



State Space Methods

Lecture 2: controllability and state feedback

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Controllability

A continuous time system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0$$

is said to be **controllable** iff for any $\xi \in \mathbb{R}^n$ there exists $u(t)$ such that for some $T > 0$, $x(T) = \xi$.

A discrete time system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = 0$$

is said to be **controllable** iff for any $\xi \in \mathbb{R}^n$ there exists $(u(0), u(1), \dots)$ such that for some $N > 0$, $x(N) = \xi$.

Controllability

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = 0$$

and iterate:

$$x(1) = Ax(0) + Bu(0)$$

\vdots

Controllability

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = 0$$

and iterate:

$$x(1) = Bu(0)$$

$$x(2) = Ax(1) + Bu(1)$$

$$\vdots$$

Controllability

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = 0$$

and iterate:

$$x(1) = Bu(0)$$

$$x(2) = ABu(0) + Bu(1)$$

$$\vdots$$

Controllability

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = 0$$

and iterate:

$$\begin{aligned}x(1) &= Bu(0) \\x(2) &= ABu(0) + Bu(1) \\x(3) &= A^2Bu(0) + ABu(1) + Bu(2) \\&\vdots\end{aligned}$$

Controllability

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = 0$$

and iterate:

$$x(1) = Bu(0)$$

$$x(2) = ABu(0) + Bu(1)$$

$$x(3) = A^2Bu(0) + ABu(1) + Bu(2)$$

$$\vdots$$

Controllability

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = 0$$

and iterate:

$$x(1) = Bu(0)$$

$$x(2) = ABu(0) + Bu(1)$$

$$x(3) = A^2Bu(0) + ABu(1) + Bu(2)$$

$$\vdots$$

$$x(n) = Ax(n-1) + Bu(n-1)$$

Controllability

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = 0$$

and iterate:

$$x(1) = Bu(0)$$

$$x(2) = ABu(0) + Bu(1)$$

$$x(3) = A^2Bu(0) + ABu(1) + Bu(2)$$

$$\vdots$$

$$x(n) = A^{n-1}Bu(0) + \dots + ABu(n-2) + Bu(n-1)$$

Controllability

Writing the equation

$$x(n) = A^{n-1}Bu(0) + \dots + ABu(n-2) + Bu(n-1)$$

in matrix form we obtain:

$$x(n) = \underbrace{(B \quad AB \quad \dots \quad A^{n-1}B)}_{\text{Controllability matrix}} \begin{pmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{pmatrix}$$

When is $x(n) = \xi$ solvable for any $\xi \in \mathbb{R}^n$?

Controllability

Theorem

A system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & (\text{continuous time}) \\ x(k+1) = Ax(k) + Bu(k) & (\text{discrete time}) \end{cases}$$

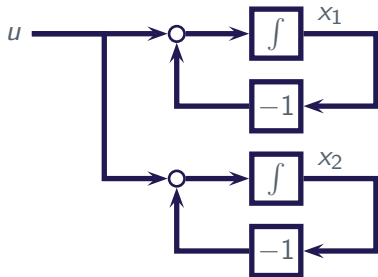
where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, is controllable if and only if

$$\text{rank} \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} = n$$

For $m = 1$ this reduces to

$$\det \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} \neq 0$$

Example: parallel connection



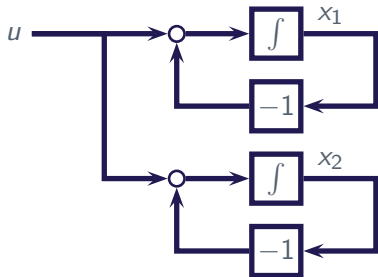
State space equations:

$$\begin{cases} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = -x_2 + u \end{cases}$$

State space equations in matrix form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

Example: parallel connection



$$\dot{x} = Ax + Bu$$

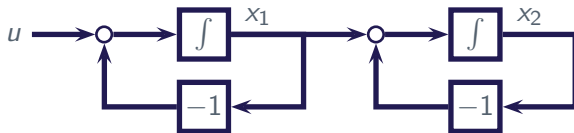
$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$C = (B \quad AB) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad \det(C) = 0$$

$$\text{rank}(C) = 1 < 2 \implies \text{uncontrollable}$$

Example: series connection



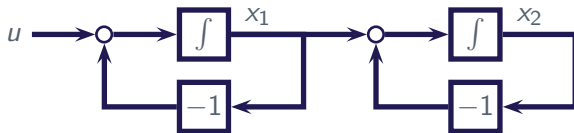
State equations:

$$\begin{cases} \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= -x_2 + x_1 \end{cases}$$

State space model in matrix form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

