11

Interaction

11.1 Introduction

So far we have focused on control of simple loops with one sensor, one actuator, and one controller. In practical applications, a control system can have many loops, sometimes thousands. In spite of this, a large control system can often be dealt with loop by loop since the interaction between the loops is negligible. There are, however, situations when there may be considerable interaction between different control loops. A typical case is when several streams are blended to obtain a desired mixture. In such a case it is clear that the loops interact. Other cases are control of boilers, paper machines, distillation towers, chemical reactors, heat exchangers, steam distribution networks, drive systems, and systems for air-conditioning. Processes that have many control variables and many measured variables are called multi-input multi-output (MIMO) systems. Because of the interactions it may be difficult to control such systems loop by loop.

A reasonably complete treatment of multivariable systems is far outside the scope of this book. In this chapter we will briefly discuss some issues in interacting loops that are of particular relevance for PID control. Section 11.2 gives simple examples that illustrate what may happen in interacting loops. In particular it is shown that controller parameters in one loop may have significant influence on dynamics of other loops. Bristol's relative gain array, which is a simple way to characterize the interactions, is also introduced. The problem of pairing inputs and outputs is discussed, and it is shown that the interactions may generate zeros of a multivariable system. In Section 11.3 we present a design method based on decoupling, which is a natural extension of the tuning methods for single-input single-output systems. Section 11.4 presents problems that occur in drive systems with parallel motors. The chapter ends with a summary and references.

11.2 Interaction of Simple Loops

In this section we will illustrate some effects of interaction in the simplest case

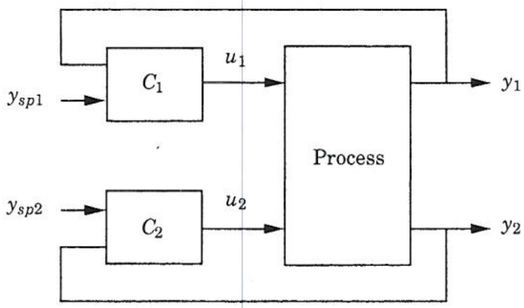

Figure 11.1 Block diagram of a system with two inputs and two outputs (TITO).

of a system with two inputs and two outputs. Such a system is called a TITO system. The system can be represented by the equations

$$\begin{aligned} Y_1(s) &= p_{11}(s)U_1(s) + p_{12}U_2(s) \\ Y_2(s) &= p_{21}(s)U_1(s) + p_{22}U_2(s), \end{aligned} \tag{11.1}$$

where $p_{ij}(s)$ is the transfer function from the j:th input to the i:th output. The transfer functions p_{11}, $p_{12}(s)$, $p_{21}(s)$, and p_{22} can be combined into the matrix

$$P(s) = \begin{pmatrix} p_{11}(s) & p_{12}(s) \\ p_{21}(s) & p_{22}(s) \end{pmatrix}, \tag{11.2}$$

which is called the transfer function or the matrix transfer function of the system. Some effects of interaction will be illustrated by an example.

EXAMPLE 11.1—EFFECTS OF INTERACTION
Consider the system described by the block diagram in Figure 11.1. The system has two inputs and two outputs. There are two controllers, the controller C_1 controls the output y_1 by the input u_1 and C_2 controls the output y_2 by the input u_2. One effect of interaction is that the tuning of one loop can influence the other loop. This is illustrated in Figure 11.2, which shows a simulation of the first loop when C_1 is a PI controller and $C_2 = k_2$ is a proportional controller.

The example shows that the gain of the second loop has a significant influence on the behavior of the first loop. The response of the first loop is good when the second loop is disconnected, $k_2 = 0$, but the system becomes more sluggish when the gain of the second loop is increased. The system is unstable for $k_2 = 0.8$.

