Digital Signal Processing
ESD-5 & IV-5/elektro E24

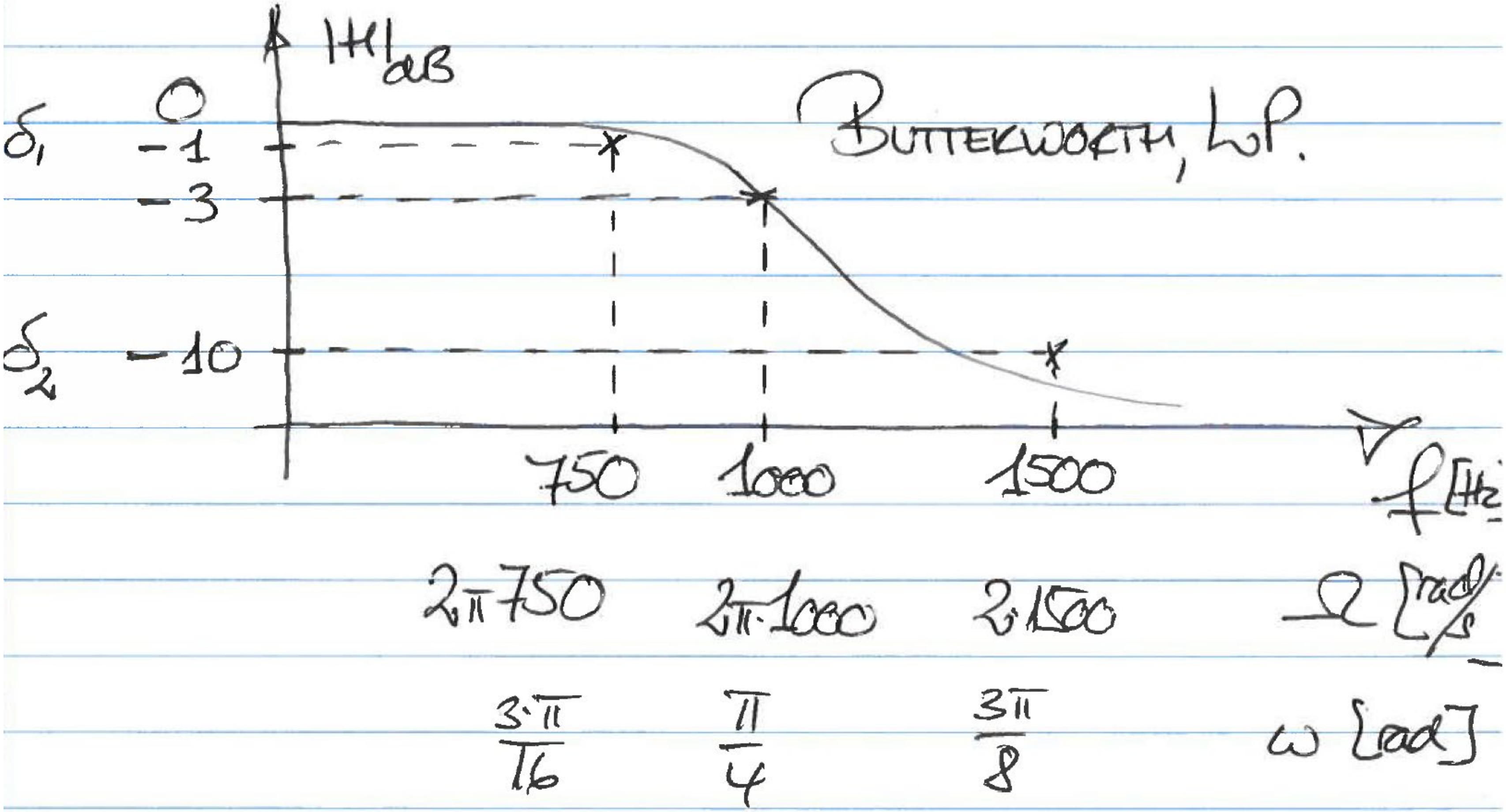
5. Filters with Finite Impulse Response (FIR) – Linear Phase, and the Window Method

Assoc. Prof. Peter Koch, AAU

Agenda for Today…

- Brief recap of the Bilinear Transformation
- Systems with General Linear Phase Response
- Filters with Finite Impulse Response (FIR)
- The Window Design Method

Brief recap of IIR design, Bilinear Transformation



$$\boxed{\omega = T \cdot \Omega}$$

First we design the Analog Prototype, Butterworth LP

$\delta_1 = -1\ dB = 0.89125$
$\delta_2 = -10\ dB = 0.31623$

$$|H_C(j\Omega)|^2 = \frac{1}{1 + ({}^{j\Omega}/_{j\Omega_c})^{2N}}$$

From this we can derive two equations in two unknowns;

$$\delta_1^2 \leq \frac{1}{1 + ({}^{\Omega_1}/_{\Omega_c})^{2N_1}} \qquad \delta_2^2 \geq \frac{1}{1 + ({}^{\Omega_2}/_{\Omega_c})^{2N_2}}$$

Since we have to use the Bilinear Transform to do $H(s) \curvearrowright H(z)$, we must insert the pre-warped values for Ω_c, Ω_1, and Ω_2.

$$\Omega' = \frac{2}{T}\tan(\frac{\omega}{2})$$

…and that leads to $N = 3$.

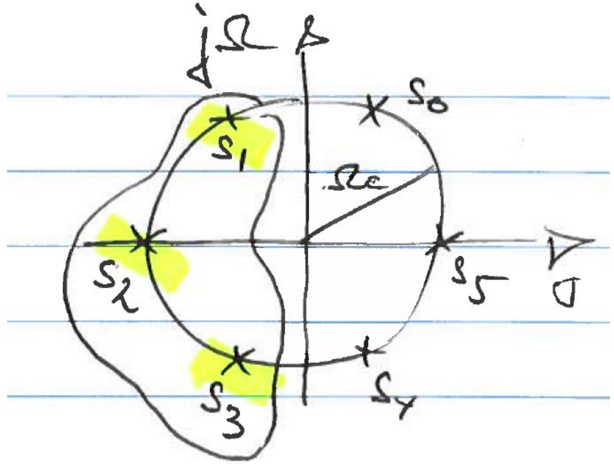
AALBORG UNIVERSITY
DENMARK

The poles in $H(s)$

$$s_1 = \Omega_c(-\frac{1}{2} + j\frac{1}{2}\sqrt{3})$$

$$s_2 = -\Omega_c$$

$$s_3 = \Omega_c(-\frac{1}{2} - j\frac{1}{2}\sqrt{3})$$



$$H(s) = \frac{G}{\prod_{k=1}^{3}(s - s_k)}$$

DC gain 0 dB; $|H(s)| = 1$ for $s = 0 \Rightarrow G = \Omega^3$

$$H(s) = \frac{\Omega^3}{(s + \Omega_c)(s^2 + \Omega_c s + {\Omega_c}^2)}$$

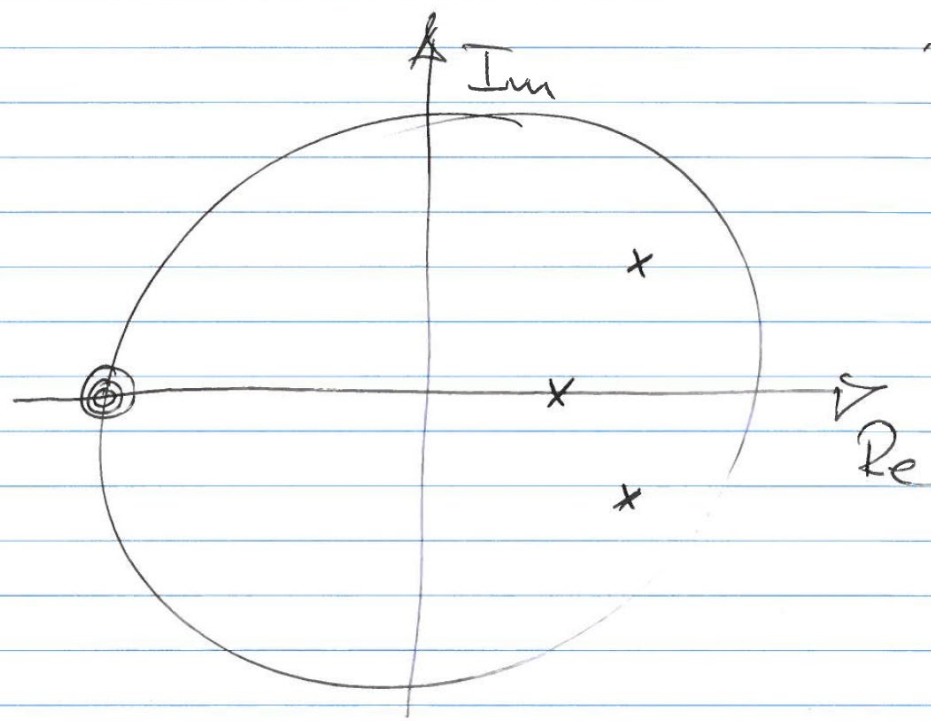
…and now we can apply the Bilinear Transform with prewarped cuf-off freq.

