Lösningsförslag tentamen 220818

1. a) Låt $z = r \cdot e^{i\theta}$ så att

$$z^6 = r^6 \cdot e^{i6\theta}.$$

Notera att

$$-64 = 64 \cdot (-1) = 64 \cdot e^{i\pi}.$$

Vi får då

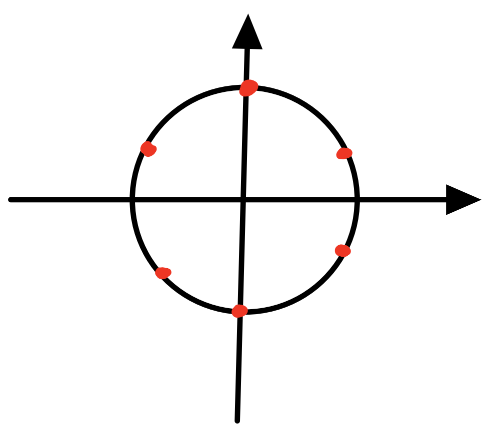
$$z^6 = -64$$

$$r^6 \cdot e^{i6\theta} = 64 \cdot e^{i\pi} = 64 \cdot e^{i(\pi + k \cdot 2\pi)}$$

$$\begin{cases} r^6 = 64, \quad r > 0 \\ 6\theta = \pi + k \cdot 2\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} r = 2 \\ \theta = \frac{\pi}{6} + k \cdot \frac{\pi}{3}, \quad k = 0, 1, 2, 3, 4, 5 \end{cases}$$

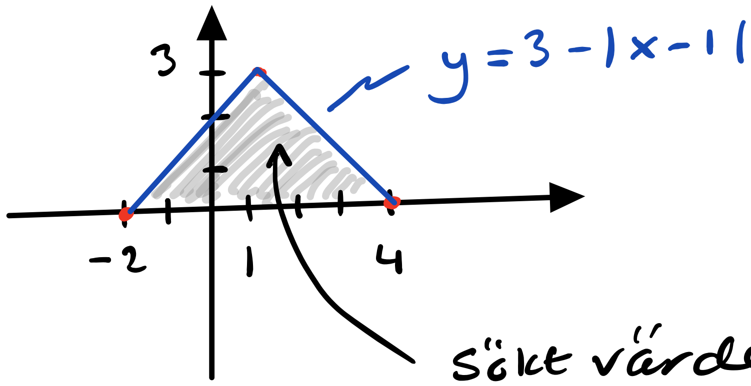
Svar: $z = 2 \cdot e^{i \cdot \left(\frac{\pi}{6} + k \cdot \frac{\pi}{3}\right)}, \quad k = 0, 1, 2, 3, 4, 5.$



b) Rita kurvan

$$y = 3 - |x-1|$$

för $-2 \le x \le 4$.



sökt värde på integralen ges av denna area.

$$A = \frac{6 \cdot 3}{2} = 9 \text{ a.e.}$$

Svar:

$$\int_{-2}^{4} (3 - |x-1|)\,dx = 9. \qquad \square$$

2. $y' = x\cdot(1-y)$, $y<1$

kan skrivas om som

$$\frac{1}{1-y}\cdot y' = x$$

$$\int \frac{1}{1-y}\,dy = \int x\,dx$$

$$-\ln(1-y) + C_1 = \frac{x^2}{2} + C_2 \quad \Big\} \; y<1$$

$$\ln(1-y) = -\frac{x^2}{2} + C_1 - C_2$$

$$1-y = e^{-\frac{x^2}{2}+C_1-C_2} = e^{-\frac{x^2}{2}}\cdot D$$

om $D = e^{C_1-C_2}$.

