

CSci5551 Introduction to Intelligent Robotics Systems

Velocity Kinematics

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Velocity Kinematics

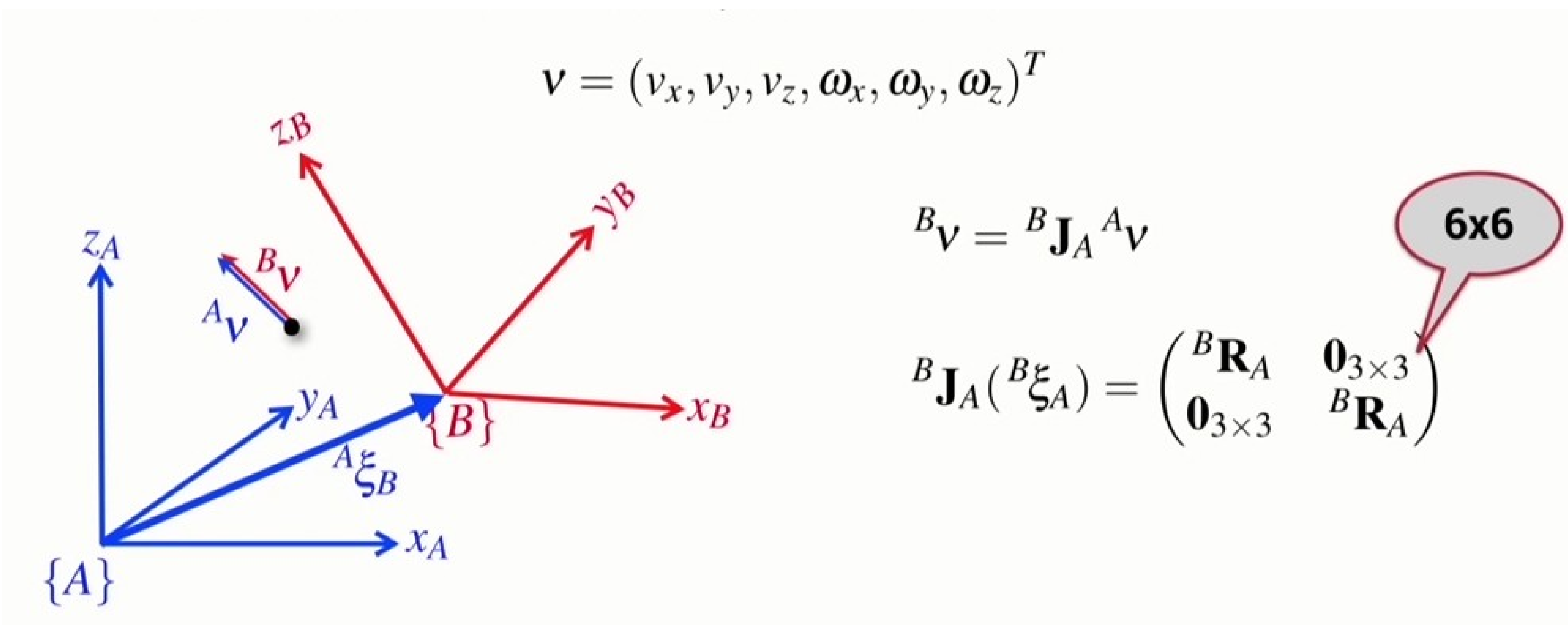
- Relating joint velocities with end-effector velocities
  - Both forward (joint to end-effector) and inverse (end-effector to joint)
- Velocities involved are both
  - Linear, v, and
  - Angular/Rotational, ω

Velocity Kinematics

- Differential equations help relate positions/joint variables to velocities
- Joint velocities == rate of change of joint values
- End-effector velocity == rate of change of joint end-effector position and orientation
  - Rate-of-change of position == linear+angular velocity
  - Rate-of-change of orientation == angular velocity

Velocities and Joints

- Angular velocities, thus, give rise to
  - Positional AND rotational changes
- Linear velocities give rise to positional changes only



Differentiation of Rotation Matrices

- Recall, that the rotation matrice R is a representation of orientation
  - The three columns are unit vectors representing <x,y,z> axes of the coordinate frame
- Differentiating R with respect to time, t, will yield the rate of change of orientations
- Focus :
  - How do we differentiate R,
  - How do propagate changes through a chain of rigid links, and represent velocities, and
  - How do we propagate it in both directions: joints to tool, and tool to joint

The background math

The Tool:
Skew Symmetric Matrices

$$S = \begin{bmatrix} 0 & -s_{12} & s_{13} \\ s_{12} & 0 & -s_{23} \\ -s_{13} & -s_{23} & 0 \end{bmatrix}$$

$$S + S^T = 0$$

Three unique elements only!
Makes it possible to represent a *vector* with a skew Symmetric Matrix!

For a 3D vector $v=[x_1, x_2, x_3]$:

$$S = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

For the principal axes, i, *j*, *and k:*

$$S_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$S_j = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$S_k = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For orthogonal matrices:

$$R(\boldsymbol{a} \times \boldsymbol{b}) = R\boldsymbol{a} \times R\boldsymbol{b}$$

The background math

A useful property of Skew Symmetric Matrices: cross-product simplification!

$$\boldsymbol{x} \times \boldsymbol{y} = S(\boldsymbol{x})\boldsymbol{y}$$

Skew Symmetric Matrices are linear, so:

$$S(\alpha \boldsymbol{a} + \beta \boldsymbol{b}) = \alpha S(\boldsymbol{a}) + \beta S(\boldsymbol{b})$$

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The background math

Consider the expression: $RS(\boldsymbol{a})R^T\boldsymbol{b}$:

$$RS(\boldsymbol{a})R^T\boldsymbol{b} = R\left[\boldsymbol{a} \times R^T\boldsymbol{b}\right] \qquad \triangleleft \cdots\cdots \quad \boldsymbol{x} \times \boldsymbol{y} = S(\boldsymbol{x})\boldsymbol{y}$$

$$= [R\boldsymbol{a}] \times \left[RR^T\boldsymbol{b}\right] \qquad \triangleleft \cdots\cdots \quad R(\boldsymbol{a} \times \boldsymbol{b}) = R\boldsymbol{a} \times R\boldsymbol{b}$$

$$= R\boldsymbol{a} \times \boldsymbol{b}$$

$$= S(R\boldsymbol{a})\boldsymbol{b} \qquad \triangleleft \cdots\cdots \quad \boldsymbol{x} \times \boldsymbol{y} = S(\boldsymbol{x})\boldsymbol{y}$$

$$\therefore, RS(\boldsymbol{a})R^T = S(R\boldsymbol{a})$$

This is the **Similarity Transform for S(a)!**
It is equivalent to the skew symmetric matrix of a vector a rotated by R

Differentiating R, *wrt* θ

Start here:

$$R(\theta)R^T(\theta) = I$$

Differentiating *wrt* θ:

$$\frac{dR(\theta)}{d\theta}R^T(\theta) + R(\theta)\frac{dR^T(\theta)}{d\theta} = 0$$

$$\text{Let,}\Sigma = \frac{dR(\theta)}{d\theta}R^T(\theta)$$

$$\text{Then,}\Sigma^T = R(\theta)\left(\frac{dR(\theta)}{d\theta}\right)^T$$

$$= R(\theta)\frac{dR^T(\theta)}{d\theta}$$

$$(AB)^T = B^T A^T$$

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Substituting...

$$\frac{dR(\theta)}{d\theta}R^T(\theta) + R(\theta)\frac{dR^T(\theta)}{d\theta} = 0$$

$$\therefore, \Sigma + \Sigma^T = 0$$

$$\text{Thus,}\Sigma = \frac{dR(\theta)}{d\theta}R^T(\theta)\text{is Skew Symmetric, and we can write:}$$

$$\Sigma R(\theta) = \frac{dR(\theta)}{d\theta}R^T(\theta)R(\theta)$$

$$\therefore, \frac{dR(\theta)}{d\theta} = \Sigma R(\theta)$$

Substituting...