Simple analysis gives insight into what happens. In the particular case the system is described by

$$\begin{aligned} Y_1(s) &= \frac{1}{(s+1)^2}U_1(s) + \frac{2}{(s+1)^2}U_2(s) \\ Y_2(s) &= \frac{1}{(s+1)^2}U_1(s) + \frac{1}{(s+1)^2}U_2(s). \end{aligned}$$

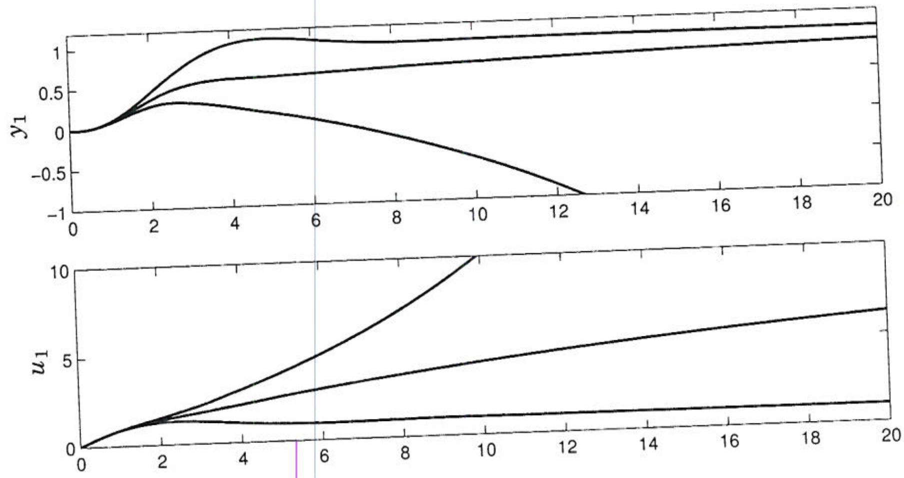

Figure 11.2 Simulation of responses to steps in set points for loop 1 of the system in Figure 11.1. Controller C_1 is a PI controller with gains $k_1 = 1$, $k_i = 1$, and the C_2 is a proportional controller with gains $k_2 = 0$, 0.8, and 1.6.

The feedback in the second loop is $U_2(s) = -k_2 Y_2(s)$. Introducing this in the second equation gives

$$U_2(s) = -\frac{k_2}{s^2 + 2s + k_2 + 1} U_1(s),$$

and insertion of this expression for $U_2(s)$ in the first equation gives

$$Y_1(s) = g_{11}^{cl}(s) U_1(s) = \frac{s^2 + 2s + 1 - k_2}{(s+1)^2(s^2 + 2s + 1 + k_2)} U_1(s).$$

This equation shows clearly that the gain k_2 in the second loop has a significant effect on the dynamics relating u_1 and y_1. The static gain is

$$g_{11}^{cl}(0) = \frac{1 - k_2}{1 + k_2}.$$

Notice that the gain decreases as k_2 increases and that the gain becomes negative for $k_2 > 1$. □

The example indicates that there is a need to have some way to determine if interactions may cause difficulties. A simple measure of interaction will now be discussed.

Bristol's Relative Gain Array

A simple way to investigate the effect of the interaction is to investigate how the static process gain of one loop is influenced by the gains in the other loops. Consider first the system with two inputs and two outputs shown in Figure 11.1. We will investigate how the static gain in the first loop is influenced by the controller in the second loop. To avoid making specific assumptions about the controller, Bristol assumed that the second loop was in perfect control, meaning that the output of the second loop is zero. It then follows from (11.1) that

$$\begin{aligned} Y_1(s) &= p_{11}(s)U_1(s) + p_{12}U_2(s) \\ 0 &= p_{21}(s)U_1(s) + p_{22}U_2(s). \end{aligned}$$

Eliminating $U_2(s)$ from the first equation gives

$$Y_1(s) = \frac{p_{11}(s)p_{22}(s) - p_{12}(s)p_{21}(s)}{p_{22}(s)} U_1(s).$$

The ratio of the static gains of loop 1 when the second loop is open and when the second loop is closed is thus

$$\lambda = \frac{p_{11}(0)p_{22}(0)}{p_{11}(0)p_{22}(0) - p_{12}(0)p_{21}(0)}. \tag{11.3}$$

Parameter λ is called *Bristol's interaction index* for TITO systems. Notice that the index refers to static conditions. In practice this can also be interpreted as interaction for low-frequency signals. There is no interaction if $p_{12}(0)p_{21}(0) = 0$, which implies that $\lambda = 1$. Small or negative values of λ indicate that there are interactions. Consider, for example, the system in Example 11.1 where the interaction index is $\lambda = -1$, which indicates that interactions pose severe difficulties.

