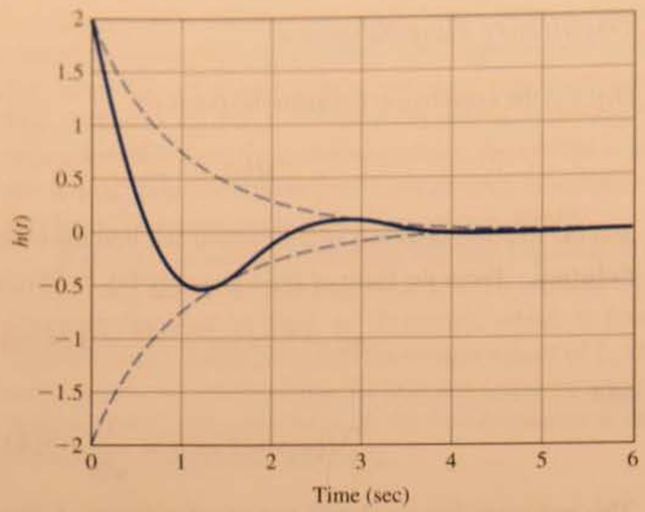
Figure 3.21
System response for Example 3.24



3.4 Time-Domain Specifications

Definitions of rise time, settling time, overshoot, and peak time

Specifications for a control system design often involve certain requirements associated with the time response of the system. The requirements for a step response are expressed in terms of the standard quantities illustrated in Fig. 3.22:

1. The **rise time** t_r is the time it takes the system to reach the vicinity of its new set point.
2. The **settling time** t_s is the time it takes the system transients to decay.
3. The **overshoot** M_p is the maximum amount the system overshoots its final value divided by its final value (and is often expressed as a percentage).
4. The **peak time** t_p is the time it takes the system to reach the maximum overshoot point.

3.4.1 Rise Time

Rise time t_r

For a second-order system, the time responses shown in Fig. 3.18(b) yield information about the specifications that is too complex to be remembered unless converted to a simpler form. By examining these curves in light of the definitions given in Fig. 3.22, we can relate the curves to the pole-location parameters ζ and ω_n. For example, all the curves rise in roughly the same time. If we consider the curve for $\zeta = 0.5$ to be an average, the rise time from $y = 0.1$ to $y = 0.9$ is approximately $\omega_n t_r = 1.8$. Thus we can say that

$$t_r \cong \frac{1.8}{\omega_n}. \tag{3.60}$$

Although this relationship could be embellished by including the effect of the damping ratio, it is important to keep in mind how Eq. (3.60) is typically used. It is accurate only for a second-order system with no zeros; for all other systems it is a rough approximation to the relationship between t_r and ω_n. Most systems being analyzed for control systems design are more complicated than the pure second-order system, so designers use Eq. (3.60) with the knowledge that it is a rough approximation only.

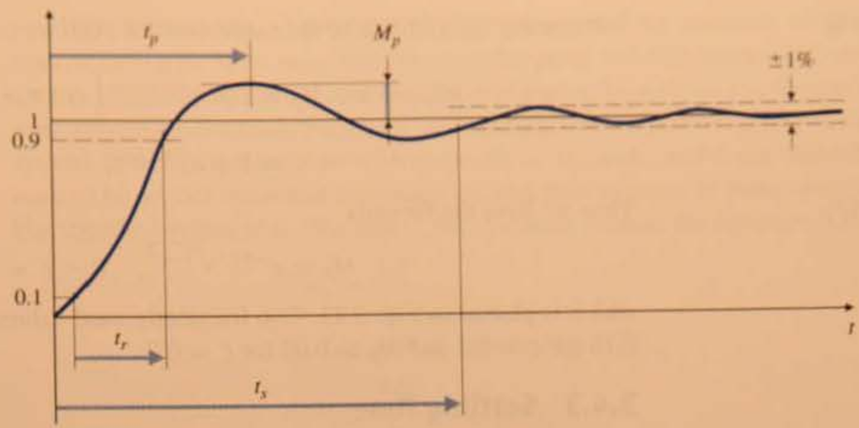


Figure 3.22
Definition of rise time t_r, settling time t_s, and overshoot M_p

3.4.2 Overshoot and Peak Time

For the overshoot M_p we can be more analytical. This value occurs when the derivative is zero, which can be found from calculus. The time history of the curves in Fig. 3.18(b), found from the inverse Laplace transform of $H(s)/s$, is

$$y(t) = 1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right), \tag{3.61}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\sigma = \zeta\omega_n$. We may rewrite the preceding equation using the trigonometric identity

$$A\sin(\alpha) + B\cos(\alpha) = C\cos(\alpha - \beta)$$

or

$$C = \sqrt{A^2+B^2} = \frac{1}{\sqrt{1-\zeta^2}},$$

$$\beta = \tan^{-1}\left(\frac{A}{B}\right) = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right),$$

Standard second-order system step response

with $A = \frac{\sigma}{\omega_d}$, $B = 1$, and $\alpha = \omega_d t$, in a more compact form as

$$y(t) = 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \beta). \tag{3.62}$$

When $y(t)$ reaches its maximum value, its derivative will be zero:

$$\dot{y}(t) = \sigma e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right) - e^{-\sigma t}(-\omega_d\sin\omega_d t + \sigma\cos\omega_d t) = 0$$

$$= e^{-\sigma t}\left(\frac{\sigma^2}{\omega_d} + \omega_d\right)\sin\omega_d t = 0.$$

This occurs when $\sin\omega_d t = 0$, so

$$\omega_d t_p = \pi$$

and thus

$$t_p = \frac{\pi}{\omega_d}. \tag{3.63}$$

Peak time t_p

Substituting Eq. (3.63) into the expression for $y(t)$, we compute

$$y(t_p) \triangleq 1 + M_p = 1 - e^{-\sigma\pi/\omega_d}\left(\cos\pi + \frac{\sigma}{\omega_d}\sin\pi\right)$$
$$= 1 + e^{-\sigma\pi/\omega_d}.$$

Overshoot M_p

Thus we have the formula

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}, \quad 0 \le \zeta < 1, \tag{3.64}$$

which is plotted in Fig. 3.23. Two frequently used values from this curve are $M_p = 0.16$ for $\zeta = 0.5$ and $M_p = 0.05$ for $\zeta = 0.7$.