$$\boxed{\frac{dR(\theta)}{d\theta} = \Sigma R(\theta)}$$

- Where do we get Σ?
- **Since Σ is a Skew Symmetric Matrix, there is some vector *k* whose Skew Symmetric Matrix representation will be equal to Σ.**
- In this case, this brings us back to our old friend,
  - the **axis-angle representation of rotation: (*k*, *θ*)**
  - **Σ** is the **Skew Symmetric Matrix representation of *k***

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In summary

$$\boxed{\frac{dR(\theta)}{d\theta} = \Sigma R(\theta)}$$

- Find the axis ***k***, rotation around which is represented by the matrix **R**
- Represent ***k*** in its **Skew Symmetric Matrix** form, $\boldsymbol{\Sigma}_{\boldsymbol{k}}$
- **Multiply** Σ and **R; you will get the first derivative of R.**

$$k = \frac{1}{2sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

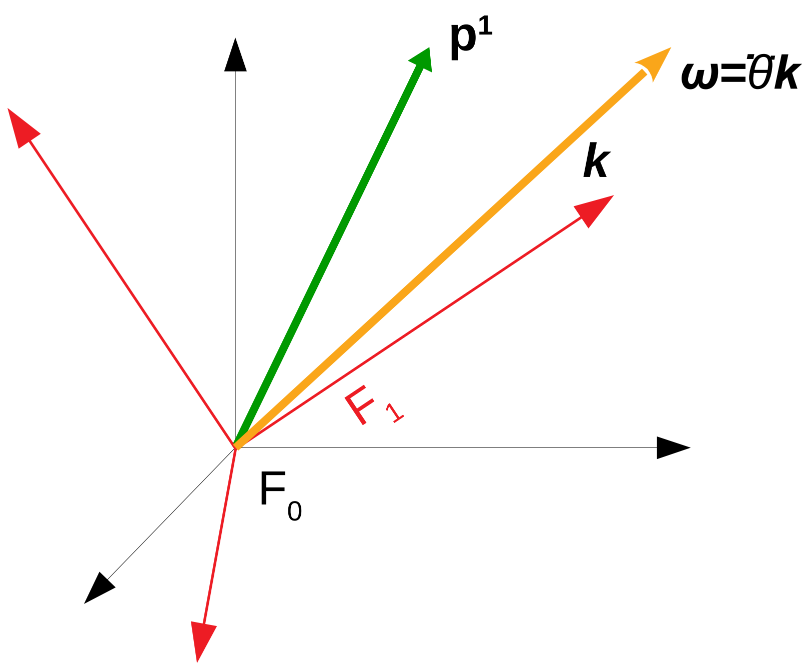
r_{ij} = element at location (i,j) of the rotation matrix **R**

Velocity Kinematics

- kinematics of particles with respect to rotating and translating frames
- In particular, consider:
  1) Velocity about a fixed point
  2) General velocity: rotation plus translation
  3) Acceleration
  4) Addition of Angular Velocities

Velocity about a fixed point

- Imagine a bar (p^1) welded to Frame F_1.
  - p^1 is constant *w.r.t.* F_1
- Then, consider the bar is rotating around frame F_0.
  - That means, Frame F_1 is rotating around F_0 as well.



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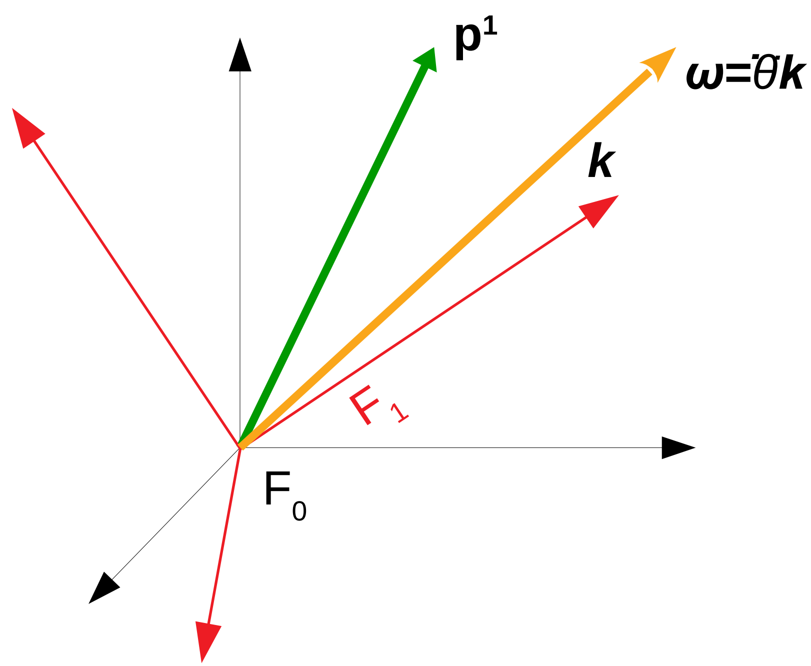
Velocity about a fixed point

- From our very early discussions, we know that
  $\mathbf{p}^0 = R_{01}\mathbf{p}^1$
- But now, we qualify R with a time factor as it is changing
  - $\mathbf{p}^0 = R_{01}(t)\mathbf{p}^1$
- Differentiating *wrt* time *t*,

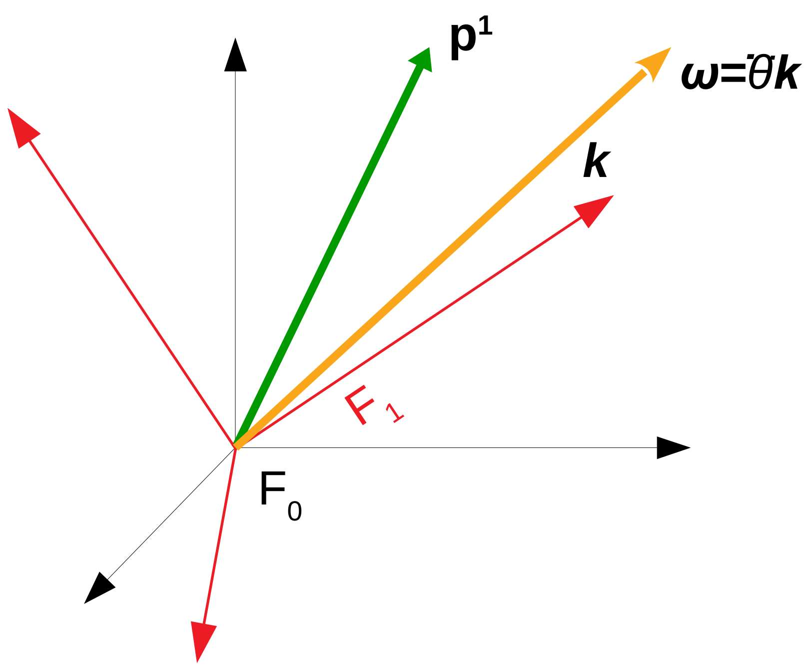
$$\dot{\boldsymbol{p}}^{\mathbf{0}} = \frac{dR_{01}(t)}{dt}\boldsymbol{p}^1$$



Velocity about a fixed point

- Representing R_{01} using the axis-angle approach, let's assume ***k*** is the rotation axis and θ is the rotation angle around ***k***.

$$
\begin{aligned}
\dot{R}_{01}(t) &= \frac{dR_{01}(t)}{dt} \\
&= \frac{dR_k(\theta)}{d\theta}\frac{d\theta}{dt} \\
&= S(k)R_k(\theta(t))\dot{\theta} \\
&= S(\omega)R_k(\theta(t))
\end{aligned}
$$

