Digital Signal Processing

ESD-5 & IV-5 (elektro), E24

9. The Discrete Fourier Transform, cont.

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The Discrete Fourier Transform, DFT

In our previous lecture we discussed the basic math associated with the Discrete Fourier Transform (DFT), which is the Fourier Transform that we want to use for calculating the spectral content of a finite-length physically observable time-discrete signal, i.e., a sequence for which we do not know any closed-form mathematical spectral representation.

Today, first we will recap the high-ligths from the previous lecture and then we will introduce some important DFT properties.

Our starting point was the Discrete Fourier Series, DFS

Analysis equation: $$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}.$$

Synthesis equation: $$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}.$$

Remember that Tilde (~) denotes priodicity – so, $\tilde{x}[n]$ as well as $\tilde{X}[k]$ are periodic sequences with period *N*…

$W_N = e^{-j(2\pi/N)}$ is the twiddle factor, with $2\pi/N$ representing the fundamental frequency – but how should we understand the term ”fundamental frequency”…?

A brief refresh on periodic signals
- in the continuous-time domain

Assume a signals being defined as a finite sum of sinusoids with different frequency, phase, and amplitude – the general form is;

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(\omega_k t + \varphi_k)$$

where ω_k is the angular frequency.

Now, a VERY IMPORTANT class of signals are those which have a PERIODIC behaviour, i.e., the signal repeats itself after a certain amount of time, T_0 sec.

$x(t) = x(t + T_0)$

It should be obvious, that a pure sinusoid, i.e., a single-tone signal, is periodic, but let's see…

$\omega_k = k\omega_0 \Rightarrow 2\pi f_k = 2\pi k f_0 \Rightarrow f_k = k f_0$

The frequency f_0 is the FUNDAMENTAL FREQUENCY; $f_0 = \frac{1}{T_0}$

$\cos(2\pi k f_0 (t+T) + \varphi) = \cos\left(2\pi k f_0 \left(t + \frac{1}{f_0}\right) + \varphi\right)$ for $T = T_0$

$= \cos(2\pi k f_0 t + 2\pi k + \varphi) = \cos(2\pi k f_0 t + \varphi)$

Yes, $\cos(*)$ represents a periodic signal – but we need more advanced signals…

Question is however, whether a multi-tone signal can also be periodic…??

The answer is YES, under the condition that $\omega_k = k\omega_0$, where ω_0 is what we denote as the FUNDAMENTAL ANGULAR FREQUENCY.

So, let's try to add N hamonically related sinusoids;

OBS…!!

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

where $f_k = k f_0 \quad k \in \mathbb{Z}$

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$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

Using Euler, we know that $x(t)$ can be re-written using the Complex Amplitude notation;

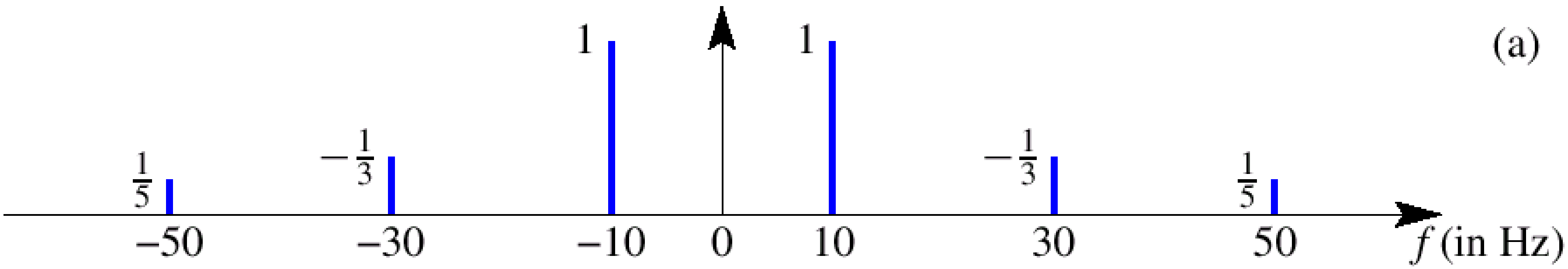
$$X_k = A_k e^{j\varphi_k}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \{\frac{1}{2} X_k e^{j2\pi f_k t} + \frac{1}{2} X_k^* e^{-j2\pi f_k t}\}$$

Main Conclusion:

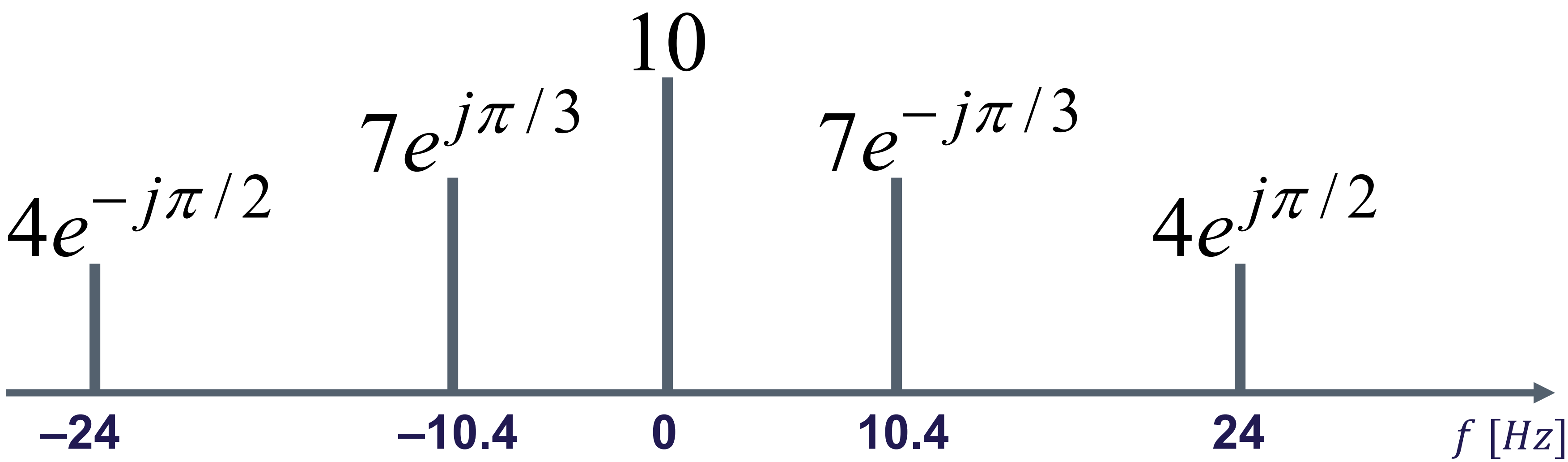
For periodic signals, all spectral lines have frequencies that are *integer* multiples of the fundamental frequency – the fundamental frequency being the lowest one of these…

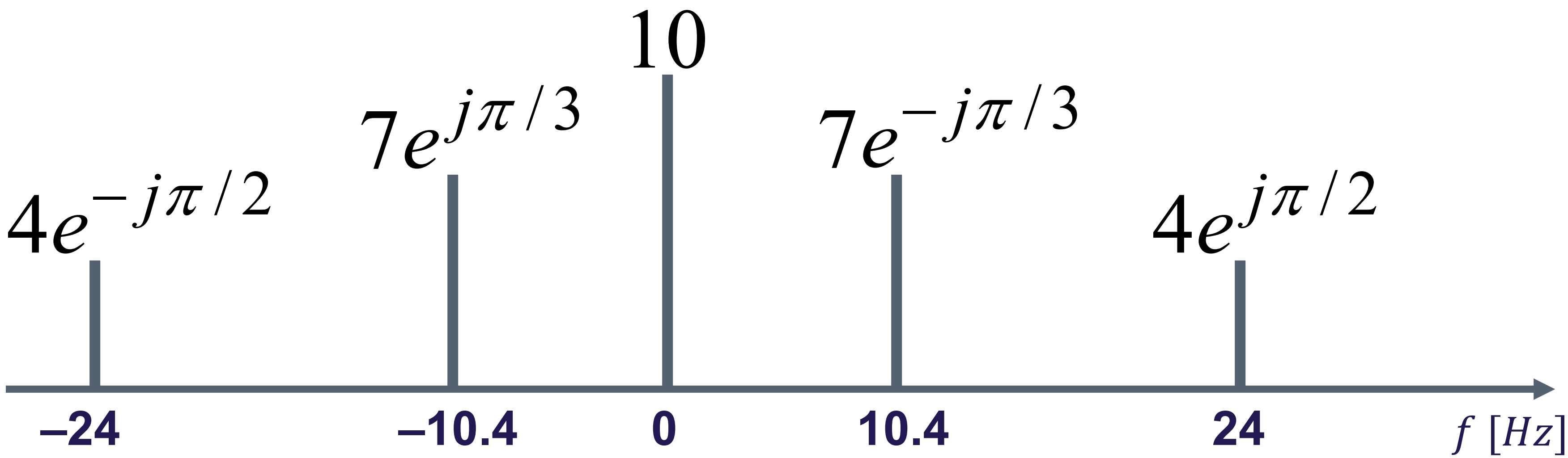
We see that the spectrum is complex conjugate symmetric, and comprizes spectral line at the harmonic frequencies f_k.