$$s = \frac{2}{T} \cdot \frac{z-1}{z+1} \quad \text{and} \quad \Omega_c' = \frac{2}{T}\tan(\frac{\omega_c}{2})$$

The digital filter $H(z)$

$$H(z) = 0.0317 \cdot \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{1 - 1.4590z^{-1} + 0.9104z^{-2} - 0.1978z^{-3}}$$

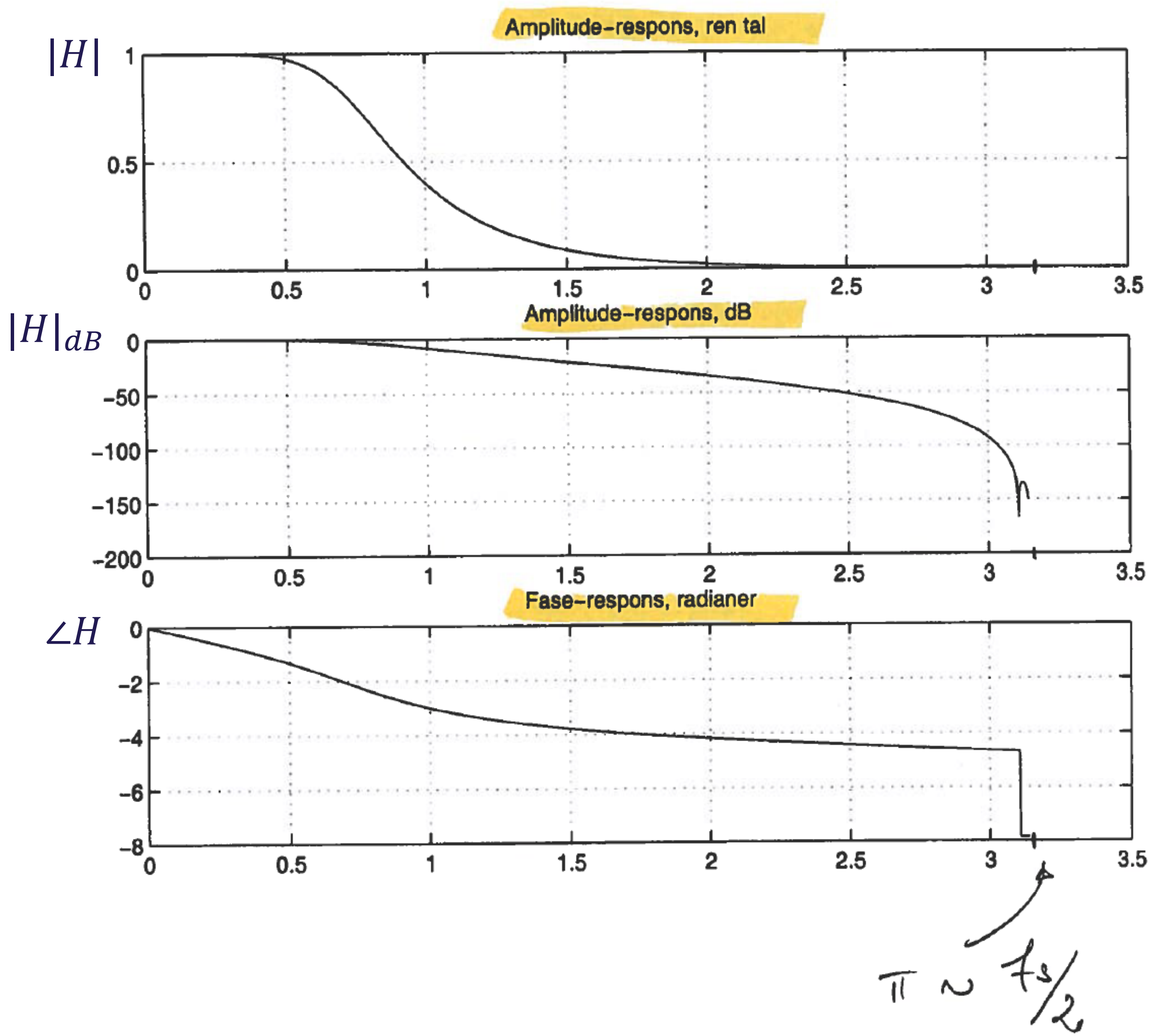
This is a 3'rd order IIR filter with 3 poles and 3 zeros (all in $z = -1$)



$z_1 = 0.4140$

$(z_2, z_3) = 0,5225 \pm j0,4525$

The Amplitude- and the Phase Response



The Difference Equation

$$H(z) = \frac{Y(z)}{X(z)} = 0.0317 \cdot \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{1 - 1.4590z^{-1} + 0.9104z^{-2} - 0.1978z^{-3}}$$

$$Y(z)(1 - 1.4590z^{-1} + 0.9104z^{-2} - 0.1978z^{-3}) = 0.0317 \cdot X(z)(1 + 3z^{-1} + 3z^{-2} + z^{-3})$$

$$Y(z) = Y(z)(1.4590z^{-1} + 0.9104z^{-2} - 0.1978z^{-3}) + 0.0317 \cdot X(z)(1 + 3z^{-1} + 3z^{-2} + z^{-3})$$

$$\mathcal{Z}^{-1}\{Y(z)\} = y[n] = (1.4590y[n-1] + 0.9104y[n-2] - 0.1978y[n-3]) + 0.0317 \cdot (x[n] + 3x[n-1] + 3x[n-2] + x[n-3])$$

$$x[n] \longrightarrow \boxed{\text{Difference equation}} \longrightarrow y[n]$$

So, for every new input sample $x[n]$, we need to conduct

- 7 multiplications
- 6 additions
- Updates of internal variables

IIR – in conclusion…

Good news

- Very few arithmetic operations
- Approximation to a well-known/understood analog filter function

Bad news

- Potentially unstable
- The phase response has to be accepted ”as is”

Might be that we can find an alternative which circumvent the bad news…??

Yes we can – and the answer is Finite Impulse Response Filters

The Finite Impulse Response Filter

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{l=1}^{N} a_l z^{-l}} = \sum_{k=0}^{M} b_k z^{-k} \quad \text{for} \quad a_l = 0 \quad \mathrm{l} = 1..\mathrm{N}$$

$$Y(z) = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \quad \Rightarrow \quad h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

