Digital Signal Processing
ESD5 & IV5-elektro, E24

3. Digital IIR Filters: The Bilinear Transformation

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Synthesis of a Digital (Discrete-Time) Filter $H(z)$ with Infinite Impulse Response (IIR)

Specification of the Effective Filter

Design of an Analog Proto-Type Filter, $H(s)$

Transform $H(s)$ into $H(z)$

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The Impulse Invariant Method

$$H_C(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \quad \longrightarrow \quad H(z) = \sum_{k=1}^{N} \frac{TA_k}{1 - e^{s_k T} z^{-1}}$$

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Evaluating the transfer function $\boldsymbol{H(z)}$

Since $H(s)$ is stable (i.e., s_k is located in the left-hand part of the s-plane), we have already seen in our previous lecture that $H(z)$ is also stable…

$$H_C(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \qquad\qquad H(z) = \sum_{k=1}^{N} \frac{TA_k}{1 - e^{s_k T} z^{-1}}$$

Poles in $H(s)$; s_k

Poles in $H(z)$; $z_k = e^{s_k T}$

$$s_k = \sigma_k + j\Omega_k$$

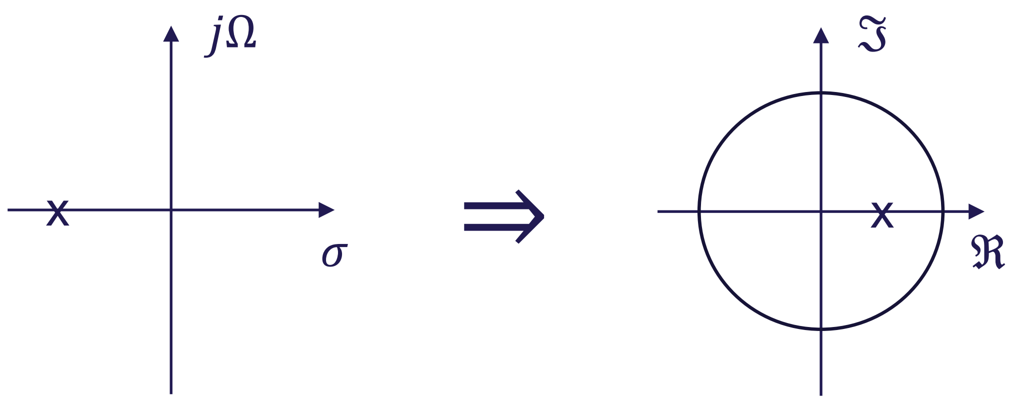
$$z_k = e^{s_k T}$$

Evaluating the transfer function $H(z)$

$$s_k = \sigma_k + j\Omega_k$$
$$z_k = e^{s_k T}$$

$$z_k = e^{(\sigma_k + j\Omega_k)T} = e^{\sigma_k T} \cdot e^{j\Omega_k T}$$

$|z_k| = e^{\sigma_k T} < 1$ for $\sigma_k < 0$



QED..!

s-plane $\qquad$ z-plane

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Evaluating the transfer function $H(z)$

- The Impulse Invariant Methods preserves stability and the order of the analog filter, but…

- There is no simple mathematical relation which describes how the complete s-plane is mapped into the z-plane.

