Digital Signal Processing ESD5/IV5

3rd lecture

Suggested Solution

1. a) $$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}}$$

$$= \frac{2}{\left(\frac{2}{T}\frac{z-1}{z+1}+1\right)\left(\frac{2}{T}\frac{z-1}{z+1}+3\right)}$$

$$= \frac{2(z+1)^2}{\left(\frac{2}{T}(z-1)+(z+1)\right)\left(\frac{2}{T}(z-1)+3(z+1)\right)}$$

$$= \frac{2(z+1)^2}{\left(\frac{2}{T}\right)^2(z-1)^2+\frac{6}{T}(z-1)(z+1)+\frac{2}{T}(z-1)(z+1)+3(z+1)^2}$$

$$= \frac{2(z+1)^2}{\left(\frac{2}{T}\right)^2(z^2+1-2z) + \frac{8}{T}(z^2-1) + 3(z^2+1+2z)}$$

$$= \frac{2(z^2+2z+1)}{\left(\left(\frac{2}{T}\right)^2 + \frac{8}{T} + 3\right)z^2 + \left(6 - 2\left(\frac{2}{T}\right)^2\right)z + \left(\left(\frac{2}{T}\right)^2 - \frac{8}{T} + 3\right)}$$

$$= \frac{2z^2+4z+2}{483z^2 - 794z + 323}$$

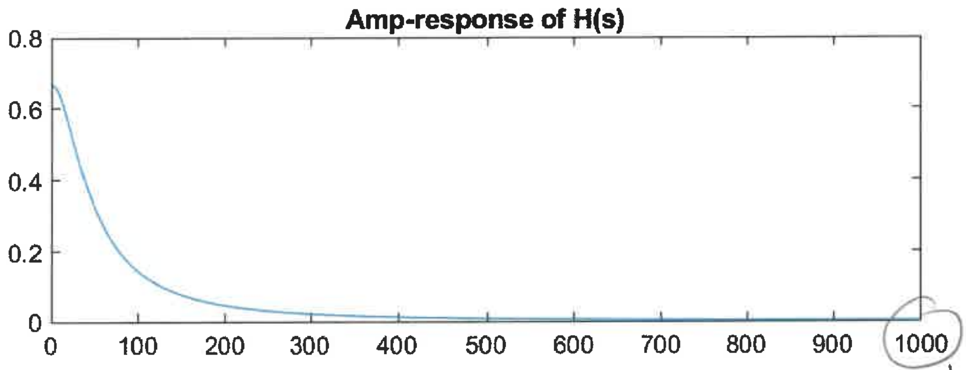
$$= \frac{2+4z^{-1}+2z^{-2}}{483 - 794z^{-1} + 323z^{-2}}$$

$$= K \cdot \frac{2+4z^{-1}+2z^{-2}}{1-1.6439z^{-1}+0{,}6687z^{-2}}, \quad K = \frac{1}{483}$$

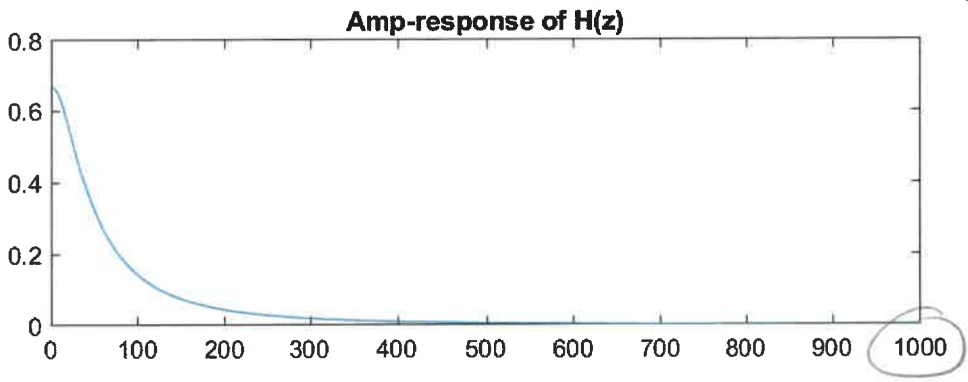
③

$$H(s) = \frac{2}{(s+1)(s+3)}$$

$$H(z) = \frac{1}{483} \cdot \frac{2 + 4z^{-1} + 2z^{-2}}{1 - 1.6439z^{-1} + 0.6687z^{-2}}$$



