ESD5 – Fall 2024
Problem Set 7 – Solutions

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Problem 1 – From the Analog to the Discrete World

To avoid oversampling, we should carefully consider all the possible cases of a_x and b_x. A complete solution considering those is given below. However, one may argue that since the problem does not specify that oversampling is a problem, we could just assume the worst-case conditions of $a_x = 8$ kHz and $b_x = 16$ kHz, which solution can also be considered right. We denote the sampling frequency as f_s and the sampling period as T_s. For simplicity, note that $T_s = \frac{1}{f_s}$; which is straightforward to be obtained.

(a) $x(t)$?

$$f_s = \begin{cases} 2a_x, & \text{if } a_x \geq 4000, \\ 8000, & \text{o/w.} \end{cases}$$

(b) $y(t)$?

$$f_s = \begin{cases} 2b_y, & \text{if } b_y \geq a_x, \\ 2a_x, & \text{o/w.} \end{cases}$$

(c) $x(t) + y(t)$?

$$f_s = \begin{cases} 2b_y, & \text{if } b_y \geq a_x \text{ and } a_x \geq 4000, \\ 2a_x, & \text{if } a_x \geq b_y \text{ and } b_y \geq 4000, \\ 8000, & \text{o/w.} \end{cases}$$

(d) $x(t)y(t)$? By ignoring the amplitudes, that is, $A_x = A_y = 1$, we have that $x(t)y(t)$ can be written as (use trigonometric identities):

$$
\begin{aligned}
x(t)y(t) = \frac{1}{2}\Big(&\cos(2\pi(a_x - b_y)t) + \cos(2\pi(a_x + b_y)t) + \sin(4\pi a_x t) \\
&+ \cos(2\pi(4000 - a_x)t) - \cos(2\pi(4000 + a_x)t) \\
&+ \sin(2\pi(4000 - b_y)t) + \sin(2\pi(4000 + b_y)t)\Big).
\end{aligned}
$$

Due to the periodicity of trigonometric functions, it turns out that we can only pay attention to the highest components: $a_x + b_y$, $a_x + 4000$, and $b_y + 4000$. In this case, it becomes easier to visualize graphically the regions, as can be seen in the figure below.

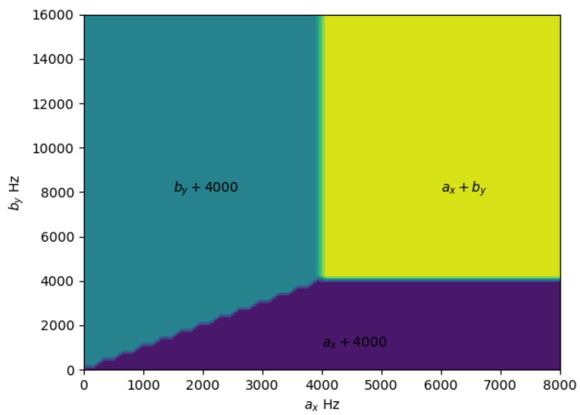

Figure 1: Decision regions for (d). The sampling frequency f_s is twice the frequency of the decision region. For example. $f_s = 2(b_y + 4000)$ in the green area.

OBS: The solutions above are based on the fact that the Fourier transform is linear.

Let us focus now on $x(t)$. Assume $A_x = 1$ and $a_x = 8$ kHz. Imagine that we observe the signal for 1 ms and we start to observe the signal at $T = 0$. Answer the following:

(e) The Nyquist sampling rate is $f_s = 16$ kHz and the sampling period is $T_s = \frac{1}{16}$ ms. Therefore, the number of samples can be computed as:

$$N = \left\lceil \frac{1 \text{ ms}}{1/16 \text{ ms}} \right\rceil = 16 \text{ samples}.$$

(f) The figure below shows the function and its 16 sampled points. The 16 values obtained were:

$[1., 0.02, 0.71, -1.76, 1.08, 0.37, -0.28, -0.85, 0.64, 0.9, -1.37, 0.26, 0.14, 1.12, -1.97, 1.]^T$

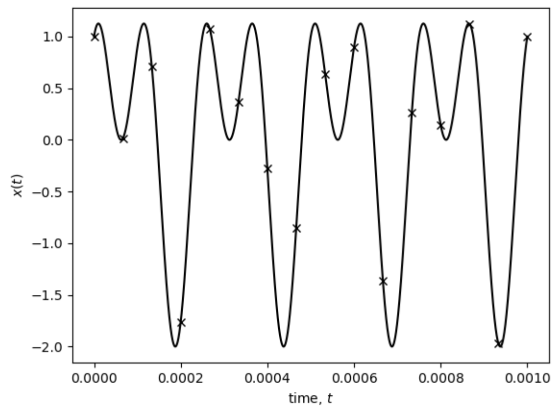

Figure 2: Function over time and its sampled values according to the Nyquist sampling theorem.

