State Space Methods

Lecture 2: controllability and state feedback

Jakob Stoustrup

Department of Electronic Systems, Automation & Control
Technical Faculty of IT and Design
Aalborg University

Email: jakob@es.aau.dk

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Controllability

A continuous time system

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \quad x(0) = 0$$

is said to be *controllable* iff for any $\xi \in \mathbb{R}^n$ there exists $u(t)$ such that for some $T > 0$, $x(T) = \xi$.

Controllability

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A discrete time system

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = 0$$

is said to be *controllable* iff for any $\xi \in \mathbb{R}^n$ there exists $(u(0), u(1), \ldots)$ such that for some $N > 0$, $x(N) = \xi$.

Controllability

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = 0$$

and iterate:

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Controllability

Writing the equation

$$x(n) = A^{n-1}Bu(0) + \ldots + ABu(n-2) + Bu(n-1)$$

in matrix form we obtain:

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$$x(n) = \begin{pmatrix} B & AB & \ldots & A^{n-1}B \end{pmatrix} \begin{pmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{pmatrix}$$

Controllability

Writing the equation

$$x(n) = A^{n-1}Bu(0) + \ldots + ABu(n-2) + Bu(n-1)$$

in matrix form we obtain:

$$x(n) = \underbrace{\begin{pmatrix} B & AB & \ldots & A^{n-1}B \end{pmatrix}}_{\text{Controllability matrix}} \begin{pmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{pmatrix}$$

When is $x(n) = \xi$ solvable for any $\xi \in \mathbb{R}^n$?

Controllability

Theorem

A system

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \ \ \textit{(continuous time)} \\ x(k+1) &= Ax(k) + Bu(k) \ \textit{(discrete time)} \end{cases}$$

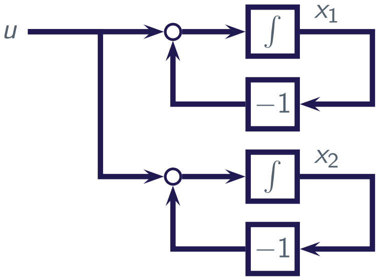
where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, *is controllable if and only if*

$$\operatorname{rank} \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} = n$$

For $m = 1$ *this reduces to*

$$\det \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} \neq 0$$

Example: parallel connection



State space equations:

$$\left\{ \begin{array}{l} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = -x_2 + u \end{array} \right\}$$

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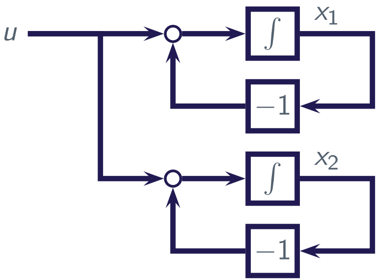

State space equations:

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State space equations in matrix form:

$$\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

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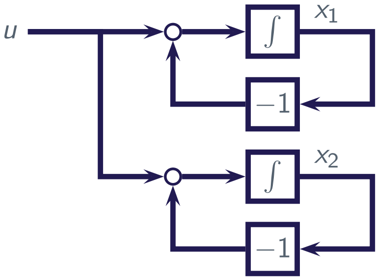

$$\dot{x} = Ax + Bu$$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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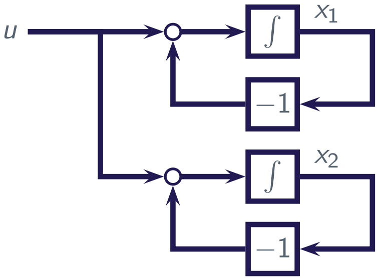

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$$\mathcal{C} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad \det(\mathcal{C}) = 0$$

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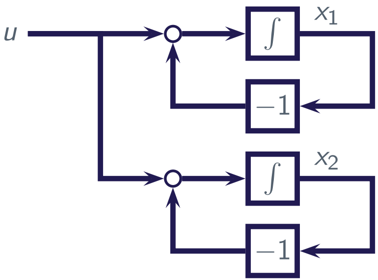

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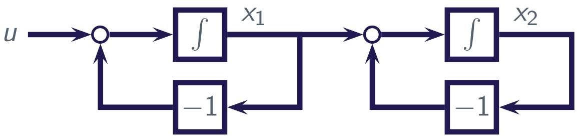
$\mathrm{rank}(\mathcal{C}) = 1 < 2 \implies$ uncontrollable

Example: series connection



State equations:

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -x_1 + u \\ \dot{x}_2 & = & -x_2 + x_1 \end{array} \right\}$$

