TENTAMENSKOD:_______________

INSTITUTIONEN FÖR
NATURVETENSKAP OCH TEKNIK

Kursens namn/kurskod	**Numeriska Metoder för civilingenjörer** **DT508G**
Examinationsmomentets namn/provkod	**Teori, 2,5 högskolepoäng (Provkod: A003)**
Datum	2020-10-27
Tid	Kl. 08:15 – 12:15

Tillåtna hjälpmedel	Skrivmateriel, miniräknare med raderat minne, kursbok, anteckningar från föreläsning.
Instruktion	Läs igenom alla frågor noga. Börja varje fråga på ett nytt svarsblad. Skriv bara på ena sidan av svarsbladet. Skriv tentamenskoden på varje svarsblad. Skriv läsligt! Det räcker att ange fem decimaler. Glöm inte att scanna lösningar och ladda up det till WISEflow
Viktigt att tänka på	Motivera väl, redovisa tydligt alla väsentliga steg, rita tydliga figurer och svara med rätt enhet. Lämna in i uppgiftsordning.
Ansvarig/-a lärare (ev. telefonnummer)	Danny Thonig (Mobil: 0727010037)
Totalt antal poäng	30
Betyg (ev. ECTS)	Skrivningens maxpoäng är 30. För betyg 3/4/5 räcker det med samt 15/22/28 poäng totalt. Detaljerna framgår av separat dokument publicerat på Blackboard.
Tentamensresultat	Resultatet meddelas på Studentforum inom 15 arbetsdagar efter tentadagen.
Övrigt	Lärare är inte på plats. Varsågod och ringa om du har frågar.

Lycka till!

1. Interpolate a cubic spline between the three points (0, 1), (2, 2) and (4, 0).

   a) What is spline interpolation and what are cubic splines? What three properties does cubic splines have at the nodes. Why is the problem inconsistent and how this can be solved? Compare spline interpolation with the Lagrange interpolation. What is the difference? [5p]
   b) From the points mentioned above, construct the spline coefficient matrices for a curvature-adjusted cubic spline and $\nu_0 = 1$ and $\nu_2 = -0.5$. [2p]
   c) Perform LU factorisation on the spline coefficient matrix obtained from b) with Cholesky method. Which other LU factorisation methods do you know? [3p]
   d) With the resulting LU factorization from c) solve the system of linear equations of this spline problem. Compare your result with $c_0 = 0.5$, $c_1 = -\frac{5}{8}$, and $c_2 = -\frac{1}{4}$. [2p]
   e) Determine all other coefficients of the cubic spline and write explicitly down the two cubic splines. [3p]

Suggested solution

a)

See book T. Sauer P. 167 - 169. Three properties: i) $S_i(x_i) = y_i$ and $S_i(x_{i+1}) = y_{i+1}$ for $i = 0,..,n-1$; ii) $S'_{i-1}(x_i) = S'_i(x_i)$ for $i = 1,...,n-1$; iii) $S''_{i-1}(x_i) = S''_i(x_i)$ for $i = 1,...,n-1$. The problem is inconsistent since for 3n unknowns in the spline interpolation, only 3n-2 equation are defined by the previous properties. Thus, two more properties need to be defined, e.g. by natural splines, where the curvature at the first and last spline at the start and end point, respectively, is zero.

In Lagrange interpolation the total set of point $(x_0, y_0), \ldots, (x_n, y_n)$ is used to design an interpolation polynomial, where in spline interpolation only two of these points are interpolated and these interpolation polynomials are "glued" together at the points (knots).

b)

$a_0 = 1$, $a_1 = 2$, and $a_2 = 0$, From $\delta_i = x_{i+1} - x_i$ and $\Delta_i = y_{i+1} - y_i$ we have $\delta_0 = 2$, and $\delta_1 = 2$. $\Delta_0 = 1$ and $\Delta_1 = -2$. Thus,

$$\begin{pmatrix} 2 & 0 & 0 \\ 2 & 8 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3\left(\frac{-2}{2} - \frac{1}{2}\right) \\ -0.5 \end{pmatrix}$$

c)

In Cholesky, $l_{ii} = u_{ii}$

$$\begin{pmatrix} 2 & 0 & 0 \\ 2 & 8 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} a & g & h \\ 0 & c & i \\ 0 & 0 & f \end{pmatrix}}_{U}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 2 & 8 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} a^2 & ag & ah \\ ba & bg + c^2 & bh + ci \\ da & dg + ec & dh + ei + f^2 \end{pmatrix}$$

