Multible Input Multible Output systems
MIMO systems

Outline

Effects of interaction between control loops

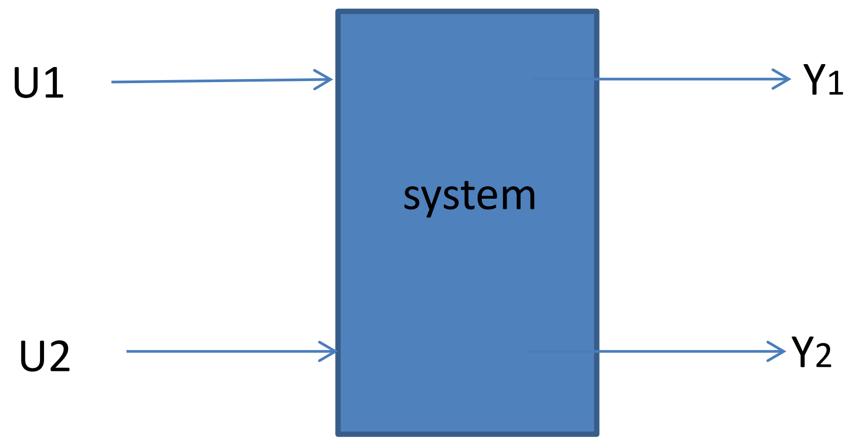
How do we find the best SISO loops

(Bristols relative gain array)

How do we eliminiate the effects of cross couplings in TITO systems - decoupling

Two input – two output systems
TITO system

Example 1:
Heat exchanger
Input: flow and temperature
Output:
Flow and temperature



Example 2:
Vehicle 2 independent wheels
Input: motor torque 1, motor torque 2
Output: velocity and direction

Example 3:
tank
Input: cold water flow, hot water flow
Output: level and temperature

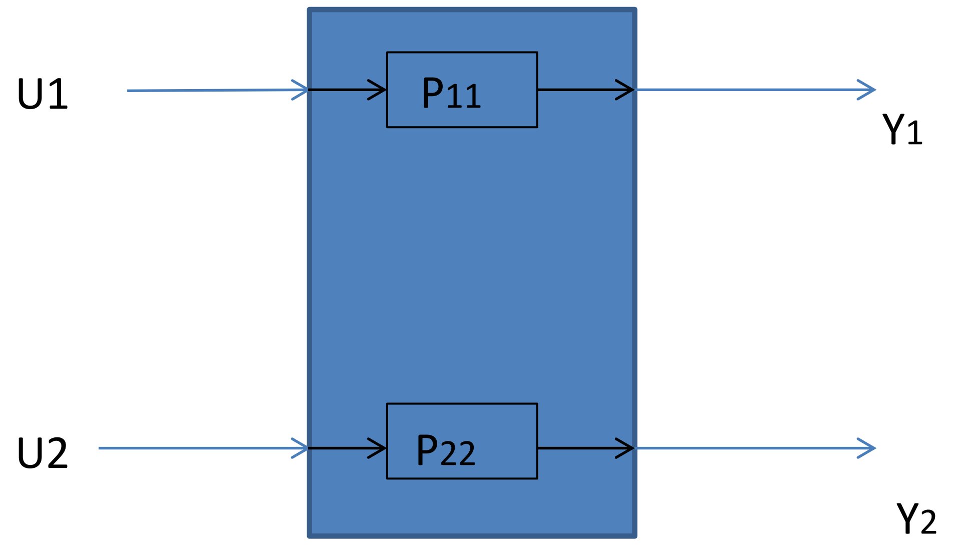
Example 4:
Crane
Input: motor torque 1, motor torque 2
Output: load position x and y direction

TITO system with independent input/output pair

System :