3.4.3 Settling Time

The final parameter of interest from the transient response is the settling time t_s. This is the time required for the transient to decay to a small value so that $y(t)$ is almost in the steady state. Various measures of smallness are possible. For illustration we will use 1% as a reasonable measure; in other cases 2% or 5% are used. As an analytic computation, we notice that the deviation of y from 1 is the product of the decaying exponential $e^{-\sigma t}$ and the circular functions sine and cosine. The duration of this error is essentially decided by the transient exponential, so we can define the settling time as that value of t_s when the decaying exponential reaches 1%:

Settling time t_s

$$e^{-\zeta\omega_n t_s} = 0.01.$$

Therefore,

$$\zeta\omega_n t_s = 4.6,$$

or

$$t_s = \frac{4.6}{\zeta\omega_n} = \frac{4.6}{\sigma}, \tag{3.65}$$

where σ is the negative real part of the pole, as may be seen from Fig. 3.17.

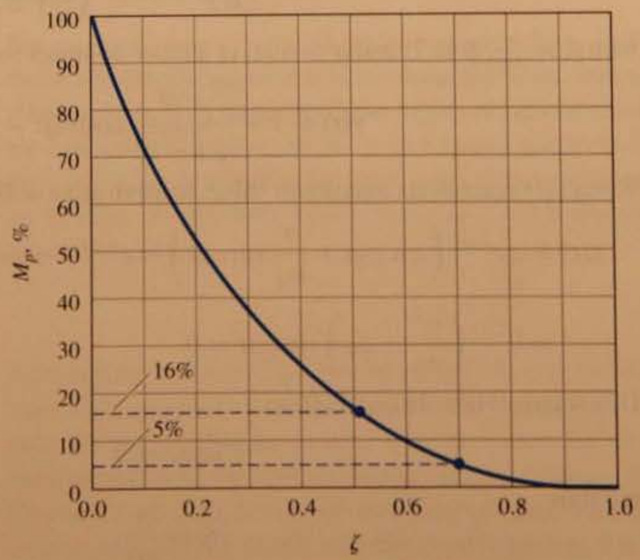


Figure 3.23
Overshoot M_p versus damping ratio ζ for the second-order system

Equations (3.60), (3.64), and (3.65) characterize the transient response of a system having no finite zeros and two complex poles and with undamped natural frequency ω_n, damping ratio ζ, and negative real part σ. In analysis and design, they are used to estimate rise time, overshoot, and settling time, respectively, for just about any system. In design synthesis we wish to specify t_r, M_p, and t_s and to ask where the poles need to be so that the actual responses are less than or equal to these specifications. For specified values of t_r, M_p, and t_s, the synthesis form of the equation is then

Design synthesis

$$\omega_n \geq \frac{1.8}{t_r}, \tag{3.66}$$

$$\zeta \geq \zeta(M_p) \quad \text{(from Fig. 3.23)}, \tag{3.67}$$

$$\sigma \geq \frac{4.6}{t_s}. \tag{3.68}$$

These equations, which can be graphed in the s-plane as shown in Fig. 3.24(a–c), will be used in later chapters to guide the selection of pole and zero locations to meet control system specifications for dynamic response.

It is important to keep in mind that Eqs. (3.66)–(3.68) are qualitative guides and not precise design formulas. They are meant to provide only a starting point for the design iteration. After the control design is complete, the time response should always be checked by an exact calculation, usually by numerical simulation, to verify whether the time specifications have actually been met. If not, another iteration of the design is required.

First-order system step response

For a first-order system,

$$H(s) = \frac{\sigma}{s+\sigma},$$

and the transform of the step response is

$$Y(s) = \frac{\sigma}{s(s+\sigma)}.$$

We see from entry 11 in Table A.2 that $Y(s)$ corresponds to

$$y(t) = (1 - e^{-\sigma t})1(t). \tag{3.69}$$

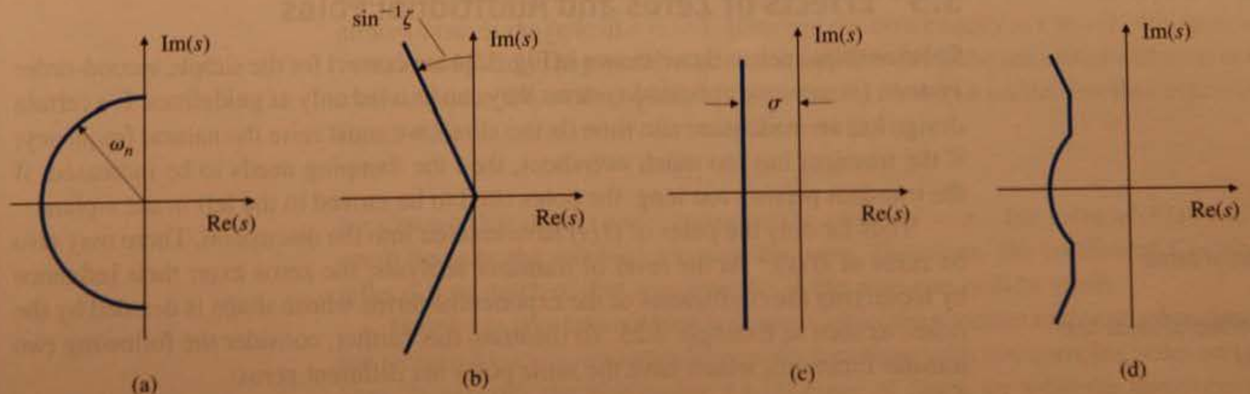


Figure 3.24
Graphs of regions in the s-plane delineated by certain transient requirements: (a) rise time; (b) overshoot; (c) settling time; (d) composite of all three requirements

Comparison with the development for Eq. (3.65) shows that the value of t_s for a first-order system is the same:

$$t_s = \frac{4.6}{\sigma}.$$

No overshoot is possible, so $M_p = 0$. The rise time from $y = 0.1$ to $y = 0.9$ can be seen from Fig. 3.13 to be

$$t_r = \frac{\ln 0.9 - \ln 0.1}{\sigma} = \frac{2.2}{\sigma}.$$

Time constant τ

However, it is more typical to describe a first-order system in terms of its time constant, which was defined in Fig. 3.13 to be $\tau = 1/\sigma$.

