Matrix Factorization

CS 556

Outline

- Eigen Decomposition
- Singular Value Decomposition (SVD)
- SVD for Recommender Systems
- Pandas Library

Orthogonal Matrix

A square matrix $A \in \mathbb{R}^{nxn}$ is an orthogonal matrix if an only if its columns are orthonormal so that: $AA^T = I = A^T A$, which implies that $A^{-1} = A^T$.

Columns of A are orthonormal if for every two columns (i, j)

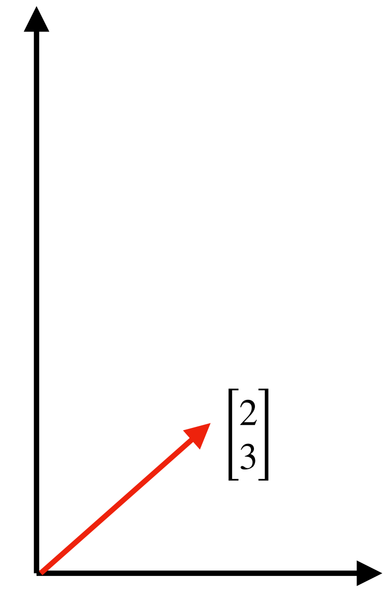
$$a_i b_j = 0 \ if \ i \neq j$$

$$a_i b_j = 1 \ if \ i = j$$

2D Transformation Matrix

What happens when we multiply a 2D matrix with a vector?

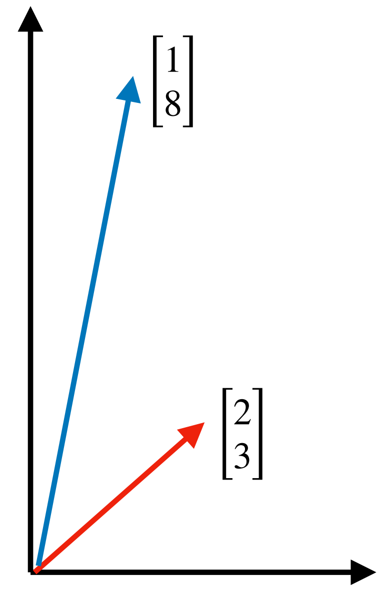
$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = ?$$



2D Transformation Matrix

What happens when we multiply a 2D matrix with a vector?

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

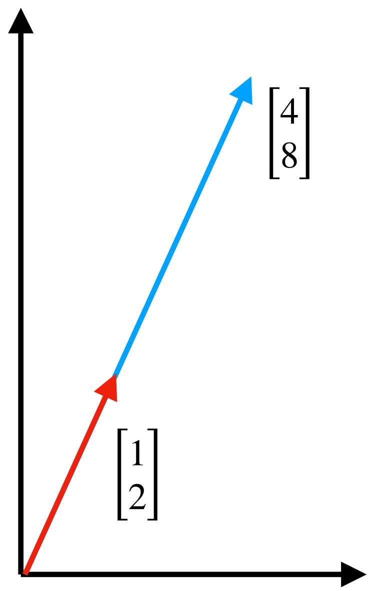


The matrix will transform the vector by rotating and stretching/shortening it

2D Transformation Matrix

What happens when we multiply a 2D matrix with a vector?

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

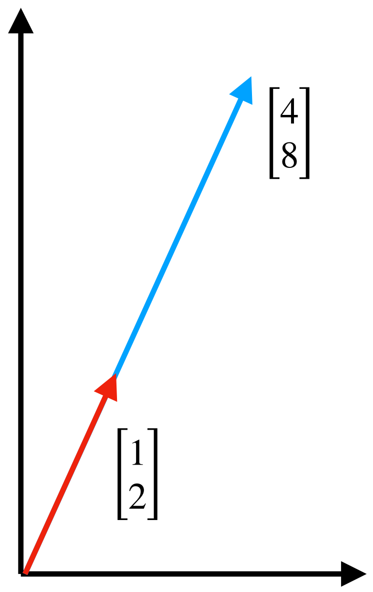


2D Transformation Matrix

What happens when we multiply a 2D matrix with a vector?

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$



2D Transformation Matrix

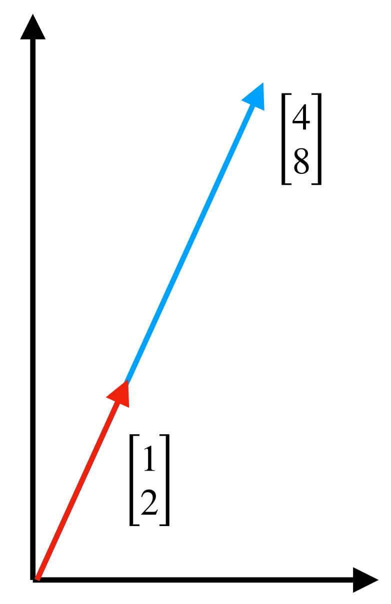
What happens when we multiply a 2D matrix with a vector?

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

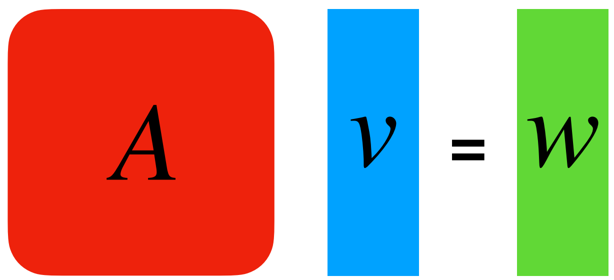
$$4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

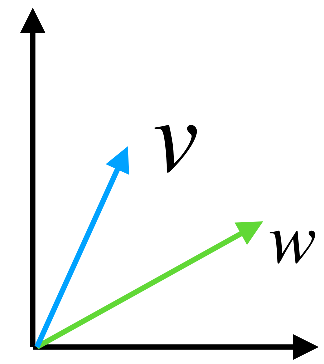
Eigen-value

Eigen-vector



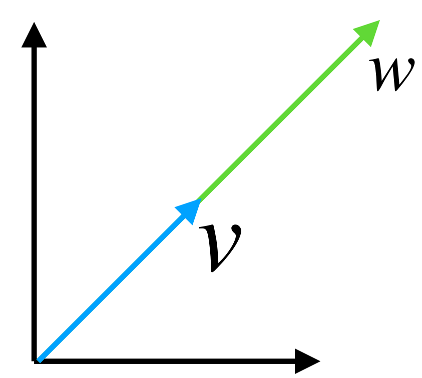
Eigenvectors & Eigenvalues





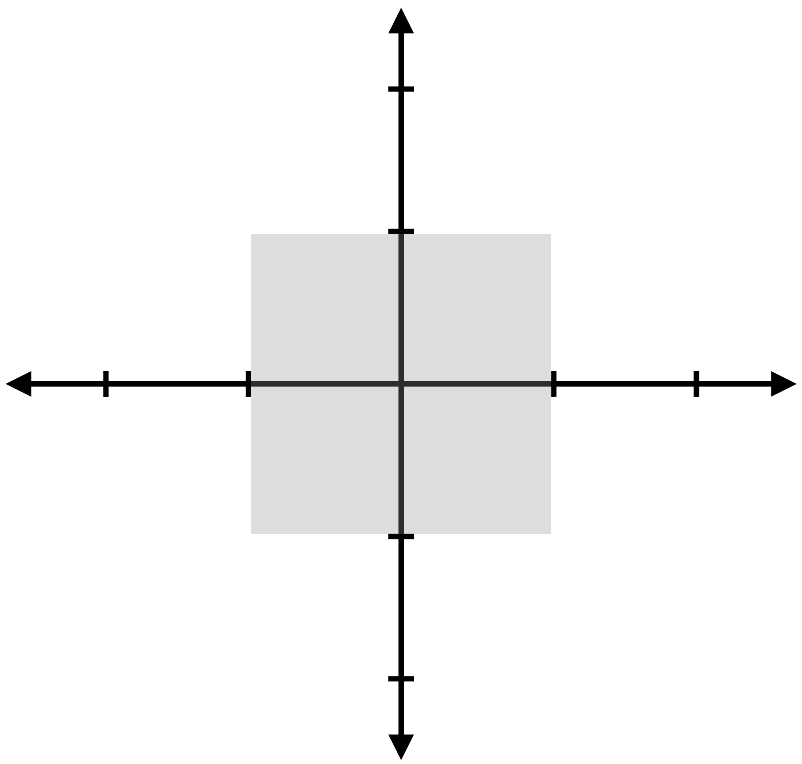
Transformation matrix A is applied to a vector **v** and outputs a vector **w**. If **w** points in the same direction as **v** (a.k.a. lies on the same 1 dimensional subspace), then **v** is an eigenvector of matrix A.