$$y = 1 - D\cdot e^{-x^2/2}$$

så villkoret $y(0)=0$ ger att

$$0 = 1 - D \Leftrightarrow D = 1.$$

Svar: $y = 1 - e^{-x^2/2}$. $\square$

3. $$\int_{-3}^{-2} x^2 \cdot \sqrt{x+3}\, dx = \left[\begin{array}{l} t=\sqrt{x+3}, \quad x = t^2-3 \\ dx = 2t\,dt \\ t(-3)=0, \quad t(-2)=1 \end{array} \right]$$

$$= \int_0^1 (t^2-3)^2 \cdot t \cdot 2t\, dt$$

$$= \int_0^1 (t^4-6t^2+9)\cdot 2t^2 dt$$

$$= \int_0^1 (2t^6-12t^4+18t^2)\, dt$$

$$= \left[\frac{2t^7}{7} - \frac{12t^5}{5} + 6t^3 \right]_0^1$$

$$= \frac{2}{7} - \frac{12}{5} + 6$$

$6 \cdot 35$
$= 180+30 = 210$

$$= \frac{10 - 84 + 210}{35} = \frac{136}{35}.$$

Svar: $\frac{136}{35}$.

$\square$

4.

$$e^t = 1 + t + \frac{t^2}{2!} + \dots \quad \text{ger att}$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2} + \dots$$

$$= 1 + 2X + x^2 \cdot \underbrace{B_1(x)}_{\text{begr. nära } x=0}$$

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \dots$$

ger att

$$\ln(1+x^3) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \dots$$

$$= x^3 + x^6 \cdot \underbrace{B_2(x)}_{\text{begr. nära } x=0}$$

$$\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots$$

ger att

$$\cos(3x) = 1 - \frac{9x^2}{2!} + \frac{81x^4}{4!} - \dots$$

$$= 1 - \frac{9x^2}{2} + x^4 \cdot \underbrace{B_3(x)}_{\text{begr. nära } x=0}$$

Detta ger att

$$\lim_{x \to 0} \frac{(e^{2x}-1)\cdot \ln(1+x^3)}{(1-\cos(3x))^2}$$

$$= \lim_{x \to 0} \frac{(\cancel{1}+2x+x^2\cdot B_1(x)-\cancel{1})\cdot(x^3+x^6\cdot B_2(x))}{\left(\cancel{1}-(\cancel{1}-\frac{9x^2}{2}+x^4\cdot B_3(x))\right)^2}$$

$$= \lim_{x \to 0} \frac{2x^4+2x^7 B_2(x)+x^5\cdot B_1(x)+x^8 B_1(x) B_2(x)}{\frac{81x^4}{4}-9x^6\cdot B_3(x)+x^8\cdot B_3(x)^2}$$

$$= \lim_{x \to 0} \frac{2+\overbrace{2x^3\cdot B_2(x)}^{\to 0}+\overbrace{x B_1(x)}^{\to 0}+\overbrace{x^4 B_1(x) B_2(x)}^{\to 0}}{\frac{81}{4}-\underbrace{9x^2 B_3(x)}_{\to 0}+\underbrace{x^4\cdot B_3(x)^2}_{\to 0}}$$

$$= \frac{2}{\frac{81}{4}} = \frac{8}{81}.$$

Svar: $\frac{8}{81}$. $\square$

5. $y = y_h + y_p$ där

y_h allm. lösn. till $y'' + y = 0$,

$y_p = y_{p_1} + y_{p_2}$,

y_{p_1} part. lösn. till $y'' + y = x^2$

y_{p_2} — " — $y'' + y = 2\cos(x)$.

$\boxed{y_h}$ Kar. ekv. $r^2 + 1 = 0$

$\Leftrightarrow r = \pm i$

så att

$y_h = A \cdot \cos(x) + B\sin(x)$.

$\boxed{y_{p_1}}$ Ansätt $y_{p_1} = ax^2 + bx + b$.

Då fås $y_{p_1}' = 2ax + b$

$y_p'' = 2a$.