The interaction index can be generalized to systems with many inputs and many outputs. The idea is to compare the static gains for one output when all other loops are open with the gains when all other outputs are zero. The result can be summarized in *Bristol's relative gain array* (RGA) which is defined as

$$R = P(0). * P^{-T}(0), \tag{11.4}$$

where $P(0)$ is the static gain of the system, $P^{-T}(0)$ the transpose of the inverse of $P(0)$, and $.*$ denotes component-wise multiplication of matrices. The element r_{ij} is the ratio between the open-loop and closed-loop static gains from the input signal u_j to the output y_i. It can be shown that the matrix R is symmetric and that all rows and columns sum to one. Notice that Bristol's relative gain array only captures the behavior of the process at low frequencies.

For the system (11.1) the relative gain array becomes

$$R = \begin{pmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{pmatrix}, \tag{11.5}$$

where λ is the interaction index (11.3). There is no interaction if $\lambda = 1$. This means that the second loop has no impact on the first loop and vice versa. If λ is between 0 and 1 the closed loop has higher gain than the open loop. The effect is most severe for $\lambda = 0.5$. If λ is larger than 1 the closed loop has lower gain than the open loop. When λ is negative the gain of the first loop changes sign when the second loop is closed. The effect of the interactions is thus severe.

Pairing

To control a system loop by loop we must first decide how the controllers should be connected, i.e., if y_1 in Figure 11.1 should be controlled by u_1 or u_2. This is called the *pairing problem*.

The relative gain array can be used as a guide for pairing. There is no interaction if $\lambda = 1$. If $\lambda = 0$ there is also no interaction, but the loops should be interchanged. The loops should be interchanged when $\lambda < 0.5$. If $0 < \lambda < 1$ the gain of the first loop increases when the second loop is closed, and if $\lambda > 1$ the closed-loop gain is less than the open-loop gain. Bristol recommended that pairing should be made so that the corresponding relative gains are positive and as close to one as possible. Pairing of signals with negative relative gains should be avoided. If the gains are outside the interval $0.67 < \lambda < 1.5$, decoupling can improve the control significantly. We illustrate pairing with an example.

EXAMPLE 11.2—PAIRING OF SIGNALS

Consider the system in Example 11.1. The static gain matrix is

$$P(0) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Its inverse is

$$P^{-1}(0) = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix},$$

and the relative gain array becomes

$$R = P(0).*P^{-T}(0) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} .* \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

which means that $\lambda = -1$. The pairing rule says that y_1 should be paired with u_2.

When $u_1 = -k_2 y_2$ the relation between u_2 and y_1 becomes

$$Y_1(s) = g_{12}^{cl}(s)U_2(s) = \frac{2s^2 + 4s + 2 + k_2}{(s+1)^2(s^2 + 2s + 1 + k_2)}U_2(s),$$

and the static gain is

$$g_{12}^{cl}(0) = \frac{2 + k_2}{1 + k_2}.$$

The gain decreases with increasing k_2, but it is never negative for $k_2 > 0$. There is interaction but not as severe as for the pairing of y_1 with u_1. The properties of the closed-loop system are illustrated in Figure 11.3. A comparison with Figure 11.2 shows that there is a drastic reduction in the interaction when the inputs are switched. □

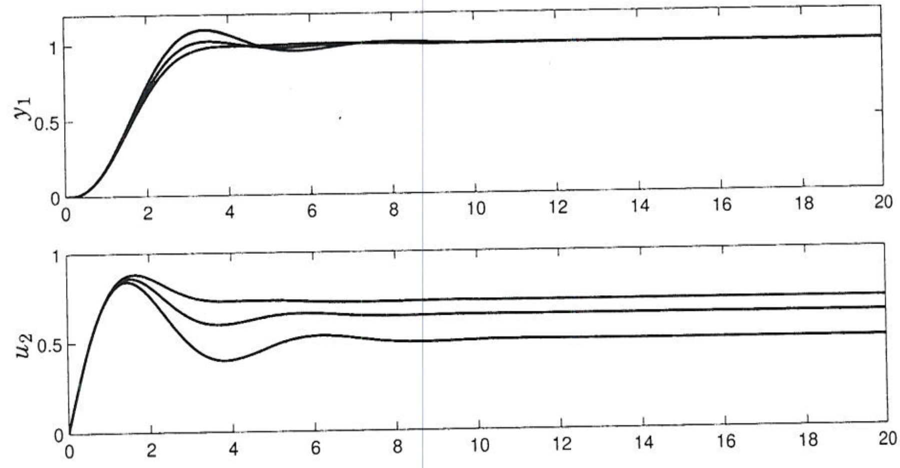

Figure 11.3 Simulation of responses to a step in the set point for y_1 of the system in Figure 11.1 when the loops are switched so that the controller for y_1 is $U_2 = C_1(s)(Y_{sp1} - Y_1)$ and the controller for y_2 is $u_1 = -k_2 y_2$ with $k_2 = 0$, 0.8, and 1.6. The controller C_1 is a PI controller with gains $k_1 = 1$, $k_i = 1$.

