TENTAMENSKOD:_______________

INSTITUTIONEN FÖR
NATURVETENSKAP OCH TEKNIK

Kursens namn/ kurskod	**Numeriska Metoder för civilingenjörer** **DT508G**
Examinationsmomentets namn/provkod	**Teori, 3,5 högskolepoäng (Provkod: A001)**
Datum	2019-11-01
Tid	Kl. 14:15 – 19:15

Tillåtna hjälpmedel	Skrivmateriel, formel blad och miniräknare med raderat minne.
Instruktion	Läs igenom alla frågor noga. Börja varje fråga på ett nytt svarsblad. Skriv bara på ena sidan av svarsbladet. Skriv tentamenskoden på varje svarsblad. Skriv läsligt!
Viktigt att tänka på	Motivera väl, redovisa alla väsentliga steg, rita tydliga figurer och svara med rätt enhet. Lämna in i uppgiftsordning.
Ansvarig/-a lärare (ev. telefonnummer)	Danny Thonig (Mobil: 0727010037)
Totalt antal poäng	60
Betyg (ev. ECTS)	Skrivningens maxpoäng är 60. För betyg 3/4/5 räcker det med samt 30/40/50 poäng totalt. Detaljerna framgår av separat dokument publicerat på Blackboard.
Tentamensresultat	Resultatet meddelas på Studentforum inom 15 arbetsdagar efter tentadagen.
Övrigt	

Lycka till!

1. The two numbers ***(i)*** x=2.9 and ***(ii)*** x=0.7 should be represented into a single precession language understandable for the computer. [10p]
   a) Find the binary $(x)_2$ and floating-point representation $fl(x)$ of the number (i) and (ii). [3p]
   b) Calculate the error (rounding and chopping error) of the floating-point representation of the number (i) and (ii). Calculate $fl(2.9-0.7)$ and draw a conclusion about arithmetic performed on a computer. [2p]
   c) Find the machine representation of (i) and (ii). What are the machine representation of $+\infty, -\infty$, and NaN? [2p]
   d) Calculate the rounding error when calculating the product $\Pi_{i=1}^{n} x_k$. [3p]

Solution:

a)

[1p] for the solution path

[1p] for binary solution $(2.9)_2 = \ldots 10.1\overline{1100}$ and $(0.7)_2 = \ldots 0.1\overline{0110}$.

[1p] for floating point solution $fl(2.9) = 1.01\overline{1100} \times 2^1$ and $fl(0.7) = 1.\overline{0110} \times 2^{-1}$.

b)

[1p] for calculating the errors.

Express the floating point number in single precession.

$fl(2.9) = 1.01110011001100110011001 \,|\, \overline{1001} \times 2^1 \approx 1.01110011001100110011010 \times 2^1$

Let error$_c$ and error$_r$ be the copping and the rounding error, respectively. Thus, error$_c = .\overline{1001} \times 2^1 \times 2^{-23}$ and error$_r = 2^1 \times 2^{-23}$. Note, already here you would have gotten [1p].

$fl(0.7) = 1.01100110011001100110011 \,|\, \overline{0011} \times 2^{-1} \approx 1.01100110011001100110011 \times 2^{-1}$

Let error$_c$ and error$_r$ be the copping and the rounding error, respectively. Thus, error$_c = .\overline{0011} \times 2^{-2} \times 2^{-23}$ and error$_r = 0$. Note, already here you would have gotten [1p].

[1p] for solving $fl(2.9-0.7)$.

Either we use the following

$fl(x+y) = fl(fl(x)+fl(y)) = fl(x(1+err_x) + y(1+err_y)) = (x(1+err_x) + y(1+err_y))(1+err_{xy})$

or

Make the two exponents equal

$fl(2.9) = 1.011\overline{1001} \times 2^1$

$fl(0.7) = 0.010\overline{1100} \times 2^1$

And subtract the mantises. Note at the fifth bit after the comma, there is a carry bit shifting a 1 to the left bit (nr. 4). Thus,

$fl(2.9-0.7) = 1.0\overline{0011} \times 2^1$

The error comes from

$1.00011001100110011001100|\overline{1100} \times 2^1$

And is

error$_c = .\overline{1100} \times 2^1 \times 2^{-23}$ and 1×2^{-23}.

Conclusion: relative error is smaller than $\frac{\varepsilon_{Mach}}{2}$.

c)

[1p] for machine representation.

Let the machine representation be like $se_1e_2\ldots e_8b_1b_2\ldots b_{11}$ in single precision.

Shift of the exponent $(2^{n^{exp}-1} - 1) + p$, where $n^{exp} = 8$ and, consequently, $(2^{n^{exp}-1} - 1) = 127$.