The Impulse Response has finite length, M+1

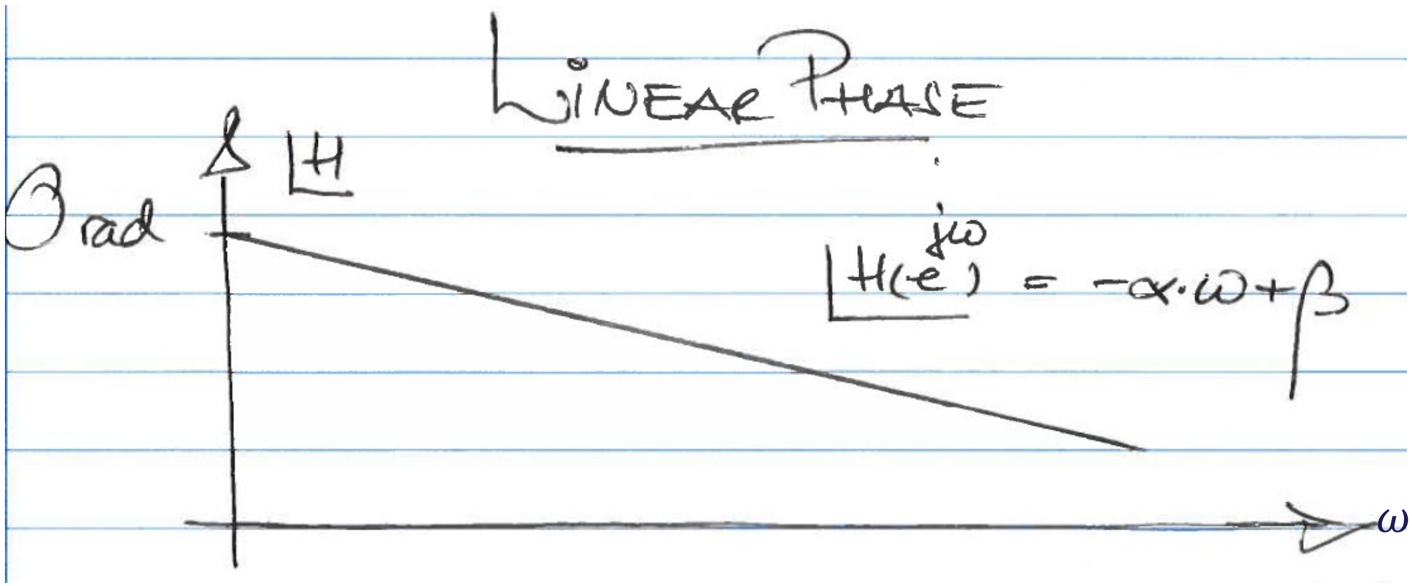
$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{\sum_{k=0}^{M} b_k z^{M-k}}{z^M}$$

We conclude that $H(z)$ has all its poles located in $z = 0$.

This means that an FIR filter is ALWAYS stable

AALBORG UNIVERSITY
DENMARK

Moreover, the FIR filter CAN be designed to have linear phase response…

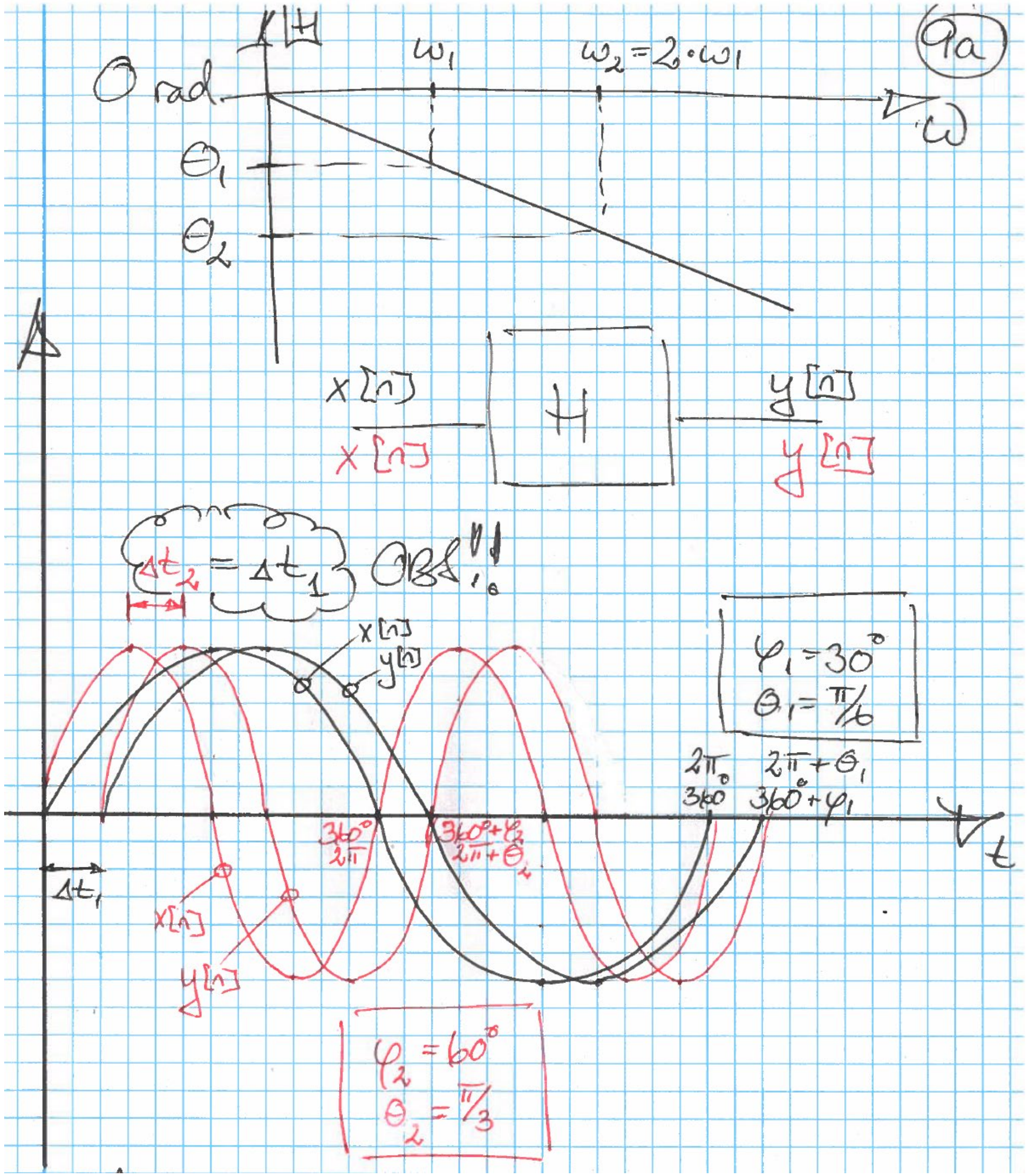


A system with a linear phase response is said to have a constant Group Delay..!

$$G(e^{j\omega}) \triangleq -\frac{d}{dt}\angle H(e^{j\omega}) \ [s]$$

AALBORG UNIVERSITY DENMARK

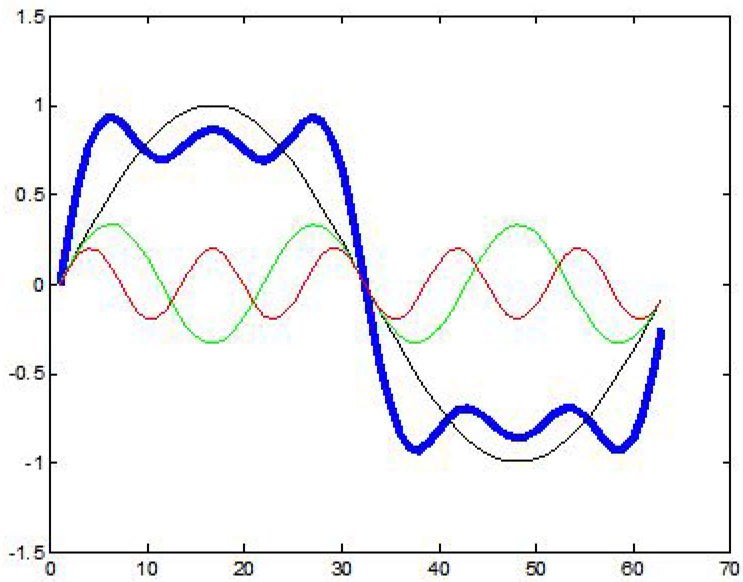
A system with linear phase response (i.e., constant group delay) shows the exact same delay to all sinusoids in the signal, no matter their frequency…

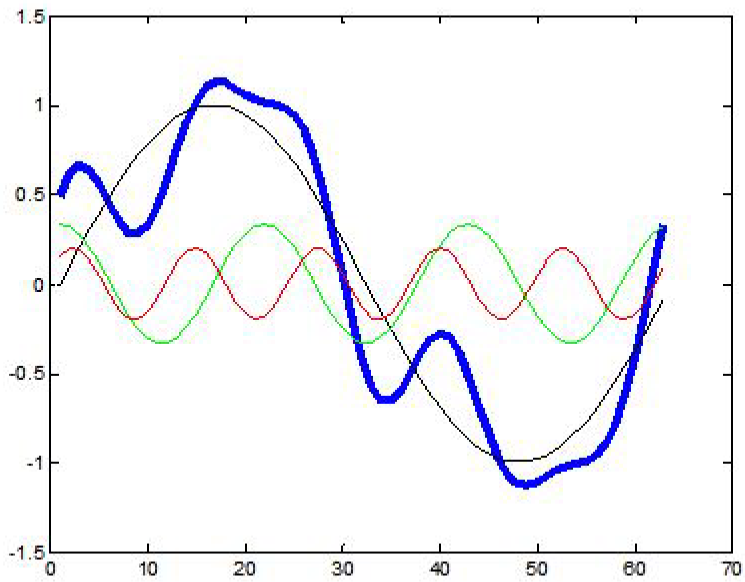


A system with linear phase response (constant group delay) is most wanted when we have to modify the spectral content of a signal, AND at the same time maintain the waveform ”as good as possible”.