EXAMPLE 3.25 *Transformation of the Specifications to the s-Plane*

Find the allowable regions in the s-plane for the poles of a transfer function of a system if the system response requirements are $t_r \leq 0.6$ sec, $M_p \leq 10\%$, and $t_s \leq 3$ sec.

Solution. Without knowing whether or not the system is second order with no zeros, it is impossible to find the allowable region accurately. Regardless of the system, we can obtain a first approximation using the relationships for a second-order system. Equation (3.66) indicates that

$$\omega_n \geq \frac{1.8}{t_r} = 3.0 \text{ rad/sec},$$

Eq. (3.67) and Fig. 3.23 indicate that

$$\zeta \geq 0.6,$$

and Eq. (3.68) indicates that

$$\sigma \geq \frac{4.6}{3} = 1.5 \text{ sec}.$$

The allowable region is anywhere to the left of the solid line in Fig. 3.25. Note that any pole meeting the ζ and ω_n restrictions will automatically meet the σ restriction.

3.5 Effects of Zeros and Additional Poles

Relationships such as those shown in Fig. 3.24 are correct for the simple second-order system; for more complicated systems they can be used only as guidelines. If a certain design has an inadequate rise time (is too slow), we must raise the natural frequency; if the transient has too much overshoot, then the damping needs to be increased; if the transient persists too long, the poles need to be moved to the left in the s-plane.

Effect of zeros

The effect of zeros near poles

Thus far only the poles of $H(s)$ have entered into the discussion. There may also be zeros of $H(s)$.[8] At the level of transient analysis, the zeros exert their influence by modifying the coefficients of the exponential terms whose shape is decided by the poles, as seen in Example 3.23. To illustrate this further, consider the following two transfer functions, which have the same poles but different zeros:

[8] We assume that $b(s)$ and $a(s)$ have no common factors. If this is not so, it is possible for $b(s)$ and $a(s)$ to be zero at the same location and for $H(s)$ to not equal zero there. The implications of this case will be discussed in Chapter 7, when we have a state-space description.

Using the formula given in Eq. (4.13), with changes in the parameter of interest, we can compute

$$S_F^{T_{cl}} = 1.0, \tag{4.20}$$

$$S_G^{T_{cl}} = \frac{1}{1 + GD_{cl}H}, \tag{4.21}$$

$$S_H^{T_{cl}} = \frac{GD_{cl}H}{1 + GD_{cl}H}. \tag{4.22}$$

Of these, the most interesting is the last. Notice that with respect to H, the sensitivity approaches unity as the loop gain grows. Therefore it is particularly important that the transfer function of the sensor be not only low in noise but also very stable in gain. Money spent on the sensor is money well spent.

4.2 Control of Steady-State Error to Polynomial Inputs: System Type

In studying the regulator problem, the reference input is taken to be a constant. It is also the case that the most common plant disturbance is a constant bias. Even in the general tracking problem the reference input is often constant for long periods of time or may be adequately approximated as if it were a polynomial in time, usually one of low degree. For example, when an antenna is tracking the elevation angle to a satellite, the time history as the satellite approaches overhead is an S-shaped curve as sketched in Fig. 4.5. This signal may be approximated by a linear function of time (called a ramp function or velocity input) for a significant time relative to the speed of response of the servomechanism. As another example, the position control of an elevator has a ramp function reference input, which will direct the elevator to move with constant speed until it comes near the next floor. In rare cases, the input can even be approximated over a substantial period as having a constant acceleration. Consideration of these cases leads us to consider steady-state errors in stable systems with polynomial inputs.

As part of the study of steady-state errors to polynomial inputs, a terminology has been developed to express the results. For example, we classify systems as to "**type**" according to the degree of the polynomial that they can reasonably track. For example, a system that can track a polynomial of degree 1 with a constant error is called *Type 1*. Also, to quantify the tracking error, several "**error constants**" are defined. In all of the following analysis, it is assumed that the systems are stable, else the analysis makes no sense at all.

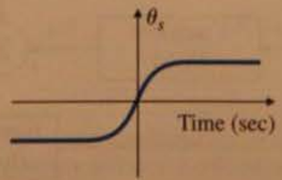


Figure 4.5
Signal for satellite tracking

4.2.1 System Type for Tracking

In the unity feedback case shown in Fig. 4.2, the system error is given by Eq. (4.8). If we consider tracking the reference input alone and set $W = V = 0$, then the equation for the error is simply

$$E = \frac{1}{1 + GD_{cl}} R = \mathcal{S}R. \tag{4.23}$$

To consider polynomial inputs, we let $r(t) = {t^k}/{k!}1(t)$ for which the transform is $R = \frac{1}{s^{k+1}}$. We take a mechanical system as the basis for a generic reference nomenclature, calling step inputs for which $k = 0$ "position" inputs, ramp inputs for which $k = 1$ are called "velocity" inputs and if $k = 2$, the inputs are called "acceleration" inputs, regardless of the units of the actual signals. Application of the Final Value Theorem to the error formula gives the result

$$\lim_{t\to\infty} e(t) = e_{ss} = \lim_{s\to 0} sE(s) \tag{4.24}$$

$$= \lim_{s\to 0} s\frac{1}{1 + GD_{cl}} R(s) \tag{4.25}$$

$$= \lim_{s\to 0} s\frac{1}{1 + GD_{cl}} \frac{1}{s^{k+1}}. \tag{4.26}$$

We consider first a system for which GD_{cl} has no pole at the origin and a step input for which $R(s) = {1}/{s}$. Thus $r(t)$ is a polynomial of degree 0. In this case, Eq. (4.26) reduces to

