Digital Signal Processing
ESD-5 and IV-5/Elektro, E24

2. Digital Filters w. Infinite Impulse Response
- The Impulse Invariant Method, again…

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	Time		Frequency
Continuous Time	$x(t)$	Laplace ⟷	$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$ $= \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\Omega)t}\,dt$ $= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$ for $\sigma = 0$ $= X(j\Omega)$ Fourier transform
	Sampling $T = \frac{1}{f_s}$ ↓ Reconstruction ↑		?? ↓
Discrete Time	$x[n]$	z transform ⟷	$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ $= \sum_{n=-\infty}^{\infty} x[n]\,(re^{j\omega})^{-n}$ $= \sum_{n=-\infty}^{\infty}(r^{-n}\cdot x[n])e^{-j\omega n}$ $= \sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n}$ for $r = 1$ $= X(e^{j\omega})$ Discrete Time FT

$$\omega = \Omega T = 2\pi f\,\frac{1}{f_s}$$

Synthesis of a Digital (Discrete-Time) Filter $H(z)$ with Infinite Impulse Response (IIR)

Specification of the Effective Filter

↓

Design of an Analog Proto-Type Filter, $H(s)$

↓

Transform $H(s)$ into $H(z)$

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The Impulse Invariant Method

Let's assume that we have established the Continuous-Time filter $H_C(s)$.

Using the Inverse Laplace transform we can now derive the impulse response;

$$h_C(t) = \mathcal{L}^{-1}\{H_C(s)\}$$

The overall idea is to generate a discrete-time system having an impulse response which (at selected time instances) is identical/invariant to $h_C(t)$.

Next, we therefore sample $h_C(t)$ at equidistant time stamps, i.e., we ”measure” the value of $h_C(t)$ at $t = nT$, where T is the sample period ($T = {}^{1}/_{f_s}$), and n is an integer;

$$h[n] = h_C(nT) \qquad -\infty < n < \infty$$

Normally, we refer to n as the sample number, and denote $h[n]$ as a sequence.

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What happens in the Frequency Domain…?

$$H\left(e^{j\omega}\right) = DTFT\{h[n]\}$$

$$= \sum_{n=-\infty}^{\infty} h[n] \cdot e^{-j\omega n}$$

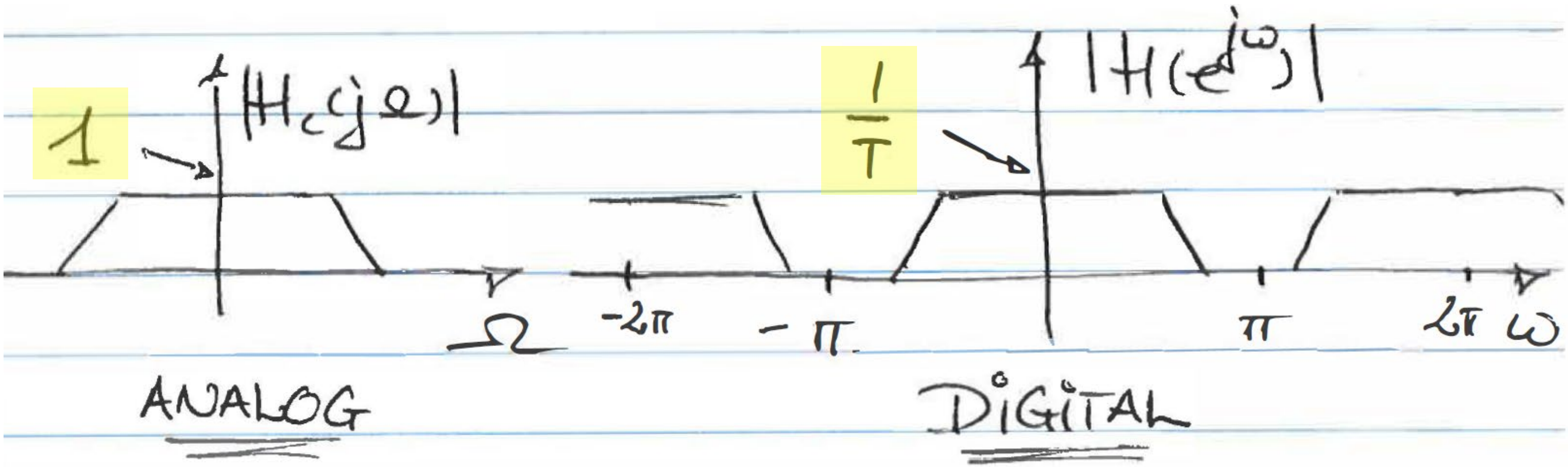
$$= \sum_{n=-\infty}^{\infty} h_C(nT) \cdot e^{-j\omega n}$$