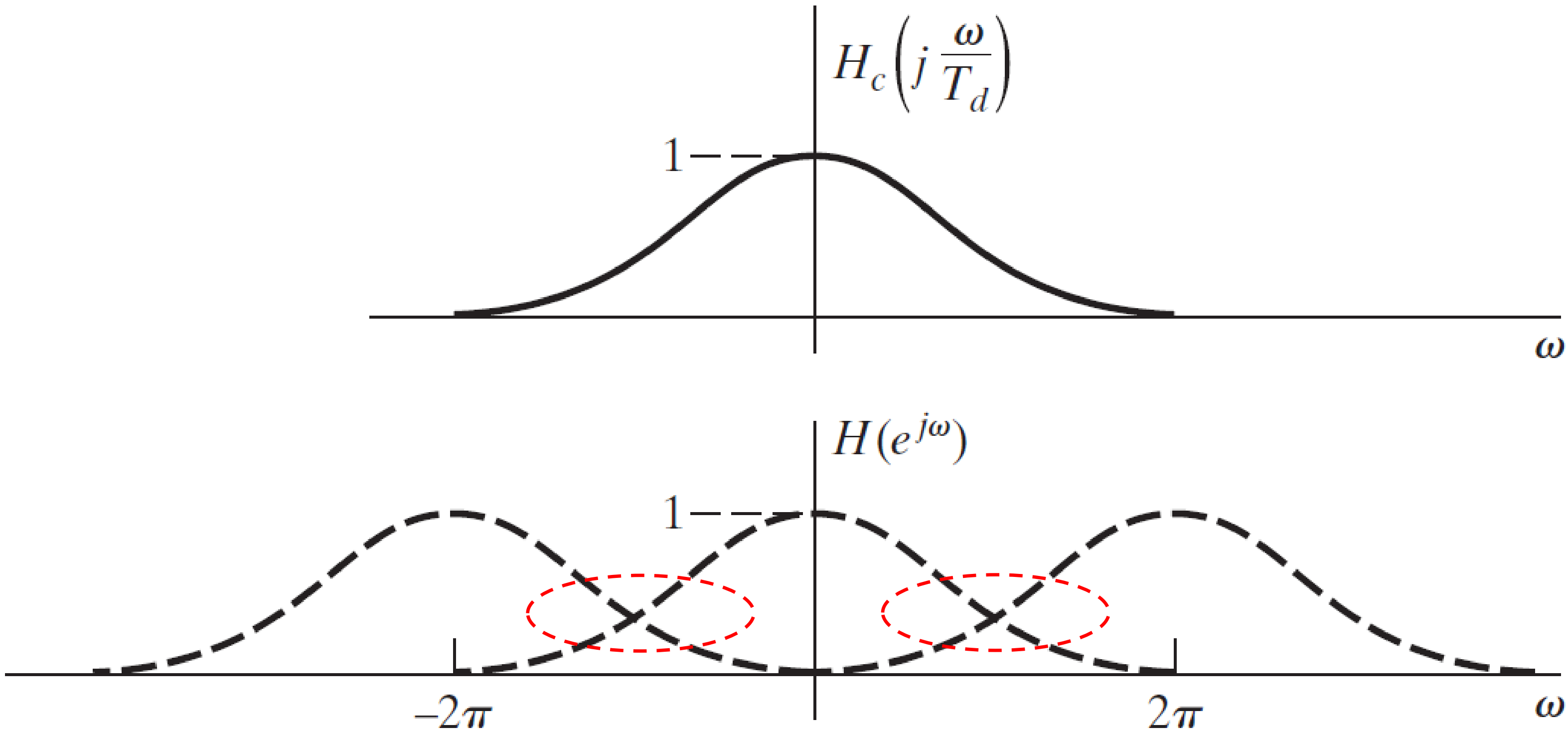
- The zeros of $H(z)$ are functions of the poles $e^{s_k T}$ and $T \cdot A_k$, and thus the zeros are being mapped from s to z differently than the poles are.

…the zeros are also not that important, but there is one other thing which may causes us some challenges…!!!

Evaluating the transfer function $H(z)$

Sampling the impulse response leads to a periodic frequency response…

If we face this fact and similarly recognizes that NO continuous-time system is (100%) band-limited, then we have the following scenario;



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Therefore, the frequency response of a digital system realized from an impulse invariant transformation is an aliased version of the frequency response of the continuous-time system from which is was derived. Thus we can write

$$H\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_C(j\Omega + jk\Omega_s) \quad \text{where} \quad \Omega_s = {}^{2\pi}/_{T}$$

In conclusion we therefore need to accept that the Impulse Invariant Method can be used for mapping $H(s)$ to $H(z)$ only in the cases where $H(s)$ is ”sufficiently” band-limited, i.e., $|H(j\Omega)| < \varepsilon$, for frequencies near half the sample frequency.

This means that the IIM cannot be used for the design of HP and BS filters…!