$$A\mathbf{v} = \mathbf{w}$$
$$\lambda\mathbf{v} = \mathbf{w}$$

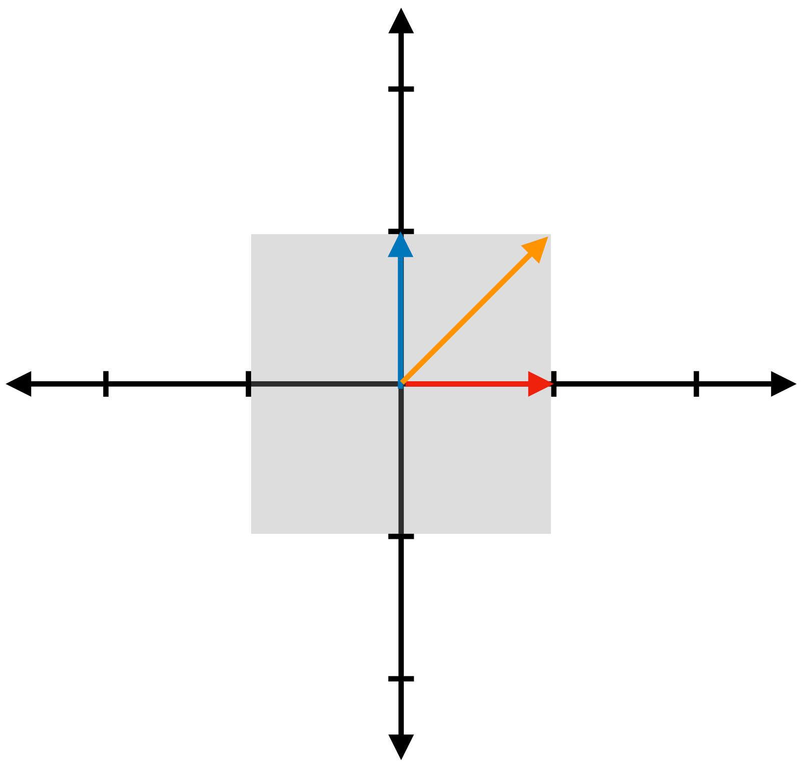


λ is an eigenvalue associated with eigenvector $\mathbf{v}$ of A

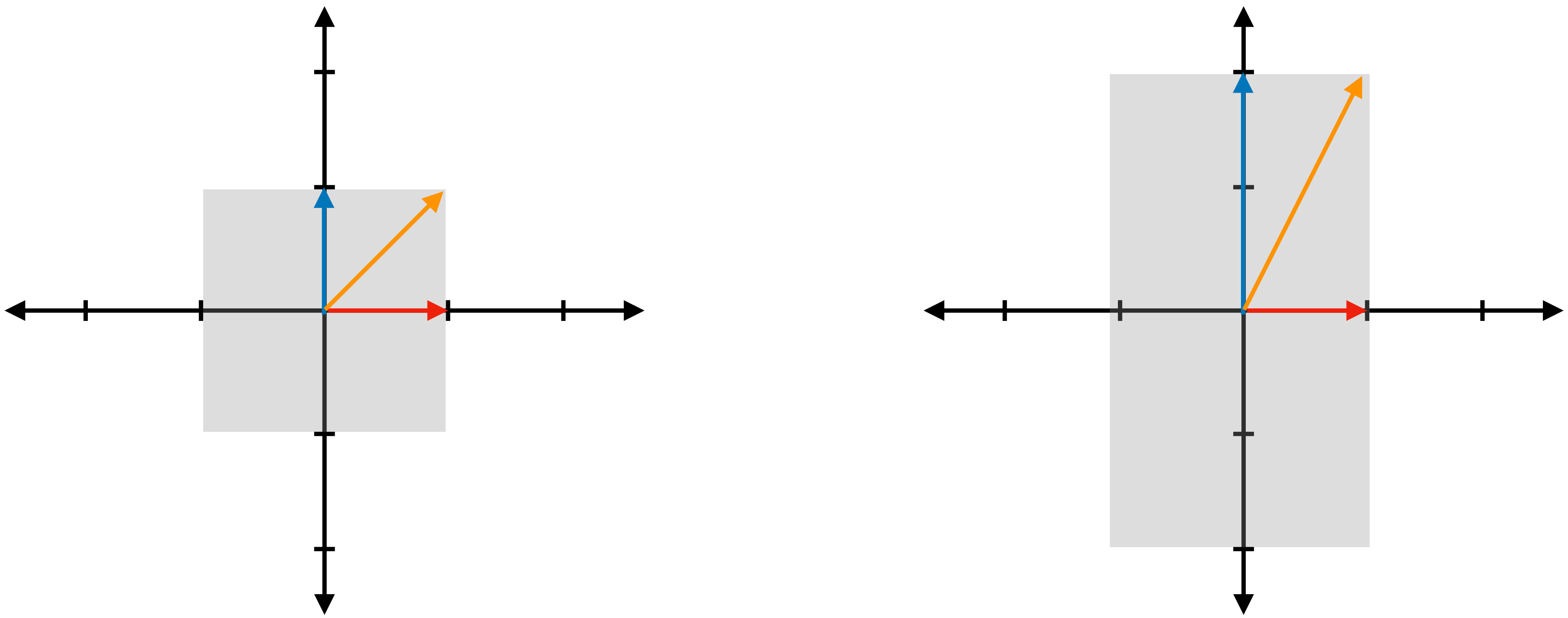
Eigenvectors & Eigenvalues



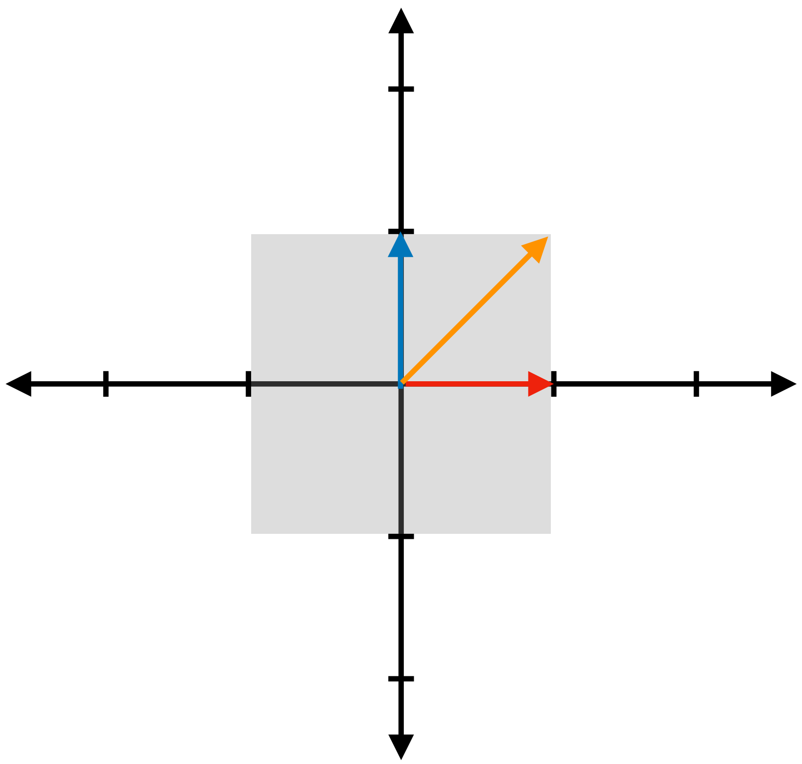
Eigenvectors & Eigenvalues

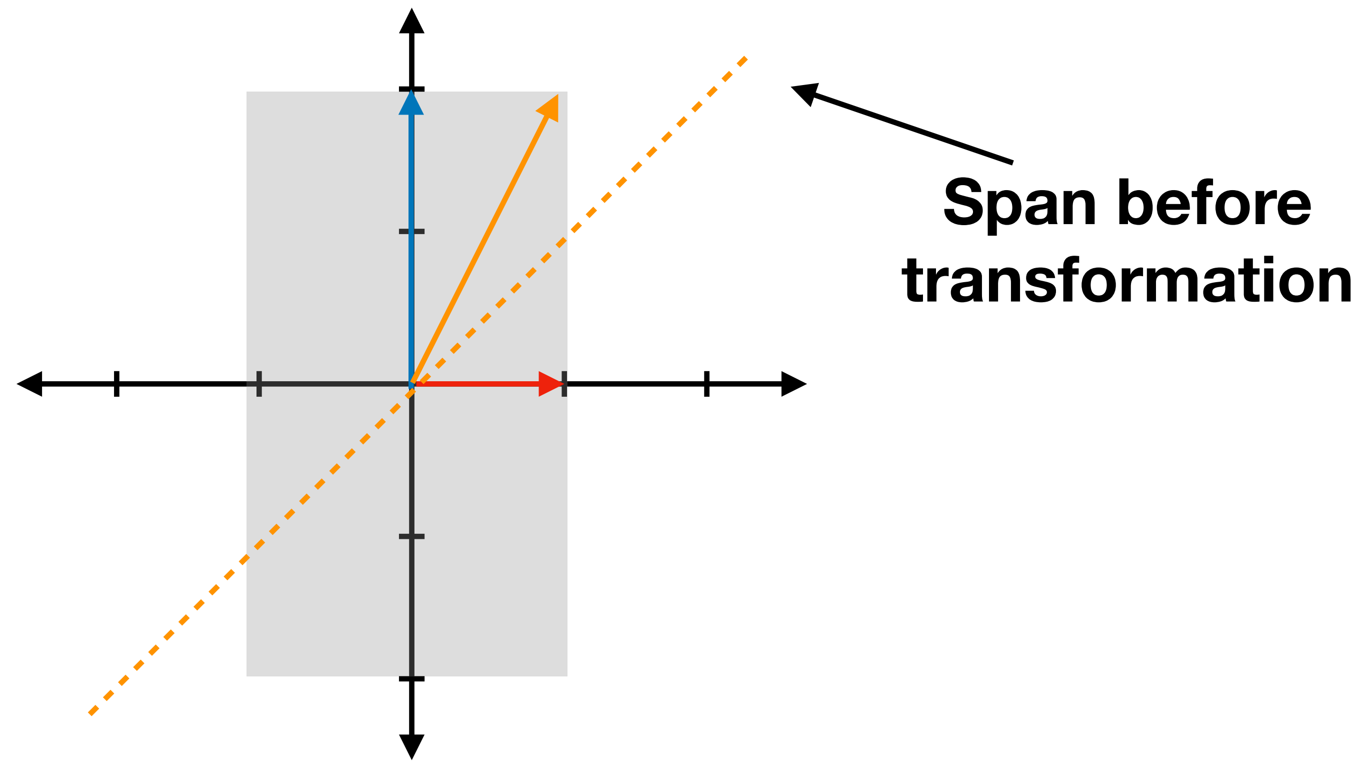


Eigenvectors & Eigenvalues



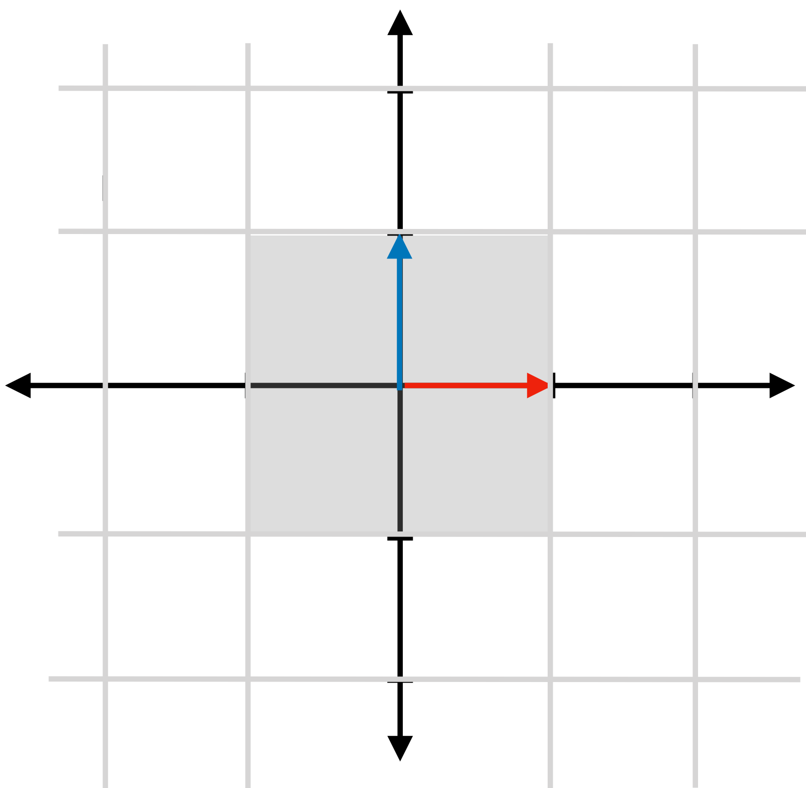
Eigenvectors & Eigenvalues