From Nyquist Sampling Theorem, we know that the spectrum of a sampled signal is periodic with period 2π (O&S Equ. 20, p. 171), and thus we can write;

$$H\left(e^{j\omega}\right) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_C(j\frac{\omega}{T} + j\frac{2\pi}{T}k) \qquad \omega = \Omega T$$

What happens in the Frequency Domain…?

$$H\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_C\left(j\frac{\omega}{T} + j\frac{2\pi}{T}k\right) \qquad \omega = \Omega T$$



Therefore, to maintain the same pass-band amplification in the discrete-time domain (digital) as we have in the coutinuous-time (analog) domain, we need to multiply with T when we sample the continuous-time impulse response.

$$h[n] = T \cdot h_C(nT)$$

The overall design procedure

1) First we need $\boldsymbol{h_C(t)}$

The outset is the proto-type filter with transfer function $H_C(s)$.

$$H_C(s) = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{l=0}^{N} \alpha_l s^l}$$

For Butterworth LP, $M = 0$, and $\beta_0 = A_0$, i.e.;

$$H_C(s) = \frac{A_0}{\sum_{l=0}^{N} \alpha_l s^l}$$

To find $h_C(t)$, we need to Invers Laplace transform $H_C(s)$ which is not an easy task, unless we manipulate the expression.

We re-write $H_C(s)$ by the use of Partial Fraction Expansion.

$$H_C(s) = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \frac{A_3}{s - s_3} + \cdots + \frac{A_N}{s - s_N} = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

Why is that a good idea…??

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The overall design procedure

$$H_C(s) = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \frac{A_3}{s - s_3} + \cdots + \frac{A_N}{s - s_N} = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

$$h_C(t) = \mathcal{L}^{-1}\{H_C(s)\} = A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 e^{s_3 t} + \cdots + A_N e^{s_N t} = \sum_{k=1}^{N} A_k e^{s_k t}\, u(t)$$

Equ. 8, p. 523

2) Now, let's sample $h_C(t)$

$$h[n] = T \cdot h_C(nT)$$

$$h[n] = T \cdot \sum_{k=1}^{N} A_k e^{s_k nT}\, u[n]$$

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The overall design procedure

3) Finally, we derive $H(z)$ by using the z-transform on $h[n]$

$$H(z) = \mathcal{Z}\{h[n]\} = \sum_{n=0}^{\infty} h[n] \cdot z^{-n} = \sum_{n=0}^{\infty} \{T \cdot \sum_{k=1}^{N} A_k e^{s_k n T}\} \cdot z^{-n}$$

$$H(z) = T \sum_{k=1}^{N} A_k \sum_{n=0}^{\infty} (e^{s_k T} z^{-1})^n$$

$$H(z) = \sum_{k=1}^{N} \frac{T A_k}{1 - e^{s_k T} z^{-1}}$$

ROC: $|z| > e^{s_k T}$

This is the general Transfer Function $H(z)$ found by transforming $H(s)$, using the Impulse Invariant Method

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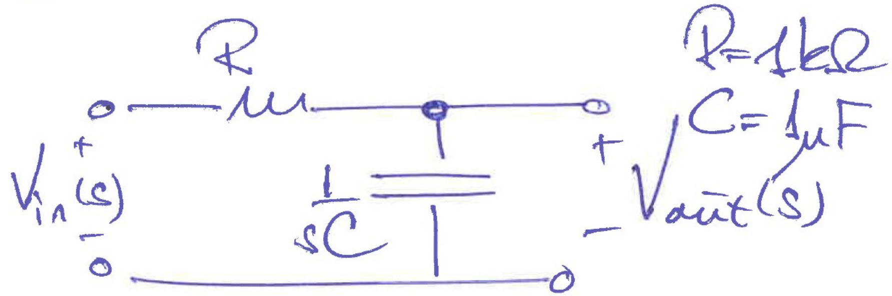
Let's have a look at our design example from Lecture #1



