
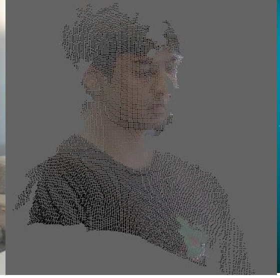


CSci 5551
Introduction to Intelligent Robotics Systems

Rotations “round ‘n round we go”

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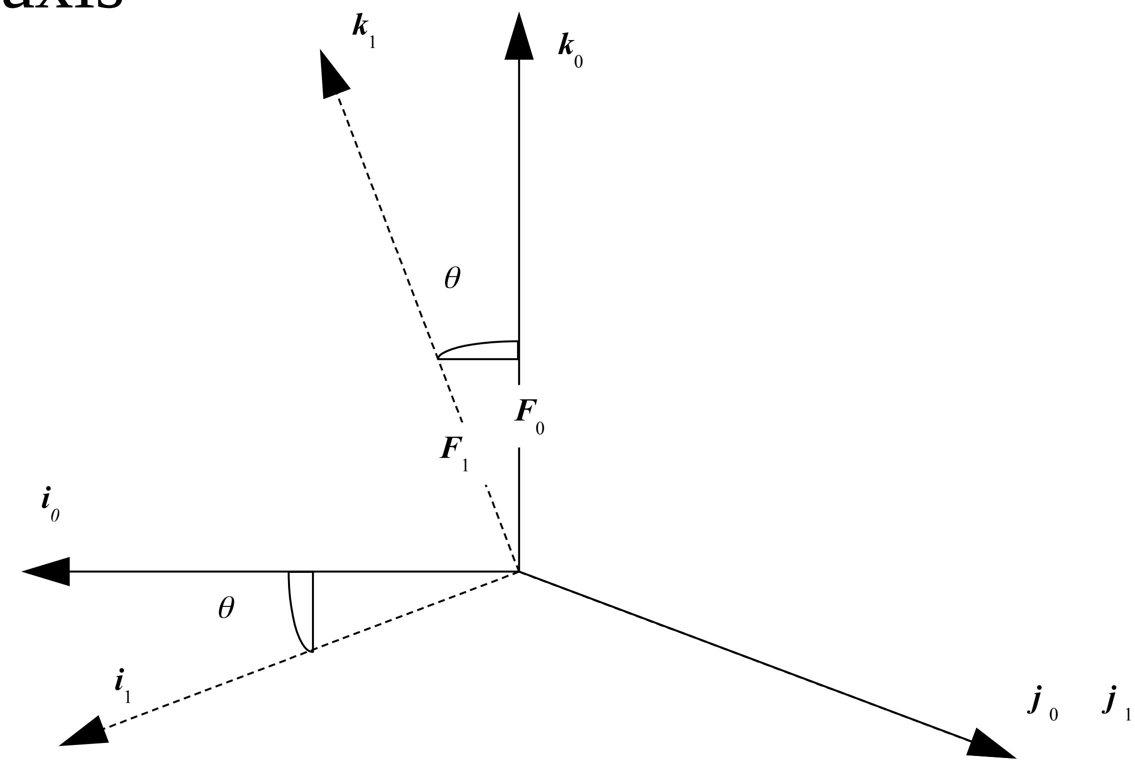
Notation Differences

