MA505G Flervariabelanalys för civilingenjörer, 7.5 hp

Formelblad ht 2020

Logaritmer

$$ {}^a\log(x\cdot y) = {}^a\log(x) + {}^a\log(y) \qquad {}^a\log\left(\frac{x}{y}\right) = {}^a\log(x) - {}^a\log(y) \qquad {}^a\log(x^y) = y\cdot {}^a\log(x) $$

Trigonometri

$$ \sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{1}{2} \qquad \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} $$

$$ \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) \qquad \cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) $$

$$ \sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y) \qquad \cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y) $$

$$ \sin(2x) = 2\sin(x)\cos(x) \qquad \cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x) $$

$$ \sin^2(x) = \frac{1}{2}(1-\cos(2x)) \qquad \cos^2(x) = \frac{1}{2}(1+\cos(2x)) $$

Standardgränsvärden

$$ \lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \lim_{x\to 0}\frac{\tan x}{x} = 1 \qquad \lim_{x\to 0}\frac{e^x-1}{x} = 1 $$

$$ \lim_{x\to 0}\frac{\arcsin x}{x} = 1 \qquad \lim_{x\to 0}\frac{\arctan x}{x} = 1 \qquad \lim_{x\to 0}\frac{\ln(1+x)}{x} = 1 $$

$$ \lim_{x\to\infty}\frac{x^\alpha}{e^x} = 0 \qquad \lim_{x\to\infty}\frac{\ln x}{x^\alpha} = 0 \quad (\text{om } \alpha > 0) \qquad \lim_{x\to 0^+} x^\alpha \ln x = 0 \quad (\text{om } \alpha > 0) $$

Deriveringsregler

$$ D\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x) \qquad D\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2} $$

$$ D\left(f\big(g(x)\big)\right) = f'\big(g(x)\big)g'(x) \qquad D\left(f^{-1}(x)\right) = \frac{1}{f'\big(f^{-1}(x)\big)} $$

Integreringsregler och elementära primitiva funktioner

Partiell integration $(F' = f)$:

$$ \int f(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx $$

Variabelbyte $(t = g(x))$:

$$ \int f(g(x))g'(x)dx = \int f(t)dt $$

$$ \int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C \qquad \int \frac{1}{x}\,dx = \ln|x| + C $$

$$ \int e^x\,dx = e^x + C \qquad \int \ln x\,dx = x\ln x - x + C $$

$$ \int \sin x\,dx = -\cos x + C \qquad \int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2} + C $$

$$ \int \cos x\,dx = \sin x + C \qquad \int \arccos x\,dx = x\arccos x - \sqrt{1-x^2} + C $$

$$ \int \tan x\,dx = -\ln|\cos x| + C \qquad \int \arctan x\,dx = x\arctan x - \ln\sqrt{1+x^2} + C $$

$$ \int \frac{x}{1+x^2}\,dx = \ln\sqrt{1+x^2} + C \qquad \int \frac{1}{\sqrt{1+x^2}}\,dx = \ln\left|x + \sqrt{1+x^2}\right| + C $$

Partialbråksuppdelning

$$ \frac{\cdots}{(x+a)(x+b)^2(x^2+c^2)} = \frac{A}{x+a} + \frac{B_1}{x+b} + \frac{B_2}{(x+b)^2} + \frac{C_1x + C_2}{x^2+c^2} $$

Maclaurinserier

$$ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots $$

$$ \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots $$

$$ (1+x)^\alpha = 1 + \alpha x + \alpha(\alpha-1)\frac{x^2}{2!} + \alpha(\alpha-1)(\alpha-2)\frac{x^3}{3!} + \ldots \qquad \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots $$

Vektorer och matriser

$$|\boldsymbol{u}| = \sqrt{u_1^2+u_2^2+u_3^2} \qquad \boldsymbol{u}\cdot\boldsymbol{v} = u_1v_1+u_2v_2+u_3v_3 \qquad \boldsymbol{u}\times\boldsymbol{v} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \tfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{aligned} &(a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}) \\ &-(a_{13}a_{22}a_{31}+a_{11}a_{23}a_{32}+a_{12}a_{21}a_{33}) \end{aligned}$$

Linjer och plan

Linje genom (x_0,y_0,z_0) med riktning (a,b,c): $(x,y,z)=(x_0,y_0,z_0)+t(a,b,c), \quad t\in\mathbb{R}$.

Plan genom (x_0,y_0,z_0) med normal (a,b,c): $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$.

Avstånd från (x_1,y_1,z_1) till planet $ax+by+cz+d=0$: $\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$.

