Homework 4

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1 Question 1 - Convolution Operation (30 points)

In this problem, we will use the convolution operation on the matrix using the 3x3 filter as shown below.

$$\begin{bmatrix} 0 & 2 & 4 & 1 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 2 & 4 & 1 & 0 & 1 \\ 2 & 0 & 5 & 2 & 2 \\ 0 & 1 & 3 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

Input Filter

Apply the convolution operation for all the following settings respectively, and write your answers in a LaTex-generated PDF file with the name FirstName LastName HW4.pdf

- Convolution with stride of 1
- Zero padding of 1 + convolution with stride of 1
- Zero padding of 2 + convolution with stride of 2
- Convolution with stride of 1 + max pooling of 3 with stride of 1
- Zero padding of 2 + convolution with stride of 1 + max pooling of 3 with stride of 1

1.1 Solution

Assume the input matrix is $A \in \mathbf{R}^{n\times n}$, the filter is $f \in \mathbf{R}^{k\times k}$, and the convolution output is $B \in \mathbf{R}^{m\times m}$. The output size m can be calculated as:

$$m = \lfloor \frac{n + 2p - k}{s} \rfloor + 1 \tag{1}$$

, where p is the padding size and s is the stride. We also denote $A[i_1 : i_2, j_1 : j_2]$ as the region of A that includes rows i_1 through $i_2 - 1$ and columns j_1 through $j_2 - 1$, inclusive.

1.1.1 Convolution with a stride of 1

In here, we have $n = 5$, $p = 0$, $k = 3$, and $s = 1$. We can calculate m as:

$$m = \lfloor \frac{5 + 2 \times 0 - 3}{1} \rfloor + 1 = 3 \tag{2}$$

Therefore, we have $O \in \mathbf{R}^{3\times3}$.

$$O[1,1] = \sum \begin{bmatrix} 0 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 4 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = \sum \begin{bmatrix} 0\times1 & 2\times0 & 4\times-1 \\ 3\times1 & 1\times0 & 1\times-1 \\ 2\times1 & 4\times0 & 1\times-1 \end{bmatrix} = \sum \begin{bmatrix} 0 & 0 & -4 \\ 3 & 0 & -1 \\ 2 & 0 & -1 \end{bmatrix} = -1$$

$$O[1,2] = \sum \begin{bmatrix} 2 & 4 & 1 \\ 1 & 1 & 0 \\ 4 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = \sum \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} = 6$$

$$O[1,3] = \sum \begin{bmatrix} 4 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = \sum \begin{bmatrix} 4 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = 4$$

$$O[2,1] = \sum \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 0 & 5 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = \sum \begin{bmatrix} 3 & 0 & -1 \\ 2 & 0 & -1 \\ 2 & 0 & -5 \end{bmatrix} = 0$$

$$O[2,2] = \sum \begin{bmatrix} 1 & 1 & 0 \\ 4 & 1 & 0 \\ 0 & 5 & 2 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = \sum \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = 3$$

Following a similar process, we have:

$$O[2,3] = 3, \quad O[3,1] = -5, \quad O[3,2] = 1, \quad O[3,3] = 5$$

Therefore, our output O is:

$$O = \begin{bmatrix} -1 & 6 & 4 \\ 0 & 3 & 3 \\ -5 & 1 & 5 \end{bmatrix}$$

1.1.2 Zero padding of 1 + convolution with stride of 1

In here, we have $n = 5$, $p = 1$, $k = 3$, and $s = 1$. We can calculate m as:

$$m = \lfloor \frac{5 + 2 \times 1 - 3}{1} \rfloor + 1 = 5 \tag{3}$$

Therefore, we have $O \in \mathbf{R}^{5\times5}$. Since do zero pad of size 1 on both sides on A, we now have a new matrix:

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 4 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 5 & 2 & 2 & 0 \\ 0 & 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can calculate O as follows:

$$O[1,1] = \sum \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 3 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = -3$$

$$O[1,2] = \sum \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ 3 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = -2$$

$$O[1,3] = \sum \begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = 2$$

$$O[1,4] = \sum \begin{bmatrix} 0 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = 4$$

$$O[1,5] = \sum \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = 1$$

Following a similar process, we have:

$$O[2,1] = \sum \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = -7$$

$$O[2,2] = -1, \quad O[2,3] = 6, \quad O[2,4] = 4, \quad O[2,5] = 1$$

$$O[3,1] = -5, \quad O[3,2] = 0, \quad O[3,3] = 3, \quad O[3,4] = 3, \quad O[3,5] = 2$$

$$O[4,1] = -5, \quad O[4,2] = -5, \quad O[4,3] = 1, \quad O[4,4] = 5, \quad O[4,5] = 4$$

$$O[5,1] = -1, \quad O[5,2] = -6, \quad O[5,3] = -3, \quad O[5,4] = 5, \quad O[5,5] = 4$$