Insättn. i $y'' + y = x^2$:

$2a + ax^2 + bx + C = x^2$

$$a\cdot x^2 + bx + 2a + c = 1\cdot x^2 + 0\cdot x + 0$$

$$\left.\begin{array}{l} a = 1 \\ b = 0 \\ 2a+c=0 \end{array}\right\} \quad a=1 \text{ i } 2a+c=0: \quad c=-2.$$

$$y_{p_1} = \underline{x^2 - 2}.$$

$\boxed{y_{p_2}}$ $\quad y_{p_2} = \mathrm{Re}(u_p)$ där u_p part. lösn.

till $\quad u'' + u = 2\cdot e^{ix}$.

Vi ansätter

$u_p = z\cdot e^{ix}$ så att

$u_p' = (z' + i\cdot z)e^{ix}$ och

$u_p'' = (z'' + 2iz' - z)\cdot e^{ix}$.

Insättn. i $\quad u'' + u = 2\cdot e^{ix}$:

$$(z'' + 2iz' - z)\cdot e^{ix} + z\cdot e^{ix} = 2\cdot e^{ix}$$

$$(z'' + 2iz')\cdot \cancel{e^{ix}} = 2\cdot \cancel{e^{ix}}$$

$$z'' + 2iz' = 2$$

$z' = \frac{2}{2i} = -i$ duger så

$z_p' = -ix$ duger.

Vi får då

$$u_p = -ix \cdot e^{ix}$$

$$= -ix \cdot (\cos(x) + i \cdot \sin(x))$$

$$= x \cdot \sin(x) - i \cdot x \cdot \cos(x).$$

$$y_{p_2} = Re(u_p) = \underline{x \cdot \sin(x)}.$$

Svar: $y = y_h + y_p$

$$= A \cdot \cos(x) + B\sin(x) + x^2 - 1 + x \cdot \sin(x).$$

$\square$

6. Notera att vi direkt kan bestämma till som explicit funktion av x.

$$\int_1^x \frac{1}{1+3t^2}\,dt = \int_1^x \frac{1}{1+(\sqrt{3}t)^2}\,dt$$

$$= \begin{bmatrix} u=\sqrt{3}t\ , \quad t=\frac{1}{\sqrt{3}}u \\ dt=\frac{1}{\sqrt{3}}\,du \\ u(1)=\sqrt{3}\ , \ u(x)=\sqrt{3}x \end{bmatrix} = \int_{\sqrt{3}}^{\sqrt{3}x} \frac{1}{1+u^2}\cdot\frac{1}{\sqrt{3}}\,du$$

$$= \left[\frac{1}{\sqrt{3}}\arctan(u)\right]_{\sqrt{3}}^{\sqrt{3}x}$$

$$= \frac{1}{\sqrt{3}}\Big(\arctan(\sqrt{3}x) - \arctan(\sqrt{3})\Big)$$

$$= \frac{1}{\sqrt{3}}\Big(\arctan(\sqrt{3}x) - \frac{\pi}{3}\Big).$$

Detta ger att

$$x \cdot f(x) = \int_1^x \frac{1}{1+3t^2}\,dt$$

$$= \frac{1}{\sqrt{3}}\left(\arctan(\sqrt{3}x) - \frac{\pi}{3}\right)$$

Svar: $f(x) = \dfrac{1}{x\cdot\sqrt{3}}\left(\arctan(\sqrt{3}x) - \dfrac{\pi}{3}\right)$.

Anmärkning: En alternativ metod är att derivera båda led och lösa DE

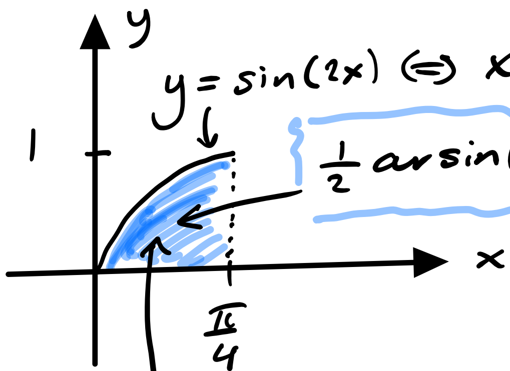
$$1\cdot f(x) + x\cdot f'(x) = \frac{1}{1+3x^2}$$

med villkoret $f(1)=0$ som fås genom insättn. av $x=1$ i ursprungligt samband.

