DISCRETE TIME SYSTEMS AND Z-TRANSFORM

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Plan for the lectures

- Discrete time signals
  - Basic sequences and operations
  - Linear systems
  - Stability, causality, time invariance
- Linear time invariant (LTI) systems
  - Inpulse response and convolution
  - Parallel and cascade system combination
- Fourier transform of LTI systems
  - Definition and conditions for existence
- Z-transform
  - Definition and region of convergence (ROC)
  - Right, left-sided and finite duration sequences
  - ROC analysis
- Inverse z-transform
  - Definition and inspection method
  - Partial fraction expansion
  - Power series expansion
- Transform analysis of LTI systems
  - Linear constant coefficient difference equations
  - Stability and causality
  - Inverse systems
  - FIR and IIR systems

Literature

Alan V. Oppenheim - Ronald W. Schafer:
Discrete-Time Signal Processing
Pearson 2014
Second or Third Edition
ISBN 10: 1-292-02572-7
ISBN 13: 978-1-292-02572-8

Lecture format

- Lecture (approx.1h45m including break) + exercise session
- Slides + large usage of blackboard
- No pre-scheduled breaks
  - Breaks are ”distributed” according to the complexity of the presented topics
- Please let me know if I erase the blackboard too quickly!

Today’s agenda

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- Discrete time signals
    - Basic sequences and operations
    - Linear systems
    - Stability, causality, time invariance
- Linear time invariant (LTI) systems
    - Impulse response and convolution
- Fourier transform of LTI systems
    - Definition and conditions for existence

Discrete time signals - Sequences

- Discrete signals can be represented as a sequence of numbers

$$x = \{x[n]\} \qquad -\infty < n < \infty$$

where n is an integer.

- In case such sequences arise from periodic sampling of an analog signal:

$$x[n] = x_a[nT_s] \qquad -\infty < n < \infty$$

where Ts is the *sampling interval* and $f_s=1/T_s$ is the *sampling frequency*.