What is the fundamental frequency in this signal…??

What about this signal – does it have a fundamental frequency …??





Remember that $f_0 = k f_k$ where $k \in \mathbb{Z}$

The fundamental frequency f_0 is the *largest* frequency such that $f_0 = k f_k \;\forall\, f_k$.

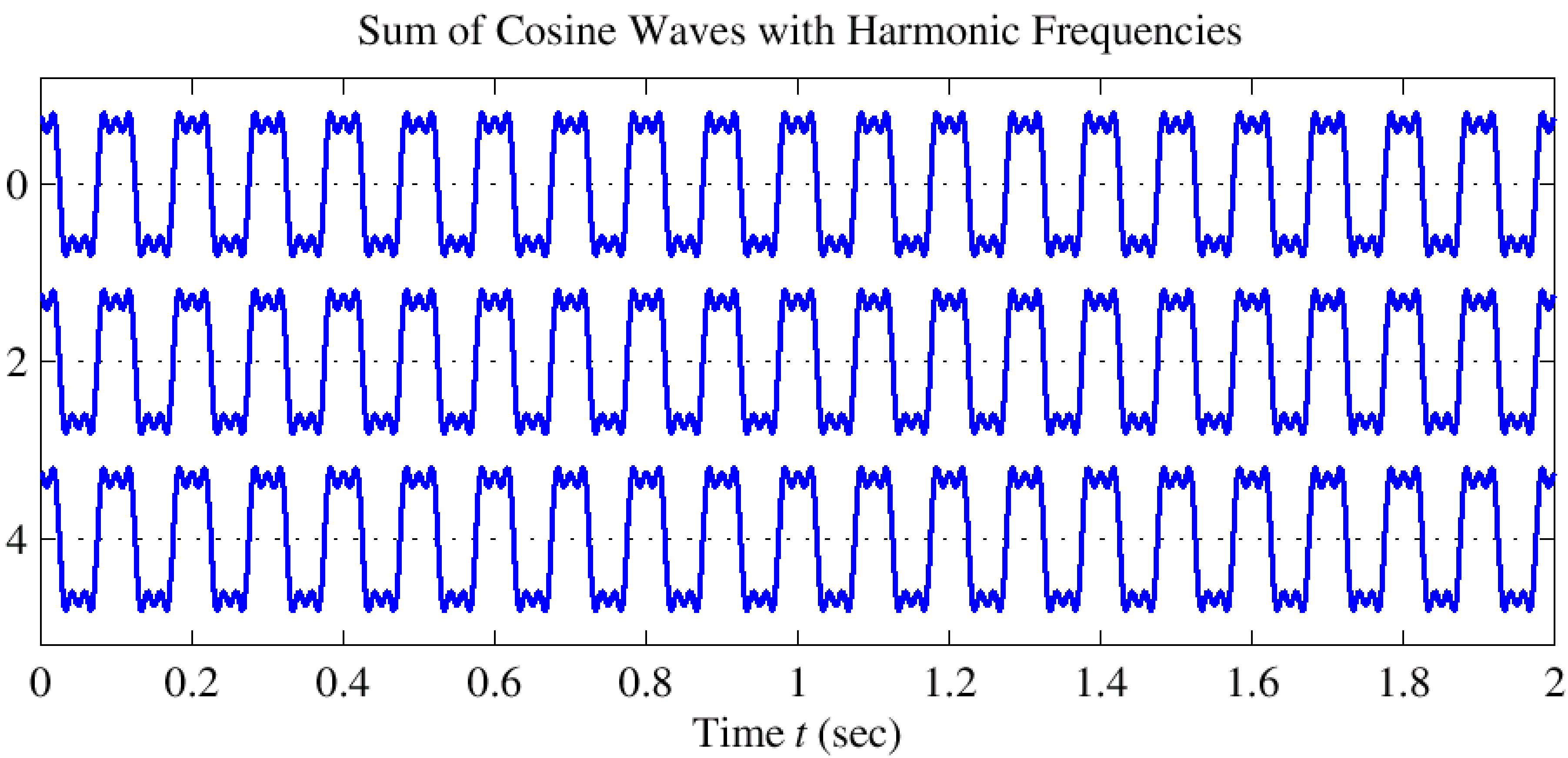
Mathematically, it means that f_0 can be found as the *Greatest Common Divisor* of a set of integers, i.e., all the f_k's in the signal; $f_0 = GCD\{f_k\}\;\; k = 1, 2, \ldots, N$

Now, in our example, one of the two frequencies, $f_1 = 10.4\,Hz$, is not an integer, but that problem can be solved;

$$f_0 = 0.1 \cdot GCD\{10 \cdot 10.4, 10 \cdot 24\} = 0.1 \cdot GCD\{104, 240\} = 0.1 \cdot 8 = 0.8\,Hz$$

Note that f_0 is NOT an integer and it is NOT in the signal (its amplitude is 0), but that is also not necessary – $0.8\,Hz$ is the fundamental frequency of the signal.

Example of a harmonic signal with 3 frequencies



What is the fundamental frequency f_0…??

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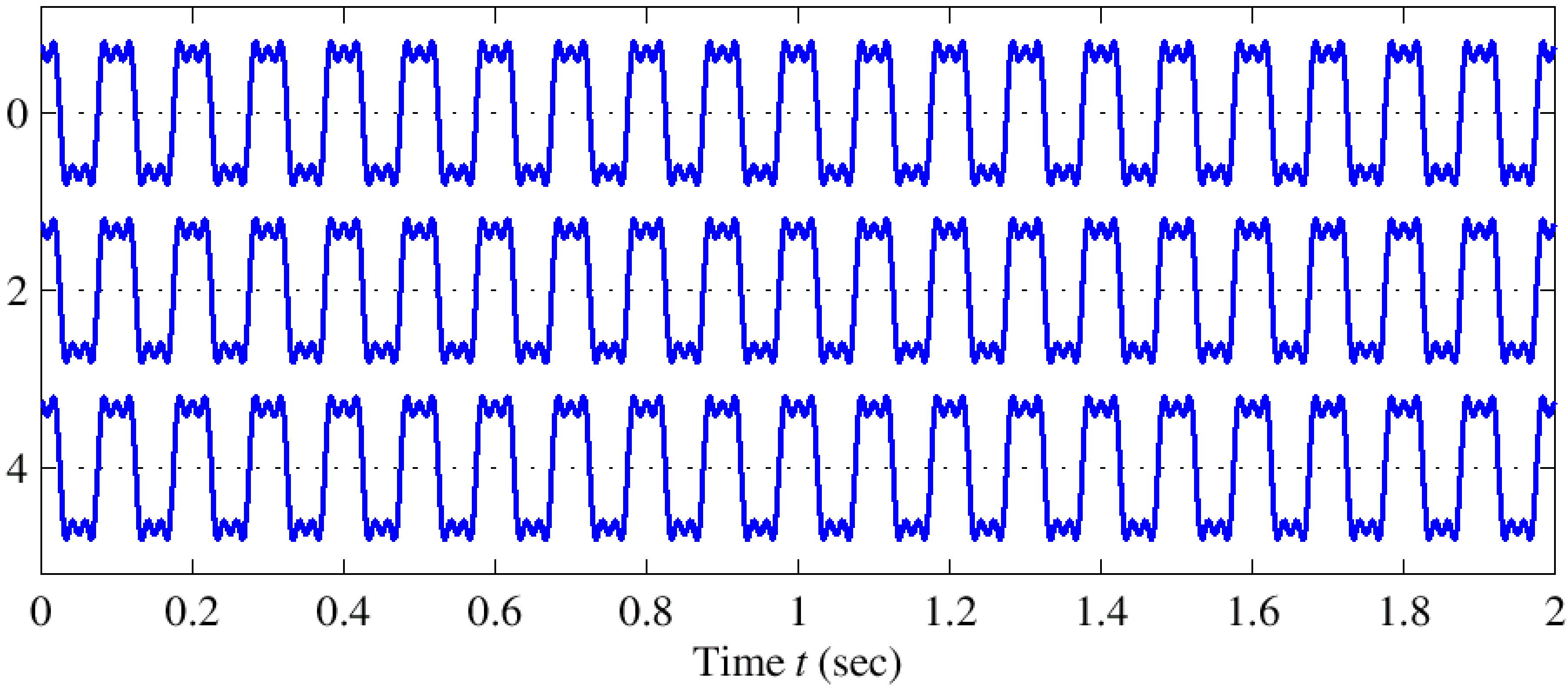
Sum of Cosine Waves with Harmonic Frequencies