We found that the impulse response $h(t) = \frac{1}{RC} e^{-t/RC} \cdot u(t) = 1000 \cdot e^{-1000t} u(t)$

Using the Laplace transform we get $H(s) = \frac{1000}{s+1000}$

This system (i.e., the analog filter) has one real pole in $s = \sigma + j\Omega = -1000$

Using the formula from p. 9, $H(z) = \sum_{k=1}^{N} \frac{TA_k}{1-e^{s_k T} z^{-1}}$, and since $N = 1$ we get

$$H(z) = \frac{\frac{1}{8000} \cdot 1000}{1 - e^{-1000/8000} z^{-1}} = \frac{0.125}{1 - 0.8825 z^{-1}}$$

So, the digital (or discrete-time) filter has the transfer function

$$H(z) = \frac{0.125}{1 - 0.8825z^{-1}} = \frac{0.125z}{z - 0.8825}$$

from which we see that it is a 1st order system with its pole at $z = 0.8825$, i.e., the pole is located *inside the unit circle*, and thus the filter is stable. The filter also has a zero in $z = 0$.

If you did the exercises from Lecture #1, you may also have found that the frequency response is

$$H\left(e^{j\omega}\right) = H(z)|_{z=e^{j\omega}} = \frac{0.125 \cdot e^{j\omega}}{e^{j\omega} - 0.8825}$$

We can now calculate / plot the amplitude response $|H\left(e^{j\omega}\right)|$

Remember that $H\left(e^{j\omega}\right)$ is a 2π periodic function. We therefore only need the information in the interval $[0;\ \pi[\ rad$.

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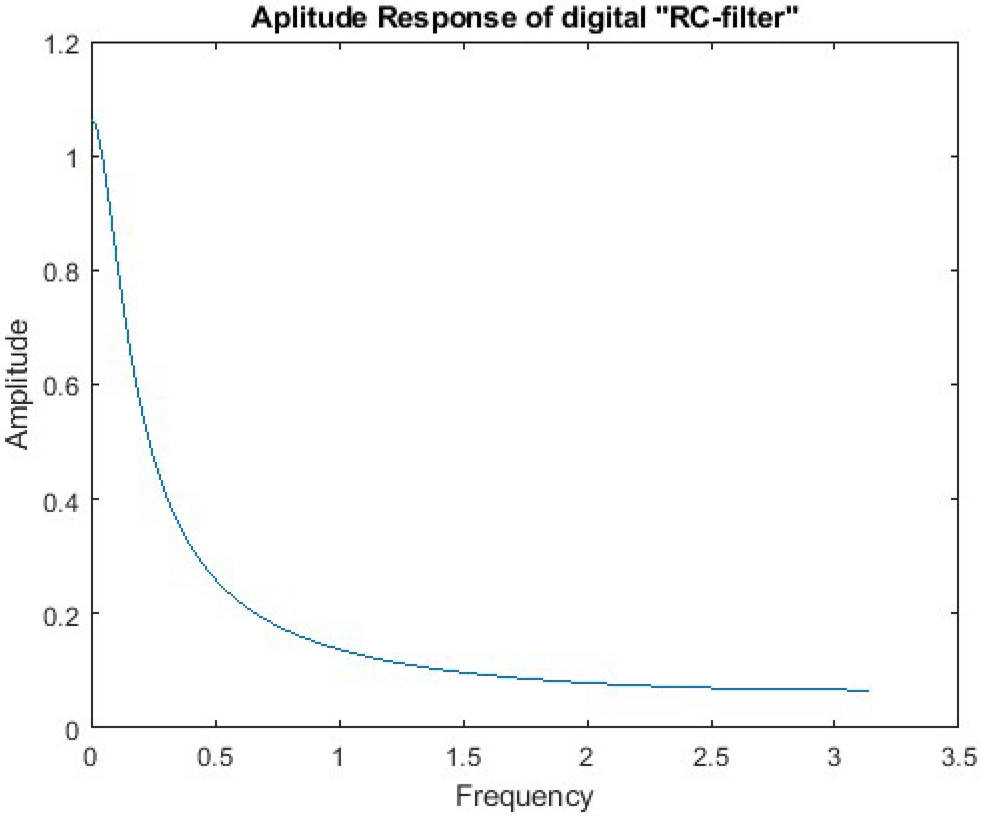
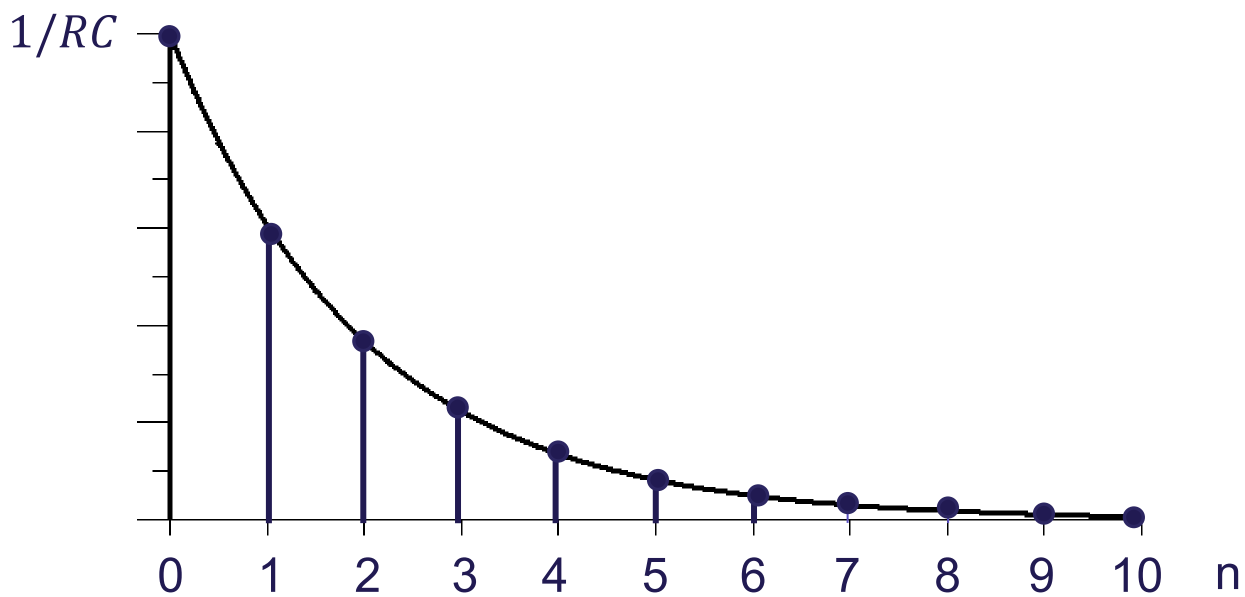