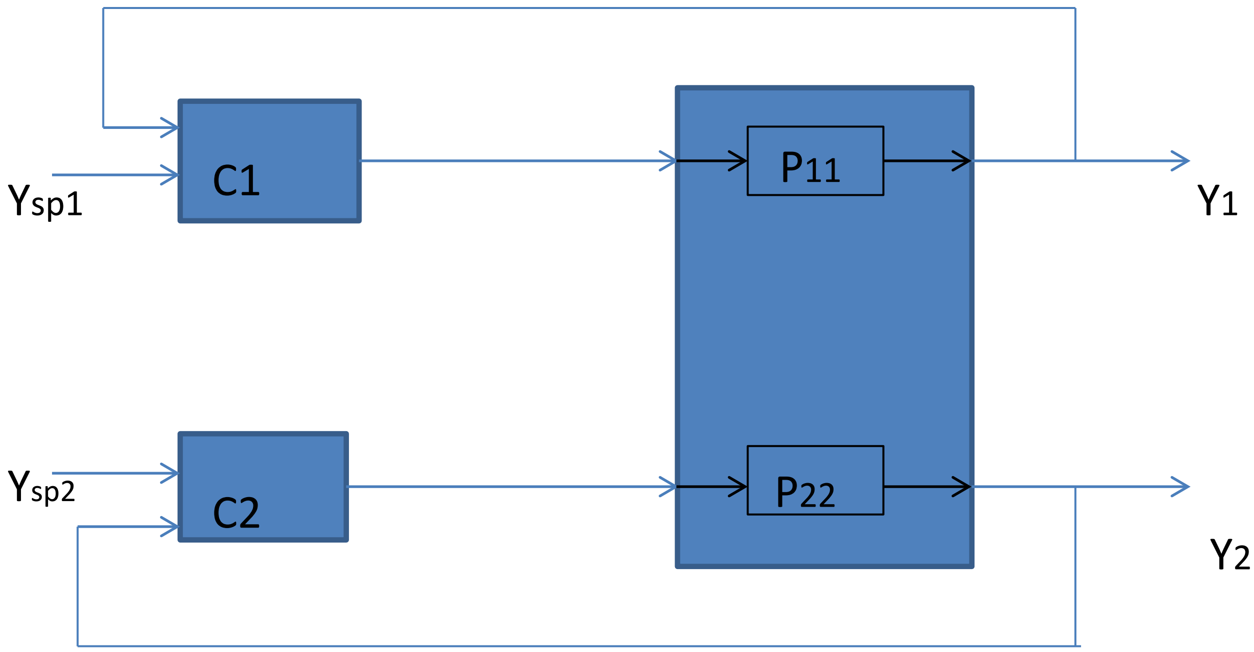
$Y1(s)=p11(s)U1(s)$

$Y2(s)=p22(s)U2(s)$



We can design two independent SISIO control systems

TITO system with independent input/output pair



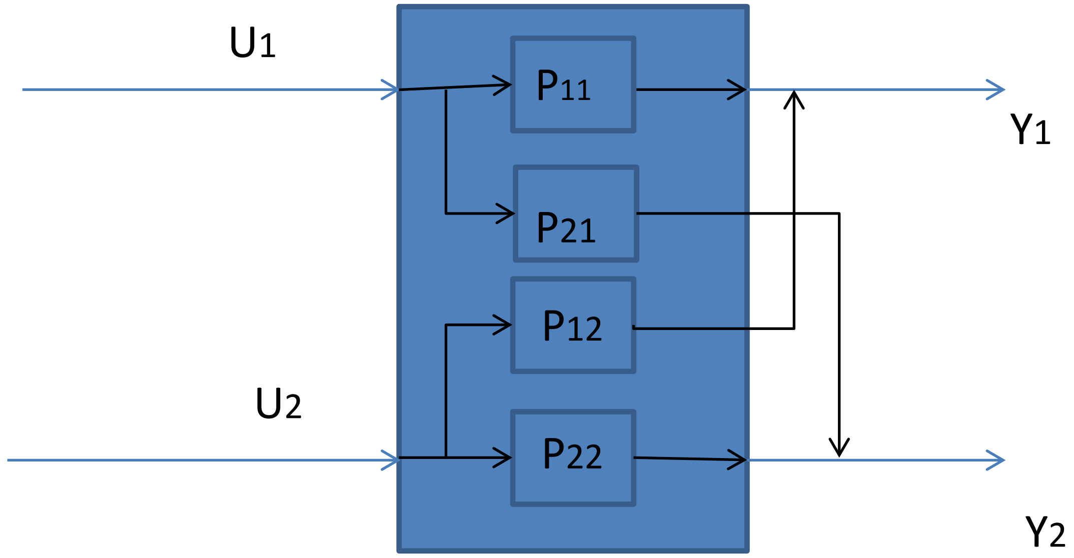
System : $Y_1(s)=P_{11}(s)U_1(s)$, $\dfrac{Y_1(s)}{Y_{sp1}(s)} = \dfrac{C_1(s)P_{11}(s)}{1+C_1(s)P_{11}(s)}$

$Y_2(s)=P_{22}(s)U_2(s)$), $\dfrac{Y_2(s)}{Y_{sp2}(s)} = \dfrac{C_2(s)P_{22}(s)}{1+C_2(s)P_{22}(s)}$

Interaction between inputs and outputs

Input: hot water, cold water

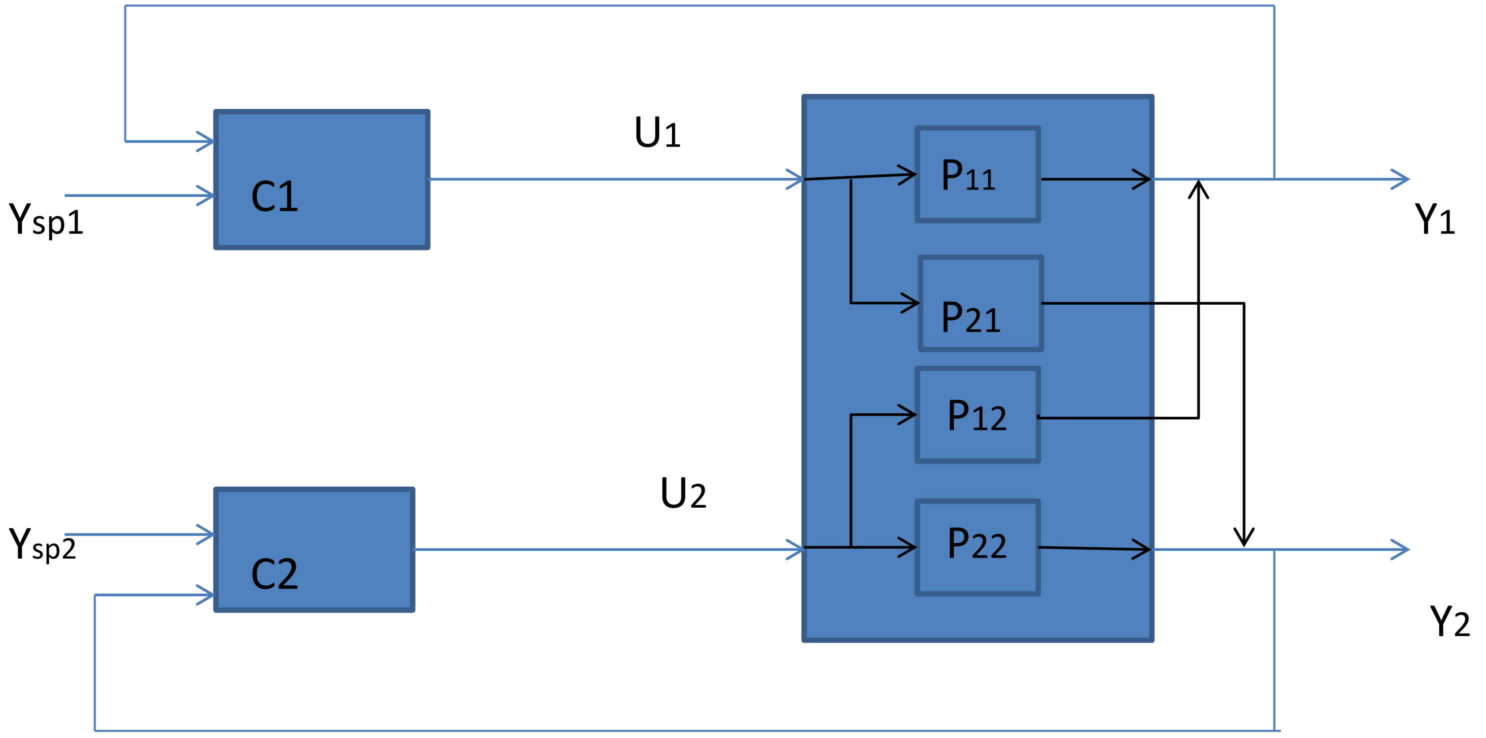
Output: temperature in tank, water in tank



