CSCE 633: Machine Learning

Lecture 28: Neural Networks: Backpropagation and Some Applications

Texas A&M University



Multilayer Perceptron: Representation

- Input: $\mathbf{x} \in \mathbb{R}^D$
- Output:

$$y \in \{0,1\}$$
 or $y \in \{1,\ldots,K\}$ (classification) $y \in \mathbb{R}$ or $y \in \mathbb{R}^K$ (regression)

- Training data: $\mathcal{D}^{train} = \{(\mathbf{x_1}, y_1), \dots, (\mathbf{x_N}, y_N)\}$
- Model: h_{W,b}(x)
 represented through forward propagation (see previous slides)
- Model parameters: weights $W^{(1)}, \ldots, W^{(L)}$ and biases $b^{(1)}, \ldots, b^{(L)}$

Multilayer Perceptron: Evaluation criterion

$$J(\mathbf{W}, \mathbf{b}, \mathcal{D}^{train}) = \frac{1}{2} ||h_{\mathbf{W}, \mathbf{b}}(\mathbf{x}) - y||_2^2 \text{ (regression)}$$

$$J(\mathbf{W}, \mathbf{b}, \mathcal{D}^{train}) = y \log h_{\mathbf{W}, \mathbf{b}}(\mathbf{x}) + (1 - y) \log(1 - h_{\mathbf{W}, \mathbf{b}}(\mathbf{x})) \text{ (classification)}$$





Multilayer Perceptron: Evaluation criterion

Regression

$$J(\mathbf{W}, \mathbf{b}) = \frac{1}{N} \sum_{n=1}^{M} \frac{1}{2} \|h_{\mathbf{W}, \mathbf{b}}(\mathbf{x}_n) - y_n\|_2^2 + \frac{\lambda}{2} \sum_{l=1}^{L-1} \sum_{i=1}^{s_l} \sum_{j=1}^{s_{l+1}} (W_{ji}^{(l)})^2$$

Classification

$$J(\mathbf{W}, \mathbf{b}) = \frac{1}{N} \sum_{n=1}^{M} (y_n \log h_{\mathbf{W}, \mathbf{b}}(\mathbf{x_n}) + (1 - y_n) \log(1 - h_{\mathbf{W}, \mathbf{b}}(\mathbf{x_n})))$$

$$+ \frac{\lambda}{2} \sum_{l=1}^{L-1} \sum_{i=1}^{s_l} \sum_{j=1}^{s_{l+1}} (W_{ji}^{(l)})^2$$

We will perform gradient descent



Gradient descent for regression

$$J(\mathbf{W}, \mathbf{b}) = \frac{1}{N} \sum_{n=1}^{M} \frac{1}{2} \|h_{\mathbf{W}, \mathbf{b}}(\mathbf{x}_n) - y_n\|_2^2 + \frac{\lambda}{2} \sum_{l=1}^{L-1} \sum_{i=1}^{s_l} \sum_{j=1}^{s_{l+1}} (W_{ji}^{(l)})^2$$

$$W_{ij}^{(I)} := W_{ij}^{(I)} - \alpha \frac{\vartheta J(\mathbf{W}, \mathbf{b})}{\vartheta W_{ij}^{(I)}}$$
$$b_i^{(I)} := b_i^{(I)} - \alpha \frac{\vartheta J(\mathbf{W}, \mathbf{b})}{\vartheta b_i^{(I)}}$$

Note: Initialize the parameters randomly → symmetry breaking

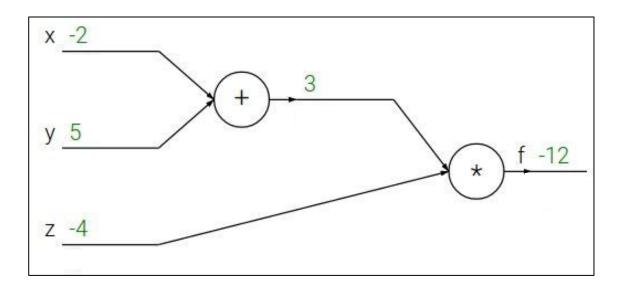
Use backpropagation to compute partial derivatives $\frac{\partial J(\mathbf{W}, \mathbf{b})}{\partial W_{ij}^{(l)}}$ and $\frac{\partial J(\mathbf{W}, \mathbf{b})}{\partial b_i^{(l)}}$



Backpropagation Example in Computational Graph¹

$$f(x,y,z) = (x+y)z$$

e.g.
$$x = -2$$
, $y = 5$, $z = -4$





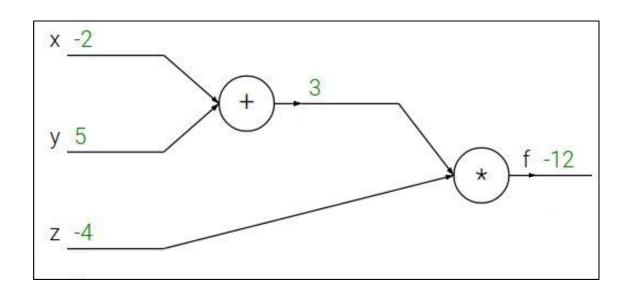


$$f(x,y,z)=(x+y)z$$

e.g.
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$$q=x+y \qquad rac{\partial q}{\partial x}=1, rac{\partial q}{\partial y}=1$$

$$f=qz \qquad \quad rac{\partial f}{\partial q}=z, rac{\partial f}{\partial z}=q$$