7. Det kan vara till hjälp att först rita området.

$\frac{1}{2}\arcsin(y) = x$ kan skrivas om som $y = \sin(2x)$.

$y = \sin(2x) \iff x = \frac{1}{2}\arcsin(y)$



$\frac{1}{2}\arcsin(y) \leq x \leq \frac{\pi}{4}\ ,\ 0 \leq y \leq 1$ D

$0 \leq y \leq \sin(2x)\ ,\ 0 \leq x \leq \frac{\pi}{4}$ D

Arean av D ges av

$$A(D) = \int_0^{\pi/4} \sin(2x)\,dx = \left[-\frac{1}{2}\cos(2x)\right]_0^{\pi/4}$$

$$= -\frac{1}{2}\underbrace{\cos\frac{\pi}{2}}_{=0} + \frac{1}{2}\underbrace{\cos 0}_{=1} = \frac{1}{2}.$$

För x_{tp} används $dA = \sin(2x)\,dx$

för y_{tp} används $dA = \left(\frac{\pi}{4} - \frac{1}{2}\arcsin(y)\right)dy$.

$\boxed{x_{tp}}$

$$x_{tp} = \frac{1}{A(D)} \int_0^{\pi/4} x \, dA$$

$$= \frac{1}{\frac{1}{2}} \cdot \int_0^{\pi/4} x \cdot \sin(2x) dx$$

$$= 2 \cdot \left(\left[-\frac{1}{2}\cos(2x) \cdot x \right]_0^{\pi/4} + \int_0^{\pi/4} \frac{1}{2}\cos(2x) \cdot 1 \, dx \right)$$

$$= 2 \cdot \left(0 + \left[\frac{1}{4}\sin(2x) \right]_0^{\pi/4} \right)$$

$$= 2 \cdot \left(\frac{1}{4} - 0 \right) = \frac{1}{2} .$$

Notera att $\frac{\pi}{4} \approx \frac{3}{4}$ så $x_{tp} = \frac{1}{2}$ ett rimligt resultat.

$\boxed{y_{tp}}$

$$y_{tp} = \frac{1}{A(D)} \int_0^1 y\,dA = \frac{1}{\frac{1}{2}} \int_0^1 y\left(\frac{\pi}{4} - \frac{1}{2}\arcsin y\right)dy$$

$$= 2\cdot\left(\frac{\pi}{4}\underbrace{\int_0^1 y\,dy}_{I_1} - \frac{1}{2}\underbrace{\int_0^1 y\cdot\arcsin(y)\,dy}_{I_2}\right)$$

$$I_1 = \int_0^1 y\,dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}.$$

$$I_2 = \int_0^1 y\cdot\arcsin y\,dy = \begin{bmatrix} t = \arcsin y \\ \sin t = y \\ dy = \cos t\,dt \\ t(0)=0,\ t(1) = \frac{\pi}{2} \end{bmatrix}$$

$$= \int_0^{\pi/2} \sin t\cdot t\cdot\cos t\,dt = \left[\sin 2t = 2\sin t\cos t\right]$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin(2t)\cdot t\,dt$$

$$= \frac{1}{2}\left(\left[-\frac{1}{2}\cos(2t)\cdot t\right]_0^{\pi/2} + \int_0^{\pi/2}\frac{1}{2}\cos(2t)\cdot 1\cdot dt\right)$$