Thus, $a = \sqrt{2}$, $b = \sqrt{2}$, $c = 2\sqrt{2}$, $d = 0$, $e = 0$, $f = \sqrt{2}$, $g = 0$, $h = 0$, $i = \dfrac{1}{\sqrt{2}}$

d)

Solving the system of equation
$LUx = b$ and thus, $Ly = b$ and $Ux = y$
Thus,

$$\begin{pmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{2} & 2\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{9}{2} \\ -0.5 \end{pmatrix}$$ and thus $y_2 = -\dfrac{1}{2\sqrt{2}}$, $y_0 = \dfrac{1}{\sqrt{2}}$, and $y_1 = -\dfrac{11}{4\sqrt{2}}$

$$\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 2\sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{11}{4\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \end{pmatrix}$$ and, thus, $x_0 = 0.5$, $x_1 = -\dfrac{5}{8}$, and $x_2 = -\dfrac{1}{4}$

e)

The other coefficients are obtained from
$d_i = \dfrac{c_{i+1} - c_i}{3\delta_i}$ and $b_i = \dfrac{\Delta_i}{\delta_i} - \dfrac{\delta_i}{3}(2c_i + c_{i+1})$
Thus $d_0 = -\dfrac{3}{16}$ and $d_1 = \dfrac{3}{48}$ as well as $b_0 = \dfrac{1}{2} - \dfrac{2}{3}(1 - \dfrac{5}{8}) = \dfrac{1}{4}$ and $b_1 = -1 - \dfrac{2}{3}(-\dfrac{5}{4} - \dfrac{1}{4}) = 0$. The splines are

$S_0(x) = 1 + \dfrac{1}{4}x + \dfrac{1}{2}x^2 - \dfrac{3}{16}x^3$

$S_1(x) = 2 + 0(x-2) - \dfrac{5}{8}(x-2)^2 + \dfrac{3}{48}(x-2)^3$

2. Suppose the following first-order linear ordinary differential equation

$$\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1$$

and $y(\frac{\pi}{4}) = 3\sqrt{2}$ and $0 \le x < 7$.

a) Solve this initial value problem (IVP) with the formula of the exact solution for first-order linear ordinary differential equation. [2p]

b) The solution of the IVP is $y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$. Schematically plot this solution. [1p]

c) We want to find the root of $y(x)$ in the interval $x \in [0,2]$. Perform two steps with the bisection method. [2p]

d) Use the result from the bisection method and perform two more steps with Newton's method. [3p]

e) In general, compare both methods — bisection and Newton method — in terms of the convergence rate and the convergence order. Do you know one other method to find the root r of a function? [2p]

f) The exact root of $y(x)$ for $x \in [0,2]$ is $r = 1.4379$. Analyze the sensitivity of the problem when adding to $y(x)$ a linear function $g(x) = \varepsilon x$ with $\varepsilon = 1 \cdot 10^{-5}$. Calculate the shift of the root Δr. [2p]

g) Determine the integral $\int y(x)\mathrm{d}x$ for $x \in [0,r]$ by composite midpoint method and $m = 3$. [3p]

Suggested solution

a)

Rewrite the differential equation to get the coefficient of the derivative a one.

$$y' + \frac{\sin(x)}{\cos(x)}y = 2\cos^2(x)\sin(x) - \frac{1}{\cos(x)}$$

$$y' + \tan(x)y = 2\cos^2(x)\sin(x) - \sec(x)$$

We apply the integration rule to solve first-order linear ordinary differential equations:
If $y' = g(x)y + h(x) \quad y(a) = y_a, x \in [a, b]$ then the solution is

$$y(x) = e^{\int g(x)\mathrm{d}x}\int e^{-\int g(x)\mathrm{d}x}h(x)\mathrm{d}x$$

So $g(x) = -\tan(x)$ and $h(x) = 2\cos^2(x)\sin(x) - \sec(x)$. We need to get the integral of the tangent and use $u = \cos(x)$. Thus

$$-\int \tan(x)\mathrm{d}x = -\int \frac{\sin(x)}{\cos(x)}\mathrm{d}x = \int \frac{1}{u}\mathrm{d}u = \ln(u) = \ln(\cos(x))$$

