Introduction to 1D FDTD

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Table of Contents

Maxwell's Equation

$$\nabla \cdot \mathbf{B} = 0 \tag{1}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{2}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4}$$

But it doesn't tell us the interaction between EM wave and material.

Constitutive Relation

$$\mathbf{D}(t) = [\varepsilon(t)] \otimes \mathbf{E}(t) \qquad (5)$$

$$\mathbf{B}(t) = [\mu(t)] \otimes \mathbf{H}(t) \qquad (6)$$

- Anisotropic material

$$\mathbf{D}(t) = [\varepsilon(t)]\mathbf{E}(t)$$

- Dispersive material

$$\mathbf{D}(t) = \varepsilon(t) \otimes \mathbf{E}(t)$$

- Linear, isoropic and non-dispersive material

$$\mathbf{D}(t) = \varepsilon\mathbf{E}(t)$$

Update Equation

Source free

$$\rho = 0$$

$$\mathbf{J} = 0$$

Linear, isoropic and non-dispersive material

$$[\varepsilon(t)] \rightarrow \varepsilon$$

$$[\mu(t)] \rightarrow \mu$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} \tag{7}$$

$$\downarrow \qquad \uparrow$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{8}$$

Yee Grid Scheme

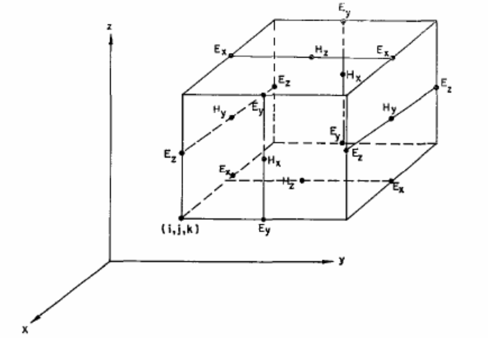


Fig. 1. Positions of various field components. The E-components are in the middle of the edges and the H-components are in the center of the faces.

- E, H arrangement make it divergence free.

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \cdot \mathbf{D} = \rho$$

- Easily calculate curl.

K. S. Yee, "Numerical solution of the initial boundary value problems involving Maxwell's equations in isotropic media," IEEE Trans. Microwave Theory and Techniques, vol. 44, pp. 61–69, 1998.

3D Formulation: Take finding $H_x^{i,j,k}|_t$ for example

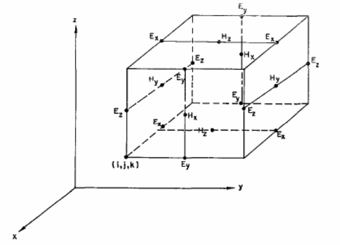


Fig. 1. Positions of various field components. The E-components are in the middle of the edges and the H-components are in the center of the faces.

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\frac{E_z^{i,j+1,k}|_t - E_z^{i,j,k}|_t}{\Delta y} - \frac{E_y^{i,j,k+1}|_t - E_y^{i,j,k}|_t}{\Delta z} = -\mu_0 \frac{H_x^{i,j,k}|_{t+\Delta t} - H_x^{i,j,k}|_{t-\Delta t}}{\Delta t} \quad (9)$$

Complete 3D Formulation

$$\frac{E_z^{i,j+1,k}\Big|_t - E_z^{i,j,k}\Big|_t}{\Delta y} - \frac{E_y^{i,j,k+1}\Big|_t - E_y^{i,j,k}\Big|_t}{\Delta z} = -\mu_0 \frac{H_x^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k}\Big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\frac{E_x^{i,j,k+1}\Big|_t - E_x^{i,j,k}\Big|_t}{\Delta z} - \frac{E_z^{i+1,j,k}\Big|_t - E_z^{i,j,k}\Big|_t}{\Delta x} = -\mu_0 \frac{H_y^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_y^{i,j,k}\Big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\frac{E_y^{i+1,j,k}\Big|_t - E_y^{i,j,k}\Big|_t}{\Delta x} - \frac{E_x^{i,j+1,k}\Big|_t - E_x^{i,j,k}\Big|_t}{\Delta y} = -\mu_0 \frac{H_z^{i,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_z^{i,j,k}\Big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\frac{H_z^{i,j+1,k}\Big|_{t+\frac{\Delta t}{2}} - H_z^{i,j,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta y} - \frac{H_y^{i,j,k+1}\Big|_{t+\frac{\Delta t}{2}} - H_y^{i,j,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta z} = -\varepsilon_0 \frac{E_x^{i,j,k}\Big|_{t+\Delta t} - E_x^{i,j,k}\Big|_t}{\Delta t}$$