Discrete time signals - Sequences

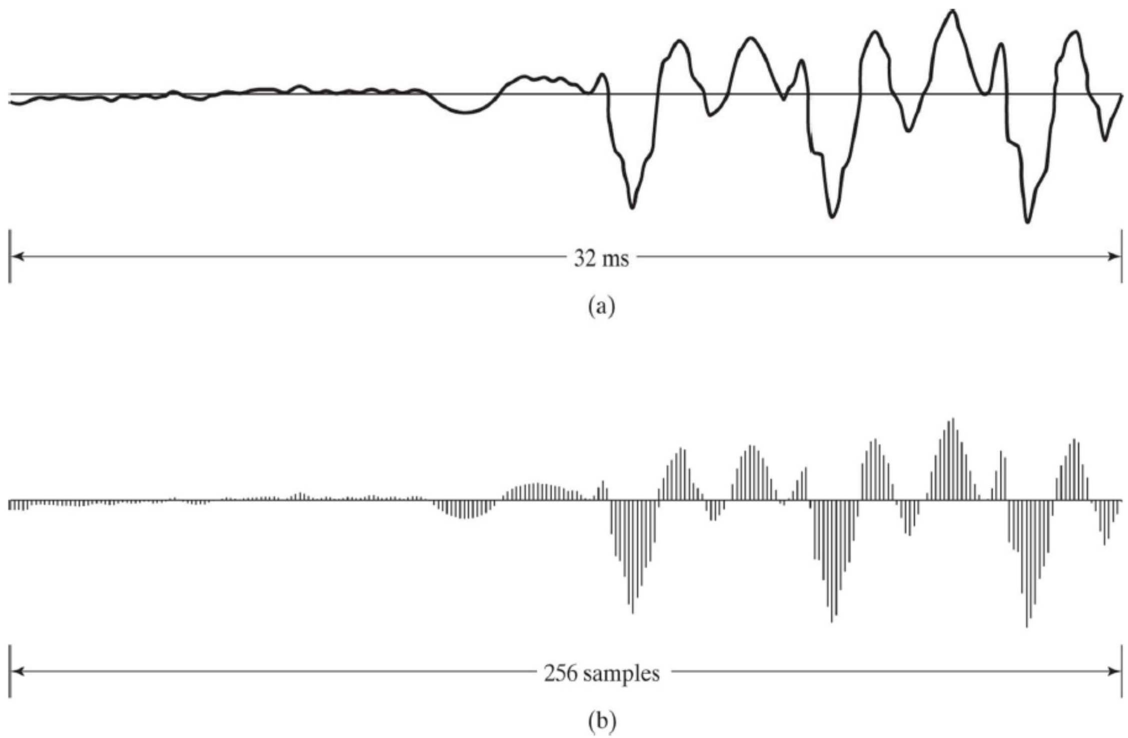


Fig 2 (13) (a) Segment of a continuous-time speech signal $x_a(t)$.
(b) Sequence of samples $x[n] = x_a(nT_s)$ obtained from the signal in part (a) with $T_s = 125\ \mu s$.

Discrete time signals - Sequences

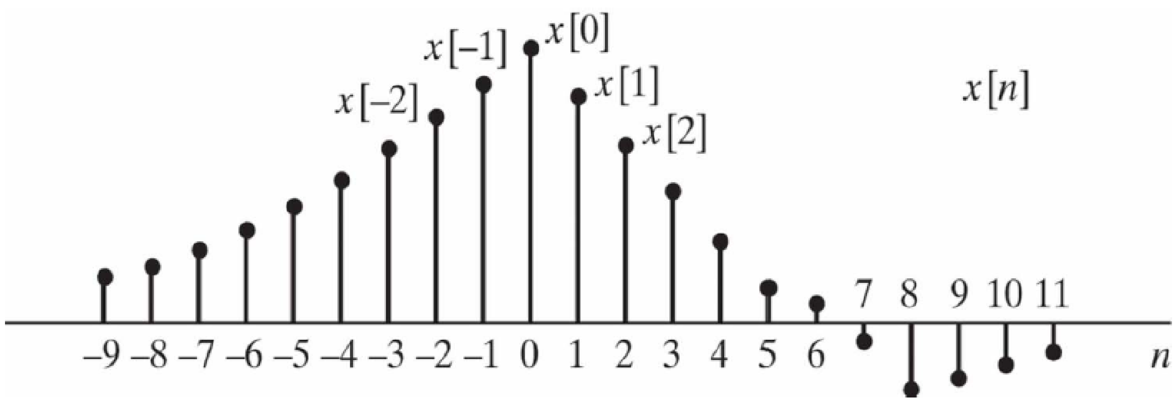


Fig 1 (13) Graphic representation of a discrete-time signal.

Basic sequences and operations

- y[n] is said to be a delayed (or shifted) version of the sequence x[n] if

  $y[n]=x[n-n_0]$, with n_0 integer

- The unit sample sequence is defined as

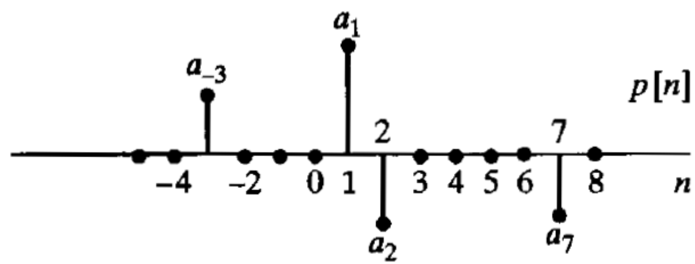
$$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0. \end{cases}$$

- An example of delayed unit sample sequence:

$$\delta[n-2] = \begin{cases} 0, & n \neq 2 \\ 1, & n = 2 \end{cases}$$

Basic sequences and operations

- An arbitrary sequence can be represented as a sum of scaled, delayed, impulses.



