Exercises module 2

A.

1) $\sum_{n=0}^{\infty} 4^n (z+1)^n \qquad z_0 = -1$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{4}$$

2) $\sum_{n=1}^{\infty} \frac{n^n}{n} (z-\pi i)^n \qquad z_0 = \pi i$

$$R = \lim_{n\to\infty} \left| \frac{\frac{n^n}{n}}{\frac{(n+1)^{n+1}}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n^n}{n} \cdot \frac{n+1}{(n+1)^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n^n}{n} \cdot \frac{1}{(n+1)^n} \right| =$$

$$= \lim_{n\to\infty} \left| \frac{1}{n} \cdot \left(\frac{n}{n+1} \right)^n \right| = \lim_{n\to\infty} \left| \frac{1}{n} \cdot \left(\frac{1}{1+\frac{1}{n}} \right)^n \right| =$$

$$= 0 \cdot \frac{1}{e} = 0$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

3) $\sum_{n=0}^{\infty} \frac{n(n-1)}{3^n}(z-i)^{2n} \qquad z_0 = i$

$$\sum_{n=0}^{\infty} a_n (z-z_0)^{2n} = \sum_{n=0}^{\infty} a_n \left[(z-z_0)^2\right]^n$$

$$(z-z_0)^2 < R$$

$$\Rightarrow \ |z-z_0| < \sqrt{R}$$

$$R = \lim_{n\to\infty} \left| \frac{\frac{n(n-1)}{3^n}}{\frac{(n+1)n}{3^{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{n-1}{n+1} \cdot 3 \right| = 3 \rightarrow \sqrt{R} = \sqrt{3}$$

4) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(z+3)^n \qquad z_0 = -3$

$$R = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n n}{4^n}}{\frac{(-1)^{n+1}(n+1)}{4^{n+1}}} \right| = \lim_{n\to\infty} \left| -4 \cdot \frac{n}{n+1} \right| = 4$$

5) $\sum_{n=1}^{\infty} \frac{2^n}{n}(4z-8)^n = \sum_{n=1}^{\infty} \frac{2^n \cdot 4^n}{n}(z-2)^n \qquad z_0 = 2$

$$R = \lim_{n\to\infty} \left| \frac{\frac{8^n}{n}}{\frac{8^{n+1}}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{1}{8} \cdot \frac{n+1}{n} \right| = \frac{1}{8}$$

6) $$\sum_{n=0}^{\infty} n!\,(2z+1)^n = \sum_{n=0}^{\infty} 2^n \cdot n!\left(z+\frac{1}{2}\right)^n$$

$$z_0 = -\frac{1}{2}$$

$$R = \lim_{n\to\infty}\left|\frac{2^n \cdot n!}{2^{n+1}(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{1}{2}\cdot\frac{n!}{(n+1)\cdot n!}\right| = 0$$

7) $$\sum_{n=0}^{\infty}\frac{(z-2i)^n}{n^n} \qquad z_0 = 2i$$

$$R = \lim_{n\to\infty}\left|\frac{\frac{1}{n^n}}{\frac{1}{(n+1)^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{n^n}\right| = \lim_{n\to\infty}\left|(n+1)\cdot\left(\frac{n+1}{n}\right)^n\right| =$$

$$= \lim_{n\to\infty}\left|(n+1)\cdot\left(1+\frac{1}{n}\right)^n\right| = \infty$$

8) $$\sum_{n=0}^{\infty}\left(\frac{1-i}{2+3i}\right)^n z^n \qquad z_0 = 0$$

$$R = \lim_{n\to\infty}\left|\frac{\left(\frac{1-i}{2+3i}\right)^n}{\left(\frac{1-i}{2+3i}\right)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2+3i}{1-i}\right| = \sqrt{\frac{13}{2}}$$

9) $\sum_{n=0}^{\infty} 16^n (z+i)^{4n}$ $\quad z_0 = -i$

$$R = \lim_{n\to\infty} \left| \frac{16^n}{16^{n+1}} \right| = \frac{1}{16} \qquad \sqrt[4]{R} = \sqrt[4]{\frac{1}{16}} = \frac{1}{2}$$

10) $\sum_{n=0}^{\infty} \frac{3n}{2^n \cdot n^2} z^n$ $\quad z_0 = 0$

$$R = \lim_{n\to\infty} \left| \frac{\frac{3}{2^n n}}{\frac{3}{2^{n+1}(n+1)}} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}}{2^n} \cdot \frac{n+1}{n} \right| = 2$$

B.

1) $\sin(2z^2)$

We know that $\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

Therefore

$$\sin(2z^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(2z^2)^{2n+1}}{(2n+1)!} = 2z^2 - \frac{(2z^2)^3}{3!} + \frac{(2z^2)^5}{5!} - \frac{(2z^2)^7}{7!} + \dots$$

$$= 2z^2 - \frac{8z^6}{6} + \frac{32z^{10}}{120} - \frac{128z^{14}}{5040} + \dots$$

2) $\frac{1}{2+z^4} = \frac{1}{2}\left(\frac{1}{1+\frac{z^4}{2}}\right)$

We know that $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad |z|<1$

Therefore

$$\frac{1}{2}\left(\frac{1}{1+\frac{z^4}{2}}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z^4}{2}\right)^n = \frac{1}{2} - \frac{z^4}{4} + \frac{z^8}{8} - \frac{z^{12}}{16} + \dots$$

$$\left|-\frac{z^4}{2}\right| < 1 \Rightarrow |z| < \sqrt[4]{2}$$

3) $\frac{1}{1+5iz} = \sum_{n=0}^{\infty} (-5iz)^n = 1 - 5iz + 25z^2 + \dots$

$$|5iz| < 1 \Rightarrow |z| < \frac{1}{5}$$

4) $\cos^2\left(\frac{z}{2}\right) = \frac{1+\cos(z)}{2} = \frac{1}{2} + \frac{1}{2}\cos(z) =$

$$= \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

5) $\sin^2(z) = \frac{1-\cos(2z)}{2} = \frac{1}{2} - \frac{1}{2}\cos(2z) =$

$$= \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \frac{(2z)^{2n}}{(2n)!}$$

C.

