
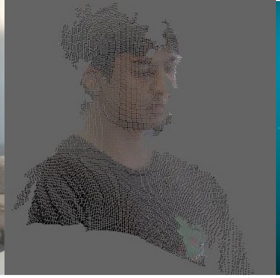


CSci5551 Introduction to Intelligent Robotics Systems

Velocity Kinematics Part 2

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Velocity Kinematics

- Relating joint velocities with end-effector velocities
  - Both forward (joint to end-effector) and inverse (end-effector to joint)
- Velocities involved are both
  - Linear, v, and
  - Angular/Rotational, ω

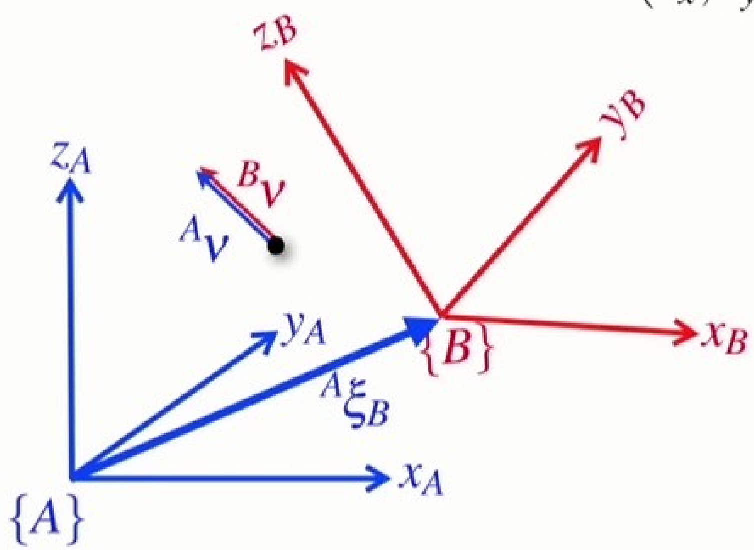
Velocity Kinematics

- Differential equations help relate <u>positions/joint variables</u> to <u>velocities</u>
- Joint velocities == rate of change of joint values
- End-effector velocity == rate of change of joint end-effector position and orientation
  - Rate-of-change of position == linear+angular velocity
  - Rate-of-change of orientation == angular velocity

Velocities and Joints

- Angular velocities, thus, give rise to
  - Positional AND rotational changes
- Linear velocities give rise to positional changes only

$$\nu = (v_x, v_y, v_z, \omega_x, \omega_y, \omega_z)^T$$



$${}^B\nu = {}^B\mathbf{J}_A \, {}^A\nu$$

$${}^B\mathbf{J}_A({}^B\xi_A) = \begin{pmatrix} {}^B\mathbf{R}_A & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & {}^B\mathbf{R}_A \end{pmatrix}$$

6x6

The Manipulator Jacobian

- A manipulator
  - can be described as a series of points, namely the end points of the links,
  - Links are each, in turn, rotating and translating with respect to the previous link
- In the world of velocity *manipulator* kinematics, the goal is to achieve a mapping, which does <u>two things</u>.

The Manipulator Jacobian

- Goal 1: Relate the rate of change of position of the manipulator end-effector to the rate of change of motion of each link.

$$\dot{\boldsymbol{q}} = \frac{dq_i}{dt} \rightarrow \dot{\boldsymbol{X}} = \begin{Bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{Bmatrix} = \begin{Bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \\ \frac{d\phi_x}{dt} \\ \frac{d\phi_y}{dt} \\ \frac{d\phi_z}{dt} \end{Bmatrix} = \begin{Bmatrix} \frac{d\boldsymbol{p}}{dt} \\ \frac{d\boldsymbol{\phi}}{dt} \end{Bmatrix}$$

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The Manipulator Jacobian

- Goal 2: Relate the differential change in position (location and orientation) of one frame due to a differential change of another frame.
  - A *superset* of Goal 1.
  - Thus, more *powerful* and *useful* solution to deduce