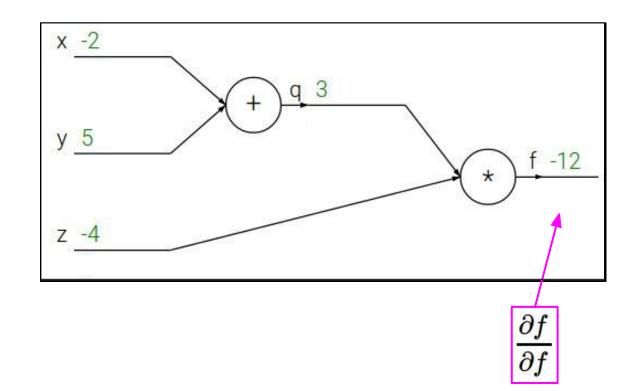


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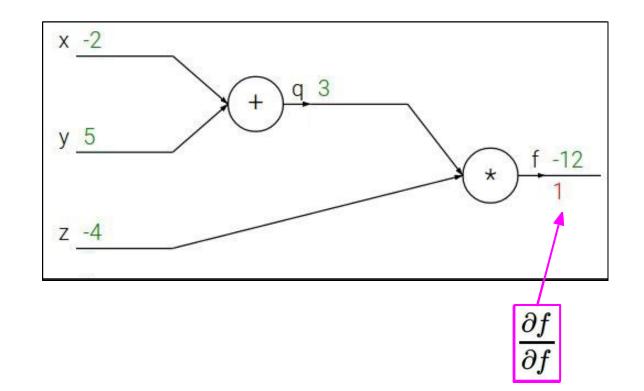


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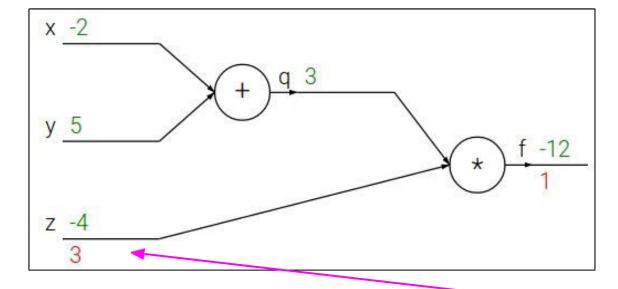


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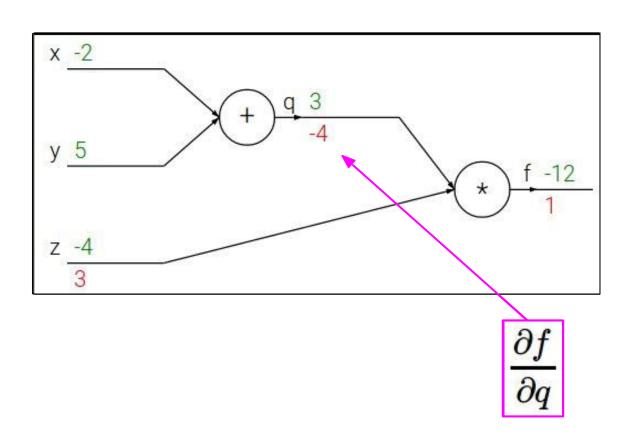
 $\frac{\partial f}{\partial z}$

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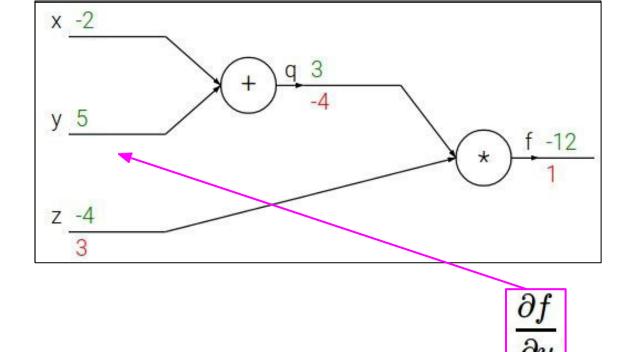


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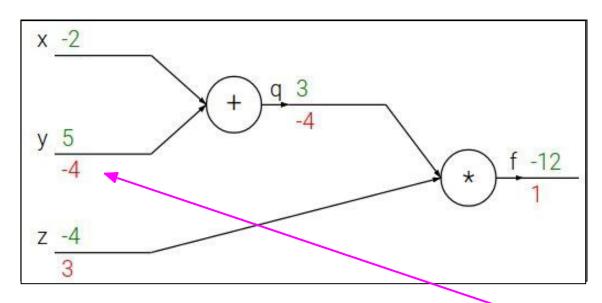
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Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$