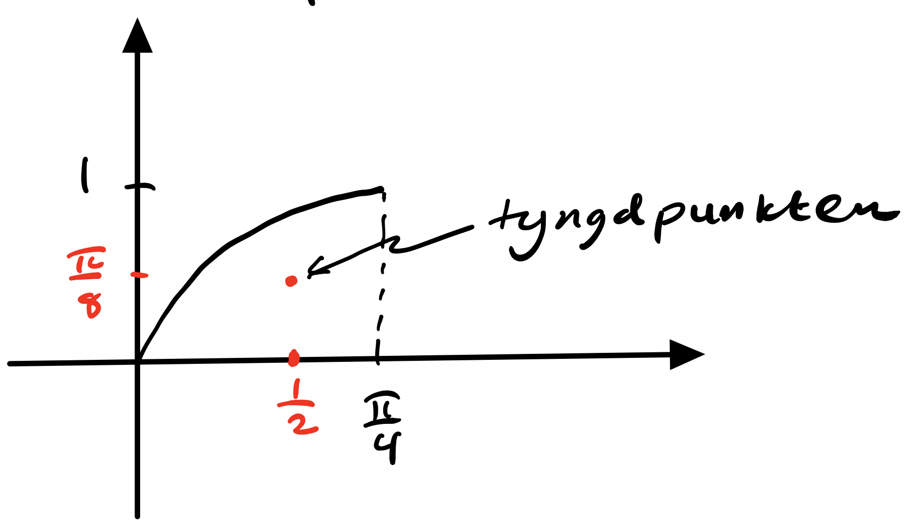
$$= \frac{1}{2}\cdot\left(-\frac{1}{2}\cdot(-1)\cdot\frac{\pi}{2} - 0 + \left[\frac{1}{4}\sin(2t)\right]_0^{\pi/2}\right)$$

$$= \frac{1}{2}\cdot\left(\frac{\pi}{4} + 0 - 0\right) = \underline{\underline{\frac{\pi}{8}}}.$$

Notera att $\frac{\pi}{8} \approx \frac{3}{8}$ som är ett rimligt resultat.

Svar: $x_{tp} = \frac{1}{2}$,

$y_{tp} = \frac{\pi}{8}$.



$\square$

8. Vi söker först $\omega(\phi)$ så att

$$\frac{d\omega}{d\phi} = -\omega \cdot \frac{\ln 2}{8\pi}$$

där $\omega(0) = \omega_0$.

$$\omega' + \frac{\ln 2}{8\pi} \cdot \omega = 0 \qquad \Big\{ \quad IF \ e^{\frac{\ln 2}{8\pi} \cdot \phi}$$

$$\frac{d}{d\omega}\left(\omega \cdot e^{\frac{\ln 2}{8\pi}\phi}\right) = 0 \cdot e^{\frac{\ln 2}{8\pi}\phi} = 0$$

$$\omega \cdot e^{\frac{\ln 2}{8\pi}\phi} = C$$

$$\omega(\phi) = C \cdot e^{-\frac{\ln 2}{8\pi}\phi}.$$

$\omega(0) = \omega_0$ gör att $C = \omega_0$.

$$\omega(\phi) = \omega_0 \cdot e^{-\frac{\ln 2}{8\pi}\phi}.$$

Bestäm nu ϕ så att $\omega(\phi) = \frac{\omega_0}{2}$:

$$\frac{\omega_0}{2} = \omega_0 \cdot e^{-\frac{\ln 2}{8\pi}\phi} \iff e^{-\frac{\ln 2}{8\pi}\phi} = \frac{1}{2}$$

$$\iff -\frac{\ln 2}{8\pi}\phi = \ln \frac{1}{2}$$

$$\Leftrightarrow$$

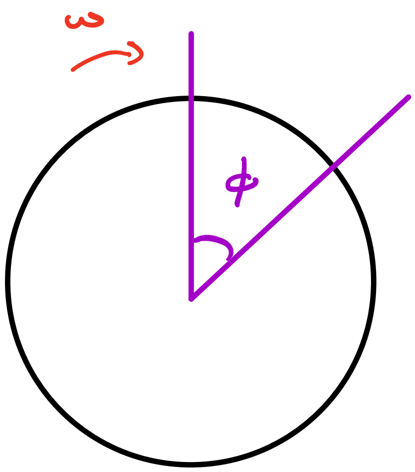
$$\cancel{-\ln 2} \cdot \frac{1}{8\pi} \cdot \phi = \cancel{-\ln 2}$$

$$\Leftrightarrow$$

$$\phi = 8\pi = 4 \cdot 2\pi$$

antal varv

Svar: 4 varv.