$$e_{ss} = \lim_{s\to 0} s\frac{1}{1 + GD_{cl}} \frac{1}{s} \tag{4.27}$$

$$= \frac{1}{1 + GD_{cl}(0)}. \tag{4.28}$$

We define this system to be *Type 0* and we define the constant, $GD_{cl}(0) \triangleq K_p$ as the "*position error constant.*" Notice that if the input should be a polynomial of degree higher than 1, the resulting error would grow without bound. A polynomial of degree 0 is the highest degree a system of *Type 0* can track at all. If $GD_{cl}(s)$ has one pole at the origin, we could continue this line of argument and consider first-degree polynomial inputs but it is quite straightforward to evaluate Eq. (4.26) in a general setting. For this case, it is necessary to describe the behavior of the controller and plant as s approaches 0. For this purpose, we collect all the terms except the pole(s) at the origin into a function $GD_{clo}(s)$, which is finite at $s = 0$ so that we can define the constant $GD_{clo}(0) = K_n$ and write the loop transfer function as

$$GD_{cl}(s) = \frac{GD_{clo}(s)}{s^n}. \tag{4.29}$$

For example, if GD_{cl} has no integrator, then $n = 0$. If the system has one integrator, then $n = 1$, and so forth. Substituting this expression into Eq. (4.26),

$$e_{ss} = \lim_{s\to 0} s \frac{1}{1 + \frac{GD_{clo}(s)}{s^n}} \frac{1}{s^{k+1}} \tag{4.30}$$

$$= \lim_{s\to 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}. \tag{4.31}$$

From this equation we can see at once that if $n > k$ then $e = 0$ and if $n < k$ then $e \to \infty$. If $n = k = 0$, then $e_{ss} = \frac{1}{1+K_0}$ and if $n = k \neq 0$, then $e_{ss} = {}^1/_{K_n}$. As we saw above, if $n = k = 0$, the input is a zero-degree polynomial otherwise known as a step or position, the constant K_o is called the "*position constant*" written as K_p, and the system is classified as "*Type 0*." If $n = k = 1$, the input is a first-degree polynomial otherwise known as a ramp or velocity input and the constant K_1 is called the "*velocity constant*" written as K_v. This system is classified "*Type 1*" (read "type one"). In a similar way, systems of *Type 2* and higher types may be defined. A clear picture of the situation is given by the plot in Fig. 4.6 for a system of Type 1 having a ramp reference input. The error between input and output of size $\frac{1}{K_v}$ is clearly marked.

Using Eq (4.29), these results can be summarized by the equations:

$$K_p = \lim_{s\to 0} GD_{cl}(s), \qquad n = 0, \tag{4.32}$$

$$K_v = \lim_{s\to 0} sGD_{cl}(s), \qquad n = 1, \tag{4.33}$$

$$K_a = \lim_{s\to 0} s^2 GD_{cl}(s), \qquad n = 2. \tag{4.34}$$

The type information can also be usefully gathered in a table of error values as a function of the degree of the input polynomial and the type of the system as shown in Table 4.1.

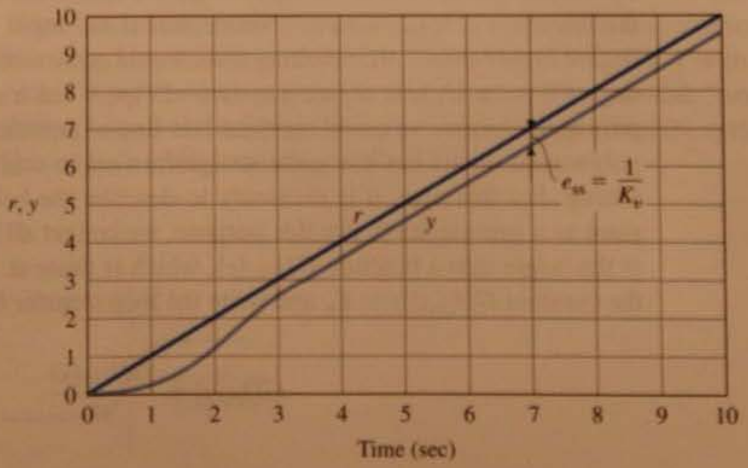


Figure 4.6
Relationship between ramp response and K_v

TABLE 4.1 Errors as a Function of System Type

Type Input	Step (position)	Ramp (velocity)	Parabola (acceleration)
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

EXAMPLE 4.1 *System Type for Speed Control*

Determine the system type and the relevant error constant for speed control with proportional feedback given by $D(s) = k_p$. The plant transfer function is $G = \frac{A}{\tau s+1}$.

Solution. In this case, $GD_{cl} = \frac{k_p A}{\tau s+1}$ and applying Eq. (4.32) we see that $n = 0$ in this case as there is no pole at $s = 0$. Thus the system is Type 0 and the error constant is a position constant given by $K_p = k_p A$.

EXAMPLE 4.2 *System Type Using Integral Control*

Determine the system type and the relevant error constant for the speed control example with proportional plus integral control having controller given by $D_c = k_p + {k_I}/{s}$. The plant transfer function is $G = \frac{A}{\tau s+1}$.

Solution. In this case, the loop transfer function is $GD_{cl}(s) = \frac{A(k_p s+k_I)}{s(\tau s+1)}$ and, as a unity feedback system with a single pole at $s = 0$, the system is immediately seen as Type 1. The velocity constant is given by Eq. (4.33) to be $K_v = \lim_{s\to 0} sGD_{cl}(s) = Ak_I$.

Robustness of system type

The definition of system type helps us to identify quickly the ability of a system to track polynomials. In the unity feedback structure, if the process parameters change without removing the pole at the origin in a Type 1 system, the velocity constant will change but the system will still have zero steady-state error in response to a constant input and will still be Type 1. Similar statements can be made for systems of Type 2 or higher. Thus we can say that system type is a **robust property** with respect to parameter changes in the unity feedback structure. Robustness is a major reason for preferring unity feedback over other kinds of control structure.