Let’s have a brief look at the math…

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Given a 1'st order HP filter

The transfer function for a 1'st order HP filter:

$$H(s) = \frac{s}{s+\Omega_c}$$

$$|H(j\Omega)| = \left|\frac{j\Omega}{j\Omega+\Omega_c}\right| = \frac{\Omega}{\sqrt{\Omega^2+\Omega_c^2}} = \begin{cases} 0, & \Omega = 0 \\ \frac{1}{\sqrt{2}}, & \Omega = \Omega_c \\ 1, & \Omega \longrightarrow \infty \end{cases}$$

The problem is that $H(s)$ **is not** on the form $\frac{A_k}{s-s_k}$ -- the s in the numerator leads to an additional term in $h(t)$ when we perform the Inverse Laplace transform.

$$H(s) = \frac{s}{s+\Omega_c} = \frac{(s+\Omega_c)+s-(s+\Omega_c)}{s+\Omega_c} = 1 - \frac{\Omega_c}{s+\Omega_c}$$

$$h(t) = \mathcal{L}^{-1}\left\{1 - \frac{\Omega_c}{s+\Omega_c}\right\} = \delta(t) - \Omega_c e^{-\Omega_c t} u(t)$$

$$h(t) = \delta(t) - \Omega_c e^{-\Omega_c t} \quad t \geq 0$$

$$h[n] = T \cdot h(nT) = "T \cdot \left(\delta[nT] - \Omega_c e^{-\Omega_c nT}\right) = \infty" \quad \text{for} \quad n = 0$$

It is not possible to sample the impulse $\delta(t)$ ... !

$$h[1] = T \cdot \left(-\Omega_c e^{-\Omega_c T}\right)$$

$$h[2] = T \cdot \left(-\Omega_c e^{-\Omega_c 2T}\right)$$

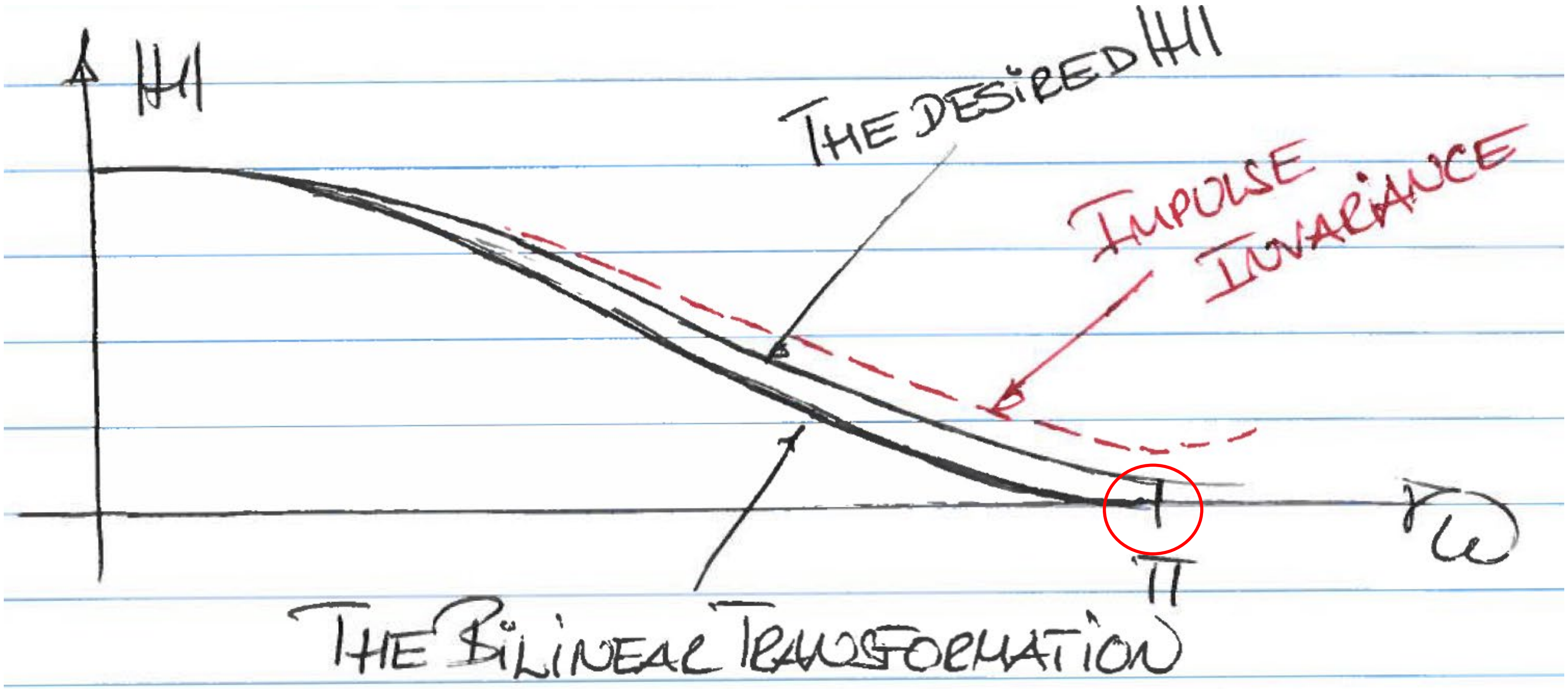
…and so on.

