# Module 3

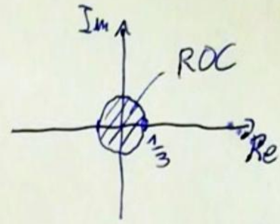
1.

a) $x[n] = \left(\frac{1}{3}\right)^n u[-n]$ $\qquad u[-n] = \begin{cases} 1 & -n \geq 0 \rightarrow n \leq 0 \\ 0 & -n < 0 \rightarrow n > 0 \end{cases}$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = \sum_{n=-\infty}^{+\infty} \left(\frac{1}{3}\right)^n u[-n] z^{-n} = \sum_{n=-\infty}^{0} \left(\frac{1}{3}\right)^n z^{-n} =$$

$$= \sum_{n=-\infty}^{0} \left(\frac{1}{3} z^{-1}\right)^n \underset{q=-n}{\uparrow}{=} \sum_{q=0}^{\infty} \left(\frac{1}{3z}\right)^{-q} = \sum_{q=0}^{\infty} (3z)^q =$$

$$= \frac{1}{1-3z} \quad \text{for } |3z| < 1 \rightarrow |z| < \frac{1}{3}$$




b) $x[n] = \delta[n]$ $\qquad \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = 1 \cdot z^{-0} = 1 \qquad \text{all } z$$

c) $x[n] = \delta[n-1]$ $\qquad \delta[n-1] = \begin{cases} 1 & n-1 = 0 \rightarrow n = 1 \\ 0 & n-1 \neq 0 \rightarrow n \neq 1 \end{cases}$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = 1 \cdot z^{-1} = z^{-1} \qquad |z| > 0$$

d) $x[n] = \delta[n+1]$ $\qquad \delta[n+1] = \begin{cases} 1 & n = -1 \\ 0 & n \neq -1 \end{cases}$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = 1 \cdot z^{-(-1)} = z$$

e) $x[n] = \left(\frac{1}{3}\right)^n (u[n] - u[n-5])$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \qquad u[n-5] = \begin{cases} 1 & n \geq 5 \\ 0 & n < 5 \end{cases}$$

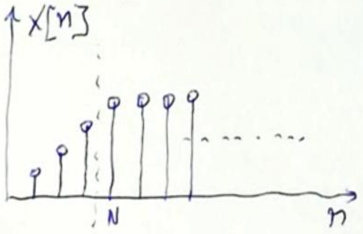
$$\rightarrow \quad u[n] - u[n-5] = \begin{cases} 1 & 0 \leq n \leq 5 \\ 0 & n < 0, \; n \geq 5 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} \left(\frac{1}{3}\right)^n (u[n] - u[n-5]) z^{-n} = \sum_{n=0}^{4} \left(\frac{1}{3}\right)^n z^{-n} = \sum_{n=0}^{4} \left(\frac{1}{3z}\right)^n =$$

$$\left| \; \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \qquad |a| < 1 \right.$$

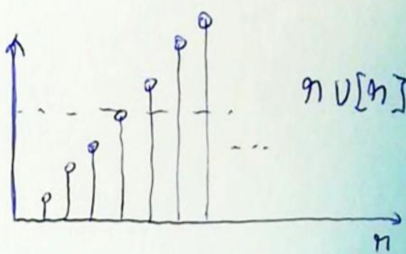
$$= \frac{1 - \left(\frac{1}{3z}\right)^5}{1 - \frac{1}{3z}} = \frac{1 - (3z)^{-5}}{1 - (3z)^{-1}} \qquad |z| > 0$$

f) $x[n] = \begin{cases} n & 0 \leq n \leq N-1 \\ N & n \geq N \end{cases}$

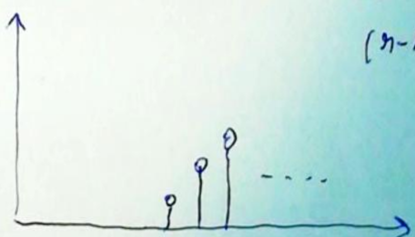



We observe that

$$x[n] = n\,u[n] - (n-N)\,u[n-N]$$




$n\,u[n]$



$(n-N)\,u[n-N]$

From the table of z-transform properties we know that

$$n\,x[n] \xrightarrow{z} -z\frac{d}{dz}X[z]$$

Since $u[n] \xrightarrow{z} \frac{1}{1-z^{-1}}$, $n\,u[n] \xrightarrow{z} -z\frac{d}{dz}\frac{1}{1-z^{-1}} =$

$$= \frac{z^{-1}}{(1-z^{-1})^2} \qquad |z|>1$$

We also know that

$$x[n-n_0] \xrightarrow{z} X(z)\,z^{-n_0}$$

$$(n-N)\,u[n-N] \xrightarrow{z} \frac{z^{-1}}{(1-z^{-1})^2}\cdot z^{-N} = \frac{z^{-N-1}}{(1-z^{-1})^2} \qquad |z|>1$$

Therefore

$$x[n] = n\,u[n] - (n-N)\,u[n-N] \xrightarrow{z} X(z) = \frac{z^{-1}-z^{-N-1}}{(1-z^{-1})^2} =$$

$$= \frac{z^{-1}(1-z^{-N})}{(1-z^{-1})^2}$$

9) $x[n] = a^{|n|} \quad 0 \leq |a| < 1$

$$a^{|n|} = \begin{cases} a^{n} & n \geq 0 \\ a^{-n} & n < 0 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{-1} a^{-n} z^{-n} + \sum_{n=0}^{\infty} a^{n} z^{-n} = \underbrace{\sum_{n=1}^{\infty} a^{n} z^{n}}_{(1)} + \underbrace{\sum_{n=0}^{\infty} a^{n} z^{-n}}_{(2)}$$

(1) $\displaystyle\sum_{n=1}^{\infty} (az)^{n} = \sum_{n=0}^{\infty} (az)^{n} - (az)^{0} = \frac{1}{1-az} - 1 = \frac{az}{1-az}$

$$|az| < 1 \rightarrow |z| < \frac{1}{|a|}$$

(2) $\displaystyle\sum_{n=0}^{\infty} (az^{-1})^{n} = \frac{1}{1-az^{-1}} \qquad |az^{-1}| < 1 \rightarrow |z| > |a|$

We therefore obtain

$$X(z) = \frac{az}{1-az} + \frac{1}{1-az^{-1}} \qquad |a| < |z| < \frac{1}{|a|}$$

h) $x[n] = \begin{cases} 1 & 0 \le n < N-1 \\ 0 & n \ge N \\ 0 & n < 0 \end{cases}$

$$X(z) = \sum_{n=0}^{N-1} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n - \sum_{n=N}^{\infty} (z^{-1})^N = \sum_{n=0}^{\infty} (z^{-1})^n - z^{-N} \sum_{n=0}^{\infty} (z^{-1})^n =$$

$$= \sum_{n=0}^{\infty} (z^{-1})^n (1 - z^{-N}) = \frac{1 - z^{-N}}{1 - z^{-1}} =$$

$$= \frac{1 - \frac{1}{z^N}}{1 - \frac{1}{z}} = \frac{\frac{z^N - 1}{z^N}}{\frac{z-1}{z}} = \frac{z^N - 1}{z^{N-1}(z-1)}$$

Note that the pole in 1 is canceled by the zero in 1.
Therefore the series converges for $z \neq 0$