$$\frac{H_x^{i,j,k+1}\Big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta z} - \frac{H_z^{i+1,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_z^{i,j,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta x} = -\varepsilon_0 \frac{E_y^{i,j,k}\Big|_{t+\Delta t} - E_y^{i,j,k}\Big|_t}{\Delta t}$$

$$\frac{H_y^{i+1,j,k}\Big|_{t+\frac{\Delta t}{2}} - H_y^{i,j,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta x} - \frac{H_x^{i,j+1,k}\Big|_{t+\frac{\Delta t}{2}} - H_x^{i,j,k}\Big|_{t+\frac{\Delta t}{2}}}{\Delta y} = -\varepsilon_0 \frac{E_z^{i,j,k}\Big|_{t+\Delta t} - E_z^{i,j,k}\Big|_t}{\Delta t}$$

1D Formulation

In one dimension case, each grid represent a infinite large slab in xy-plane where simulation cell locate at z-axis, which also implies following relation,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0.$$

So previous 3D formulation result reduce to

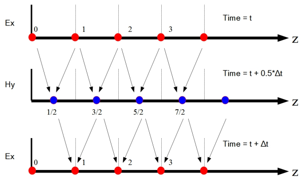
$$\frac{H_y^k\left(t+\frac{\Delta t}{2}\right) - H_y^{k-1}\left(t+\frac{\Delta t}{2}\right)}{\Delta z} = -\varepsilon_0 \frac{E_x^k(t+\Delta t) - E_x^k(t)}{\Delta t}$$

$$\frac{E_x^{k+1}(t) - E_x^k(t)}{\Delta z} = -\mu_0 \frac{H_y^k\left(t+\frac{\Delta t}{2}\right) - H_y^k\left(t-\frac{\Delta t}{2}\right)}{\Delta t} \tag{10}$$

1D Update Equation

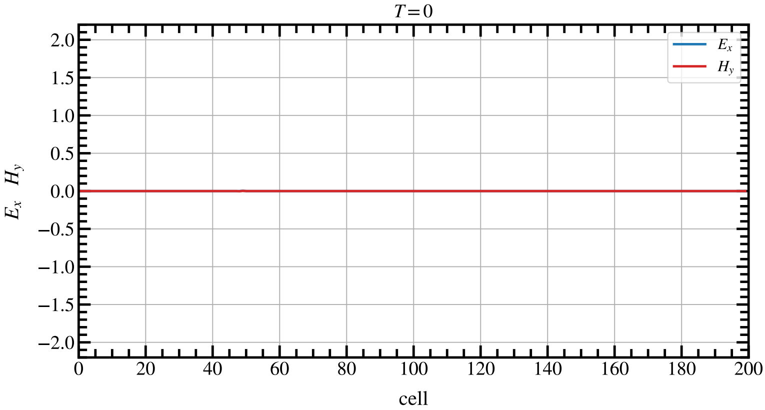
$$E_x^k(t+\Delta t) = E_x^k(t) + \left(\varepsilon_0 \frac{\Delta t}{\Delta z}\right)\left(H_y^k\left(t+\frac{\Delta t}{2}\right) - H_y^{k-1}\left(t+\frac{\Delta t}{2}\right)\right)$$

$$H_y^k\left(t+\frac{\Delta t}{2}\right) = H_y^k\left(t-\frac{\Delta t}{2}\right) - \left(\mu_0 \frac{\Delta t}{\Delta z}\right)\left(E_x^{k+1}(t) - E_x^k(t)\right)$$