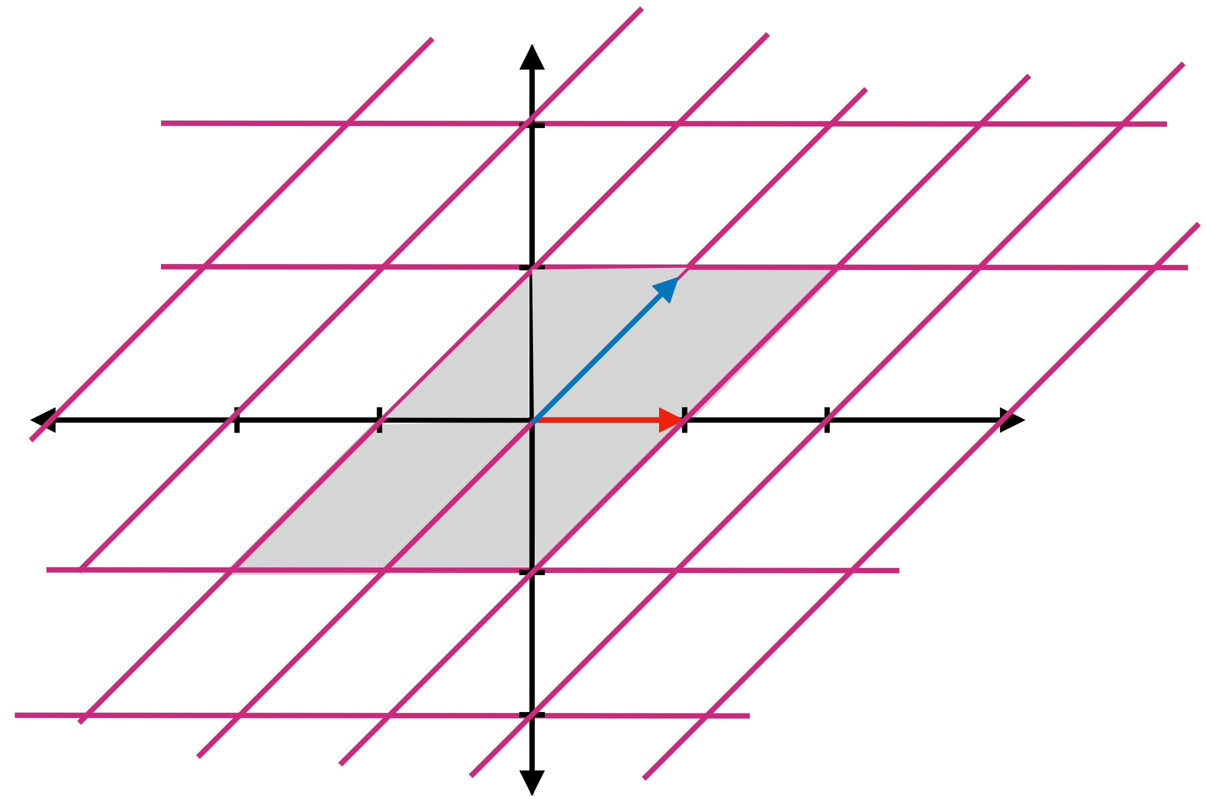
2 eigenvectors with values 1 and 2

Eigenvectors & Eigenvalues



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

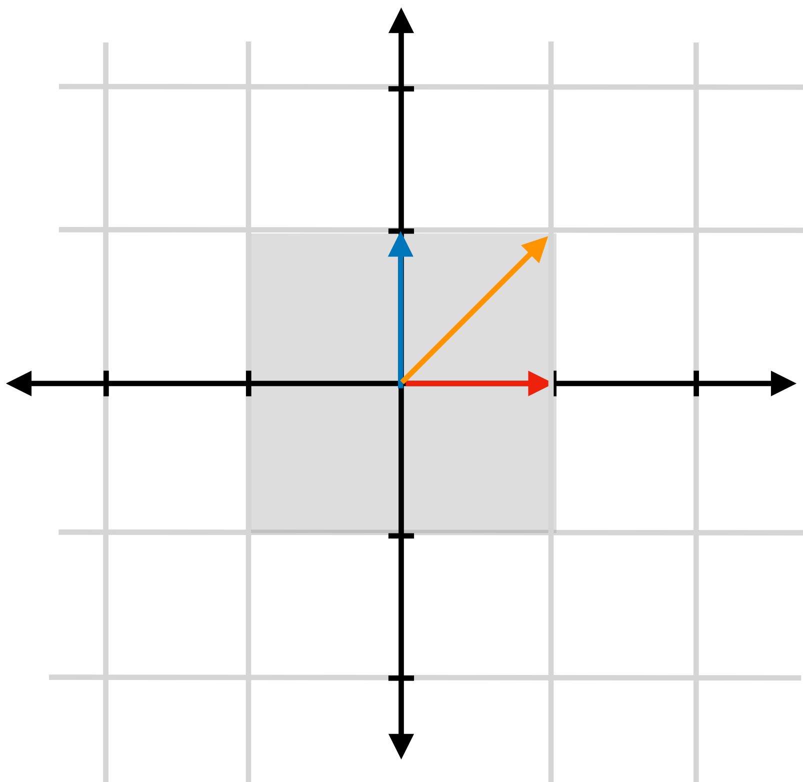
Natural Basis



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

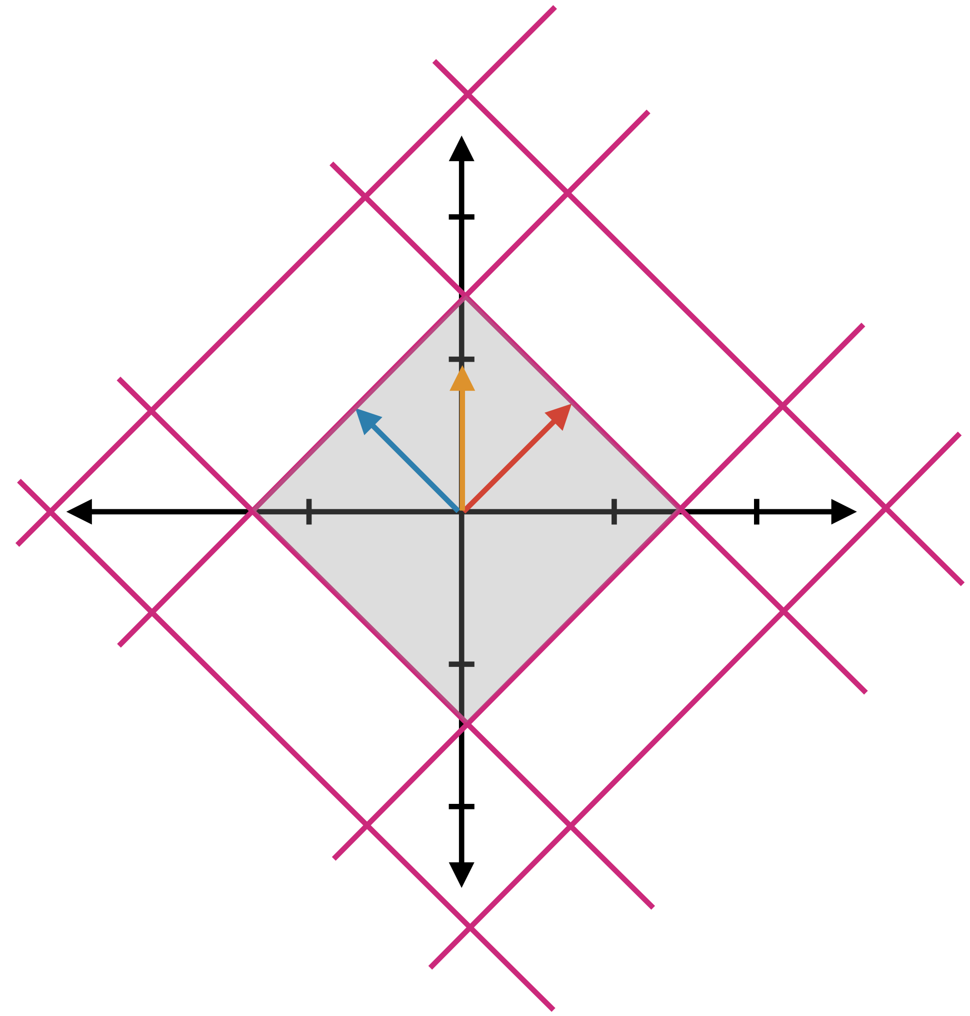
Shear Transformation

Eigenvectors & Eigenvalues



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