$$\mathrm{R}_{01} \equiv^0_1 R$$

$$R_{AB} \equiv^A_B R$$

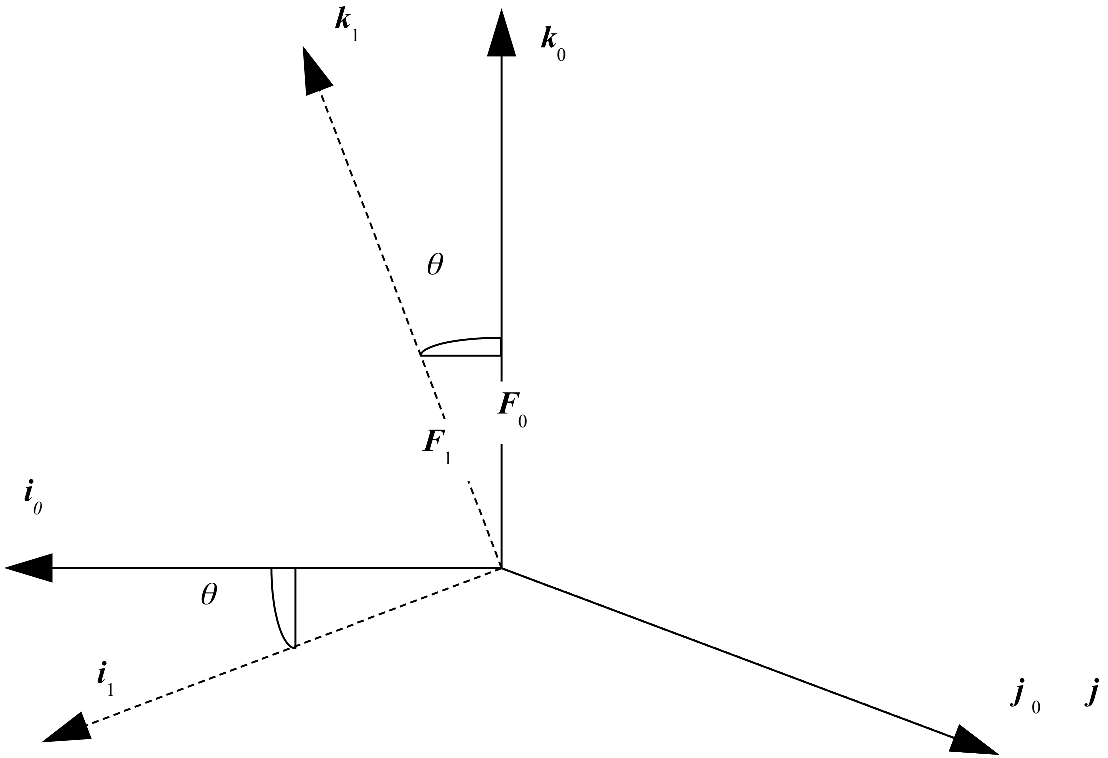
Exercise

Find the rotation matrix which relates frame F_1 to frame F_0, given that F_1 is established by a rotation of θ about the y_0 axis (the axis along j_0).



Step 1 – compute the elements of R

$$i_o \cdot i_1 = \cos\theta$$
$$i_o \cdot j_1 = 0$$
$$i_o \cdot k_1 = \cos(\pi/2 - \theta) = \text{-} \sin\pi/2 \sin(-\theta) = \sin(\theta)$$
$$j_o \cdot i_1 = 0$$
$$j_o \cdot j_1 = 1$$
$$j_o \cdot k_1 = 0$$
$$k_o \cdot i_1 = \cos(\pi/2 + \theta) = \text{-}\sin(\theta)$$
$$k_o \cdot j_1 = 0$$
$$k_o \cdot k_1 = \cos\theta$$