$$p[n] = a_{-3}\delta[n+3] + a_1\delta[n-1] + a_2\delta[n-2] + a_7\delta[n-7].$$

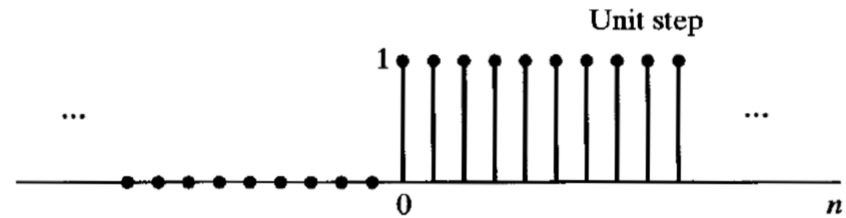
- More generally, any sequence can be expressed as:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Basic sequences and operations

- The unit step sequence is defined as

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$$



- The unit step is related to the impulse by $u[n] = \sum_{k=-\infty}^{n} \delta[k];$ or $u[n] = \sum_{k=0}^{\infty} \delta[n-k].$

- Conversely, the impulse sequence can be expressed as the first backward difference of the unit step sequence:

$$\delta[n] = u[n] - u[n-1].$$

Discrete time systems

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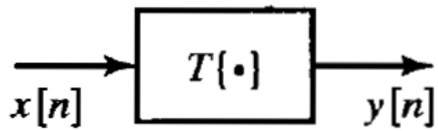
- Definition
- Properties
  - Linearity
  - Time invariance
  - Causality
  - Stability

Discrete time systems

- A discrete-time system is an operator that maps an input sequence x[n] to an output sequence y[n]

$$y[n] = T\{x[n]\}$$



- Examples of operators:

  - Delay $$y[n] = x[n - n_{delay}] \qquad -\infty < n < \infty$$

  - Moving average $$y[n] = \frac{1}{M1 + M2 + 1} \sum_{k=-M1}^{M2} x[n-k]$$

  - FIR filter $$y[n] = \frac{1}{\sum_{k=0}^{M} b_k} \sum_{k=0}^{M} b_k \cdot x[n-k]$$

Linear discrete time systems

- The class of linear system is defined by the principle of superposition.

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$$

$$T\{a \cdot x[n]\} = a \cdot T\{x[n]\} = a \cdot y[n]$$

$$\Downarrow$$

$$T\{a \cdot x_1[n] + b \cdot x_2[n]\} = a \cdot T\{x_1[n]\} + b \cdot T\{x_2[n]\} = a \cdot y_1[n] + b \cdot y_2[n]$$

Linear discrete time systems

- The accumulator system

$$y[n] = \sum_{k=-\infty}^{n} x[k] \quad \Longrightarrow \quad \text{Linear system}$$

- Consider the following

$$w[n] = \log_{10}(|x[n]|). \quad \Longrightarrow \quad \text{Non-Linear system}$$

Time invariant discrete systems

- A time invariant system is a system for which a time delay/shift of the input sequence causes a corresponding shift in the output sequence.

$$x_1[n] = x[n - n_0] \Rightarrow y_1[n] = y[n - n_0].$$

- The accumulator is a time invariant system.
- Compressor is a non-time invariant system

$$y[n] = x[Mn], \qquad -\infty < n < \infty,$$

Causal discrete time systems

- A system is causal if, for every choice of n_0, the output sequence at the index $n=n_0$ depends only on the input sequence values for $n<=n_0$.

- Forward difference system

$$y[n] = x[n+1] - x[n]. \Rightarrow$$

Non- causal

- Backward difference system

$$y[n] = x[n] - x[n-1], \Rightarrow$$

Causal

Stable discrete time systems

- A system is stable if and only if every bounded input sequence produces a bounded output sequence.

$$|x[n]| \leq B_x < \infty, \qquad \text{for all } n. \quad \Rightarrow \quad |y[n]| \leq B_y < \infty, \qquad \text{for all } n.$$

- Examples

$$y[n] = \sum_{k=-\infty}^{n} u[k] \quad \Rightarrow$$ Not stable

$$y[n] = x[n - n_d], \quad \Rightarrow$$ Stable

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k] \quad \Rightarrow$$ Stable

What about $y[n] = \log_{10}(|x[n]|)$?

Linear time invariant systems

- Linear systems → principle of superposition
- General sequence can be expressed as a linear combination of delayed and scaled unit pulses → a linear system can be completely characterized by its impulse response.

$$y[n] = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} \Rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h_k[n].$$

- If only linearity is imposed, $h_k[n]$ depends on both *k* and *n*.
- Time invariance: if h[n] is the response to delta[n], then the response to δ[n-k] is h[n-k].