For 2.9, $p = 1$ and thus $e_1e_2\ldots e_8 = 10000000$.

$$mach(2.9) = 0|10000000|01110011001100110011010$$

For 0.7, $p = -1$ and thus $e_1e_2\ldots e_8 = 01111110$

$$mach(0.7) = 0|01111110|01100110011001100110011$$

[1p] for machine representation of $+\infty$,$-\infty$, and NaN

$$+\infty = 0\,\underbrace{11...1}_{8}\,\underbrace{00...0}_{23}$$

$$-\infty = 1\,\underbrace{11...1}_{8}\,\underbrace{00...0}_{23}$$

$$NaN = 1\,\underbrace{11...1}_{8}\,\underbrace{XX\ldots X}_{23}$$

d)

The solution can be constructed recursively. Let consider (this gives [1p])

$$fl(x_{n-1} \cdot x_n) = fl(fl(x_{n-1}) \cdot fl(x_n)) = fl\left(x_{n-1}x_n(1+\delta_{n-1})(1+\delta_n)\right) = x_{n-1}x_n(1+\delta_{n-1})(1+\delta_n)(1+\delta'_{n-1})$$

Multiplying one more

$$fl(fl(x_{n-2}) \cdot fl(x_{n-1} \cdot x_n)) = fl\left(x_{n-2}(1+\delta_{n-2})x_{n-1}x_n(1+\delta_{n-1})(1+\delta_n)(1+\delta_{n-1,n})\right) = x_{n-2}x_{n-1}x_n(1+\delta_{n-2})(1+\delta_{n-1})(1+\delta_n)(1+\delta'_{n-1})(1+\delta'_{n-2})$$

Continue the series (this gives [1p])

$$fl(\Pi_{k=1}^n x_k) = fl(fl(x_1) \cdot fl(\Pi_{k=2}^n x_k)) = \Pi_{k=1}^n x_k \left(\Pi_{k=1}^n(1+\delta_k)\right)\left(\Pi_{k=2}^n(1+\delta'_{k-1})\right)$$

Thus (this gives [1p]),

$$\frac{fl(|\Pi_{k=1}^n x_k) - \Pi_{k=1}^n x_k|}{|\Pi_{k=1}^n x_k|} = |1 - \left(\Pi_{k=1}^n(1+x_k)\right)\left(\Pi_{k=2}^n(1+\delta'_{k-1})\right)|$$

2. The population dynamics of three competing species $x_1, x_2,$ and x_3 can be described by [10p]

$$x_1'(t) = x_1(t)\left[1 - x_1(t) - \alpha x_2(t) - \beta x_3(t)\right]$$

$$x_2'(t) = x_2(t)\left[1 - x_2(t) - \beta x_1(t) - \alpha x_3(t)\right]$$

$$x_3'(t) = x_3(t)\left[1 - x_3(t) - \alpha x_1(t) - \beta x_2(t)\right],$$

where α and β are parameters measuring the influence that the species have on each other. Find a stable solution ($x_1'(t) = x_2'(t) = x_3'(t) = 0$) of the scaled populations $\boldsymbol{x} = (x_1(t), x_2(t), x_3(t))$ using fixed-point method.

a) Fix-point method is one of multiple methods to obtain a solution. Describe fixed-point method for functions, first, with one variable and, second, with multiple variables. What is a cobweb? Under which conditions converges the fixed-point method to a unique solution? What is the order of convergence? Into what classes fixed point can be categories? What is the sensitivity of the fixed-point method? [4p]

b) Consider the stable solution of the three population dynamics problem above. Define a fixed-point problem and check when a unique stable solution exists. How does the stability of the solution depend on α and β? [2p]

c) If $\alpha = 0.3$ and $\beta = 0.6$, estimate the two solution $\boldsymbol{x} = (x_1(t), x_2(t), x_3(t))$ in the set described by $0.5 \le x_1(t) \le 1$, $0.0 \le x_2(t) \le 1$, and $0.5 \le x_3(t) \le 1$. Here, calculate the first three steps of the iteration. What is the rate of convergence? Can at least one fixed point expressed directly in terms of α and β? [3p]

d) Describe one other method that can be applied to obtain the solution of the above problem. Indicate why this method is maybe more applicable for the problem. [1p]

Solution:

a)

See text books, Sauer, Page 30-43 as well as Burden, Page 55-65 and Page 642 - 650.
[1p] Definition fixed-point and fixed-point method
[1p] Multidimensional definition (just one sentence like "it is similar to 1d case" would have been enough)
[1p] Convergence criteria $|g'(r)| < 1$ and linear convergence.
[1p] Classes of fixed points and sensitivity.

b)