Another form of the formula for the error constants can be developed directly in terms of the closed-loop transfer function $\mathcal{T}(s)$. From Fig. 4.4 the transfer function including a sensor transfer function is

$$\frac{Y(s)}{R(s)} = \mathcal{T}(s) = \frac{GD}{1+GDH}, \tag{4.35}$$

and the system error is

$$E(s) = R(s) - Y(s) = R(s) - \mathcal{T}(s)R(s).$$

The reference-to-error transfer function is thus

$$\frac{E(s)}{R(s)} = 1 - \mathcal{T}(s),$$

and the system error transform is

$$E(s) = [1 - \mathcal{T}(s)]R(s).$$

We assume that the conditions of the Final Value Theorem are satisfied, namely that all poles of $sE(s)$ are in the LHP. In that case the steady-state error is given by applying the Final Value Theorem to get

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s[1 - \mathcal{T}(s)]R(s). \tag{4.36}$$

If the reference input is a polynomial of degree k, the error transform becomes

$$E(s) = \frac{1}{s^{k+1}}[1 - \mathcal{T}(s)]$$

and the steady-state error is given again by the Final Value Theorem:

$$e_{ss} = \lim_{s\to 0} s\frac{1 - \mathcal{T}(s)}{s^{k+1}} = \lim_{s\to 0} \frac{1 - \mathcal{T}(s)}{s^k}. \tag{4.37}$$

As before, the result of evaluating the limit in Eq. (4.37) can be zero, a nonzero constant, or infinite and if the solution to Eq. (4.37) is a nonzero constant, the system is referred to as *Type k*. Notice that a system of Type 1 or higher has a closed-loop DC gain of 1.0, which means that $\mathcal{T}(0) = 1$ in these cases.

EXAMPLE 4.3 *System Type for a Servo with Tachometer Feedback*

Consider an electric motor position control problem including a non-unity feedback system caused by having a tachometer fixed to the motor shaft and its voltage (which is proportional to shaft speed) is fed back as part of the control. The parameters are

$$G(s) = \frac{1}{s(\tau s + 1)},$$
$$D(s) = k_p,$$
$$H(s) = 1 + k_t s,$$
$$F(s) = 1.$$

Determine the system type and relevant error constant with respect to reference inputs.

Solution. The system error is

$$\begin{aligned}
E(s) &= R(s) - Y(s) \\
&= R(s) - \mathcal{T}(s)R(s) \\
&= R(s) - \frac{DG(s)}{1 + HDG(s)}R(s) \\
&= \frac{1 + (H(s) - 1)DG(s)}{1 + HDG(s)}R(s).
\end{aligned}$$

The steady-state system error from Eq. (4.37) is

$$e_{ss} = \lim_{s\to 0} sR(s)[1 - \mathcal{T}(s)].$$

For a polynomial reference input, $R(s) = {}^{1}/_{s^{k+1}}$ and hence

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} \frac{[1-\mathcal{T}(s)]}{s^k} = \lim_{s\to 0} \frac{1}{s^k} \frac{s(\tau s+1) + (1 + k_t s - 1)k_p}{s(\tau s+1) + (1+k_t s)k_p} \\ &= 0, \qquad k = 0 \\ &= \frac{1 + k_t k_p}{k_p}, \qquad k = 1; \end{aligned}$$

therefore the system is Type 1 and the velocity constant is $K_v = \frac{k_p}{1+k_t k_p}$. Notice that if $k_t > 0$, perhaps to improve stability or dynamic response, the velocity constant is smaller than with simply the unity feedback value of k_p. The conclusion is that if tachometer feedback is used to improve dynamic response, the steady-state error is usually increased.

4.2.2 System Type for Regulation and Disturbance Rejection

A system can also be classified with respect to its ability to reject polynomial disturbance inputs in a way analogous to the classification scheme based on reference inputs. The transfer function from the disturbance input $W(s)$ to the error $E(s)$ is

$$\frac{E(s)}{W(s)} = \frac{-Y(s)}{W(s)} = T_w(s) \tag{4.38}$$

because, if the reference is equal to zero, the output is the error. In a similar way as for reference inputs, the system is Type 0 if a step disturbance input results in a nonzero constant steady-state error and is Type 1 if a ramp disturbance input results in a steady-state value of the error that is a non zero constant, etc. In general, following the same approach used in developing Eq. (4.31), we assume that a constant n and a function $T_{o,w}(s)$ can be defined with the properties that $T_{o,w}(0) = {}^{1}/_{K_{n,w}}$ and that the disturbance-to-error transfer function can be written as

$$T_w(s) = s^n T_{o,w}(s). \tag{4.39}$$

Then the steady-state error to a disturbance input, which is a polynomial of degree k, is

$$\begin{aligned} y_{ss} &= \lim_{s\to 0}\left[sT_w(s)\frac{1}{s^{k+1}}\right] \\ &= \lim_{s\to 0}\left[T_{o,w}(s)\frac{s^n}{s^k}\right]. \end{aligned} \tag{4.40}$$

From Eq. (4.40), if $n > k$, then the error is zero and if $n < k$, the error is unbounded. If $n = k$, the system is type k and the error is given by ${}^{1}/_{K_{n,w}}$.

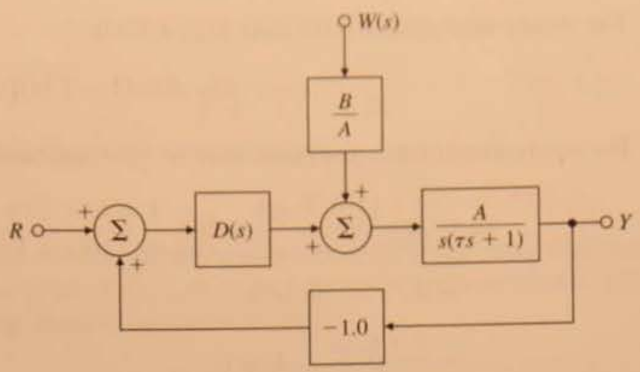


Figure 4.7
DC motor with unity feedback

EXAMPLE 4.4

System Type for a DC Motor Position Control

Consider the simplified model of a DC motor in unity feedback as shown in Fig. 4.7, where the disturbance torque is labeled $W(s)$. This case was considered in Example 2.11.

(a) Use the controller

$$D(s) = k_p, \tag{4.41}$$

and determine the system type and steady-state error properties with respect to disturbance inputs.

(b) Let the controller transfer function be given by

$$D(s) = k_p + \frac{k_I}{s}, \tag{4.42}$$

and determine the system type and the steady-state error properties for disturbance inputs.