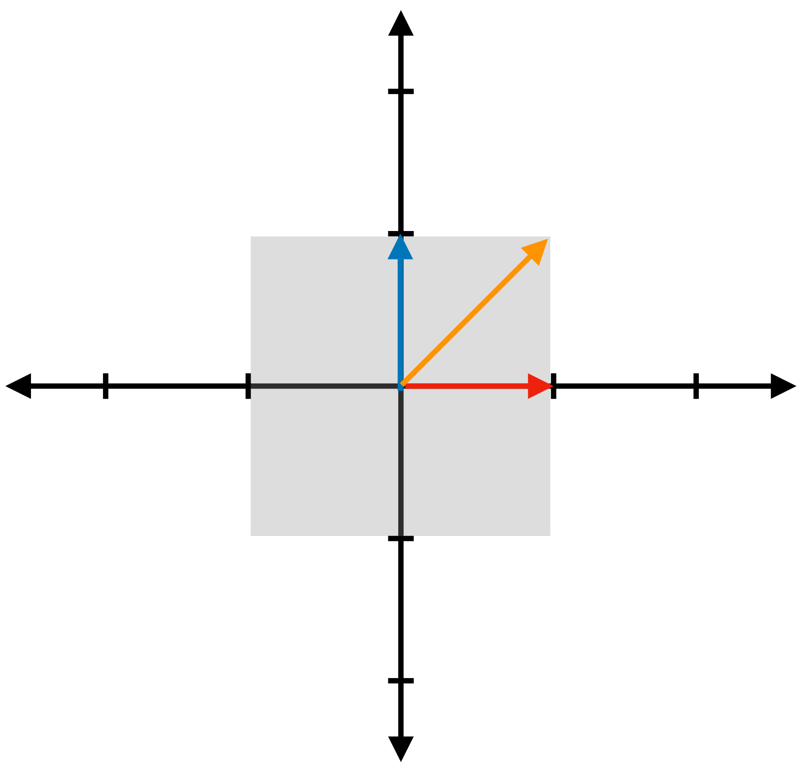
Natural Basis

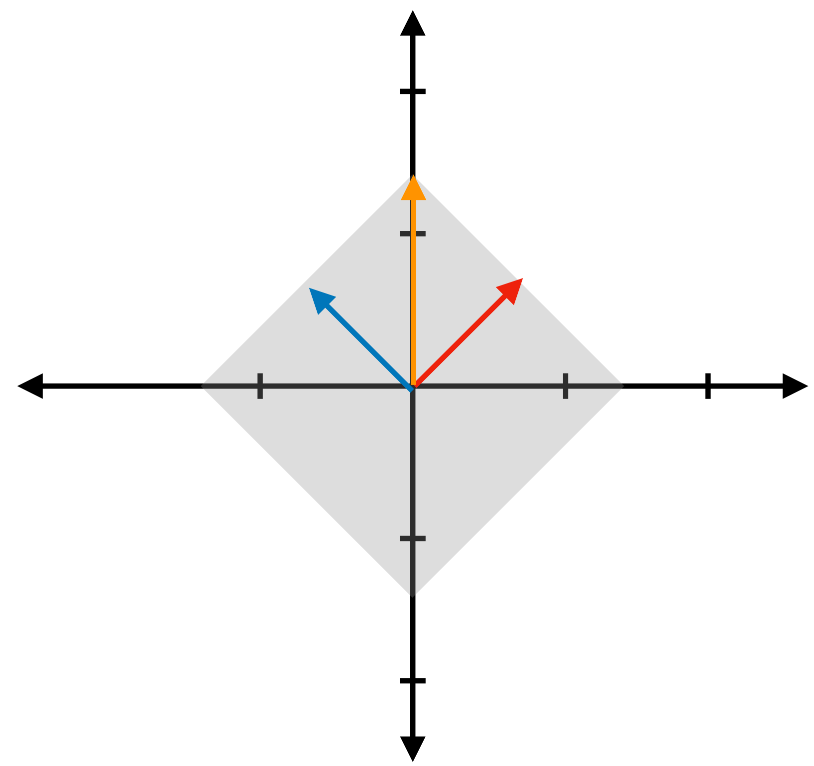


$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \theta = \frac{\pi}{4}$$

Rotation

Eigenvectors & Eigenvalues





0 Eigenvectors

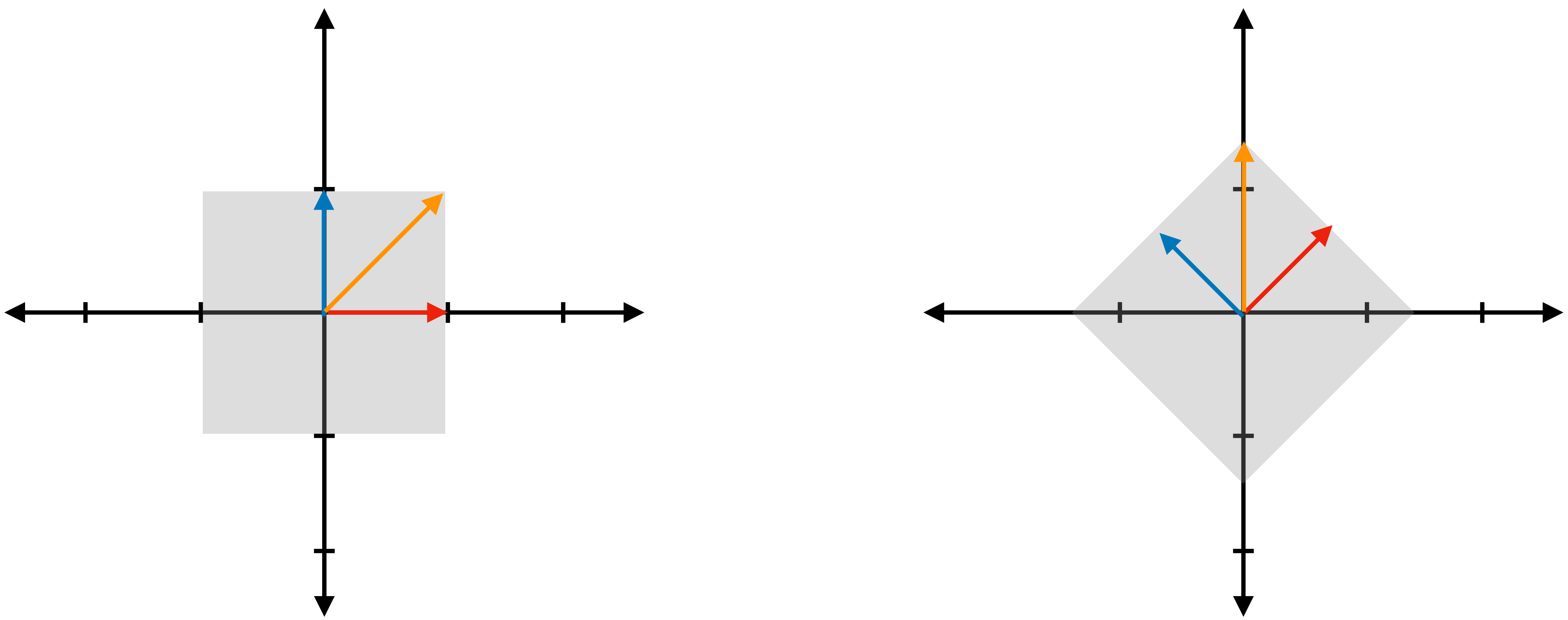
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Natural Basis

$$\begin{bmatrix} cos\,\theta & -\,sin\,\theta \\ sin\,\theta & cos\,\theta \end{bmatrix} \qquad \theta = \frac{\pi}{4}$$