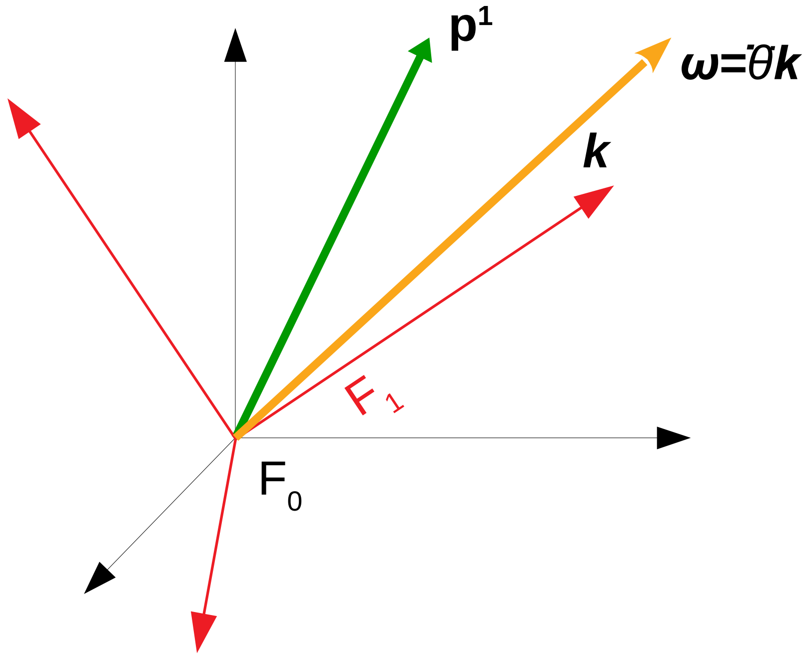


ω is the angular <u>velocity</u> around ***k***

Velocity about a fixed point

- ω is the angular <u>velocity</u> around ***k***
  - Describes the angular rotation of F_1 around the rotation axis ***k***
- Finally, plugging all values: $\dot{\boldsymbol{p}}^0 = S(\boldsymbol{\omega})R_{01}\boldsymbol{p}^1$
- Equivalently:

$$\underline{\boldsymbol{p}}^0 = \boldsymbol{\omega} \times \underline{\boldsymbol{p}}$$



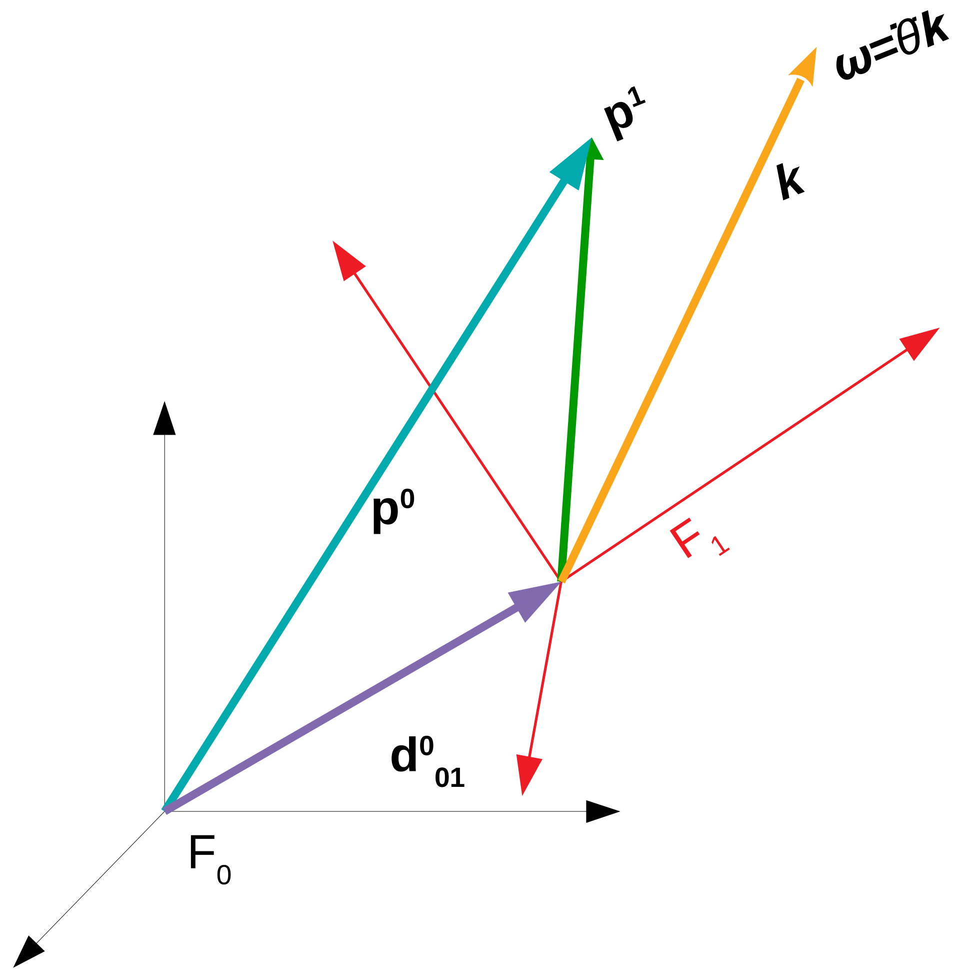
General Velocity: Rotation+Translation

- Imagine the same bar (p^1) welded to Frame F_1
  - p^1 still constant *w.r.t.* F_1
  - But F_1 is both translating rotating *wrt* F_0
- We have

  $$\mathbf{p}^0 = R_{01}(t)\mathbf{p}^1 + d^0_{01}$$

- Differentiating:

$$\begin{aligned}\dot{\boldsymbol{p}}^0 &= \dot{R}_{01}\boldsymbol{p}^1 + \dot{\boldsymbol{d}}^0_{01} \\ &= S(\boldsymbol{\omega})R_{01}\boldsymbol{p}^1 + \dot{\boldsymbol{d}}^0_{01}\end{aligned}$$



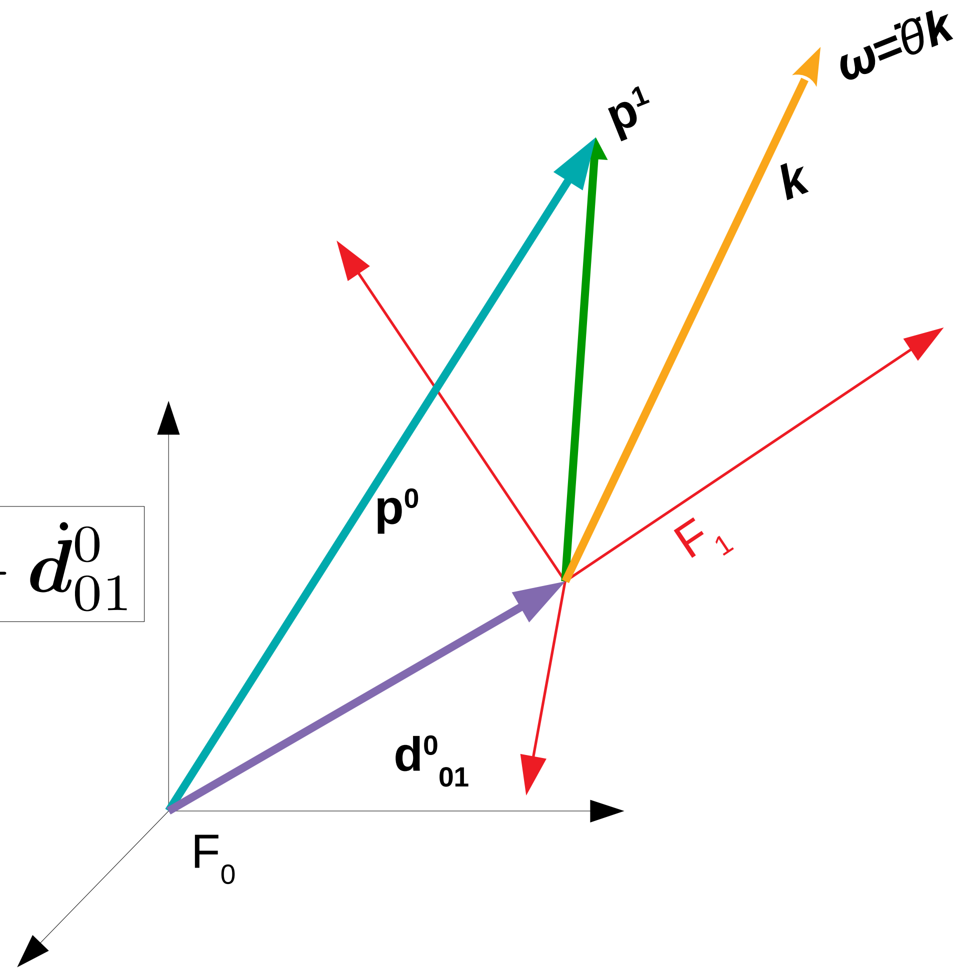
Generalizing Further

- Imagine the same bar p^1 **not** welded to Frame F_1
  - p^1 **not** constant *w.r.t.* F_1
  - But F_1 is both translating rotating *wrt* F_0
- We have

$$\dot{\boldsymbol{p}}^0 = S(\boldsymbol{\omega})\boxed{R_{01}\boldsymbol{p}^1} + \boxed{R_{01}\dot{\boldsymbol{p}}^1 + \dot{\boldsymbol{d}}^0_{01}}$$

