INSTITUTIONEN FÖR
NATURVETENSKAP OCH TEKNIK

Kursens namn/kurskod	**Numeriska Metoder för civilingenjörer DT508G**
Examinationsmomentets namn/provkod	**Teori, 2,5 högskolepoäng (Provkod: A003)**
Datum	2021-01-08
Tid	Kl. 08:15 – 12:15

Tillåtna hjälpmedel	Skrivmateriel, miniräknare med raderat minne, kursbok, anteckningar från föreläsning.
Instruktion	Läs igenom alla frågor noga. Börja varje fråga på ett nytt svarsblad. Skriv bara på ena sidan av svarsbladet. Skriv tentamenskoden på varje svarsblad. Skriv läsligt! Det räcker att ange fem decimaler. Glöm inte att scanna lösningar och ladda up det till WISEflow
Viktigt att tänka på	Motivera väl, redovisa tydligt alla väsentliga steg, rita tydliga figurer och svara med rätt enhet. Lämna in i uppgiftsordning.
Ansvarig/-a lärare (ev. telefonnummer)	Danny Thonig (Mobil: 0727010037)
Totalt antal poäng	30
Betyg (ev. ECTS)	Skrivningens maxpoäng är 30. För betyg 3/4/5 räcker det med samt 15/22/28 poäng totalt. Detaljerna framgår av separat dokument publicerat på Blackboard.
Tentamensresultat	Resultatet meddelas på Studentforum inom 15 arbetsdagar efter tentadagen.
Övrigt	Lärare är inte på plats. Varsågod och ringa om du har frågar.

Lycka till!

1.

a) Determine the order of convergence and the convergence rate for the modified Newton method

$$x_{n+1} = x_n - m\,\frac{f(x_n)}{f'(x_n)}$$

Under the conditions $f(x) \in C^{m+1}(m \geq 1)$, $f(r) = f'(r) = f'' = \ldots = f^{(m-1)}(r) = 0$, and $f^{(m)}(r) \neq 0$, where r is the root. **Tip:** Use Taylor series of $f(x_n)$ and $f'(x_n)$ around r. [7p]

b) Perform two iterations of the method for $f(x) = e^x - x - 1$ with the initial guess $x_0 = 1$ and show that it converges to the root $r = 0$. [3p]

Solutions:

a)

The order of convergence is at least 2. The Taylor series of $f(x_n)$ around r is

$$f(x_n) = f(r) + f'(r)(x_n - r) + \ldots + \frac{f^{(m)}(r)}{m!}(x_n - r)^m + \frac{f^{(m+1)}(c_n)}{(m+1)!}(x_n - r)^{m+1}$$, where

c_n is a point between r and x_n, and only two terms can be different from zero:

$$f(x_n) = \frac{f^{(m)}(r)}{m!}(x_n - r)^m + \frac{f^{(m+1)}(c_n)}{(m+1)!}(x_n - r)^{m+1}$$

Analogously, from the Taylor series for $f'(x_n)$, we obtain

$$f'(x_n) = \frac{f^{(m)}(r)}{(m-1)!}(x_n - r)^{m-1} + \frac{f^{(m+1)}(c_n^*)}{m!}(x_n - r)^m$$

where c_n and c_n^* is a point between r and x_n, not necessarily equal to each other. Forming the difference $r - x_{n+1}$, we get with the iteration formula

$$r - x_{n+1} = r - x_n + m\,\frac{f(x_n)}{f'(x_n)} = \frac{f'(x_n)(r - x_n) + mf(x_n)}{f'(x_n)}$$

Using the expressions from above for $f(x_n)$ and $f'(x_n)$ and simplifying gives

$$r - x_{n+1} = \frac{-\frac{f^{(m+1)}(c_n^*)}{m!}(x_n - r)^2 + m\frac{f^{(m+1)}(c_n)}{(m+1)!}(x_n - r)^2}{\frac{f^{(m)}(r)}{(m-1)!} + \frac{f^{(m+1)}(c_n^*)}{m!}(x_n - r)}$$

In the limit

$$\lim_{n\to\infty} \frac{|r - x_{n+1}|}{|r - x_n|^2} = \lim_{n\to\infty} \left| \frac{-\frac{f^{(m+1)}(c_n^*)}{m!} + m\frac{f^{(m+1)}(c_n)}{(m+1)!}}{\frac{f^{(m)}(r)}{(m-1)!} + \frac{f^{(m+1)}(c_n^*)}{m!}(x_n - r)} \right|$$

By assuming x_n, c_n, and c_n^* converges to r as n approaches infinity and using the continuity of $f^{(m)}(x)$ and $f^{(m+1)}(x)$, we get

$$\lim_{n\to\infty} \frac{|r - x_{n+1}|}{|r - x_n|^2} = \frac{\frac{(m-1)!}{m!}\left(1 - \frac{m}{m+1}\right)\left|f^{(m+1)}(r)\right|}{f^{(m)}(r)} = \frac{1}{m(m+1)}\left|\frac{f^{(m+1)}(r)}{f^{(m)}(r)}\right|$$

So the exponent in the denominator of the left side is the order of convergence and it is equal to 2. The rate of convergence is given by the expression on the right side.

b)
First we have to determine the multiplicity, which is obviously 2. Thus, we have to analyze

$x_{n+1} = x_n - 2\dfrac{f(x_n)}{f'(x_n)}$. $f(x) = e^x - x - 1, f(x)' = e^x - 1$.Thus
Iteration 1: $f(x_0) = e - 2, f'(x_0) = e - 1, x_1 = 0.1640$
Iteration 2: $f(x_1) = 0.0142, f'(x_1) = 0.1782, x_2 = 0.0045$