We are looking for the problem, when
$0 = x_1(t)\left[1 - x_1(t) - \alpha x_2(t) - \beta x_3(t)\right]$
$0 = x_2(t)\left[1 - x_2(t) - \beta x_1(t) - \alpha x_3(t)\right]$
$0 = x_3(t)\left[1 - x_3(t) - \alpha x_1(t) - \beta x_2(t)\right],$
Which can be rewritten in a fix-point problem, e.g. as (gives [1p])
$x_1 = \sqrt{x_1 - \alpha x_1 x_2 - \beta x_1 x_3}$
$x_2 = \sqrt{x_2 - \beta x_2 x_1 - \alpha x_2 x_3}$

$x_3 = \sqrt{x_3 - \alpha x_3 x_1 - \beta x_3 x_2}$,

Which is $\boldsymbol{x} = G(\boldsymbol{x})$, where $\boldsymbol{x} = \left(x_1, x_2, x_3\right)$. A stable solution exists, if $||D_x G||_\infty < 1$ (actually already if $\left|\dfrac{\partial G_i}{\partial x_j}\right| < 1$). Thus, we have to determine the Jacobian

$$D_{\boldsymbol{x}} G = \begin{pmatrix} \dfrac{1 - \alpha x_2 - \beta x_3}{2\sqrt{x_1 - \alpha x_1 x_2 - \beta x_1 x_3}} & \dfrac{-\alpha x_1}{2\sqrt{x_1 - \alpha x_1 x_2 - \beta x_1 x_3}} & \dfrac{-\beta x_1}{2\sqrt{x_1 - \alpha x_1 x_2 - \beta x_1 x_3}} \\ \dfrac{-\beta x_2}{2\sqrt{x_2 - \beta x_2 x_1 - \alpha x_2 x_3}} & \dfrac{1 - \beta x_1 - \alpha x_3}{2\sqrt{x_2 - \beta x_2 x_1 - \alpha x_2 x_3}} & \dfrac{-\alpha x_2}{2\sqrt{x_2 - \beta x_2 x_1 - \alpha x_2 x_3}} \\ \dfrac{-\alpha x_3}{2\sqrt{x_3 - \alpha x_3 x_1 - \beta x_3 x_2}} & \dfrac{-\beta x_3}{2\sqrt{x_3 - \alpha x_3 x_1 - \beta x_3 x_2}} & \dfrac{1 - \alpha x_1 - \beta x_2}{2\sqrt{x_3 - \alpha x_3 x_1 - \beta x_3 x_2}} \end{pmatrix}$$

(this gives [1p])

The infinity norm is defined as
$||A||_\infty = \max_i \sum_j |a_{ij}|$.

One solution is when $x_1 = x_2 = x_3 = x$. Then $(1 - (1 + \alpha + \beta)x)x = 0$
Which has the solution $x = 0$ and $x = \dfrac{1}{1 + \alpha + \beta}$. Obvious, the first fix-point is not a stable fixed point.

c)

With the starting point $\boldsymbol{x}_0 = \left(0.5, 0, 0.5\right)$ we get
First step: $(5.91608 \cdot 10^{-1}, 0, 6.51920 \cdot 10^{-1})$
Second step: $(6.00166 \cdot 10^{-1}, 0, 7.32268 \cdot 10^{-1})$
Third step: $(5.80066 \cdot 10^{-1}, 0, 7.74870 \cdot 10^{-1})$

The rate of convergence is given by the Jacobian above

With the starting point $\boldsymbol{x}_0 = (1, 1, 1)$ we get
First step: $(3.16228 \cdot 10^{-1}, 3.16228 \cdot 10^{-1}, 3.16228 \cdot 10^{-1})$
Second step:$(4.75634 \cdot 10^{-1}, 4.75634 \cdot 10^{-1}, 4.75634 \cdot 10^{-1})$
Third step:$(5.215641 \cdot 10^{-1}, 5.215641 \cdot 10^{-1}, 5.215641 \cdot 10^{-1})$

This gives [1p].

Yes, one solution can be expressed by $x = \dfrac{1}{1 + \alpha + \beta}$, where $x_1 = x_2 = x_3 = x$
(This gives [1p]). Thus, the exact solution for the latter solution is
$x = 1/1.9 = 0.526318.$
The rate of convergence is given by the Jacobian above, which is for this case (gives [1p])

$$\lim_{i\to\infty}\frac{x_{i+1}}{x_i} = D_xG = \left\|\begin{pmatrix} \frac{1-0.3x_2-0.6x_3}{2\sqrt{x_1-0.3x_1x_2-0.6x_1x_3}} & \frac{-0.3x_1}{2\sqrt{x_1-0.3x_1x_2-0.6x_1x_3}} & \frac{-0.6x_1}{2\sqrt{x_1-0.3x_1x_2-0.6x_1x_3}} \\ \frac{-0.6x_2}{2\sqrt{x_2-0.6x_2x_1-0.3x_2x_3}} & \frac{1-0.6x_1-0.3x_3}{2\sqrt{x_2-0.6x_2x_1-0.3x_2x_3}} & \frac{-0.3x_2}{2\sqrt{x_2-0.6x_2x_1-0.3x_2x_3}} \\ \frac{-0.3x_3}{2\sqrt{x_3-0.3x_3x_1-0.6x_3x_2}} & \frac{-0.6x_3}{2\sqrt{x_3-0.3x_3x_1-0.6x_3x_2}} & \frac{1-0.3x_1-0.6x_2}{2\sqrt{x_3-0.3x_3x_1-0.6x_3x_2}} \end{pmatrix}\right\|_\infty < 1$$