Rotation

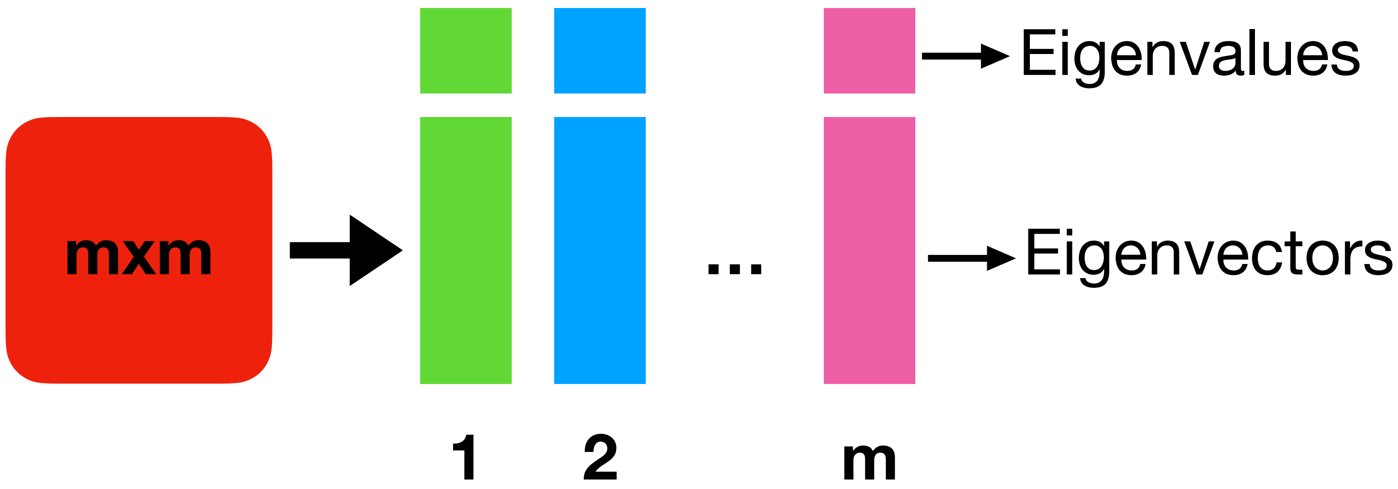
Eigenvectors & Eigenvalues



0 Eigenvectors

Check the vectors that lie on the same span after transformation and measure how much their magnitudes change

Eigen Decomposition



Eigen-decomposition is the factorization of a matrix into a canonical form, whereby the matrix is represented in terms of its eigenvalues and eigenvectors.

Finding Eigenvalues

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$A\mathbf{v} - \lambda\mathbf{v} = 0$$

$$(A - \lambda)\mathbf{v} = 0$$

$$(A - \lambda I)\mathbf{v} = 0$$

It has nontrivial null space.
It must be singular

$$(A - \lambda I)\mathbf{v} = 0$$

$$det(A - \lambda I) = 0$$

$$det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\lambda^2 - (a + d)\lambda + ad - bc = 0$$

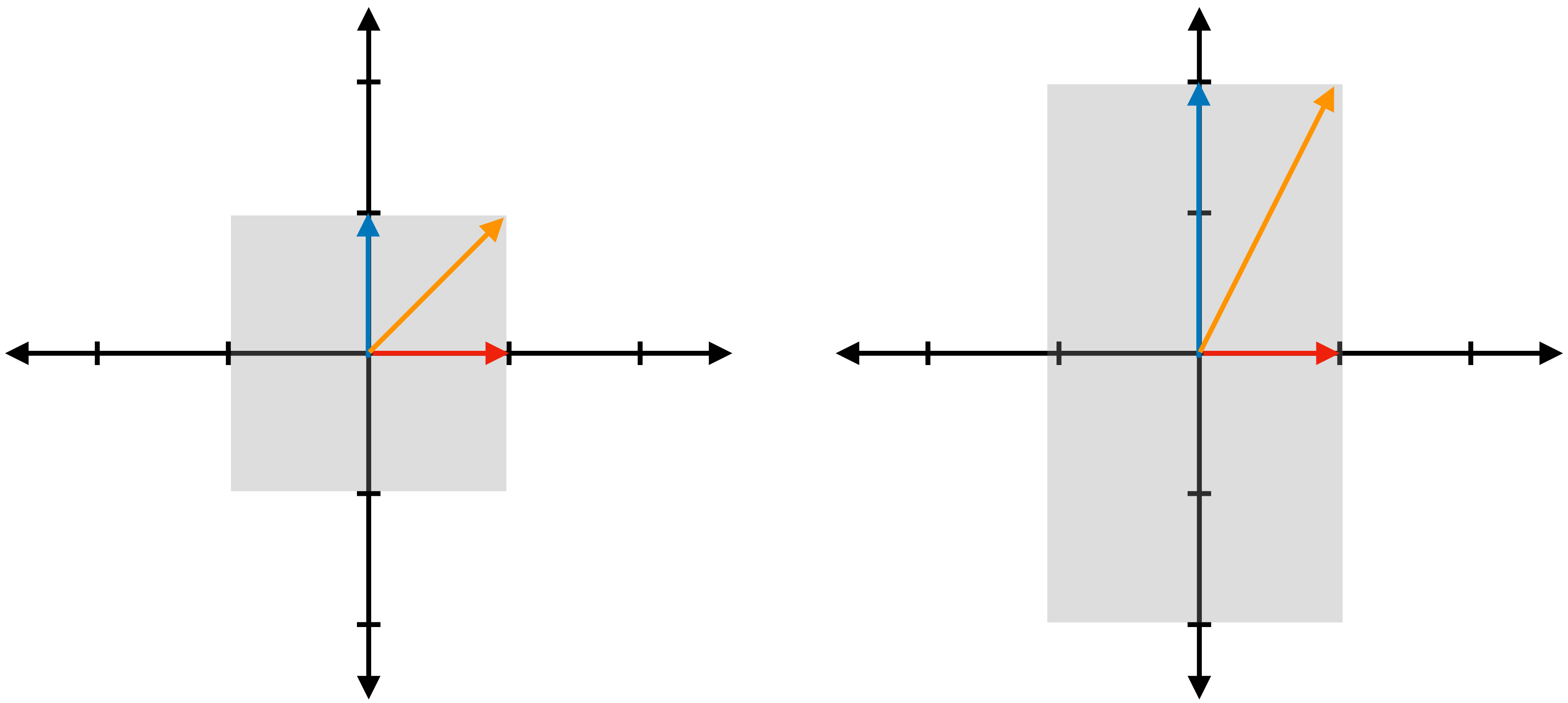
$$\lambda^2 - Tr(A)\lambda + det(A) = 0$$

Characteristic
Polynomial

Finding Eigenvectors

1. Find all eigenvalues λ
2. For each λ , find $v \in N(A - \lambda I)$
   - Find a vector v that is in the null space of the matrix $(A - \lambda I)$

Example 1



$$a = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0$$

$$(1-\lambda)(2-\lambda) = 0$$

$$\lambda = 1, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 \end{bmatrix} = 0, \quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

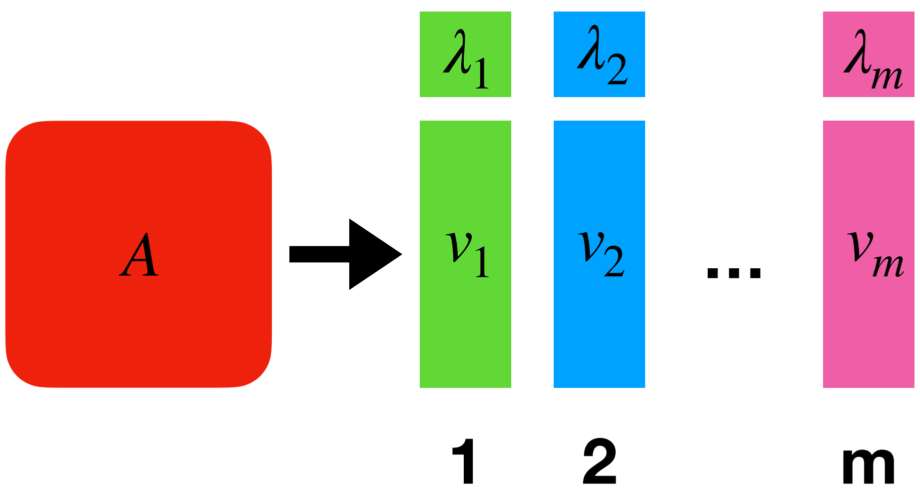
$$\lambda = 2, \quad \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ 0 \end{bmatrix} = 0, \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Find all Eigenvalues & Eigenvectors

In Class Exercise

$$\begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}$$

Diagonalization



$$
\begin{aligned}
Av_1 &= \lambda_1 v_1 \\
Av_2 &= \lambda_2 v_2 \\
&\vdots \\
Av_m &= \lambda_m v_m
\end{aligned}
$$

$$
\begin{aligned}
Av_1 &= \lambda_1 v_1 \\
Av_2 &= \lambda_2 v_2 \\
Av_3 &= \lambda_3 v_3
\end{aligned}
\longrightarrow
\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}
\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}
=
\begin{bmatrix} v_{11}\lambda_1 & v_{12}\lambda_2 & v_{13}\lambda_3 \\ v_{21}\lambda_1 & v_{22}\lambda_2 & v_{23}\lambda_3 \\ v_{31}\lambda_1 & v_{32}\lambda_2 & v_{33}\lambda_3 \end{bmatrix}
\longrightarrow AV = V\Lambda
$$

$$AV = V\Lambda \longrightarrow \boxed{A = V\Lambda V^{-1}}$$

Finding a set of basis vectors V such that the original matrix A is diagonal in that basis space, assuming that the columns in V are linearly independent.