Chain rule:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial q} \frac{\partial q}{\partial y}$$

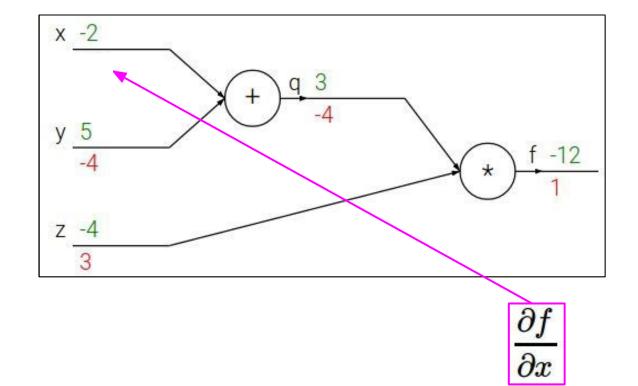


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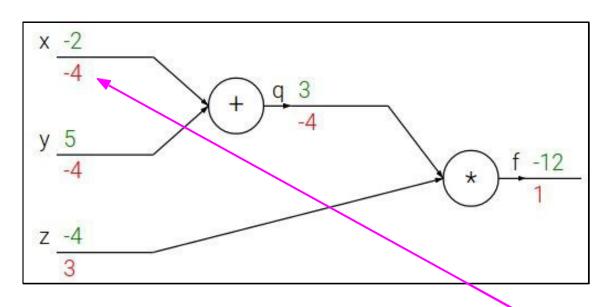
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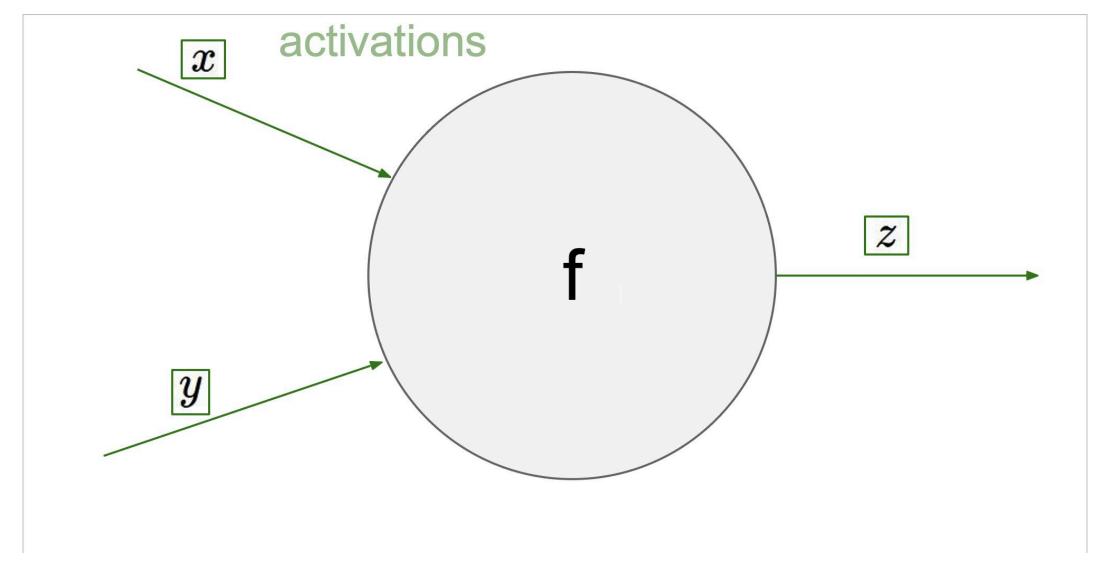