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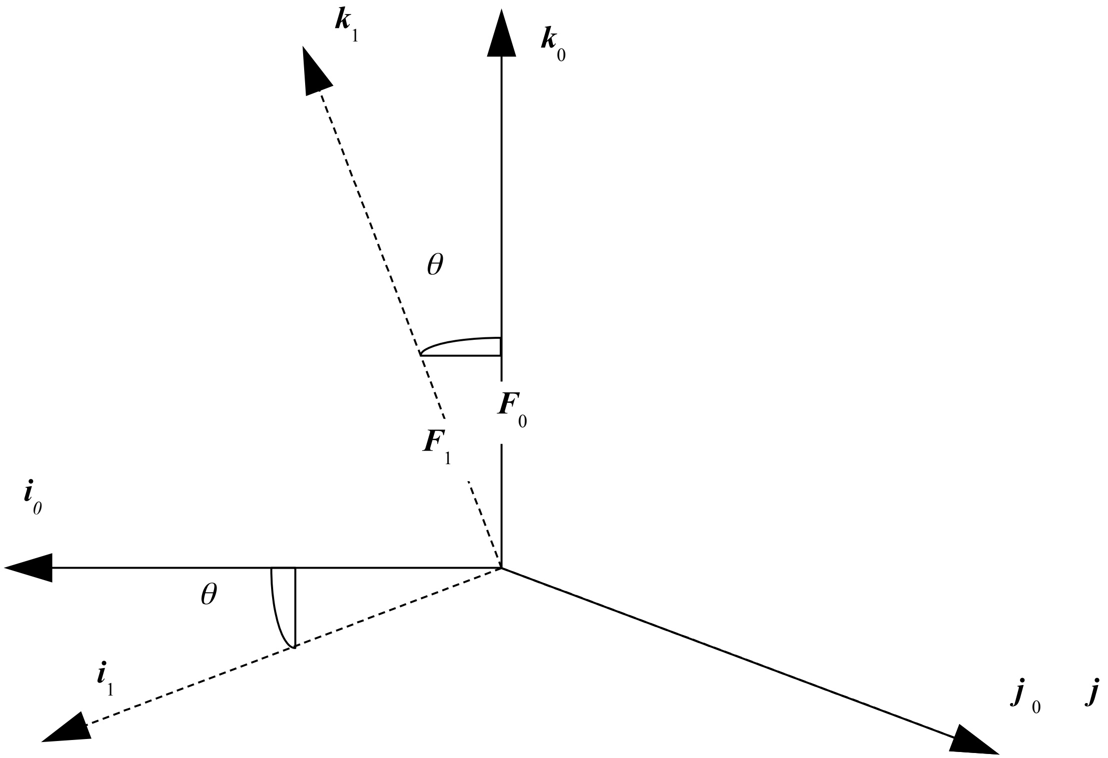
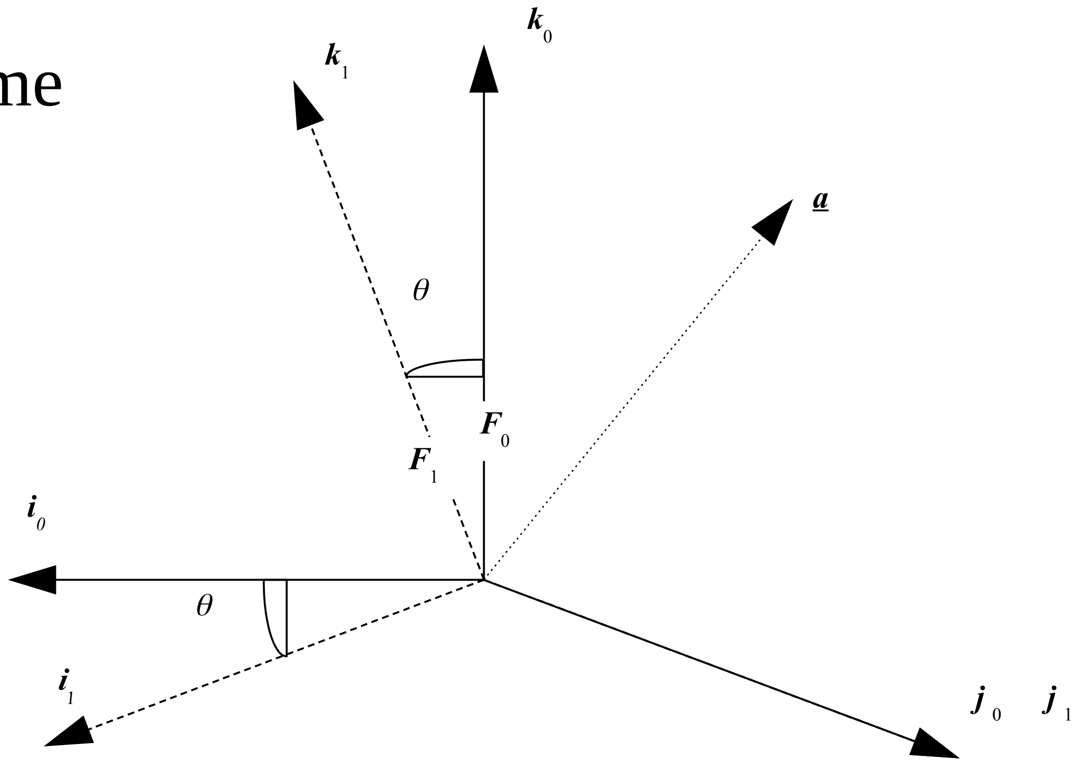
Step 2 – represent R