If, however, we assume that we somehow can sample $\delta(t)$, then it means that $h[0]$ becomes the predominant sample in the impulse response.

$$H(z) = \mathrm{Z}\{h[n]\} = \sum_{n=0}^{\infty} h[n] z^{-n} \approx h[0]$$

Caused by the aliasing phenomenon, this $H(z)$ is a very poor approximation to a HP filter…!!!

The cure… Another transformation which avoids aliasing



Aliasing is eliminated if we can limit the complete $H\left(e^{j\omega}\right)$ to within one period

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The Bilinear Transformation

The overall idea…

$-\infty < \Omega < \infty$ is mapped into one iteration on the Unit Circle, i.e., $-\pi < \omega < \pi$

Of course, trying to do this, we certainly ask for trouble, but let's try…

The way that we want to do it, is to substitute s in $H(s)$ with a function in z.

$$H(z) = H(s)|_{s=f(z)}$$

Question now is, can we derive such a function f…???

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The Bilinear Transformation

$z = e^{j\omega}$ (on the Unit Circle)

$\omega = T\Omega$ (relation between continuous- and discrete-time frequency)

$s = j\Omega$ (the continuous-time frequency axis)

$$z = e^{jT\frac{s}{j}} = e^{sT}$$

$$\ln(z) = sT$$

$$s = \frac{1}{T}\ln(z) = \frac{1}{T} \cdot 2\left\{\frac{z-1}{z+1} + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \frac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \cdots\right\}$$

$$s \cong \frac{2}{T} \cdot \frac{z-1}{z+1}$$

$$\boxed{s = f(z) = \frac{2}{T} \cdot \frac{z-1}{z+1}}$$

This is the Bilinear Transformation, Equ. 18, p. 528

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How is the s-plane mapped to the z-plane…??

$$s = \frac{2}{T} \cdot \frac{z-1}{z+1}$$

$$z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s} = \frac{1 + \frac{T}{2}(\sigma + j\Omega)}{1 - \frac{T}{2}(\sigma + j\Omega)}$$

Using this expression, we can now investigate three important parameters;

1) Origo in the s-plane

$$s = 0 \;\Rightarrow\; z = 1$$

What can we conclude…?

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How is the s-plane mapped to the z-plane…??

$$z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s} = \frac{1 + \frac{T}{2}(\sigma + j\Omega)}{1 - \frac{T}{2}(\sigma + j\Omega)}$$

2) Left-hand side of the s-plane

$$Re\{s\} = \sigma < 0$$

$$|z| = \frac{\sqrt{(1 + \frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}}{\sqrt{(1 - \frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}} < 1$$

What can we conclude…?

How is the s-plane mapped to the z-plane…??