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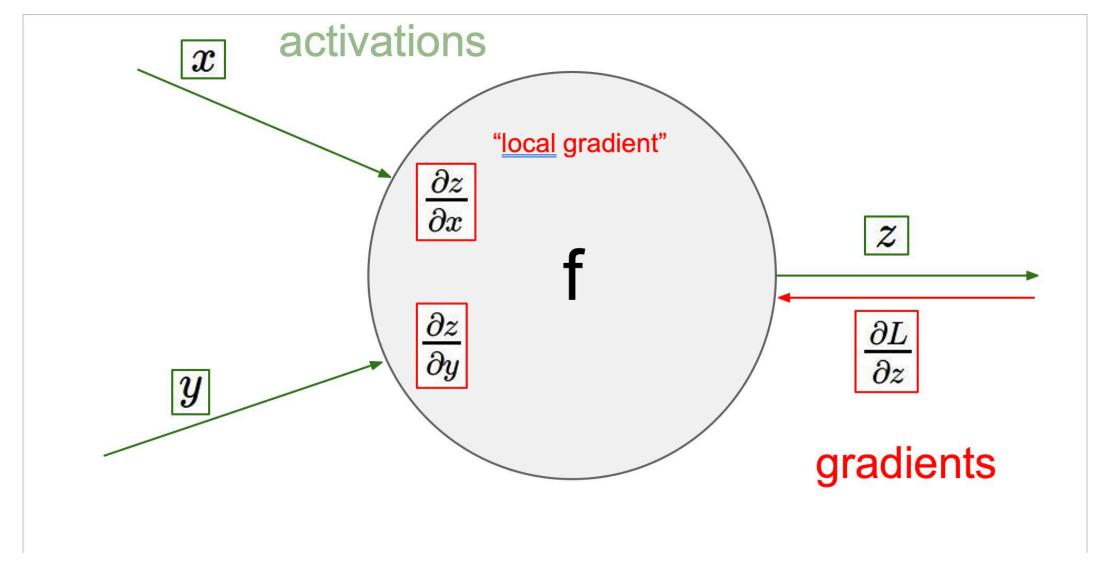
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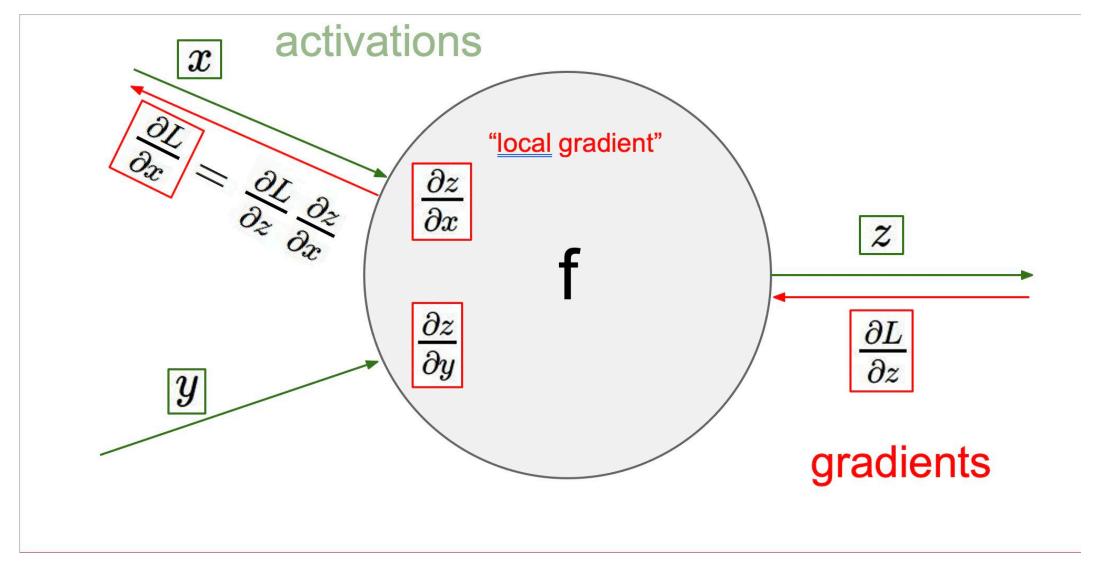
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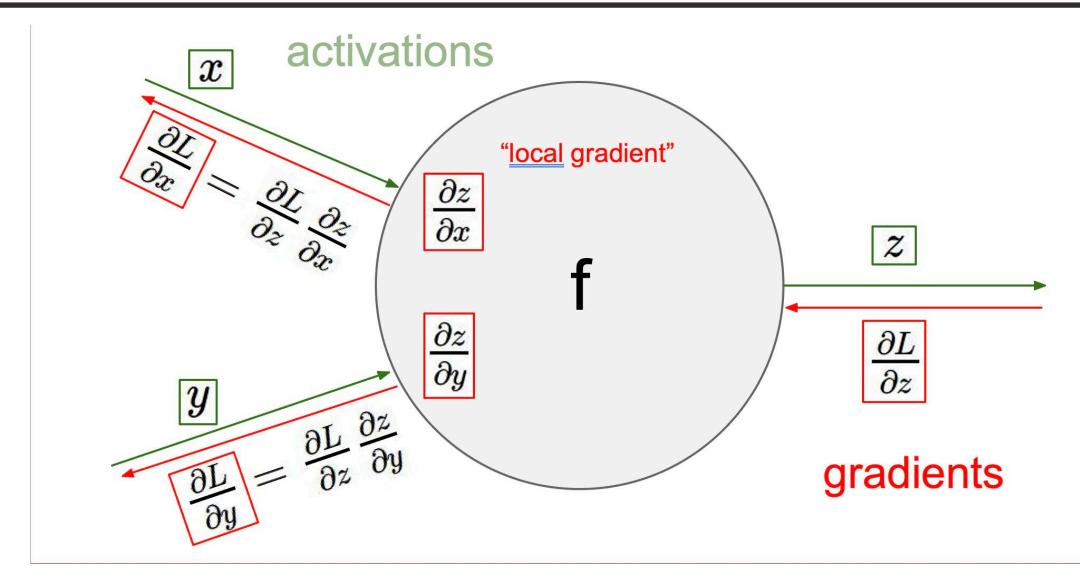
 $\frac{\partial f}{\partial x}$













