TENTAMENSKOD:________________

INSTITUTIONEN FÖR
NATURVETENSKAP OCH TEKNIK

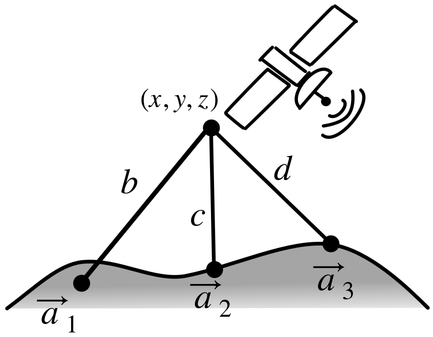
Kursens namn/kurskod	**Numeriska Metoder för civilingenjörer** **DT508G**
Examinationsmomentets namn/provkod	**Teori, 2,5 högskolepoäng (Provkod: A003)**
Datum	2021-10-27
Tid	Kl. 14:15 – 19:15

Tillåtna hjälpmedel	Skrivmateriel, miniräknare med raderat minne, kursscript, kursböker.
Instruktion	Läs igenom alla frågor noga. Börja varje fråga på ett nytt svarsblad. Skriv bara på ena sidan av svarsbladet. Skriv tentamenskoden på varje svarsblad. Skriv läsligt! Det räcker att ange tre decimaler. Glöm inte att scanna lösningar och ladda up det till WISEflow
Viktigt att tänka på	Motivera väl, redovisa tydligt alla väsentliga steg, rita tydliga figurer och svara med rätt enhet. Lämna in i uppgiftsordning.
Ansvarig/-a lärare (ev. telefonnummer)	Danny Thonig (Mobil: 0727010037)
Totalt antal poäng	40
Betyg (ev. ECTS)	Skrivningens maxpoäng är 40. För betyg 3/4/5 räcker det med samt 20/28/36 poäng totalt. Detaljerna framgår av separat dokument publicerat på Blackboard.
Tentamensresultat	Resultatet meddelas på Studentforum inom 15 arbetsdagar efter tentadagen.
Övrigt	Lärare är inte på plats. Varsågod och ringa om du har frågar.

Lycka till!

Task 1 — A satellite problem

ESA want to correct the position of one of their satellites and, thus, need first to know the position of this satellite. It can be calculated based on the distances b, c, and d from the satellite to three markers on earth (see Figure). We want to use Broyden's method 2 to find the position (x, y, z) of the satellite.



Use $\vec{a}_1 = (0,0,0), \vec{a}_2 = (3,7,1),\ \vec{a}_3 = (5,1,3)$, $b = 3$, $c = 4$, and $d = 5$.

It should be noticed that the basic problem of the satellite and the drone are the same. You were asked to apply different methods (Broyden 1 and Broyden 2). Detailed numerical results can be extracted from the attached Matlab file.

a) Find the function $\vec{F}$ of the satellite problem in the form $\vec{F}(\vec{x}) = 0$ and calculate $\vec{F}(\vec{x}_0)$. The initial guess is $\vec{x}_0 = (1,1,1)$. **Tip:** Remember how the distance between two points is defined.

Solution suggestion

The distance d between two points $\vec{a}$ and $\vec{b}$ is defined

$$\sqrt{(a_x - b_x)^2 + (a_y - b_y)^2 + (a_z - b_z)^2} = d$$

And, thus, we construct from three point $\vec{a}_1, \vec{a}_2$, and $\vec{a}_3$ to one common point (x, y, z) a multidimensional function $F : \mathbb{R}^3 \to \mathbb{R}^3$

$$\vec{F}(\vec{x}) = \begin{pmatrix} \sqrt{(x - a_{1,x})^2 + (y - a_{1,y})^2 + (z - a_{1,z})^2} - b \\ \sqrt{(x - a_{2,x})^2 + (y - a_{2,y})^2 + (z - a_{2,z})^2} - c \\ \sqrt{(x - a_{3,x})^2 + (y - a_{3,y})^2 + (z - a_{3,z})^2} - d \end{pmatrix} = 0$$

For exam version v1, v4, v5, $\vec{F}(\vec{x}_0) = (-1.2679, 2.3246, -0.5279)$

For exam version v2, v3, v6, $\vec{F}(\vec{x}_0) = (-4.7639, 2.6023, 2.3246)$

b) For the initial matrix $\boldsymbol{B}_0$ in the Broyden method, we need to calculate the Jacobian at the point $\vec{x}_0$. Calculate $D\vec{F}(x_0)$.