Example: A signal is the sum of three sinusoids having frequencies *f*, 3*f* and 5*f*.

If this signal is the input to an LTI system with linear phase, then these frequencies will see the same delay, and thus at the output we will also observe a signal which has *approximately* the same waveform as the signal on the input. The three components will potentially be scaled differently in amplitude – thus ”approximately”…





If the same signal is fed into an LTI system with non-linear phase, i.e., frequency dependent group delay, then the components will be delayed differently, and thus the waveform cannot be exactly maintained.

Frequency Response for Linear Phase Systems

$H(e^{j\omega}) = H(z)$ for $z = e^{j\omega}$

$$\boxed{H(e^{j\omega}) = A(e^{j\omega}) \cdot e^{-j(\alpha\omega - \beta)}}$$

Let's assume our systems has this Frequency Response, where $A(e^{j\omega})$ is a real function.

Amplitude Response

$$|H(e^{j\omega})| = |A(e^{j\omega}) \cdot e^{-j(\alpha\omega-\beta)}| = |A(e^{j\omega})| \cdot |e^{-j(\alpha\omega-\beta)}| = |A(e^{j\omega})|$$

Phase Response

$$\angle H(e^{j\omega}) = \arg\{A(e^{j\omega}) \cdot [\cos(\alpha\omega - \beta) - j\sin(\alpha\omega - \beta)]$$

$$\angle H(e^{j\omega}) = \arg\{A(e^{j\omega})\} + \arctan\left(\frac{-\sin(\alpha\omega - \beta)}{\cos(\alpha\omega - \beta)}\right)$$

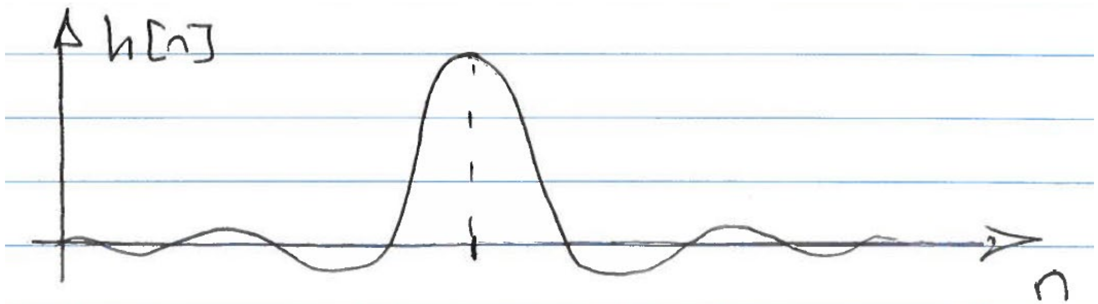
$$\angle H(e^{j\omega}) = \begin{cases} -(\alpha\omega - \beta) & A(e^{j\omega}) \geq 0 \\ -(\alpha\omega - \beta) - \pi & A(e^{j\omega}) < 0 \end{cases}$$

Equations for straight lines…!

AALBORG UNIVERSITY
DENMARK

Symmetric Impulse Response

We will now show that a system, which has a **Symmetric Impulse Response**, has a **Linear Phase Response**



1) A system with general linear phase

$$H(e^{j\omega}) = A(e^{j\omega})\cos(\beta - \alpha\omega) + jA(e^{j\omega})\sin(\beta - \alpha\omega)$$

Equ. 128, p.341

2) A system with general transfer function

$H(e^{j\omega}) = H(z)$ for $z = e^{j\omega}$ and thus for a real sequence $h[n]$ we have;

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] \cdot e^{-j\omega n} = \sum_{n=-\infty}^{\infty} h[n]\cos(\omega n) - j\sum_{n=-\infty}^{\infty} h[n]\sin(\omega n)$$

Equ. 129, p.341

AALBORG UNIVERSITY
DENMARK

Symmetric Impulse Response

For 1) and 2) we now calculate the argument, **and** derive $\tan(\cdot)$ on both sides;

1) $$\tan\{\angle H(e^{j\omega})\} = \tan\{\arctan(\frac{\sin(\beta - \alpha\omega)}{\cos(\beta - \alpha\omega)})\} = \frac{\sin(\beta - \alpha\omega)}{\cos(\beta - \alpha\omega)}$$

2) $$\tan\{\angle H(e^{j\omega})\} = \tan\{\arctan(\frac{-\sum h[n]\sin(\omega n)}{\sum h[n]\cos(\omega n)})\} = \frac{-\sum h[n]\sin(\omega n)}{\sum h[n]\cos(\omega n)}$$

We see here that both left-hand sides are identical, i.e.,

$$\frac{\sin(\beta - \alpha\omega)}{\cos(\beta - \alpha\omega)} = \frac{-\sum h[n]\sin(\omega n)}{\sum h[n]\cos(\omega n)} \Rightarrow$$

$$\sum_{n=-\infty}^{\infty} h[n]\cos(\omega n)\sin(\beta - \alpha\omega) = -\sum_{n=\infty}^{\infty} h[n]\sin(\omega n)\cos(\beta - \alpha\omega)$$

To this we apply the trigonometric identity; $\sin(x)\cos(y) = \frac{1}{2}\{\sin(x-y) + \sin(x+y)\}$

$$\sum_{n=-\infty}^{\infty} h[n]\sin(\beta + (n-\alpha)\omega) = 0$$ which should hold for $\forall\ \omega$. Equ. 130, p.341

Symmetric Impulse Response

$$\sum_{n=-\infty}^{\infty} h[n]\sin(\beta + (n-\alpha)\omega) = 0$$ which should hold for $\forall\ \omega$.

Conclusion: This expression is a necessary condition for Linear Phase Response in a system with a general transfer function.

Our next task is to find values for α, β, and $h[n]$ such that this condition holds $\forall\ \omega$.

Without proof, the following values fulfills the condition;

- $h[n] = h[2\alpha - n] = h[M - n]$, where
- $\alpha = \frac{M}{2} \Rightarrow 2\alpha = M,\quad M$ is an integer, even or odd.
- $\beta = \begin{cases} 0 \\ \pi \end{cases}$ or $\beta = \begin{cases} \pi/2 \\ 3\pi/2 \end{cases}$