Implementation

For each node i in output layer L

•
$$\delta_i^{(L)} = (\alpha_i^{(L)} - y_n)f'(z_i^{(L)})$$

• For each node *i* in layer $I = L - 1, L - 2, \dots, 2$

• Hidden nodes:
$$\delta_i^{(l)} = \left(\sum_{j=1}^{s_{l+1}} W_{ji}^{(l)} \delta_j^{(l+1)}\right) f'(z_i^{(l)})$$

Compute the desired partial derivatives as:

$$\frac{\partial J(\mathbf{W}, \mathbf{b})}{\partial W_{ij}^{(I)}} = \alpha_j^{(I)} \delta_i^{(I+1)}$$
$$\frac{\partial J(\mathbf{W}, \mathbf{b})}{\partial b_i^{(I)}} = \delta_i^{(I+1)}$$

• Update the weights as:

$$W_{ij}^{(I)} := W_{ij}^{(I)} - \alpha \frac{\vartheta J(\mathbf{W}, \mathbf{b})}{\vartheta W_{ij}^{(I)}}$$
$$b_i^{(I)} := b_i^{(I)} - \alpha \frac{\vartheta J(\mathbf{W}, \mathbf{b})}{\vartheta b_i^{(I)}}$$



Gradient Descent (GD)

Algorithm 1 Batch Gradient Descent at Iteration k

Require: Learning rate ϵ_k

Require: Initial Parameter θ

1: while stopping criteria not met do

2: Compute gradient estimate over N examples:

3:
$$\hat{\mathbf{g}} \leftarrow +\frac{1}{N} \nabla_{\theta} \sum_{i} L(f(\mathbf{x}^{(i)}; \theta), \mathbf{y}^{(i)})$$

4: Apply Update: $\theta \leftarrow \theta - \epsilon \hat{\mathbf{g}}$

5: end while

Positive: Gradient estimates are stable

 Negative: Need to compute gradients over the entire training for one update





Stochastic Gradient Descent (SGD)

Algorithm 2 Stochastic Gradient Descent at Iteration k

Require: Learning rate ϵ_k

Require: Initial Parameter θ

- 1: while stopping criteria not met do
- 2: Sample example $(\mathbf{x}^{(i)}, \mathbf{y}^{(i)})$ from training set
- 3: Compute gradient estimate:
- 4: $\hat{\mathbf{g}} \leftarrow +\nabla_{\theta} L(f(\mathbf{x}^{(i)}; \theta), \mathbf{y}^{(i)})$
- 5: Apply Update: $\theta \leftarrow \theta \epsilon \hat{\mathbf{g}}$
- 6: end while
 - \bullet ϵ_k is learning rate at step k
 - Sufficient condition to guarantee convergence:

$$\sum_{k=1}^{\infty} \epsilon_k = \infty \text{ and } \sum_{k=1}^{\infty} \epsilon_k^2 < \infty$$



• GD versus SGD

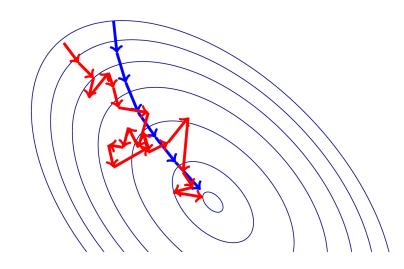
Batch Gradient Descent:

$$\hat{\mathbf{g}} \leftarrow +\frac{1}{N} \nabla_{\theta} \sum_{i} L(f(\mathbf{x}^{(i)}; \theta), \mathbf{y}^{(i)})$$
$$\theta \leftarrow \theta - \epsilon \hat{\mathbf{g}}$$

SGD:

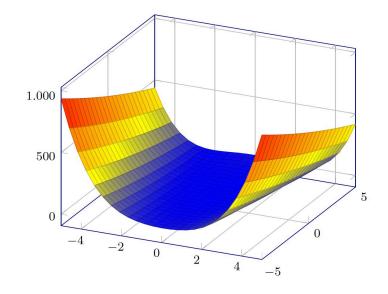
$$\hat{\mathbf{g}} \leftarrow +\nabla_{\theta} L(f(\mathbf{x}^{(i)}; \theta), \mathbf{y}^{(i)})$$

$$\theta \leftarrow \theta - \epsilon \hat{\mathbf{g}}$$





- Momentum
 - The Momentum method is a method to accelerate learning using SGD.
 - In particular SGD suffers in the following scenarios:
 - Error surface has high curvature
 - Small but consistent gradients
 - Noisy gradients



 Gradient Descent would move quickly down the walls, but very slowly through the valley floor