We use further
$-\ln(\cos(x)) = \ln(\cos(x)^{-1}) = \ln(\sec(x))$, $e^{\ln f(x)} = f(x)$, and the trigonometric relation $\sin(2x) = 2\cos(x)\sin(x)$

$$y(x) = \cos(x)\int \sec(x)(2\cos^2(x)\sin(x) - \sec(x))\mathrm{d}x = \cos(x)\int 2\cos(x)\sin(x) - \sec(x)^2\mathrm{d}x$$

$$y(x) = \cos(x)\int \sin(2x) - \frac{1}{\cos(x)^2}\mathrm{d}x = \cos(x)\left(-\frac{1}{2}\cos(2x) - \tan(x) + c\right) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + c\cos(x)$$

From the initial condition we can fix the constant c

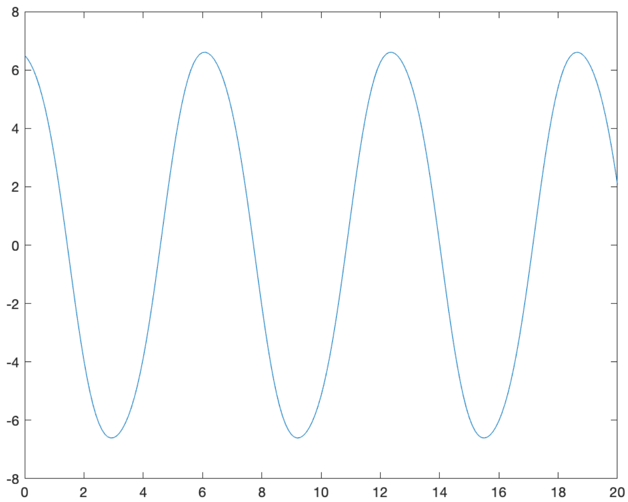
$$3\sqrt{2} = y(\frac{\pi}{4}) = -\frac{1}{2}\cos(\frac{\pi}{4})\cos(\frac{\pi}{2}) - \sin(\frac{\pi}{4}) + c\cos(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2} + c\frac{\sqrt{2}}{2}$$

And, thus, $c = 7$.

So the solution is

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$$

b)



c)

Let focus on the interval $x \in [0,2]$.

To start the bisection, we check $y(0) = 6.5$, $y(2) = -3.95833$ and the starting condition $y(0)y(2) < 0$ is fulfilled.

1. Step

$c = \dfrac{a+b}{2} = 1$ and $y(1) = 3.05307$. Thus, $y(1)y(2) < 0$ and $a = c$.

2. step

$c = \dfrac{a+b}{2} = 1.5$ and $y(1.5) = -0.46732$. Thus, $b = c$.

d)

Determine the derivative

$$y'(x) = -\frac{1}{2}(-\sin(x)\cos(2x) - \cos(x)\sin(2x)2) - \cos(x) - 7\sin(x)$$

$$y'(x) = -\frac{1}{2}(\sin(x)^2 - 5\cos(x)^2)\sin(x) - \cos(x) - 7\sin(x)$$

Newtons method is

$x_{i-1} = x_i - \dfrac{y(x_i)}{y'(x_i)}$ with $x_0 = 1.5$ from bisection method.

1. Step

$y(1.5) = -0.46732$

$y'(1.5) = -7.53698$
and
$x_1 = 1.43800$

2. Step
$y(1.43800) = -4.69068 \cdot 10^{-4}$
$y'(1.5) = -7.51424$
and
$x_1 = 1.43794$

e)

	Bisection method	Newton method
Order of convergence	Linear	Quadratic
Rate of convergence	1/2	f“/2f’

Other method: e.g. fixed point method.

f)

$$\Delta r = \frac{\varepsilon g(r)}{y'(r)} = -1.91359 \cdot 10^{-11}$$

e)

$$h = \frac{r}{m} = \frac{r}{3}$$

Thus, the midpoints are
$x_0 = \dfrac{r}{6}$, $x_1 = \dfrac{r}{2}$, and $x_2 = \dfrac{5r}{6}$

$$y(x_0) = 6.13161, \quad y(x_1) = 4.55903, \quad y(x_2) = 1.75028$$

and the integral from midpoint method is

$$I = h\sum_{i=0}^{2} y(x_i) = \frac{r}{3}12.44092 = 5.96293$$