We clearly see that the filter has Butterworth characteristic – as expected.

However, the DC gain is not $0\ dB$ as we would normally see for a simple 1st order RC lowpas filter.

What might be the issue here…??

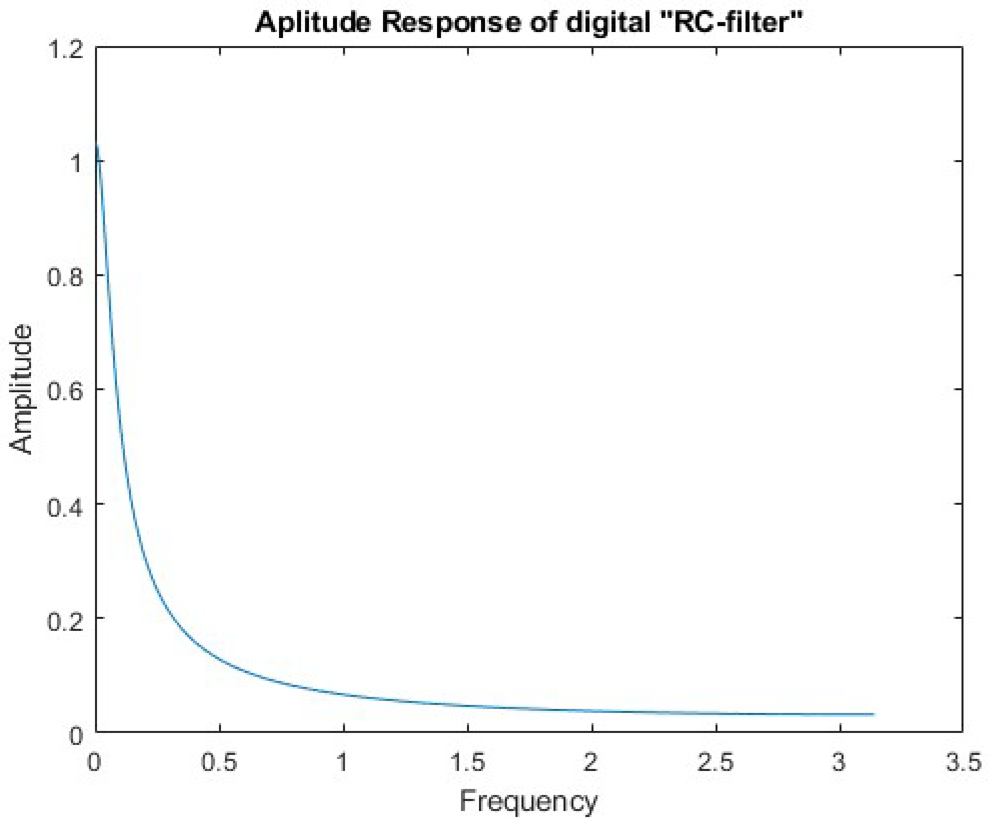
The problem is the sample frequency applied… In our case we have used $f_s = 8\ kHz$, but it seems that it is a too low value.