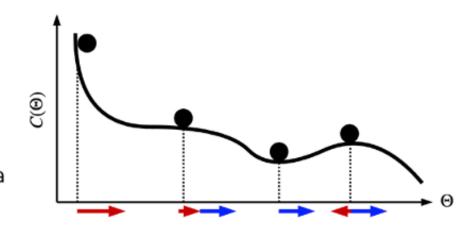


Update rule in SGD:

$$\Theta^{(t+1)} \leftarrow \Theta^{(t)} - \eta \mathbf{g}^{(t)}$$

where $oldsymbol{g}^{(t)} =
abla_{\Theta} C(\Theta^{(t)})$

 Gets stuck in local minima or saddle points

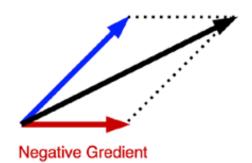


Momentum: make the same movement $v^{(t)}$ in the last iteration, corrected by negative gradient:

$$\mathbf{v}^{(t+1)} \leftarrow \lambda \mathbf{v}^{(t)} - (1-\lambda)\mathbf{g}^{(t)}$$

$$\Theta^{(t+1)} \leftarrow \Theta^{(t)} + \eta v^{(t+1)}$$

 $\mathbf{v}^{(t)}$ is a moving average of $-\mathbf{g}^{(t)}$





• Popular Solver Examples: AdGrad, RMSProp, Adam

$$\begin{array}{c} \mathsf{SGD}\colon\theta\leftarrow\theta-\epsilon\hat{\mathbf{g}}\\ \mathsf{Momentum}\colon\,\mathbf{v}\leftarrow\alpha\mathbf{v}-\epsilon\hat{\mathbf{g}}\;\mathsf{then}\;\theta\leftarrow\theta+\mathbf{v}\\ \mathsf{Nesterov}\colon\,\mathbf{v}\leftarrow\alpha\mathbf{v}-\epsilon\nabla_{\theta}\Bigg(L(f(\mathbf{x}^{(i)};\theta+\alpha\mathbf{v}),\mathbf{y}^{(i)})\Bigg)\;\;\mathsf{then}\;\theta\leftarrow\theta+\mathbf{v}\\ \mathsf{AdaGrad}\colon\,\mathbf{r}\leftarrow\mathbf{r}+\mathbf{g}\odot\mathbf{g}\;\;\mathsf{then}\;\Delta\theta-\leftarrow\frac{\epsilon}{\delta+\sqrt{\mathbf{r}}}\odot\mathbf{g}\;\;\mathsf{then}\;\theta\leftarrow\theta+\Delta\theta\\ \mathsf{RMSProp}\colon\,\mathbf{r}\leftarrow\rho\mathbf{r}+(1-\rho)\hat{\mathbf{g}}\odot\hat{\mathbf{g}}\;\;\mathsf{then}\;\Delta\theta\leftarrow-\frac{\epsilon}{\delta+\sqrt{\mathbf{r}}}\odot\hat{\mathbf{g}}\;\;\mathsf{then}\;\theta\leftarrow\theta+\Delta\theta\\ \mathsf{Adam}\colon\,\hat{\mathbf{s}}\leftarrow\frac{\mathbf{s}}{1-\rho_1^t},\hat{\mathbf{r}}\leftarrow\frac{\mathbf{r}}{1-\rho_2^t}\;\;\mathsf{then}\;\Delta\theta=-\epsilon\frac{\hat{\mathbf{s}}}{\sqrt{\hat{\mathbf{r}}}+\delta}\;\;\mathsf{then}\;\theta\leftarrow\theta+\Delta\theta \end{array}$$

AdaGrad

- Idea: Downscale a model parameter by square-root of sum of squares of all its historical values
- Parameters that have large partial derivative of the loss -> learning rates for them are rapidly declined
- Some interesting theoretical properties

Algorithm 4 AdaGrad

Require: Global Learning rate ϵ , Initial Parameter θ , δ

Initialize $\mathbf{r} = 0$

- 1: while stopping criteria not met do
- 2: Sample example $(\mathbf{x}^{(i)}, \mathbf{y}^{(i)})$ from training set
- 3: Compute gradient estimate: $\hat{\mathbf{g}} \leftarrow +\nabla_{\theta} L(f(\mathbf{x}^{(i)}; \theta), \mathbf{y}^{(i)})$
- 4: Accumulate: $\mathbf{r} \leftarrow \mathbf{r} + \hat{\mathbf{g}} \odot \hat{\mathbf{g}}$
- 5: Compute update: $\Delta \theta \leftarrow -\frac{\epsilon}{\delta + \sqrt{\mathbf{r}}} \odot \hat{\mathbf{g}}$
- 6: Apply Update: $\theta \leftarrow \theta + \Delta \theta$
- 7: end while





RMSProp

 AdaGrad might shrink the learning rate too aggressively, we can adapt it to perform better by accumulating an exponentially decaying average of the gradient