Diagonalization

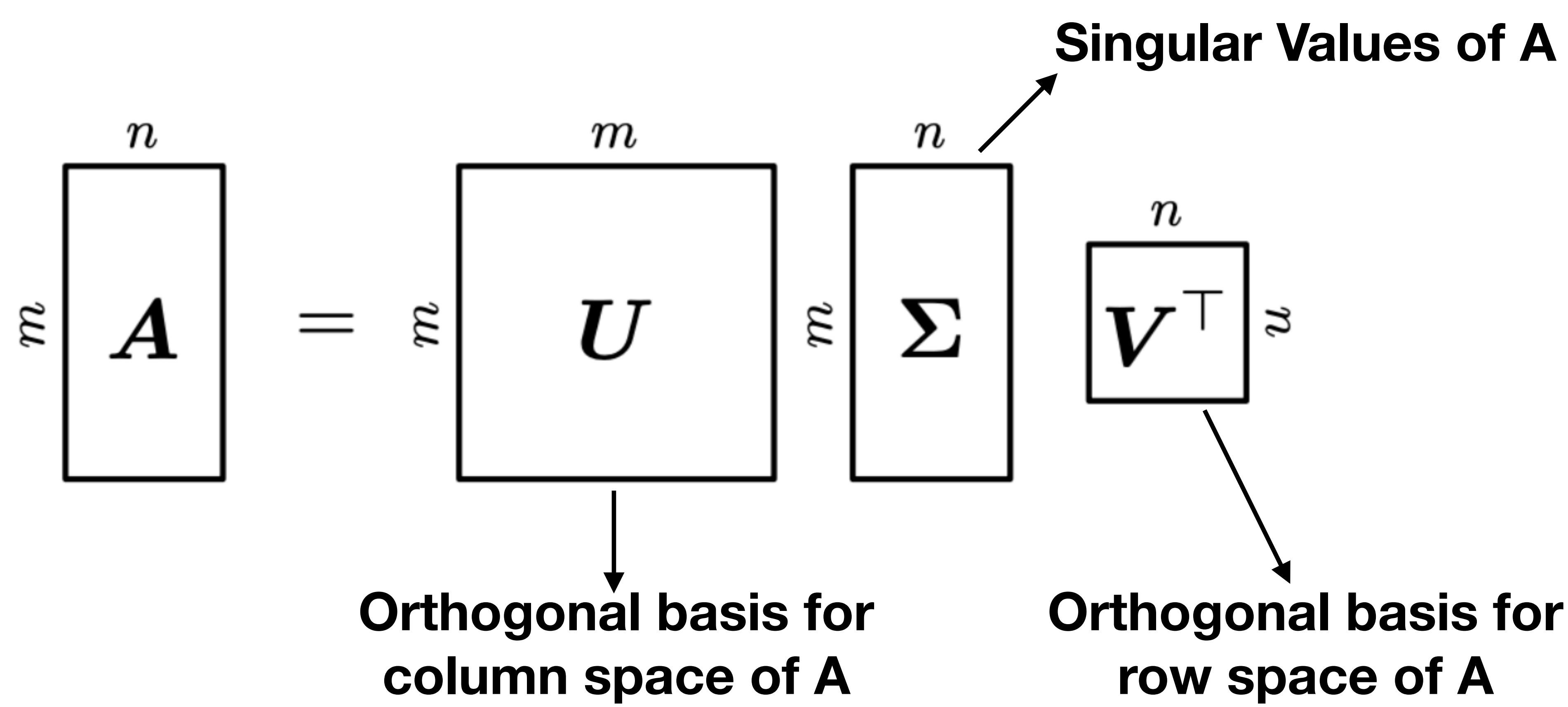
Example

$$A = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$$

Singular Value Decomposition

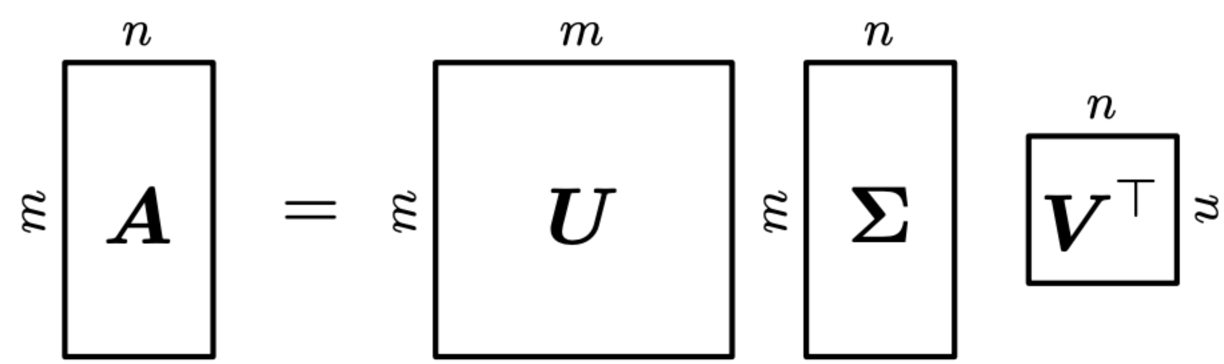
SVD

The goal of SVD is to decompose a matrix A as the product of 3 other matrices $A = U\Sigma V^T$, where matrix V and U are orthogonal matrices and Σ is a diagonal matrix.



SVD Theorem

Theorem 4.22 (SVD Theorem). *Let* $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ *be a rectangular matrix of rank* $r \in [0, \min(m,n)]$*. The SVD of* $\boldsymbol{A}$ *is a decomposition of the form*



(4.64)

with an orthogonal matrix $\boldsymbol{U} \in \mathbb{R}^{m \times m}$ *with column vectors* $\boldsymbol{u}_i$*,* $i = 1, \ldots, m$*, and an orthogonal matrix* $\boldsymbol{V} \in \mathbb{R}^{n \times n}$ *with column vectors* $\boldsymbol{v}_j$*,* $j = 1, \ldots, n$*. Moreover,* $\boldsymbol{\Sigma}$ *is an* $m \times n$ *matrix with* $\Sigma_{ii} = \sigma_i \geqslant 0$ *and* $\Sigma_{ij} = 0,\ i \neq j$*.*

How to compute SVD?

$$A = U\Sigma V^T$$

$$A^TA = (U\Sigma V^T)^T U\Sigma V^T$$

$$A^TA = V\Sigma^T U^T U \Sigma V^T$$

$$A^TA = V\Sigma^T I \Sigma V^T$$

Since **V** is Column Orthonormal
$V^T = V^{-1}$

$$A^TA = V\Sigma^T \Sigma V^T$$

$$A^TA = V\Sigma^2 V^T$$

To find V compute eigen-decomposition of A^TA where V will be the eigenvectors of A^TA and Σ^2 are the eigenvalues of A^TA

$$AA^T = U\Sigma^2 U^T$$

To find U compute eigen-decomposition of AA^T where U will be the eigenvectors of AA^T and Σ^2 are the eigenvalues of AA^T

Compute SVD Exercise

Compute SVD for matrix A:

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

SVD in action

User-Movie Rating

				
John 	5	1	3	5
Tom 	?	?	?	2
Alice 	4	?	3	?

Image by: Di Noia, Tommaso & Ostuni, Vito. (2015). Recommender Systems and Linked Open Data. 10.1007/978-3-319-21768-0_4.