Using the above exact solution, only three terms have to be evaluated:

$$\frac{1-0.157895-0.315789}{2\sqrt{(1-0.157895-0.315789)0.526316}} = 0.5, \frac{-0.157895}{2\sqrt{(1-0.157895-0.315789)0.526316}} = 0.15, \frac{-0.315789}{2\sqrt{(1-0.157895-0.315789)0.526316}} = 0.30$$

$$\lim_{i\to\infty}\frac{x_{i+1}}{x_i} = D_xG = \left\|\begin{pmatrix} 0.5 & -0.15 & -0.30 \\ -0.30 & 0.5 & -0.15 \\ -0.15 & 0.30 & 0.5 \end{pmatrix}\right\|_\infty = 0.95 < 1$$

d)

See text book. E.g. Newton method, Broyden methods : T. Sauer Page 131-137.
[1p] for just mention the method (although required, no explanation needed)

3. The statistic in the table describes the annual average gold price between 1990 and 2010, in U.S. dollars per troy ounce. In 2012, a troy ounce of gold had its maximum by an annual average price of around 1668 U.S. dollars. [10p]

i	Year	Average gold price
1	1990	383
2	1995	384
3	2000	279
4	2005	444
5	2010	1224
6	2015	1160

To obtain a functional trend, the data should be approximated by least squares polynomials of degrees 2.

a) Formulate the problem into a system of equations $Ax = b$. Why it is impossible to determine an exact solution? What is the least square solution? [3p]

b) Is the matrix $A^T A$ of the normal equation well conditioned? Note: it is needed to determine the inverse of the matrix. [3p]

c) Solve the normal equation for the problem. [2p]

d) Determine from the least square solution the year in which the gold price had its maximum and compare with the given number above. [2p]

Solution:

a)

A polynomial of order 2 is
$P(x) = a_2 x^2 + a_1 x + a_0$

Assuming a least square fit, using the years probably overloads the problem. This, we can use the first column of integers.

Plug in the integers in the polynomial as x value and replacing $P(x)$ by the values of the gold prices, we obtain

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \\ 36 & 6 & 1 \end{pmatrix} x = \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 383 \\ 384 \\ 279 \\ 444 \\ 1224 \\ 1160 \end{pmatrix}. \text{ [1p]}$$

There are more equations (6) than unknowns (3) and no solution exists [1p]. However, least square problem can be defined, that minimizes the residual $r = ||b - A\overline{x}||$. For more information on the least square method, see book. [1p]

For the least square problem, the following equation has to be solved
$A^T A \overline{x} = A^T b$

b)

Thus,

$$A^TA = \begin{pmatrix} 1 & 4 & 9 & 16 & 25 & 36 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \\ 36 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 2275 & 441 & 91 \\ 441 & 91 & 21 \\ 91 & 21 & 6 \end{pmatrix}$$

$(A^TA)^{-1} = \begin{pmatrix} 0.02679 & -0.1875 & 0.25 \\ -0.1875 & 1.36964 & -1.95 \\ 0.25 & -1.95 & 3.2 \end{pmatrix}$ [1p].

[1p] definition of the condition.

$cond(A^TA) = ||A^TA||\ ||(A^TA)^{-1}|| = 1.054701 \cdot 10^4$

It is, in principle, a well conditioned problem and solvable.

c)

$$A^Tb = \begin{pmatrix} 83894 \\ 16844 \\ 3874 \end{pmatrix}$$

Thus, the solution of the least square problem is

$$\bar{x} = \begin{pmatrix} 5.74107 \\ -21.41607 \\ 52.45 \end{pmatrix} \cdot 10^1$$

d)

To get the maximum, one has to calculate the derivative of the polynomial

$P'(x) = 2 \cdot a_2 x + a_1 = 0$

$x = -\dfrac{a_1}{2a_2}$ and it is a maximum if $2 \cdot a_2 < 0$ (gives [1p])

From the least square solution, $x = 1.86516$, and from the linear correlation between the integer and the year, say $y = 1985 + x \cdot 5$, we obtain the extremum of the least square solution at year 1994.32581 (this gives [1p]), which is not the maximum, but a minimum of the gold price development. So the maximum is at 2015.