Solution. (a) The closed-loop transfer function from W to E (where $R = 0$) is

$$\begin{aligned} T_w(s) &= \frac{-B}{s(\tau s+1)+Ak_p} \\ &= s^0 T_{o,w}, \\ n &= 0, \\ K_{o,w} &= \frac{-Ak_p}{B}. \end{aligned}$$

Applying Eq. (4.40) we see that the system is Type 0 and the steady-state error to a unit step torque input is $e_{ss} = {}^{-B}/_{Ak_p}$. From the earlier section, this system is seen to be Type 1 for reference inputs and illustrates that system type can be different for different inputs to the same system.

(b) For this controller the disturbance error transfer function is

$$T_w(s) = \frac{-Bs}{s^2(\tau s+1)+(k_p s+k_I)A}, \tag{4.43}$$

$$n = 1, \tag{4.44}$$

$$K_{n,w} = \frac{Ak_I}{-B}, \tag{4.45}$$

and therefore the system is Type 1 and the error to a unit ramp disturbance input will be

$$e_{ss} = \frac{-B}{Ak_I}. \tag{4.46}$$

Truxal's Formula for the Error Constants

Truxal (1955) derived a formula for the velocity constant of a Type 1 system in terms of the closed-loop poles and zeros, a formula that connects the steady-state error to the system's dynamic response. Since control design often requires a trade-off between these two characteristics, Truxal's formula can be useful to know. Its derivation is quite direct. Suppose the closed-loop transfer function $\mathcal{T}(s)$ of a Type 1 system is

$$\mathcal{T}(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}. \tag{4.47}$$

Since the steady-state error in response to a step input in a Type 1 system is zero, the DC gain is unity; thus

$$\mathcal{T}(0) = 1. \tag{4.48}$$

The system error is given by

$$E(s) \triangleq R(s) - Y(s) = R(s)\left[1 - \frac{Y(s)}{R(s)}\right] = R(s)[1 - \mathcal{T}(s)]. \tag{4.49}$$

The system error due to a unit ramp input is given by

$$E(s) = \frac{1-\mathcal{T}(s)}{s^2}. \tag{4.50}$$

Using the Final Value Theorem, we get

$$e_{ss} = \lim_{s\to 0}\frac{1-\mathcal{T}(s)}{s}. \tag{4.51}$$

Using L'Hôpital's rule we rewrite Eq. (4.51) as

$$e_{ss} = -\lim_{s\to 0}\frac{d\mathcal{T}}{ds} \tag{4.52}$$

or

$$e_{ss} = -\lim_{s\to 0}\frac{d\mathcal{T}}{ds} = \frac{1}{K_v}. \tag{4.53}$$

Equation (4.53) implies that $1/K_v$ is related to the slope of the transfer function at the origin, a result that will also be shown in Section 6.1.2. Using Eq. (4.48), we can rewrite Eq. (4.53) as

$$e_{ss} = -\lim_{s\to 0}\frac{d\mathcal{T}}{ds}\frac{1}{\mathcal{T}} \tag{4.54}$$

or

$$e_{ss} = -\lim_{s\to 0}\frac{d}{ds}[\ln \mathcal{T}(s)]. \tag{4.55}$$

Substituting Eq. (4.47) into Eq. (4.55), we get

$$e_{ss} = -\lim_{s\to 0}\frac{d}{ds}\left\{\ln\left[K\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}\right]\right\} \tag{4.56}$$

$$= -\lim_{s \to 0} \frac{d}{ds} \left[K + \sum_{i=1}^{m} \ln(s - z_i) - \sum_{i=1}^{m} \ln(s - p_i) \right] \tag{4.57}$$

or

$$\frac{1}{K_v} = -\frac{d \ln T}{ds}\bigg|_{s=0} = \sum_{i=1}^{n} -\frac{1}{p_i} + \sum_{i=1}^{m} \frac{1}{z_i}. \tag{4.58}$$

We observe from Eq. (4.58) that K_v increases as the closed-loop poles move away from the origin. Similar relationships exist for other error coefficients, and these are explored in the problems.

EXAMPLE 4.5 *Truxal's Formula*

Truxal's formula

A third-order Type 1 system has closed-loop poles at $-2 \pm 2j$ and -0.1. The system has only one closed-loop zero. Where should the zero be if a $K_v = 10$ is desired?

Solution. From Truxal's formula we have,

$$\frac{1}{K_v} = -\frac{1}{-2+2j} - \frac{1}{-2-2j} - \frac{1}{-0.1} + \frac{1}{z}$$

or

$$0.1 = 0.5 + 10 + \frac{1}{z},$$

$$\frac{1}{z} = 0.1 - 0.5 - 10,$$

$$= -10.4$$

Therefore, the closed-loop zero should be at $z = 1/_{-10.4} = -0.0962$.

4.3 The Three-Term Controller: PID Control

In later chapters we will study three general analytic and graphical design techniques based on the root locus, the frequency response, and the state space formulation of the equations. Here we describe a control method having an older pedigree that was developed through long experience and by trial and error. Starting with simple proportional feedback, engineers early discovered integral control action as a means of eliminating bias offset. Then, finding poor dynamic response in many cases, an "anticipatory" term based on the derivative was added. The result is called the three-term or PID controller and has the transfer function[3]

$$D(s) = k_p + \frac{k_I}{s} + k_D s, \tag{4.59}$$

where k_p is the proportional term, k_I is the integral term, and k_D is the derivative term. We'll discuss them in turn.

[3]The derivative term alone makes this transfer function nonproper and impractical. However adding a high-frequency pole to make the term proper only slightly modifies the performance.

4.3.1 Proportional Control (P)

When the feedback control signal is linearly proportional to the system error, we call the result **proportional feedback**. This was the case for the feedback used in the controller of speed in Section 4.1 for which the controller transfer function is

$$\frac{U(s)}{E(s)} = D_{cl}(s) = k_p. \tag{4.60}$$

If the plant is second order, as, for example, for a motor with nonnegligible inductance, then the plant transfer function can be written as

$$G(s) = \frac{A}{s^2 + a_1 s + a_2}. \tag{4.61}$$

In this case, the characteristic equation with proportional control is

$$1 + k_p G(s) = 0, \tag{4.62}$$

$$s^2 + a_1 s + a_2 + k_p = 0. \tag{4.63}$$

The designer can control the constant term in this equation, which determines the natural frequency, but cannot control the damping of the equation. The system is Type 0 and if k_p is made large to get adequately small steady-state error, the damping may be much too low for satisfactory transient response with proportional control alone.