Multivariable Zeros

In Section 4.3 we found that right-half plane zeros imposed severe restriction on the achievable performance. For single-input single-output systems the zeros can be found by inspection. For multivariable systems zeros can, however, also be created by interaction. One definition of zeros that also works for multivariable systems is that the zeros are the poles of the inverse system. The zeros of the system (11.1) are given by

$$\det P(s) = p_{11}(s)p_{22}(s) - p_{12}(s)p_{21}(s) = 0. \tag{11.6}$$

Zeros in the right half plane are of particular interest because they impose limitations on the achievable performance. We illustrate this with an example.

EXAMPLE 11.3—ROSENBROCK'S SYSTEM
Consider a system with the transfer function

$$P(s) = \begin{pmatrix} p_{11}(s) & p_{12}(s) \\ p_{21}(s) & p_{22}(s) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{s+1} & \dfrac{2}{s+3} \\ \dfrac{1}{s+1} & \dfrac{1}{s+1} \end{pmatrix}. \tag{11.7}$$

The dynamics of the subsystems are very benign. There are no dynamics limitations in control of any individual loop. The relative gain array is

$$R = \begin{pmatrix} 1 & 2/3 \\ 1 & 1 \end{pmatrix} .* \begin{pmatrix} 3 & -3 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix},$$

which shows that there are significant interactions. Using the rules for pairing we find that it is reasonable to pair u_1 with y_1 and u_2 with y_2. Since $\lambda > 1.5$

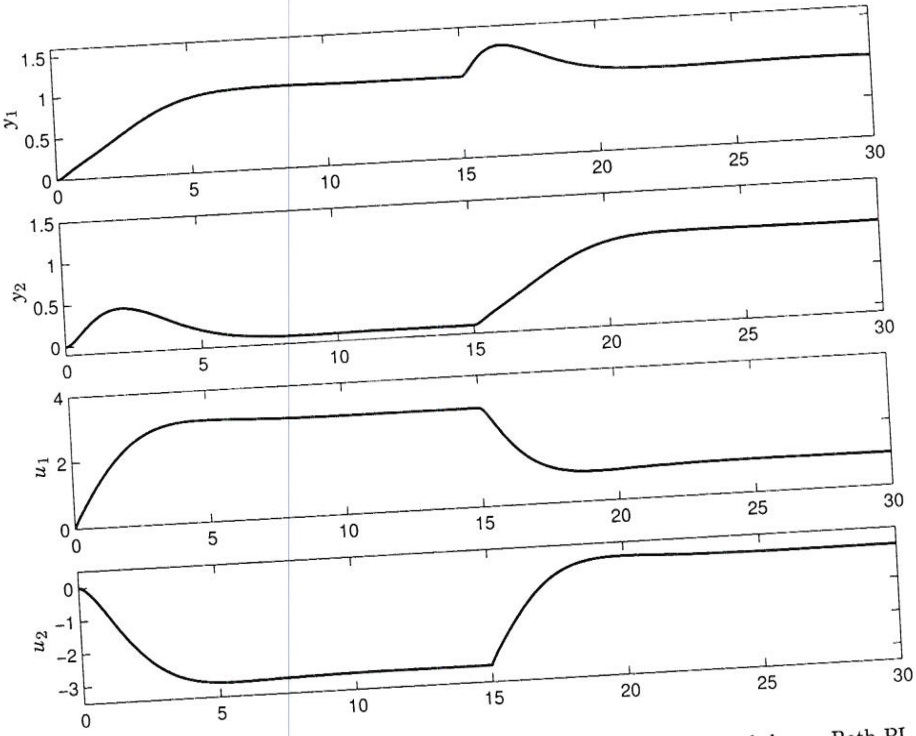

Figure 11.4 Step responses of the process (11.7) with PI control of both loops. Both PI controllers have gains $k = 2$ and $k_i = 2$. A step in y_{sp1} is first applied at time 0, and a step in y_{sp2} is then applied at time 15.

we can expect difficulties because of the interaction. It follows from (11.6) that the zeros of the system are given by

$$\det P(s) = \frac{1}{s+1}\left(\frac{1}{s+1} - \frac{2}{s+3}\right) = \frac{1-s}{(s+1)^2(s+3)} = 0.$$

There is a zero at $s = 1$ in the right half plane, and we can therefore expect difficulties when control loops are designed to have bandwidth larger than $\omega_0 = 1$.