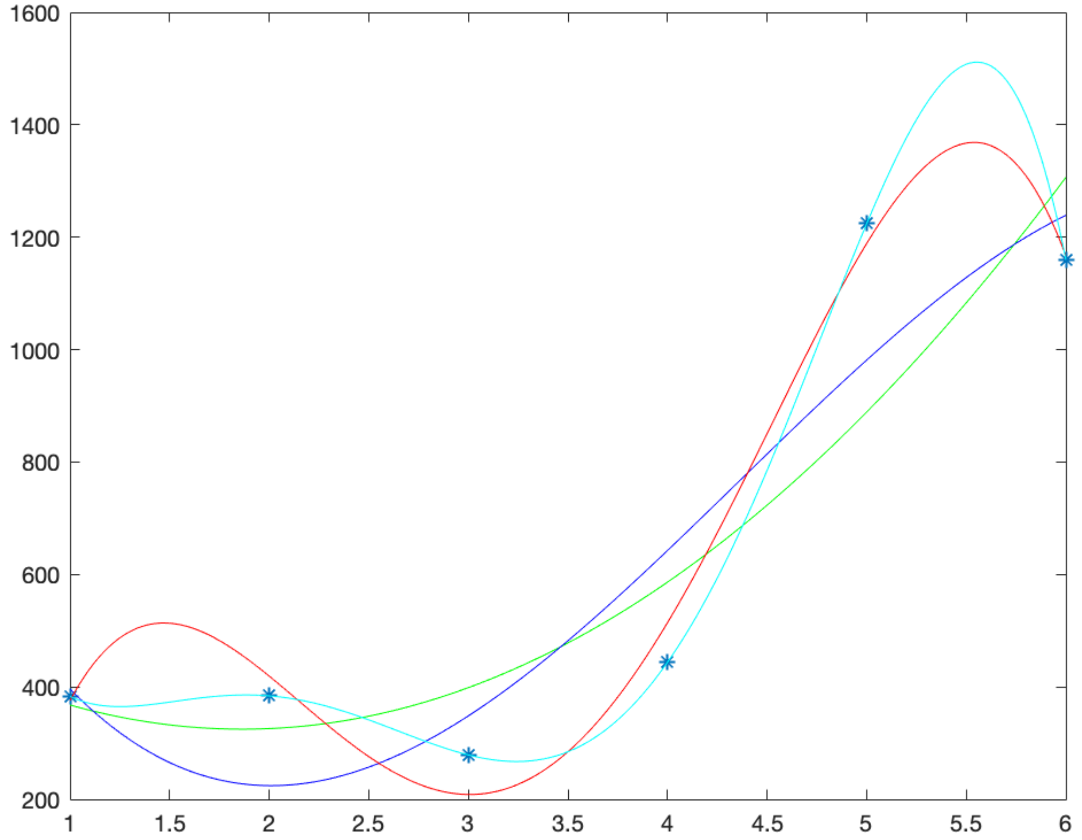

Figure: Green - Least square solution with 2nd. order polynomial, Blue - Least square solution with 3rd. order polynomial, Red- Least square solution with 4th. order polynomial, Cyan - Least square solution with 5th. order polynomial

4. A persymmetric matrix is a matrix that is symmetric about both diagonals; that is, an $N \times N$ matrix $A = (a_{ij})$ is per symmetric if $a_{ij} = a_{ji} = a_{N+1-i,N+1-j}$, for all $i = 1,2,...,N$ and $j = 1,2,...,N$. A number of problems in communication theory have solutions that involve the eigenvalues and eigenvectors of matrices that are persymmetric form. Hence, lets consider the 3×3 persymmetric matrix [10p]

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

a) Use the Geršgorin Circle Theorem to show that if A is the matrix given above and λ is its minimal eigenvalue, then $|\lambda - 4| = \rho(A - 4I)$, where ρ denotes the spectral radius. [3p]

b) Use the initial guess of the $v = (-1,1,-1)^t$ and calculate three steps of the power iteration method to obtain the largest eigenvalue. [2p]

c) Use the initial guess of the $v = (1,1,1)^t$ and calculate three steps of the <u>inverse</u> power iteration to find one of the remaining eigenvalues. Use the property of similar matrices and make a guess on the parameter s from the Geršgorin Circle Theorem. [3p]

d) The above matrix has Toepliz form, say [2p]

$$A = \begin{pmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & \ddots & a & \ddots & \\ & & & & \ddots & \\ & & & & c & a \end{pmatrix},$$

where the eigenvalues are exactly given by

$\lambda_k = a + 2\sqrt{bc}\cos\left(\dfrac{k\pi}{(N+1)}\right), k = 1,...,N$. Find the exact eigenvalues and related eigenvectors of A (using the above mentioned formula) and determine the relative error of the eigenvector between the analytical solution and the numerical obtained from part b) and c). Use the norm $||.||_\infty$.