2. Find the polynomial that interpolates $f(0) = 0, f(1) = 1, f(2) = 0, f(3) = 1,$ $f(4) = 0$ using Newton divided differences. Use the Newton table to generate the necessary divided differences. [6p]

Solutions:

0	0				
		1			
1	1		-1		
		-1		2/3	
2	0		1		-1/3
		1		-2/3	
3	1		-1		
		-1			
4	0				

Thus,

$$f(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$$
$$+(x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)(x - x_2)(x - x_3)f[x_0, x_1, x_2, x_3, x_4]$$
$$f(x) = x - x(x - 1) + \frac{2}{3}x(x - 1)(x - 2) - \frac{1}{3}x(x - 1)(x - 2)(x - 3)$$

3. Consider the matrix

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & -4 \end{pmatrix}$$

   a) Use Geršgorin's circle theorem to analyze the locations of eigenvalues of A. If all eigenvalues are real, where do theses eigenvalues locate? Please give as good estimated eigenvalues as possible. [3p]
   b) Perform one steps of shifted inverse power iteration, where the shift s should be selected by the approximated smallest eigenvalues found in a). The initial guess is $x_0 = (0,0.5,1.0)$. [2p]

Solutions:

a) Geršgorin's circles:
$R_1 = \{z \in \mathbb{C} : ||z - 4|| \leq 1\}$

$R_2 = \{z \in \mathbb{C} : ||z|| \leq 2\}$
$R_3 = \{z \in \mathbb{C} : ||z+4|| \leq 2\}$

b)
the smallest eigenvalue will be in the region $-6 \leq \lambda \leq -2$. Lets e.g. take $s = -4$. Note that you were free to use any norm possible.

1. Normalize x_0: $||x_0||_2 = \frac{\sqrt{5}}{2}$ and, thus, $u = \frac{1}{\sqrt{5}}(0,1,2)$.

2. $(\boldsymbol{A} - s\boldsymbol{I})v = u$ and, thus, the system of linear equations need to be solved with e.g. Gauss elimination methods (not shown here).

$$\begin{pmatrix} 8 & 1 & 0 \\ 1 & 4 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$v = (-0.1278, 1.0222, 3.5138)^T$ and $\lambda = v^T u = 3.6$.

4. The Chebyshev polynomials of second kind are defined as
$$U_n(x) = \frac{1}{n+1} T'_{n+1}(x), n \geq 0$$
where $T_{n+1}(x)$ is the Chebyshev polynomial of first kind.
   a) Using the form $T_n(x) = \cos(n \arccos(x))$, $x \in [-1,1]$, derive a similar expression for $U_n(x)$. [2p]
   b) Show that the Chebyshev Polynomial of the second kind satisfies the recursion $U_0(x) = 1$, $U_1(x) = 2x$, and $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$. [3p]
   c) The Chebyshev polynomials of second kind are orthogonal with respect to the inner product
$$< f, g > = \int_{-1}^{1} f(x)g(x)\sqrt{1-x^2}\mathrm{d}x.$$
   Demonstrate this by using composite trapezoid method with $m = 2$ and for $< U_3, U_3 >$ and $< U_3, U_2 >$. [4p]

Solutions:

a)
The derivative of $\arccos(x)$ is
$$\partial_x \arccos(x) = -\frac{1}{\sqrt{1-x^2}}.$$
Thus,
$$T'_{n+1}(x) = \frac{(n+1)\sin((n+1)\arccos(x))}{\sqrt{1-x^2}} \text{ and } U_n(x) = \frac{\sin((n+1)\arccos(x))}{\sqrt{1-x^2}}.$$
If we replace $x = \cos\theta$, we have
$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$$

b)
Lets take the recursion of the Chebyshev polynomial of first kind

$T_0 = 1, T_1 = x, T_2 = 2x^2 - 1$ or in general $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

Then,

$U_{n+1}(x) = \frac{1}{n+2}T'_{n+2}(x) = \frac{1}{n+2}\partial_x(2xT_{n+1}(x) - T_n(x)) = \frac{1}{n+2}(2T_{n+1}(x) + 2xT'_{n+1}(x) - T'_n(x)) = \frac{1}{n+2}(2T_{n+1}(x) + 2x(n+1)U_n(x) - nU_{n-1}(x))$

Further, it is $T_n(x) = U_n(x) - xU_{n-1}(x)$. Thus,

$U_{n+1}(x) = \frac{1}{n+2}(2(U_{n+1}(x) - xU_n(x)) + 2x(n+1)U_n(x) - nU_{n-1}(x))$

$\frac{n}{n+2}U_{n+1}(x) = \frac{n}{n+2}(2xU_n(x) - U_{n-1}(x))$

$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$

c)

Composite trapezoid says apply relation from $[-1,0]$ and $[0,1]$. Thus $h = 1$

$U_2 = 4x^2 - 1 U_3 = 8x^3 - 4x$.

Thus, we have to solve

$\int_{-1}^{1} \underbrace{(8x^3 - 4x)^2\sqrt{1-x^2}}_{F_1(x)}\,\mathrm{d}x$ and $\int_{-1}^{1} \underbrace{(8x^3 - 4x)(4x^2 - 1)\sqrt{1-x^2}}_{F_2(x)}\,\mathrm{d}x$

And the integrals are approximated by

$\frac{1}{2}(F_i(-1) + 2F_i(0) + F_i(1))$

So for the first integral we get 0 and for the second we also get 0.