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The Manipulator Jacobian

- For Goal 1, the question becomes to find a map *J* such that

$$\dot{\boldsymbol{X}} = J\dot{\boldsymbol{q}}$$

- For Goal 2, the question becomes to find a map *K* such that for two frames F_0 and F_1:

$$dX^b = \begin{bmatrix} d\boldsymbol{p}^b \\ d\boldsymbol{\phi}^b \end{bmatrix} = K_{ba} \begin{bmatrix} d\boldsymbol{p}^b \\ d\boldsymbol{\phi}^b \end{bmatrix}$$

The Manipulator Jacobian

- Expression of a manipulator Jacobian for a robot with both prismatic and revolute joints is as follows:

$$\begin{pmatrix} \boldsymbol{v}_e^0 \\ \boldsymbol{\omega}_e^0 \end{pmatrix} = \begin{bmatrix} \dots & \boldsymbol{z}_i^0 \times \boldsymbol{a}_{ie}^0 & z_j^0 & \dots \\ \dots & \boldsymbol{z}_i^0 & \dots 0 & \dots \end{bmatrix} \begin{bmatrix} \vdots \\ \dot{q}_{i+1} \\ \vdots \\ \dot{q}_{j+1} \\ \vdots \end{bmatrix} = \mathrm{J}_o \dot{\boldsymbol{q}}$$

The Manipulator Jacobian

- The top three rows affect linear velocity (**velocity Jacobian J_v**)
- The bottom three rows affect angular velocity (**angular velocity Jacobian J_ω**)

$$\begin{pmatrix} \boldsymbol{v}_e^0 \\ \boldsymbol{\omega}_e^0 \end{pmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{\boldsymbol{q}}$$

- From the end-effector frame, F_e,

$$\begin{pmatrix} \boldsymbol{v}_e^e \\ \boldsymbol{\omega}_e^e \end{pmatrix} = \mathrm{R}_{e0} \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{\boldsymbol{q}}$$

The Analytical Jacobian

- based on a *minimal representation* for the *orientation* of the end-effector frame
- Let $X = \begin{array}{c} p(q) \\ \alpha(q) \end{array}$ denote the end-effector pose where
  - *p(q):* vector from base-frame origin to end-effector frame origin
  - *α(q)*: *minimal representation* of end-effector orientation relative to the base frame
- The analytical Jacobian is thus

$$\dot{X} = \left[\begin{array}{c} \dot{p} \\ \dot{\alpha} \end{array} \right] = J_\alpha(q)\dot{q}$$

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The Analytical Jacobian

- *minimal representation=ZYZ Euler angle representation*
- It can be shown that if $R=R_{z,\varphi} R_{y,\theta} R_{z,\Psi}$ (Euler angle representation), then $\dot{R} = S(\omega)R$, where

$$\omega = \begin{bmatrix} c_\psi s_\theta \dot{\phi} - s_\psi \dot{\theta} \\ s_\psi s_\theta \dot{\psi} + c_\psi \theta \\ \dot{\psi} + c_\theta \dot{\psi} \end{bmatrix} = \begin{bmatrix} c_\psi s_\theta & -s_\psi & 0 \\ s_\psi s_\theta & c_\psi & 0 \\ c_\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = T(\alpha)\dot{\alpha}$$

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With the Geometric Jacobian

$$\begin{aligned} J(q)\dot{q} &= \begin{bmatrix} v \\ \omega \end{bmatrix} \\ &= \begin{bmatrix} \dot{p} \\ T(\alpha)\dot{\alpha} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & T(\alpha) \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{\alpha} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & T(\alpha) \end{bmatrix} J_\alpha \end{aligned}$$

Thus the analytical Jacobian, Ja (q), may be computed from the geometric Jacobian as

$$J_a(q) = \begin{bmatrix} I & 0 \\ 0 & T(\alpha)^{-1} \end{bmatrix} J(q)$$

provided that *det*(*T*(*α*))≠0.