Therefore, our output O is:

$$O = \begin{bmatrix} -3 & -2 & 2 & 4 & 1 \\ -7 & -1 & 6 & 4 & 1 \\ -5 & 0 & 3 & 3 & 2 \\ -5 & -5 & 1 & 5 & 4 \\ -1 & -6 & -3 & 5 & 4 \end{bmatrix}$$

1.1.3 Zero padding of 2 + convolution with stride of 2

In here, we have $n = 5$, $p = 2$, $k = 3$, and $s = 2$. We can calculate m as:

$$m = \lfloor \frac{5 + 2 \times 2 - 3}{2} \rfloor + 1 = 4 \tag{4}$$

Therefore, we have $O \in \mathbf{R}^{4\times4}$. Since do zero pad of size 2 on both sides on A, we now have a new matrix:

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 5 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can calculate O as follows:

$$O[1,1] = \sum A'[1:4, 1:4] \odot f = \sum \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = 0$$

$$O[1,2] = \sum A'[1:4, 3:6] \odot f = \sum \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 4 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = -4$$

$$O[1,3] = \sum A'[1:4, 5:8] \odot f = \sum \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = 4$$

$$O[1,4] = \sum A'[1:4, 7:10] \odot f = \sum \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = 0$$

Following a similar process, we have:

$$O[2,1] = \sum A'[3:6, 1:4] \odot f = \sum \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = -5$$

$$O[2,2] = \sum A'[3:6, 3:6] \odot f = \sum \begin{bmatrix} 0 & 0 & 4 \\ 3 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = -1$$

$$O[2,3] = 4, \quad O[2,4] = 2$$

$$O[3,1] = -4, \quad O[3,2] = -5, \quad O[3,3] = 5, \quad O[3,4] = 4$$

$$O[4,1] = 0, \quad O[4,2] = -3, \quad O[4,3] = 2, \quad O[4,4] = 1$$

Therefore, our output O is:

$$O = \begin{bmatrix} 0 & -4 & 4 & 0 \\ -5 & -1 & 4 & 2 \\ -4 & -5 & 5 & 4 \\ 0 & -3 & 2 & 1 \end{bmatrix}$$

1.1.4 Convolution with stride of 1 + max pooling of 3 with stride of 1

If we have a convolution with a stride of 1, according to the first question calculated previously, we have intermediate matrix value:

$$O' = \begin{bmatrix} -1 & 6 & 4 \\ 0 & 3 & 3 \\ -5 & 1 & 5 \end{bmatrix}$$

If we perform a max pool of size 3 with the stride of 1 on O', the result O we get is:

$$O[1,1] = \max(O[1:4, 1:4]) = 6$$

Therefore, our output O is:

$$O = \begin{bmatrix} 6 \end{bmatrix} \in \mathbf{R}^{1\times 1}$$

1.1.5 Zero padding of 2 + convolution with stride of 1 + max pooling of 3 with stride of 1

In here, we have $n = 5$, $p = 2$, $k = 3$, and $s = 1$. We can calculate m as:

$$m = \lfloor \frac{5 + 2 \times 2 - 3}{1} \rfloor + 1 = 7 \tag{5}$$

Therefore, we have intermediate matrix $O' \in \mathbf{R}^{7\times 7}$. Since do zero pad of size 2 on both sides on A, we now have a new matrix:

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 5 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can calculate O' as follows:

$$O'[1,1] = \sum A'[1:4, 1:4] \odot f = \sum \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = 0$$

$$O'[1,2] = \sum A'[1:4, 2:5] \odot f = \sum \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} = -2$$

After performing convolution on the whole matrix A', we have O' as:

$$O' = \begin{bmatrix} 0 & -2 & -4 & 1 & 4 & 1 & 0 \\ -3 & -3 & -2 & 2 & 4 & 1 & 1 \\ -5 & -7 & -1 & 6 & 4 & 1 & 2 \\ -7 & -5 & 0 & 3 & 3 & 2 & 4 \\ -4 & -5 & -5 & 1 & 5 & 4 & 4 \\ -2 & -1 & -6 & -3 & 5 & 4 & 3 \\ 0 & -1 & -3 & -1 & 2 & 2 & 1 \end{bmatrix}$$

Next, we perform a max pool of size 3 with stride 1, the result O we get is:

$$O[1,1] = \max(O[1:4, 1:4]) = 0$$
$$O[1,2] = \max(O[1:4, 2:5]) = 6$$
$$O[1,3] = \max(O[1:4, 3:6]) = 6$$
$$O[1,4] = \max(O[1:4, 4:7]) = 6$$
$$O[1,5] = \max(O[1:4, 5:8]) = 4$$
$$O[2,1] = \max(O[2:5, 1:4]) = 0$$

We keep following the same procedure and our output O is:

$$O = \begin{bmatrix} 0 & 6 & 6 & 6 & 4 \\ 0 & 6 & 6 & 6 & 4 \\ 0 & 6 & 6 & 6 & 5 \\ 0 & 3 & 5 & 5 & 5 \\ 0 & 1 & 5 & 5 & 5 \end{bmatrix}$$