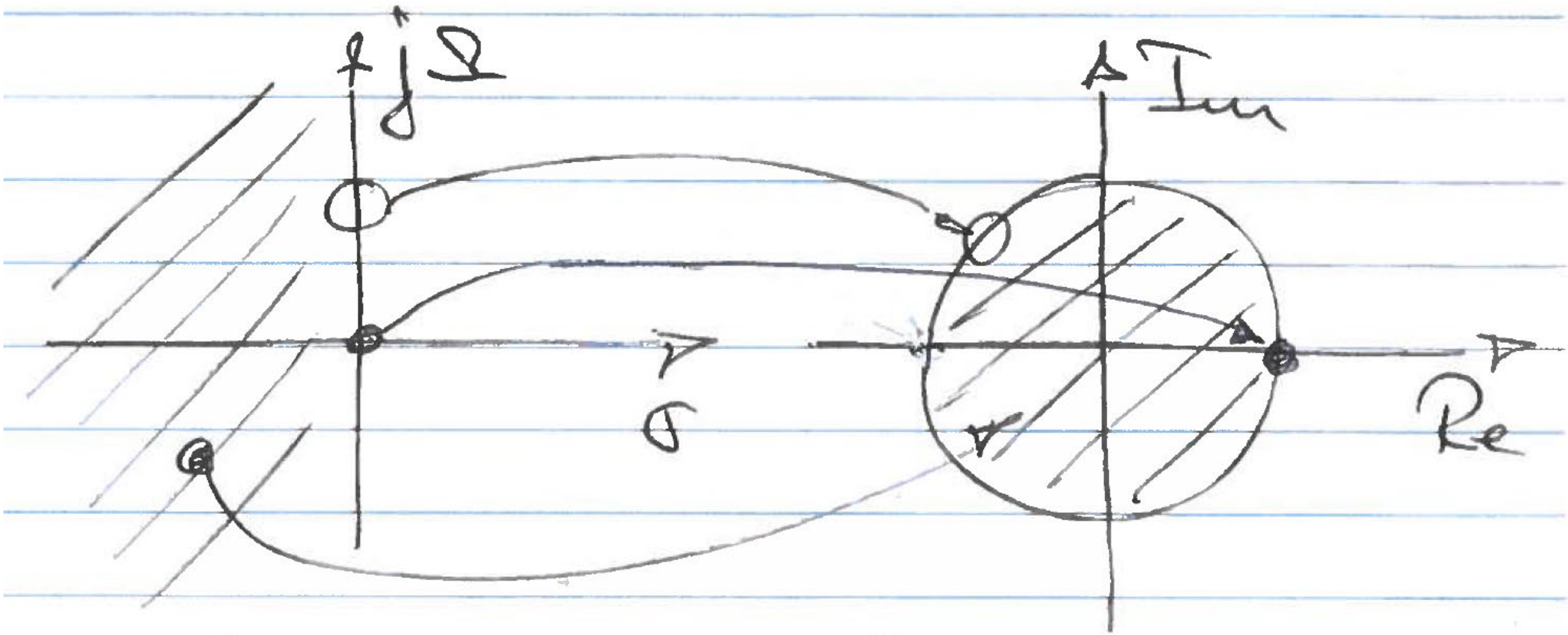
3) $j\Omega$-axis in the s-plane

$$z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s} = \frac{1 + \frac{T}{2}(\sigma + j\Omega)}{1 - \frac{T}{2}(\sigma + j\Omega)}$$

$$\sigma = 0 \quad \Rightarrow \quad |z| = 1 \;\; \forall \;\; \Omega$$

What can we conclude…?

How is the *s*-plane mapped to the *z*-plane…??



Now, what is the relation between Ω and ω, using the Bilinear Transformation

$$s = \sigma + j\Omega = \frac{2}{T} \cdot \frac{z-1}{z+1} = \frac{2}{T}\frac{e^{j\omega}-1}{e^{j\omega}+1} \quad \text{for} \quad \mathrm{z} = e^{j\omega}$$

In this expression, we now utilize

1) that $e^{j\omega} = e^{j\frac{\omega}{2}}e^{j\frac{\omega}{2}}$ and $1 = e^{j\frac{\omega}{2}}e^{-j\frac{\omega}{2}}$

2) that $\sin(\omega) = \frac{e^{j\omega}-e^{-j\omega}}{2j}$ and $\cos(\omega) = \frac{e^{j\omega}+e^{-j\omega}}{2}$

thus deriving the expression;