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J_v Using Diferentiation

- Differentiate the Transformation matrix T_{0t} directly, *wrt* each joint!
- Example: 2-link planar manipulator

$$T_{02} = A_{01}A_{12} = \begin{bmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) & 0 & a_1 c\theta_1 + a_2 c(\theta_1+\theta_2) \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2) & 0 & a_1 s\theta_1 + a_2 s(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{d}_{0e}^{0} = \begin{bmatrix} a_1 c\theta_1 + a_2 c(\theta_1+\theta_2) \\ a_1 s\theta_1 + a_2 s(\theta_1+\theta_2) \\ 0 \end{bmatrix}.$$

J_v Using Diferentiation

$$\frac{\partial \mathbf{d}_{0e}^0}{\partial \theta_1} = \begin{bmatrix} -a_1 s\theta_1 - a_2 s(\theta_1 + \theta_2) \\ a_1 c\theta_1 + a_2 c(\theta_1 + \theta_2) \\ 0 \end{bmatrix}.$$

$$\frac{\partial \mathbf{d}_{0e}^0}{\partial \theta_2} = \begin{bmatrix} -a_2 s(\theta_1 + \theta_2) \\ a_2 c(\theta_1 + \theta_2) \\ 0 \end{bmatrix}.$$

$$J_v = \begin{bmatrix} \frac{\partial \mathbf{d}_{0e}^0}{\partial \theta_1} & \frac{\partial \mathbf{d}_{0e}^0}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} -a_1 s\theta_1 - a_2 s(\theta_1 + \theta_2) & -a_2 s(\theta_1 + \theta_2) \\ a_1 c\theta_1 + a_2 c(\theta_1 + \theta_2) & a_2 c(\theta_1 + \theta_2) \\ 0 & 0 \end{bmatrix}$$

Jacobian Singularities

- configurations where the Jacobian loses rank.
- Singularities of the matrix *T(α)* are called representational singularities.
- *T(α)* is invertible provided sin(θ)≠0.
- Note that the 6×3 Jacobian $J(\mathbf{q})$ defines a mapping:

$$\dot{\boldsymbol{X}} = J(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- Infinitesimally this defines a linear transformation

$$d\boldsymbol{X} = J(\boldsymbol{q})d\boldsymbol{q}$$

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Jacobian Singularities

$$\boldsymbol{dX} = J(\boldsymbol{q})\boldsymbol{dq}$$

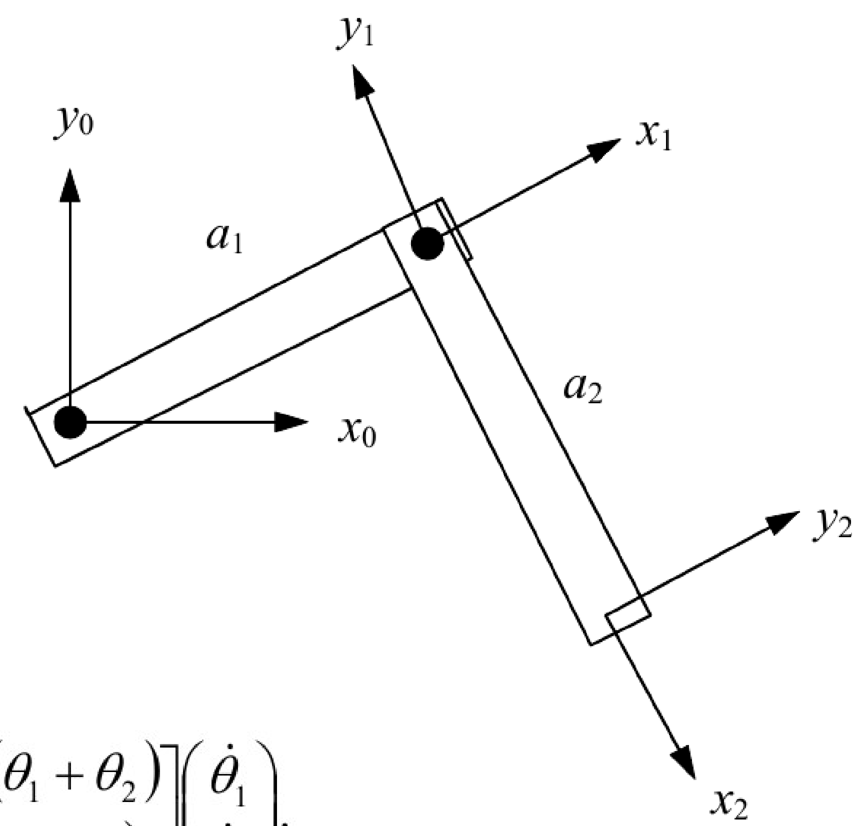
- Linear transformations are between the differentials $d\mathbf{q}$ and $d\boldsymbol{X}$.
  - Think as directions in R^6 , and R^3 , respectively.
- The Jacobian is a function of the configuration $\boldsymbol{q}$, thus configurations for which the rank of $\boldsymbol{J}$ decreases are of special significance == **singular configurations**