Both input affects both output

$$Y1(s)=p11(s)U1(s)+P12(s)U2(s)$$
$$Y2(s)=p21(s)U1(s)+p22(s)U2(s)$$

Interaction in simple loops



Transfer function of a TITO system

$Y_1(s)=P_{11}(s)U_1(s)+P_{12}(s)U_2(s)$

$Y_2(s)=P_{21}(s)U_1(s)+P_{22}(s)U_2(s)$

$p_{ij}(s)$ is the transfer function from input j to output i

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

P(s)=transfer function of the system

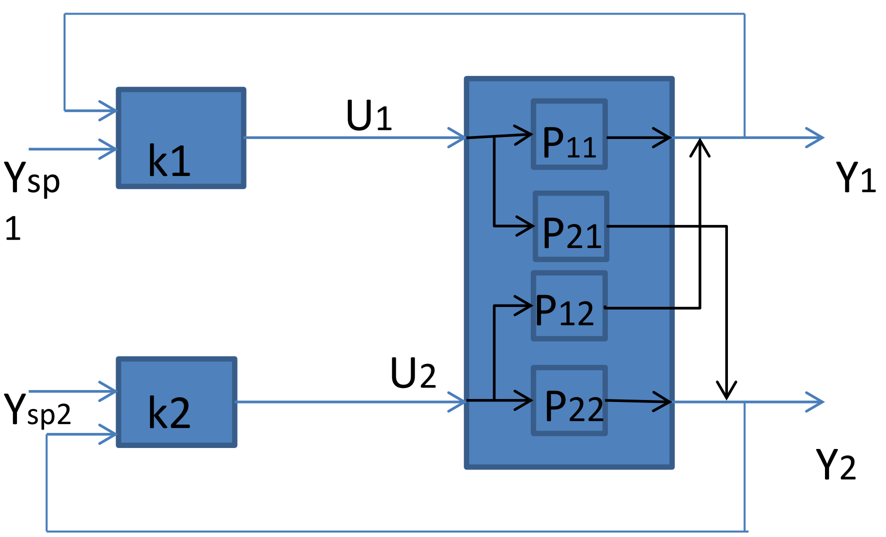
= the transfer matrix of the system

Effects of interaction

U's, Y's and P's are functions of s

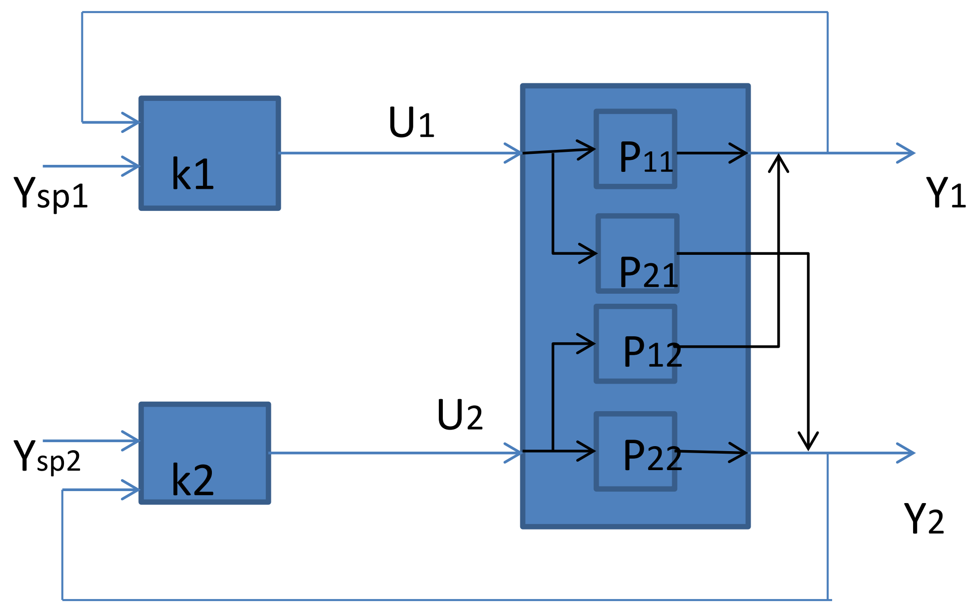
- $Y_1 = P_{11}U_1 + P_{12}U_2$ (1)
- $Y_2 = P_{21}U_1 + P_{22}U_2$ (2)
- Feedback loop 1: $U_1 = -k_1 Y_1$
- Feedback loop 2: $U_2 = -k_2 Y_2 \Rightarrow Y_2 = \frac{-U_2}{k_2}$
- This is introduced in eq. 2
- $U_2 = -\frac{P_{21}k_2}{P_{22}k_2+1}U_1$
- This is introduced in eq 1
- $\frac{Y_1}{U_1} = \frac{P_{11}P_{22}k_2 + P_{11} - P_{12}P_{21}k_2}{P_{22}k_2+1}$

$\frac{Y_1}{U_1}$ depends on all sub systems



Effects of interaction- example

- $Y_1(s) = \frac{1}{(s+1)^2}U_1(s)+\frac{2}{(s+1)^2}U_2(s)$ 1
- $Y_2(s) = \frac{1}{(s+1)^2}U_1(s)+\frac{1}{(s+1)^2}U_2(s)$ 2
- Feed back loop 1: $U_1(s) = -k_1 Y_1(s)$
- Feed back loop 2: $U_2(s) = -k_2 Y_2(s)$ => $Y_2(s) = \frac{-U_2(s)}{k_2}$,
- This is introduced in eq. 2
- $U_2(s) = -\frac{k_2}{s^2+2s+k_2+1}U_1(s)$
- We will find $\frac{Y_1}{U_1}$



Effects of interaction

$$U_2(s) = -\frac{k_2}{s^2 + 2s + k_2 + 1} U_1(s)$$

- $U_2(s)$ is insertet in eq 1
- $Y_1(s) = \frac{s^2+2s+1-k_2}{(s+1)^2(s^2+2s+1+k_2)} U_1(s)$
- $Y_1(0) = \frac{1-k_2}{1+k_2}$
- The static gain in k_2 has effect on the dynamics relating to U_1 and Y_1.
- $The\ gain\ decrease\ as$ k_2 increase, the gain is negative for $\mathrm{k}_2 > 1$

Will it be better to let
U1 control Y2
and U2 control Y1 ??