AALBORG UNIVERSITY
DENMARK

Symmetric Impulse Response

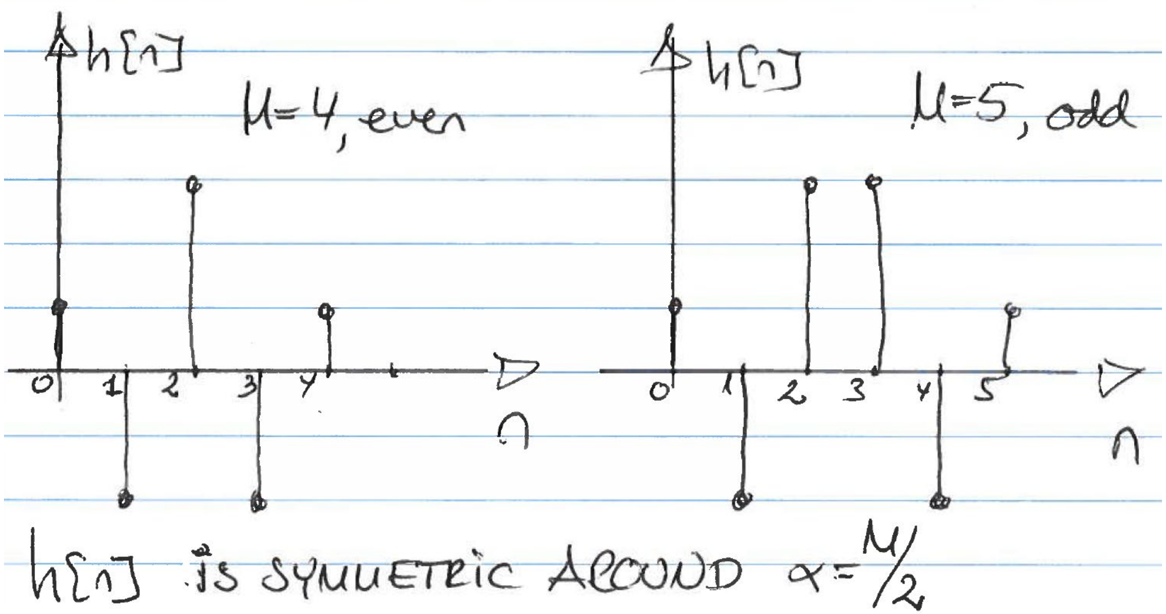
Let's now assume that the system $h[n]$ is causal, then

$$\sum_{n=0}^{\infty} h[n]\sin(\beta + (n-\alpha)\omega) = 0$$ which should hold for $\forall\ \omega$.

If, at the same time, the requirement for symmetric impulse response should hold, then

$h[n] = 0$ outside the interval $0 \le n \le M$ where $n = \frac{M}{2} = \alpha$ is the symmetry point.

From this we conclude, that the system $h[n]$ is Finite Impulse Response, FIR…!!!



Finite Symmetric Impulse Response, FIR

Linear Phase Response is achievable for Symmetric Impulse Response,

$$h[n] = h[M - n]$$

…and also for Anti-Symmetric Impulse Response,

$$h[n] = -h[M - n]$$

Combined with M being either *even* or *odd*, provides four possible FIR filter types;

$h[n]$ \ M	Even	Odd
Symmetric $\beta = 0$ or π	I	II
Anti-symmetric $\beta = {}^{\pi}/_{2}$ or ${}^{3\pi}/_{2}$	III	IV

The Frequency Response

For a Type I FIR-filter, the impulse response is symmetric and M is even, i.e., the symmetry point $\alpha = \frac{M}{2}$ is an integer. If we assume that $\beta = 0$, then the frequency response $H(e^{j\omega}) = A(e^{j\omega}) \cdot e^{-j(\alpha\omega-\beta)} = H(e^{j\omega}) = A(e^{j\omega}) \cdot e^{-j\alpha\omega}$

For this case, it can be shown, that $H(e^{j\omega})$ has the form;

$$H(e^{j\omega}) = \{h[{}^{M}/_{2}] + 2\sum_{k=1}^{M/2} h[{}^{M}/_{2} + k]\cos(\omega k)\} \cdot e^{-j\omega^{M}/_{2}}$$

Equ. 140a, p. 343

See next handwritten slide for the overall calculation…

GENEREL FREKVENSRESPONS FOR FIR SYSTEM $\quad H(e^{j\omega}) = \sum_{n=0}^{M} h[n] \cdot e^{-j\omega n}$

NY VARIABEL k: $n = M/2 + k \Rightarrow k = n - M/2$

$$H(e^{j\omega}) = \sum_{k=-M/2}^{M/2} h[M/2+k] \cdot e^{-j\omega(M/2+k)}$$

$\Downarrow$

$$H(e^{j\omega}) = e^{-j\omega M/2} \sum_{k=-M/2}^{M/2} h[M/2+k]\, e^{-j\omega k}$$

$\Downarrow$

$$H(e^{j\omega}) = e^{-j\omega M/2} \left\{ \sum_{k=-M/2}^{-1} h[M/2+k]\, e^{-j\omega k} + h[M/2] + \sum_{k=1}^{M/2} h[M/2+k] \cdot e^{-j\omega k} \right\}$$

$\Downarrow$

$$H(e^{j\omega}) = e^{-j\omega M/2} \left\{ \sum_{k=1}^{M/2} h[M/2+k] \cdot e^{j\omega k} + h[M/2] + \sum_{k=1}^{M/2} h[M/2+k]\, e^{-j\omega k} \right\}$$

Lovligt at vende summationsfortegn pga. symmetri i impuls-responsen.

$\Downarrow$

$$H(e^{j\omega}) = e^{-j\omega M/2} \left\{ h[M/2] + \sum_{k=1}^{M/2} h[M/2+k] \cdot \underbrace{\left(e^{j\omega k} + e^{-j\omega k}\right)}_{2\cos(\omega k) \text{ via Eulers identitet}} \right\}$$

$\Downarrow$

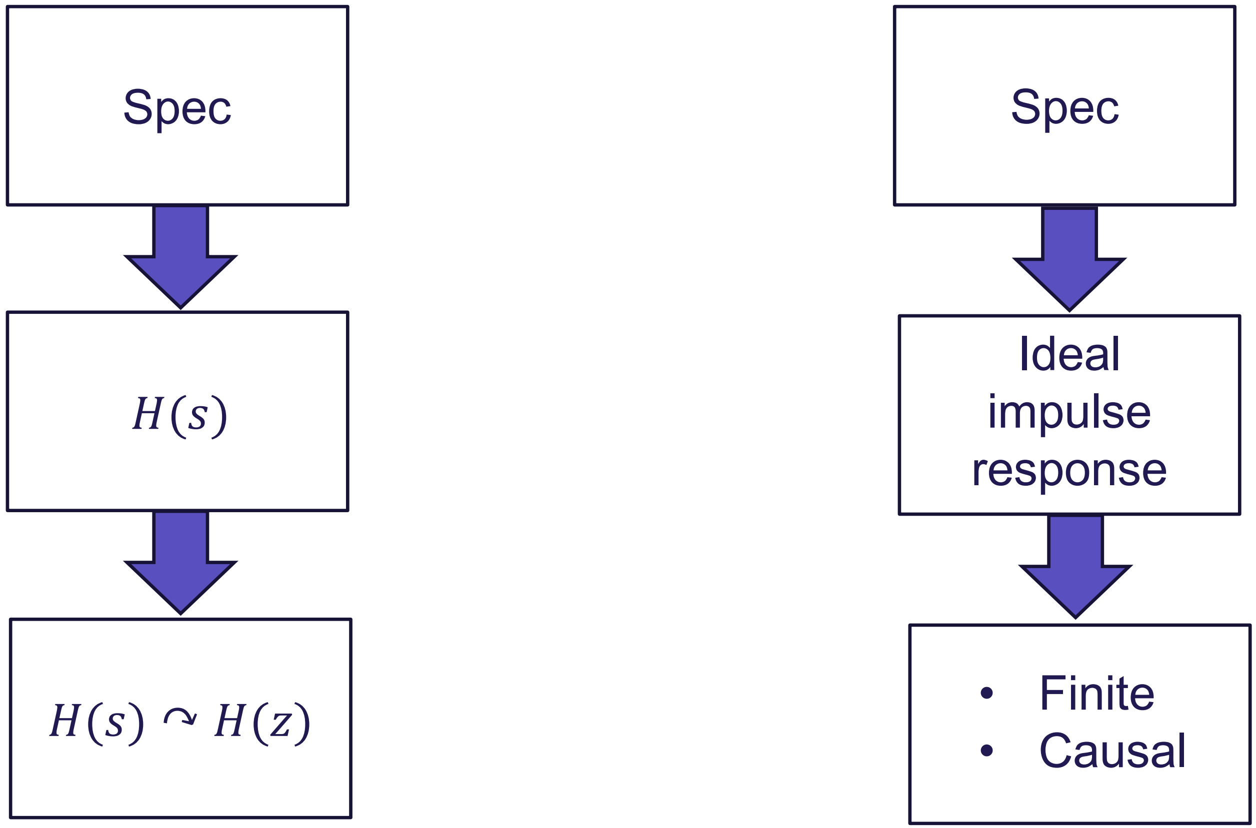
$$H(e^{j\omega}) = e^{-j\omega M/2} \left\{ h[M/2] + 2 \cdot \sum_{k=1}^{M/2} h[M/2+k] \cdot \cos(\omega k) \right\}$$

