Homework 1

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1 Problem 1: Gradient Calculation (8 points)

In this question, you are required to calculate gradients for 2 scalar functions.

1. Calculate the gradient of the function $f(x, y) = x^2 + \ln(y) + xy + y^3$. What is the gradient value for $(x, y) = (10, -10)$?

2. Calculate the gradient of the function $f(x, y, z) = \tanh(x^3 y^3) + \sin(z^2)$. What is the gradient value for $(x, y, z) = (-1, 0, \pi/2)$?

1.1 Solution 1

Given we have:

$$f(x, y) = x^2 + \ln(y) + xy + y^3 \tag{1}$$

According to the definition of gradient of a function:

$$\nabla f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x + y \\ \frac{1}{y} + x + 3y^2 \end{bmatrix} \tag{2}$$

For the gradient value of (x, y) at $(10, -10)$, we will have:

$$\nabla f(10, -10) = \begin{bmatrix} 2(10) + (-10) \\ \frac{1}{(-10)} + 10 + 3(-10)^2 \end{bmatrix} = \begin{bmatrix} 10 \\ 309.9 \end{bmatrix} \tag{3}$$

1.2 Solution 2

Given we have:

$$f(x, y, z) = \tanh(x^3 y^3) + \sin(z^2) \tag{4}$$

According to the definition of gradient of a function:

$$\nabla f(x, y, z) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} (1 - \tanh^2(x^3 y^3)) \times 3y^3 x^2 \\ (1 - \tanh^2(x^3 y^3)) \times 3x^3 y^2 \\ \cos(z^2) \times 2z \end{bmatrix} \tag{5}$$

For the gradient value of (x, y, z) at $(-1, 0, \pi/2)$, we will have:

$$\nabla f(-1, 0, \pi/2) = \begin{bmatrix} (1 - \tanh^2(0)) \times 0 \\ (1 - \tanh^2(0)) \times 0 \\ \pi \cos(\frac{\pi^2}{4}) \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \\ -2.454 \end{bmatrix} \tag{6}$$

2 Problem 2: Matrix Multiplication (8 points)

In this question, you are required to perform matrix multiplication.

1. $\begin{bmatrix} 10 \\ -5 \\ 2 \\ 8 \end{bmatrix} \begin{bmatrix} 0 & 3 & 0 & 1 \end{bmatrix} =?$

2. $\begin{bmatrix} 1 & -1 & 6 & 7 \\ 9 & 0 & 8 & 1 \\ -8 & 1 & 2 & 3 \\ 10 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 & 0 \\ 0 & -1 & 1 \\ -3 & 0 & 4 \\ 3 & 4 & 7 \end{bmatrix} =?$

2.1 Solution 1

Let $A = \begin{bmatrix} 10 \\ -5 \\ 2 \\ 8 \end{bmatrix} \in \mathbb{R}^{4\times 1}$ and $B = \begin{bmatrix} 0 & 3 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{1\times 4}$. When we perform the matrix multiplication $A \times B$, we are expected to have the result matrix $C \in \mathbb{R}^{4\times 4}$. We can denote C as:

$$C = \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} \\ C_{2,1} & C_{2,2} & C_{2,3} & C_{2,4} \\ C_{3,1} & C_{3,2} & C_{3,3} & C_{3,4} \\ C_{4,1} & C_{4,2} & C_{4,3} & C_{4,4} \end{bmatrix} \tag{7}$$

According to matrix multiplication, we have:

$$C_{(1,1)} = A_{(1,*)} \times B_{(*,1)} = [10] \times [0] = 0 \tag{8}$$
$$C_{(1,2)} = A_{(1,*)} \times B_{(*,2)} = [10] \times [3] = 30 \tag{9}$$
$$C_{(1,3)} = A_{(1,*)} \times B_{(*,3)} = [10] \times [0] = 0 \tag{10}$$
$$C_{(1,4)} = A_{(1,*)} \times B_{(*,4)} = [10] \times [1] = 10 \tag{11}$$
$$C_{(2,1)} = A_{(2,*)} \times B_{(*,1)} = [-5] \times [0] = 0 \tag{12}$$
$$C_{(2,2)} = A_{(2,*)} \times B_{(*,2)} = [-5] \times [3] = -15 \tag{13}$$
$$C_{(2,3)} = A_{(2,*)} \times B_{(*,3)} = [-5] \times [0] = 0 \tag{14}$$
$$C_{(2,4)} = A_{(2,*)} \times B_{(*,4)} = [-5] \times [1] = -5 \tag{15}$$
$$C_{(3,1)} = A_{(3,*)} \times B_{(*,1)} = [2] \times [0] = 0 \tag{16}$$
$$C_{(3,2)} = A_{(3,*)} \times B_{(*,2)} = [2] \times [3] = 6 \tag{17}$$
$$C_{(3,3)} = A_{(3,*)} \times B_{(*,3)} = [2] \times [0] = 0 \tag{18}$$
$$C_{(3,4)} = A_{(3,*)} \times B_{(*,4)} = [2] \times [1] = 2 \tag{19}$$
$$C_{(4,1)} = A_{(4,*)} \times B_{(*,1)} = [8] \times [0] = 0 \tag{20}$$
$$C_{(4,2)} = A_{(4,*)} \times B_{(*,2)} = [8] \times [3] = 24 \tag{21}$$
$$C_{(4,3)} = A_{(4,*)} \times B_{(*,3)} = [8] \times [0] = 0 \tag{22}$$
$$C_{(4,4)} = A_{(4,*)} \times B_{(*,4)} = [8] \times [1] = 8 \tag{23}$$
$$\tag{24}$$