How do we find the best SISO loops
(Bristols relative gain array)

Bristols Relative Gain Array

Investigate how the static process gain of one loop is infuenced by the gains in other loops

- We investigate how the the static gain in the first loop is affected by the controller in the second loop.
- Assume: Second loop in perfect control, $Y_2(s)$= $Y_{2\ sp}$(s) => $Y_2(s) = 0$ for $Y_{2\ sp}(s) = 0$.

$Y_1(s)=P_{11}(s)U_1(s)+P_{12}(s)U_2(s)$

$0 =P_{21}(s)U_1(s)+P_{22}(s)U_2(s)$

Eliminating $U_2(s)$ gives

$$Y_1(s) = \frac{P_{11}(s)P_{22}(s) - P_{12}(s)P_{21}(s)}{P_{22}(s)} U_1(s)$$

Bristols interaction index ʎ for TITO systems

- ʎ =the ratio of the static gains of loop 1 when the second loop is open and when the second loop is closed
- ʎ= $\frac{static\ gains\ of\ loop\ 1\ for\ open\ loop\ in\ loop\ 2}{static\ gain\ of\ loop\ 1\ for\ closed\ loop\ in\ loop\ 2}$
- $ʎ = \frac{P_{11}(0)}{\frac{P_{11}(0)P_{22}(0)-P_{12}(0)P_{21}(0)}{P_{22}(0)}} = \frac{P_{11}(0)P_{22}(0)}{P_{11}(0)P_{22}(0)-P_{12}(0)P_{21}(0)}$

- Interaction for static or low frequency signals
- $P_{12}(0)P_{21}(0)$=0 => ʎ=1 no interaction

RGA – Bristols relative gain array

- Compare the static gain for one output when all other loops are open with the gains when all other outputs are zero.
- $R = P(0) .* P^{-T}(0)$
- P(0) : the static gain for the system
- $P^{-T}(0)$: the transposed of the inverse of P(0)
- .* : component-wise multiplication
- r_{ij} : the ratio between the open loop and closed loop gain from input u_j to output y_i
- R: symetric, all rows and columns sum to 1.

RGA for TITO system

- $\mathrm{R} = \begin{bmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{bmatrix}$ ʎ is the interaction index
- ʎ =1 : no interaction, second loop has no impact on first loop
- 0<ʎ<1 : closed loop has higher gain than open loop
- ʎ >1 : closed loop has lower gain than open loop
- ʎ <0 :gain of first loop changes sign when second loop is closed.

Pairing : Decide how inputs and outputs should be connected in control loops using RGA

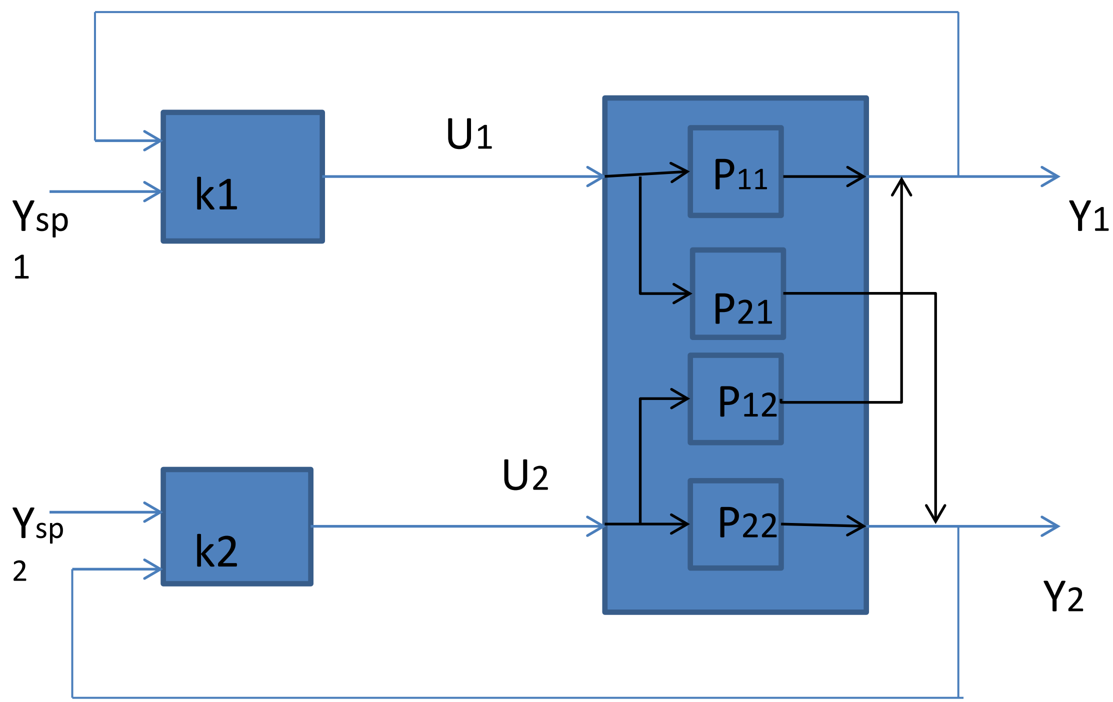
- $\Lambda = 1$ no interaction
- $\Lambda = 0$ no interaction but loops should be interchanged
- $\Lambda < 0.5$ loops should be interchanged
- $0 < \Lambda < 1$ closed loop gains <open loop gains
- Corresponding relative gains should be positive and close to 1
- Pairing of signals with negative relative gains schould be avoided.
- If gains are outside the interval $0.67 < \Lambda < 1.5$ decoupling can improve the control significantly.

Example

- $Y_1(s) = \frac{1}{(s+1)^2}U_1(s) + \frac{2}{(s+1)^2}U_2(s)$
- $Y_2(s) = \frac{1}{(s+1)^2}U_1(s) + \frac{1}{(s+1)^2}U_2(s)$

Is this input output relation the best ??