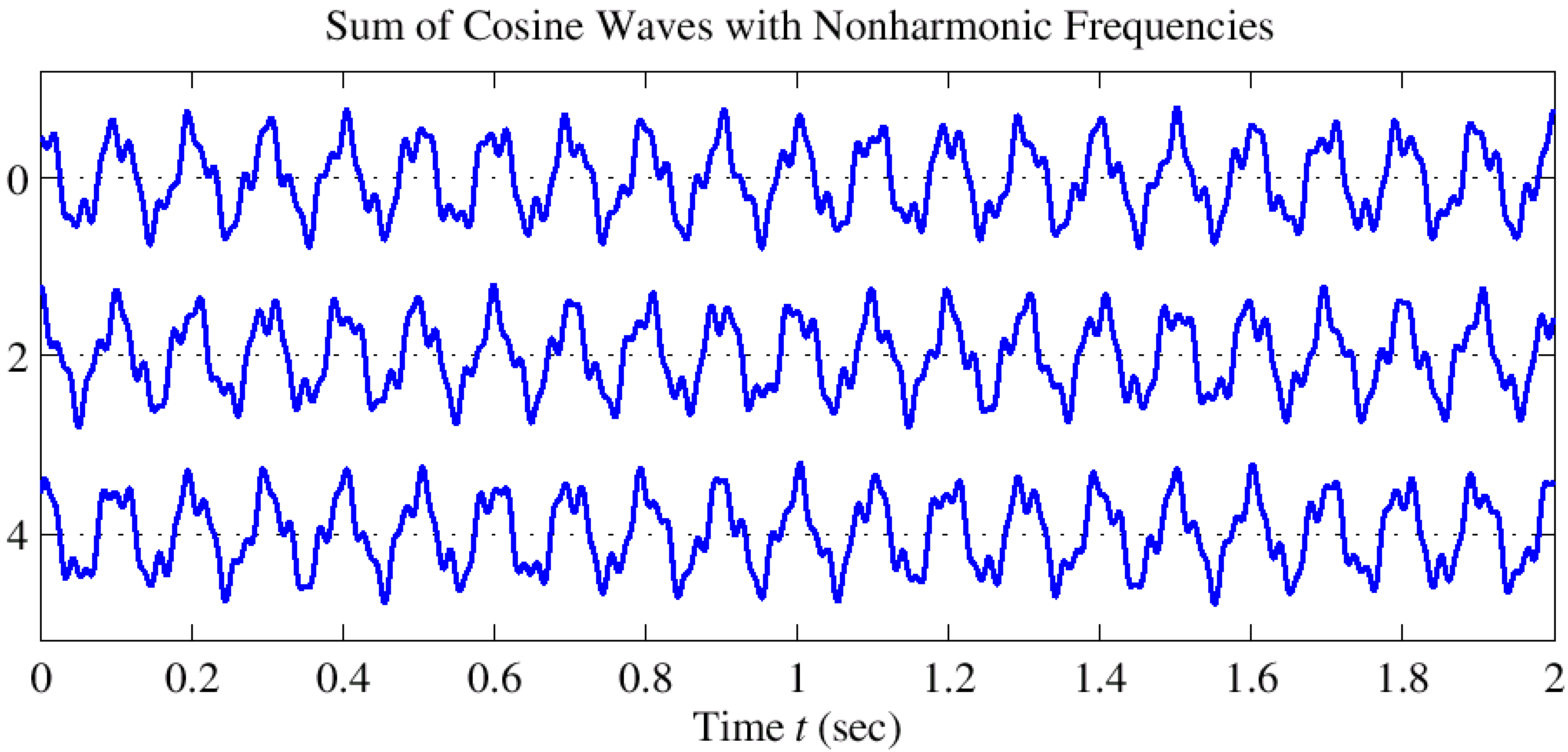


Since the signal is periodic, we can read the period from the graph:

$T_0 = 0.1\,sec \quad \Rightarrow \quad f_0 = 10\,Hz.$

$$x(t) = 2\cos(2\pi 10 t) - \frac{2}{3}\cos(2\pi 30 t) + \frac{2}{5}\cos(2\pi 50 t)$$

Example of a signal comprising non-harmonic frequencies



We clearly see that the signal in NOT periodic…!!

$$x(t) = 2\cos(2\pi 10t) - \frac{2}{3}\cos\left(2\pi(20\sqrt{2})t\right) + \frac{2}{5}\cos(2\pi(30\sqrt{3})t)$$

$$20\sqrt{2} = 28.28 \sim 30$$
$$30\sqrt{3} = 51.96 \sim 50$$

The frequencies of the last two terms are irrational numbers, and thus it is not possible to find a GCD..!!!

So, it is not possible to find the DFS of this signal – there is no fundamental freq.

Important properties of the DFS

Linearity:

$$\left.\begin{array}{l}\tilde{x}_1[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_1[k] \\ \tilde{x}_2[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_2[k]\end{array}\right\} \text{yields}$$

$$a\,\tilde{x}_1[n] + b\,\tilde{x}_2[n] \overset{DFS}{\longleftrightarrow} a\,\tilde{X}_1[k] + b\,\tilde{X}_2[k]$$

Time-shift:

$$\tilde{x}[n-m] \overset{DFS}{\longleftrightarrow} W_N^{km}\,\tilde{X}[k]$$ Phase shift

Frequency-shift:

$$\tilde{X}[k-l] \overset{DFS}{\longleftrightarrow} W_N^{-nl}\,\tilde{x}[n]$$ Heterodyne

Another important property which we DID NOT introduce last time: Periodic Convolution

Let $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ be two periodic sequences, each with period N and with discrete Fourier series coefficients denoted by $\tilde{X}_1[k]$ and $\tilde{X}_2[k]$, respectively. If we form the product

$$\tilde{X}_3[k] = \tilde{X}_1[k]\tilde{X}_2[k],$$

then the periodic sequence $\tilde{x}_3[n]$ with Fourier series coefficients $\tilde{X}_3[k]$ is

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m].$$

This is an interesting, but not surprising result; Remember that multiplication in frequency results in convolution in time – and the right-hand side looks very much like the well-known convolution sum, as we know from aperiodic discrete convolution.

There are two important differences though…

1) the sum is only over N samples, and
2) $\tilde{x}_3[n]$ is periodic

This is known as periodic convolution – see all the math on p. 659.

Periodic Convolution, DFS

$$\tilde{X}_3[k] = \tilde{X}_1[k]\tilde{X}_2[k]$$

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

$$\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k]\tilde{X}_2[k]$$

The periodic convolution of periodic sequences thus corresponds to multiplication of the corresponding periodic sequences of Fourier series coefficients.

Let's have a closer look at $\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$

which calls for the product of sequences $\tilde{x}_1[m]$ and $\tilde{x}_2[n-m] = \tilde{x}_2[-(m-n)]$ viewed as functions of m with n fixed. This is the same as for an aperiodic convolution, but with the following two major differences:

1. The sum is over the finite interval $0 \le m \le N-1$.
2. The values of $\tilde{x}_2[n-m]$ in the interval $0 \le m \le N-1$ repeat periodically for m outside of that interval.

