State Space Methods

Lecture 1: State space models

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State space models

A linear third order system in continuous time with two inputs and two outputs has a state space model of the following form:

$$
\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_{11}u_1 + b_{12}u_2 \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_{21}u_1 + b_{22}u_2 \\
\dot{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + b_{31}u_1 + b_{32}u_2 \\
\hline
y_1 &= c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + d_{11}u_1 + d_{12}u_2 \\
y_2 &= c_{21}x_1 + c_{22}x_2 + c_{23}x_3 + d_{21}u_1 + d_{22}u_2
\end{aligned}
$$

where x_1, x_2, x_3 are called the *states*, u_1, u_2 are called the *inputs*, and y_1, y_2 are called the *outputs*.

State space models

In matrix form, a continuous time state space model can be written as:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

where:

$$
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad , \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad , \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
$$

$$
A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad , \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}
$$

$$
C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix} \quad , \quad D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}
$$

State space models

In matrix form, a continuous time state space model can be written as:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

Similarly, a discrete time state space model can be written as:

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k) + Du(k)
\end{aligned}
$$

Choosing state variables

For a physical system, the number of states required is typically equal to the number of 'energy storages', and a possible choice of state variables is often those variables, that 'represent' energy storage.

Choosing state variables

Linear component	Recommended variable
capacitor	voltage
electrical coil	current
spring	length
mass (kinetic)	velocity
mass (potential)	elevation
inertia wheel	angular velocity
plane spring	winding angle
heat storage	temperature
gas accumulator	pressure

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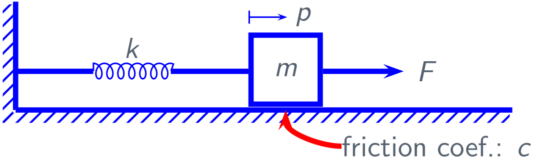
Example: mass-spring-damper



The force F is considered as input, and the mass velocity v is considered as output of this system.
The system is of second order, since it has one mass which can contain both kinetic and potential energy.

Example: mass-spring-damper

A possible selection of states are the position p and the velocity v. The derivative of v is given by Newton's second law:

$$m\dot{v} = -k \cdot p - c \cdot v + F \quad \Longrightarrow$$
$$\dot{v} = -\frac{k}{m} \cdot p - \frac{c}{m} \cdot v + \frac{1}{m} \cdot F$$

The derivative of p is simply given by:

$$\dot{p} = v$$

Example: mass-spring-damper

Thus, we have the following state space model:

$$\begin{aligned} \begin{pmatrix} \dot{p} \\ v \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} p \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} F \\ v &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ v \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} F \end{aligned}$$

which is indeed of the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

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State space model → transfer fct.

Taking Laplace transforms of the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

yields

$$\begin{aligned} sx(s) &= Ax(s) + Bu(s) \\ y(s) &= Cx(s) + Du(s) \end{aligned}$$

rearranging, we obtain:

$$\begin{aligned} (sI - A)\,x(s) &= Bu(s) \\ y(s) &= Cx(s) + Du(s) \end{aligned}$$

Premultiplying with $(sI - A)^{-1}$ on either side of the system equation, results in

$$\begin{aligned} x(s) &= (sI - A)^{-1} Bu(s) \\ y(s) &= Cx(s) + Du(s) \end{aligned}$$

State space model → transfer fct.

Premultiplying with $(sI - A)^{-1}$ on either side of the system equation, results in

$$\begin{aligned} x(s) &= (sI - A)^{-1} Bu(s) \\ y(s) &= Cx(s) + Du(s) \end{aligned}$$

Finally, we obtain:

$$\begin{aligned} x(s) &= (sI - A)^{-1} Bu(s) \\ y(s) &= C\,(sI - A)^{-1} Bu(s) + Du(s) \end{aligned}$$

Consequently,

$$y(s) = G(s)u(s)\,, \quad \text{where:}$$

$$G(s) = C\,(sI - A)^{-1} B + D$$

Example: mass-spring-damper

For the spring-mass-damper system with $m = 1$, $c = 3$, $k = 2$, the state space representation is:

$$\begin{pmatrix} \dot{p} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} p \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F$$

$$v = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ v \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} F$$

Thus, the transfer function becomes:

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

Example: mass-spring-damper

Thus, the transfer function becomes:

$$
\begin{aligned}
G(s) &= C\,(sI - A)^{-1}\,B + D \\
&= \begin{pmatrix} 0 & 1 \end{pmatrix} \left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s & -1 \\ 2 & s+3 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{s^2 + 3s + 2} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{s}{s^2 + 3s + 2}
\end{aligned}
$$

Transfer fct. $\rightarrow$ state space model

Consider the transfer function $g(s) = \frac{1}{s^2+a_1 s+a_2}$. From the relationship

$$y(s) = \frac{1}{s^2 + a_1 s + a_2} u(s)$$

we infer

$$s^2 y(s) + a_1 s y(s) + a_2 y(s) = u(s)$$

Taking inverse Laplace transform, this becomes:

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_2 y(t) = u(t)$$

Transfer fct. → state space model

$$\ddot{y}(t) + a_1\dot{y}(t) + a_2 y(t) = u(t)$$

A possible choice of states is: $x_1 = y$, $x_2 = \dot{y}$. With this choice, the system equations become:

$$\begin{aligned}\dot{x}_1 &= \dot{y} = x_2 \\ \dot{x}_2 &= \ddot{y} = -a_1\dot{y} - a_2 y + u = -a_2 x_1 - a_1 x_2 + u\end{aligned}$$

In matrix form, we obtain:

$$\begin{aligned}\begin{pmatrix}\dot{x}_1 \\ \dot{x}_2\end{pmatrix} &= \begin{pmatrix}0 & 1 \\ -a_2 & -a_1\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix} + \begin{pmatrix}0 \\ 1\end{pmatrix} u \\ y &= \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix} + \begin{pmatrix}0\end{pmatrix} u\end{aligned}$$

or

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

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Poles of state space models

With

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

we have that:

$$G(s) \to \infty \text{ for } s \to p \quad \Rightarrow \quad \det\left(pI - A\right) = 0$$

Hence,

$$p \text{ is a pole for } G(s) \;\Rightarrow\; p \text{ is an eigenvalue for } A$$

Example: mass-spring-damper

For the mass-spring-damper system, the A matrix was:

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

which has the characteristic polynomial:

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{vmatrix} \\ &= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) \end{aligned}$$

Thus, the system has poles in $\{-1, -2\}$.

Zeros of state space models

With

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

we have that:

$$\begin{aligned}
G(z)u = 0 \quad &\Rightarrow \quad C\,(zI - A)^{-1}\,Bu + Du = 0 \\
\Rightarrow \quad & C\xi + Du = 0\,,\ \xi = (zI - A)^{-1}\,Bu \\
\Rightarrow \quad & C\xi + Du = 0\,,\ (A - zI)\,\xi + Bu = 0 \\
\Rightarrow \quad & \begin{pmatrix} A - zI & B \\ C & D \end{pmatrix} \begin{pmatrix} \xi \\ u \end{pmatrix} = 0
\end{aligned}$$

Thus, z is a zero for $G(s)$ $\Rightarrow$

$$\begin{pmatrix} A - zI & B \\ C & D \end{pmatrix} \quad \text{does not have full column rank}$$

Example: mass-spring-damper

For the mass-spring-damper system, zeros must satisfy:

$$\begin{vmatrix} A - zI & B \\ C & D \end{vmatrix} = 0$$

or

$$\begin{vmatrix} -z & 1 & 0 \\ -2 & -3 - z & 1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -z & 1 \\ 0 & 1 \end{vmatrix} \cdot (-1) = z = 0$$

Hence, the system has a zero in the origin.

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State space transformations

State space representations are not unique!
Given one model:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

another model can be obtained by a non-singular transformation of the state vector:

$$x = T\xi\,, \quad \xi = T^{-1}x$$

State space transformations

Introducing this in the state space model, we obtain:

$$\begin{aligned} T\dot{\xi} &= AT\xi + Bu \\ y &= CT\xi + Du \end{aligned}$$

or, equivalently

$$\begin{aligned} \dot{\xi} &= T^{-1}AT\xi + T^{-1}Bu \\ y &= CT\xi + Du \end{aligned}$$

State space transformations

$$
\begin{aligned}
\dot{\xi} &= T^{-1}AT\xi + T^{-1}Bu \\
y &= CT\xi + Du
\end{aligned}
$$

Thus, a new state space model of the form

$$
\begin{aligned}
\dot{\xi} &= \tilde{A}\xi + \tilde{B}u \\
y &= \tilde{C}\xi + \tilde{D}u
\end{aligned}
$$

where

$\tilde{A} = T^{-1}AT$	$\tilde{B} = T^{-1}B$
$\tilde{C} = CT$	$\tilde{D} = D$

has been obtained.

Example: mass-spring-damper

For the mass-spring-damper system, we change basis using the following transformation matrix:

$$T = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

This gives the following new state space representation:

$$\begin{aligned} \dot{\xi} &= \tilde{A}\xi + \tilde{B}u \\ y &= \tilde{C}\xi + \tilde{D}u \end{aligned}$$

with

Example: mass-spring-damper

$$
\begin{aligned}
\tilde{A} = T^{-1}AT &= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \\
\tilde{B} = T^{-1}B &= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
\tilde{C} = CT &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -1 & -2 \end{pmatrix} \\
\tilde{D} = D &= 0
\end{aligned}
$$

Example: mass-spring-damper

Transfer matrix:

$$\begin{aligned}
\tilde{G}(s) &= \tilde{C}\left(sI - \tilde{A}\right)^{-1}\tilde{B} + \tilde{D} \\
&= \begin{pmatrix} -1 & -2 \end{pmatrix} \begin{pmatrix} s+1 & 0 \\ 0 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0 \\
&= -\frac{1}{s+1} + 2 \cdot \frac{1}{s+2} = \frac{-(s+2) + 2(s+1)}{(s+1)(s+2)} \\
&= \frac{s}{s^2 + 3s + 2} \\
&= G(s)
\end{aligned}$$