Vector from rotation axis to **p** in F_0

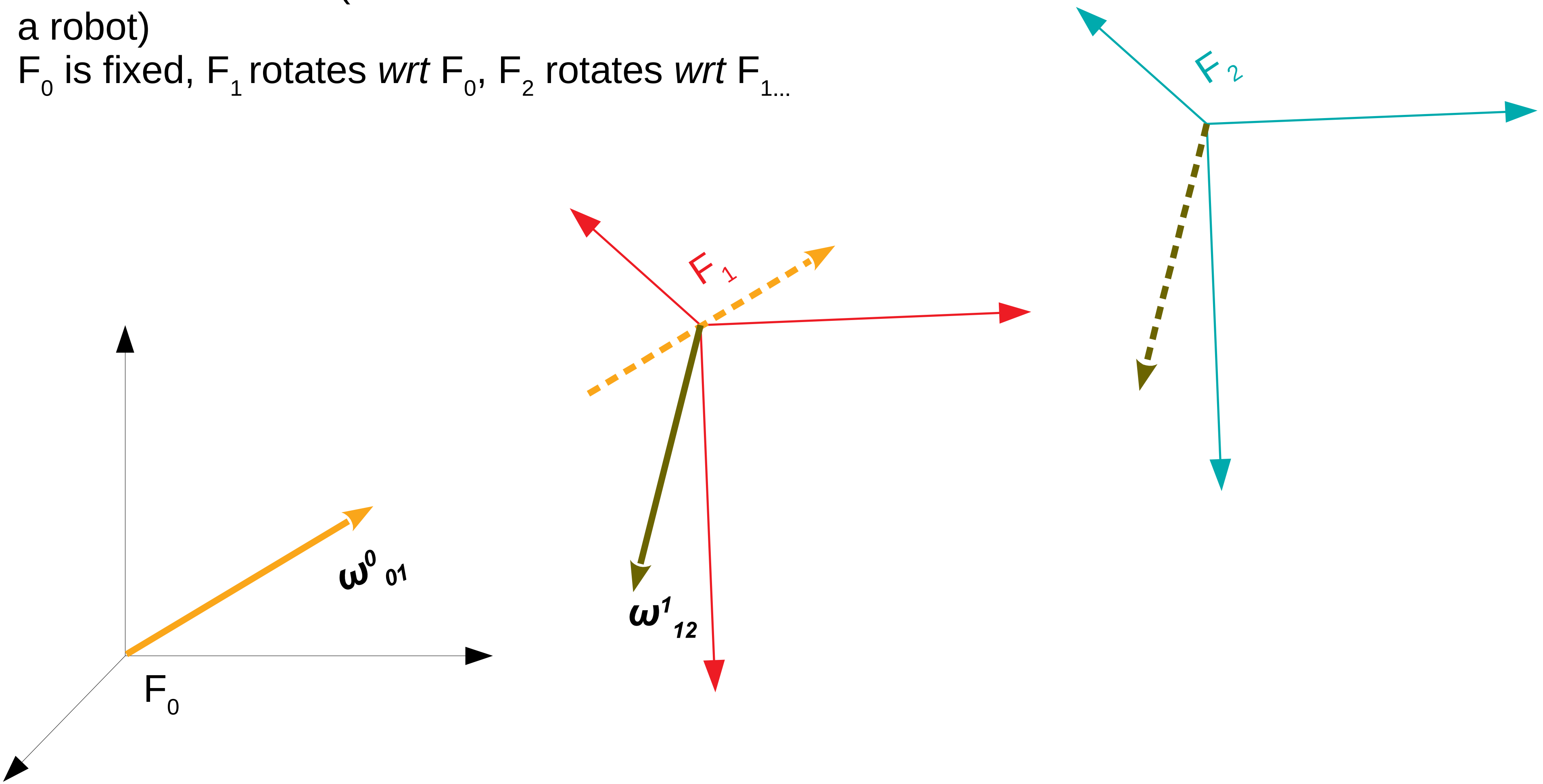
Linear velocity of **p** in F_0



Adding Angular Velocities

A series of frames (like those attached to links of a robot)

F_0 is fixed, F_1 rotates *wrt* F_0, F_2 rotates *wrt* F_{1}...



Adding Angular Velocities

- As before:
  $R_{02} = R_{01} R_{12}$

- Differentiating:

$$
\begin{aligned}
\dot{R}_{02} &= \dot{R}_{01} R_{12} + R_{01} \dot{R}_{12} \\
&= S(\boldsymbol{\omega}_{01}^{0}) R_{01} R_{12} + R_{01} S(\boldsymbol{\omega}_{12}^{1}) R_{12} \\
&= S(\boldsymbol{\omega}_{01}^{0}) R_{02} + R_{01} S(\boldsymbol{\omega}_{12}^{1}) R_{01}^{T} R_{01} R_{12} \\
&= S(\boldsymbol{\omega}_{01}^{0}) R_{02} + S(R_{01} \boldsymbol{\omega}_{12}^{1}) R_{02} \\
&= S(\boldsymbol{\omega}_{01}^{0} + R_{01} \boldsymbol{\omega}_{12}^{1})) R_{02}
\end{aligned}
$$

$R^T R = I$

$R S(\boldsymbol{a}) R^T = S(R\boldsymbol{a})$

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Adding Angular Velocities

- But we know that (slide 16):
  $\dot{R}_{02} = S(\boldsymbol{\omega}_{02}^0)R_{02}$
- Equating:
  $$\boldsymbol{\omega}_{02}^0 = \boldsymbol{\omega}_{01}^0 + R_{01}\boldsymbol{\omega}_{12}^1$$
- In general:
  $$\boldsymbol{\omega}_{0n}^0 = \boldsymbol{\omega}_{01}^0 + R_{01}\boldsymbol{\omega}_{12}^1 + R_{02}\boldsymbol{\omega}_{23}^2 + \cdots + R_{0,n-1}\boldsymbol{\omega}_{n-1,n}^{n-1}$$

The Manipulator Jacobian

- A manipulator
  - can be described as a series of points, namely the end points of the links,
  - Links are each, in turn, rotating and translating with respect to the previous link
- In the world of velocity *manipulator* kinematics, the goal is to achieve a mapping, which does <u>two things</u>.