5Hz



$f_s/2 = 5\,Hz$

- De to amp.-responser har nøjagtig samme DC-forstærkning
- Sammenligner man for $f = 0.5\,Hz$ findes at $|H(j\Omega)| = 0.1417$ og $|H(e^{j\omega})| = 0.1400$
- Sammenligner man for $f = f_s/2 = 5\,Hz$ findes at $|H(j\Omega)| = 0.0020$ og $|H(e^{j\omega})| = 0.0000$

2) $H(s) = \dfrac{\Omega_c}{s+\Omega_c}$ Bestem $H(z)$ for sampleperiode lig T. og $\omega_c = \pi/4$

a) Da vi ønsker den „digitale knæk-frekvens" ω_c beliggende ved $\pi/4$, må vi først beregne den pre-warpede analoge knækfrekvens;

$$\Omega_c' = \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right) = \frac{2}{T}\cdot\tan\left(\frac{\pi}{8}\right)$$

Nu anvendes den bilineære transformation

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1},\ \Omega_c=\Omega_c'}$$

$$H(z) = \frac{\Omega_c'}{\frac{2}{T}\frac{z-1}{z+1}+\Omega_c'}$$

$$= \frac{\frac{2}{T}\tan\left(\frac{\pi}{8}\right)}{\frac{2}{T}\frac{z-1}{z+1}+\frac{2}{T}\tan\left(\frac{\pi}{8}\right)}$$

Bemærk at sampleperioden kan forkortes ud.

$$= \frac{\tan(\frac{\pi}{8})(z+1)}{(z-1) + \tan(\frac{\pi}{8})(z+1)}$$

$$= \frac{\tan(\frac{\pi}{8})(1+z^{-1})}{(1+\tan(\frac{\pi}{8})) + (\tan(\frac{\pi}{8})-1)z^{-1}}$$

$$= \frac{0.4142(1+z^{-1})}{1.4142 - 0.5854z^{-1}}$$

$$= \frac{0.2929(1+z^{-1})}{1 - 0.4142z^{-1}}$$

Bemærk at jeg her har bragt overføringsfunktionen på formen $H(z) = \frac{B(z)}{A(z)}$ hvor $A(z) = 1 + \sum_{k=1}^{N} a_k z^{-k}$

Denne form er særdeles anvendelig, når vi senere ønsker at konvertere til tidsdomænet.

b) Ligger 3dB frekvense, der hvor vi ~~ønsker~~ ...?

$$H(e^{j\omega}) = 0.2929 \frac{1 + e^{-j\omega}}{1 - 0.4142 e^{-j\omega}}$$

$$H(e^{j\frac{\pi}{4}}) = 0.2929 \frac{1 + e^{-j\frac{\pi}{4}}}{1 - 0.4142 e^{-j\frac{\pi}{4}}}$$

$$H(e^{j\frac{\pi}{4}}) = 0.2929 \cdot \frac{\left(1 + \cos\left(\frac{\pi}{4}\right)\right) + j\sin\left(\frac{\pi}{4}\right)}{1 - 0.4142\cos\left(\frac{\pi}{4}\right) + j\,0.4142\sin\left(\frac{\pi}{4}\right)}$$

$\Downarrow$

$$\left|H(e^{j\frac{\pi}{4}})\right| = 0.2929 \cdot \frac{\sqrt{(1.7071)^2 + (0.7071)^2}}{\sqrt{(0.7071)^2 + (0.2929)^2}}$$

$$= 0.2929 \cdot \frac{1.8478}{0.7654}$$

$$= 0.7071 \text{ (dvs } 1/\sqrt{2}\text{)}$$

Så OK...!

3) $$H(s) = \frac{s}{s+\Omega_c}$$

Anvend den bilineære transformation til beregning af $H(z)$. $T = 1/150$ s og det effektive filter skal ha' 3dB frekvens ved 30Hz

a) Med 3dB frekvensen ved 30Hz og en sample frekvens på 150Hz, må $\omega_c = \frac{2\pi}{5}$. Med udgangspunkt heri kan vi nu beregne, hvor Ω_c skal være beliggende FØR vi anvender den bilineære transformation, dvs. vi skal bestemme den pre-warpede analoge knækfrekvens;

$$\Omega_c' = \frac{2}{T}\cdot\tan\left(\frac{\omega_c}{2}\right) = 300\cdot\tan\left(\frac{\frac{2\pi}{5}}{2}\right)$$

$$= 217{,}9 \text{ rad/sec.}$$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1},\ \Omega_c=\Omega_c'}$$

$$H(z) = \frac{\frac{2}{T}\frac{z-1}{z+1}}{\frac{2}{T}\frac{z-1}{z+1} + \Omega_c'}$$