Consider, for example, the problem of controlling the variable y_1. If the second loop is open we can achieve very fast response with a PI controller. When the second loop is closed there will, however, be severe performance limitations due to the interactions, and the control loop has to be detuned. Figure 11.4 shows responses obtained with controllers having gains $k = 2$ and $k_i = 2$ in both loops. In the figure we have first made a unit step in the set point of the first controller and then a set-point change in the second controller. The figure shows that there are considerable interactions. The system becomes unstable if the gain is increased by a factor of 3. □

Example 11.3 illustrates that an innocent-looking multivariable system may have zeros in the right half plane. The opposite is also possible, as is illustrated by the next example.

EXAMPLE 11.4—BENEFICIAL INTERACTION
Consider the system

$$P(s) = \begin{pmatrix} p_{11}(s) & p_{12}(s) \\ p_{21}(s) & p_{22}(s) \end{pmatrix} = \begin{pmatrix} \dfrac{s-1}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \\ \dfrac{-6}{(s+1)(s+2)} & \dfrac{s-2}{(s+1)(s+2)} \end{pmatrix}. \qquad (11.8)$$

The system has the relative gain array

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which indicates that y_1 should be paired with u_1 and that y_2 should be paired with u_2. The multivariable system has no zeros. We thus have the interesting situation that there are severe limitations to control either the first or the second loop individually because of the right-half plane zeros in the elements p_{11} and p_{22}. Since the multivariable system does not have any right-half plane zeros it is possible to control the multivariable system with high bandwidth. This is illustrated in Figure 11.4, where both loops are controlled with PI controllers having gains $k = 100$ and $k_i = 2000$. Notice the fast response of the system. One difficulty is, however, that the system becomes unstable if one of the loops is broken.

□

11.3 Decoupling

Decoupling is a simple way to deal with the difficulties created by interactions between loops. The idea is to design a controller that reduces the effects of the interaction. Ideally, changes in one set point should only affect the corresponding process output. This can be accomplished by a precompensator that mixes the signals sent from the controller to the process inputs. The details will be given for systems with two inputs and two outputs, but the method can be applied to signals with many inputs and many outputs.

Assume that the process has the transfer function (11.2) and that $P(0)$ is nonsingular. We first introduce a static decoupler $u = D\bar{u}$, where D is a constant matrix

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}.$$

The transfer function from $\bar{u}$ to y is then given by $P(s)D$. The choice

$$D = P^{-1}(0) = \frac{1}{\det P(0)} \begin{pmatrix} p_{22}(0) & -p_{12}(0) \\ -p_{21}(0) & p_{11}(0) \end{pmatrix} \qquad (11.9)$$

makes $P(0)D$ the identity matrix. The system $P(s)D$ is thus statically decoupled, and the coupling is small for low frequencies. The coupling remains small if the system is controlled by decoupled controllers, provided that the bandwidths of the control loops are sufficiently small.

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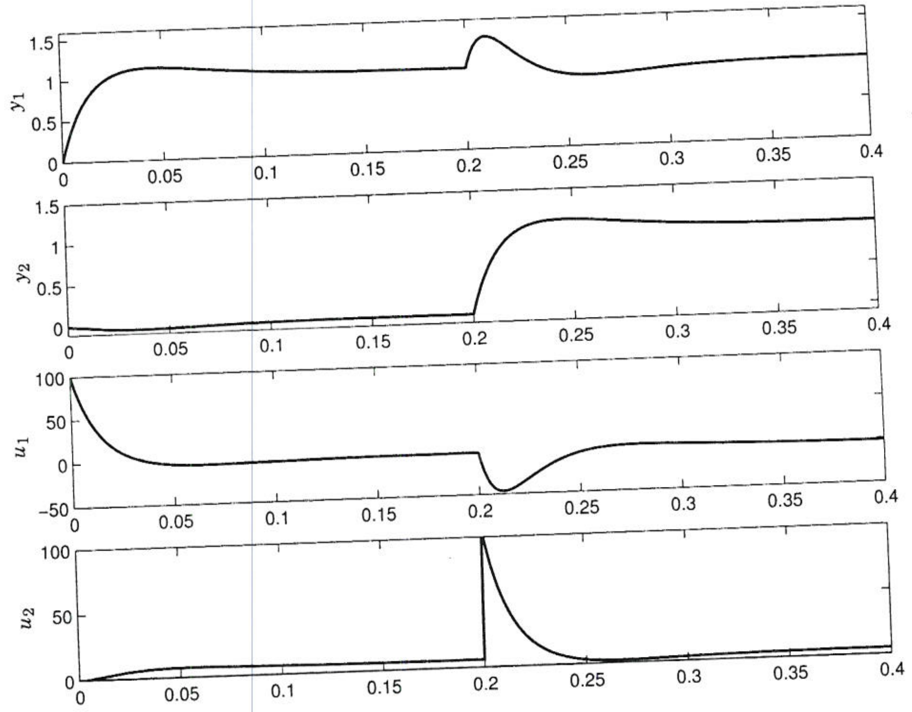