LIGN 140a
p. 343

$\Downarrow$

$$H(e^{j\omega}) = A(e^{j\omega}) \cdot e^{-j\omega M/2}$$

Next; How do we design an FIR filter…



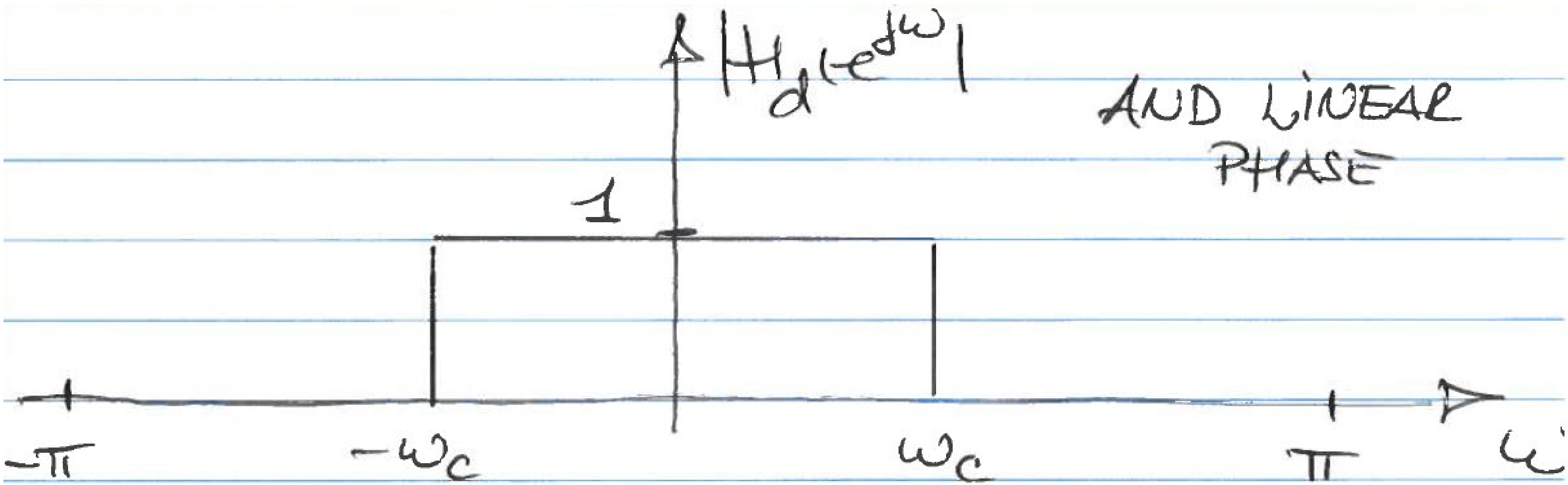
IIR-filter

FIR-filter

AALBORG UNIVERSITY
DENMARK

The Window Design Method

The starting point is the IDEAL / DESIRED frequency response



$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n]\, e^{-j\omega n}$$ Fourier Series

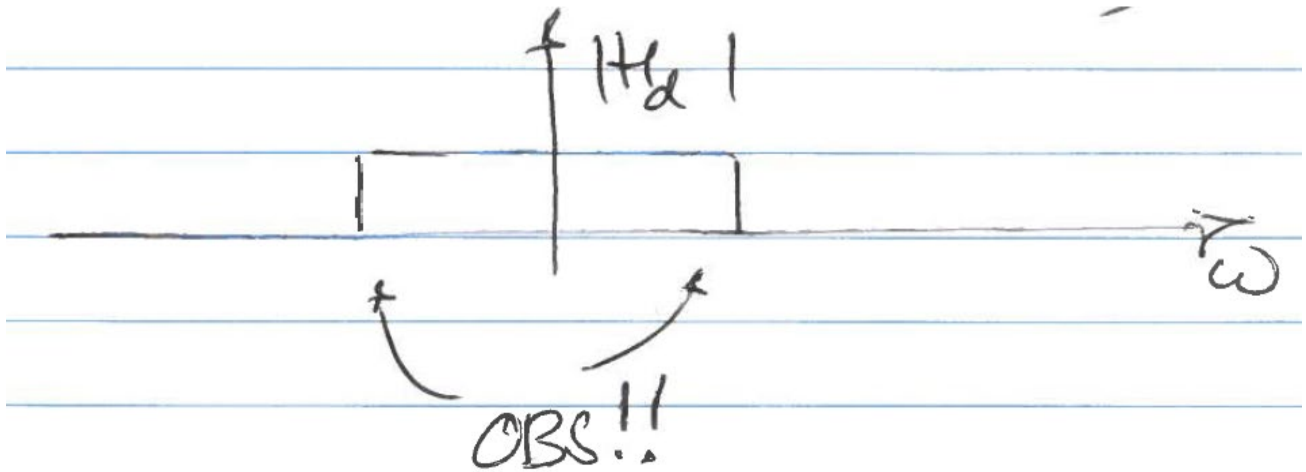
We are now seeking the Fourier Coefficients;

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n}\, d\omega$$ Equ. 53 & 54, p. 557

AALBORG UNIVERSITY
DENMARK

The Window Design Method

We are facing a problem here…



These discontinuities lead to a much unwanted situation – $h_d[n]$ becomes non-causal AND infinite…!!

In order to alleviate this problem, we TRUNCATE $h_d[n]$, i.e.,

$$h[n] = \begin{cases} h_d[n] & 0 \le n \le M \\ 0 & otherwise \end{cases}$$

where $h[n]$ is the impulse response that we want to implement – it is both causal and finite

Truncation by a Window Function in the time domain, $w[n]$

$$h[n] = h_d[n] \cdot w[n]$$

$$w[n] = \begin{cases} f[n] & 0 \le n \le M \\ 0 & otherwise \end{cases}$$

Initially, this seems as a good idea, but… Multiplication in the Time Domain is equivalent to Convolution in the Frequency Domain.

$$H\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\Theta}\right) W\left(e^{j(\omega-\Theta)}\right) d\Theta$$