4.3.2 Proportional Plus Integral Control (PI)

Proportional plus integral control

Adding an integral term to the controller to get the automatic reset effect results in the **proportional plus integral** control equation in the time domain:

$$u(t) = k_p e + k_I \int_{t_0}^{t} e(\tau)\, d\tau, \tag{4.64}$$

for which the $D_{cl}(s)$ in Fig. 4.2 becomes

$$\frac{U(s)}{E(s)} = D_{cl}(s) = k_p + \frac{k_I}{s}. \tag{4.65}$$

Introduction of the integral term raises the type to Type 1 and the system can therefore reject completely constant bias disturbances. For example, consider PI control in a speed control example, where the plant is described by

$$Y = \frac{A}{\tau s + 1}(U + W). \tag{4.66}$$

The transform equation for the controller is

$$U = k_p(R - Y) + k_I \frac{R - Y}{s}, \tag{4.67}$$

and the system transform equation with this controller is

$$(\tau s + 1)Y = A\left(k_p + \frac{k_I}{s}\right)(R - Y) + AW, \tag{4.68}$$

and, if we multiply by s and collect terms,

$$(\tau s^2 + (Ak_p + 1)s + Ak_I)Y = A(k_p s + k_I)R + sAW. \tag{4.69}$$

Because the PI controller includes dynamics, use of this controller will change the dynamic response. This we can understand by considering the characteristic equation given by

$$\tau s^2 + (Ak_p + 1)s + Ak_I = 0. \tag{4.70}$$

The two roots of this equation may be complex and, if so, the natural frequency is $\omega_n = \sqrt{Ak_I/\tau}$ and the damping ratio is $\zeta = \frac{Ak_p+1}{2\tau\omega_n}$. These parameters may both be determined by the controller gains. On the other hand, if the plant is second order, described by

$$G(s) = \frac{A}{s^2 + a_1 s + a_2}, \tag{4.71}$$

then the characteristic equation of the system is

$$1 + \frac{k_p s + k_I}{s}\frac{A}{s^2 + a_1 s + a_2} = 0, \tag{4.72}$$

$$s^3 + a_1 s^2 + a_2 s + Ak_p s + Ak_I = 0. \tag{4.73}$$

In this case, the controller parameters can be used to set two of the coefficients but not the third. For this we need derivative control.

4.3.3 PID Control

The final term in the classical controller is derivative control, **D**. An important effect of this term is that it gives a sharp response to suddenly changing signals. Because of this, the "**D**" term is sometimes introduced into the feedback path as shown in Fig. 4.8(a). This could be either a part of the standard controller or could describe a velocity sensor such as a tachometer on the shaft of a motor. The closed-loop characteristic equation is the same as if the term were in the forward path as given by Eq. (4.59) and drawn in Fig. 4.8(b). It is important to notice that the *zeros* from the reference to the output are

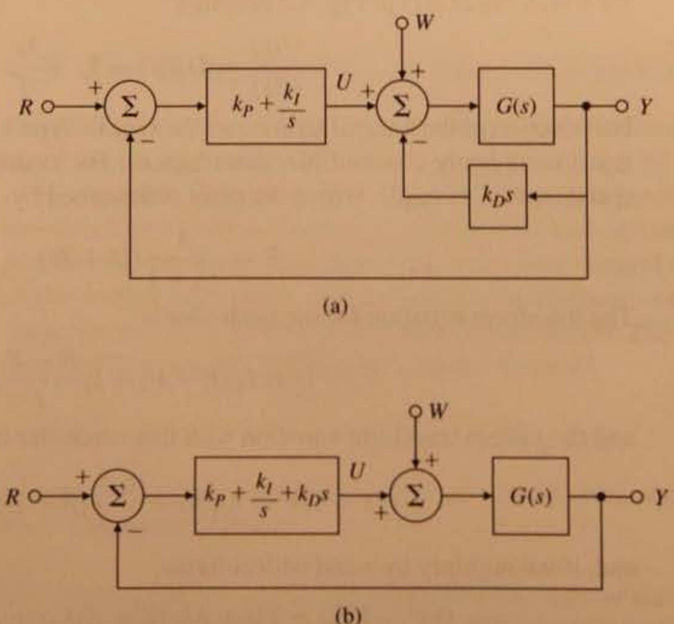


Figure 4.8
Block diagram of the PID controller: (a) with the D-term in the feedback path; and (b) with the D-term in the forward path

different in the two cases. With the derivative in the feedback path, the reference is not differentiated, which is how the undesirable response to sudden changes is avoided.

To illustrate the effect of a derivative term on PID control, consider speed control but with the second-order plant. In that case, the characteristic equation is

$$s^2 + a_1 s + a_2 + A(k_p + \frac{k_I}{s} + k_D s) = 0,$$
$$s^3 + a_1 s^2 + a_2 s + A(k_p s + k_I + k_D s^2) = 0. \tag{4.74}$$

Collecting terms results in

$$s^3 + (a_1 + Ak_D)s^2 + (a_2 + Ak_p)s + Ak_I = 0. \tag{4.75}$$

The point here is that this equation, whose three roots determine the nature of the dynamic response of the system, has three free parameters in k_p, k_I, and k_D and that by selection of these parameters, the roots can be uniquely and, in theory, arbitrarily determined. Without the derivative term, there would be only two free parameters, but with three roots, the choice of roots of the characteristic equation would be restricted. To illustrate the effect more concretely, a numerical example is useful.

EXAMPLE 4.6 *PID Control of Motor Speed*

Consider the DC motor speed control with parameters[4]

$$J_m = 1.13 \times 10^{-2}\ \text{N·m·sec}^2/\text{rad}, \quad b = 0.028\ \text{N·m·sec/rad}, \quad L_a = 10^{-1}\text{henry},$$
$$R_a = 0.45\ \text{ohms}, \quad K_t = 0.067\ \text{N·m/amp}, \quad K_e = 0.067\ \text{volt·sec/rad} \tag{4.76}$$

These parameters were defined in Example 2.11 in Chapter 2. Use the controller parameters

$$k_p = 3, \quad k_I = 15\ \text{sec}, \quad k_D = 0.3\ \text{sec}. \tag{4.77}$$

Discuss the effects of P, PI, and PID control on the responses of this system to steps in the disturbance torque and steps in the reference input. Let the unused controller parameters be zero.

