

$$\dot{\hat{x}}_2 = A_{21}y + A_{22}\hat{x}_2 + B_2u + L(A_{12}\hat{x}_2 - A_{12}x_2)$$

*is guaranteed to give an estimate  $\hat{x}_2$  which converges to  $x_2$  at a rate given by the eigenvalues of  $A_{22} + LA_{12}$ .*

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# Reduced order observer based control

Based on the estimates of a reduced order observer, the feedback law becomes:

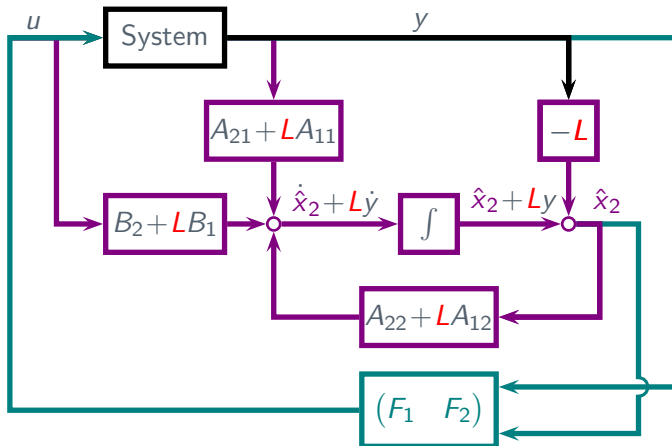
$$u = F \begin{pmatrix} y \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \end{pmatrix} \begin{pmatrix} y \\ \hat{x}_2 \end{pmatrix} = F_1 y + F_2 \hat{x}_2$$

The resulting closed loop system has poles equal to the eigenvalues of the two matrices:

$$A + BF \quad \text{and} \quad A_{22} + LA_{12}$$

This is the reduced order version of the *separation theorem*!

# Reduced order observer based control





# Algorithm for reduced order control

1. Design a state feedback matrix  $F$ , such that the eigenvalues of  $A + BF$  corresponds to desired poles.
2. Transform, if necessary, the system to a form where the output equation has the form

$$y = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For a single output system, transformation to observable canonical form is one possible choice.

# Algorithm for reduced order control

3. Partition the transformed system matrices:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = T^{-1}AT, \quad \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = T^{-1}B$$
$$\begin{pmatrix} F_1 & F_2 \end{pmatrix} = FT$$

where  $T$  is the transform matrix.

4. Design  $L$  such that  $A_{22} + LA_{12}$  is a stable matrix with desired observer poles as eigenvalues.

# Algorithm for reduced order control

4. Design  $L$  such that  $A_{22} + LA_{12}$  is a stable matrix with desired observer poles as eigenvalues.
5. Construct the reduced order observer:

$$\begin{aligned}\dot{\hat{x}}_2 + L\dot{y} &= (A_{22} + LA_{12})\hat{x}_2 + (A_{21} + LA_{11})y \\ &\quad + (B_2 + LB_1)u\end{aligned}$$

6. Close the loop by the feedback law:

$$u = F_1y + F_2\hat{x}_2$$

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# Example: reduced order control

We consider again the system

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y &= \begin{pmatrix} -3 & 2 \end{pmatrix} x\end{aligned}$$

1. We have already computed a state feedback which assigns poles in  $\{-4, -5\}$ :

$$F = \begin{pmatrix} 42 & -30 \end{pmatrix}$$



## Example: reduced order control

1. We have already computed a state feedback which assigns poles in  $\{-4, -5\}$ :

$$F = \begin{pmatrix} 42 & -30 \end{pmatrix}$$

2.  $CT = \begin{pmatrix} I & 0 \end{pmatrix}$  can be achieved by transforming to observable canonical form, which is obtained by:

$$T = \begin{pmatrix} -5 & 2 \\ -7 & 3 \end{pmatrix}$$

## Example: reduced order control

3. Partitioning gives:

$$\left( \begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right) = T^{-1}AT = \left( \begin{array}{c|c} -3 & 1 \\ \hline -2 & 0 \end{array} \right)$$

$$\left( \begin{array}{c} B_1 \\ B_2 \end{array} \right) = T^{-1}B = \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$$

$$\left( \begin{array}{c|c} F_1 & F_2 \end{array} \right) = FT = \left( \begin{array}{c|c} 0 & -6 \end{array} \right)$$