State space model in matrix form:

$$\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

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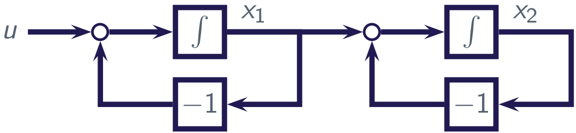

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Controllability analysis

$$\mathcal{C} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

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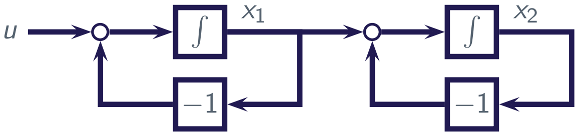

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$$\mathcal{C} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \det(\mathcal{C}) = 1 \neq 0$$

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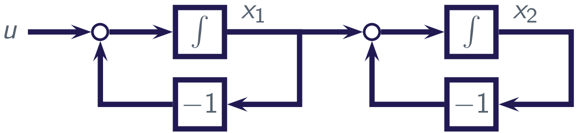

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$$\mathcal{C} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \det(\mathcal{C}) = 1 \neq 0$$

$\text{rank}(\mathcal{C}) = 2 \implies$ controllable

Contents

Controllable canonical form

Any controllable *single input* system can be written in the form:

$$\dot{x}_c = A_c x_c + B_c u\,, \quad x_c \in \mathbb{R}^n\,,\, u \in \mathbb{R}$$

where

$$A_c = \left(\begin{array}{c|c} \multicolumn{2}{c}{a^T} \\ \hline I_{n-1} & 0_{(n-1)\times 1} \end{array} \right)\,, \quad B_c = \left(\begin{array}{c} 1 \\ \hline 0_{(n-1)\times 1} \end{array} \right)$$

and where $a \in \mathbb{R}^{n\times 1}$, $a^T = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$. It can be shown that

$$\det(\lambda I - A_c) = \lambda^n - a_1 \lambda^{n-1} - \ldots - a_n$$

Controllable canonical form

For $n = 3$ the controllable canonical form becomes:

$$A_c = \left(\begin{array}{cc|c} a_1 & a_2 & a_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right), \; B_c = \left(\begin{array}{c} 1 \\ \hline 0 \\ 0 \end{array}\right)$$

which is indeed controllable:

$$\mathcal{C}_c = \begin{pmatrix} B_c & A_c B_c & A_c^2 B_c \end{pmatrix} = \begin{pmatrix} 1 & a_1 & a_1^2 + a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$\det(\mathcal{C}) = 1 \neq 0 \implies$ system is controllable.

Controllable canonical form

Given a state space model of a controllable system:

$$\dot{x} = Ax + Bu\,, \quad x \in \mathbb{R}^n\,,\, u \in \mathbb{R}$$

we wish to find a basis transformation $x = Tx_c$, such that:

$$\dot{x}_c = A_c x_c + B_c u\,, \quad x_c \in \mathbb{R}^n\,,\, u \in \mathbb{R}$$

where $A_c = T^{-1}AT$ and $B_c = T^{-1}B$, is in controllable canonical form.

Controllable canonical form

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where $A_c = T^{-1}AT$ and $B_c = T^{-1}B$, is in controllable canonical form.
We can solve for T^{-1} by rewriting these equations as

$$A_c T^{-1} = T^{-1}A \quad \text{and} \quad B_c = T^{-1}B$$

Controllable canonical form

We consider $n = 3$, and introduce the following notation for the rows of T^{-1}

$$T^{-1} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad s_1, s_2, s_3 \in \mathbb{R}^{1\times n}$$

Then we can rewrite the transformation equations $A_c T^{-1} = T^{-1} A$ and $T^{-1} B = B_c$ as:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A \, , \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Controllable canonical form

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A\,, \ \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

yields:

Controllable canonical form

Writing out these equations:

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yields:

$$\left\{ \boxed{s_1 = s_2 A} \right., \quad \left\{ \qquad \right\}$$

Controllable canonical form

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ \boxed{0 \quad 1 \quad 0} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} s_1 \\ s_2 \\ s_3 \end{matrix}} \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ \boxed{s_3} \end{pmatrix} A \,, \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

yields:

$$\left\{ \begin{matrix} s_1 = s_2 A \\ \boxed{s_2 = s_3 A} \end{matrix} \right\} \,, \quad \left\{ \qquad \qquad \right\}$$

Controllable canonical form

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A \, , \begin{pmatrix} \boxed{s_1} \\ s_2 \\ s_3 \end{pmatrix} B = \begin{pmatrix} \boxed{1} \\ 0 \\ 0 \end{pmatrix}$$

yields:

$$\left\{ \begin{array}{l} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\} , \left\{ \begin{array}{c} \boxed{s_1 B = 1} \\ \\ \\ \end{array} \right\}$$