Figure 11.5 Step responses of PI control of the process (11.8) when both loops are closed. PI controllers with gains $k = 100$ and $k_i = 2000$ are used in both loops.

Assuming that the controllers are PID controllers we find that the statically decoupled controller is described by

$$\begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} \bar{c}_1(s)Y_{sp1}(s) - c_1(s)Y_1(s) \\ \bar{c}_2(s)Y_{sp2}(s) - c_2(s)Y_2(s) \end{pmatrix},$$

where U is the control signal, Y the process output, and Y_{sp} the set point. The controllers are PID controllers with set-point weighting, hence,

$$c_i = k_{Pi} + \frac{k_{Ii}}{s} + k_{Di}s, \quad \bar{c}_i = b_i k_{Pi} + \frac{k_{Ii}}{s},$$

where b_i is the set-point weight. The set-point weights influence the interaction between the loops. Choosing $b_i = 0$ gives the smallest interaction.

The Decoupled System

The transfer function of the decoupled system is $Q(s) = P(s)D$, where

$$
\begin{aligned}
q_{11}(s) &= \frac{p_{11}(s)p_{22}(0) - p_{12}(s)p_{21}(0)}{\det P(0)}\\
q_{12}(s) &= \frac{p_{12}(s)p_{11}(0) - p_{12}(0)p_{11}(s)}{\det P(0)}\\
q_{21}(s) &= \frac{p_{21}(s)p_{22}(0) - p_{21}(0)p_{22}(s)}{\det P(0)}\\
q_{22}(s) &= \frac{p_{22}(s)p_{11}(0) - p_{21}(s)p_{12}(0)}{\det P(0)}.
\end{aligned}
$$

It follows from the construction that $Q(0)$ is the identity matrix. A Taylor series expansion of the transfer function $Q(s)$ for small $|s|$ gives

$$
Q(s) \approx \begin{pmatrix} 1 & \kappa_{12}s \\ \kappa_{21}s & 1 \end{pmatrix}
$$

for some constants κ_{12} and κ_{21}. For low frequencies ω, the diagonal elements of $Q(s)$ are equal to one, and the off-diagonal elements are proportional to s. If the bandwidth of the decentralized PID controller is sufficiently low, the off-diagonal terms will thus be small, and the system will be approximately decoupled. The closed-loop system can be described by

$$
\begin{pmatrix} 1 + q_{11}c_1 & q_{12}c_2 \\ q_{21}c_1 & 1 + q_{22}c_2 \end{pmatrix} Y = \begin{pmatrix} q_{11}\bar{c}_1 & q_{12}\bar{c}_2 \\ q_{21}\bar{c}_1 & q_{22}\bar{c}_2 \end{pmatrix} Y_{sp},
$$

where the dependency on s is suppressed to simplify the notation. This equation can be written as