User Movie Rating

- Overfit representation of user tases

Alice

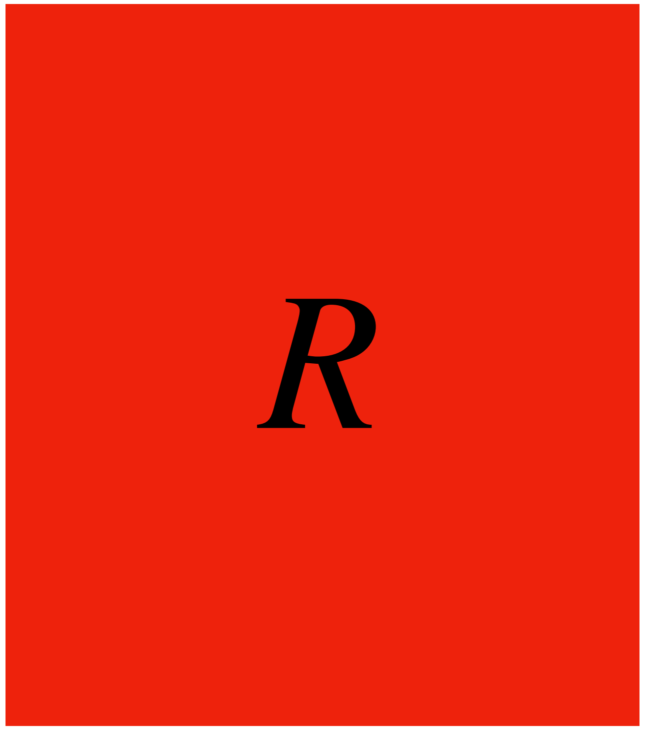
John

- Computational Complexity, potentially poor results

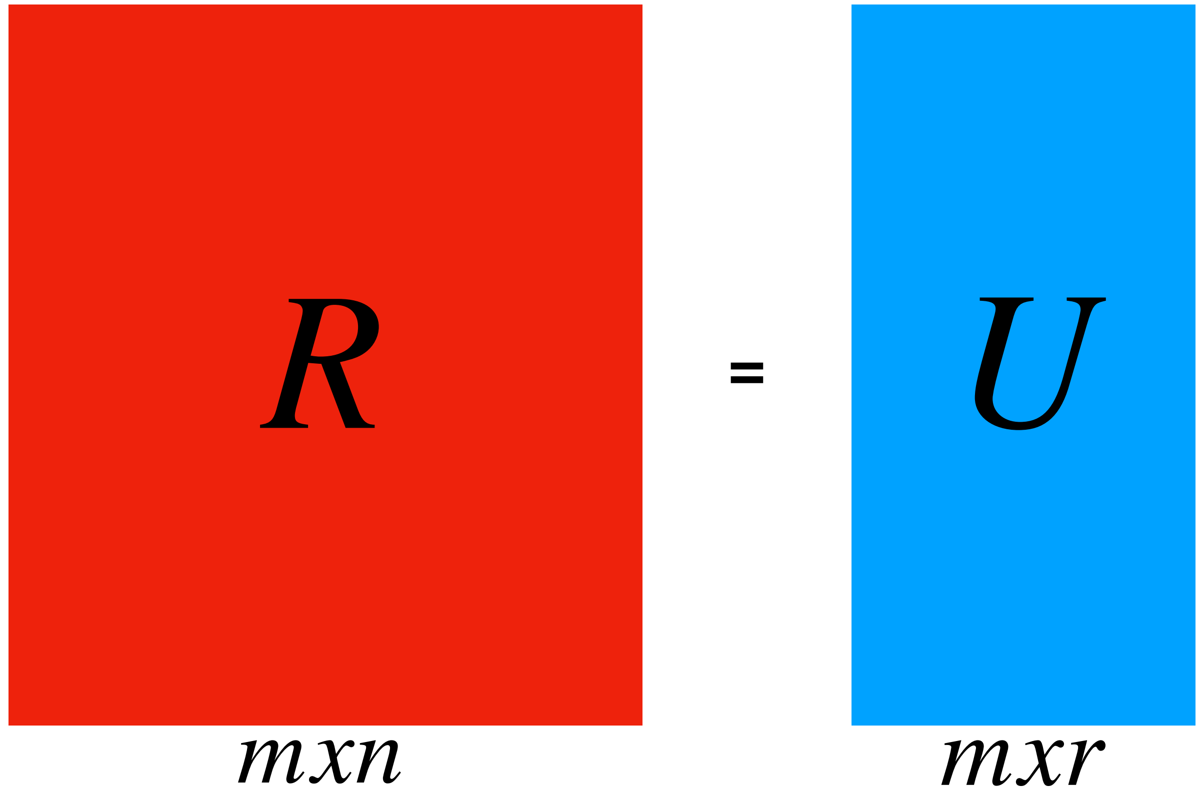
- **Goal**: A more compact representation of users tastes and items descriptions
  - Represent our tastes not in terms of the products we like and dislike, but in terms of higher level attributes

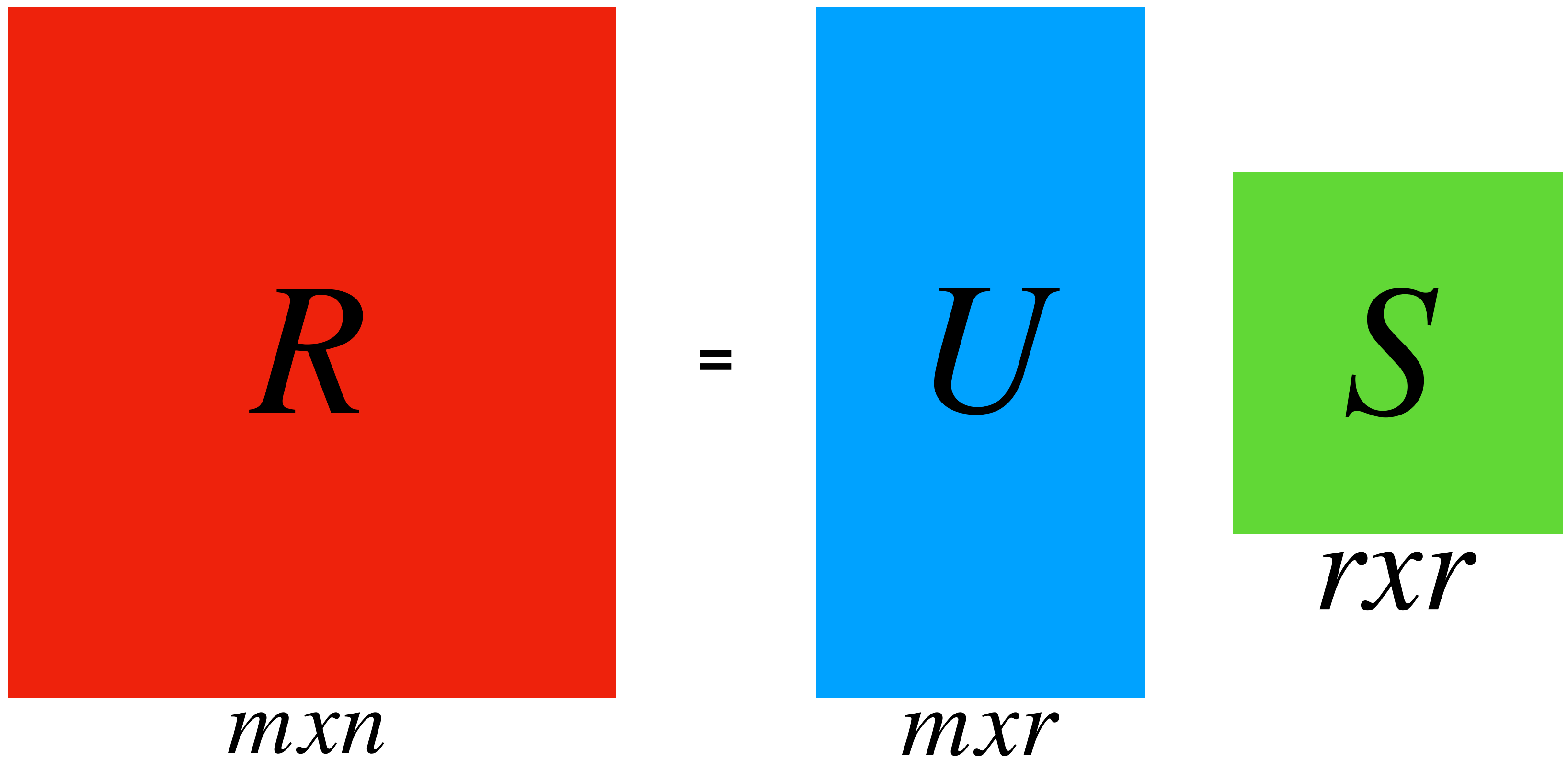
How to create these representations?

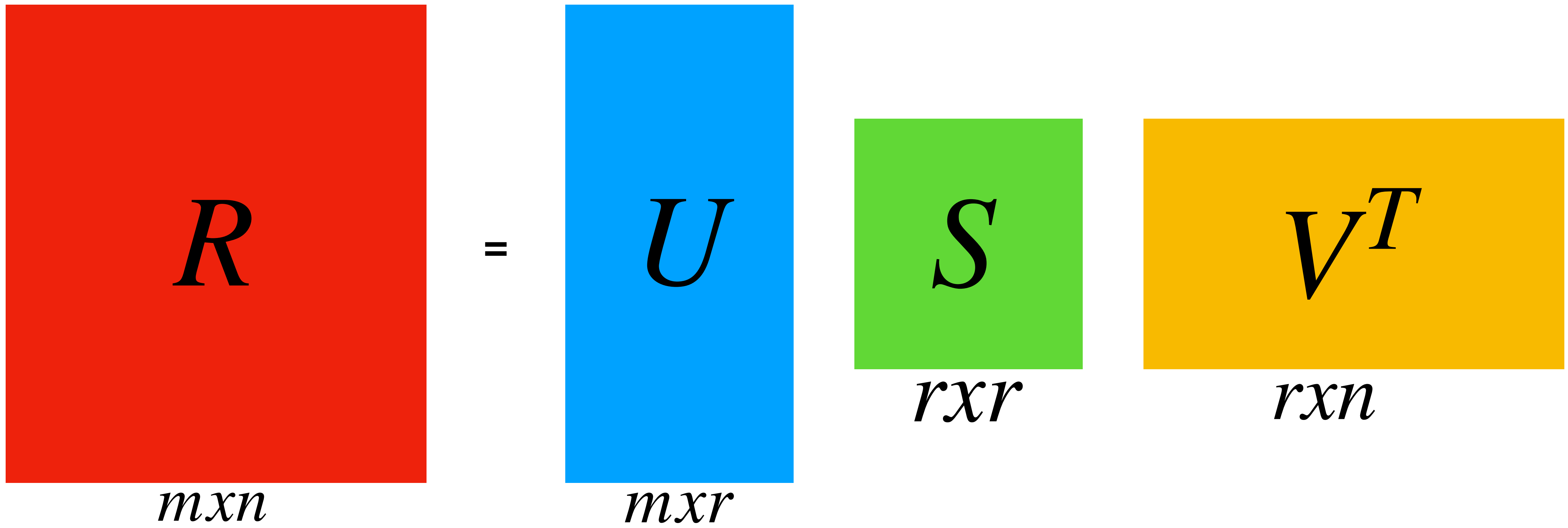
- Singular Value Decomposition (SVD)
- Intuitive Description: Reduce space to a smaller *taste space* that is compact and robust