Controllable canonical form

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A \,, \begin{pmatrix} s_1 \\ \boxed{s_2} \\ s_3 \end{pmatrix} B = \begin{pmatrix} 1 \\ \boxed{0} \\ 0 \end{pmatrix}$$

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Controllable canonical form

Writing out these equations:

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yields:

$$\left\{ \begin{aligned} s_1 &= s_2 A \\ s_2 &= s_3 A \end{aligned} \right\} \,, \quad \left\{ \begin{array}{c} s_1 B = 1 \\ s_2 B = 0 \\ \boxed{s_3 B = 0} \end{array} \right\}$$

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yields:

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Controllable canonical form

Combining the equations

$$\left\{ \begin{array}{l} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\}, \left\{ \begin{array}{l} s_1 B = 1 \\ s_2 B = 0 \\ s_3 B = 0 \end{array} \right\}$$

we obtain

$$s_3 \begin{pmatrix} B & AB & A^2 B \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

yielding the solution

$$s_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \mathcal{C}^{-1}, \; s_2 = s_3 A, \; s_1 = s_2 A$$

for nonsingular $\mathcal{C} = \begin{pmatrix} B & AB & A^2 B \end{pmatrix}$.

Example: companion form

We consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y &= \begin{pmatrix} -3 & 2 \end{pmatrix} x \end{aligned}$$

having the controllability matrix

$$\mathcal{C} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 3 & -7 \end{pmatrix}, \quad \det(\mathcal{C}) = 1 \neq 0$$

Example: companion form

We compute the rows of T^{-1} by

$$s_2 = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathcal{C}^{-1} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -7 & 5 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -3 & 2 \end{pmatrix}$$

$$s_1 = s_2 A = \begin{pmatrix} -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} = \begin{pmatrix} 2 & -1 \end{pmatrix}$$

Thus,

$$T^{-1} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

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Thus,

$$T^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \quad \Longrightarrow \quad T = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

Example: companion form

Eventually, we have

$$
\begin{aligned}
A_c = T^{-1}AT &= \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \\
&= \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}
\end{aligned}
$$

and

Example: companion form

Eventually, we have

$$A_c = T^{-1}AT = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$
$$= \left(\begin{array}{c|c} -3 & -2 \\ \hline 1 & 0 \end{array}\right)$$

and

$$B_c = T^{-1}B = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Example: companion form

Eventually, we have

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$$B_c = T^{-1}B = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \left(\begin{array}{c} 1 \\ \hline 0 \end{array}\right)$$

Example: companion form

Eventually, we have

$$A_c = T^{-1}AT = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

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Combining these two equations, we obtain:

$$\dot{x} = Ax + BFx = (A + BF)x$$

Thus, the result of a state feedback is a system with a modified system matrix, and thus with modified poles.

State feedback

For a single input system in companion form, a state feedback takes a particular simple form:

$$A_c = \left(\begin{array}{cc|c} a_1 & a_2 & a_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right), \; B_c = \left(\begin{array}{c} 1 \\ \hline 0 \\ 0 \end{array}\right)$$

Applying the feedback $u = Fx$ with

$$F_c = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}$$

We obtain:

$$A_c + B_c F_c = \left(\begin{array}{cc|c} a_1 & a_2 & a_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) + \left(\begin{array}{c} 1 \\ \hline 0 \\ 0 \end{array}\right) \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}$$

$$= \left(\begin{array}{cc|c} a_1 + f_1 & a_2 + f_2 & a_3 + f_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

State feedback

Thus, the characteristic polynomium has been changed from

$$\det(\lambda I - A_c) = \lambda^n - a_1\lambda^{n-1} - \ldots - a_n$$

to

$$\begin{aligned}\det(\lambda I - (A_c + B_c F_c)) = \\ \lambda^n - (a_1 + f_1)\lambda^{n-1} - \ldots - (a_n + f_n)\end{aligned}$$

By choosing $f_1, \ldots, f_n$ appropriately, *any* closed loop pole configuration can be obtained. This is known as *pole assignment*.

Algorithm for pole assignment

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ be given.

1. Choose desired closed loop polynomial
   $\det(\lambda I - (A + BF)) = \lambda^n + a_{\mathrm{cl},1}\lambda^{n-1} + \ldots + a_{\mathrm{cl},n}$.

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5. Compute resulting feedback gain $F = F_c T^{-1}$.

Example: pole assignment

We consider again the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y &= \begin{pmatrix} -3 & 2 \end{pmatrix} x \end{aligned}$$

for which we would like to move the poles to $\{-4, -5\}$.

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5. $F = F_c T^{-1} = \begin{pmatrix} -6 & -18 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 42 & -30 \end{pmatrix}$

Example: pole assignment