Solution suggestion

The Jacobian is defined as

$$D\overrightarrow{F}(\overrightarrow{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix}$$ and, thus,

$$D\overrightarrow{F}(\overrightarrow{x}) = \begin{pmatrix} \frac{x-a_{1,x}}{\sqrt{(x-a_{1,x})^2+(y-a_{1,y})^2+(z-a_{1,z})^2}} & \frac{y-a_{1,y}}{\sqrt{(x-a_{1,x})^2+(y-a_{1,y})^2+(z-a_{1,z})^2}} & \frac{z-a_{1,z}}{\sqrt{(x-a_{1,x})^2+(y-a_{1,y})^2+(z-a_{1,z})^2}} \\ \frac{x-a_{2,x}}{\sqrt{(x-a_{2,x})^2+(y-a_{2,y})^2+(z-a_{2,z})^2}} & \frac{y-a_{2,y}}{\sqrt{(x-a_{2,x})^2+(y-a_{2,y})^2+(z-a_{2,z})^2}} & \frac{z-a_{2,z}}{\sqrt{(x-a_{2,x})^2+(y-a_{2,y})^2+(z-a_{2,z})^2}} \\ \frac{x-a_{3,x}}{\sqrt{(x-a_{3,x})^2+(y-a_{3,y})^2+(z-a_{3,z})^2}} & \frac{y-a_{3,y}}{\sqrt{(x-a_{3,x})^2+(y-a_{3,y})^2+(z-a_{3,z})^2}} & \frac{z-a_{3,z}}{\sqrt{(x-a_{3,x})^2+(y-a_{3,y})^2+(z-a_{3,z})^2}} \end{pmatrix}$$

. From here, we can calculate $D\overrightarrow{F}(\overrightarrow{x}_0)$.

For exam version v1, v4, v5: $D\overrightarrow{F}(\overrightarrow{x}_0) = \begin{pmatrix} 0.5774 & 0.5774 & 0.5774 \\ -0.3162 & -0.9487 & 0 \\ -0.8944 & 0 & -0.4472 \end{pmatrix}$

For exam version v2, v3, v6: $D\overrightarrow{F}(\overrightarrow{x}_0) = \begin{pmatrix} 0 & 0.8944 & 0.4472 \\ -0.4650 & -0.8137 & -0.3487 \\ -0.9487 & -0.3162 & 0 \end{pmatrix}$

c) Perform LU factorization on the matrix $D\overrightarrow{F}(x_0)$ using Doolittle's method and do matrix inversion based on this LU factorization.

Solution suggestion

Doolittle's method

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{pmatrix}$$

Crouts method

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} l_{11} & u_{12}l_{11} & u_{13}l_{11} \\ l_{21} & u_{12}l_{21}+l_{22} & u_{13}l_{21}+u_{23}l_{22} \\ l_{31} & u_{12}l_{31}+l_{32} & u_{13}l_{31}+u_{23}l_{32}+l_{33} \end{pmatrix}$$

Cholesky

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} u_{11} & 0 & 0 \\ l_{21} & u_{22} & 0 \\ l_{31} & l_{32} & u_{33} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} u_{11}^2 & u_{12}u_{11} & u_{13}u_{11} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22}^2 & l_{21}u_{13}+u_{23}u_{22} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33}^2 \end{pmatrix}$$

The LU factorisation with the respective method of the respective matrix from b) can be obtained from solving the above equations. Note: The Crouts method in the given system was not possible to solve, but a row exchange in $D\vec{F}(x_0)$ solved the problem; so you were asked to solve a $PA = LU$. If this step was not obvious, solving the problem with another method was also ok.