Equ. 58, p.558

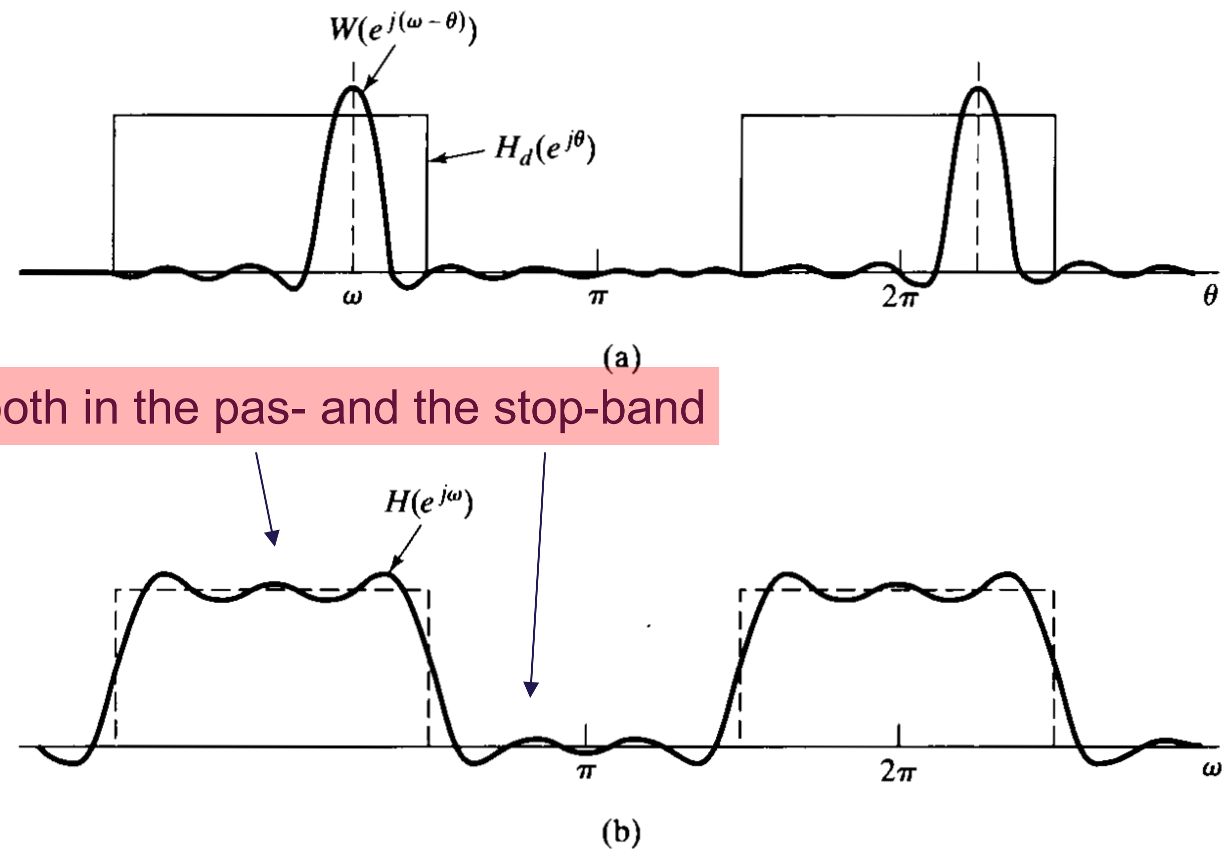
Note that this is a periodic convolution…

So, the desired frequency response is convolved with the Fourier transform of the window function…

AALBORG UNIVERSITY
DENMARK

Truncation by a Window Function in the time domain, $w[n]$

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\Theta})W(e^{j(\omega-\Theta)})d\Theta$$



Ripples both in the pas- and the stop-band

Figure 7.19 (a) Convolution process implied by truncation of the ideal impulse response. (b) Typical approximation resulting from windowing the ideal impulse response.

Truncation by a Window Function in the time domain, $w[n]$

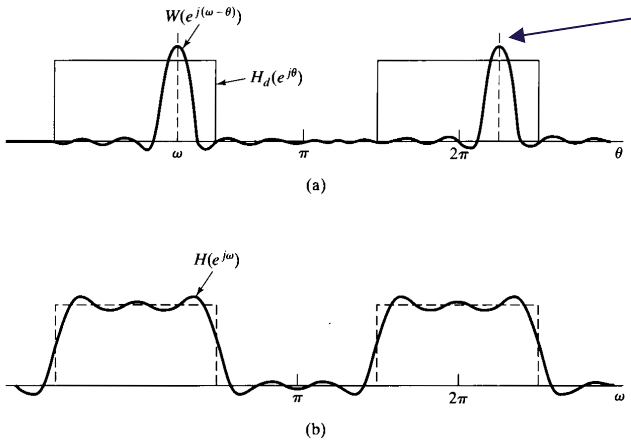


Figure 7.19 (a) Convolution process implied by truncation of the ideal impulse response. (b) Typical approximation resulting from windowing the ideal impulse response.

Ideally, we would like the Fourier Transform of the window function to be an impulse train…

In such a case, $H\left(e^{j\omega}\right) = H_d(e^{j\omega})$

If $W\left(e^{j\omega}\right)$ should be an impulse train, then $w[n]$ will also be an impulse train, i.e., $w[n] = 1, \forall\, n$, and thus there is no truncation of $h_d[n]$.

We need a compromise…!!!

AALBORG UNIVERSITY
DENMARK

The Window Function $w[n]$

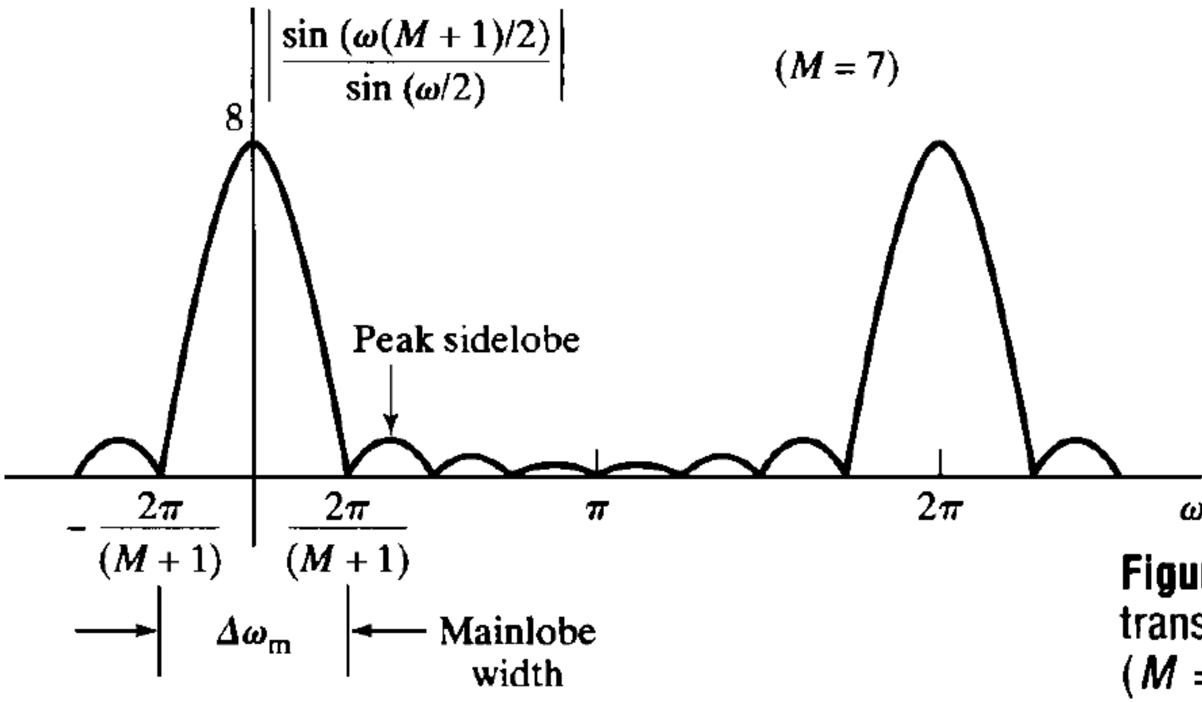


Figure 7.20 Magnitude of the Fourier transform of a rectangular window ($M = 7$).

We are searching for a Window Function with

- narrow Main-lobe (impulse train)
- small Side-lobes (small ripples)

The Window Function $w[n]$

Rectangular

$$w[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

Bartlett (triangular)

$$w[n] = \begin{cases} 2n/M, & 0 \le n \le M/2, \\ 2 - 2n/M, & M/2 < n \le M, \\ 0, & \text{otherwise} \end{cases}$$

