State Space Methods

Lecture 3: observability, observers, and observer based control

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Observability

A continuous time system

$$\dot{x}(t) = Ax(t)\,, \quad y(t) = Cx(t)$$

is said to be *observable* iff $y(t) \equiv 0 \Rightarrow x(t) \equiv 0.$

A discrete time system

$$x(k+1) = Ax(k)\,, \quad y(k) = Cx(k)$$

is said to be *observable* iff $y(k) \equiv 0 \Rightarrow x(k) \equiv 0.$

Observability

We consider the discrete time system

$$x(k+1) = Ax(k)\,, \quad y(k) = Cx(k)\,, \quad x(0) = x_0$$

and iterate:

$$\begin{aligned} x(0) &= x_0 & \qquad y(0) &= Cx_0 \end{aligned}$$

Observability

We consider the discrete time system

$$x(k+1) = Ax(k)\,, \quad y(k) = Cx(k)\,, \quad x(0) = x_0$$

and iterate:

$$\begin{array}{rclcrcl} x(0) & = & x_0 & \qquad & y(0) & = & Cx_0 \\ x(1) & = & Ax(0) & & & & \end{array}$$

Observability

We consider the discrete time system

$$x(k+1) = Ax(k)\,, \quad y(k) = Cx(k)\,, \quad x(0) = x_0$$

and iterate:

$$\begin{aligned} x(0) &= x_0 & \qquad y(0) &= Cx_0 \\ x(1) &= Ax_0 & \qquad y(1) &= CAx_0 \\ x(2) &= Ax(1) & & \end{aligned}$$

Observability

We consider the discrete time system

$$x(k+1) = Ax(k)\,, \quad y(k) = Cx(k)\,, \quad x(0) = x_0$$

and iterate:

$$\begin{array}{rclcrcl} x(0) & = & x_0 & \qquad & y(0) & = & Cx_0 \\ x(1) & = & Ax_0 & & y(1) & = & CAx_0 \\ x(2) & = & A^2x_0 & & y(2) & = & CA^2x_0 \end{array}$$

Observability

We consider the discrete time system

$$x(k+1) = Ax(k)\,, \quad y(k) = Cx(k)\,, \quad x(0) = x_0$$

and iterate:

$$\begin{array}{rcl} x(0) & = & x_0 \\ x(1) & = & Ax_0 \\ x(2) & = & A^2x_0 \\ & \vdots & \\ x(n-1) & = & Ax(n-2) \end{array} \qquad \begin{array}{rcl} y(0) & = & Cx_0 \\ y(1) & = & CAx_0 \\ y(2) & = & CA^2x_0 \end{array}$$

Observability

We consider the discrete time system

$$x(k+1) = Ax(k)\,, \quad y(k) = Cx(k)\,, \quad x(0) = x_0$$

and iterate:

$$\begin{array}{rcl@{\qquad}rcl}
x(0) & = & x_0 & y(0) & = & Cx_0 \\
x(1) & = & Ax_0 & y(1) & = & CAx_0 \\
x(2) & = & A^2x_0 & y(2) & = & CA^2x_0 \\
 & \vdots & & & & \\
x(n-1) & = & A^{n-1}x_0 & y(n-1) & = & CA^{n-1}x_0
\end{array}$$

Observability

Writing the equations

$$y(k) = CA^k x_0\,,\; k = 0, \ldots, n-1$$

in matrix form we obtain:

$$\underbrace{\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}}_{\text{Observability matrix}} x_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Observability

Writing the equations

$$y(k) = CA^k x_0 \, , \; k = 0, \ldots , n-1$$

in matrix form we obtain:

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} x_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

When is this equation solvable for some $x_0 \neq 0$?

Observability

Theorem

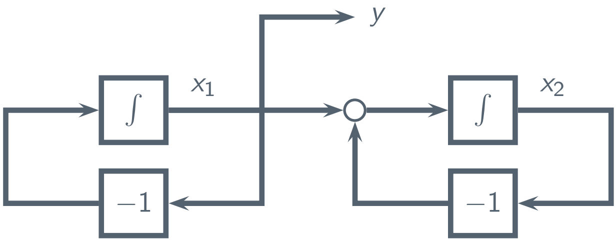
A system

continuous time	*discrete time*
$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases}$	$\Sigma : \begin{cases} x(k+1) = Ax(k) \\ y(k) = Cx(k) \end{cases}$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, is observable if and only if

$$\operatorname{rank} \mathcal{O} = \operatorname{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n$$

Example: series connection



State and output equations:

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -x_1 \\ \dot{x}_2 & = & -x_2 + x_1 \\ y & = & x_1 \end{array} \right\}$$

State space model:

$$\dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Example: series connection

For the state space matrices:

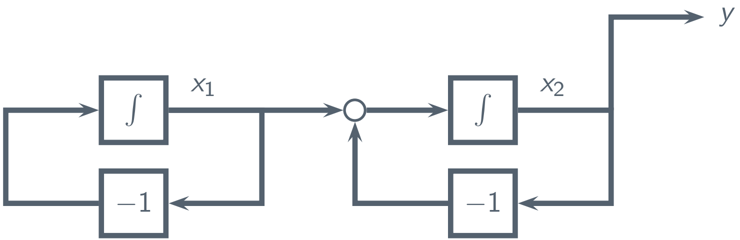
$$A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

the observability matrix $\mathcal{O}$ becomes:

$$\mathcal{O} = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

$\det \mathcal{O} = 0 \implies$ system is unobservable.

Example: series connection



State and output equations:

$$
\left\{
\begin{array}{rcl}
\dot{x}_1 & = & -x_1 \\
\dot{x}_2 & = & -x_2 + x_1 \\
y & = & x_2
\end{array}
\right\}
$$

State space model:

$$
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
$$

$$
y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
$$

Example: series connection

For the state space matrices:

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

the observability matrix $\mathcal{O}$ becomes:

$$\mathcal{O} = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$\det \mathcal{O} = -1 \neq 0 \implies$ system is observable.

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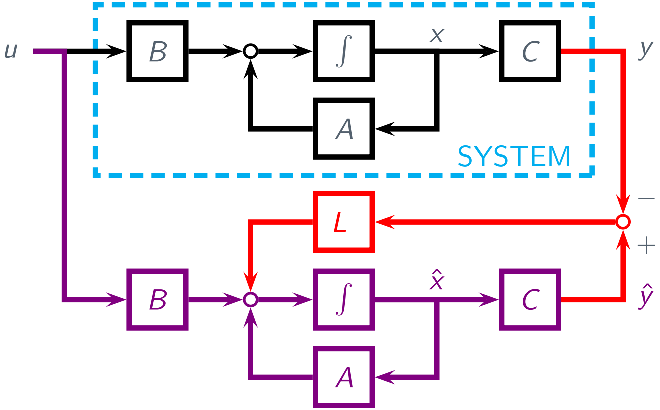
The full order observer



The full order observer

System:
$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned}
$$

Observer:
$$
\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\
\hat{y} &= C\hat{x}
\end{aligned}
$$

Error, $e = \hat{x} - x$:

$$
\begin{aligned}
\dot{e} = \dot{\hat{x}} - \dot{x} &= A\hat{x} + Bu + L(C\hat{x} - y) - (Ax + Bu) \\
&= A(\hat{x} - x) + L(C\hat{x} - Cx) \\
&= (A + LC)(\hat{x} - x) = (A + LC)e
\end{aligned}
$$

The full order observer

Theorem

A full order observer for the system

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned}
$$

with observer gain L is stable, if and only if the eigenvalues of the matrix $A + LC$ all have negative real part.

Moreover, such an L always exists, if (A, C) is observable.

Observable canonical form

Any observable *single output* system can be written in the form:

$$\dot{x}_o = A_o x_o \,, \quad y = C_o x_o \,, \quad x_o \in \mathbb{R}^n \,, \, y \in \mathbb{R}$$

where

$$A_o = \left(\begin{array}{c|c} a & \begin{array}{c} I_{n-1} \\ \hline 0_{1\times(n-1)} \end{array} \end{array} \right) \,, \quad C_o = \left(\begin{array}{c|c} 1 & 0_{1\times(n-1)} \end{array} \right)$$

and where $a \in \mathbb{R}^{n\times 1}$, $a^T = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix}$.

It can be shown that

$$\det(\lambda I - A_o) = \lambda^n - a_1 \lambda^{n-1} - \ldots - a_n$$

Observable canonical form

For $n = 3$ the observable canonical form becomes:

$$A_o = \left(\begin{array}{c|cc} a_1 & 1 & 0 \\ a_2 & 0 & 1 \\ \hline a_3 & 0 & 0 \end{array}\right), \quad C_o = \left(\begin{array}{c|cc} 1 & 0 & 0 \end{array}\right)$$

which is indeed observable:

$$\mathcal{O}_o = \begin{pmatrix} C_o \\ C_o A_o \\ C_o A_o^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_1^2 + a_2 & a_1 & 1 \end{pmatrix}$$

$\det(\mathcal{O}) = 1 \neq 0 \implies$ system is observable.