9. Primitiva till $f(x) = ?$

$$\int \frac{1}{\sqrt{x}(x-1)}\,dx = \left[\begin{array}{l} t = \sqrt{x}\ , \quad x = t^2 \\ dx = 2t\,dt \end{array}\right]$$

$$= \int \frac{1}{\cancel{t}(t^2-1)} \cdot 2\cancel{t}\,dt = \int \frac{2}{t^2-1}\,dt$$

$$= \left[\text{part. bråksuppd. } \frac{2}{t^2-1} = \frac{1}{t-1} - \frac{1}{t+1} \right]$$

$$= \int \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt = \ln|t-1| - \ln|t+1| + C$$

$$= \ln\left|\frac{t-1}{t+1}\right| + C = \left[t = \sqrt{x} \right]$$

$$= \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C.$$

Visa att $f(x)$ avtagande då $x>1$.

$$f'(x) = -\frac{1}{(\sqrt{x}(x-1))^2}\cdot\left(\frac{1}{2\sqrt{x}}\cdot(x-1)+\sqrt{x}\cdot 1\right)$$

$$= -\frac{1}{x(x-1)^2}\cdot\frac{x-1+2x}{2\sqrt{x}}$$

$$\underbrace{= -\frac{1}{x(x-1)^2}\cdot\frac{3x-1}{2\sqrt{x}}}_{<0 \text{ om } x>\frac{1}{3}} \quad >0 \text{ om } x>\frac{1}{3}$$

Vi får speciellt att $f'(x)<0$ om $x>1$. Detta är till hjälp då $\sum_{k=2}^{\infty} f(k)$ skall uppskattas eftersom $f(x)$ positiv och avtagande då $x\geq 2$:

$$\int_2^n f(x)\,dx + f(n) \le \sum_{k=2}^{n} f(k) \le \int_2^n f(x)\,dx + f(2)$$

för alla heltal $n \geq 3$.

$$f(2) = \frac{1}{\sqrt{2}\cdot 1} = \frac{1}{\sqrt{2}} \quad \text{och}$$

$$f(n) = \frac{1}{\sqrt{n}(n-1)} \to 0 \text{ då } n \to \infty.$$

Vi får då

$$\int_2^\infty f(x)\,dx \le \sum_{k=2}^{\infty} f(k) \le \int_2^\infty f(x)\,dx + \frac{1}{\sqrt{2}}.$$

Återstår att bestämma

$$\int_2^\infty f(x)\,dx.$$

Notera att

$$\int_2^\infty f(x)\,dx = \lim_{R\to\infty} \int_2^R f(x)\,dx$$

$$= \lim_{R\to\infty} \left[\ln\left| \frac{\sqrt{x}-1}{\sqrt{x}+1} \right| \right]_2^R$$

$$= \lim_{R\to\infty} \left(\ln \frac{\sqrt{R}-1}{\sqrt{R}+1} - \ln \frac{\sqrt{2}-1}{\sqrt{2}+1} \right)$$

$$\overset{(*)}{=} 0 + \ln \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$\overset{(**)}{=} 2 \cdot \ln(\sqrt{2}+1).$$

(*) $\dfrac{\sqrt{R}-1}{\sqrt{R}+1} = \dfrac{1-\frac{1}{\sqrt{R}}}{1+\frac{1}{\sqrt{R}}} \to \dfrac{1-0}{1+0} = 1$

så att $\ln \dfrac{\sqrt{R}-1}{\sqrt{R}+1} \to \ln 1 = 0$.

(**) $\dfrac{\sqrt{2}+1}{\sqrt{2}-1} = \dfrac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)(\sqrt{2}+1)} = \dfrac{(\sqrt{2}+1)^2}{1}.$

Detta ger att

$$2\ln(\sqrt{2}+1) \leq \sum_{k=2}^{\infty} \frac{1}{\sqrt{k}(k-1)} \leq 2\ln(\sqrt{2}+1) + \frac{1}{\sqrt{2}}.$$

$\square$