$$s = \frac{2}{T} j \tan(\frac{\omega}{2})$$

Now, what is the relation between Ω and ω, using the Bilinear Transformation

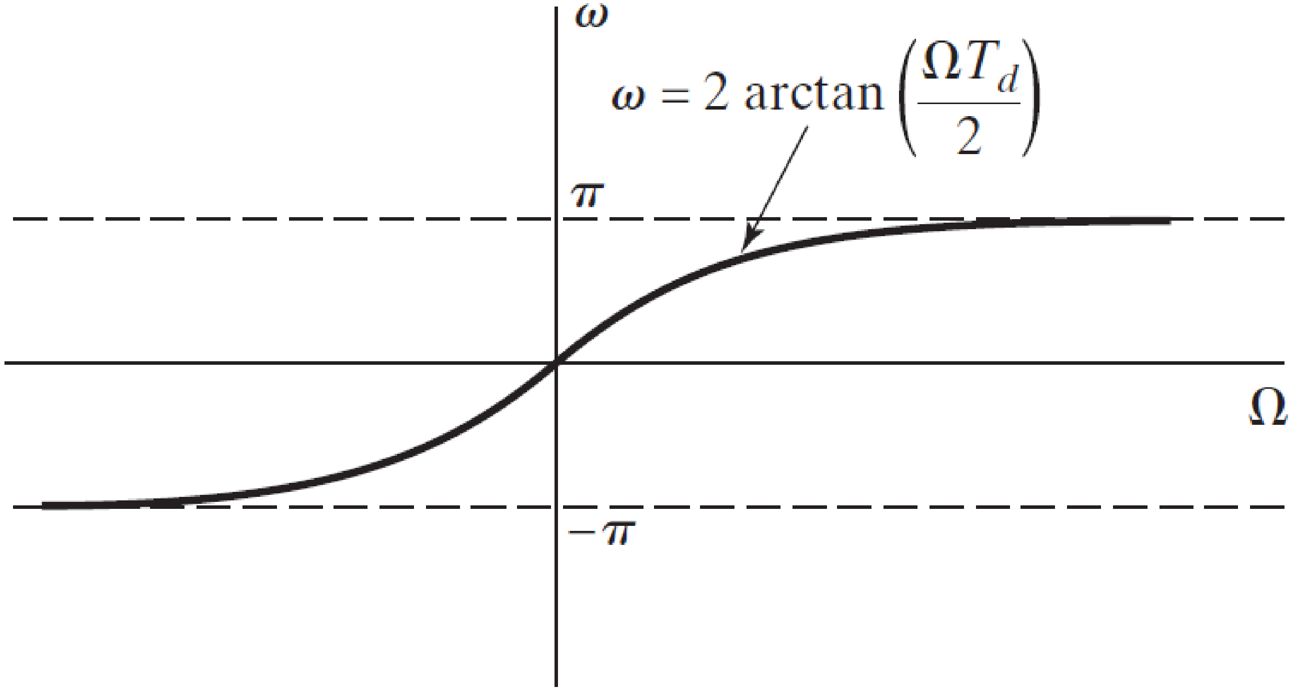
$$s = \frac{2}{T} j \tan\left(\frac{\omega}{2}\right)$$

Since the right-hand side of this expression is purely imaginary, we conclude

$$\Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right)$$ and thus $$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$ Equ. 26 & 27, p. 529

This expression tells us, that the mapping from the s-plane to the z-plane is an $arctan$ function, which is highly non-linear….!!

Now, what is the relation between Ω and ω, using the Bilinear Transformation



Arctan is ”almost” linear for values of Ω close to 0, but becomes more and more non-linear as $|\Omega|$ increases – so, good news and bad news. Let’s have a look…

$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$ is defined in $]-\pi;\pi[$ which essentially is good news because it confirms our initial requirement – make a transform which maps $j\Omega$ onto $]-\pi;\pi[$ thus AVOIDING ALIASING…!!!