Algorithm 5 RMSProp

Require: Global Learning rate ϵ , decay parameter ρ , δ

Initialize $\mathbf{r} = 0$

- 1: while stopping criteria not met do
- 2: Sample example $(\mathbf{x}^{(i)}, \mathbf{y}^{(i)})$ from training set
- 3: Compute gradient estimate: $\hat{\mathbf{g}} \leftarrow +\nabla_{\theta} L(f(\mathbf{x}^{(i)}; \theta), \mathbf{y}^{(i)})$
- 4: Accumulate: $\mathbf{r} \leftarrow \rho \mathbf{r} + (1 \rho)\hat{\mathbf{g}} \odot \hat{\mathbf{g}}$
- 5: Compute update: $\Delta \theta \leftarrow -\frac{\epsilon}{\delta + \sqrt{\mathbf{r}}} \odot \hat{\mathbf{g}}$
- 6: Apply Update: $\theta \leftarrow \theta + \Delta \theta$
- 7: end while



- Adam
 - Adam is like RMSProp with Momentum but with bias correction terms for the first and second moments

Algorithm 7 RMSProp with Nesterov

Require: ϵ (set to 0.0001), decay rates ρ_1 (set to 0.9), ρ_2 (set to 0.9), θ , δ

Initialize moments variables $\mathbf{s}=0$ and $\mathbf{r}=0$, time step t=0

- 1: while stopping criteria not met do
- 2: Sample example $(\mathbf{x}^{(i)}, \mathbf{y}^{(i)})$ from training set
- Compute gradient estimate: $\hat{\mathbf{g}} \leftarrow + \nabla_{\theta} L(f(\mathbf{x}^{(i)}; \theta), \mathbf{y}^{(i)})$
- 4: $t \leftarrow t + 1$
- 5: Update: $\mathbf{s} \leftarrow \rho_1 \mathbf{s} + (1 \rho_1)\hat{\mathbf{g}}$
- 6: Update: $\mathbf{r} \leftarrow \rho_2 \mathbf{r} + (1 \rho_2) \hat{\mathbf{g}} \odot \hat{\mathbf{g}}$
- 7: Correct Biases: $\hat{\mathbf{s}} \leftarrow \frac{\mathbf{s}}{1-\rho_1^t}, \hat{\mathbf{r}} \leftarrow \frac{\mathbf{r}}{1-\rho_2^t}$
- 8: Compute Update: $\Delta \theta = -\epsilon \frac{\hat{\mathbf{s}}}{\sqrt{\hat{\mathbf{r}}} + \delta}$
- 9: Apply Update: $\theta \leftarrow \theta + \Delta \theta$
- 10: end while



Outline

- Perceptron
- Approximating linear functions
- Activation Function
- Backpropagation
- Optimization
- Neural Network Training and Design





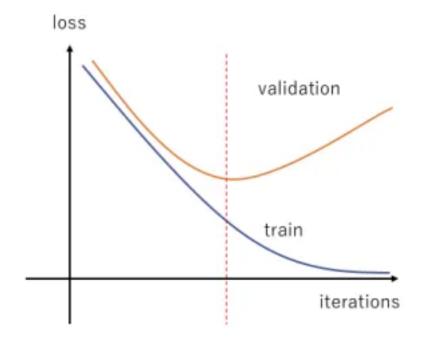
Minibatch

- Potential Problem: Gradient estimates can be very noisy
- Obvious Solution: Use larger mini-batches (In theory, growingly larger)

- Advantage: Computation time per update does not depend on number of training examples.
- This allows convergence on extremely large datasets

• "Large Scale Learning with Stochastic Gradient Descent", Leon Bottou.

Challenge: Overfitting

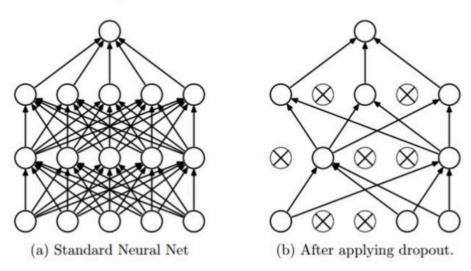




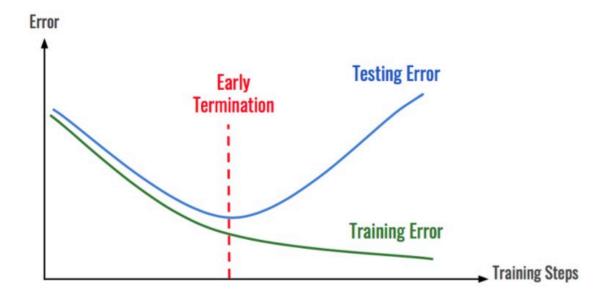
Dropout

How to avoid overfitting

- An alternative method that complements the above is dropout
- While training, dropout keeps a neuron active with some probability p (a hyperparameter), or sets it to zero otherwise



Early stop





Batch Normalization

- In ML, we assume future
 data will be drawn from
 same probability distribution
 as training data
- For a hidden layer, after training, the earlier layers have new weights and hence may generate a new distribution for the next hidden layer
- We want to reduce this internal covariate shift for the benefit of later layers

Input: Values of x over a mini-batch: $\mathcal{B} = \{x_{1...m}\}$;