1) $\frac{1}{z}$, center $z_0 = i$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

$$f(z) = \frac{1}{z}\bigg|_{z=i} = \frac{1}{i} \qquad a_0 = \frac{1}{i} = -i$$

$$f'(z) = -\frac{1}{z^2}\bigg|_{z=i} = 1 \qquad a_1 = 1$$

$$f''(z) = \frac{2}{z^3}\bigg|_{z=i} = -\frac{2}{i} \qquad a_2 = \frac{1}{2}\cdot\left(-\frac{2}{i}\right) = -\frac{1}{i} = i$$

$$f'''(z) = -\frac{6}{z^4}\bigg|_{z=i} = -6 \qquad a_3 = \frac{1}{6}\cdot(-6) = -1$$

$$f^{IV}(z) = \frac{24}{z^5}\bigg|_{z=i} = \frac{24}{i} \qquad a_4 = \frac{1}{24}\cdot\frac{24}{i} = -i$$

$\vdots$

$$f(z) = -i + (z-i) + i(z-i)^2 - (z-i)^3 - i(z-i)^4 + \dots$$

2) $\frac{1}{1-z}$ center $z_0 = i$

$$f(z) = \frac{1}{1-z}\bigg|_{z=i} = \frac{1}{1-i}$$

$$f'(z) = \frac{1}{(1-z)^2}\bigg|_{z=i} = \frac{1}{(1-i)^2}$$

$$f''(z) = \frac{2}{(1-z)^3}\bigg|_{z=i} = \frac{2}{(1-i)^3}$$

$$f'''(z) = \frac{3\cdot 2}{(1-z)^4}\bigg|_{z=i} = \frac{3\cdot 2}{(1-i)^4}$$

$$f^{IV}(z) = \frac{4\cdot 3\cdot 2}{(1-z)^5}\bigg|_{z=i} = \frac{4\cdot 3\cdot 2}{(1-i)^5}$$

$\vdots$

$$\Rightarrow f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}\bigg|_{z=i} = \frac{n!}{(1-i)^{n+1}}$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{(1-i)^{n+1}}$$

$$f(z) = \frac{1}{1-i} + \frac{1}{(1-i)^2}(z-i) + \frac{2}{(1-i)^3}(z-i)^2 + \frac{6}{(1-i)^4}(z-i)^3 +$$

$$+ \frac{24}{(1-i)^5}(z-i)^4 + \dots$$

3) $\sin(z)$, center $z_0 = \frac{\pi}{2}$

$$f(z) = \sin(z)\Big|_{\frac{\pi}{2}} = 1 \qquad a_0 = 1$$

$$f'(z) = \cos(z)\Big|_{\frac{\pi}{2}} = 0 \qquad a_1 = 0$$

$$f''(z) = -\sin(z)\Big|_{\frac{\pi}{2}} = -1 \qquad a_2 = -\frac{1}{2}$$

$$f'''(z) = -\cos(z)\Big|_{\frac{\pi}{2}} = 0 \qquad a_3 = 0$$

$$f^{IV}(z) = \sin(z)\Big|_{\frac{\pi}{2}} = 1 \qquad a_4 = \frac{1}{24}$$

$$\vdots$$

$$f(z) = 1 - \frac{1}{2}\left(z - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(z - \frac{\pi}{2}\right)^4 + \dots$$

4) e^{-z} $\quad z_0 = -2$

$$f(z) = e^{-z}\Big|_{z=-2} = e^{2}$$

$$f'(z) = -e^{-z}\Big|_{z=-2} = -e^{2}$$

$$f''(z) = e^{-z}\Big|_{z=-2} = e^{2}$$

$$f'''(z) = -e^{-z}\Big|_{z=-2} = -e^{2}$$

$$f^{IV}(z) = e^{-z}\Big|_{z=-2} = e^{2}$$

$$\vdots$$

$$\rightarrow f^{(n)}(z) = (-1)^n e^2 \rightarrow a_n = \frac{(-1)^n}{n!} e^2$$

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n e^2}{n!}(z+2)^n = e^2 - e^2(z+2) + \frac{e^2}{2}(z+2)^2 + $$

$$- \frac{e^2}{6}(z+2)^3 + \ldots$$

5) $2\sin\left(\pi z+\frac{\pi}{2}\right)$, center $z_0=0$

We know that, for $z_0=0$:

$$\sin(z)=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$$

Therefore:

$$f(z)=2\sum_{n=0}^{\infty}(-1)^n\frac{\left(\pi z+\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}=$$

$$=2\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{(2n+1)!}\left(z+\frac{1}{2}\right)^{2n+1}=$$

$$=2\left[\pi\left(z+\frac{1}{2}\right)-\frac{\pi^3}{6}\left(z+\frac{1}{2}\right)^3+\frac{\pi^5}{120}\left(z+\frac{1}{2}\right)^5+\dots\right.$$