Observable canonical form

Consider a system:

$$\dot{x} = Ax\,, \quad y = Cx\,, \quad x \in \mathbb{R}^n\,, \quad y \in \mathbb{R}$$

For $n = 3$, the observable canonical form for this system can be found through the following procedure:

1. Compute $t_3 = \mathcal{O}^{-1}\begin{pmatrix}0\\0\\1\end{pmatrix}$ where $\mathcal{O} = \begin{pmatrix}C\\CA\\CA^2\end{pmatrix}$
2. Compute $t_2 = At_3$, $t_1 = At_2$.
3. Define $T = \begin{pmatrix}t_1 & t_2 & t_3\end{pmatrix}$
4. The state space matrices for the observable canonical form are now given by $A_o = T^{-1}AT$, and $C_o = CT$.

Example: observable canonical form

We consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y &= \begin{pmatrix} -3 & 2 \end{pmatrix} x \end{aligned}$$

having the observability matrix

$$\mathcal{O} = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}, \quad \det(\mathcal{O}) = -1 \neq 0$$

Example: observable canonical form

We compute the columns of T by

$$t_2 = \mathcal{O}^{-1}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$$
$$t_1 = At_2 = \begin{pmatrix}2 & -3\\4 & -5\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}-5\\-7\end{pmatrix}$$

Thus,

$$T = \begin{pmatrix}-5 & 2\\-7 & 3\end{pmatrix} \quad\Longrightarrow\quad T^{-1} = \begin{pmatrix}-3 & 2\\-7 & 5\end{pmatrix}$$

Example: observable canonical form

Eventually, we have

$$A_o = T^{-1}AT = \begin{pmatrix} -3 & 2 \\ -7 & 5 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ -7 & 3 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} -3 & 1 \\ \hline -2 & 0 \end{array}\right) \Rightarrow \det(\lambda I - A) = \lambda^2 + 3\lambda + 2$$

and

$$C_o = CT = \begin{pmatrix} -3 & 2 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ -7 & 3 \end{pmatrix} = \left(\begin{array}{c|c} 1 & 0 \end{array}\right)$$

Example: observable canonical form

Eventually, we have

$$A_o = T^{-1}AT = \begin{pmatrix} -3 & 2 \\ -7 & 5 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ -7 & 3 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} -3 & 1 \\ \hline -2 & 0 \end{array} \right) \Rightarrow \det(\lambda I - A) = (\lambda + 1)(\lambda + 2)$$

and

$$C_o = CT = \begin{pmatrix} -3 & 2 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ -7 & 3 \end{pmatrix} = \left(\begin{array}{c|c} 1 & 0 \end{array} \right)$$

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Observer gain design

For a single output system in observable canonical form, an observer state matrix takes a particular simple form:

$$A_o = \left(\begin{array}{c|cc} a_1 & 1 & 0 \\ a_2 & 0 & 1 \\ \hline a_3 & 0 & 0 \end{array} \right), \; C_o = \left(\begin{array}{c|cc} 1 & 0 & 0 \end{array} \right)$$

Applying the observer gain

$$L_o = \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix}$$

Observer gain design

we obtain:

$$
\begin{aligned}
A_o + L_o C_o &= \left(\begin{array}{c|cc} a_1 & 1 & 0 \\ a_2 & 0 & 1 \\ \hline a_3 & 0 & 0 \end{array}\right) + \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix} \left(\begin{array}{c|cc} 1 & 0 & 0 \end{array}\right) \\
&= \left(\begin{array}{c|cc} a_1+\ell_1 & 1 & 0 \\ a_2+\ell_2 & 0 & 1 \\ \hline a_3+\ell_3 & 0 & 0 \end{array}\right)
\end{aligned}
$$

Observer gain design

Thus, the characteristic polynomium has been changed from

$$\det(\lambda I - A_o) = \lambda^n - a_1\lambda^{n-1} - \ldots - a_n$$

to

$$\det(\lambda I - (A_o + L_o C_o)) = \lambda^n - (a_1 + \ell_1)\lambda^{n-1} - \ldots - (a_n + \ell_n)$$

By choosing $\ell_1, \ldots, \ell_n$ appropriately, *any* observer pole configuration can be obtained. This is known as *observer pole assignment*.

Observer pole assignment

Let $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{1 \times n}$ be given.