Parameters to be learned: γ , β Output: $\{y_i = \mathrm{BN}_{\gamma,\beta}(x_i)\}$ $\mu_{\mathcal{B}} \leftarrow \frac{1}{m} \sum_{i=1}^m x_i \qquad \text{// mini-batch mean}$ $\sigma_{\mathcal{B}}^2 \leftarrow \frac{1}{m} \sum_{i=1}^m (x_i - \mu_{\mathcal{B}})^2 \qquad \text{// mini-batch variance}$ $\widehat{x}_i \leftarrow \frac{x_i - \mu_{\mathcal{B}}}{\sqrt{\sigma_{\mathcal{B}}^2 + \epsilon}} \qquad \text{// normalize}$ $y_i \leftarrow \gamma \widehat{x}_i + \beta \equiv \mathrm{BN}_{\gamma,\beta}(x_i) \qquad \text{// scale and shift}$

Algorithm 1: Batch Normalizing Transform, applied to activation x over a mini-batch.



Batch Normalization

- First three steps are just like standardization of input data, but with respect to only the data in mini-batch.
- We can take derivative and incorporate the learning of last step parameters into backpropagation.
- Note last step can completely un-do previous 3 steps
- But even if so, this un-doing is driven by the later layers, not the earlier layers; later layers get to "choose" whether they want standard normal inputs or not

Neural Network Design

How to chose the number of layers and nodes

- No general rule of thumb, this depends on:
 - Amount of training data available
 - Complexity of the function that is trying to be learned
 - Number of input and output nodes
- If data is linearly separable, you don't need any hidden layers at all
- Start with one layer and hidden nodes proportional to input size
- Gradually increase





Hyperparameter Tuning

- Learning rate: how much to update the weight during optimization
- Number of epochs: number of times the entire training set pass through the neural network
- Batch size: the number of times the entire training set pass through the neural network
- Activation function: the function that introduces non-linearity to the model (e.g. sigmoid, tanh, ReLU, etc.)
- Number of hidden layers and units
- Weight initialization: Uniform distribution usually works well
- Dropout for regularization: probability of dropping a unit
 We can perform grid or randomized search over all parameters





Challenge

High memory requirements

- Memory is used to store input data, weight parameters and activations as an input propagates through the network
- Activations from a forward pass must be retained until they can be used to calculate the error gradients in the backwards pass
- Example: 50-layer neural network
 - 26 million weight parameters, 16 million activations in the forward pass
 - 168MB memory (assuming 32-bit float)

Parallelize computations with GPU (graphics processing units)





Challenge

Backpropagation does not work well

- Deep networks trained with backpropagation (without unsupervised pretraining) perform worse than shallow networks
- Gradient is progressively getting more dilute
 - Weight correction is minimal after moving back a couple of layers
- High risk of getting "stuck" to local minima
- In practice, a small portion of data is labelled

Perform pretraining to mitigate this issue

	train.	valid.	test
DBN, unsupervised pre-training	0%	1.2%	1.2%
Deep net, auto-associator pre-training	0%	1.4%	1.4%
Deep net, supervised pre-training	0%	1.7%	2.0%
Deep net, no pre-training	.004%	2.1%	2.4%
Shallow net, no pre-training	.004%	1.8%	1.9%

(Bengio et al., NIPS 2007)





Unsupervised Pretraining

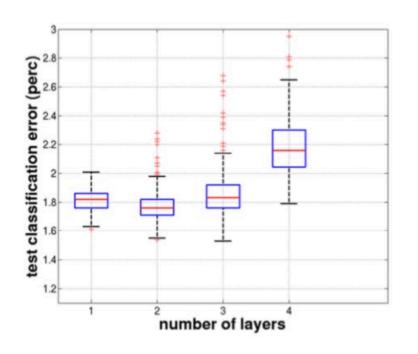
- This idea came into play when research studies found that a DNN trained on a particular task (e.g. object recognition) can be applied on another domain (e.g. object subcategorization) giving state-of-the-art results
- 1st part: Greedy layer-wise unsupervised pre-training
 - Each layer is pre-trained with an unsupervised learning algorithm
 - Learning a nonlinear transformation that captures the main variations in its input (the output of the previous layer)
- 2nd part: Supervised fine-tuning
 - The deep architecture is fine-tuned with respect to a supervised training criterion with gradient-based optimization
- We will examine the deep belief networks and stacked autoencoders

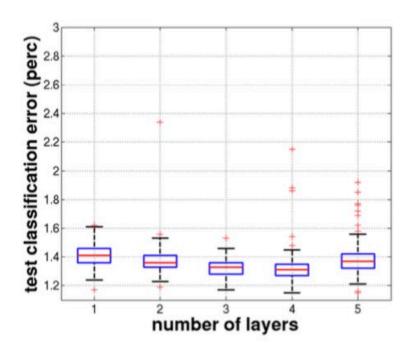
Unusual form of regularization: minimizing variance and introducing bias towards configurations of the parameter space that are useful for unsupervised learning





Unsupervised Pretraining





Without pre-training

With pre-training

[Source: Erhan et al., 2010]