Although the sampled impulse respose $h[n]$ clearly is an exponentially decaying function, it does not capture all the information in the continuous-time impulse response $h(t)$…, of course..! So, let's make a new experiment where $f_s = 16\ kHz$.

The sample frequency is now increased with a factor of two, i.e., $f_s = 16\ kHz$

$$H(z) = \frac{\frac{1}{16000} \cdot 1000}{1 - e^{-1000/16000} z^{-1}} = \frac{0.0625}{1 - 0.9394 z^{-1}} = \frac{0.0625z}{z - 0.9394}$$



With the increased sample frequency, we now get a better approximation to the expected DC gain – but is it the same filter as before…???

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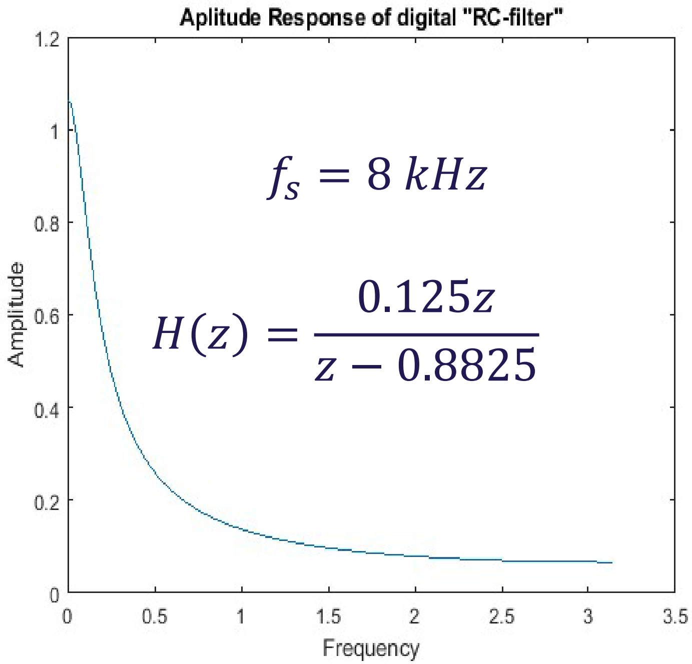
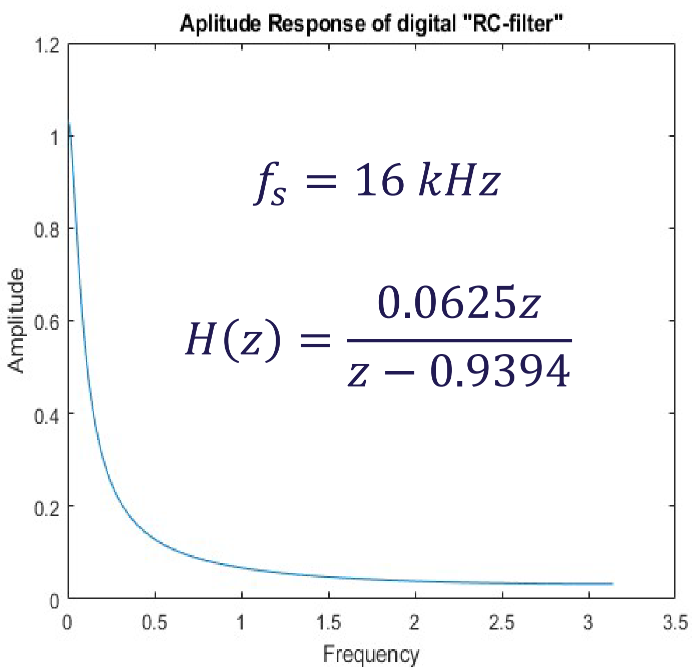