For the inverse, we apply $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$ or rewritten $\boldsymbol{A}\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right) = \left(\vec{b}_1, \vec{b}_2, \vec{b}_3\right) = \left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right)$, and, thus, we have to solve three linear systems of equations. This become simple todo, when using the LU factorization and $\boldsymbol{U}\vec{x}_i = \vec{y}_i$ and $\boldsymbol{L}\vec{y}_i = \vec{b}_i$.

E.g. the solution for v1 in Doolittle method is

$$\boldsymbol{L} = \begin{pmatrix} 1.0000 & 0 & 0 \\ -0.5477 & 1.0000 & 0 \\ -1.5492 & -1.4142 & 1.0000 \end{pmatrix} \text{ and } \boldsymbol{U} = \begin{pmatrix} 0.5774 & 0.5774 & 0.5774 \\ 0 & -0.6325 & 0.3162 \\ 0 & 0 & 0.8944 \end{pmatrix}$$

And the inverse is

$$\boldsymbol{B}_0 = \begin{pmatrix} -1.2990 & -0.7906 & -1.6771 \\ 0.4330 & -0.7906 & 0.5590 \\ 2.5981 & 1.5811 & 1.1180 \end{pmatrix}$$

In other versions of the exam you were asked to solve $\boldsymbol{A}_0\vec{s} = D\vec{F}(\vec{x}_0)\vec{s} = \vec{F}(\vec{x}_0)$, which can be done accordingly $\boldsymbol{U}\vec{s} = \vec{y}$ and $\boldsymbol{L}\vec{y} = \vec{F}(\vec{x}_0)$.

d) Determine the condition of the matrix on the infinite norm $||\cdot||_\infty$. Is it a good conditioned problem?

Solution suggestion

This task was present only in versions were Broyden 2 was applied.

The condition is $cond = ||\boldsymbol{B}_0||_\infty ||D\vec{F}(\vec{x}_0)||_\infty$ which is

v1, v4, $cond = 9.1751$

v2, v3, $cond = 187.0372$

Both setups were well conditioned.

Note, the initial matrix $\boldsymbol{B}_0$ is $\begin{pmatrix} -1.2990 & -0.7906 & -1.6771 \\ 0.4330 & -0.7906 & 0.5590 \\ 2.5981 & 1.5811 & 1.1180 \end{pmatrix}$.

e) Perform two steps of Broyden's method 2 with the above determined initial guesses.

Solution suggestion

Broyden method 1

$\vec{x}_0$, $\boldsymbol{A}_0$, and $\boldsymbol{A}_0^{-1}$ are determined before in task a-d.

$\vec{x}_{k+1} = \vec{x}_k - \boldsymbol{A}_k^{-1}\vec{F}(\vec{x}_k)$

$\overrightarrow{\delta}_{k+1} = \overrightarrow{x}_{k+1} - \overrightarrow{x}_k$
$\overrightarrow{\Delta}_{k+1} = \overrightarrow{F}(\overrightarrow{x}_{k+1}) - \overrightarrow{F}(\overrightarrow{x}_k)$
And

$$\boldsymbol{A}_{k+1} = \boldsymbol{A}_k + \frac{(\overrightarrow{\Delta}_{k+1} - \boldsymbol{A}_k\overrightarrow{\delta}_{k+1})\overrightarrow{\delta}^T_{k+1}}{\overrightarrow{\delta}^T_{k+1}\overrightarrow{\delta}_{k+1}}$$