The Manipulator Jacobian

- Goal 1: Relate the rate of change of position of the manipulator end-effector to the rate of change of motion of each link.

$$\dot{\boldsymbol{q}} = \frac{dq_i}{dt} \rightarrow \dot{\boldsymbol{X}} = \left\{ \begin{array}{c} \boldsymbol{v} \\ \boldsymbol{\omega} \end{array} \right\} = \left\{ \begin{array}{c} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \\ \frac{d\phi_x}{dt} \\ \frac{d\phi_y}{dt} \\ \frac{d\phi_z}{dt} \end{array} \right\} = \left\{ \begin{array}{c} \frac{d\boldsymbol{p}}{dt} \\ \frac{d\boldsymbol{\phi}}{dt} \end{array} \right\}$$

The Manipulator Jacobian

- Goal 2: Relate the differential change in position (location and orientation) of one frame due to a differential change of another frame.
  - A *superset* of Goal 1.
  - Thus, more *powerful* and *useful* solution to deduce

The Manipulator Jacobian

- For Goal 1, the question becomes to find a map J such that

$$\dot{\boldsymbol{X}} = J\dot{\boldsymbol{q}}$$

- For Goal 2, the question becomes to find a map K such that for two frames F_0 and F_{1}:

$$dX^b = \begin{bmatrix} d\boldsymbol{p}^b \\ d\boldsymbol{\phi}^b \end{bmatrix} = K_{ba} \begin{bmatrix} d\boldsymbol{p}^b \\ d\boldsymbol{\phi}^b \end{bmatrix}$$

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Finding J

- Use the concept of infinitesimal rotations
- If $\phi \to d\phi$ ($d\phi \to 0$), then $\cos\phi \to 1$ and $\sin\phi \to d\phi$.
- This implies, for example, rotation around the Z axis

$$R_z(\phi) = \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

becomes

$$R(d\phi_z) = \begin{bmatrix} 1 & -d\phi_z & 0 \\ d\phi_z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

About Infinitesimal Rotations

- Commutative: R($d\phi_z$)R($d\phi_y$) = R($d\phi_y$)R($d\phi_z$)
  This also means

$$R(d\phi_x, d\phi_y, d\phi_z) = \begin{bmatrix} 1 & -d\phi_z & d\phi_y \\ d\phi_z & 1 & -d\phi_x \\ -d\phi_y & d\phi_x & 1 \end{bmatrix}$$

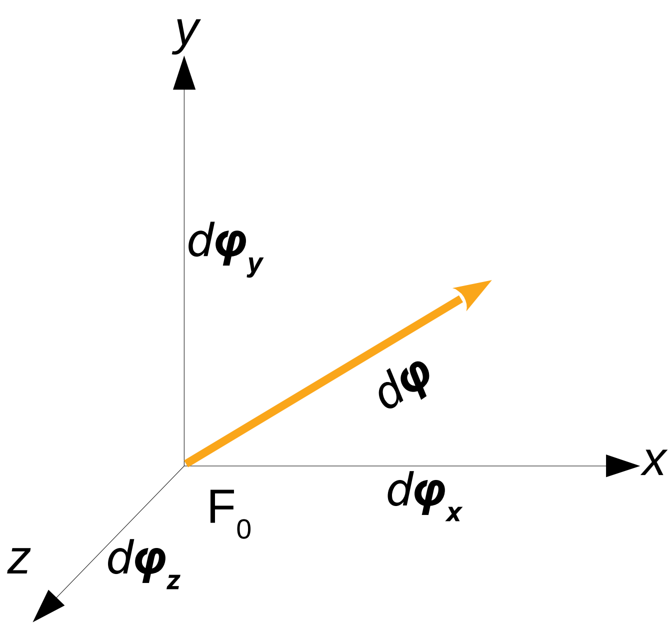
- Additive:

$$R(d\phi_x, d\phi_y, d\phi_z) + R(d\psi_x, d\psi_y, d\psi_z) = R(d\phi_x + d\psi_x, d\phi_y + d\psi_y, d\phi_z + d\psi_z)$$

About Infinitesimal Rotations

- Can be expressed as a vector in $\Re^3$:

$$d\boldsymbol{\phi} = \begin{bmatrix} d\phi_x \\ d\phi_y \\ d\phi_z \end{bmatrix}$$



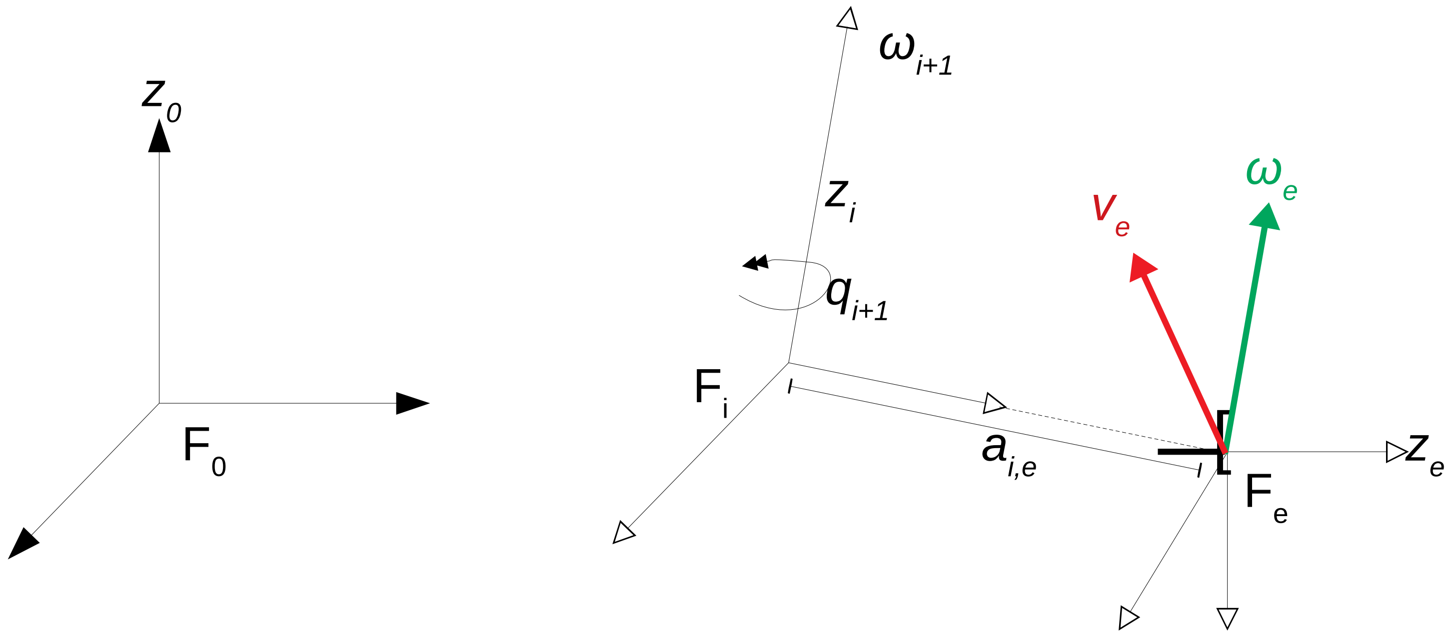
$$\frac{d\phi}{|d\phi|} = \text{direction of infinitesimal rotation}$$

$$|d\phi| = \text{magnitude of infinitesimal rotation}$$

$$\dot{\phi} = \frac{d\phi}{dt} = \boldsymbol{\omega}$$

Careful: Not true for non-infinitesimal rotations!

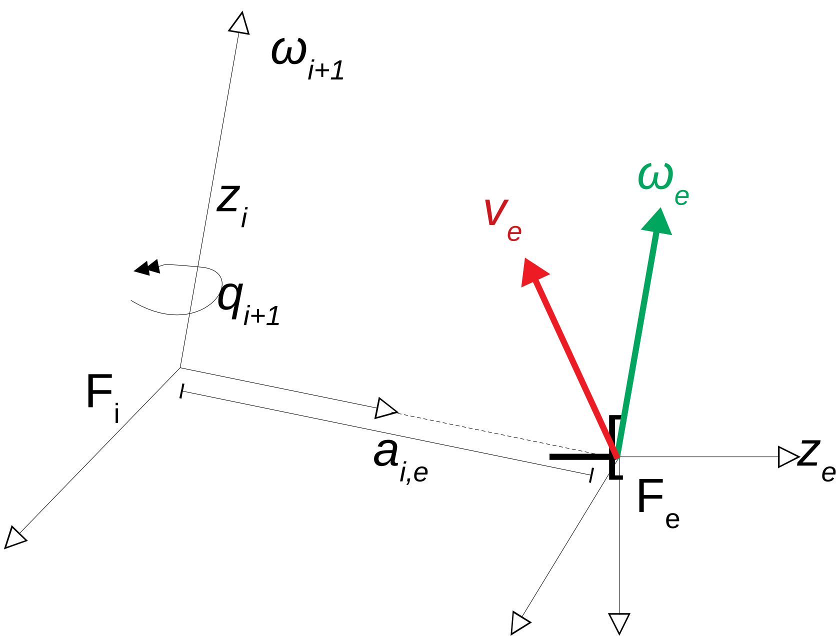
Formulating the Jacobian: The Geometric Approach



