TENTAMENSKOD:________________

ÖREBRO UNIVERSITET

INSTITUTIONEN FÖR
NATURVETENSKAP OCH TEKNIK

Kursens namn/kurskod	**Numeriska Metoder för civilingenjörer** **DT508G**
Examinationsmomentets namn/provkod	**Teori, 2,5 högskolepoäng (Provkod: A003)**
Datum	2021-01-08
Tid	Kl. 08:15 – 12:15

Tillåtna hjälpmedel	Skrivmateriel, miniräknare med raderat minne, kursbok, anteckningar från föreläsning.
Instruktion	Läs igenom alla frågor noga. Börja varje fråga på ett nytt svarsblad. Skriv bara på ena sidan av svarsbladet. Skriv tentamenskoden på varje svarsblad. Skriv läsligt! Det räcker att ange fem decimaler. Glöm inte att scanna lösningar och ladda up det till WISEflow
Viktigt att tänka på	Motivera väl, redovisa tydligt alla väsentliga steg, rita tydliga figurer och svara med rätt enhet. Lämna in i uppgiftsordning.
Ansvarig/-a lärare (ev. telefonnummer)	Danny Thonig (Mobil: 0727010037)
Totalt antal poäng	30
Betyg (ev. ECTS)	Skrivningens maxpoäng är 30. För betyg 3/4/5 räcker det med samt 15/22/28 poäng totalt. Detaljerna framgår av separat dokument publicerat på Blackboard.
Tentamensresultat	Resultatet meddelas på Studentforum inom 15 arbetsdagar efter tentadagen.
Övrigt	Lärare är inte på plats. Varsågod och ringa om du har frågar.

Lycka till!

1.

a) Determine the order of convergence and the convergence rate for Steffensen's method

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f[x_n + f(x_n)] - f(x_n)}$$

Under the conditions $f(x) \in C^2$, $f(r) = 0$, and $f'(r) \neq 0$, where r is the root. **Tip:** Use Taylor series of $f[x_n + f(x_n)]$ around x_n and Taylor series of $f(r)$ around x_n. [7p]

b) Perform two iterations of the method for $f(x) = e^x - x - 1$ with the initial guess $x_0 = 1$ and show that it converges to the root $r = 0$. [3p]

Solutions:

a)

The convergence is at least quadratic. To prove it, consider the Taylor series of $f(x_n + f(x_n))$ around x_n:

$$f(x_n + f(x_n)) = f(x_n) + f'(x_n)f(x_n) + \frac{f''(c_n)}{2}f(x_n)^2$$

Where c_n is a point between x_n and $x_n + f(x_n)$. Therefore,

$$\frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)} = \frac{f(x_n)}{f'(x_n) + \frac{f''(c_n)}{2}f(x_n)}$$

From the difference $r - x_{n+1}$, we obtain

$$r - x_{n+1} = r - x_n + \frac{f(x_n)}{f'(x_n) + \frac{f''(c_n)}{2}f(x_n)} = \frac{f(x_n) + f'(x_n)(r - x_n) + \frac{f''(c_n)}{2}f(x_n)(r - x_n)}{f'(x_n) + \frac{f''(c_n)}{2}f(x_n)}$$

The Taylor series of $f(r)$ around x_n gives us

$$0 = f(r) = f(x_n) + f'(x_n)(r - x_n) + \frac{f''(c_n^*)}{2}(r - x_n)^2$$

Where c_n^* is a point between x_n and r, not necessarily equal to c_n.

$0 = f(r) = f(x_n) + f'(c_n^+)(r - x_n)$

Where c_n^+ is another point between x_n and r, not necessarily equal to c_n or c_n^*. Putting all together

$$r - x_{n+1} = \frac{\frac{f''(c_n^*)}{2}(r - x_n)^2 + \frac{f''(c_n)}{2}f'(c_n^+)(r - x_n)^2}{f'(x_n) + \frac{f''(c_n)}{2}f(x_n)}$$

In the limit of n approaching infinity, we utilize the continuity of $f'(x)$ and $f''(x)$ to get

$$\lim_{n\to\infty} \frac{|r - x_{n+1}|}{|r - x_n|^2} = \lim_{n\to\infty} \left| \frac{\frac{f''(c_n^*)}{2} + \frac{f''(c_n)}{2}f'(c_n^+)}{f'(x_n) + \frac{f''(c_n)}{2}f(x_n)} \right| = \frac{1}{2}\left|\frac{f''(r)}{f'(r)}\right| \left|1 + f'(r)\right|$$

b)

$f(x) = e^x - x - 1$

Iteration 1: $f(x_0) = e - 2, f(x_0 + f(x_0)) = f(1 + e - 2) = 2.8567, x_1 = 0.7587$

Iteration 2: $f(x_1) = 0.3768, f(x_1 + f(x_1)) = 0.9774, x_2 = 0.5223$

2. Find the polynomial that interpolates $f(0)=0, f(1)=1, f(2)=0, f(3)=1$, $f(4)=0$ using Neville's method. Use the Neville's table to generate the necessary coefficients. [6p]