(g) One way to do so is the following:

- Since we have 2 bits, we can just represent 4 voltage (magnitude) levels. These are associated with the following: $'00'$, $'01'$, $'10'$, $'11'$.
- Find the minimum and maximum values of the function $x(t)$: $x_{\min}(t) = -2$ and $x_{\max}(t) = 1.12$. Associate those as follows: $'00' \leftarrow x_{\min}(t)$ and $'11' \leftarrow x_{\max}(t)$.
- Find the remaining points by finding the resolution:

$$\Delta = \frac{1.12 - (-2)}{4 - 1} = 1.04.$$

- Now, $'01' \leftarrow -0.96 = x_{\min} + \Delta$ and $'10' \leftarrow 0.08 = x_{\min} + 2\Delta$.
- The last step is: go through all 16 sampled magnitudes, for each one find which one of the quantization levels is closer to it. Substitute the original value with the quantized one. An illustrative figure of this process is shown below.

(optional)

$$\text{MSE} = \frac{1}{N}\sum_{n=1}^{N} e_n^2 = 0.0074.$$

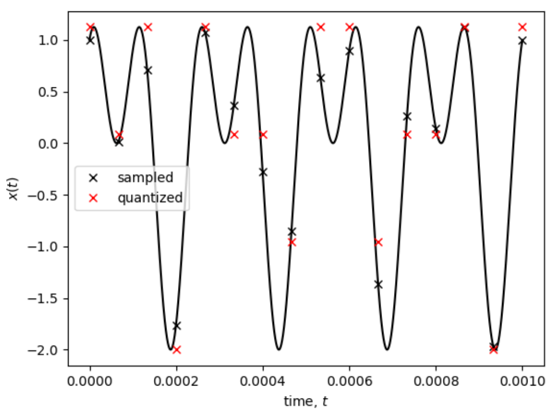

Figure 3: (g) – Quantized samples.

Problem 2 – Conventional AM modulation

The classical AM occurs as follows:

1. Multiply the baseband signal by a sinusoid carrier signal;
2. Add the carrier;

These steps can be summarized as:

$$s(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t).$$

By substituting the values, we have:

$$s(t) = 10\,[1 + 0.5(0.5\cos(2\pi 100t) + \cos(2\pi 200t))]\cos(2\pi 2000t).$$

The spectrum is given by

$$S(f) = \frac{A_c}{2}[\delta(f - f_c) + \delta(f + f_c)] + \frac{k_a A_c}{2}[M(f - f_c) + M(f + f_c)]$$

The spectrum of the AM baseband modulating signal is:

$$M(f) = 0.25[\delta(f - 100) + \delta(f + 100)] + 0.5[\delta(f - 200) + \delta(f + 200)].$$

By substituting all values, we have:

$$\begin{aligned} S(f) = {} & 5[\delta(f - 2000) + \delta(f + 2000)] + \\ & + 2.5\{0.25[\delta(f - 2100) + \delta(f - 1900)] + 0.5[\delta(f - 2200) + \delta(f + 1800)]\} \\ & + 2.5\{0.25[\delta(f + 1900) + \delta(f + 2100)] + 0.5[\delta(f - 1800) + \delta(f + 2200)]\}. \end{aligned}$$

Figures showing the AM modulated signal over time and frequency are available below.

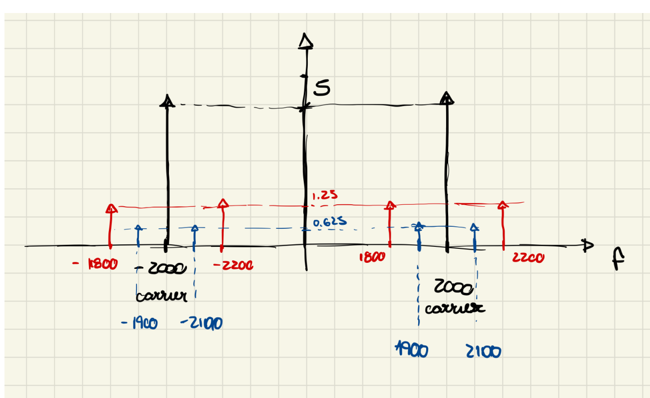

Figure 4: Expected theoretical spectrum.

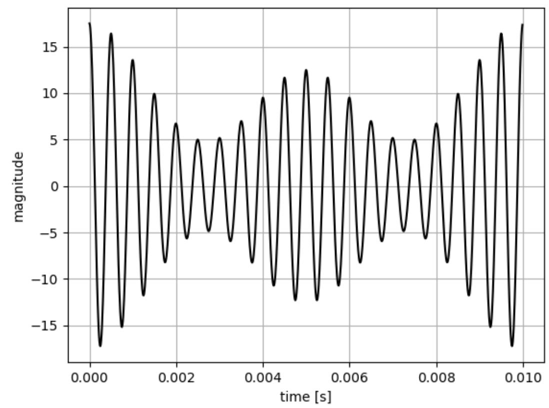

Figure 5: AM signal over time. For the fans of Artic Monkeys, hope you can appreciate the album cover now :)