Example: series connection



$$A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Controllability analysis

$$\mathcal{C} = (B \quad AB) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \det(\mathcal{C}) = 1 \neq 0$$

$\text{rank}(\mathcal{C}) = 2 \implies \text{controllable}$

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Controllable canonical form

Any controllable *single input* system can be written in the form:

$$\dot{x}_c = A_c x_c + B_c u, \quad x_c \in \mathbb{R}^n, \quad u \in \mathbb{R}$$

where

$$A_c = \left(\begin{array}{c|c} a^T & \\ \hline I_{n-1} & 0_{(n-1) \times 1} \end{array} \right), \quad B_c = \left(\begin{array}{c} 1 \\ 0_{(n-1) \times 1} \end{array} \right)$$

and where $a \in \mathbb{R}^{n \times 1}$, $a^T = (a_1 \ a_2 \ \dots \ a_n)$. It can be shown that

$$\det(\lambda I - A_c) = \lambda^n - a_1 \lambda^{n-1} - \dots - a_n$$

Controllable canonical form

For $n = 3$ the controllable canonical form becomes:

$$A_c = \left(\begin{array}{cc|c} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right), \quad B_c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which is indeed controllable:

$$\mathcal{C}_c = (B_c \quad A_c B_c \quad A_c^2 B_c) = \begin{pmatrix} 1 & a_1 & a_1^2 + a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$\det(\mathcal{C}) = 1 \neq 0 \implies$ system is controllable.

Controllable canonical form

Given a state space model of a controllable system:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}$$

we wish to find a basis transformation $x = Tx_c$, such that:

$$\dot{x}_c = A_c x_c + B_c u, \quad x_c \in \mathbb{R}^n, \quad u \in \mathbb{R}$$

where $A_c = T^{-1}AT$ and $B_c = T^{-1}B$, is in controllable canonical form.

We can solve for T^{-1} by rewriting these equations as

$$A_c T^{-1} = T^{-1}A \quad \text{and} \quad B_c = T^{-1}B$$

Controllable canonical form

We consider $n = 3$, and introduce the following notation for the rows of T^{-1}

$$T^{-1} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad s_1, s_2, s_3 \in \mathbb{R}^{1 \times n}$$

Then we can rewrite the transformation equations $A_c T^{-1} = T^{-1} A$ and $T^{-1} B = B_c$ as:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A, \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Controllable canonical form

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A, \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

yields:

Controllable canonical form

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A, \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

yields:

$$\left\{ s_1 = s_2 A \right\}, \quad \left\{ \begin{array}{l} \\ \\ \end{array} \right\}$$

Controllable canonical form

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A, \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

yields:

$$\left\{ \begin{array}{l} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\}, \quad \left\{ \begin{array}{l} \\ \end{array} \right\}$$

Controllable canonical form

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A, \quad \begin{pmatrix} \boxed{s_1} \\ s_2 \\ s_3 \end{pmatrix} B = \begin{pmatrix} \boxed{1} \\ 0 \\ 0 \end{pmatrix}$$

yields:

$$\left\{ \begin{array}{l} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\}, \quad \left\{ \begin{array}{l} \boxed{s_1 B = 1} \end{array} \right\}$$

Controllable canonical form

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A, \quad \begin{pmatrix} s_1 \\ \boxed{s_2} \\ s_3 \end{pmatrix} B = \begin{pmatrix} 1 \\ \boxed{0} \\ 0 \end{pmatrix}$$

yields:

$$\left\{ \begin{array}{l} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\}, \quad \left\{ \begin{array}{l} s_1 B = 1 \\ \boxed{s_2 B = 0} \end{array} \right\}$$

Controllable canonical form

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A, \quad \begin{pmatrix} s_1 \\ s_2 \\ \boxed{s_3} \end{pmatrix} B = \begin{pmatrix} 1 \\ 0 \\ \boxed{0} \end{pmatrix}$$

yields:

$$\left\{ \begin{array}{l} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\}, \quad \left\{ \begin{array}{l} s_1 B = 1 \\ s_2 B = 0 \\ \boxed{s_3 B = 0} \end{array} \right\}$$

Controllable canonical form

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A, \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

yields:

$$\left\{ \begin{array}{l} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\}, \quad \left\{ \begin{array}{l} s_1 B = 1 \\ s_2 B = 0 \\ s_3 B = 0 \end{array} \right\}$$

Controllable canonical form

Combining the equations

$$\left\{ \begin{array}{l} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\}, \left\{ \begin{array}{l} s_1 B = 1 \\ s_2 B = 0 \\ s_3 B = 0 \end{array} \right\}$$

we obtain

$$s_3 (B \quad AB \quad A^2 B) = (0 \quad 0 \quad 1)$$

yielding the solution

$$s_3 = (0 \quad 0 \quad 1) C^{-1}, \quad s_2 = s_3 A, \quad s_1 = s_2 A$$

for nonsingular $C = (B \quad AB \quad A^2 B)$.