4. The observer pole is chosen as  $-5$  :

$$A_{22} + LA_{12} = 0 + L1 = -5 \implies L = -5$$

## Example: reduced order control

5. The reduced order observer equation:

$$\dot{\hat{x}}_2 + L\dot{y} = (A_{22} + LA_{12})\hat{x}_2 + (A_{21} + LA_{11})y \\ + (B_2 + LB_1)u$$

becomes:

$$\dot{\hat{x}}_2 + (-5)\dot{y} = (0 - 5 \cdot 1)\hat{x}_2 + (-2 + (-5) \cdot (-3))y \\ + (1 + (-5) \cdot 0)u$$

or

$$\dot{\hat{x}}_2 - 5\dot{y} = -5\hat{x}_2 + 13y + u$$



## Example: reduced order control

6. The feedback law becomes:

$$u = F_1 y + F_2 \hat{x}_2 = 0y + (-6)\hat{x}_2 = -6\hat{x}_2$$

Taking Laplace transform of the observer eq.:

$$\dot{\hat{x}}_2 - 5\dot{y} = -5\hat{x}_2 + 13y + u$$

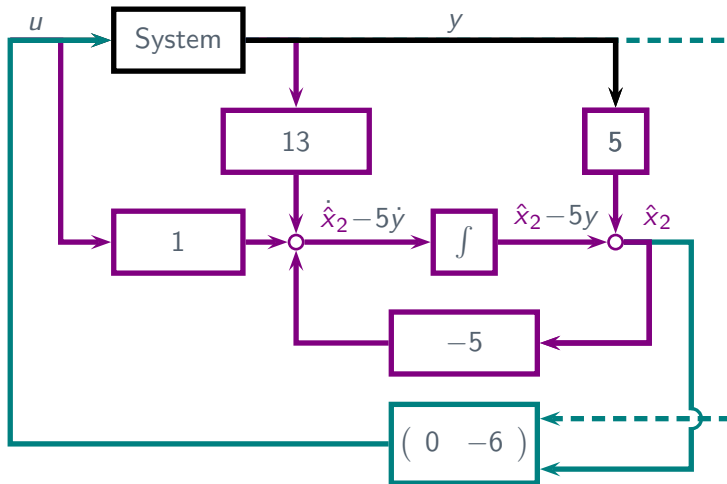
and substituting the feedback law gives:

$$s\hat{x}_2 - 5sy = -5\hat{x}_2 + 13y - 6\hat{x}_2$$

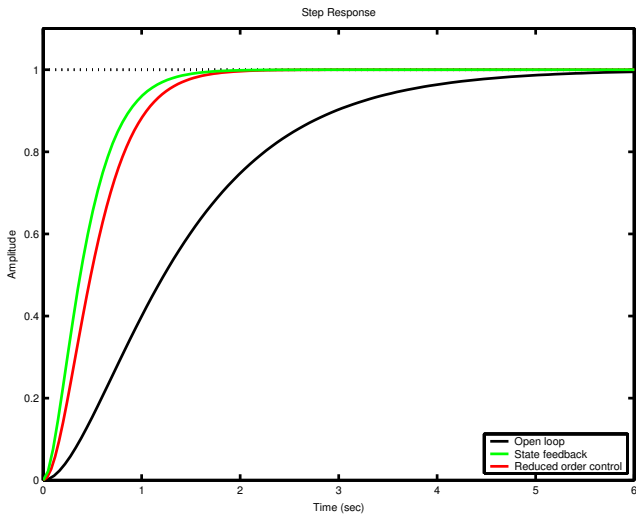
which implies:

$$(s + 11)\hat{x}_2 = (5s + 13)y \Rightarrow \boxed{u = -6\hat{x}_2 = -6\frac{5s+13}{s+11}y}$$

# Example: reduced order control



# Example: reduced order control



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# Integral control

We consider a state space system of the form:

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

for which we wish to design a feedback law:

$$u(t) = Fx(t) + F_I x_I(t)$$

where

$$x_I(t) = \int_0^t y(\tau) - r(\tau) d\tau$$

or

$$\dot{x}_I(t) = y(t) - r(t)$$

# Integral control

The equations:

$$\begin{aligned}\dot{x} &= Ax + Bu \\ \dot{x}_I &= y - r \\ y &= Cx\end{aligned}$$

can be combined into an extended state model:

$$\begin{aligned}\begin{pmatrix} \dot{x} \\ \dot{x}_I \end{pmatrix} &= \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ -I \end{pmatrix} r \\ y &= (C \ 0) \begin{pmatrix} x \\ x_I \end{pmatrix}\end{aligned}$$

for which the feedback law becomes:

$$u = Fx + F_I x_I = \begin{pmatrix} F & F_I \end{pmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix}$$

# Integral control

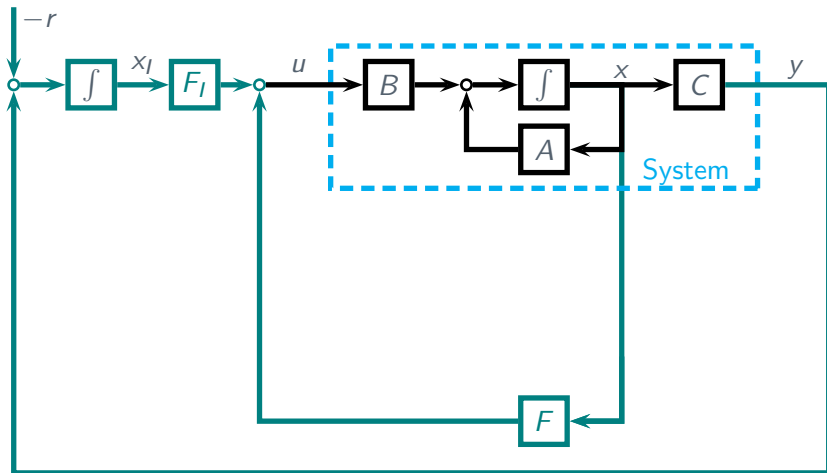
Thus, the integral control problem has been reduced to a conventional state feedback problem:

$$\begin{aligned}\dot{x}_e &= A_e x_e + B_e u \\ y &= C_e x_e\end{aligned}$$

for which we have to design a state feedback  $u = F_e x_e$ , where:

$$\begin{aligned}F_e &= ( \textcolor{teal}{F} \quad \textcolor{teal}{F}_I ) , \quad x_e = \begin{pmatrix} x \\ x_I \end{pmatrix} \\ A_e &= \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} , \quad B_e = \begin{pmatrix} B \\ 0 \end{pmatrix} , \quad C_e = ( \quad C \quad 0 \quad )\end{aligned}$$

# Integral control





# Integral control

If the states are unavailable for feedback, they can be estimated by e.g. a full order observer:

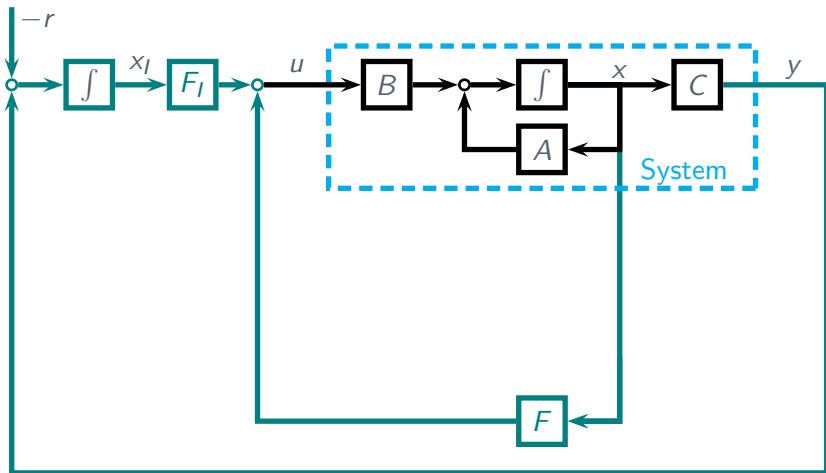
$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x}\end{aligned}$$

where  $L$  is chosen such that  $A + LC$  is stable with desirable eigenvalues.