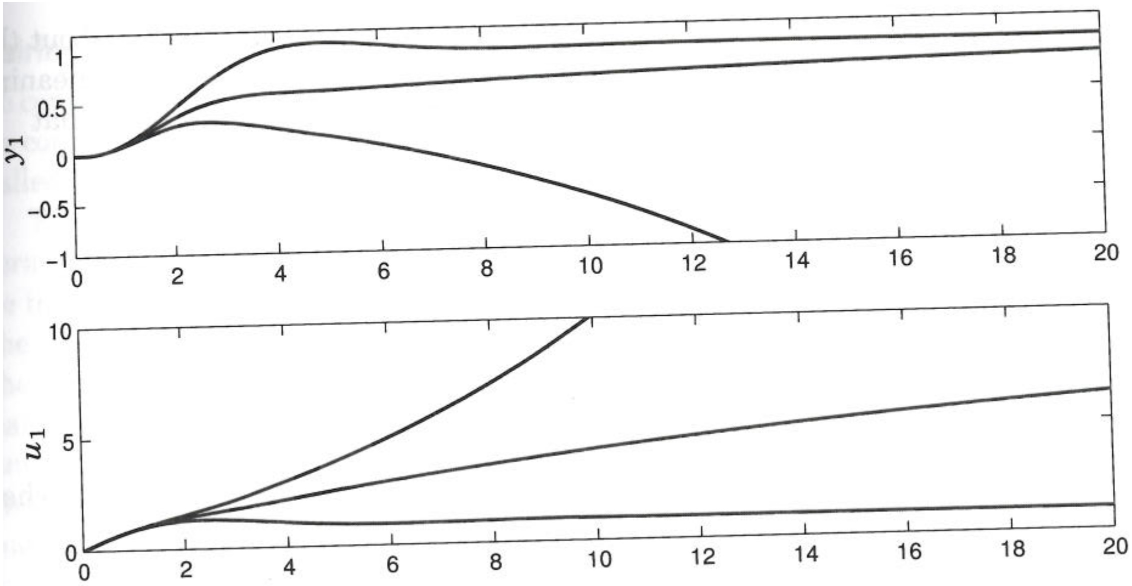
$P_{11}(0) = 1$
- $P_{12}(0) = 2$
- $P_{21}(0) = 1$
- $P_{22}(0) = 1$



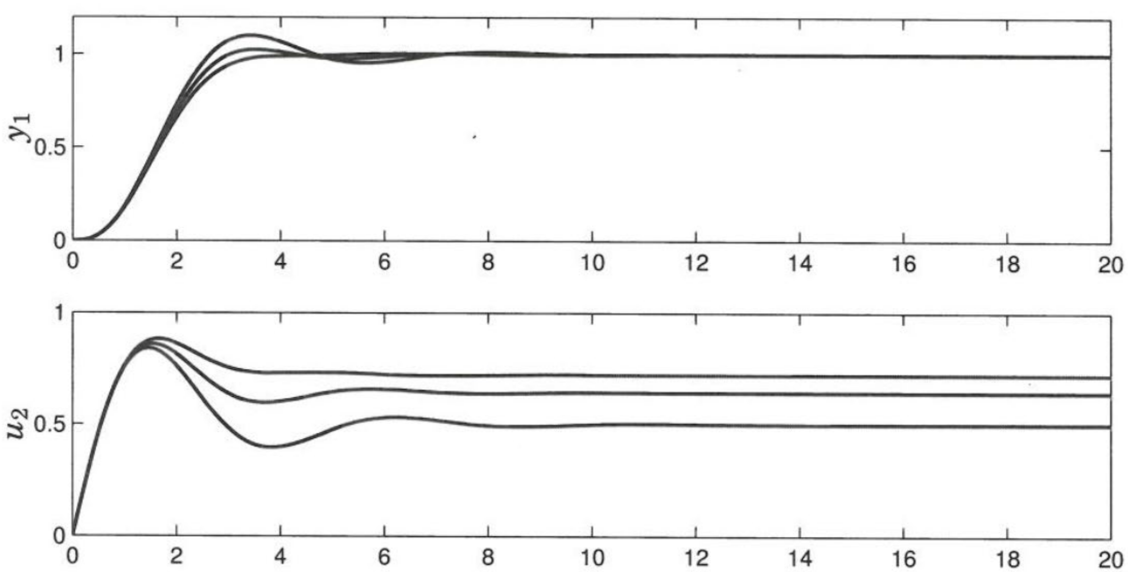
Example

- $P(0) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $\quad P^{-1}(0) = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$
- $R = P(0).* P^{-T}(0) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} .* \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$
- ʌ = -1 => y_1 should be paired with u_2.
- For closed loop: $u_1 = -k_2 y_2$ =>
- $Y_1(s) = g_{12}^{cl}(s) U_2(s) = \frac{2s^2+4s+2+k_2}{(s+1)^2(s^2+2s+1+k_2)} U_2\,(s)$
- Static : $g_{12}^{cl}(0) = \frac{2+k_2}{1+k_2}$
- g_{12} increase for decreasing k_2, g_{12} is positive for k_2 >0.
- The interaction is smaller

Impact of switching loops



Simulations of responses to step in set-points for loop 1.
original loops



Simulations of responses to step in set-points for loop 1.
switched loops

BREAK

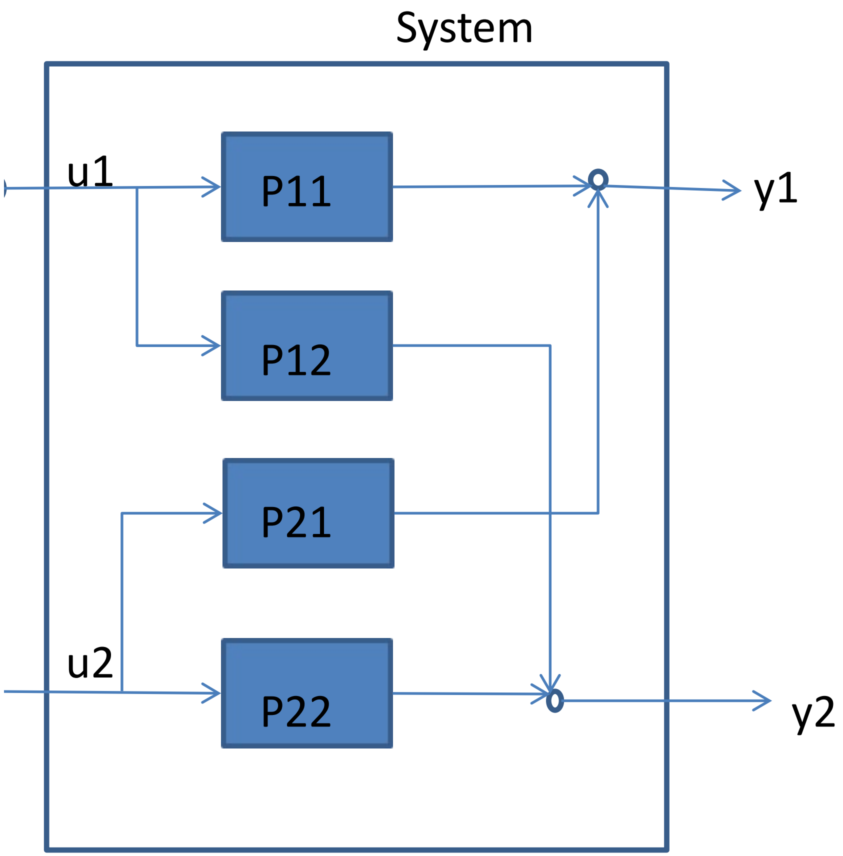
Decoupling – design of controllers that reduce the effects of interaction between loops