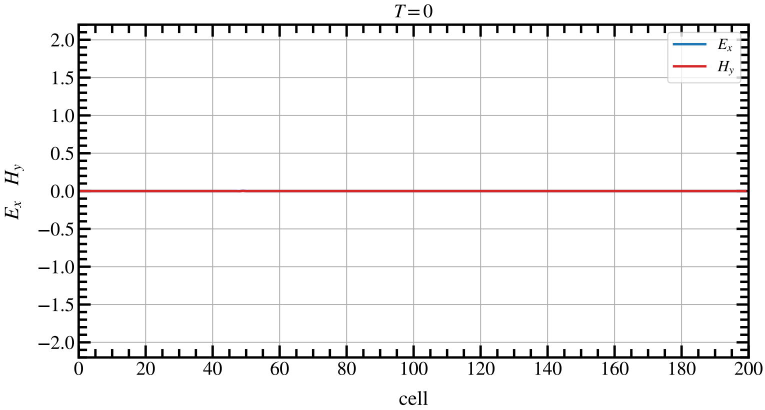


https://zhnotes.files.wordpress.com/2013/08/1dgridarrangement.jpg

Example1: Hard Source



Example2: Soft Source



Problem at Boundary

Look at the update equations again.

$$E_x^k(t+\Delta t) = E_x^k(t) + \left(\varepsilon_0 \frac{\Delta t}{\Delta z}\right)\left(H_y^k\left(t+\frac{\Delta t}{2}\right) - H_y^{k-1}\left(t+\frac{\Delta t}{2}\right)\right)$$

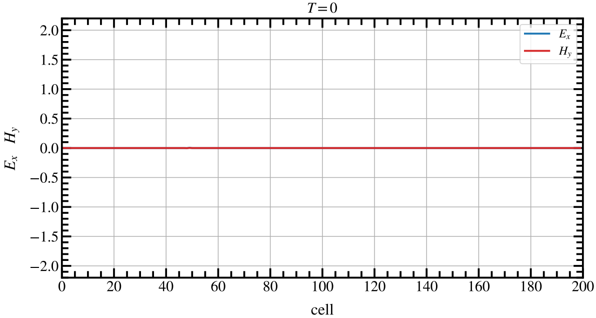
$$H_y^k\left(t+\frac{\Delta t}{2}\right) = H_y^k\left(t-\frac{\Delta t}{2}\right) - \left(\mu_0 \frac{\Delta t}{\Delta z}\right)\left(E_x^{k+1}(t) - E_x^k(t)\right)$$

Dirichlet Boundary Condition

Let $E_x^{k=N} = 0$ and $H_y^{k=1} = 0$ at the boundary.

$$E_x^k(t+\Delta t) = E_x^1(t) + \left(\varepsilon_0 \frac{\Delta t}{\Delta z}\right)\left(H_y^1\left(t+\frac{\Delta t}{2}\right) - 0\right)$$

$$H_y^N\left(t+\frac{\Delta t}{2}\right) = H_y^N\left(t-\frac{\Delta t}{2}\right) - \left(\mu_0 \frac{\Delta t}{\Delta z}\right)\left(0 - E_x^N(t)\right)$$

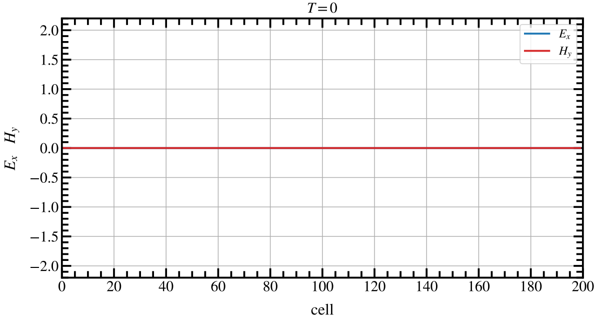


Periodic Boundary Condition

Let $E_x^{k=N+1} = E_y^{k=1}$ and $H_x^{k=0} = H_y^{k=N}$ at the boundary.