$$f_s = 8\ kHz$$

$$H(z) = \frac{0.125z}{z - 0.8825}$$



$$f_s = 16\ kHz$$

$$H(z) = \frac{0.0625z}{z - 0.9394}$$

$$\omega_{3dB} = 0.1252\ rad$$

$$f_{3dB} \cong 159\ Hz$$

IMPORTANT LESSON…!!

Since everything in the discrete-time domain is normalized to the sample frequency, the filter (essentially) stays the same for varying sample frequency (or sample period T).

Therefore, changing the sample frequency in the Impulse Invariant Method does not change the overall shape of the amplitude response.

But…

$$\omega_{3dB} = 0.0625\ rad$$

$$f_{3dB} \cong 159\ Hz$$

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Evaluating the transfer function $\boldsymbol{H(z)}$

Since $H(s)$ is stable, we could hope for $H(z)$ also being stable – let's find out.

$$H_C(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \qquad H(z) = \sum_{k=1}^{N} \frac{TA_k}{1 - e^{s_k T} z^{-1}}$$

Poles in $H(s)$; s_k

Poles in $H(z)$; $z_k = e^{s_k T}$

$$\frac{1}{s - s_k} \longrightarrow \frac{1}{1 - e^{s_k T} z^{-1}}$$ for simple poles.

$s_k = \sigma_k + j\Omega_k$

$z_k = e^{s_k T}$

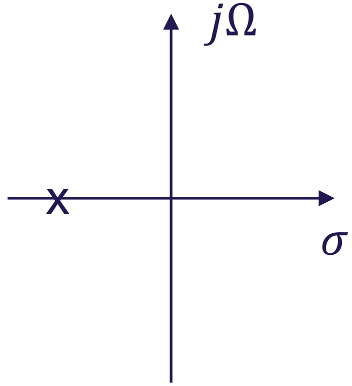
Evaluating the transfer function $H(z)$

$$s_k = \sigma_k + j\Omega_k$$

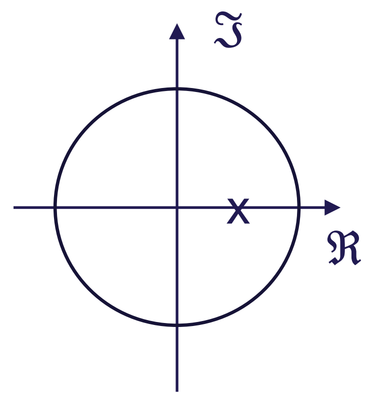
$$z_k = e^{s_k T}$$

$$z_k = e^{(\sigma_k + j\Omega_k)T} = e^{\sigma_k T} \cdot e^{j\Omega_k T}$$

$|z_k| = e^{\sigma_k T} < 1$ for $\sigma_k < 0$



$\Rightarrow$



QED..!

s-plane

z-plane

Evaluating the transfer function $H(z)$

- There is NO simple mathematical relation which describes how the complete s-plane is mapped into the z-plane.