Solutions:

$$Q_{11}=\frac{(x-0)1-(x-1)0}{1-0}=x,\ Q_{21}=\frac{(x-1)0-(x-2)1}{2-1}=-(x-2)$$

$$Q_{31}=\frac{(x-2)1-(x-3)0}{3-2}=(x-2), Q_{41}=\frac{(x-3)0-(x-4)1}{4-3}=-(x-4)$$

$$Q_{22}=\frac{(x-0)Q_{21}-(x-2)Q_{11}}{2-0}=\frac{-x(x-2)-(x-2)x}{2}=-x(x-2)$$

$$Q_{32}=\frac{(x-1)Q_{31}-(x-3)Q_{21}}{3-1}=(x-2)^2$$

$$Q_{42}=\frac{(x-2)Q_{41}-(x-4)Q_{31}}{4-2}=\frac{-(x-2)(x-4)-(x-4)(x-2)}{2}=-(x-4)(x-2)$$

$$Q_{33}=\frac{(x-0)Q_{32}-(x-3)Q_{22}}{3-0}=\frac{x(x-2)^2+(x-3)x(x-2)}{3}=\frac{1}{3}x(x-2)(2x-5)$$

$$Q_{43}=\frac{(x-1)Q_{42}-(x-4)Q_{32}}{4-1}=\frac{-(x-1)(x-4)(x-2)-(x-4)(x-2)^2}{3}=-\frac{1}{3}(x-2)(2x-3)(x-4)$$

$$Q_{44}=\frac{(x-0)Q_{43}-(x-4)Q_{33}}{4-0}=\frac{1}{4}(-\frac{1}{3}x(x-2)(2x-3)(x-4)-\frac{1}{3}x(x-2)(x-4)(2x-5))=-\frac{1}{3}x(x-2)^2(x-4)$$

3. Consider the matrix

$$A=\begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}$$

   a) Use Geršgorin's circle theorem to analyze the locations of eigenvalues of A. If all eigenvalues are real, where do theses eigenvalues locate? Please give as good estimated eigenvalues as possible. [3p]
   b) Perform two steps of power iteration to find the biggest eigenvalue and compare with your finding in a). The initial guess is $x_0=(1,-1,0)$. [2p]

Solutions:

a) Geršgorin's circles:

$R_1=\left\{z\in\mathbb{C}: ||z-4||\le 1\right\}$

$R_2=\left\{z\in\mathbb{C}: ||z-4||\le 2\right\}$

$R_3=\left\{z\in\mathbb{C}: ||z-4||\le 2\right\}$

b)

Note that you were free to use any norm possible.

1. Normalize x_0: $||x_0||_2$=$\sqrt{2}$ and, thus, $u = \frac{1}{\sqrt{2}}(1, -1,0)$.

2. $v = \boldsymbol{A}u$
$v = \frac{5}{\sqrt{2}}(1, -1,0)^T$ and $\lambda = v^T u = 5$.

3. After normalizing v it will be the same as u from step 1. So the second iteration step is the same as the first.

4. The Chebyshev polynomials of second kind are defined as

$$U_n(x) = \frac{1}{n+1}T'_{n+1}(x), n \geq 0$$

where $T_{n+1}(x)$ is the Chebyshev polynomial of first kind.

a) Using the form $T_n(x) = \cos(n \arccos(x)), x \in [-1,1]$, derive a similar expression for $U_n(x)$. [2p]

b) Show that the Chebyshev Polynomial of the second kind satisfies the recursion $U_0(x) = 1$, $U_1(x) = 2x$, and $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$. [3p]

c) The Chebyshev polynomials of second kind are orthogonal with respect to the inner product

$$< f, g > = \int_{-1}^{1} f(x)g(x)\sqrt{1-x^2}\mathrm{d}x.$$

Demonstrate this by using composite trapezoid method with $m = 2$ and for $< U_3, U_3 >$ and $< U_2, U_4 >$. [4p]

Solutions:

a)
The derivative of $\arccos(x)$ is
$\partial_x \arccos(x) = -\frac{1}{\sqrt{1-x^2}}$.
Thus,
$T'_{n+1}(x) = \frac{(n+1)\sin((n+1)\arccos(x))}{\sqrt{1-x^2}}$ and $U_n(x) = \frac{\sin((n+1)\arccos(x))}{\sqrt{1-x^2}}$.
If we replace $x = \cos\theta$, we have
$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$

b)
Lets take the recursion of the Chebyshev polynomial of first kind
$T_0 = 1, T_1 = x, T_2 = 2x^2 - 1$ or in general $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

Then,
$U_{n+1}(x) = \frac{1}{n+2}T'_{n+2}(x) = \frac{1}{n+2}\partial_x(2xT_{n+1}(x) - T_n(x)) = \frac{1}{n+2}(2T_{n+1}(x) + 2xT'_{n+1}(x) - T'_n(x)) = \frac{1}{n+2}(2T_{n+1}(x) + 2x(n+1)U_n(x) - nU_{n-1}(x))$

Further, it is $T_n(x) = U_n(x) - xU_{n-1}(x)$. Thus,

$$U_{n+1}(x) = \frac{1}{n+2}(2(U_{n+1}(x) - xU_n(x)) + 2x(n+1)U_n(x) - nU_{n-1}(x))$$
$$\frac{n}{n+2}U_{n+1}(x) = \frac{n}{n+2}(2xU_n(x) - U_{n-1}(x))$$
$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

c)
Composite trapezoid says apply relation from $[-1,0]$ and $[0,1]$. Thus $h = 1$
$U_2 = 4x^2 - 1$, $U_3 = 8x^3 - 4x$, $U_4 = 16x^4 - 12x^2 + 1$

Thus, we have to solve
$\int_{-1}^{1} \underbrace{(8x^3 - 4x)^2\sqrt{1-x^2}}_{F_1(x)}\,\mathrm{d}x$ and $\int_{-1}^{1} \underbrace{(4x^2-1)(16x^4 - 12x^2 + 1)\sqrt{1-x^2}}_{F_2(x)}\,\mathrm{d}x$
And the integrals are approximated by
$$\frac{1}{2}(F_i(-1) + 2F_i(0) + F_i(1))$$

So for the first integral we get 0 and for the second we also get -1.