$$R_{01}=\begin{bmatrix} i_0\cdot i_1 & i_0\cdot j_1 & i_0\cdot k_1 \\ j_0\cdot i_1 & j_0\cdot j_1 & j_0\cdot k_1 \\ k_0\cdot i_1 & k_0\cdot j_1 & k_0\cdot k_1 \end{bmatrix}=\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$



Exercise #2

Given the two frames $\boldsymbol{F}_0$ and $\boldsymbol{F}_1$ described in the previous exercise, and given a physical vector $\underline{\boldsymbol{a}}$ described as vector $\boldsymbol{a}^1$ in frame $\boldsymbol{F}_1$, find the corresponding vector $\boldsymbol{a}^0$ in frame $\boldsymbol{F}_0$.



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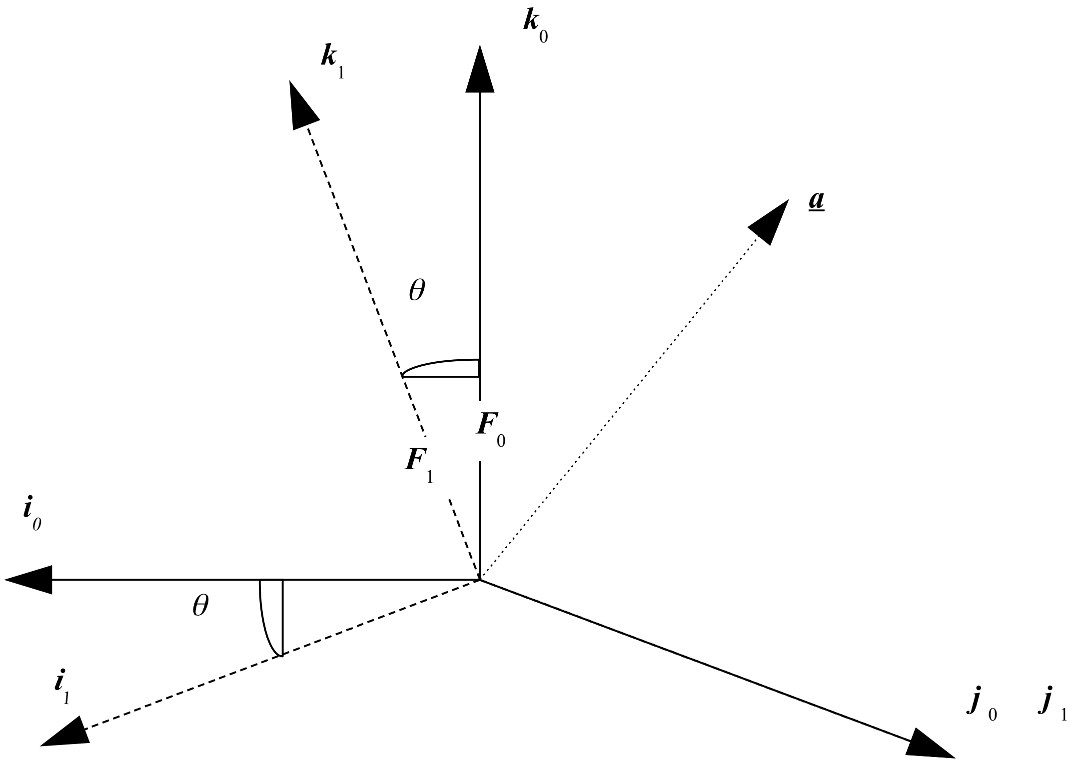
Formulate first