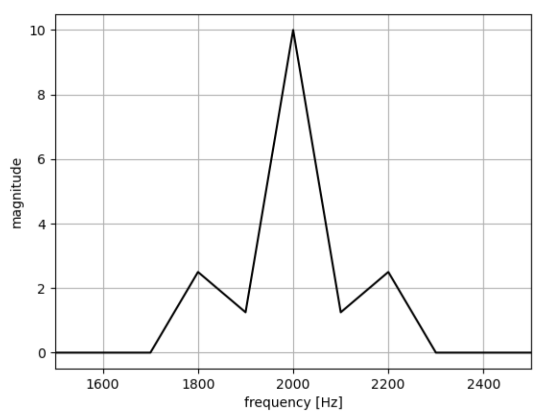

Figure 6: Spectrum obtained numerically via FFT. The power normalization takes into account that there is no left FFT plot, this is why is twice the expected plot.

Problem 3 – 4-PAM

(a) $'00'$, $'01'$, $'10'$, $'11'$.

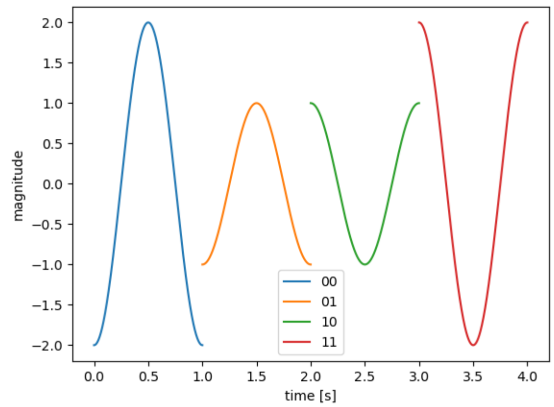

Figure 7: (b) – Signal waveforms.

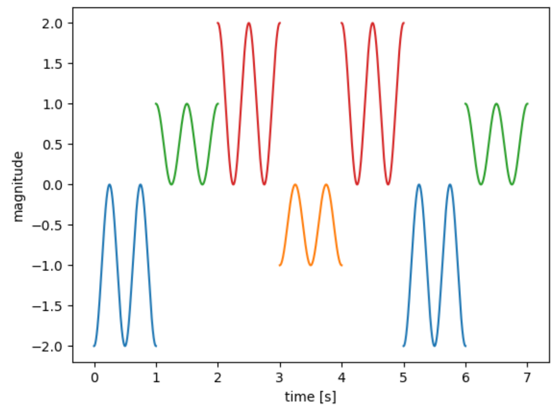

Figure 8: (c) – Sending message "00101101110010". Remember to multiply the signal waveforms by the carrier cosine.

(d) First, we need to determine the energy of the used pulse:

$$\mathcal{E}_g = \int_0^T g_T^2(t)dt = \int_0^T \cos^2(2\pi t)dt = \frac{4\pi T + \sin(4\pi T)}{8\pi}.$$

Since $T = 1$, we have $\mathcal{E}_g = 0.5$. Now, we need to obtain the basis function of the PAM, which process can be found on pg. 374 - Proakis, J. G., Salehi, M. (2001).

Communication Systems Engineering. Upper Saddle River, NJ, USA: Prentice-Hall. ISBN: 0130617938. The basis function $\psi(t)$ is

$$\psi(t) = \frac{2}{\sqrt{\mathcal{E}_g}} g_T(t).$$

Thus, at the receiver, we need to solve the following:

$$\int_0^T r(t)\psi(t)dt = A_m \frac{2}{\sqrt{\mathcal{E}_g}} \int_0^T g_T^2(t)\cos^2(2\pi t)dt + \frac{2}{\sqrt{\mathcal{E}_g}} \int_0^T n(t) g_T(t)dt.$$

We divide the solution of the above integral into two: the signal part and the noise part. The signal part yields:

$$A_m \frac{2}{\sqrt{\mathcal{E}_g}} \int_0^T g_T^2(t)\cos^2(2\pi t)dt = A_m \frac{2}{\sqrt{\mathcal{E}_g}} \int_0^T \cos^4(2\pi t)dt \approx A_m 0.5303.$$

The noise part simply results in $n = 0.5$, which is specified by the exercise. Therefore, we have the following received energy:

$$\int_0^T r(t)\psi(t)dt \approx A_m 0.5303 + 0.5$$

Since the true signal sent was $'00'$, the received value is approximately $-2 \cdot 0.5303 + 0.5 = -1.5606$.

(e) Based on the maximum-a-posterior principle, the received demodulated signals would be: $[-1.20, -2.40, 1.49, 2.10] \rightarrow [01, 00, 10, 11]$ (just find the closest symbol using the TRUE mapping $[-2, -1, 1, 2] \rightarrow [00, 01, 10, 11]$). Assuming that the sequence of true transmitted symbols was: $[00, 00, 11, 11]$, the percentage of error was 50%.