TITO system with interaction
from u1 to y1 and y2
from u2 to y1 and y2

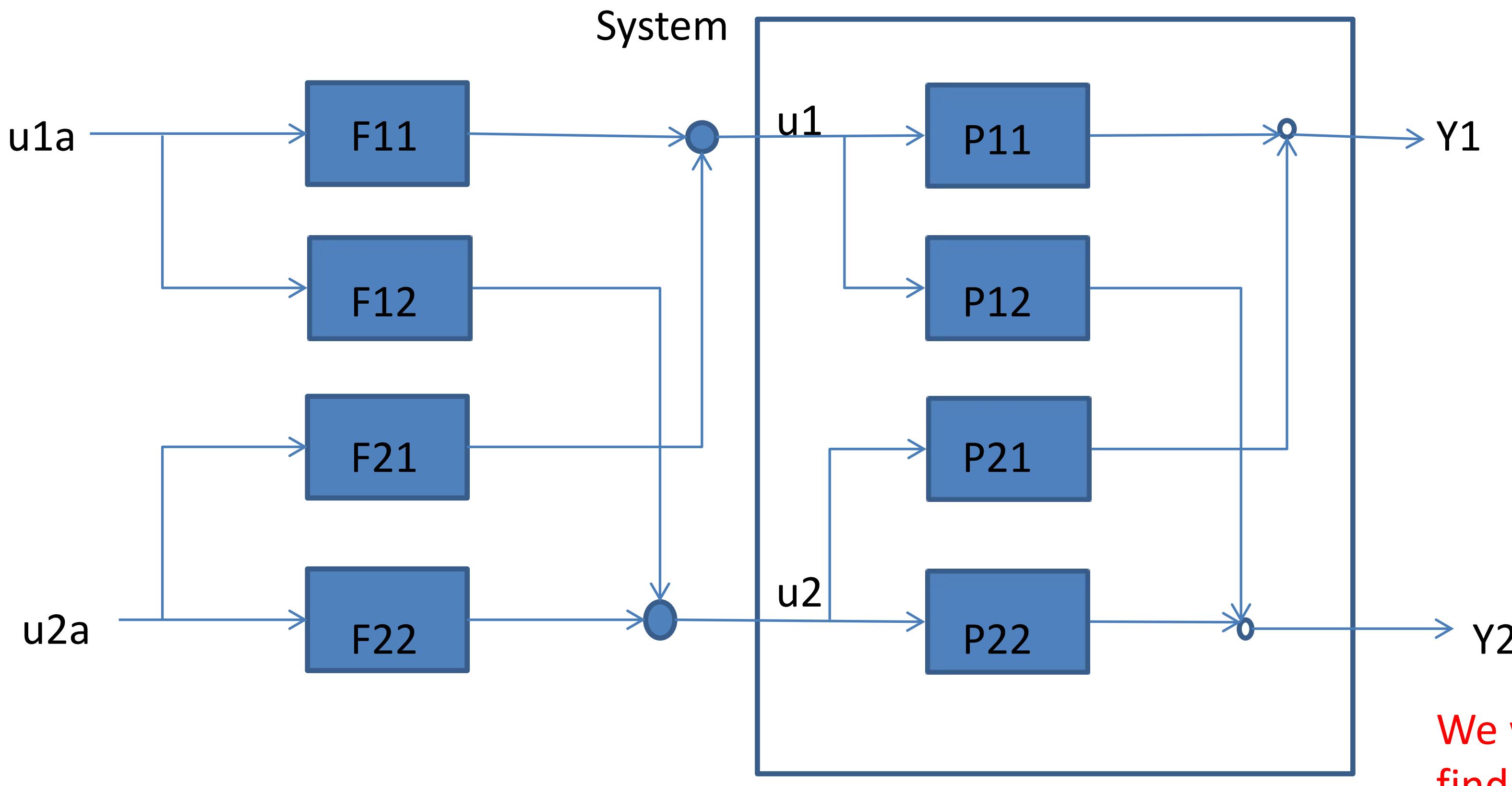
The dynamic coupling factor Q is

$$Q = \frac{P_{21}P_{12}}{P_{11}P_{22}}$$

Decoupling eliminates the effect of the interaction from u1 to y2
And from u2 to y1