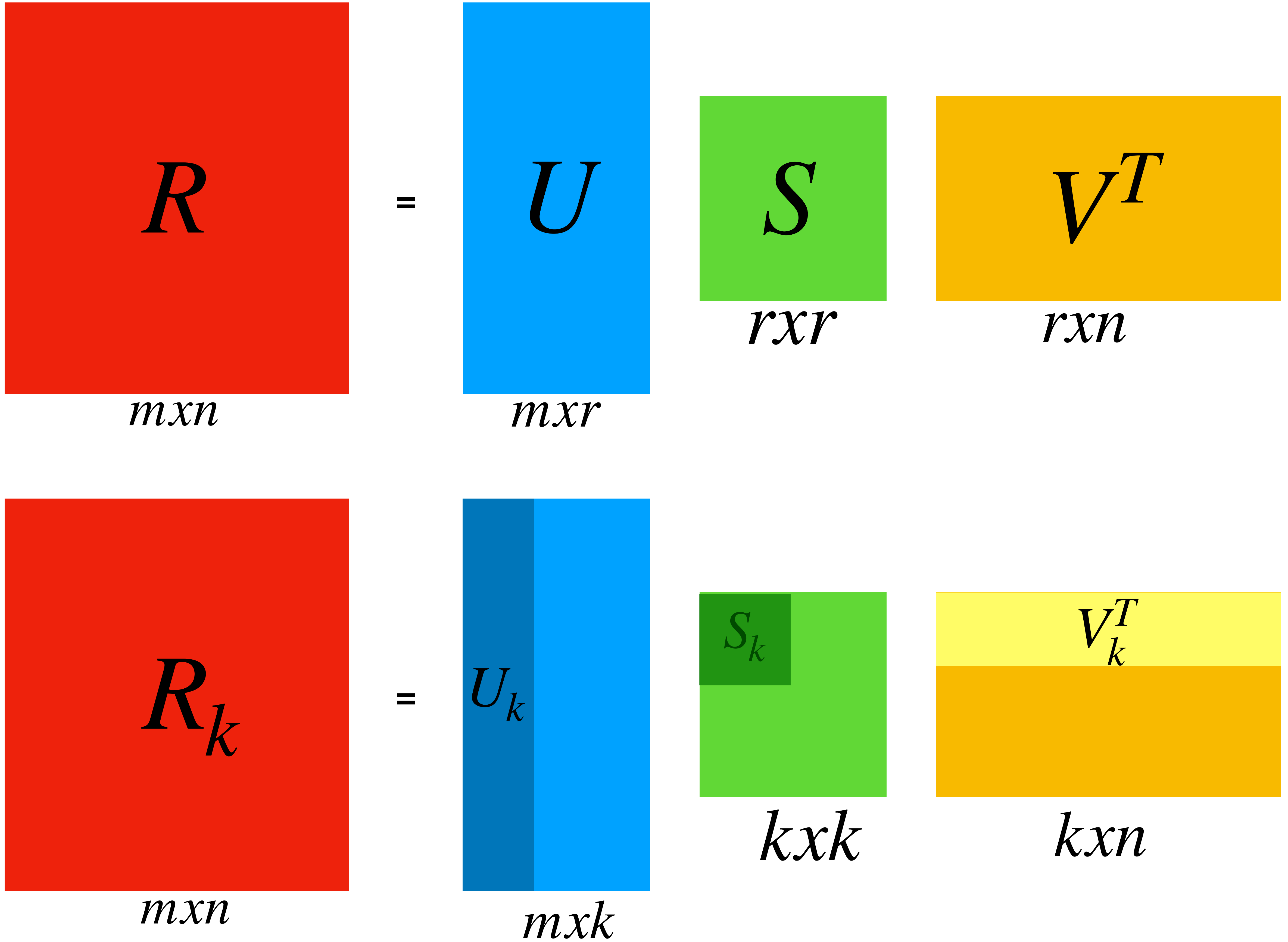


mxn

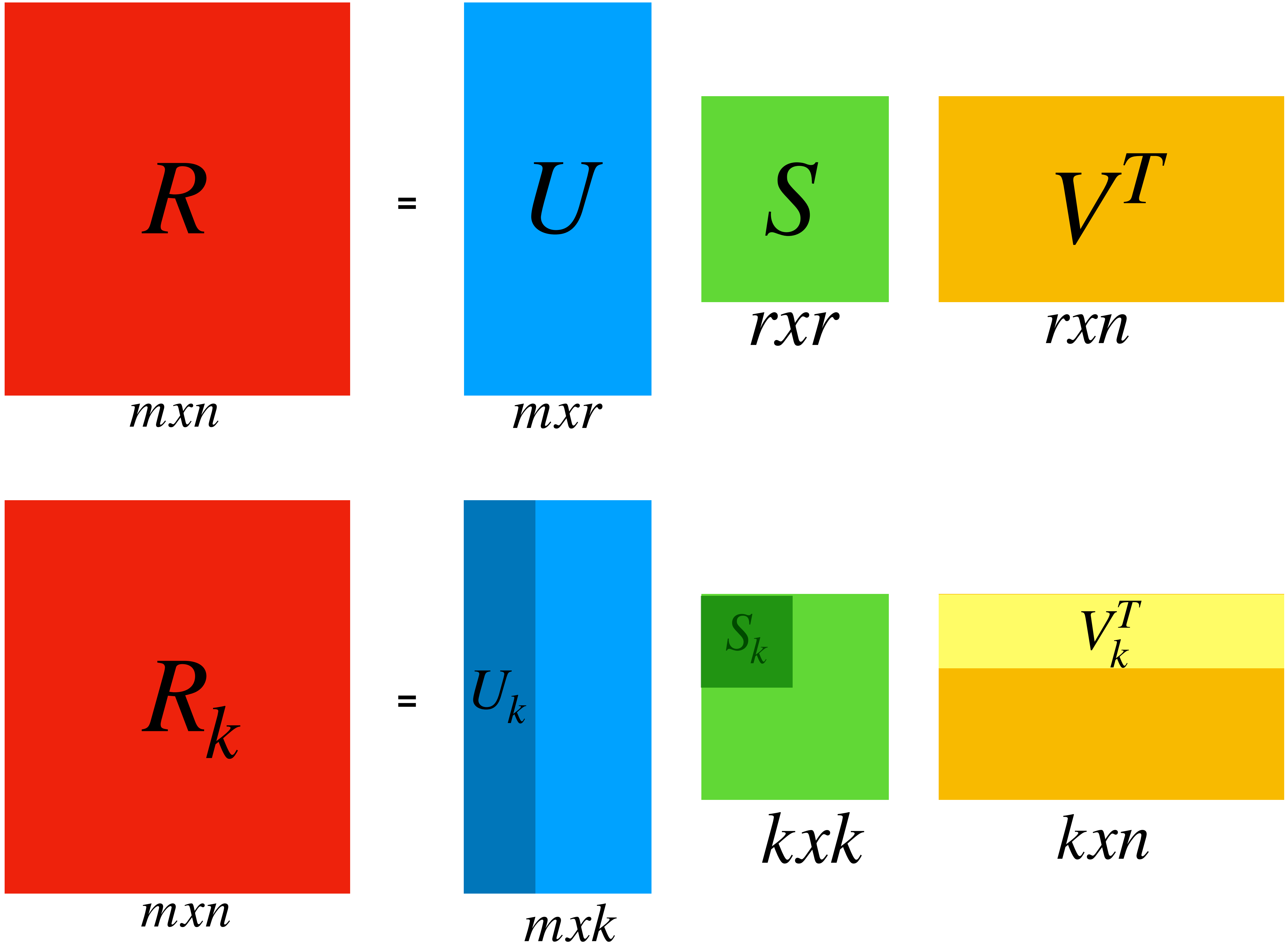

R
=
U
mxn
mxr


R
=
U
S
rxr
mxn
mxr


R
=
U
S
V^T
rxr
rxn
mxn
mxr



R
=
U
S
V^T
rxr
rxn
mxn
mxr
R_k
=
U_k
S_k
V_k^T
kxk
kxn
mxn
mxk



The reconstructed matrix $R_k = U_k S_k V_k^T$

is the closest $rank - k$ matrix to original matrix R.

Singular Value Decomposition

$$R = USV^T$$

- R *is* $m \times n$ *rating matrix*
- U *is* $m \times k$ *user feature affinity matrix*
- V *is* $n \times k$ *item feature relevance matrix*
- S *is* $k \times k$ *diagonal feature weight matrix*
- *Exists for any real* R

Rating Matrix

	The Notebook	Captain America	Seven	Forrest Gump
User 1	5	-	-	5
User 2	-	5	-	3
...				
User m	1	5	5	-

User Feature Matrix

	Drama	Thriller	Action	Fantasy
User 1	0.7	0.5	0.1	0.1
User 2	0	0.05	0.75	0.2
User m	0.4	0.3	0.2	0.1

Feature Diagonal

	Drama	Thriller	Action	Fantasy
Drama	0.4	0	0	0
Thriller	0	0.3	0	0
Action	0	0	0.27	0
Fantasy	0	0	0	0.05

Movie Feature Matrix

	Drama	Thriller	Action	Fantasy
The notebook	0.9	0	0	0
Captain America	0	0.1	0.7	0.9
Seven	0.5	0.5	0.1	0
Forrest Gump	0.98	0	0	0.03

Rating Matrix

	The Notebook	Captain America	Seven	Forrest Gump
User 1	5	???	-	5
User 2	-	5	-	3
...				
User m	1	5	5	-

User Feature Matrix

	Drama	Thriller	Action	Fantasy
User 1	0.7	0.2	0.1	0
User 2	0	0.05	0.75	0.2
User m	0.4	0.3	0.2	0.1

Feature Diagonal

	Drama	Thriller	Action	Fantasy
Drama	0.3	0	0	0
Thriller	0	0.27	0	0
Action	0	0	0.4	0
Fantasy	0	0	0	0.05

Movie Feature Matrix

	Drama	Thriller	Action	Fantasy
The notebook	1.0	0	0	0
Captain America	0	0.1	0.7	0.9
Seven	0.5	0.5	0.1	0
Forrest Gump	0.98	0	0	0.03

$$R(User_1,\ Captain\ America) = U_{User_1} S V^T_{Captain\ America}$$

$$R(User_1,\ Captain\ America) = 0.0344$$

Questions???