Case 1: Rotational Joint **only,** q_{i+1}

Why?: So that we can isolate the effects of each type of joint on end-effector velocities

Geometric Jacobian: Revolute Joints



For joint q_{i+1} :

$$\boldsymbol{v}_e^i = \boldsymbol{\omega}_{i,i+1} \times \boldsymbol{a}_{ie} = \boldsymbol{z}_i \times \boldsymbol{a}_{ie}\dot{q}_{i+1}$$

$$\boldsymbol{\omega}_e^i = \boldsymbol{z}_i\dot{q}_{i+1}$$

Summing up effects of all joints, when all joints are **revolute**:

$$\boldsymbol{v}_e = \sum_{i=0}^{e-1} \boldsymbol{\omega}_{i,i+1} \times \boldsymbol{a}_{ie} = \sum_{i=0}^{e-1} \boldsymbol{z}_i \times \boldsymbol{a}_{ie}\dot{q}_{i+1}$$

$$\boldsymbol{\omega}_e = \sum_{i=0}^{e-1} \boldsymbol{z}_i\dot{q}_{i+1}$$

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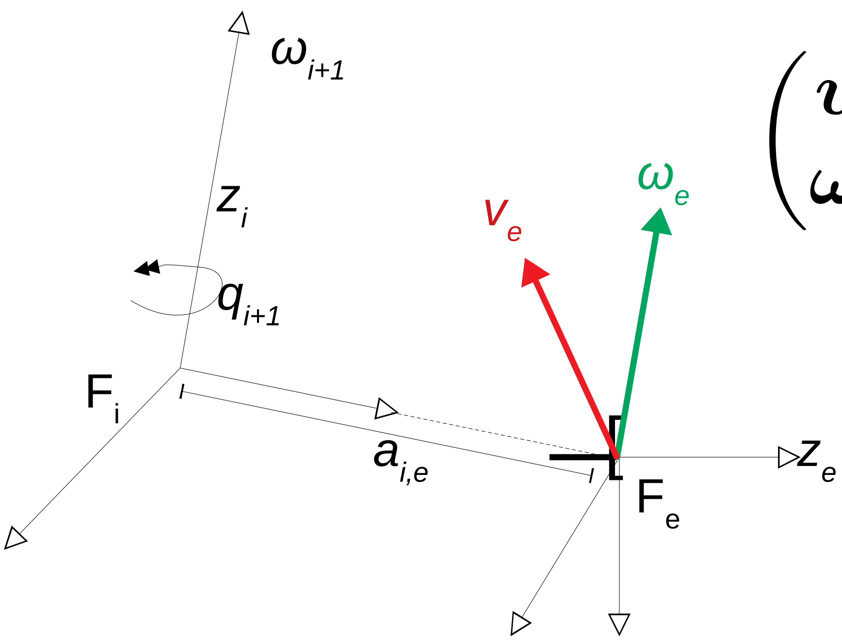
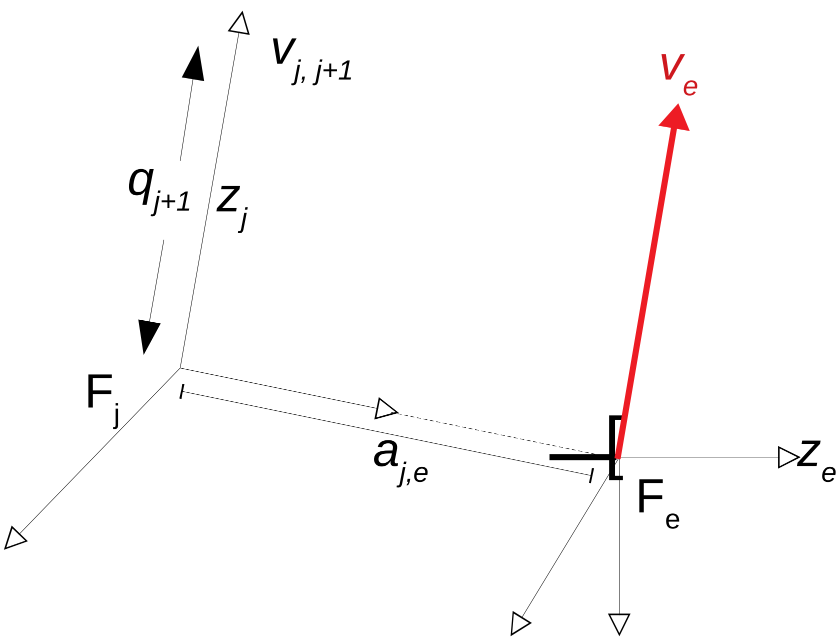
Geometric Jacobian: Revolute Joints



In Matrix form:

$$\begin{pmatrix} \boldsymbol{v}_e^0 \\ \boldsymbol{\omega}_e^0 \end{pmatrix} = \begin{bmatrix} \cdots & \boldsymbol{z}_i^0 \times \boldsymbol{a}_{ie}^0 & \cdots \\ \cdots & \boldsymbol{z}_i^0 & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ \dot{q}_{i+1} \\ \vdots \end{bmatrix}$$

Geometric Jacobian: Prismatic Joints



For joint q_{j+1} :

$$\boldsymbol{v}_e^j = \boldsymbol{z}_j \dot{q}_{j+1}$$

$$\boldsymbol{\omega}_e^j = 0$$

Summing up effects of all joints, when all joints are **prismatic**:

$$\boldsymbol{v}_e = \sum_{j=0}^{e-1} \boldsymbol{z}_j \dot{q}_{j+1}$$

$$\boldsymbol{\omega}_e = 0$$

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Finally, the Manipulator Jacobian

- Expression of a manipulator Jacobian for a robot with both prismatic and revolute joints is as follows:

$$\begin{pmatrix} \boldsymbol{v}_e^0 \\ \boldsymbol{\omega}_e^0 \end{pmatrix} = \begin{bmatrix} \cdots & \boldsymbol{z}_i^0 \times \boldsymbol{a}_{ie}^0 & z_j^0 & \cdots \\ \cdots & \boldsymbol{z}_i^0 & \dots 0 & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ \dot{q}_{i+1} \\ \vdots \\ \dot{q}_{j+1} \\ \vdots \end{bmatrix} = \mathrm{J}_o \dot{\boldsymbol{q}}$$

The Manipulator Jacobian

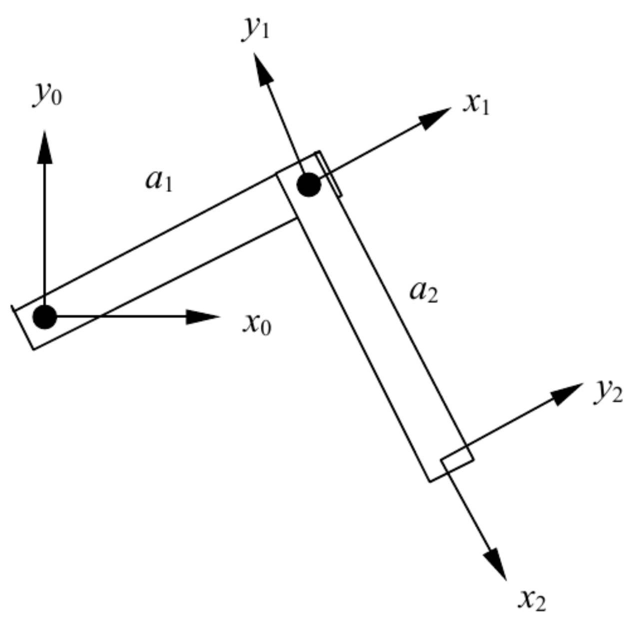
- The top three rows affect linear velocity (**velocity Jacobian J_v**)
- The bottom three rows affect angular velocity (**angular velocity Jacobian J_ω**)

$$\begin{pmatrix} \boldsymbol{v}_e^0 \\ \boldsymbol{\omega}_e^0 \end{pmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{\boldsymbol{q}}$$