From our previous discussion:

$$
\begin{aligned}
\underline{a} &= F_1^T a^1 \\
a^0 &= F_0 \cdot \underline{a} \\
a^0 &= F_0 \cdot F_1^T a^1 \\
a^0 &= R_{01} a^1
\end{aligned}
$$



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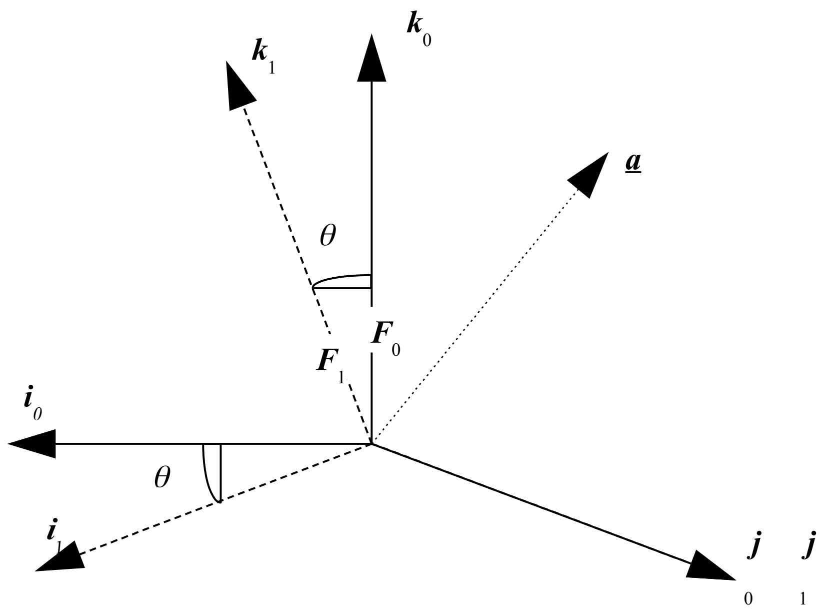
Now...

We already know R_{01}

$$R_{01} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Substituting for R_{01}

$$a^0 = \begin{bmatrix} a_{1x}\cos\theta + a_{1z}\sin\theta \\ a_{1y} \\ -a_{1x}\sin\theta + a_{1z}\cos\theta \end{bmatrix}$$



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Basic Rotation Matrices

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Composition of Rotations

Establish how rotations can be composed to represent a <u>series of rotations</u>.

We know:

$$\mathbf{p^0} = \mathbf{R_{01}} \mathbf{p^1}$$

And by that logic:

$$\mathbf{p^1} = \mathbf{R_{12}} \mathbf{p^2}$$

Combining:

$$\mathbf{p^0} = \mathbf{R_{01}} \mathbf{R_{12}} \mathbf{p^2}$$

Composition of Rotations

Thus we can state that, given two consecutive **current frame** rotations R_{01} and R_{12} (namely, a rotation between $\boldsymbol{F_0}$ and $\boldsymbol{F_1}$, followed by a rotation between $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$), the rotation between $\boldsymbol{F_0}$ and $\boldsymbol{F_2}$ can be represented by:

$$R_{02} = R_{01} R_{12}$$

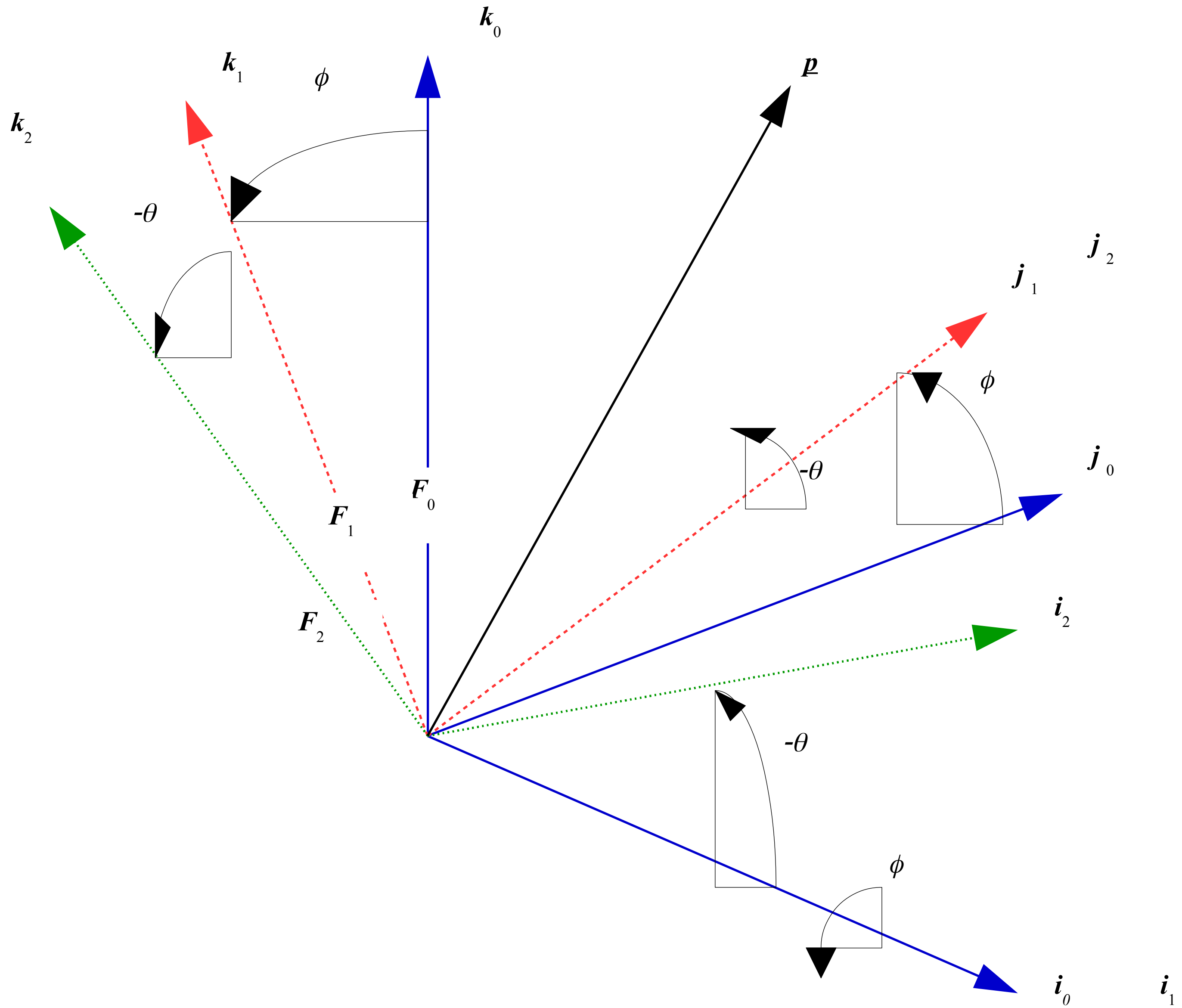
In general, **current frame** rotations are denoted by