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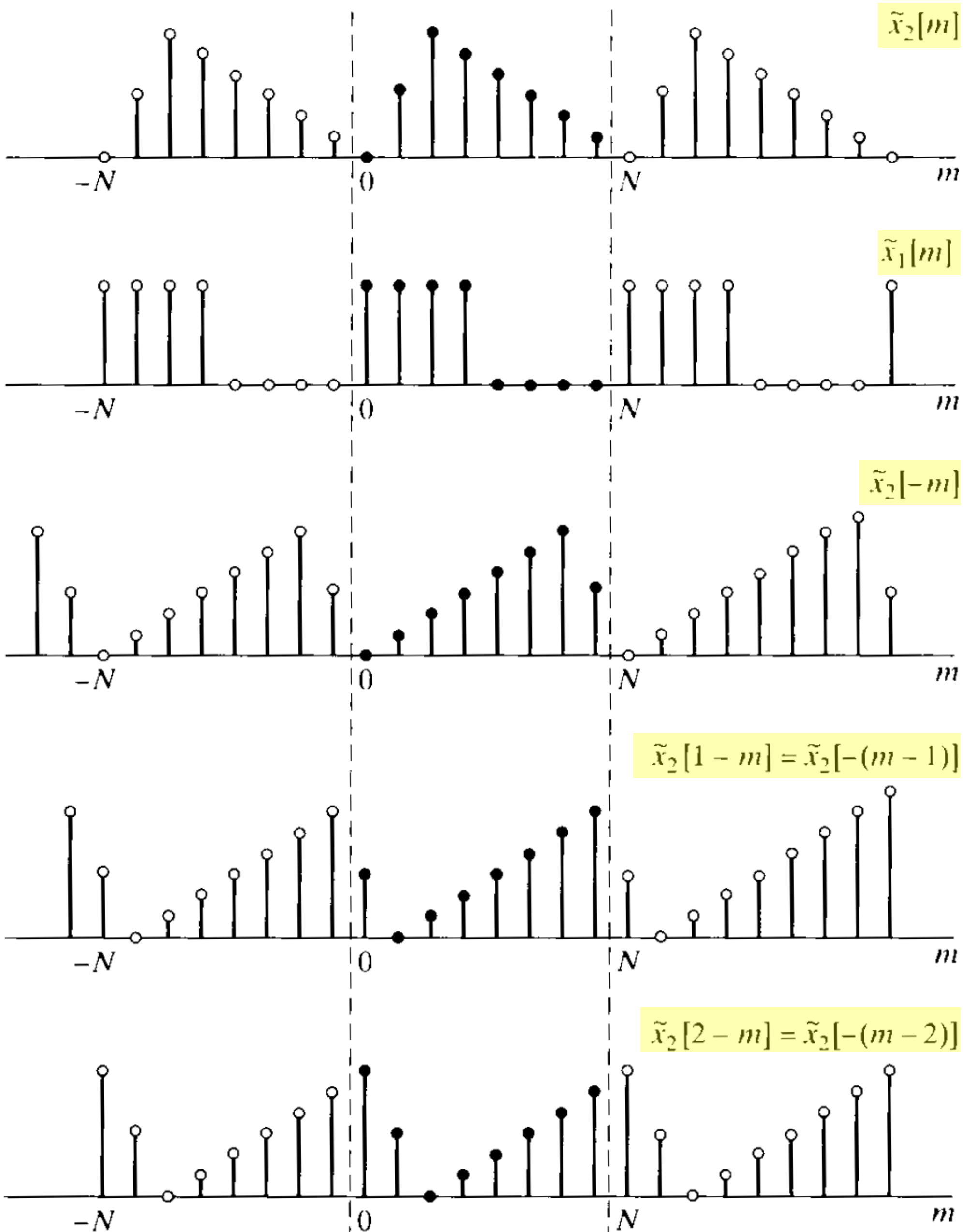
Let's see an example…

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

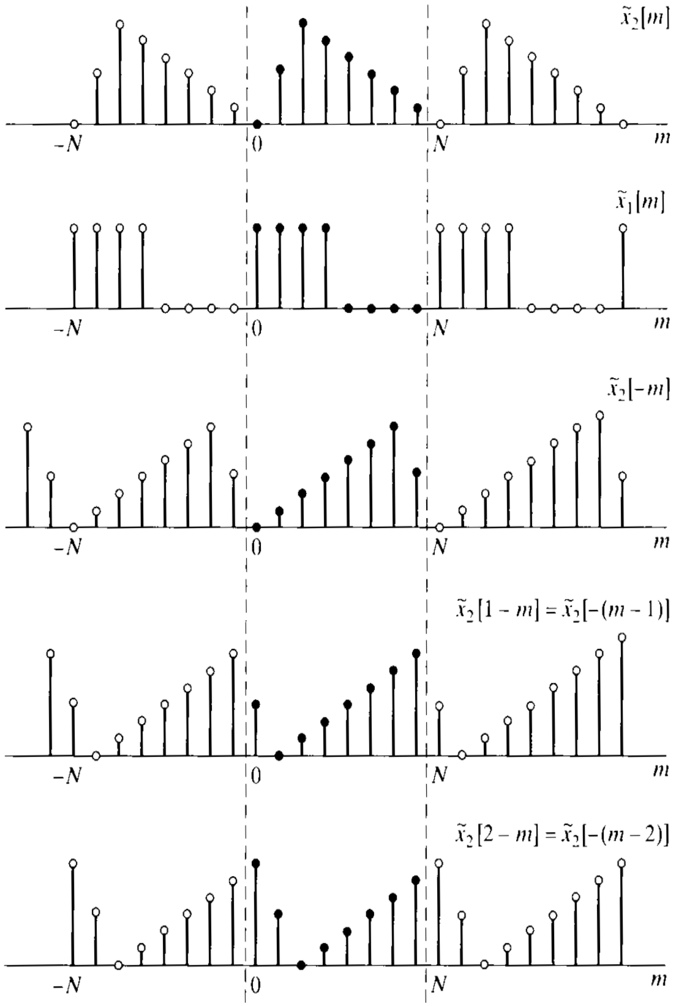


Two periodic sequences $\tilde{x}_2[n]$ and $\tilde{x}_1[n]$

Here $\tilde{x}_2[n]$ is flipped – we may choose any period around which we do the flip since the sequence is infinite

Here $\tilde{x}_2[n]$ is delayed one sample - notice the samples entering from the left side due to the periodic nature of $\tilde{x}_2[n]$.

Here $\tilde{x}_2[n]$ is delayed two samples



Two periodic sequences $\tilde{x}_2[n]$ and $\tilde{x}_1[n]$

Here $\tilde{x}_2[n]$ is flipped – we may choose any period around which we do the flip since the sequence is infinite

Here $\tilde{x}_2[n]$ is delayed one sample

Here $\tilde{x}_2[n]$ is delayed two samples

Note that $\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$ will repeat itself outside $0 \leq n \leq N-1$ and thus $\tilde{x}_3[n]$ is periodic with period *N.*

SUMMARY OF PROPERTIES OF THE DFS

	Periodic Sequence (Period N)	DFS Coefficients (Period N)
1.	$\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
2.	$\tilde{x}_1[n], \tilde{x}_2[n]$	$\tilde{X}_1[k], \tilde{X}_2[k]$ periodic with period N
3.	$a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
4.	$\tilde{X}[n]$	$N\tilde{x}[-k]$
5.	$\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
6.	$W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
7.	$\sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
8.	$\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1}\tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
9.	$\tilde{x}^*[n]$	$\tilde{X}^*[-k]$
10.	$\tilde{x}^*[-n]$	$\tilde{X}^*[k]$
11.	$\mathcal{R}e\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
12.	$j\mathcal{J}m\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$
13.	$\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
14.	$\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j\mathcal{J}m\{\tilde{X}[k]\}$
	Properties 15–17 apply only when $x[n]$ is real.	
15.	Symmetry properties for $\tilde{x}[n]$ real.	$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\} \\ \mathcal{J}m\{\tilde{X}[k]\} = -\mathcal{J}m\{\tilde{X}[-k]\} \\ \lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert \\ \sphericalangle\tilde{X}[k] = -\sphericalangle\tilde{X}[-k] \end{cases}$
16.	$\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
17.	$\tilde{x}_0[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[-n])$	$j\mathcal{J}m\{\tilde{X}[k]\}$

Last time we also discussed sampling of the Discrete-Time Fourier Transform (DTFT)

The idea is that most real-life signals $x[n]$ are not naturally periodic – and that's a major problem as related to Fourier analysis since periodicity was a prerequisite for DFS.