- From the end-effector frame, F_e,

$$\begin{pmatrix} \boldsymbol{v}_e^e \\ \boldsymbol{\omega}_e^e \end{pmatrix} = \mathrm{R}_{e0} \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{\boldsymbol{q}}$$

Example: 2-link Planar Manipulator



Joint	θ_i	d_i	a_i	α_i
1	θ_1	0	a_1	0
2	θ_2	0	a_2	0

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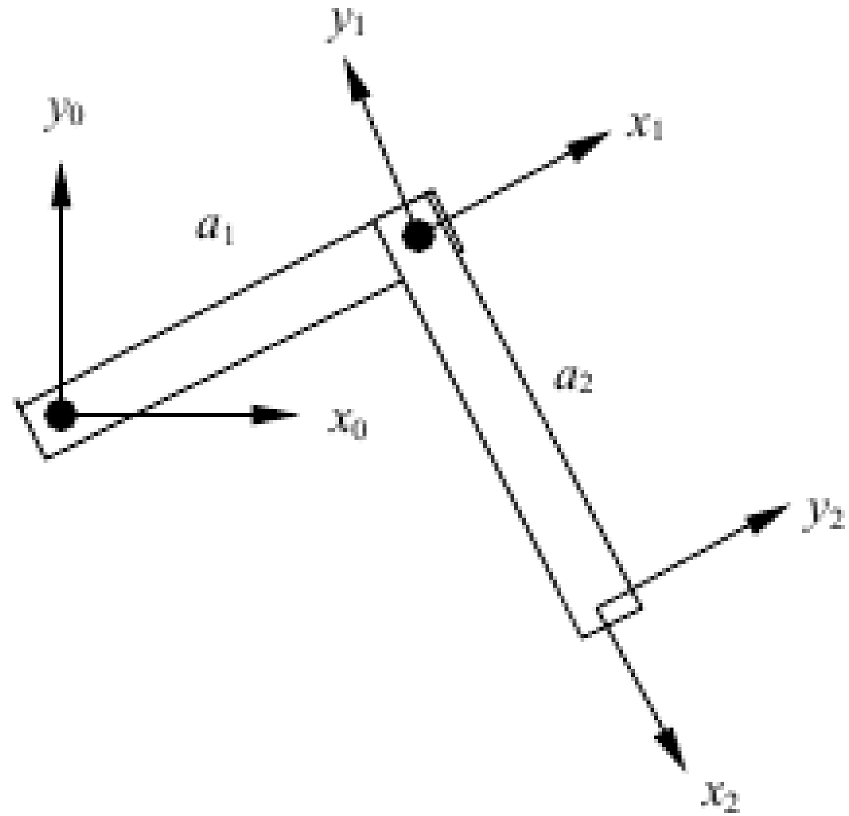
Example: 2-link Planar Manipulator

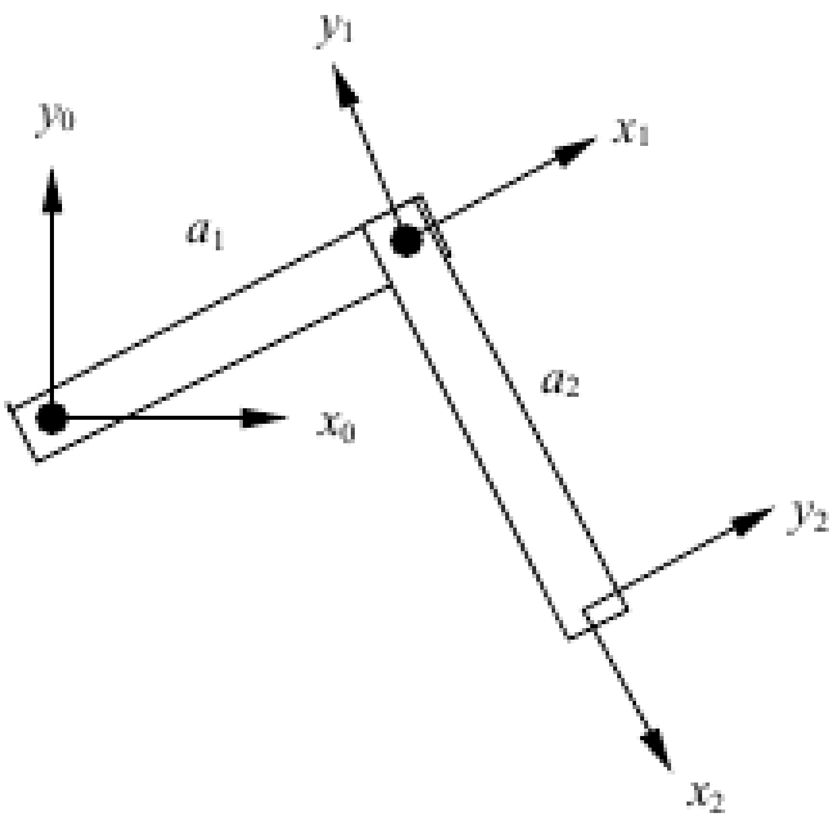
$$A_{01} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_1 c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1 s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$



$$A_{12} = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$T_{02} = A_{01}A_{12} = \begin{bmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) & 0 & a_1 c\theta_1 + a_2 c(\theta_1+\theta_2) \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2) & 0 & a_1 s\theta_1 + a_2 s(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example: 2-link Planar Manipulator



- Fin d $\boldsymbol{z}_{\mathbf{i}}^{\mathbf{0}}$, a_{ie}^{0}, for i=0, ..., n-1, where *n* is the number of joints.
- By definition: $$z_0^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
- For i>0, we can look at the column vectors of $A_{0,i}$ to find the orientation of $\boldsymbol{x}_i^0$,$\boldsymbol{y}_i^0$ and $\boldsymbol{z}_i^0$.
- In other words $\boldsymbol{z}_i^0$ is the <u>first three elements</u> of the <u>3rd column of $A_{0,i}$</u>.

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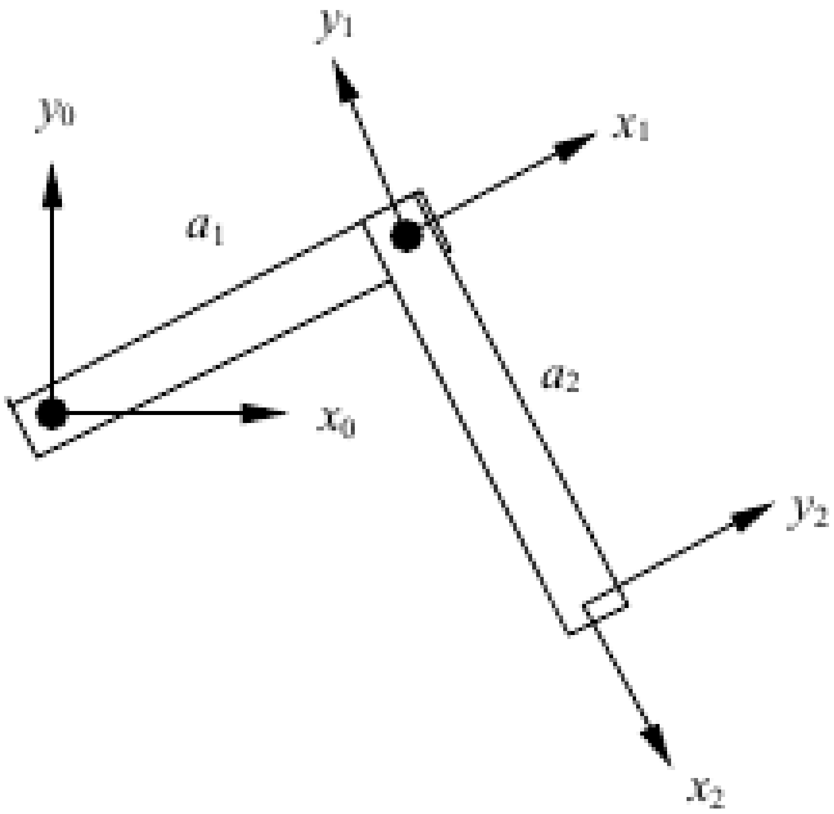
Example: 2-link Planar Manipulator