Decoupling - structure



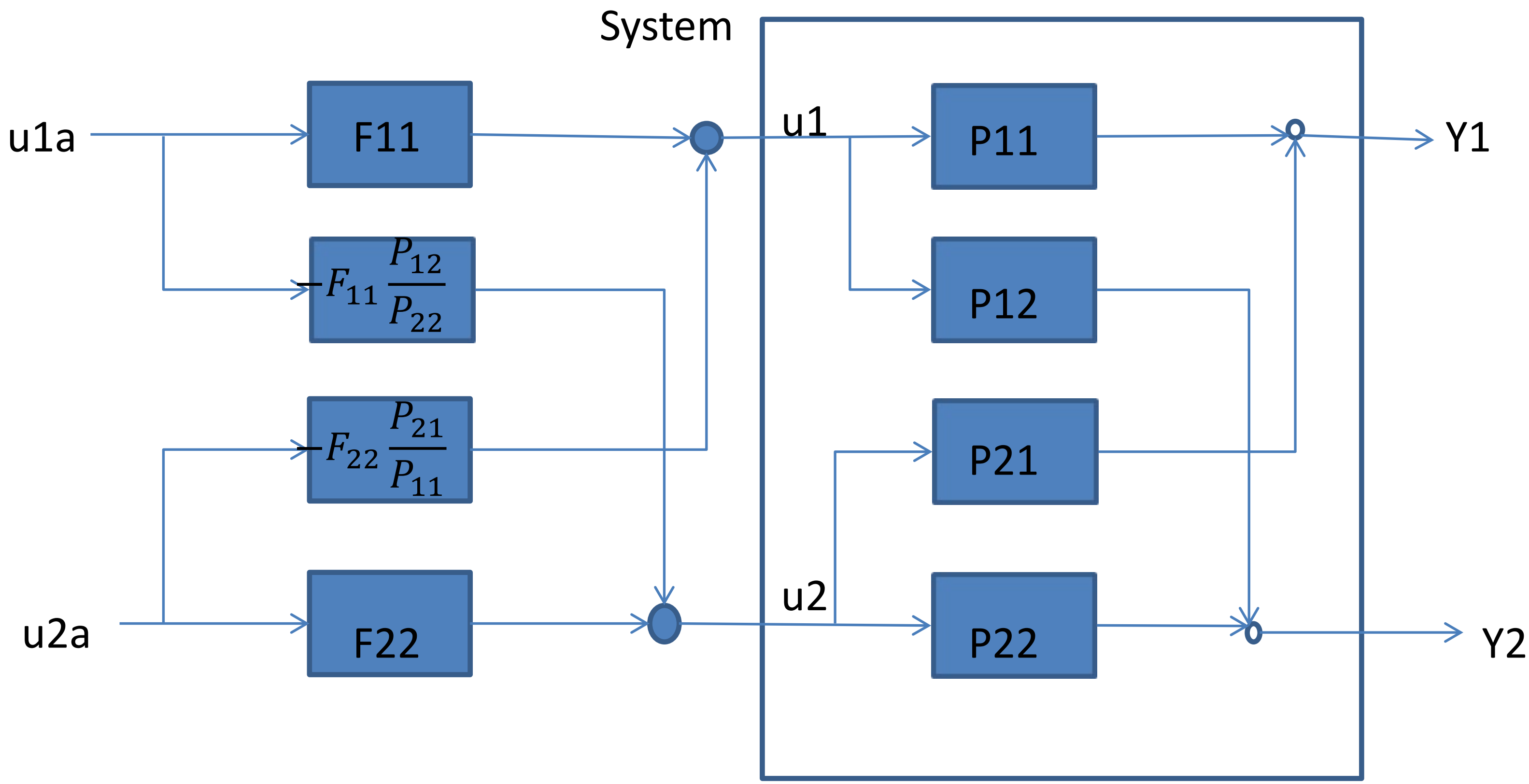
We want to find $F_{11}, F_{12}, F_{21}, F_{22}$

Decoupling if:

$$\frac{Y_2}{u_{1a}} = F_{12}P_{22} + F_{11}P_{12} = 0 \Rightarrow F_{12} = -F_{11}\frac{P_{12}}{P_{22}}$$

$$\frac{Y_1}{u_{2a}} = F_{21}P_{11} + F_{22}P_{21} = 0 \Rightarrow F_{21} = -F_{22}\frac{P_{21}}{P_{11}}$$