1. Choose desired observer polynomial $\det(\lambda I - (A + LC)) = \lambda^n + a_{\text{obs},1}\lambda^{n-1} + \ldots + a_{\text{obs},n}$.
2. Determine T, such that $A_o = T^{-1}AT$ and $C_o = CT$ are in observable canonical form.
3. Determine open loop polynomial $\det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$
4. Define $L_o = \begin{pmatrix} a_1 - a_{\text{obs},1} \\ \vdots \\ a_n - a_{\text{obs},n} \end{pmatrix}$.
5. Compute resulting observer gain $L = TL_o$.

Example: pole assignment

We consider again the system

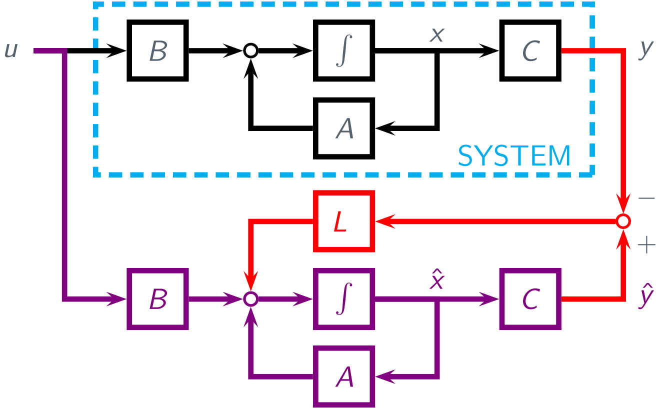
$$\begin{aligned} \dot{x} &= \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y &= \begin{pmatrix} -3 & 2 \end{pmatrix} x \end{aligned}$$

for which we would like to assign observer poles to $\{-4, -5\}$, i.e. to design L such that $A + LC$ has eigenvalues in $\{-4, -5\}$.

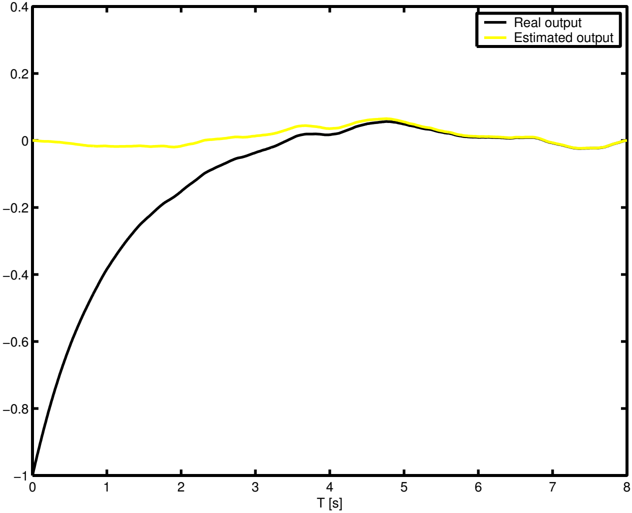
Example: pole assignment

1. Desired closed loop polynomial: $\lambda^2 + 9\lambda + 20$
2. $T = \begin{pmatrix} -5 & 2 \\ -7 & 3 \end{pmatrix} \Rightarrow A_o = \left(\begin{array}{c|c} -3 & 1 \\ \hline -2 & 0 \end{array}\right), C_o = \left(1 \,\middle|\, 0\right)$
3. Open loop polynomial: $\lambda^2 + 3\lambda + 2$
4. $L_o = \begin{pmatrix} 3-9 \\ 2-20 \end{pmatrix} = \begin{pmatrix} -6 \\ -18 \end{pmatrix}$
5. $L = TL_o = \begin{pmatrix} -5 & 2 \\ -7 & 3 \end{pmatrix} \begin{pmatrix} -6 \\ -18 \end{pmatrix} = \begin{pmatrix} -6 \\ -12 \end{pmatrix}$