Solution

a)

According to the Geršgorin Circle Theorem, two circles can be defined by

$R_1 = \{z \in \mathbb{C} \,|\, |z-2| < 1\}$ and $R_1 = \{z \in \mathbb{C} \,|\, |z-2| < 2\}$. Thus, all eigenvalues of A are between $0 \le \mu \le 4$. Lets consider the similar matrix $A - 4I$ which has the similar eigenvalues as A with $\mu' = \mu - 4$. The spectral radius is defined as the $\rho(A) = \max_i |\lambda_i|$. ([1.5p] already from calculation of the Circles.)

Thus,

$$\rho(A-4I) = max(|\mu - 4|) = \max(4-\mu) = 4 - \min \mu = 4 - \lambda = |\lambda - 4|.$$

b)

The power iteration is

For i

$$u = \frac{v}{||v||}$$
$$v = Au$$
$$\lambda = u^T v$$

End

You obtained also points without normalisation of the eigenvectors, but you have to be careful when calculating the eigenvalue. [1p] for the algorithm, [1p] for the solutions.

In the following we use both the infinity norm, which is $||x||_\infty = \max_i |x_i|$, and

$$||x||_2 = \sqrt{\sum_i x_i^2}$$

First step

$||v||_\infty = 1$ or $||v||_2 = \sqrt{3}$

With $||v||_\infty$

$$v = (-3, 4, -3)$$

And with $||v||_2$

$$v = (-\sqrt{3}, \frac{4}{\sqrt{3}}, -\sqrt{3})$$

Second step

$||v||_\infty = 4$ or $||v||_2 = \sqrt{\frac{34}{3}}$

With $||v||_\infty$

$$v = (-\frac{5}{2}, \frac{7}{2}, -\frac{5}{2})$$

And with $||v||_2$

$$u = \frac{1}{\sqrt{34}}(-3, 4, -3)$$

$$v = \frac{1}{\sqrt{34}}(-10, 14, -10)$$

Third step

$||v||_\infty = \frac{7}{2}$ or $||v||_2 = \sqrt{\frac{396}{34}}$

With $||v||_\infty$

$$u = (-\frac{5}{7}, 1, -\frac{5}{7})$$

$v=(-\frac{17}{7},\frac{24}{7},-\frac{17}{7})$

And with $||v||_2 =$
$u=\frac{1}{\sqrt{396}}(-10,14,-10)$
$v=\frac{1}{\sqrt{396}}(-34,48,-34)$

The eigenvalue after three steps for $||v||_\infty$ is $\lambda=\frac{338}{49}$.
The eigenvalue after three steps for $||v||_2$ is $\lambda=\frac{1352}{396}$.

c)

This part will be not shown in so much in detail (since it is similar to the one above).
For i

$u=\frac{v}{||v||}$
$(A-sI)^{-1}v=u$
$\lambda'=u^Tv$ and $\lambda=\frac{1}{\lambda'+s}$

End
[1p] for matrix inversion, [1p] for the algorithm and [1p] for solution.

Depend also strongly on the norm.
For
$||v||_\infty$ and $s=0$

Step 1: $v=(1.50000,2.00000,1.50000)$
Step 2: $v=(1.25000,1.75000,1.25000)$
Step 3: $v=(1.21429,1.71429,1.21429)$

$\lambda=0.289941$

$||v||_2$ and $s=0$

Step 1: $v=(0.86603,1.15470,0.86603)$
Step 2: $v=(0.85749,1.20049,0.85749)$
Step 3: $v=(0.85428,1.20605,0.85428)$

$\lambda=0.58580$
$||v||_\infty$ and $s=\frac{5}{2}$
Step 1: $v=(-0.28571,-0.85714,-0.28571)$
Step 2: $v=(0.47619,0.09524,0.47619)$
Step 3: $v=(0.17143,-1.08571,0.17143)$

$\lambda=10.45455$

$||v||_2$ and $s = \dfrac{5}{2}$

Step 1: $v = (-0.16496, -0.49487, -0.16496)$

Step 2: $v = (0.43073, 0.08615, 0.43073)$

Step 3: $v = (0.12002, -0.76015, 0.12002)$

$\lambda = 18.72727$

d)

$$\lambda_k = 2 + 2\cos\left(\frac{k\pi}{4}\right)$$
$$\lambda_1 = 3.41421 = 2 + \sqrt{2}$$
$$\lambda_2 = 2$$
$$\lambda_3 = 0.58579 = 2 - \sqrt{2}$$

Calculating this gives already [1.5p]