Therefore, if we instead

- find the DTFT of $x[n]$, i.e., $X(e^{j\omega})$, we then have a periodic function with period 2π
- Next we sample one period of $X(e^{j\omega})$ (in the frequency domain)
- and finally let these samples represent one period of the DFS sequence

then we have what we are looking for, namly;

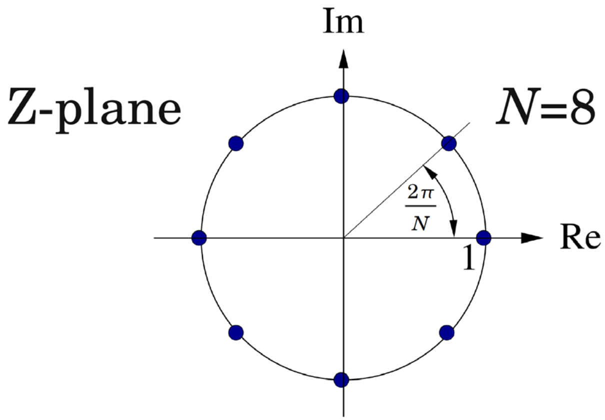
…a periodic sequence of Fourier Series coefficients.

Remember that we concluded, that this sequence can be viewed as one separate period, or it can be seen as an infinite series of consecutive periods – it's the same…

The math behind this trick… once again

Remember that the DTFT of a sequence $x[n]$, i.e., $X(e^{j\omega})$, is identically equal to the z-transform $X(z)$ on the unit circle; $z = e^{j\omega}$.

We now consider *N* equally spaces point on the unit circle, in this example *N=8*;



We next sample $X(z)$ in the points $z = e^{j\left(\frac{2\pi}{N}\right)k}$ which leads to a sampling of the DTFT

Tilde denotes "periodicity" in *k* with period *N*

$$X(z)\big|_{z=e^{j\left(\frac{2\pi}{N}\right)k}} = X(e^{j\frac{2\pi}{N}k}) = X\left(e^{j\omega_k}\right) = \tilde{X}[k] \qquad 0 \leq k \leq N-1$$

The DFS coefficients

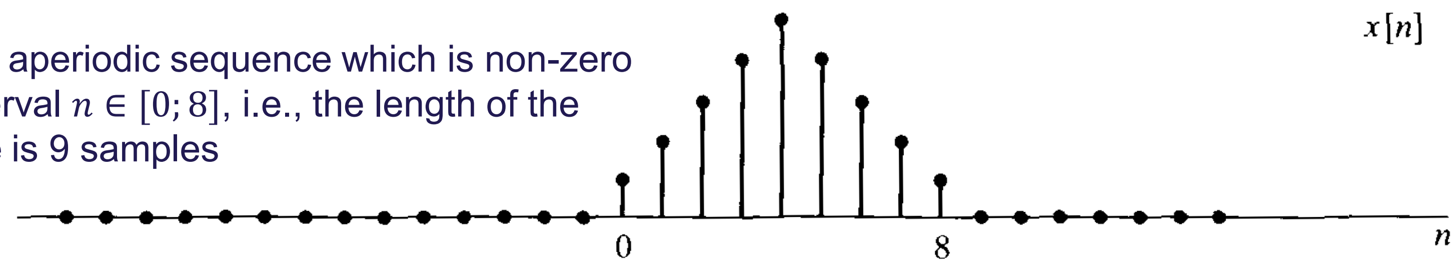
Question now is, whether this equal sign holds always…

$$\tilde{X}[k] = X(z)\Big|_{z=e^{j\left(\frac{2\pi}{N}\right)k}} = X\left(e^{j\left(\frac{2\pi}{N}\right)k}\right) = X(e^{j\omega_k}) \qquad 0 \le k \le N-1$$

This expression represents an N-periodic sequence of samples which could be the sequence of Discrete Fourier Series coefficients of a sequence $\tilde{x}[n]$.

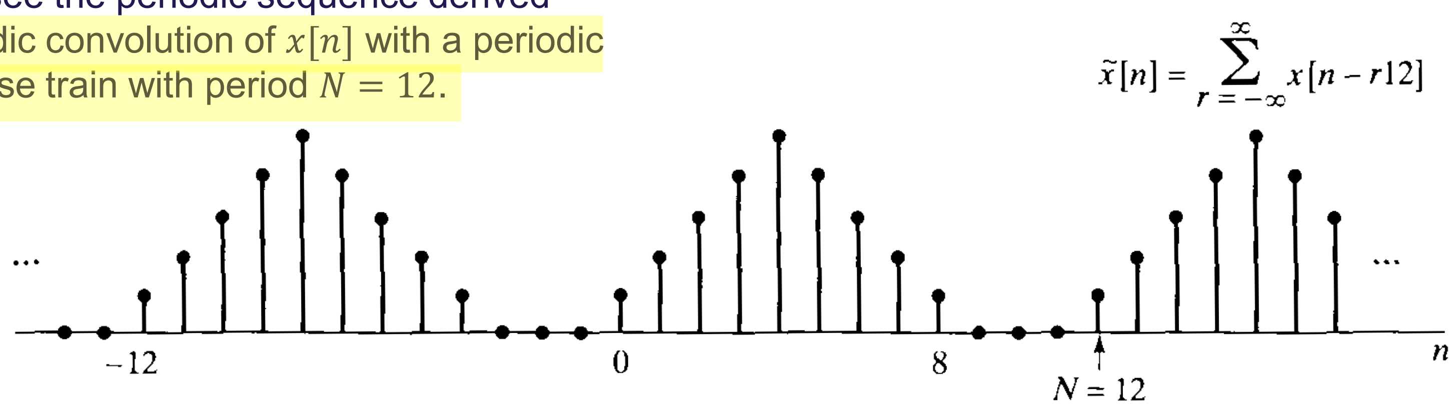
On p. 667 you'll find the math leading to the conclusion that $\tilde{x}[n]$, which corresponds to $\tilde{X}[k]$ obtained by sampling $X(z)$, is formed from $x[n]$ by adding together an infinite number of shifted replicas of $x[n]$. The shifts are all positive and negative integer multiples of *N.*

$x[n]$ is an aperiodic sequence which is non-zero in the interval $n \in [0; 8]$, i.e., the length of the sequence is 9 samples



(a)

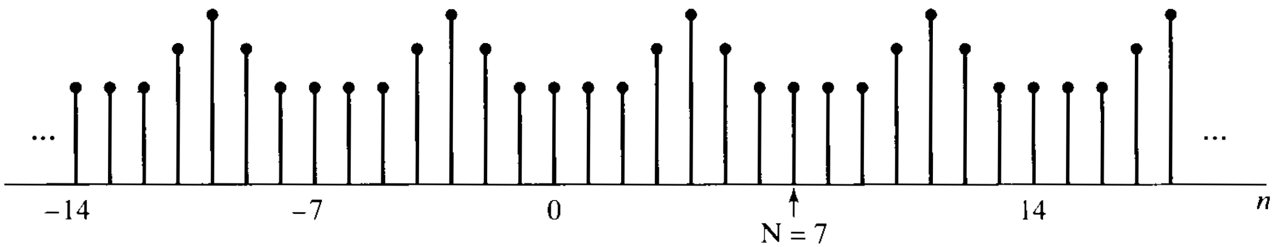
Here we see the periodic sequence derived by aperiodic convolution of $x[n]$ with a periodic unit-impulse train with period $N = 12$.