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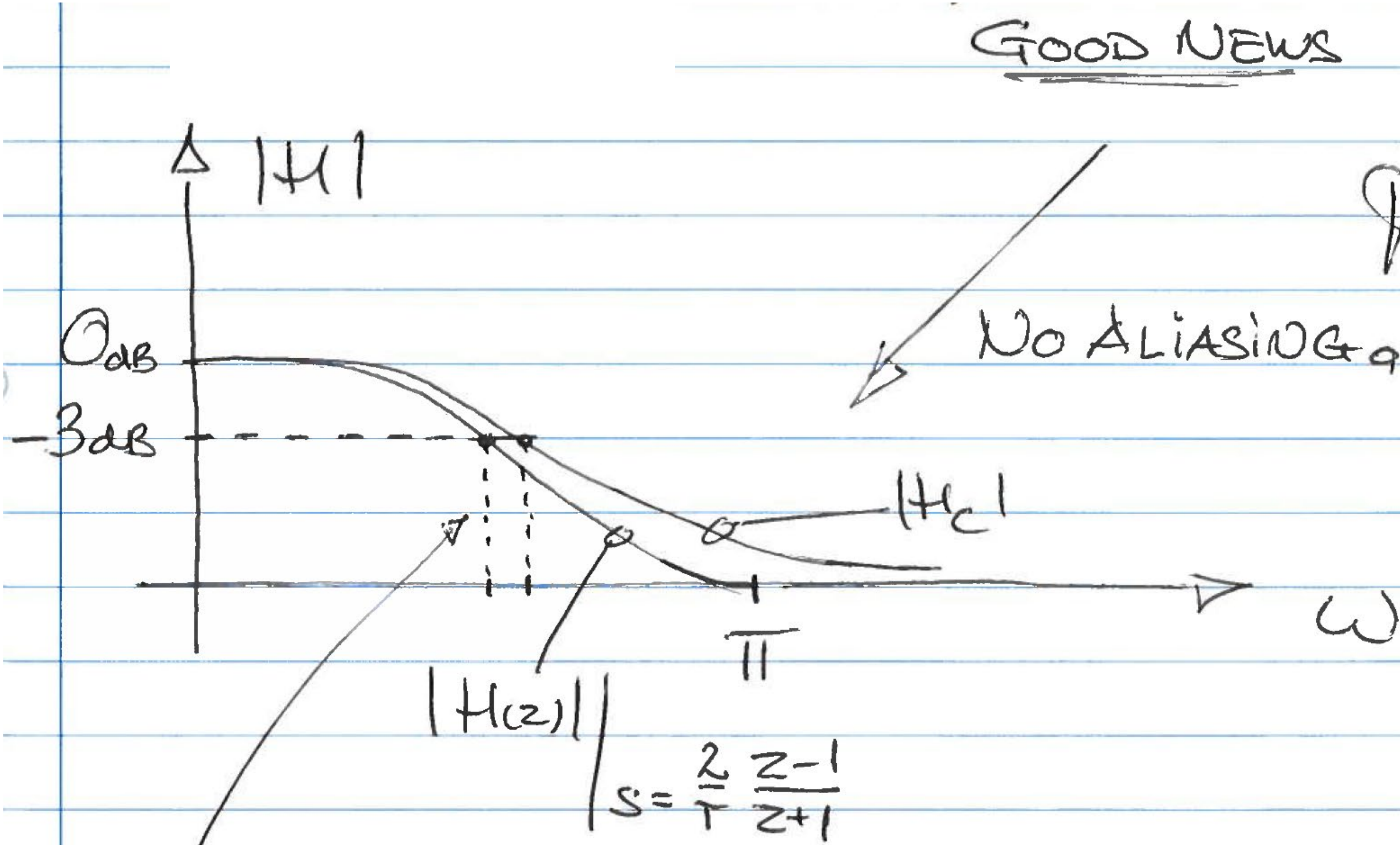
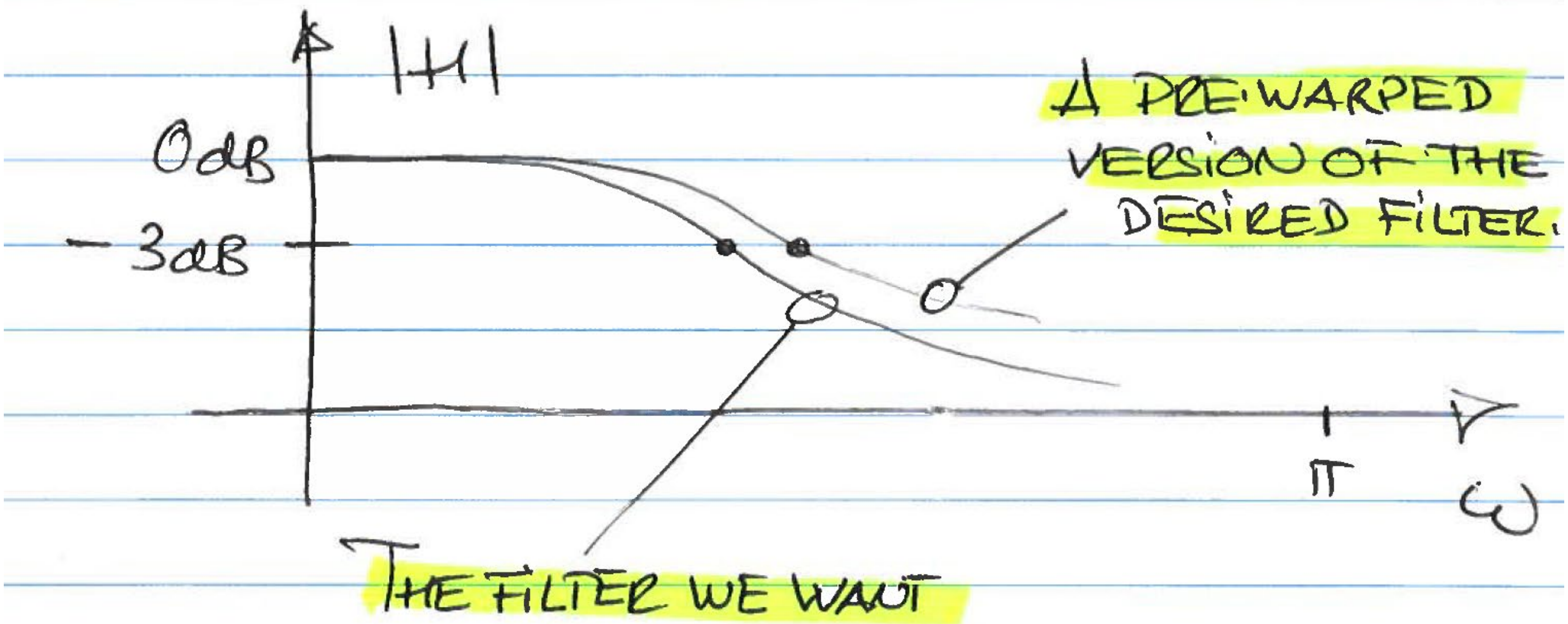
No aliasing in the Bilinear Transform