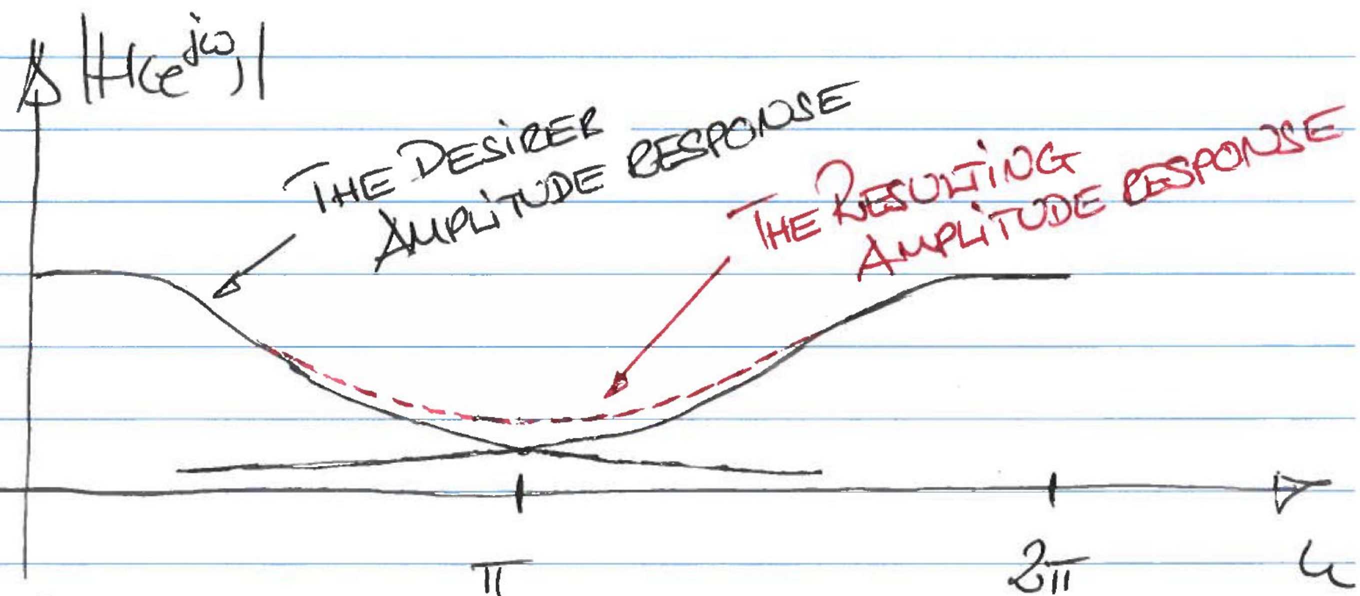
- The zeros of $H(z)$ are functions of the poles $e^{s_k T}$ and $T \cdot A_k$, and thus the zeros are being mapped from s to z differently than the poles are.

…the locations of the zeros are also not that important, but there is another issue which may causes us some trouble…!!!

Evaluating the transfer function $H(z)$

As we saw previously, slide #6, sampling the impulse response leads to a periodic frequency response…

If we relate this circumstance with the fact that NO continuous-time system is (100%) band-limited, then we are facing the following scenario;



SO, THERE IS AN OVERLAP AMONG CONSECUTIVE VERSIONS OF THE DESIRED AMPLITUDE RESPONSE

ALIASING

Therefore, the frequency response of a digital system realized from an impulse invariant transformation is an aliased version of the frequency response of the continuous-time system from which is was derived. Thus we can write

$H\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_C(j\Omega + jk\Omega_s)$ where $\Omega_s = {}^{2\pi}/_{T}$ See also p.6

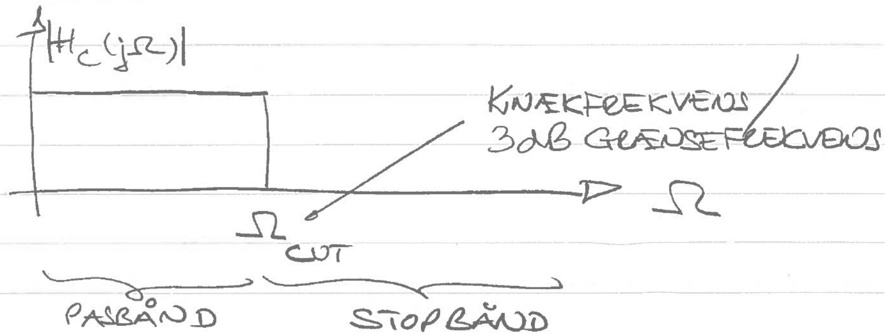
In conclusion we therefore need to accept that the Impulse Invariant Method can be used for mapping $H(s)$ to $H(z)$ only in the cases where $H(s)$ is "sufficiently" band-limited, i.e., $|H(j\Omega)| < \varepsilon$ for frequencies near half the sample frequency.

This means that the IIM cannot be used for the design of HP and BS filters…!

One possible way to overcome this problem is the derive $H_{LP}(z)$ for a Low Pass proto-type filter $H_{LP}(s)$, and then do the transform $H_{LP}(z) \curvearrowright H_{HP}(z)$.