Broyden method 2
$\overrightarrow{x}_0$ and $\boldsymbol{B}_0$ are determined before in task a-d.
$\overrightarrow{x}_{k+1} = \overrightarrow{x}_k - \boldsymbol{B}_k\overrightarrow{F}(\overrightarrow{x}_k)$
$\overrightarrow{\delta}_{k+1} = \overrightarrow{x}_{k+1} - \overrightarrow{x}_k$
$\overrightarrow{\Delta}_{k+1} = \overrightarrow{F}(\overrightarrow{x}_{k+1}) - \overrightarrow{F}(\overrightarrow{x}_k)$
And

$$\boldsymbol{B}_{k+1} = \boldsymbol{B}_k + \frac{(\overrightarrow{\delta}_{k+1} - \boldsymbol{B}_k\overrightarrow{\Delta}_{k+1})\overrightarrow{\delta}^T_{k+1}\boldsymbol{B}_k}{\overrightarrow{\delta}^T_{k+1}\boldsymbol{B}_k\overrightarrow{\Delta}_{k+1}}$$

Details on each step will be not shown. Please run the attached Matlab file for more informations.

f) What is the order of convergence and the convergence rate of Broyden's method in general? How does it compare to Newton's method.

Solution suggestion

The order of convergence in Broydens method is super linearly ($\alpha = 1.62$) and the rate is

$$S = \left|\frac{f''}{2f'}\right|^{\alpha-1}$$

Compared to Newton (order is $\alpha = 2$, quadratic) and the rate is

$$S = \left|\frac{f''}{2f'}\right|$$

Thus, Broyden is expected to converge slower.

g) Do you know a different method than Broyden's or Newton method to solve this mathematical problem? Describe one other method in terms of algorithm, order of convergence, and rate of convergence.

Solution suggestion

See text book, e.g. Bisection method, Fixed Point method, Secant Method, but also Jacobi method or Gauss-Seidel.

Task 2 — Common problem of mathematical physics

A common problem of mathematical physics is that of solving the Fredholm-Volterra integral equation:

$$\phi(x) = f(x) + \lambda\int_a^b K(x,t)\phi(t)\mathrm{d}t + \mu\int_a^x L(x,t)\phi(t)\mathrm{d}t$$

where the functions $f(x)$, $L(x,t)$, and $K(x,t)$ are given and the goal is to obtain $\phi(x)$. If $\mu = 0$ then the equation is called linear Fredholm integral equation, but if $\lambda = 0$ then the equation is called linear Volterra integral equation.

a) Develop based on your knowledge about numerical integration an actual numerical formalism that solves the Fredholm-Volterra integral using composite, $n = 2$ Trapezoidal method. Analyse the error of the method.

Solution suggestion

There were two different tasks, one with $n = 2$ Trapezoid and one with Simpson (with restriction on some integral)

Both have in common to split the interval $[a, b]$ into three points $t_0 = x_0 = a$, $t_1 = x_1 = \dfrac{a+b}{2}$, and $t_2 = x_2 = b$. Thus $h = \dfrac{b-a}{2}$. So we solve the problem at the same points at which we approach the integral.

Thus, for the trapezoid

$$\phi(x_0) = f(x_0) + \lambda\frac{h}{2}\left(K(x_0,t_0)\phi(t_0) + 2K(x_0,t_1)\phi(t_1) + K(x_0,t_2)\phi(t_2)\right)$$

$$\phi(x_1) = f(x_1) + \lambda\frac{h}{2}\left(K(x_1,t_0)\phi(t_0) + 2K(x_1,t_1)\phi(t_1) + K(x_1,t_2)\phi(t_2)\right) + \mu\frac{h}{2}\left(L(x_1,t_0)\phi(t_0) + L(x_1,t_1)\phi(t_1)\right)$$

$$\phi(x_2) = f(x_2) + \lambda\frac{h}{2}\left(K(x_2,t_0)\phi(t_0) + 2K(x_2,t_1)\phi(t_1) + K(x_2,t_2)\phi(t_2)\right) + \mu\frac{h}{2}\left(L(x_2,t_0)\phi(t_0) + 2L(x_2,t_1)\phi(t_1) + L(x_2,t_2)\phi(t_2)\right)$$