$$
Y = \bar{H} Y_{sp},
$$

where

$$
\begin{aligned}
\bar{h}_{11} &= \frac{q_{11}\bar{c}_1(1 + q_{22}c_2) - q_{12}q_{21}\bar{c}_1c_2}{(1 + q_{11}c_1)(1 + q_{22}c_2) - q_{12}q_{21}c_1c_2}\\
\bar{h}_{12} &= \frac{q_{12}\bar{c}_2(1 + q_{22}c_2) - q_{12}q_{22}\bar{c}_2c_2}{(1 + q_{11}c_1)(1 + q_{22}c_2) - q_{12}q_{21}c_1c_2}\\
\bar{h}_{21} &= \frac{q_{21}\bar{c}_1(1 + c_1q_{11}) - q_{11}q_{21}c_1\bar{c}_1}{(1 + q_{11}c_1)(1 + q_{22}c_2) - q_{12}q_{21}c_1c_2}\\
\bar{h}_{22} &= \frac{q_{22}\bar{c}_2(1 + q_{11}c_1) - q_{12}q_{21}c_1\bar{c}_2}{(1 + q_{11}c_1)(1 + q_{22}c_2) - q_{12}q_{21}c_1c_2}.
\end{aligned}
$$

Since we designed the controllers so that the interactions are small, the term $q_{12}q_{21}$ is smaller than $q_{11}q_{22}$. The matrix $\bar{H}$ can then be approximated by

$$
\bar{H} \approx H = \begin{pmatrix} \dfrac{q_{11}\bar{c}_1}{1 + q_{11}c_1} & \dfrac{q_{12}\bar{c}_2}{1 + q_{11}c_1} \\ \dfrac{q_{21}\bar{c}_1}{1 + q_{22}c_2} & \dfrac{q_{22}\bar{c}_2}{1 + q_{22}c_2} \end{pmatrix}.
$$

The diagonal elements of H are the same as for SISO control design. The standard methods for design of PI controllers presented in Chapters 6 and 7 can be used to find the controllers c_1 and c_2. By analysing the off-diagonal elements we can estimate how severe the interactions are. The controllers may have to be detuned to make sure that the interactions are tolerable. The interaction can be reduced arbitrarily by making the control loops sufficiently slow. The interaction analysis also gives the performance loss due to the interaction. If much performance is lost it is advisable to consider other design methods.

Estimating Effects of Interaction

A simple way to estimate the effects of the interactions will now be developed. The off-diagonal elements of H are given by

$$h_{12} = \frac{q_{12}\bar{c}_2}{1+q_{11}c_1}$$
$$h_{21} = \frac{q_{21}\bar{c}_1}{1+q_{22}c_2}.$$

Notice that $q_{11}(0) = q_{22}(0) = 1$ and that $q_{12} \approx \kappa_{12}s$ and $q_{21}(s) \approx \kappa_{21}s$ for small s. Since the controllers have integral action, we have for small s

$$h_{12}(s) \approx \frac{\kappa_{12}k_{I2}s}{k_{I1}}, \qquad h_{21}(s) \approx \frac{\kappa_{21}k_{I1}s}{k_{I2}}.$$

The interaction is thus very small at low frequencies, and we can thus guarantee that the interaction is arbitrarily small by having sufficiently slow controllers. To estimate the maximum of the interaction, we observe that

$$h_{12} = q_{12}\bar{c}_2 S_1, \quad h_{21} = q_{21}\bar{c}_1 S_2,$$

where $S_1 = (1+q_{11}c_1)^{-1}$ and $S_2 = (1+q_{22}c_2)^{-1}$ are the sensitivity functions for the loops when the interaction is neglected. A crude estimate of the interaction terms is thus

$$\max_\omega |h_{12}(i\omega)| \approx |\kappa_{12}|k_{I2}M_{s1}$$
$$\max_\omega |h_{21}(i\omega)| \approx |\kappa_{21}|k_{I1}M_{s2},$$

where M_{s1} and M_{s2} are the maximum sensitivities of the individual loops and where we have also used the estimate

$$q_{12}(s) \approx \kappa_{12}s, \quad q_{21}(s) \approx \kappa_{21}s$$

and

$$\bar{c}_1 \approx k_{I1}/s, \quad \bar{c}_2 \approx k_{I2}/s.$$

The interaction can thus be captured by the interaction indices

$$\kappa_1 = |\kappa_{12}k_{I2}|M_{s1}, \quad \kappa_2 = |\kappa_{21}k_{I1}|M_{s2}. \tag{11.10}$$

The index κ_1 describes how the second loop influences the first loop, and κ_2 describes how the first loop influences the second loop. Note that the term κ_{12} depends on the system and the integral gain k_{I2} in the second loop. Interaction can thus be reduced by making the integral gains lower. The estimates are not precise because of the approximations made. They are not reliable when there is a significant difference in the bandwidths of the loops.