Solution. Figure 4.9(a) illustrates the effects of P, PI, and PID feedback on the step disturbance response of the system. Note that adding the integral term increases the oscillatory behavior but eliminates the steady-state error and that adding the derivative term reduces the oscillation while maintaining zero steady-state error. Figure 4.9(b) illustrates the effects of P, PI, and PID feedback on the step reference response with similar results. The step responses can be computed by forming the numerator and denominator coefficient vectors (in descending powers of s) and using the step function in MATLAB.®

[4]These values have been scaled to measure time in milliseconds by multiplying the true L_a and J_m by 1000 each.

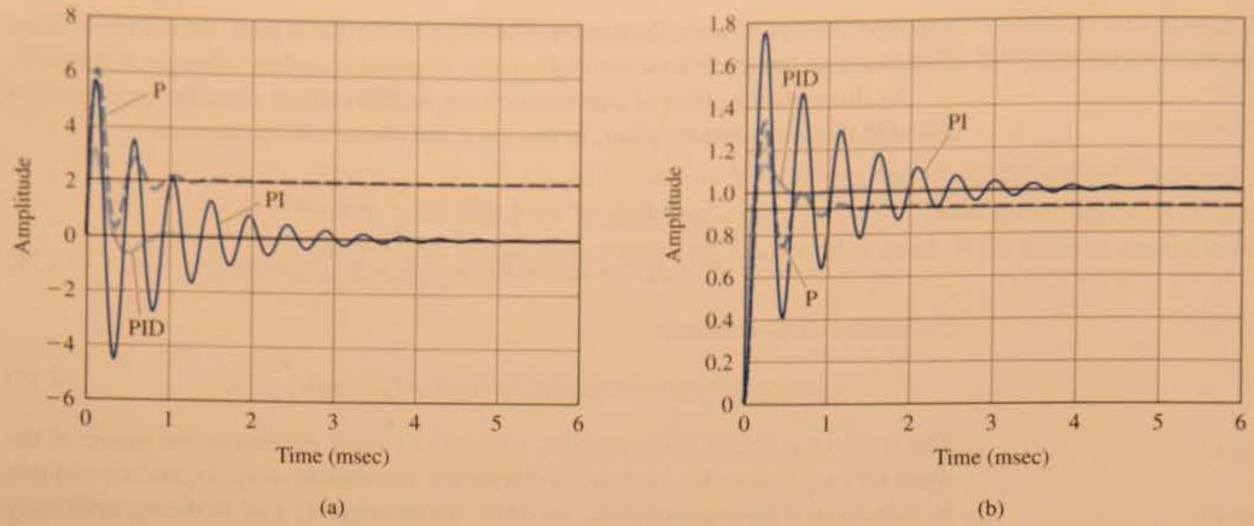


Figure 4.9
Responses of P, PI, and PID control to (a) step disturbance input (b) step reference input

EXAMPLE 4.7

PI Control for a DC Motor Position Control

Consider the simplified model of a DC motor in unity feedback as shown in Fig. 4.7 where the disturbance torque is labeled $W(s)$. Let the sensor be $-h$ rather than -1.

(a) Use the proportional controller

$$D(s) = k_p \tag{4.78}$$

and determine the system type and steady-state error properties with respect to disturbance inputs.

(b) Let the control be PI as given by

$$D(s) = k_p + \frac{k_I}{s} \tag{4.79}$$

and determine the system type and the steady-state error properties for disturbance inputs.

Solution. (a) The closed-loop transfer function from W to E (where $R = 0$) is

$$\begin{aligned} T_w(s) &= \frac{-B}{s(\tau s + 1) + Ak_p h} \\ &= s^0 T_{o,w}, \\ n &= 0, \\ K_{o,w} &= \frac{-Ak_p h}{B}. \end{aligned}$$

Applying Eq. (4.40) we see that the system is Type 0 and the steady-state error to a unit step torque input is $e_{ss} = {}^{-B}/_{Ak_{ph}}$. From the earlier section, this system is seen to be Type 1 for reference inputs and illustrates that system type can be different for different inputs to the same system. However, in this case the system is Type 0 for reference inputs.

(b) If the controller is PI, the disturbance error transfer function is

$$T_w(s) = \frac{-Bs}{s^2(\tau s + 1) + (k_p s + k_I)Ah}, \tag{4.80}$$

$$n = 1, \tag{4.81}$$

$$K_{n,w} = \frac{Ak_I h}{-B}, \tag{4.82}$$

and therefore the system is Type 1 and the error to a unit ramp disturbance input in this case will be

$$e_{ss} = \frac{-B}{Ak_I h}. \tag{4.83}$$

EXAMPLE 4.8

Satellite Attitude Control

Consider the model of a satellite attitude control system shown in Fig. 4.10 (a) where

J = moment of inertia,

W = disturbance torque,

K = sensor and reference gain,

$D(s)$ = the compensator.

With equal input filter and sensor scale factors, the system with PD control can be redrawn with unity feedback as in Fig. 4.10(b) and with PID control drawn as in Fig. 4.10(c). Assume that the control results in a stable system and determine the system types and error responses to disturbances of the control system for

(a) System Fig. 4.10(b) Proportional plus derivative control where $D(s) = k_P + k_D s$

(b) System Fig. 4.10(c) Proportional plus integral plus derivative control where $D = k_p + {}^{k_I}/_{s} + k_D s$.[5]

Solution. (a) We see from inspection of Fig. 4.10(b) that with two poles at the origin in the plant, the system is Type 2 with respect to reference inputs. The transfer function from disturbance to error is

$$T_w(s) = \frac{1}{Js^2 + k_D s + k_p} \tag{4.84}$$

$$= T_{o,w}(s) \tag{4.85}$$

for which $n = 0$ and $K_{o,w} = k_p$. The system is Type 0 and the error to a unit disturbance step is ${}^{1}/_{k_p}$.

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[5] Notice that these controller transfer functions have more zeros than poles and are therefore not practical. In practice, the derivative term would have a high-frequency pole, which has been omitted for simplicity in these examples.