Example: companion form

We consider the system

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y &= \begin{pmatrix} -3 & 2 \end{pmatrix} x\end{aligned}$$

having the controllability matrix

$$\mathcal{C} = (B \quad AB) = \begin{pmatrix} 2 & -5 \\ 3 & -7 \end{pmatrix}, \quad \det(\mathcal{C}) = 1 \neq 0$$

Example: companion form

We compute the rows of T^{-1} by

$$s_2 = (0 \ 1) C^{-1} = (0 \ 1) \begin{pmatrix} -7 & 5 \\ -3 & 2 \end{pmatrix} = (-3 \ 2)$$

$$s_1 = s_2 A = (-3 \ 2) \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} = (2 \ -1)$$

Thus,

$$T^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \implies T = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

Example: companion form

Eventually, we have

$$\begin{aligned} A_c &= T^{-1}AT = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \\ &= \left(\begin{array}{cc|c} -3 & -2 & \\ 1 & & 0 \end{array} \right) \end{aligned}$$

and

$$B_c = T^{-1}B = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example: companion form

Eventually, we have

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and

$$B_c = T^{-1}B = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example: companion form

Eventually, we have

$$\begin{aligned} A_c &= T^{-1}AT = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \\ &= \left(\begin{array}{cc|c} -3 & -2 & \\ 1 & & 0 \end{array} \right) \Rightarrow \det(\lambda I - A) = (\lambda + 1)(\lambda + 2) \end{aligned}$$

and

$$B_c = T^{-1}B = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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State feedback

For a state space model

$$\dot{x} = Ax + Bu$$

a *state feedback* is a feedback of the form

$$u = Fx$$

Combining these two equations, we obtain:

$$\dot{x} = Ax + BFx = (A + BF)x$$

Thus, the result of a state feedback is a system with a modified system matrix, and thus with modified poles.

State feedback

For a single input system in companion form, a state feedback takes a particular simple form:

$$A_c = \left(\begin{array}{ccc|c} a_1 & a_2 & a_3 & \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right), \quad B_c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Applying the feedback $u = Fx$ with

$$F_c = (f_1 \quad f_2 \quad f_3)$$

We obtain:

$$\begin{aligned} A_c + B_c F_c &= \left(\begin{array}{ccc|c} a_1 & a_2 & a_3 & \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (f_1 \quad f_2 \quad f_3) \\ &= \left(\begin{array}{ccc|c} a_1 + f_1 & a_2 + f_2 & a_3 + f_3 & \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) \end{aligned}$$

State feedback

Thus, the characteristic polynomial has been changed from

$$\det(\lambda I - A_c) = \lambda^n - a_1 \lambda^{n-1} - \dots - a_n$$

to

$$\begin{aligned} \det(\lambda I - (A_c + B_c F_c)) = \\ \lambda^n - (a_1 + f_1) \lambda^{n-1} - \dots - (a_n + f_n) \end{aligned}$$

By choosing f_1, \dots, f_n appropriately, *any* closed loop pole configuration can be obtained. This is known as *pole assignment*.



Algorithm for pole assignment

Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$ be given.

1. Choose desired closed loop polynomial
 $\det(\lambda I - (A + BF)) = \lambda^n + a_{cl,1}\lambda^{n-1} + \dots + a_{cl,n}$.
2. Determine T , such that $A_c = T^{-1}AT$ and $B_c = T^{-1}B$ are in controllable canonical form.
3. Determine open loop polynomial
 $\det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \dots + a_n$
4. Define $F_c = (a_1 - a_{cl,1} \quad \dots \quad a_n - a_{cl,n})$.
5. Compute resulting feedback gain $F = F_c T^{-1}$.

Example: pole assignment

We consider again the system

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y &= \begin{pmatrix} -3 & 2 \end{pmatrix} x\end{aligned}$$

for which we would like to move the poles to $\{-4, -5\}$.

1. Desired closed loop polynomial: $\lambda^2 + 9\lambda + 20$
2. $T = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \Rightarrow A_c = \left(\begin{array}{cc|c} -3 & -2 & 1 \\ 1 & 0 & 0 \end{array} \right), B_c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
3. Open loop polynomial: $\lambda^2 + 3\lambda + 2$
4. $F_c = (3 - 9 \quad 2 - 20) = (-6 \quad -18)$
5. $F = F_c T^{-1} = (-6 \quad -18) \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = (42 \quad -30)$

Example: pole assignment