Taylors formel i två variabler

$$f(a+h,b+k) = f(a,b)+f'_x(a,b)h+f'_y(a,b)k+\frac{1}{2}\big(f''_{xx}(a,b)h^2+2f''_{xy}(a,b)hk+f''_{yy}(a,b)k^2\big)+|(h,k)|^3B(h,k).$$

Variabelbyte i dubbelintegraler

$$\iint_D f(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_E f(g(u,v),h(u,v))\left|\frac{d(x,y)}{d(u,v)}\right|\mathrm{d}u\mathrm{d}v, \qquad \text{där } \frac{d(x,y)}{d(u,v)} = \begin{vmatrix} g'_u & g'_v \\ h'_u & h'_v \end{vmatrix}.$$

Polära koordinater

Planpolära koordinater

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad \frac{d(x,y)}{d(r,\varphi)} = r$$

Rymdpolära koordinater

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad \frac{d(x,y,z)}{d(r,\theta,\varphi)} = r^2\sin\theta$$

Kurvor och ytor

Parameterkurva $\boldsymbol{r}(t)=(x(t),y(t),z(t))$: enhetstangent $\boldsymbol{T}=\boldsymbol{r}'(t)/|\boldsymbol{r}'(t)|$, bågelement $\mathrm{d}s=|\boldsymbol{r}'(t)|dt$, orienterat bågelement $\mathrm{d}\boldsymbol{r}=\boldsymbol{T}\mathrm{d}s=\boldsymbol{r}'(t)\,\mathrm{d}t$.

Parameteryta $\boldsymbol{r}(s,t)=(x(s,t),y(s,t),z(s,t))$: enhetsnormal $\boldsymbol{n}=\boldsymbol{r}'_s\times\boldsymbol{r}'_t/|\boldsymbol{r}'_s\times\boldsymbol{r}'_t|$, areaelement $\mathrm{d}S=|\boldsymbol{r}'_s\times\boldsymbol{r}'_t|\mathrm{d}s\,\mathrm{d}t$, orienterat areaelement $\mathrm{d}\boldsymbol{S}=\boldsymbol{n}\,\mathrm{d}S=(\boldsymbol{r}'_s\times\boldsymbol{r}'_t)\mathrm{d}s\,\mathrm{d}t$.

Funktionsyta $z=f(x,y)$:

$\boldsymbol{r}(x,y)=(x,y,f(x,y))$, $\mathrm{d}S=\sqrt{(f'_x)^2+(f'_y)^2+1}\,\mathrm{d}x\mathrm{d}y$, $\boldsymbol{n}\,\mathrm{d}S=(-f'_x,-f'_y,1)\,\mathrm{d}x\mathrm{d}y$.

Sfär med radie R:

$\boldsymbol{r}(\theta,\varphi)=(R\sin\theta\cos\varphi,R\sin\theta\sin\varphi,R\cos\theta)$, $\mathrm{d}S=R^2\sin\theta\mathrm{d}\theta\mathrm{d}\varphi$, $\boldsymbol{n}\,\mathrm{d}S=\boldsymbol{r}(\theta,\varphi)R\sin\theta\mathrm{d}\theta\mathrm{d}\varphi$.

Cylinderyta med radie R:

$\boldsymbol{r}(\varphi,z)=(R\cos\varphi,R\sin\varphi,z)$, $\mathrm{d}S=R\mathrm{d}\varphi\,\mathrm{d}z$, $\boldsymbol{n}\,\mathrm{d}S=(\cos\varphi,\sin\varphi,0)R\mathrm{d}\varphi\,\mathrm{d}z$.

Gradient, divergens, rotation

Om $f(x,y,z)$ och $\boldsymbol{F}(\boldsymbol{r})=\big(P(x,y,z),Q(x,y,z),R(x,y,z)\big)$ är givna i kartesiska koordinater:

$$\begin{aligned} \operatorname{grad} f &= (f'_x,f'_y,f'_z) = (\partial_x,\partial_y,\partial_z)f = \nabla f \\ \operatorname{div}\boldsymbol{F} &= P'_x+Q'_y+R'_z = (\partial_x,\partial_y,\partial_z)\cdot(P,Q,R) = \nabla\cdot\boldsymbol{F} \\ \operatorname{rot}\boldsymbol{F} &= (R'_y-Q'_z,P'_z-R'_x,Q'_x-P'_y) = (\partial_x,\partial_y,\partial_z)\times(P,Q,R) = \nabla\times\boldsymbol{F} \end{aligned}$$

Integralsatser

Om randkurvorna/-ytorna är slutna och positivt orienterade:

Greens formel: $\displaystyle\int_{\partial D} P\,\mathrm{d}x+Q\,\mathrm{d}y = \iint_D (Q'_x-P'_y)\,\mathrm{d}x\mathrm{d}y$

Stokes sats: $\displaystyle\int_{\partial\Gamma} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \iint_\Gamma (\operatorname{rot}\boldsymbol{F})\cdot\boldsymbol{n}\,\mathrm{d}S$

Gauss sats: $\displaystyle\iint_{\partial K} \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S = \iiint_K \operatorname{div}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$