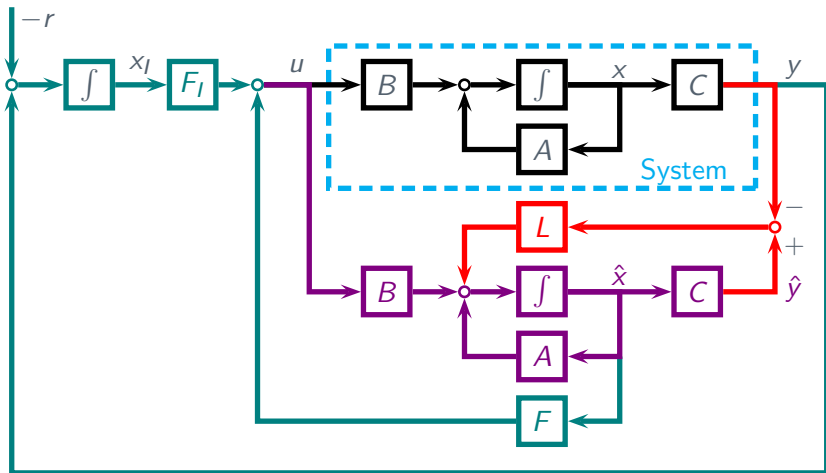
**Separation result:** The closed loop poles of such an observer based integral control scheme consist of the eigenvalues of

$$A_e + B_e F_e \quad \text{and of} \quad A + LC$$

# Integral control



# Integral control



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## Example: integral control

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for which we have already computed an observer gain assigning poles in  $\{-4, -5\}$  :

$$L = \begin{pmatrix} -6 \\ -12 \end{pmatrix}$$

## Example: integral control

The extended system becomes:

$$A_e = \left( \begin{array}{c|c} A & 0 \\ \hline C & 0 \end{array} \right) = \left( \begin{array}{cc|c} 2 & -3 & 0 \\ 4 & -5 & 0 \\ \hline -3 & 2 & 0 \end{array} \right)$$

$$B_e = \left( \begin{array}{c} B \\ 0 \end{array} \right) = \left( \begin{array}{c} 2 \\ 3 \\ 0 \end{array} \right)$$

$$C_e = ( C \mid 0 ) = ( -3 \quad 2 \mid 0 )$$

## Example: integral control

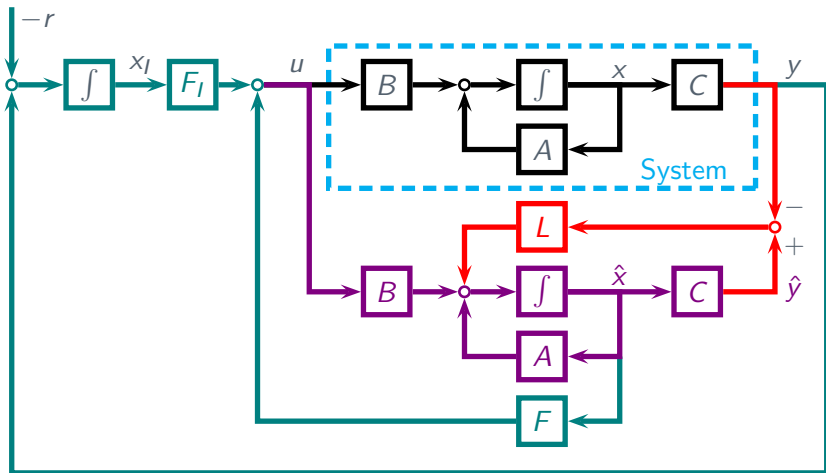
Using e.g. controllable canonical form, an extended state feedback can be found, which assigns poles in  $\{-3, -4, -5\}$  :

$$\begin{aligned} F_e &= (117 \quad -81 \quad -60) \\ \Rightarrow F &= (117 \quad -81) , \quad F_I = -60 \end{aligned}$$

The resulting controller can be shown to have the transfer function:

$$-\frac{1}{6s} \cdot \frac{55s^2 + 207s + 200}{s^2 + 18s + 119}$$

# Integral control





# Example: integral control