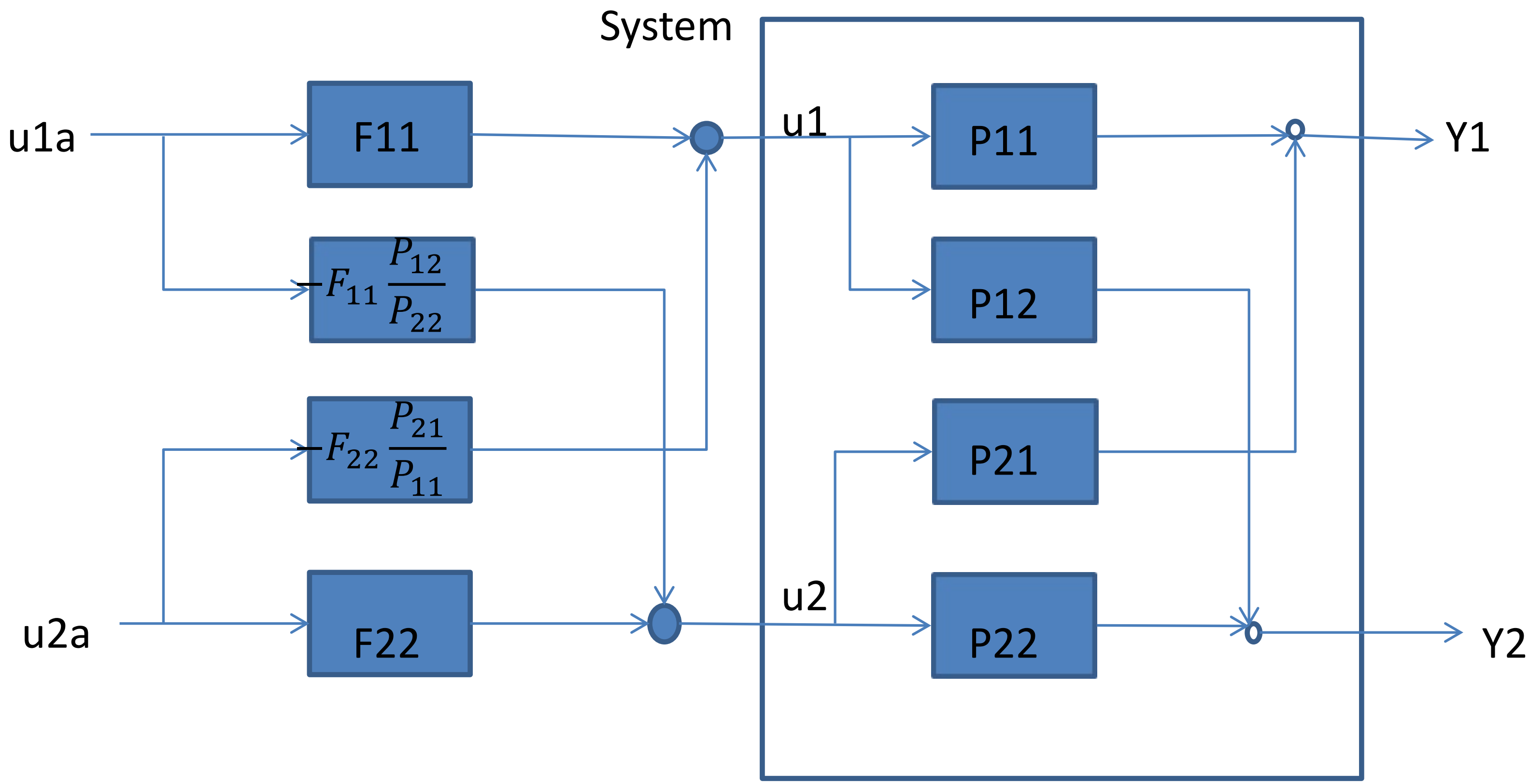
Decoupling X-couplings = 0



$$\frac{Y_2}{u_{1a}} = -F_{11}\frac{P_{12}}{P_{22}}P_{22} + F_{11}P_{12} = 0$$

$$\frac{Y_1}{u_{2a}} = -F_{22}\frac{P_{21}}{P_{11}}P_{11} + F_{22}P_{21} = 0$$

Decoupling – direct couplings



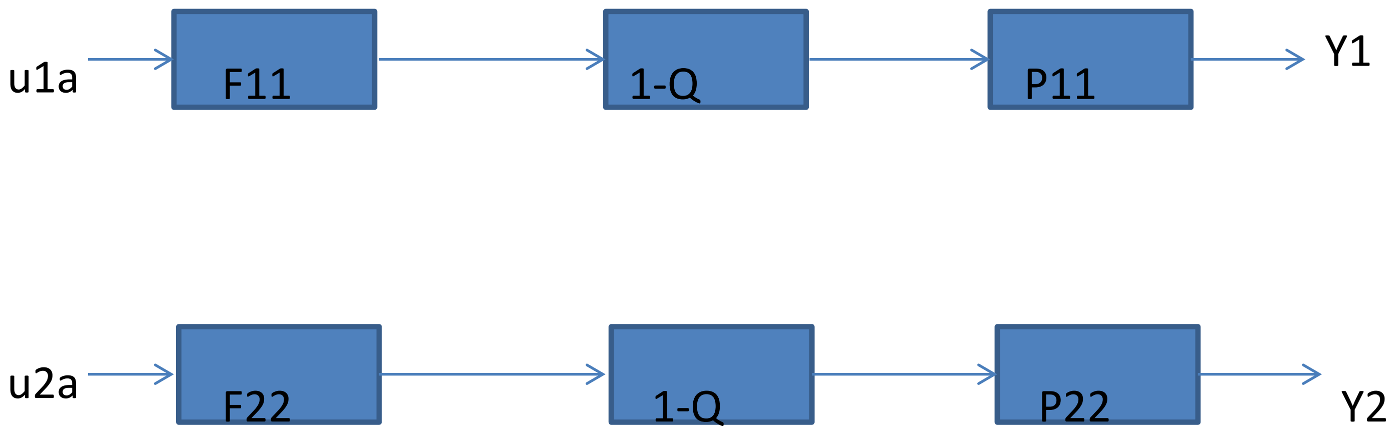
$$\frac{Y_1}{u_{1a}} = -F_{11}\frac{P_{12}}{P_{22}}P_{21} + F_{11}P_{11} = \left(1 - \frac{P_{12}P_{21}}{P_{22}P_{11}}\right)P_{11}F_{11} = (1-Q)P_{11}F_{11}$$

$$\frac{Y_2}{u_{2a}} = -F_{22}\frac{P_{21}}{P_{11}}P_{12} + F_{22}P_{22} = (1 - \frac{P_{12}P_{21}}{P_{22}P_{11}})P_{22}F_{22} = (1-Q)P_{22}F_{22}$$

SISO loops

- $\frac{Y_1}{u_{1a}} = \quad (1-Q)F_{11}P_{11}$
- $\frac{Y_2}{u_{2a}} \quad = (1-Q)F_{22}P_{22}$
- $\frac{Y_2}{u_{1a}} \quad = 0$
- $\frac{Y_1}{u_{2a}} = \quad 0$

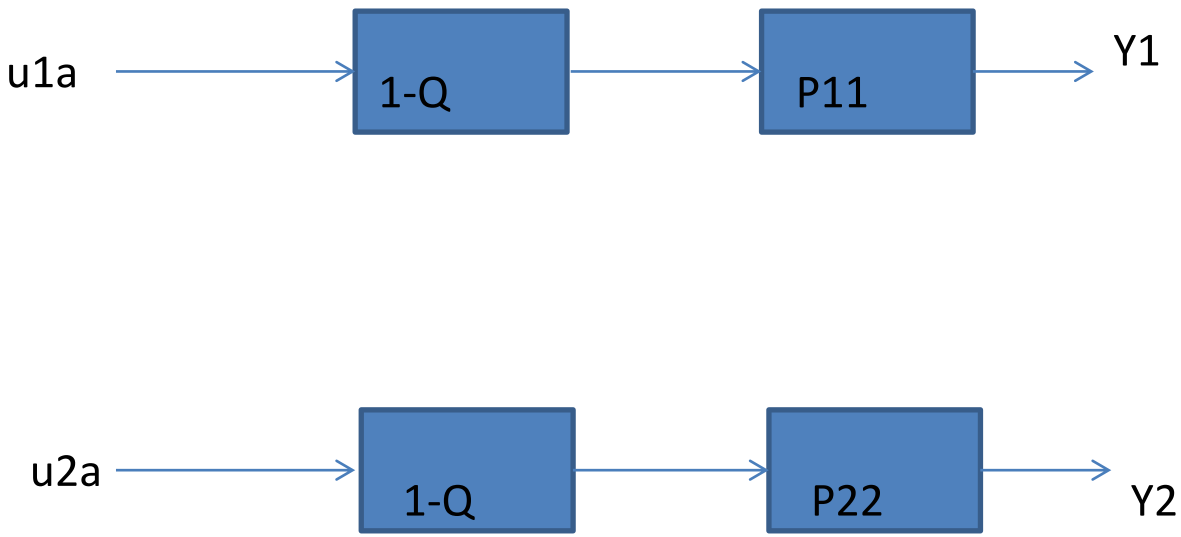
$$Q = \frac{P_{21}P_{12}}{P_{11}P_{22}}$$



How to choose F

- $F_{12} = -F_{11}\frac{P_{12}}{P_{22}}$
- $F_{21} = -F_{22}\frac{P_{21}}{P_{11}}$
- 2 equations 4 unknown, we can choose 2
- F_{11} and F_{22} are chosen to be 1 then F_{12} and F_{21} can be calculated

Decoupled system



The controllers can be designed using ordinary SISO rules, but you need to know good models.