$$R_{ik} = R_{i,i+1} R_{i+1,i+2} \cdots R_{k-1,k}$$

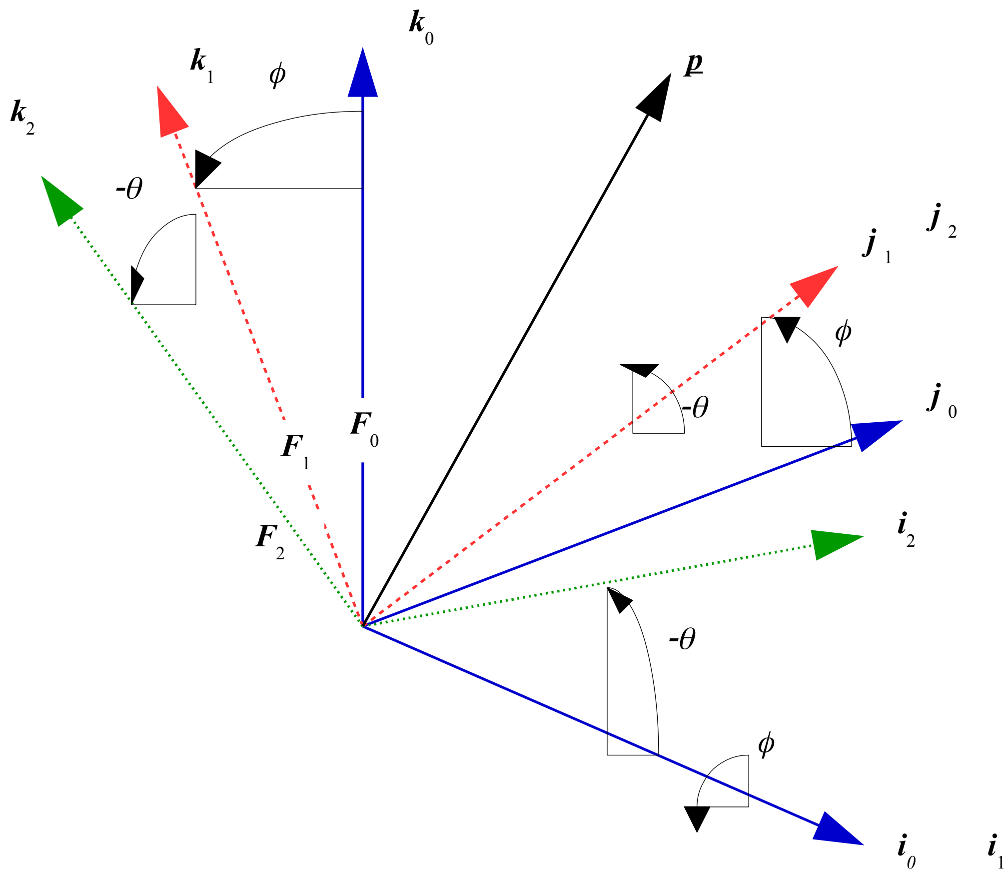
An example of rotation composition

- Find the rotation matrix R which represents a rotation of φ about x_0 ($\boldsymbol{i}_0$) followed by a rotation of -θ about the new y_1 ($\boldsymbol{j}_1$)

An example of rotation composition



An example of rotation composition



- Find the rotation matrix R which represents a rotation of φ about x_0 ($\boldsymbol{i}_0$) followed by a rotation of -θ about the new y_1 ($\boldsymbol{j}_1$)

Two successive "current frame" rotations

$$R_{02} = R_x(\phi) R_y(-\theta)$$

Where:

$$R_{02} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} c\,\theta & 0 & -s\,\theta \\ -s\,\phi s\,\theta & c\,\phi & -s\,\phi c\,\theta \\ c\,\phi s\,\theta & s\,\phi & c\,\phi c\,\theta \end{bmatrix}$$

Important to Remember

Please note: the sequence of the matrix multiplication is crucial.