We will look into this later…

BUTTERWORTH LP. FILTER



DET ER KUN I MATEMATIKKEN AT VI KAN KONSTRUERE DET IDEELLE FILTER $\longrightarrow$

VI MÅ ~~SØGE~~ SØGE EN TILNÆRMELSE

EEN MULIGHED (BLANDT MANGE)

BUTTERWORTH

$$|H_c(j\Omega)|^2 = \frac{1}{1 + \left(j\Omega / j\Omega_c\right)^{2N}}$$

$(\text{AMPLITUDE-RESPONS})^2$

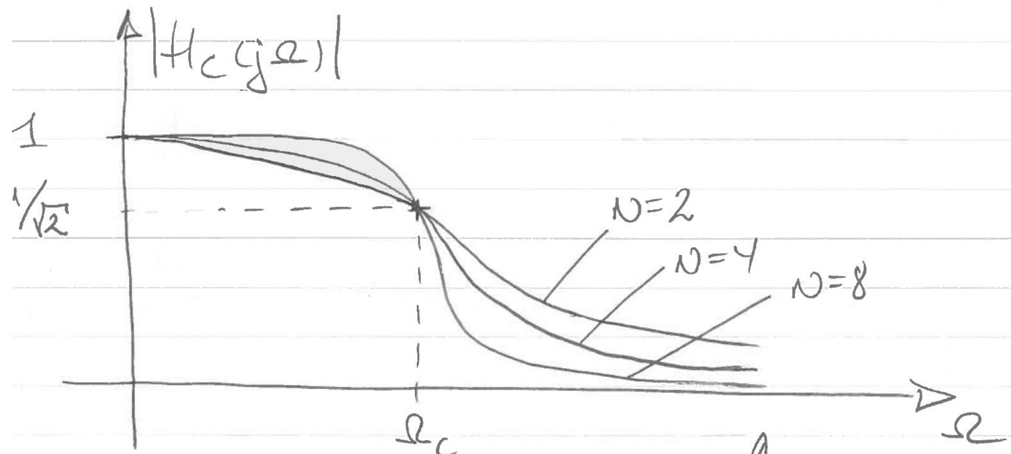
N ER FILTER-ORDEN

BUTTERWORTH KARAKTERISTIKA ;

1) $|H_c(j\Omega)|^2 = 1 \Big|_{\Omega=0} \quad \forall N$

2) $|H_c(j\Omega)|^2 = \frac{1}{2} \Big|_{\Omega=\Omega_c} \quad \forall N$

3) MONOTON AFTAGENDE I PAS- OG STOP·BÅND



$20 \cdot \log 1/\sqrt{2} \cong -3\,dB$ OBS!

- JO HØJERE ORDEN, JO BEDRE TILNÆRMELSE TIL DET IDEELLE FILTER

∿ JO HØJERE ORDEN, JO SMALLERE TRANSITION FRA PAS- TIL STOP·BÅND

- BUTTERWORTH HAR MAKSIMAL FLAD AMPLITUDE-KARAKTERISTIK — INGEN RIPPEL

POL-PLACERING ;

$$|H_c(s)|^2 = H_c(s) \cdot H_c(-s) = \frac{1}{1+(s/j\Omega_c)^{2N}}$$

$$1 + (s/j\Omega_c)^{2N} = 0$$

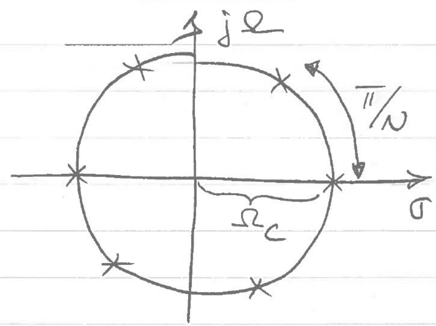
$$\Downarrow$$

$$s/j\Omega_c = \sqrt[2N]{-1}$$

$$\Downarrow$$

$$s_k = (-1)^{1/2N}(j\Omega_c) = \Omega_c \cdot e^{(j\pi/2N)(2k+N-1)}$$

for $k = 0, 1, \ldots, 2N-1$



$H_c(s)$ KONSTRUERES UD FRA POLERNE i VENSTRE HALVPLAN (STABIL) ;

$$H_c(s) = \frac{G}{\prod_{k=1}^{N}(s-s_k)}$$

Aff!