$$E_x^1\left(t+\Delta t\right) = E_x^1\left(t\right) + \left(\varepsilon_0 \frac{\Delta t}{\Delta z}\right)\left(H_y^1\left(t+\frac{\Delta t}{2}\right) - H_y^N\left(t+\frac{\Delta t}{2}\right)\right)$$

$$H_y^N\left(t+\frac{\Delta t}{2}\right) = H_y^N\left(t-\frac{\Delta t}{2}\right) - \left(\mu_0 \frac{\Delta t}{\Delta z}\right)\left(E_x^1\left(t\right) - E_x^N\left(t\right)\right)$$



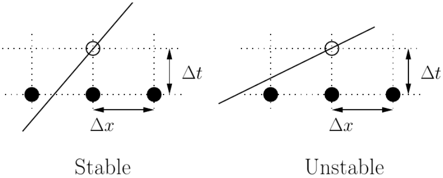
Grid Resolution and Courant Condition

- Before start of the simulation, you have to decide grid resolution which is either
  - $\Delta x \approx$ smallest length of concerning target or
  - $\Delta x \leq \frac{1}{10}\lambda_{min}$.
- Next you need to decide update time, for stability $c_0 \Delta t$ must less than the Δx we chosen above (for 1D)

$$c_0 \Delta t \leq \frac{\Delta x}{\sqrt{3}} \tag{11}$$

Letting

$$\Delta t = \frac{\Delta x}{2c_0} \tag{12}$$



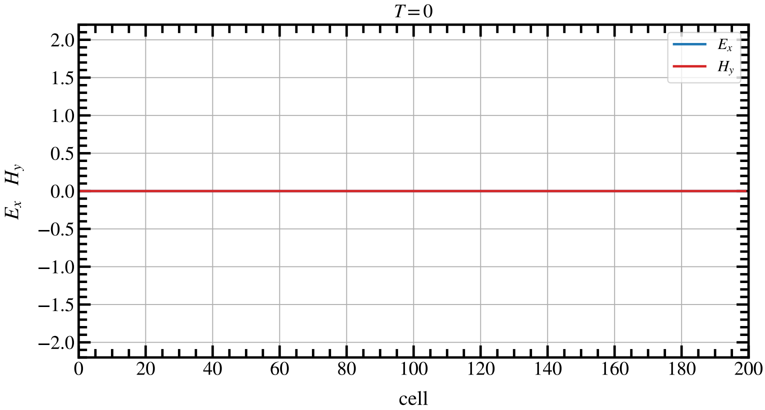
(b) CFL condition

Perfect Boundary Condition

- A disturbance of E at $z = k$ takes only affect adjacent H field in the next iteration, and it takes two time step to affect adjacent E field.

$$E_x^{N_z+1}\Big|_t = E_x^{N_z}\Big|_{t-2\Delta t} \tag{13}$$

Perfect Boundary Condition



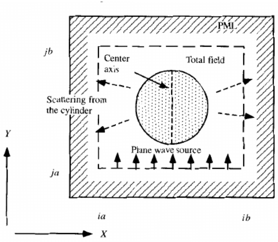
TF/SF

Total field side:

$$E_x^k\left(t+\Delta t\right)=E_x^k\left(t\right)+\left(\varepsilon_0\frac{\Delta t}{\Delta z}\right)\left(H_y^k\left(t+\frac{\Delta t}{2}\right)-\left(H_y^{k-1}\left(t+\frac{\Delta t}{2}\right)+H_y^{k-1(source)}\left(t+\frac{\Delta t}{2}\right)\right)\right)$$

Scatter field side:

$$H_y^{k-1}\left(t+\frac{\Delta t}{2}\right)=H_y^{k-1}\left(t-\frac{\Delta t}{2}\right)-\left(\mu_0\frac{\Delta t}{\Delta z}\right)\left(\left(E_x^k\left(t\right)-E_x^{k(source)}\left(t\right)\right)-E_x^{k-1}\left(t\right)\right)$$