Jacobian Singularities

- Why care about Singularities ?
  - represent configurations from which certain directions of motion may be unattainable
  - bounded end-effector velocities may correspond to unbounded joint velocities
  - bounded end-effector forces and torques may correspond to unbounded joint torques
  - usually (but not always) correspond to points on the boundary of the manipulator workspace, that is, to points of maximum reach of the manipulator
  - correspond to points in the manipulator workspace that may be unreachable under small perturbations of the link parameters, such as length, offset, etc.
  - there will not exist a unique solution to the inverse kinematics problem; in such cases there may be no solution or there may be infinitely many solutions.

Singularity- Example

- If we only consider the non-zero (planar) directions of motion (x,y) of the manipulator, which are achievable as a result of the joint motion, then we have the following relationship :



$$v^0 = \begin{pmatrix} \dot{x}_e \\ \dot{y}_e \end{pmatrix}^0 = J_{(2\times 2)0}\dot{q}$$

$$= \begin{bmatrix} -a_1 s\theta_1 - a_2 s(\theta_1+\theta_2) & -a_2 s(\theta_1+\theta_2) \\ a_1 c\theta_1 + a_2 c(\theta_1+\theta_2) & a_2 c(\theta_1+\theta_2) \end{bmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}.$$

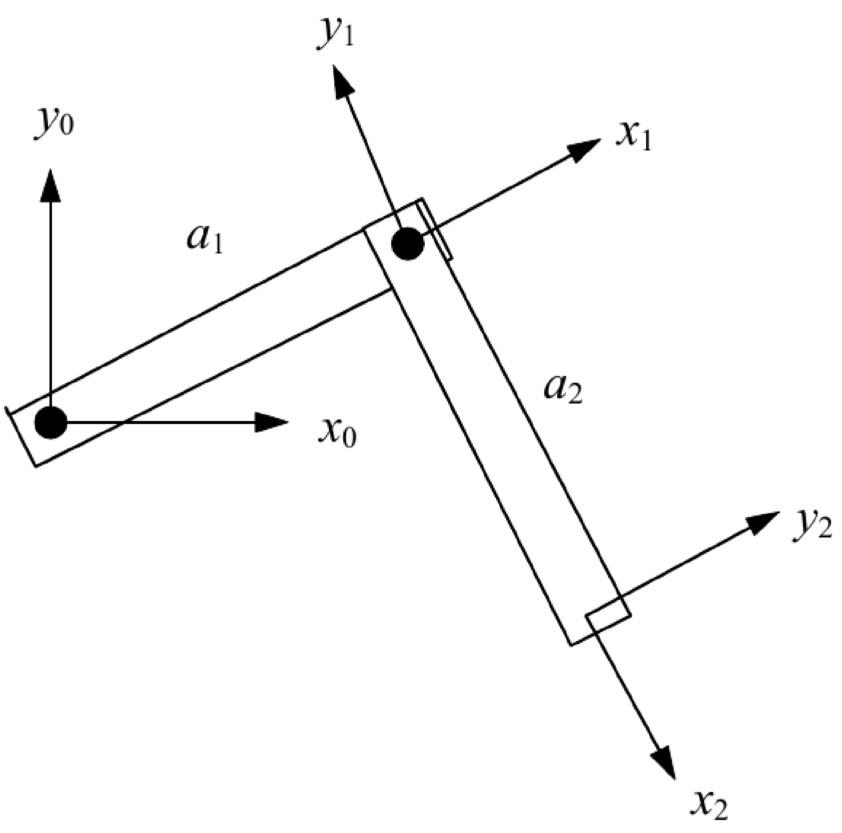
Singularity- Example

- Here $J_{(2x2)}$ is the mapping from the two joint space degrees of freedom to the two work-space degrees of freedom. Because we have a square matrix, we can now write (dropping the (2x2) subscript):