Igen, vi ser, at sample-perioden kan forkortes ud

——— eller vi kunne også prøve blot at fortsætte vores beregninger;

$$H(z) = \frac{\frac{2}{T}(z-1)}{\frac{2}{T}(z-1) + \Omega_c'(z+1)}$$

$$= \frac{\frac{2}{T}(z-1)}{\left(\frac{2}{T} + \Omega_c'\right)z + \left(\Omega_c' - \frac{2}{T}\right)}$$

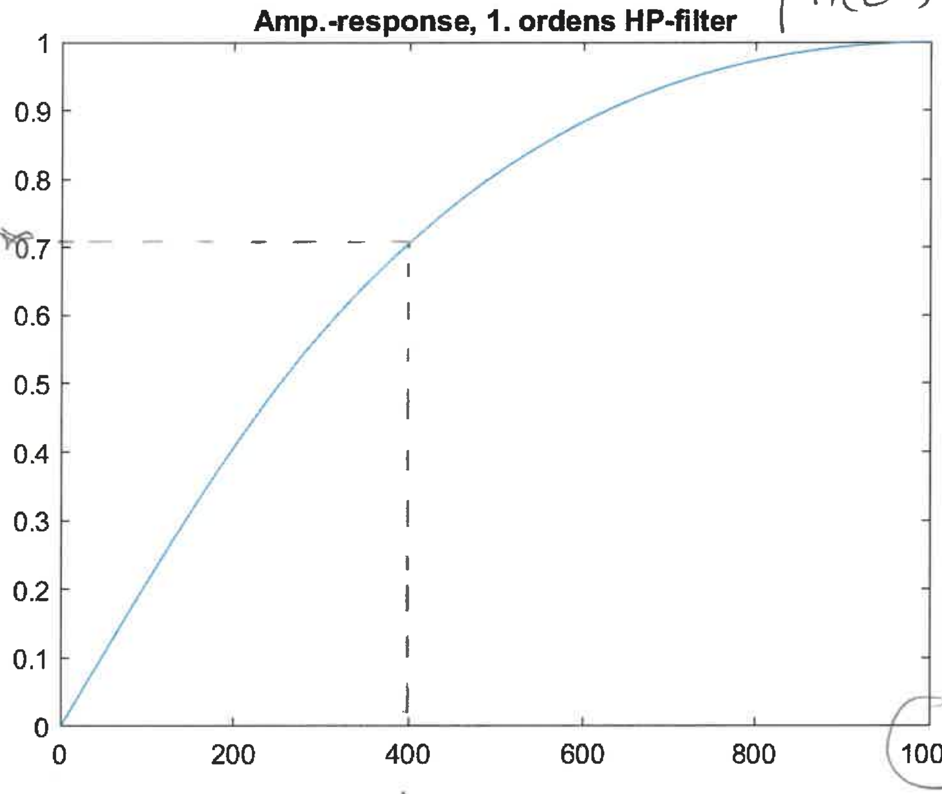
$$= \frac{\frac{2}{T}}{\frac{2}{T} + \Omega_c'} \cdot \frac{1 - z^{-1}}{1 + \left(\frac{\Omega_c' - \frac{2}{T}}{\frac{2}{T} + \Omega_c'}\right) z^{-1}}$$

$$= \frac{300}{300 + 217{,}9} \cdot \frac{1 - z^{-1}}{1 + \frac{217{,}9 - 300}{217{,}9 + 300} \cdot z^{-1}}$$

$$= 0.5792 \cdot \frac{1 - z^{-1}}{1 - 0.1584 z^{-1}}$$

$|H(e^{j\omega})|$

$1/\sqrt{2}$



$f_s/2 = 75$ Hz

~ 30Hz

$$\left|H\left(e^{j\frac{2\pi}{5}}\right)\right| = 1/\sqrt{2}$$

c) Filterets differensligning:

$$H(z) = \frac{Y(z)}{X(z)} = 0.5792 \cdot \frac{1 - z^{-1}}{1 - 0{,}1584 z^{-1}}$$

$\Downarrow$

$$Y(z)\left(1 - 0.1584 z^{-1}\right) = 0.5792 X(z)\left(1 - z^{-1}\right)$$

$\Downarrow$

$$Y(z) = 0.1584 Y(z) z^{-1} + 0.5792 X(z) - 0.5792 X(z) z^{-1}$$

$\mathcal{Z}^{-1} \Downarrow$

$$y[n] = 0.1584 y[n-1] + 0.5792 x[n] - 0.5792 x[n-1]$$