Therefore, the result of $A \times B$ is:

$$C = \begin{bmatrix} 0 & 30 & 0 & 10 \\ 0 & -15 & 0 & -5 \\ 0 & 6 & 0 & 2 \\ 0 & 24 & 0 & 8 \end{bmatrix} \tag{25}$$

2.2 Solution 2

Let $A = \begin{bmatrix} 1 & -1 & 6 & 7 \\ 9 & 0 & 8 & 1 \\ -8 & 1 & 2 & 3 \\ 10 & 4 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}$ and $B = \begin{bmatrix} 6 & 2 & 0 \\ 0 & -1 & 1 \\ -3 & 0 & 4 \\ 3 & 4 & 7 \end{bmatrix} \in \mathbb{R}^{4\times3}$. When we perform the matrix multiplication $A \times B$, we are expected to have the result matrix $C \in \mathbb{R}^{4\times3}$. We can denote C as:

$$C = \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \\ C_{4,1} & C_{4,2} & C_{4,3} \end{bmatrix} \tag{26}$$

According to matrix multiplication, we have:

$$\begin{align}
C_{(1,1)} = A_{(1,*)} \times B_{(*,1)} = \begin{bmatrix} 1 & -1 & 6 & 7 \end{bmatrix} \times \begin{bmatrix} 6 & 0 & -3 & 3 \end{bmatrix} = 6 + 0 - 18 + 21 = 9 \tag{27} \\
C_{(1,2)} = A_{(1,*)} \times B_{(*,2)} = \begin{bmatrix} 1 & -1 & 6 & 7 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 0 & 4 \end{bmatrix} = 2 + 1 + 0 + 28 = 31 \tag{28} \\
C_{(1,3)} = A_{(1,*)} \times B_{(*,3)} = \begin{bmatrix} 1 & -1 & 6 & 7 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 4 & 7 \end{bmatrix} = 0 - 1 + 24 + 49 = 72 \tag{29} \\
C_{(2,1)} = A_{(2,*)} \times B_{(*,1)} = \begin{bmatrix} 9 & 0 & 8 & 1 \end{bmatrix} \times \begin{bmatrix} 6 & 0 & -3 & 3 \end{bmatrix} = 54 + 0 - 24 + 3 = 33 \tag{30} \\
C_{(2,2)} = A_{(2,*)} \times B_{(*,2)} = \begin{bmatrix} 9 & 0 & 8 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 0 & 4 \end{bmatrix} = 18 + 0 + 0 + 4 = 22 \tag{31} \\
C_{(2,3)} = A_{(2,*)} \times B_{(*,3)} = \begin{bmatrix} 9 & 0 & 8 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 4 & 7 \end{bmatrix} = 0 + 0 + 32 + 7 = 39 \tag{32} \\
C_{(3,1)} = A_{(3,*)} \times B_{(*,1)} = \begin{bmatrix} -8 & 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 6 & 0 & -3 & 3 \end{bmatrix} = -48 + 0 - 6 + 9 = -45 \tag{33} \\
C_{(3,2)} = A_{(3,*)} \times B_{(*,2)} = \begin{bmatrix} -8 & 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 0 & 4 \end{bmatrix} = -16 - 1 + 0 + 12 = -5 \tag{34} \\
C_{(3,3)} = A_{(3,*)} \times B_{(*,3)} = \begin{bmatrix} -8 & 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 4 & 7 \end{bmatrix} = 0 + 1 + 8 + 21 = 30 \tag{35} \\
C_{(4,1)} = A_{(4,*)} \times B_{(*,1)} = \begin{bmatrix} 10 & 4 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 6 & 0 & -3 & 3 \end{bmatrix} = 60 + 0 + 0 + 3 = 63 \tag{36} \\
C_{(4,2)} = A_{(4,*)} \times B_{(*,2)} = \begin{bmatrix} 10 & 4 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 0 & 4 \end{bmatrix} = 20 - 4 + 0 + 4 = 20 \tag{37} \\
C_{(4,3)} = A_{(4,*)} \times B_{(*,3)} = \begin{bmatrix} 10 & 4 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 4 & 7 \end{bmatrix} = 0 + 4 + 0 + 7 = 11 \tag{38} \\
\tag{39}
\end{align}$$

Therefore, the result of $A \times B$ is:

$$C = \begin{bmatrix} 9 & 31 & 72 \\ 33 & 22 & 39 \\ -45 & -5 & 30 \\ 63 & 20 & 11 \end{bmatrix} \tag{40}$$