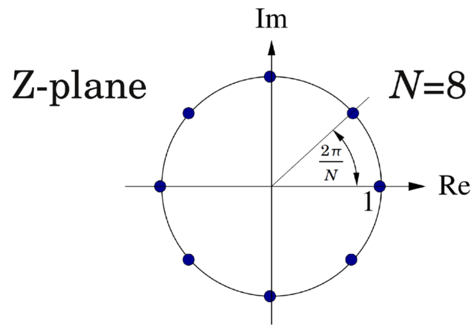
What happens if the period is less than 9 (in the example)..?

Here we have the same sequence $x[n]$, but now the period $N = 7$

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - r7]$$



Basically what we see here is "an overlap in the time domain" which can be considered as "aliasing" – the period N is too small, i.e., there are too few samples in the frequency domain..



Consequently, time domain aliasing can be avoided only if $x[n]$ has finite length, just as frequency domain aliasing can be avoided only for signals that have bandlimited Fourier transforms.

OBS...!!

And now… The Discrete Fourier Transform

We begin by considering a finite-length sequence $x[n]$ of length N samples such that $x[n] = 0$ outside the range $0 \leq n \leq N-1$. In many instances, we will want to assume that a sequence has length N even if its length is $M \leq N$. In such cases, we simply recognize that the last $(N-M)$ samples are zero. To each finite-length sequence of length N, we can always associate a periodic sequence

This is known as ”zero padding”

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN].$$

So, N is the length of the period, and r is number of the period.

$$\tilde{x}[n] = x[(n \text{ modulo } N)]$$

For convenience, we will use the notation $((n))_N$ to denote (n modulo N)

$$\tilde{x}[n] = x[((n))_N]$$

These two sequences are identical only when $x[n]$ has a length which is less than or equal to N - if it is longer, then the upper expression will use samples from ”the next period” which are not $((n))_N$.

See slide no. 21.

Now, the sequence of discrete Fourier series coefficients $\tilde{X}[k]$ of the periodic sequence $\tilde{x}[n]$ is itself a periodic sequence with period N. To maintain a duality between the time and frequency domains, we will choose the Fourier coefficients that we associate with a finite-duration sequence to be a finite-duration sequence corresponding to one period of $\tilde{X}[k]$. This finite-duration sequence, $X[k]$, will be referred to as the discrete Fourier transform (DFT). Thus, the DFT, $X[k]$, is related to the DFS coefficients, $\tilde{X}[k]$, by

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\tilde{X}[k] = X[(k \text{ modulo } N)] = X[((k))_N].$$

So, the DFS is an infinite periodic sequence, whereas the DFT represents only one period of this sequence.

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The DFS vs. the DFT

Discrete **F**ourier **S**eries –
Analysis and Synthesis

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn},$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

Discrete **F**ourier **T**ransformation –
Analysis and Synthesis

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] W_N^{kn}, & 0 \le k \le N-1, \\ 0, & \text{otherwise,} \end{cases}$$

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

Generally, the DFT analysis and synthesis equations are written as follows:

Analysis equation: $$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn},$$

Synthesis equation: $$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}.$$

That is, the fact that $X[k] = 0$ for k outside the interval $0 \le k \le N-1$ and that $x[n] = 0$ for n outside the interval $0 \le n \le N-1$ is implied, but not always stated explicitly.

However, since the DFT represents one period of the DFS, the inherent periodicity is always present…!!

It means essentially that the DFT spectrum $X[k]$ is also periodic with period 2π despite that it represents a finite length sequence $x[n]$.

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OBS…!!!

Some Properties of the DFT
- Circular Shift of a Sequence (without proof)

From the DTFT properties we remember; ”time shift ⇔ phase shift”;

$$x[n-n_d] \quad (n_d \text{ an integer}) \longleftrightarrow e^{-j\omega n_d}X(e^{j\omega})$$

Now we will consider the operation in the time domain that corresponds to multiplying the DFT coefficients of a finite-length sequence $x[n]$ by the linear phase factor $e^{-j(2\pi k/N)m}$. Specifically, let $x_1[n]$ denote the finite-length sequence for which the DFT is $e^{-j(2\pi k/N)m}X[k]$; i.e., if

$$x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k],$$

Altså – hvilken operation får vi i tids-domænet, når vi i frekvens-domænet ganger med W_N^{km}…??

then we are interested in $x_1[n]$ such that

$$x_1[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_1[k] = e^{-j(2\pi k/N)m}X[k].$$

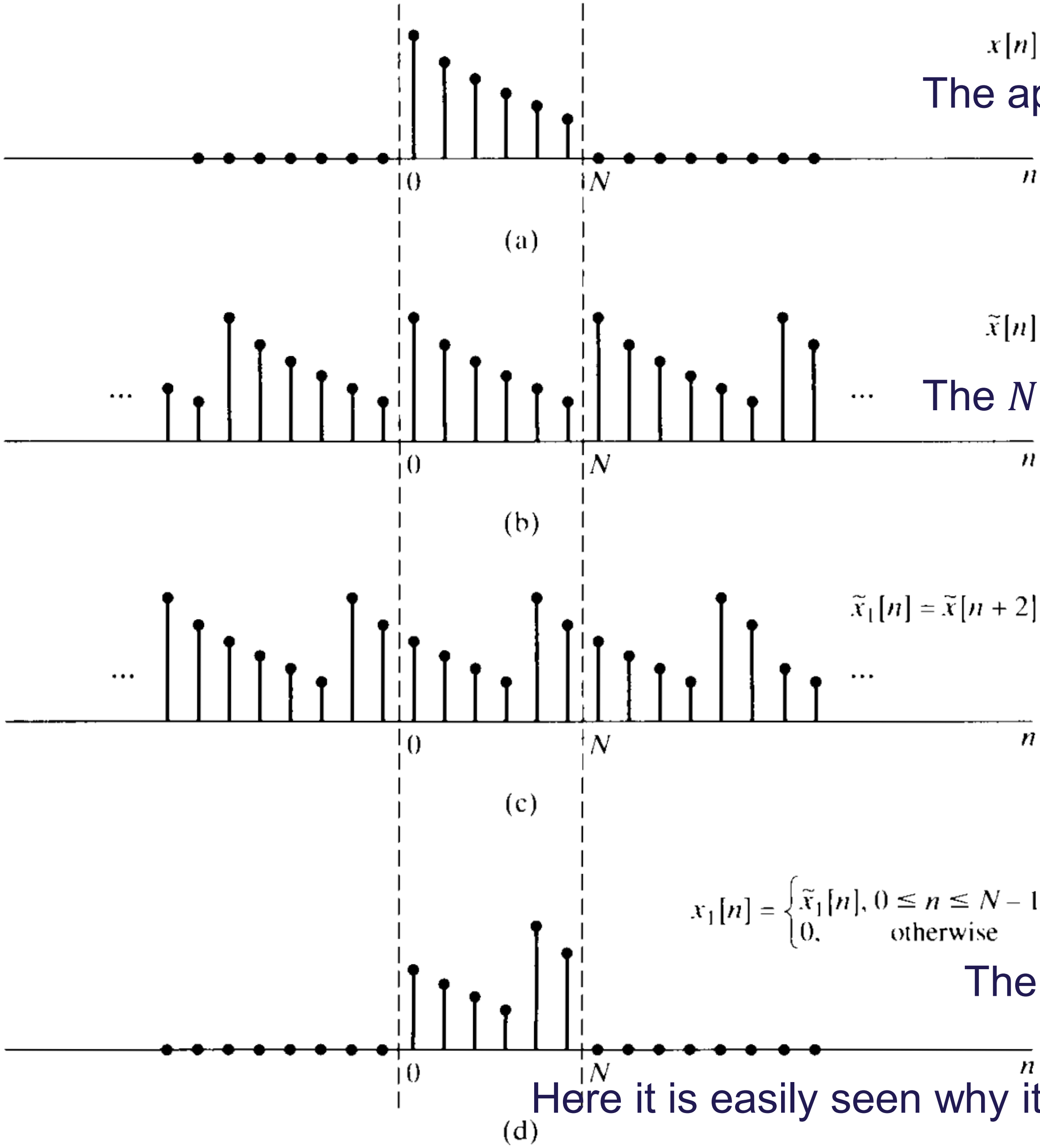
Since the N-point DFT represents a finite-duration sequence of length N, both $x[n]$ and $x_1[n]$ must be zero outside the interval $0 \le n \le N-1$, and consequently, $x_1[n]$ cannot result from a simple time shift of $x[n]$. - but rather a ”Circular Time Shift” of $x[n]$.