The full order observer



Example: obs. pole assignment



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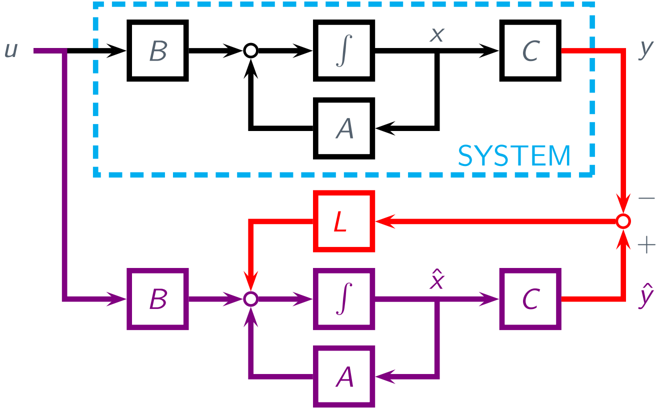
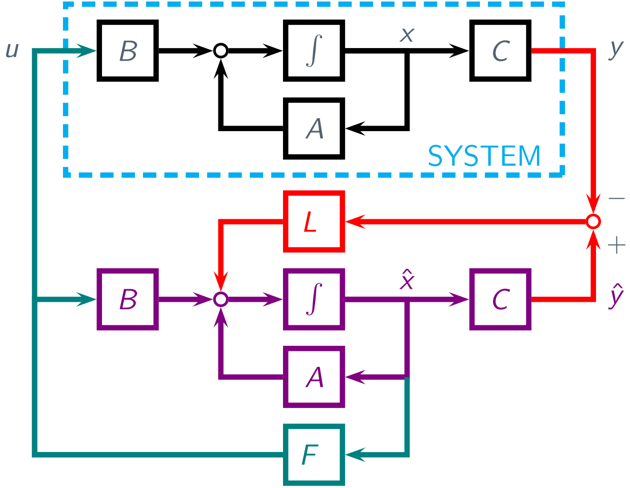
Observer based control



Observer based control



Observer based control

System:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Observer:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

Feedback: $u = F\hat{x}$

Error, $e = \hat{x} - x$:

$$\begin{aligned} \dot{e} &= \dot{\hat{x}} - \dot{x} \\ &= A\hat{x} + BF\hat{x} + L(C\hat{x} - y) - (Ax + BF\hat{x}) \\ &= A(\hat{x} - x) + L(C\hat{x} - Cx) \\ &= (A + LC)(\hat{x} - x) = (A + LC)e \end{aligned}$$

The separation principle

Combining the two equations:

$$\begin{aligned}\dot{x} &= Ax + Bu = Ax + BF\hat{x} = Ax + BF(e + x) \\ &= (A + BF)x + BFe\end{aligned}$$

and

$$\dot{e} = (A + LC)e$$

gives:

$$\begin{pmatrix} \dot{x} \\ e \end{pmatrix} = \begin{pmatrix} A + BF & BF \\ 0 & A + LC \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix}$$

The separation principle

Theorem

An observer based controller for the system

$$\begin{aligned} \dot{x} &= Ax + Bu \quad , \quad x \in \mathbb{R}^n \\ y &= Cx \end{aligned}$$

with observer gain L and feedback gain F results in $2n$ closed loop poles, coinciding with the eigenvalues of the two matrices:

$$A + BF \quad \text{and} \quad A + LC$$

Example: observer based control

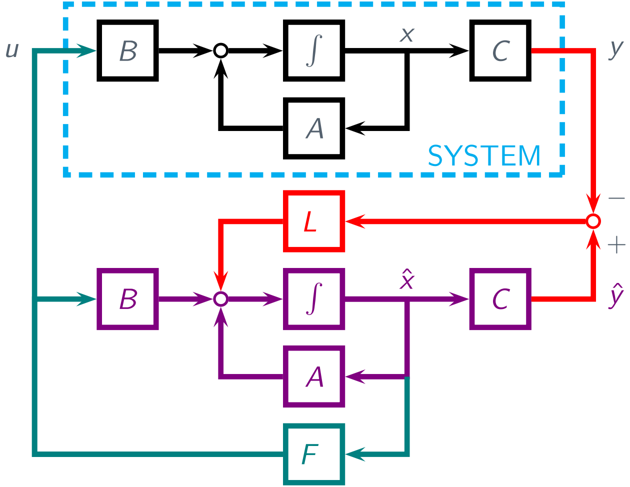
We consider again the system

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\
y &= \begin{pmatrix} -3 & 2 \end{pmatrix} x
\end{aligned}
$$

for which we apply an observer based controller with

$$
L = \begin{pmatrix} -6 \\ -12 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 42 & -30 \end{pmatrix}
$$

Observer based control



Example: observer based control

The transfer function of the controller becomes:

$$K(s) = -F\,(sI - A - BF - LC)^{-1}\,L$$
$$= -108\frac{s + \frac{7}{3}}{s^2 + 15s + 74}$$

The closed loop transfer function becomes:

$$G(s)\,(I - K(s)G(s))^{-1} = \frac{s^2 + 15s + 74}{(s+5)^2(s+4)^2}$$

Example: observer based control