BUT NOW THE 3 dB FREQ MOVES DOWNWARDS!

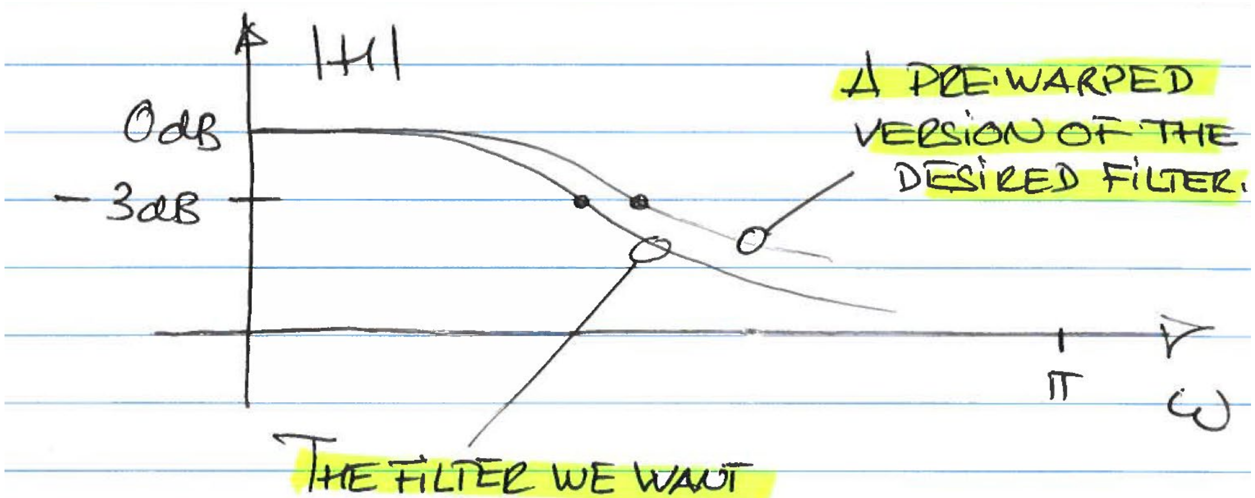
No aliasing, but instead a distorted frequency mapping…

We may eliminate this distortion by introducing a PRE-DISTORTION, called a PRE-WARPING



So, the idea here is that we move UPWARDS one critical frequency, which during Bilinear transformation is then moved DOWNWARDS and thus located exactly where we want it to be located.

The Bilinear Transform – Prewarping



$$\Omega_{C,new} = \frac{2}{T} \cdot \tan\left(\frac{\omega_C}{2}\right)$$

(see slide #19)

This pre-warping is done on the continuous-time filter $H(s)$ prior to it being transformed into $H(z)$.

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