The eigenvectors can be obtained from

$(A - \lambda_k I)v_k = 0$

Thus,

$$\begin{pmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In general it can be shown that

$v^k = \left(\sqrt{\dfrac{b}{c}}\sin\left(\dfrac{1\pi k}{N+1}\right), \ldots, \sqrt{\dfrac{b}{c}}\sin\left(\dfrac{N\pi k}{N+1}\right)\right)$. Here, $\sqrt{\dfrac{b}{c}} = 1$ and, thus,

$$v_1 = (\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}),\ v_2 = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}),\ v_3 = (-\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2})$$

5. The fundamental theorem of interpolation says that each function can be approximated by a polynomial, such as the Lagrange polynomial. Based on this, solve the following part problems [10p]

   a) Derive the Three-Point Endpoint (two formulas) and the Three-point Midpoint Formula for the derivative $f'(x)$ using equally spaced point x_0, $x_1 = x_0 + h$, and $x_2 = x_0 + 2h$. You can check the results with the one on the formula sheet. [4p]
   b) Calculate the error of the three-point formulas found above. What is the order of approximation. [2p]
   c) Based on the results of a) determine each missing entry in the following above. [4p]

x	$f(x)$	$f'(x)$
8.1	16.9441	
8.3	17.56492	
8.5	18.19056	
8.7	18.82091	

Solution:

a)

[1p] for (or Taylor expansion method. [1p] also when the way to the solution was mentioned)

$$f(x) = \sum_{k=0}^{2} L_k(x) f(x_k) + \frac{1}{3!} f^{(3)}(c(x)) \Pi_{k=0}^{2}(x - x_k).$$

Lets focus on $L_k(x)$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \text{ an, thus, } L_0'(x) = \frac{2x - x_1 - x_2}{(x_0-x_1)(x_0-x_2)}$$
$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \text{ an, thus, } L_1'(x) = \frac{2x - x_0 - x_2}{(x_1-x_0)(x_1-x_2)}$$
$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \text{ an, thus, } L_2'(x) = \frac{2x - x_0 - x_1}{(x_2-x_0)(x_2-x_1)}$$

Thus

$$f'(x) = \sum_{k=0}^{2} L_k'(x) f(x_k) + \frac{1}{6}\left(\frac{d}{dx} f^{(3)}(c(x))\right) \Pi_{k=0}^{2}(x - x_k) + \frac{1}{6}\left(\frac{d}{dx} \Pi_{k=0}^{2}(x - x_k)\right) f^{(3)}(c(x))$$

At $x = x_j$, $\dfrac{d}{dx} f^{(3)}(c(x_j)) = 0$ and, thus,

$$f'(x_j) = \sum_{k=0}^{2} L_k'(x_j) f(x_k) + \frac{1}{6} \Pi_{k=0, k\neq j}^{2}(x_j - x_k) f^{(3)}(c(x))$$

Or (the following gives [1p])

$$f'(x_j) = f(x_0)\frac{2x_j - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} + f(x_1)\frac{2x_j - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} + f(x_2)\frac{2x_j - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} + \frac{1}{6}\Pi_{k=0,k\neq j}^{2}(x_j - x_k)f^{(3)}(c(x))$$

Let use $x_1 = x_0 + h$ and $x_2 = x_0 + 2h$ and neglecting for now the error. Thus

$$f'(x_0) = \frac{1}{h}\left(-\frac{3}{2}f(x_0) + 2f(x_0 + h) - \frac{1}{2}f(x_0 + 2h)\right)$$

(1)

$$f'(x_0 + h) = \frac{1}{h}\left(-\frac{1}{2}f(x_0) + \frac{1}{2}f(x_0 + 2h)\right)$$

replacing $x_0 + h \to x_0$, we get $f'(x_0) = \dfrac{1}{h}\left(-\dfrac{1}{2}f(x_0 - h) + \dfrac{1}{2}f(x_0 + h)\right)$

(2)$f'(x_0 + 2h) = \dfrac{1}{h}\left(\dfrac{1}{2}f(x_0) - 2f(x_0 + h) + \dfrac{3}{2}f(x_0 + 2h)\right)$

Replacing $x_0 + 2h \to x_0$, we get

$f'(x_0) = \dfrac{1}{h}\left(\dfrac{1}{2}f(x_0 - 2h) - 2f(x_0 - h) + \dfrac{3}{2}f(x_0)\right)$ (3)

The midpoint formula gives [1p] and the end-point formulas gives another [1p].

b)

The error is already given above in general. Thus, the error for (1) is $\dfrac{h^2}{3}f^{(3)}(c_0)$. The error for (2) is $-\dfrac{h^2}{6}f^{(3)}(c_1)$, and for (3) it is $\dfrac{h^2}{3}f^{(3)}(c_2)$ (this gives [1p]). The order of approximation is $O(h^2)$. This gives [1p]

c)