See all the math argument on p. 676-677

Thus, the finite-length sequence $x_1[n]$ whose DFT is given by $X_1[k] = e^{-j(2\pi k/N)m}X[k]$ is

$$x_1[n] = \begin{cases} \tilde{x}_1[n] = x[((n-m))_N], & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

Circular Shift of a Sequence – An Example



The aperiodic sequence

The N-periodic counterpart

The periodic sequence advanced 2 samples

The aperiodic counterpart

Here it is easily seen why it is called ”circular shift”

Circular Convolution

Here we consider two *finite-duration* sequences $x_1[n]$ and $x_2[n]$, both of length N, with DFTs $X_1[k]$ and $X_2[k]$, respectively, and we wish to determine the sequence $x_3[n]$ for which the DFT is $X_3[k] = X_1[k]X_2[k]$.

Specifically, $x_3[n]$ corresponds to one period of $\tilde{x}_3[n]$, which is given by $\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$

$$x_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m], \qquad 0 \le n \le N-1,$$

Periodic Convolution, see. slide 14-17

or, equivalently,

$$x_3[n] = \sum_{m=0}^{N-1} x_1[((m))_N]x_2[((n-m))_N], \qquad 0 \le n \le N-1.$$

Since $((m))_N = m$ for $0 \le m \le N-1$, and thus $x_3[n]$ can be re-written as;

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N], \qquad 0 \le n \le N-1.$$

This is known as an N-point Circular Convolution

N-point Circular Convolution $x_3[n] = x_1[n] \; Ⓝ \; x_2[n]$

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N], \qquad 0 \le n \le N-1$$

Looks like a linear convolution as we know it, but it differs in two important ways;

- The sequence x_2 is ”circularly time reversed” with respect to x_1.
- The sequence x_2 is ”circularly shifted” with respect to x_1.

”Circularly Time Reversed” – what does that actually mean…????

(here n=0)

$y[n] = x[((n\text{-}m))_6]$

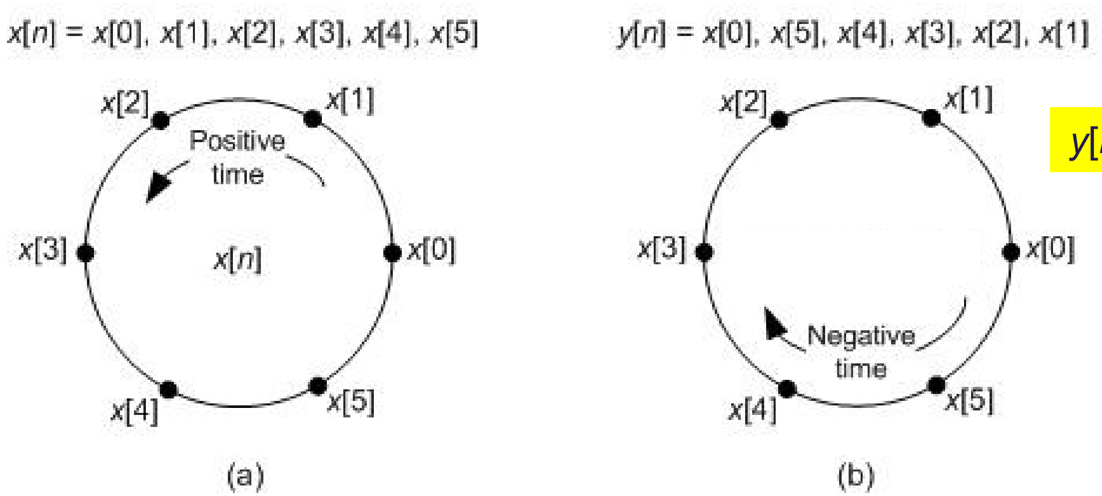


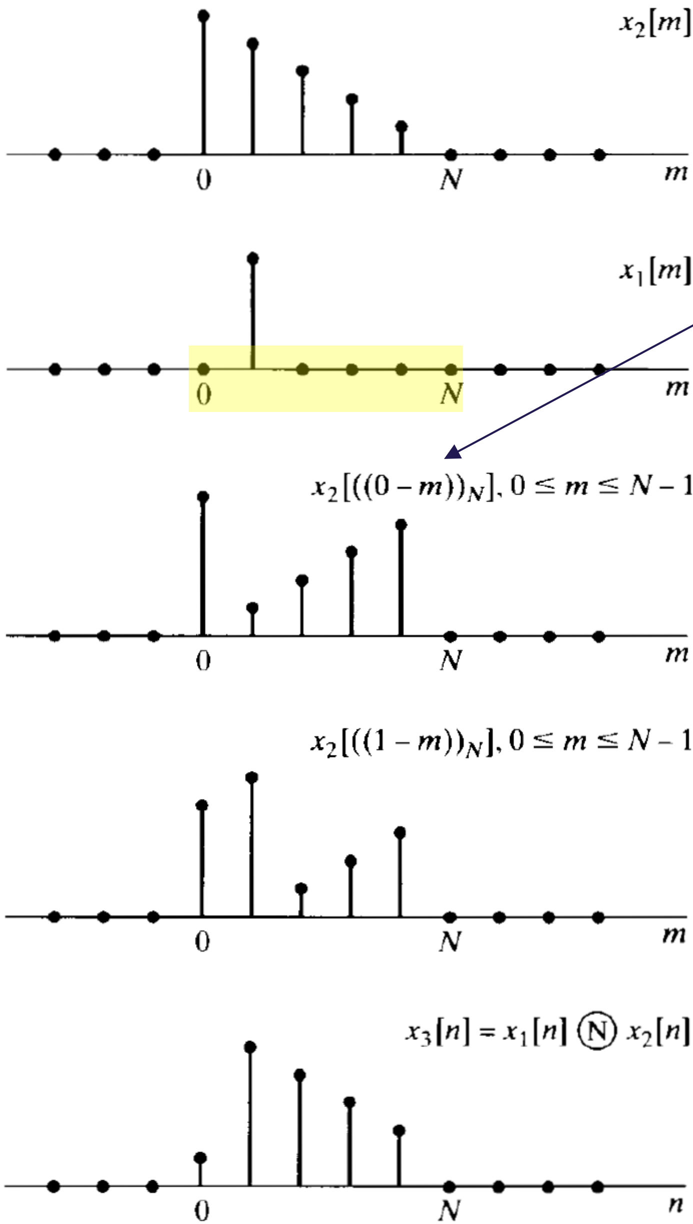
FIGURE A-1. Graphical description of circular time-reversal: (a) an x[n] time sequence; (b) y[n] sequence equal to a circular reversed x[n].

Circular convolution – an example;
convolution with $x_1[n] = \delta[n-1]$

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N], \qquad 0 \le n \le N-1$$

$$x_1[n] = \begin{cases} 0, & 0 \le n < n_0, \\ 1, & n = n_0, \\ 0, & n_0 < n \le N-1. \end{cases}$$