TF/SF Derivation

By assuming

$$E_x(t) = sin(\omega t - kz)$$

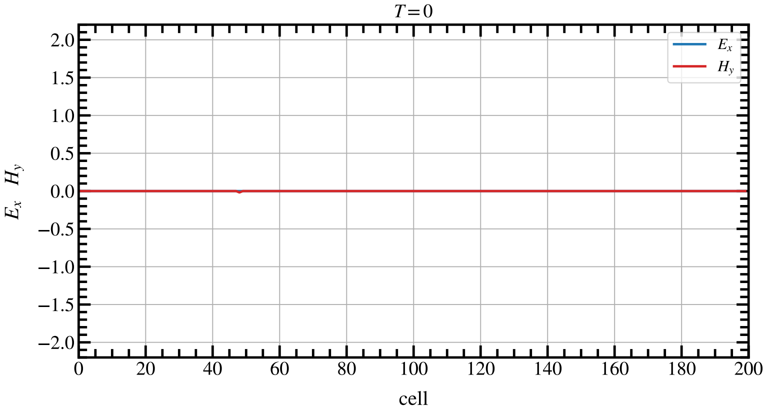
$$H_y(t) = sin(\omega t - kz - \phi)$$

Substitute back into Maxwell's equation. We have the following relation about

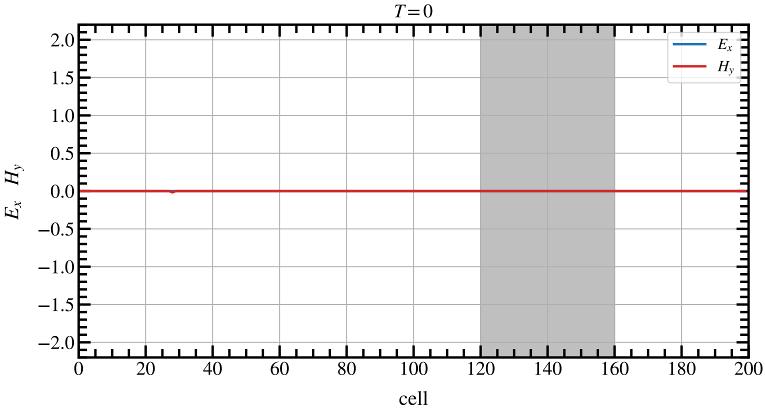
$$H_y^{k-1(source)}\left(t + \frac{\Delta t}{2}\right) = \sqrt{\frac{\varepsilon_{source}}{\mu_{source}}} f\left(t + \frac{n\Delta z}{2c_0} - \frac{\Delta t}{2}\right) \quad (14)$$

$$E_x^{k(source)}(t) = f(t) \quad (15)$$

TF/SF Example 1: Simple Soft Source



TF/SF Example 2: Lossy Dielectric Medium



Derivation

Goal: Transform $f(t)$ into $F(f)$. Method:

$$F(f) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi ft} dt \tag{16}$$

and can be approximate by

$$F(f) \approx \sum_{n=1}^{m} f(n\Delta t) e^{-j2\pi nf\Delta t} \Delta t \tag{17}$$

turn into efficient computer code.

$$k_n = (e^{-j2\pi f\Delta t})$$

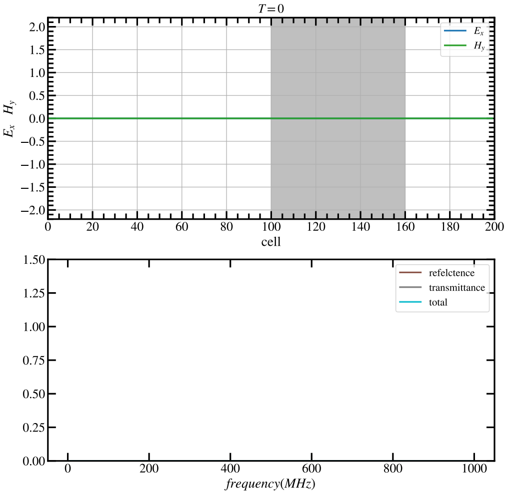
$$do\ n\ =\ 1,\ m$$

$$F(f) = F(f) + f(n\Delta t){k_n}^n$$

$$end\ do$$

$$F(f) = F(f)\Delta t$$

Example: Reflectance and Transmittance



References

- Stability of Finite Difference Methods - MIT
- https://empossible.net/academics/emp5304/