First, notice that $h = 0.2$. For each result you get [1p].

i) use (1) $f'(8.1) = \dfrac{1}{0.2}\left(-\dfrac{3}{2}16.9441 + 2 \cdot 17.56492 - \dfrac{1}{2}18.19056\right) = 3.09205$

ii) Use (2) $f'(8.3) = \dfrac{1}{0.2}\left(-\dfrac{1}{2}16.9441 + \dfrac{1}{2}18.19056\right) = 3.11615$

iii) Use (2) $f'(8.5) = \dfrac{1}{0.2}\left(-\dfrac{1}{2}17.56492 + \dfrac{1}{2}18.82091\right) = 3.139975$

iv) Use (3) $f'(8.7) = \dfrac{1}{0.2}\left(\dfrac{1}{2}17.56492 - 2 \cdot 18.19056 + \dfrac{3}{2}18.82091\right) = 3.163525$

6. Decide whether the following statements are true or false. Explain the reason for your answer: [10p]

a) The degree of precision of the Simpson 3/8 rule is 3. [1p]

b) The evaluation points x_i of the Gauss Quadrature are the roots of the Hermit polynomial. [1p]

c) The set of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ should be interpolated by a polynomial. There are infinite possible polynomials of degree $n-1$ to achieve this interpolation. [1p]

d) The roots of the Chebyshev Polynomial are equidistant and minimise the interpolation error. [1p]

e) The intermediate value theorem is also valid for discontinuous functions. [1p]

f) The backward error is the amount we would need to change the approximated solution to make it correct. [1p]

g) In complete pivoting methods, which are applied to solve systems of linear equations, both column and row are interchanges in the k-th step according to the maximum in the element $a_{ij}, i, j = k, k+1,...,n$. [1p]

h) The iterative method $x_{k+1} = Ax_k + b$ converges and has a unique solution when the spectral radius $\rho(A) > 1$ and b is arbitrary. [1p]

i) In Broyden's method 1, A_i is the best approximation available at step i to the inverse Jacobian matrix. [1p]

j) The convergence rate of the power iteration is defined by the smallest eigenvalue divided by the largest. [1p]

Solution:

You get already [0.5p] if true or false Is correct.

a) [1p] True. The order of the polynomial that is integrated exact by the 3/8 Simpsons rule is 3.

b) [1p] False. Evaluation points x_i are the roots of the Legendre polynomial.

c) [1p] False. The main theorem of interpolation says that there is only one polynomial.

d) [1p] False. The roots of the Chebyshev Polynomial not equidistant. The roots are $x_i = cos\left(\frac{(2i-1)\pi}{2n}\right)$.

e) [1p] False. The intermediate value theorem is only valid for continues function. With discontinues functions and with $c \in [a, b]$ there can be an $f(c)$ which is not between $f(a)$ and $f(b)$.

f) [1p] False. The forward error is is the amount we would need to change the approximated solution to make it correct.

g) [1p] True. Compared to Partial Pivoting and scaled pivoting that exchanges only the rows, in complete pivoting both rows and columns are exchanged.

h) [1p] False. For the iterative methods that solve linear systems of equations, the spectral radius has to be smaller than one.

i) [1p] False. Broyden’s method 1 approximates the Jacobian matrix and not the inverse, which would be Broyden’s method 2.

j) [1p] False. The convergence rate is given the second largest divided by the largest eigenvalue.

Formula sheet

Machine representation for single precision

$se_1e_2\dots e_8b_1b_2\dots b_{23}$

Fixed point method

$x_{k+1} = g(x_k)$

Normal equation

$A^TA\bar{x} = A^Tb$

Residual

$r = b - Ax$

Order n polynomial

$P_n(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$

Geršgorin Circle

$$R_i = \left\{ z \in \mathbb{C} \mid \; |z - a_{ii}| < \sum_{j=1, j\neq i}^{n} |a_{ij}| \right\}$$

Polynomial approach of a function f

$$f(x) = \sum_{k=1}^{n} f(x_k)L_k(x) + \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{n!} f^{(n)}(c(x))$$

$$L_k(x) = \frac{(x-x_1)(x-x_2)\dots\overline{(x-x_k)}\dots(x-x_n)}{(x_k-x_1)(x_k-x_2)\dots\overline{(x_k-x_k)}\dots(x_k-x_n)}$$

Three-point Endpoint and Midpoint formula

$$f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right]$$

$$f'(x_0) = \frac{1}{2h}\left[-f(x_0-h) + f(x_0+h)\right]$$

$$f'(x_0) = \frac{1}{2h}\left[f(x_0-2h) - 4f(x_0-h) + 3f(x_0)\right]$$