Finite length sequence



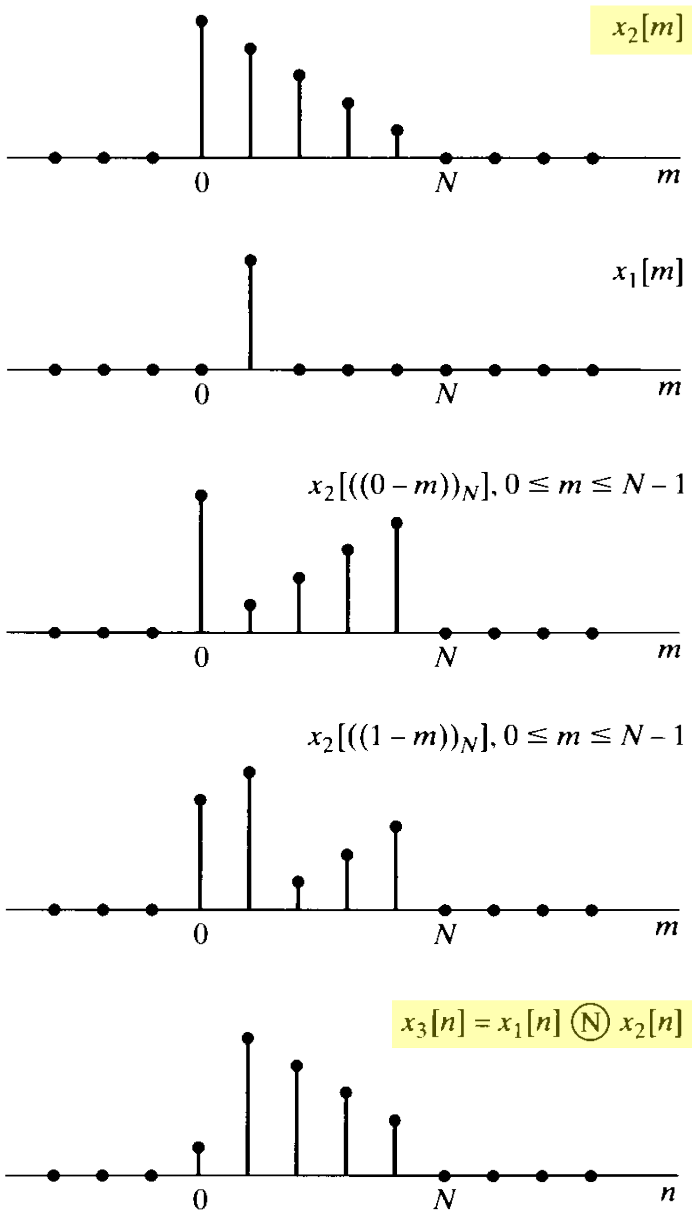
Here x_2 is circularly time reversed AND circular shifted 0 samples

Here x_2 is circularly time reversed AND circular shifted 1 samples

etc….

Continuing this shifting process for n=0..$N-1$ and for every n multiply with $x_1[n]$. Finally, add the N resulting sequences together to get $x_3[n]$

What else can we learn from this example…??



Phase shift…

The DFT of $x_1[n]$ is $X_1[k] = W_N^{kn_0}$

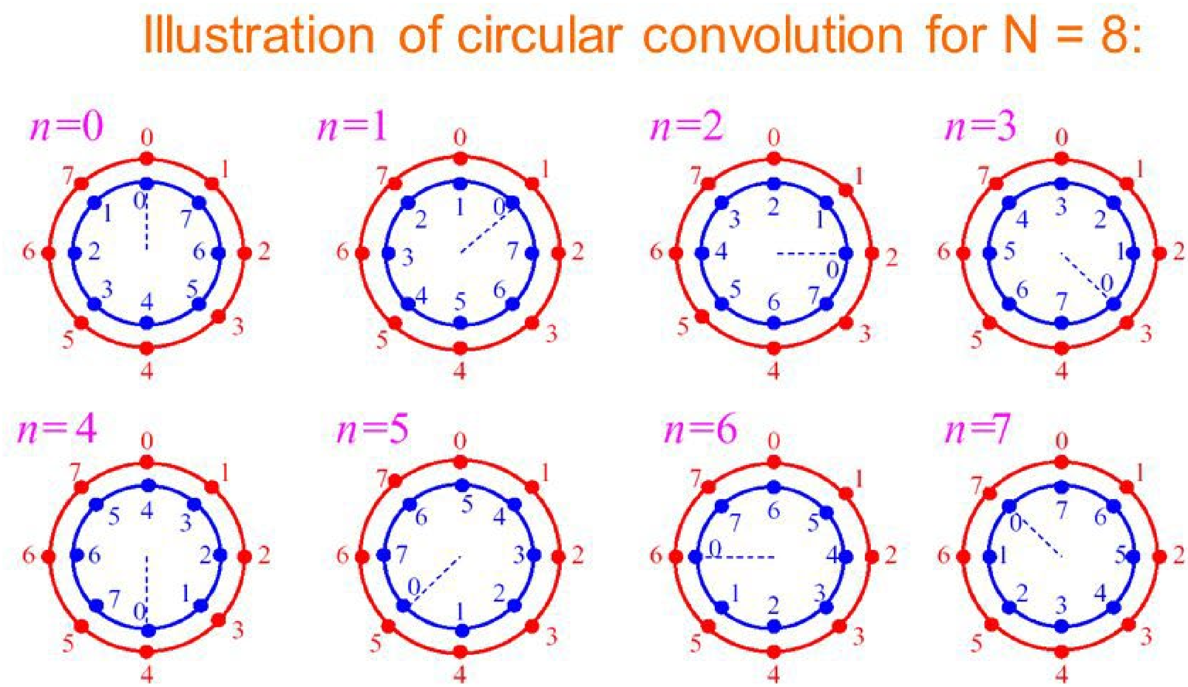
Since convolution in the time domain is equivalent to multiplication in the frequency domain, we have;

$$X_3[k] = W_N^{kn_0} X_2[k]$$

Now, the finite-length sequence $x_3[n]$ which corresponds to $X_3[k]$, is then seen to be $x_2[n]$ circular shifted right by $n_0 = 1$ sample.

An illustrative explanation of Circular Convolution

The idea is to represent the two finite length sequences on two concentric circles – one linearly (blue) and the other circularly time reversed (red).
Then these two circles are rotated relative to each other, and for every shift the appropriate sample values are multiplied and added.



When is circular convolution needed / useful…??

Given two length N sequences$x_1[n]$ and$x_2[n]$.

If $X_1[k] = DFT\{x_1[n]\}$ and $X_2[k] = DFT\{x_2[n]\}$ then

$$X_1[k]X_2[k] = DFT\{x_1[n] \, \textcircled{N} \, x_2[n]\}$$

Computation of the convolution in the freqency domain. An example…

Efficient Frequency Domain Computation: For large FIR filters or systems where frequency domain filtering is faster than time domain, the DFT can be used to compute convolution by transforming both $x[n]$ and $h[n]$ into the frequency domain, multiplying them element-wise, and then taking the inverse DFT.

However, because the DFT inherently performs circular convolution, the outputs align directly only if both sequences are periodic or truncated to fit the same length.

Therefore, if the input signal is longer than the filter response, we can **divide the input into blocks** of the same length as the filter and use circular convolution on each block. This technique is known as **overlap-save** or **overlap-add** methods, where circular convolution is applied on each block, and the overlapping segments are added appropriately to maintain the correct linear convolution result.

You may want yourself to study these efficient methods (for large N) if needed…

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Circular convolution is a commutative operation

Likewise linear convolution, circular convolution is a commutative operation.

$$x_3[n] = x_1[n] \circledN x_2[n] \qquad x_3[n] = \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

$$x_3[n] = x_2[n] \circledN x_1[n] \qquad x_3[n] = \sum_{m=0}^{N-1} x_2[m] x_1[((n-m))_N]$$

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	Finite-Length Sequence (Length N)	N-point DFT (Length N)	
1.	$x[n]$	$X[k]$	
2.	$x_1[n], x_2[n]$	$X_1[k], X_2[k]$	
3.	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$	Linear
4.	$X[n]$	$Nx[((-k))_N]$	Duality
5.	$x[((n-m))_N]$	$W_N^{km} X[k]$	Time shift
6.	$W_N^{-\ell n} x[n]$	$X[((k-\ell))_N]$	Frequency shift (heterodyne)
7.	$\sum_{m=0}^{N-1} x_1(m) x_2[((n-m))_N]$	$X_1[k]X_2[k]$	Circular convolution in time
8.	$x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1(\ell) X_2[((k-\ell))_N]$	Circular convolution in frequency
9.	$x^*[n]$	$X^*[((-k))_N]$	
10.	$x^*[((-n))_N]$	$X^*[k]$	
11.	$\mathcal{R}e\{x[n]\}$	$X_{\text{ep}}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$	Complex Conjugated Symmetry
12.	$j\mathcal{J}m\{x[n]\}$	$X_{\text{op}}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$	
13.	$x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$	
14.	$x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$	
	Properties 15–17 apply only when $x[n]$ is real.		
15.	Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{J}m\{X[k]\} = -\mathcal{J}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \sphericalangle\{X[k]\} = -\sphericalangle\{X[((-k))_N]\} \end{cases}$	
16.	$x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$	
17.	$x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$	