Hanning

$$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

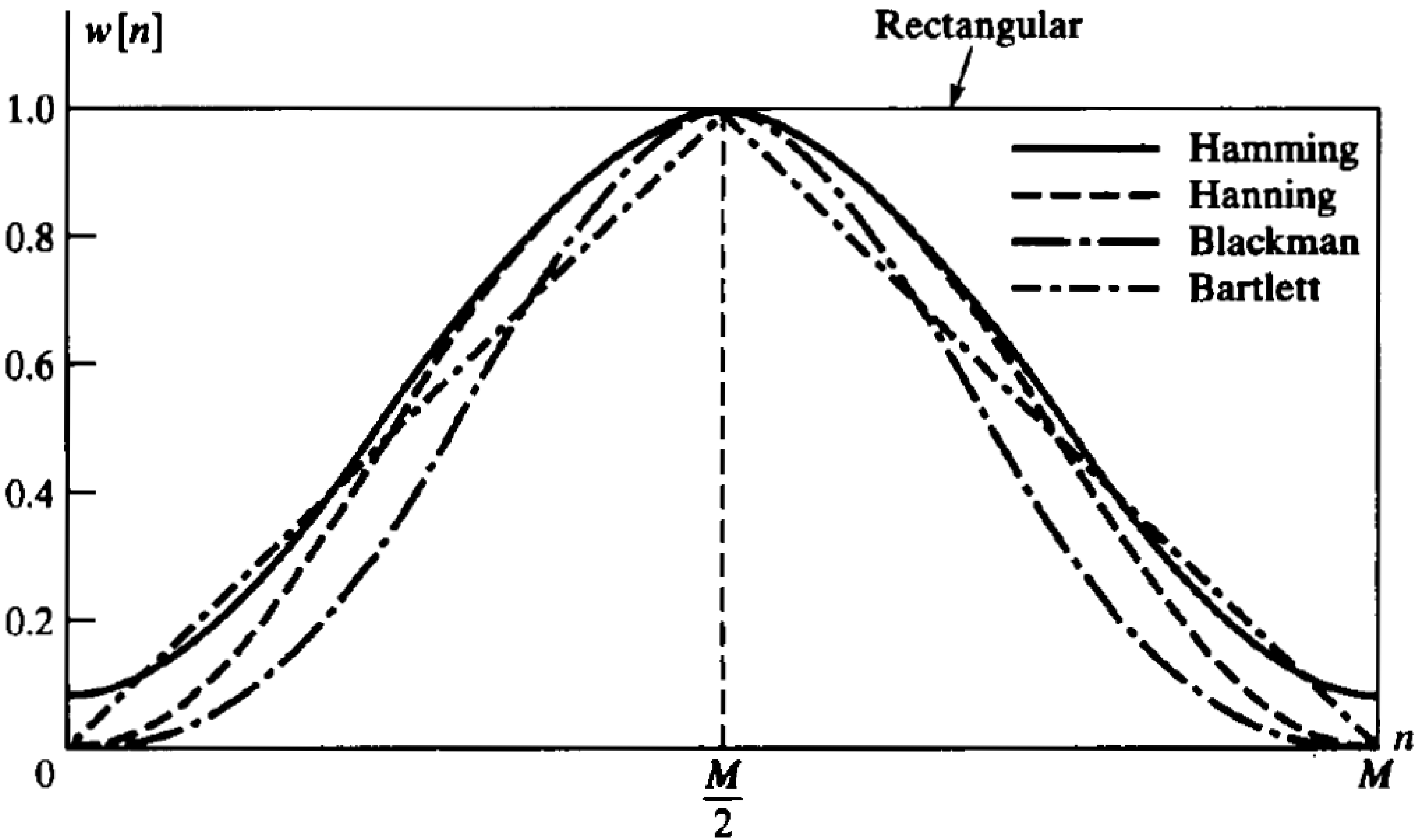
Hamming

$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

Blackman

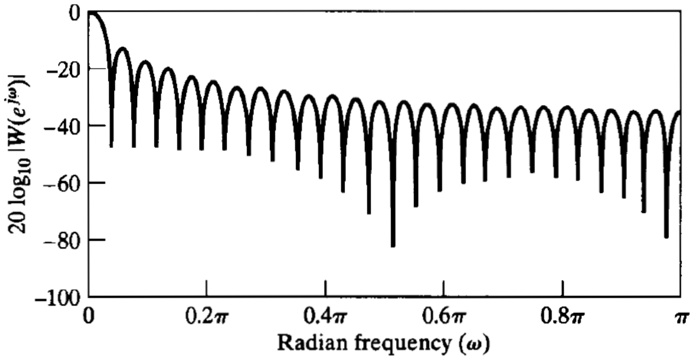
$$w[n] = \begin{cases} 0.42 - 0.5\cos(2\pi n/M) + 0.08\cos(4\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

The Window Function $w[n]$

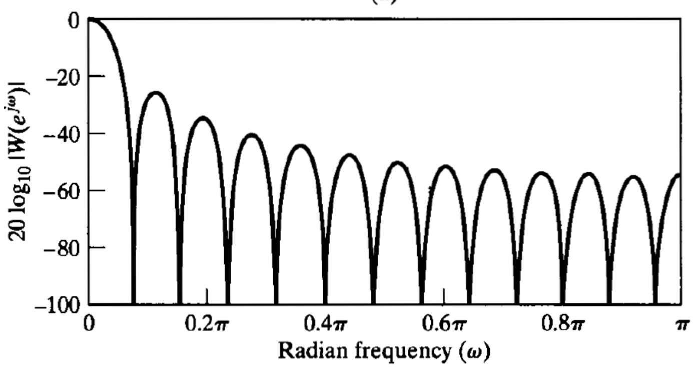


AALBORG UNIVERSITY
DENMARK

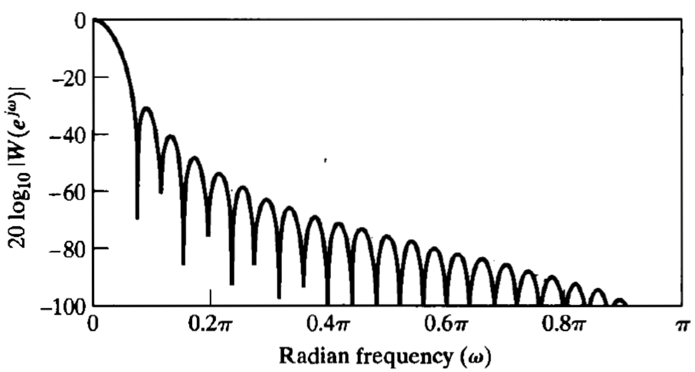
The Window Function $w[n]$ – in the frequency domain



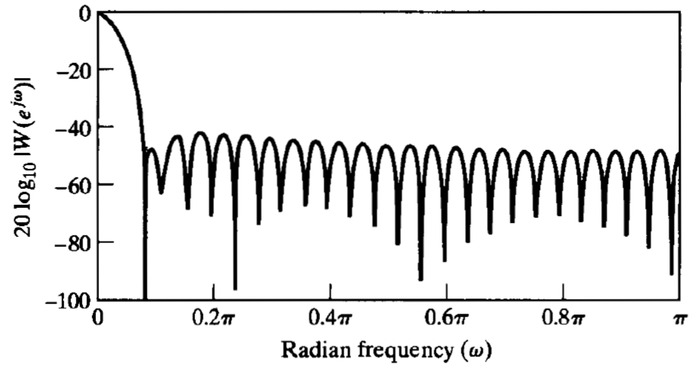
(a)



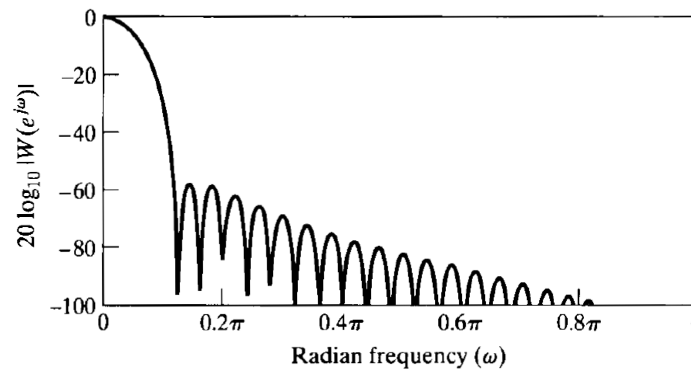
(b)



(c)



(d)



(e)

Figure 7.22 Fourier transforms (log magnitude) of windows of Figure 7.21. with $M = 50$. (a) Rectangular. (b) Bartlett. (c) Hanning. (d) Hamming. (e) Blackman. (e) Blackman.

Design Procedure – trial and error…

1. Specify $|H_d(e^{j\omega})|$ and require a linear phase response
2. Find $h_d[n]$ by IDTFT
3. Choose window function $w[n]$
4. Find expression for $h[n] = h_d[n] \cdot w[n]$
5. Espress $H(e^{j\omega})$ as a function of the FILTER ORDER M, p.20
6. For selected M plot $|H(e^{j\omega})|$ and check if spec's are fulfilled
7. If OK then we are done…
8. Otherwise, choose another $w[n]$ and/or M, and try again.