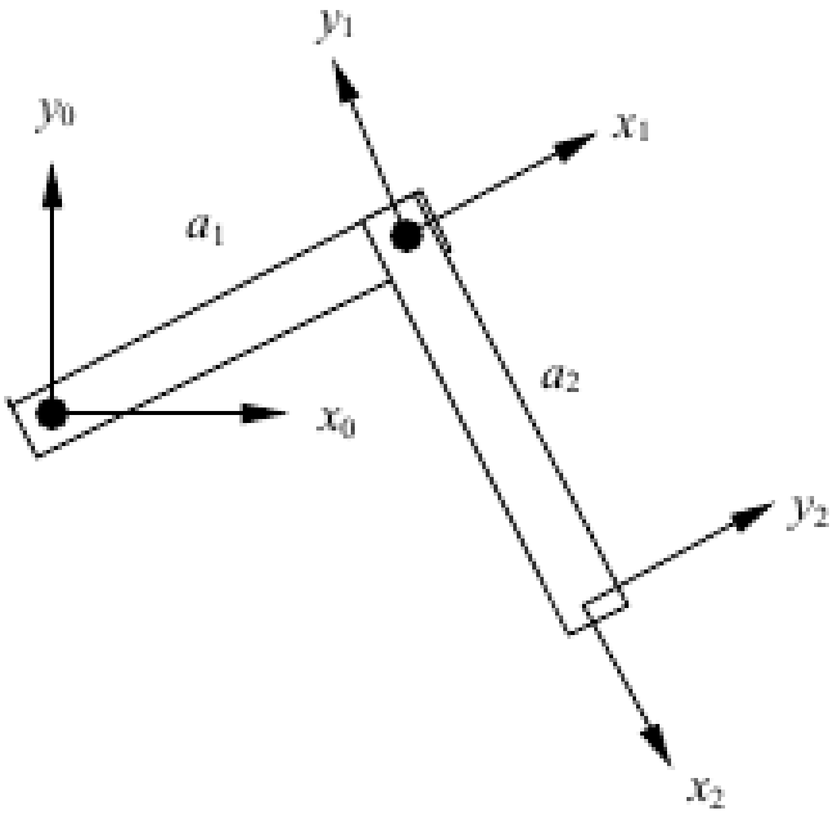
$$A_{0i} = \begin{bmatrix} \boldsymbol{n}_i^0 & \boldsymbol{s}_i^0 & \boldsymbol{a}_i^0 & \boldsymbol{d}_i^0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\boldsymbol{z}_i^0 = \boldsymbol{a}_i^0$$

In this specific instance, from $A_{0,1}$:

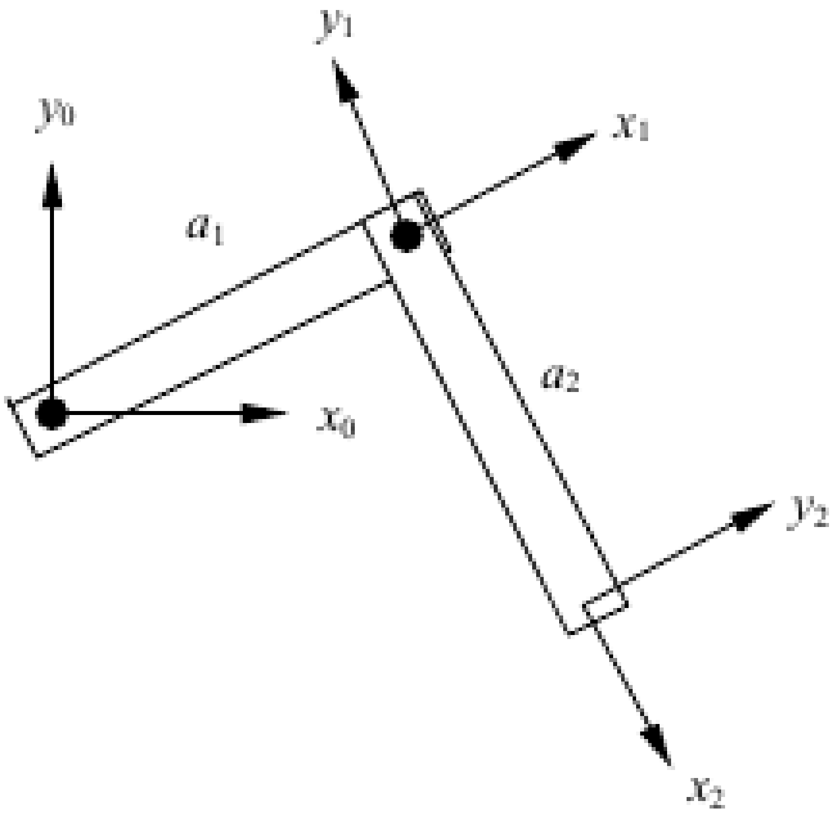
$$\boldsymbol{z}_1^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Example: 2-link Planar Manipulator



- $\boldsymbol{a}_{ie}^{0}$: look at the 4th column of the Homogeneous transformations, $A_{0,i}$ and, $T_{0,e}$, specifically, $\boldsymbol{d}_i^0$ and $\boldsymbol{d}_e^0$.
- $\boldsymbol{d}_i^0$ is the location of the ith frame (usually at the center of the i+1th joint).
- $\boldsymbol{d}_e^0$ is the location of the end-effector.
- $\boldsymbol{a}_{ie}^{0}$ is the distance between $\boldsymbol{d}_i^0$ and $\boldsymbol{d}_e^0$: $\boldsymbol{a}_{ie}^{0}=\boldsymbol{d}_e^{0}-\boldsymbol{d}_i^{0}$
- Also, $\boldsymbol{a}_{0e}^{0}=\boldsymbol{d}_e^{0}$

Example: 2-link Planar Manipulator



- In this specific case:

$$a_{0e}^0 = \begin{bmatrix} a_1 c\theta_1 + a_2 c(\theta_1 + \theta_2) \\ a_1 s\theta_1 + a_2 s(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$a_{1e}^0 = \begin{bmatrix} a_2 c(\theta_1 + \theta_2) \\ a_2 s(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

Example: 2-link Planar Manipulator

- Finally, the Jacobian

$$J_0 = \begin{bmatrix} \boldsymbol{z}_0^0 \times \boldsymbol{d}_{0e}^0 & \boldsymbol{z}_1^0 \times \boldsymbol{d}_{1e}^0 \\ \boldsymbol{z}_0^0 & \boldsymbol{z}_1^0 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_1 c\theta_1 + a_2 c(\theta_1 + \theta_2) \\ a_1 s\theta_1 + a_2 s(\theta_1 + \theta_2) \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 c(\theta_1 + \theta_2) \\ a_2 s(\theta_1 + \theta_2) \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -a_1 s\theta_1 - a_2 s(\theta_1 + \theta_2) & -a_2 s(\theta_1 + \theta_2) \\ a_1 c\theta_1 + a_2 c(\theta_1 + \theta_2) & a_2 c(\theta_1 + \theta_2) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Example: 2-link Planar Manipulator

- The final equation:

$$\begin{bmatrix} \boldsymbol{v}_e^0 \\ \boldsymbol{\omega}_e^0 \end{bmatrix} = \begin{bmatrix} -a_1 s\theta_1 - a_2 s(\theta_1 + \theta_2) & -a_2 s(\theta_1 + \theta_2) \\ a_1 c\theta_1 + a_2 c(\theta_1 + \theta_2) & a_2 c(\theta_1 + \theta_2) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$