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

convolution sum

$$y[n] = x[n] * h[n].$$

Linear time invariant systems

- Convolution operation is commutative

$$x[n] * h[n] = h[n] * x[n].$$

- Convolution operation distributes over addition

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

Linear time invariant systems

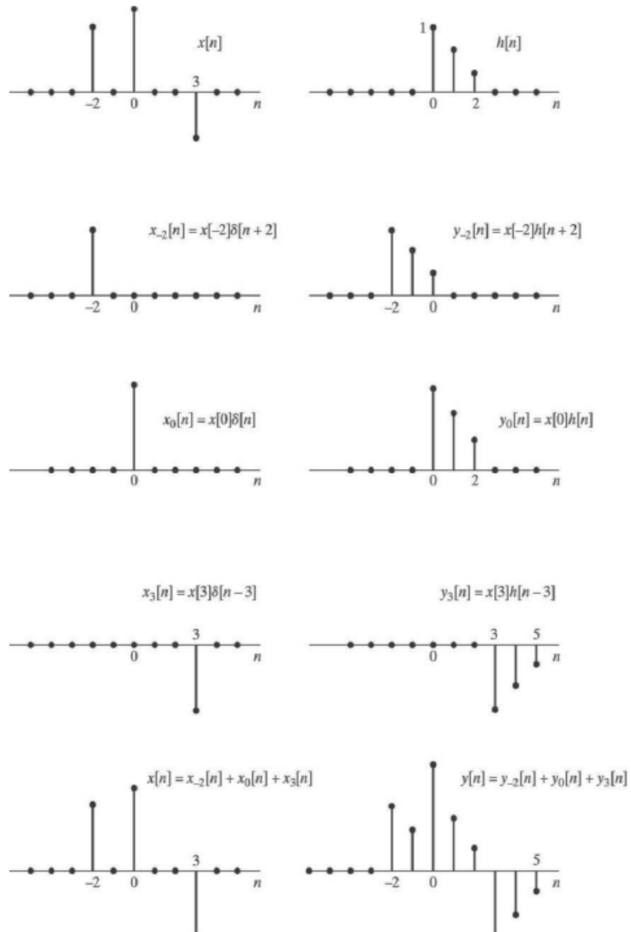


Fig 8 (27)

Representation of the output of an LTI system as the superposition of responses to individual samples of the input.

$$y[n] = x[n] * h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \qquad \text{for all } n.$$

Linear time invariant systems

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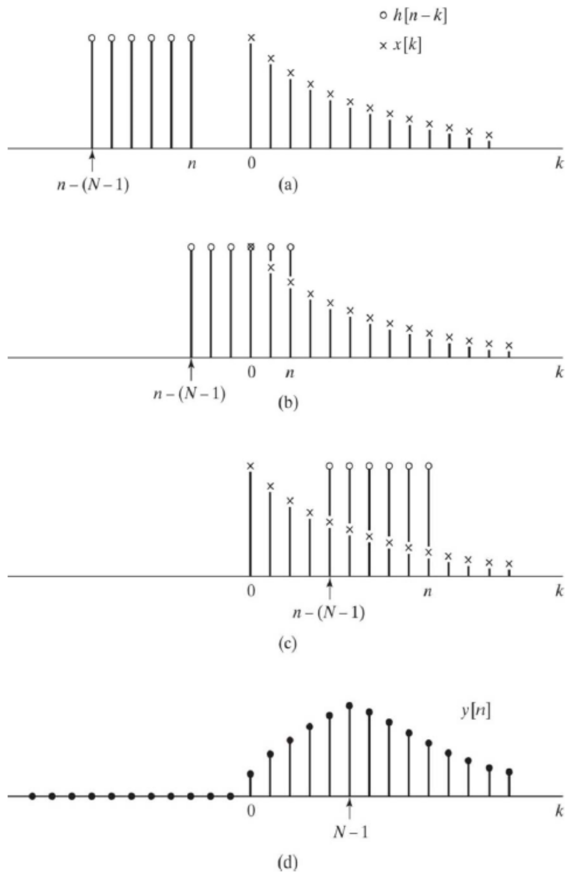

Fig 10 (30)

Sequence involved in computing a discrete convolution.