$$\dot{\boldsymbol{q}} = J_0^{-1} \boldsymbol{v}^0$$

- W e can find the inverse of J_0 using the formula from linear algebra:

$$J_0^{-1} = \frac{1}{\det J_0} \mathrm{Adj}\, J_0$$



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Singularity- Example

- In this case:

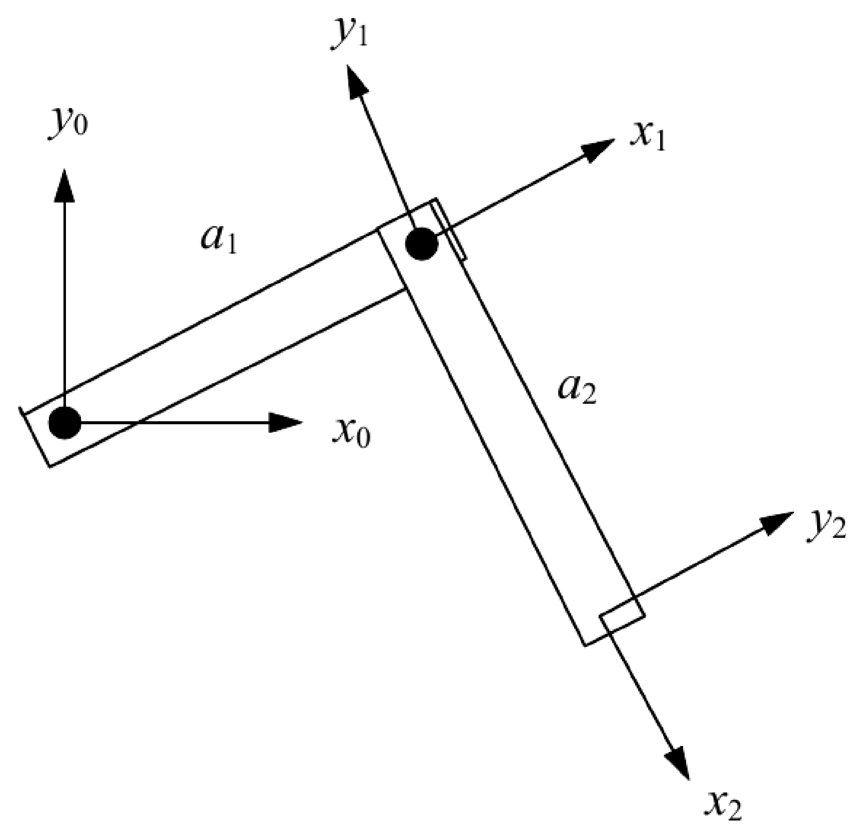
$$\det J_0 = a_1 a_2 \sin\theta_2$$

- If *det* $J_0 = 0$, then J_0^{-1} is undefined.
- This results in a singular configuration case of the planar manipulator, with **two** possibilities:

  1) $\theta_2 = 0$, arm stretched straight out

  2) $\theta_2 = \pi$, arm folded over.

- No unique solution for $\dot{\mathbf{q}}$



Singularity- Example

- When *det* $\boldsymbol{J}_0 = 0$, the matrix $\boldsymbol{J}_0$ is less than full rank; that is, the columns of $\boldsymbol{J}$ are linearly dependent.
- For example, for $\theta_2 = 0$:

$$J_0 = \begin{bmatrix} -a_2 s(\theta_1) & -a_2 s(\theta_1) \\ a_2 c(\theta_1) & a_2 c(\theta_1) \end{bmatrix}$$