And this gives a system of linear equations

$$-\begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix} = \begin{pmatrix} \lambda\frac{h}{2}K(x_0,x_0) - 1 & \lambda h K(x_0,x_1) & \lambda\frac{h}{2}K(x_0,x_2) \\ \lambda\frac{h}{2}K(x_1,x_0) + \mu\frac{h}{2}L(x_1,x_0) & \lambda h K(x_1,x_1) + \mu\frac{h}{2}L(x_1,x_1) - 1 & \lambda\frac{h}{2}K(x_1,x_2) \\ \lambda\frac{h}{2}K(x_2,x_0) + \mu\frac{h}{2}L(x_2,x_0) & \lambda h K(x_2,x_1) + \mu h L(x_2,x_1) & \lambda\frac{h}{2}K(x_2,x_2) + \mu\frac{h}{2}L(x_2,x_2) - 1 \end{pmatrix}\begin{pmatrix} \phi(x_0) \\ \phi(x_1) \\ \phi(x_2) \end{pmatrix}$$

Thus, for the Simpson

$$\phi(x_0) = f(x_0) + \lambda\frac{h}{3}\left(K(x_0,t_0)\phi(t_0) + 4K(x_0,t_1)\phi(t_1) + K(x_0,t_2)\phi(t_2)\right)$$

$$\phi(x_1) = f(x_1) + \lambda\frac{h}{3}\left(K(x_1,t_0)\phi(t_0) + 4K(x_1,t_1)\phi(t_1) + K(x_1,t_2)\phi(t_2)\right) + \mu\frac{h}{2}\left(L(x_1,t_0)\phi(t_0) + L(x_1,t_1)\phi(t_1)\right)$$

$$\phi(x_2) = f(x_2) + \lambda\frac{h}{3}\left(K(x_2,t_0)\phi(t_0) + 4K(x_2,t_1)\phi(t_1) + K(x_2,t_2)\phi(t_2)\right) + \mu\frac{h}{3}\left(L(x_2,t_0)\phi(t_0) + 4L(x_2,t_1)\phi(t_1) + L(x_2,t_2)\phi(t_2)\right)$$

In the second equation we used for the third term the trapezoid method, as suggested by the task. Thus,

$$-\begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix} = \begin{pmatrix} \lambda\frac{h}{3}K(x_0,x_0) - 1 & \lambda\frac{4h}{3}K(x_0,x_1) & \lambda\frac{h}{3}K(x_0,x_2) \\ \lambda\frac{h}{3}K(x_1,x_0) + \mu\frac{h}{2}L(x_1,x_0) & \lambda\frac{4h}{3}K(x_1,x_1) + \mu\frac{h}{2}L(x_1,x_1) - 1 & \lambda\frac{h}{3}K(x_1,x_2) \\ \lambda\frac{h}{3}K(x_2,x_0) + \mu\frac{h}{3}L(x_2,x_0) & \lambda\frac{4h}{3}K(x_2,x_1) + \mu\frac{4h}{3}L(x_2,x_1) & \lambda\frac{h}{3}K(x_2,x_2) + \mu\frac{h}{3}L(x_2,x_2) - 1 \end{pmatrix}\begin{pmatrix} \phi(x_0) \\ \phi(x_1) \\ \phi(x_2) \end{pmatrix}$$

b) Solve the following equation by your proposed method using $n = 2$ panels of width h, where $h = \dfrac{b-a}{n}$:

$$\phi(x) = \pi x^2 + \int_0^{\pi} 3(0.5\sin(3x) - tx^2)\phi(t)\mathrm{d}t.$$

Tip: you obtain the solution by solving a system of equations. You can apply Gauss elimination method here.

Solution suggestion

The respective function can be included in the above matrix by the reader. See Matlab file for more details.

c) Draw your result $\phi(x)$ from b) and compare to the exact solution $\phi(x) = -0.5\sin(3x)$.

Solution suggestion

The respective function and solution of b) can be done by the reader. See Matlab file for more details.