(a) – (c): The sequences $x[k]$ and $h[n-k]$ as a function of k for different values of n. (Only nonzero samples are shown.)

(d): Corresponding output sequence as a function of n.

h[n] = u[n] – u[n-N]

$$x[n] = \begin{cases} a^n, for\ n \geq 0 \\ 0, for\ n < 0 \end{cases}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \qquad \text{for all } n.$$

Fourier transform

- The Fourier transform of the function *x(t)* is defined as

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

- The inverse Fourier transform is defined as

$$x(t) = \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

Fourier transform

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Physical interpretation: spectrum of a signal

Time domain $\xrightarrow{\text{Fourier transform}}$ Frequency domain

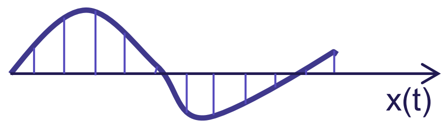
Time domain $\xleftarrow{\text{Inverse Fourier transform}}$ Frequency domain

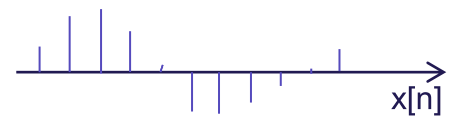
- The Fourier transform returns the amplitude and phase of sinusoidal signals at the different frequencies that compose the time domain signal.
- The Inverse Fourier transform returns the time domain signal which is composed of the sinusoidal signals at different amplitude and phases of the frequency domain representation.

Fourier transform

- What about the spectrum of a discrete signal, obtained by sampling x(t)?

time





frequency $$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

??

- It can be shown that the spectrum of a discrete time signal can be calculated as

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

Discrete time Fourier transform (DTFT)

Fourier transform

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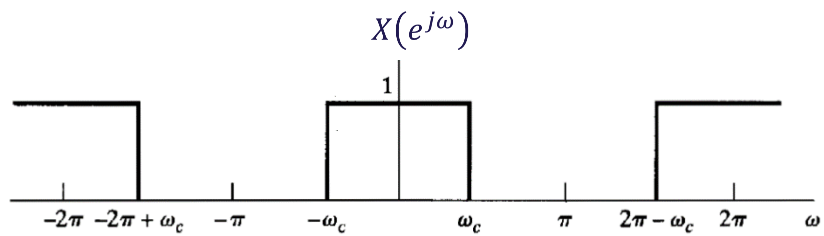
- A common notation for $X(\omega)$ is $X(e^{j\omega})$.
- $X(e^{j\omega})$ is periodic of period 2π → periodic spectrum



- Since $X(e^{j\omega})$ is 2π – periodic, the inverse Fourier transform can be calculated by integrating over a single period, e.g. [- π, π].

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Fourier transform

- The frequency response of a linear time invariant system is the Fourier transform of the impulse response,

$$\mathrm{H}(\omega) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n}$$

$$\Downarrow$$

$$\mathrm{h}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

Fourier transform

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- What is the condition of existence of the Fourier transform?

Determining the class of signals that can be represented by

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega,$$

is equivalent to considering the convergence of the infinite sum in $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$

$$|X(e^{j\omega})| < \infty \qquad \text{for all } \omega, \quad \Rightarrow \quad |X(e^{j\omega})| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]|\,|e^{-j\omega n}|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

If x[n] is absolutely summable, than the Fourier transform exists.

Fourier transform

- Example: does the Fourier transform of the following sequence exist?

Let $x[n] = a^n u[n]$. The Fourier transform of this sequence is

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= \frac{1}{1 - ae^{-j\omega}} \qquad \text{if } |ae^{-j\omega}| < 1 \quad \text{or} \quad |a| < 1.$$

Clearly, the condition $|a| < 1$ is the condition for the absolute summability of $x[n]$; i.e.,

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} < \infty \qquad \text{if } |a| < 1. \tag{2.140}$$