$$R_x(\phi)R_y(\theta) \neq R_y(\theta)R_x(\phi)$$

Unless

$$R_x \equiv R_y(\theta) \equiv I$$

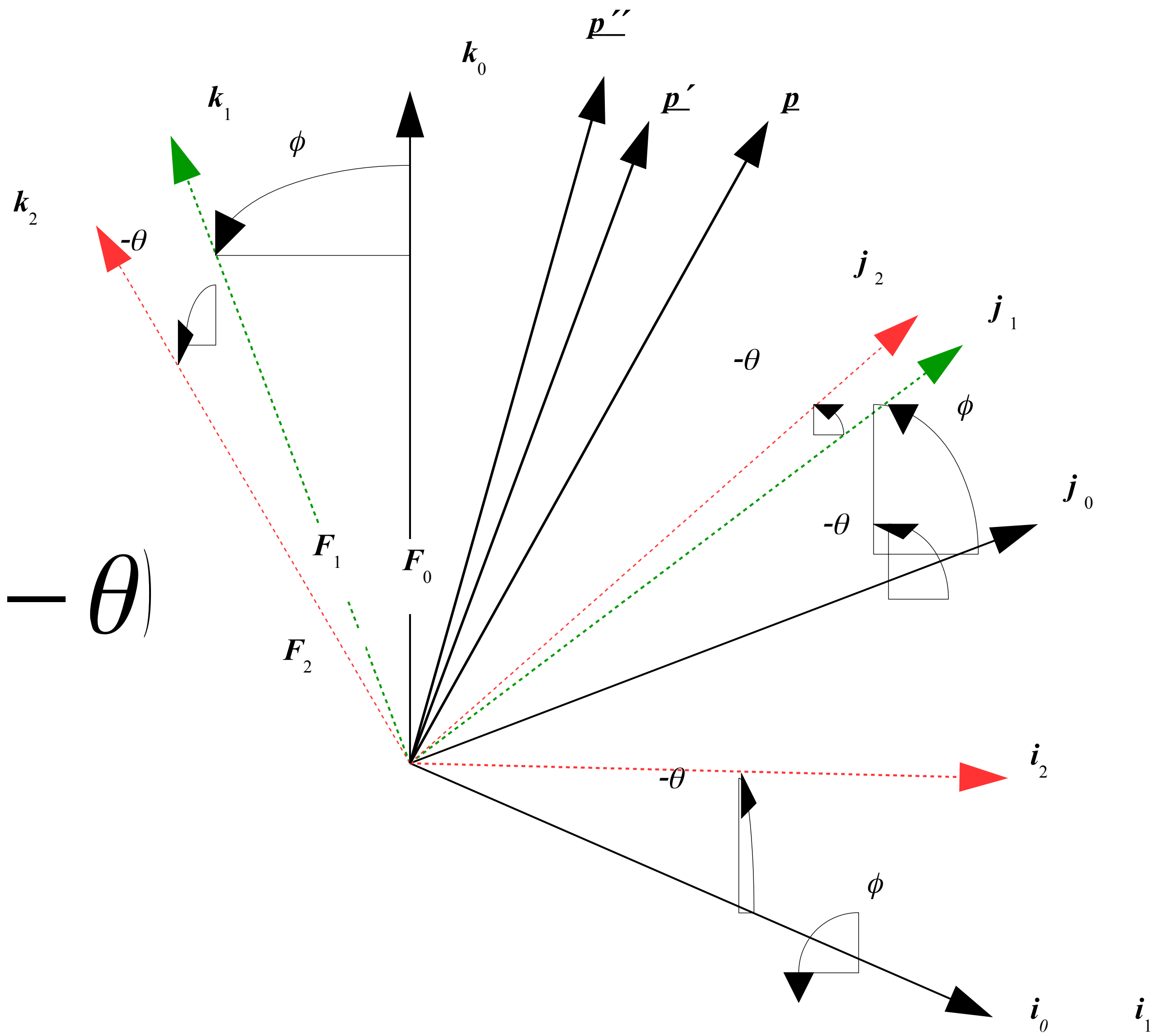
Fixed Frame Rotations

In this case we are considering rotating a vector (and its attached frame) about multiple axes of the **same fixed frame**.

Fixed Frame Rotation - Example Problem

Rotate the vector ***p*** about x_0 by φ (to a location ***p'***) followed by a rotation about y_0 by -θ to ***p''***.

$$R_{02} = R_x(\phi)\, R_y(-\theta)$$



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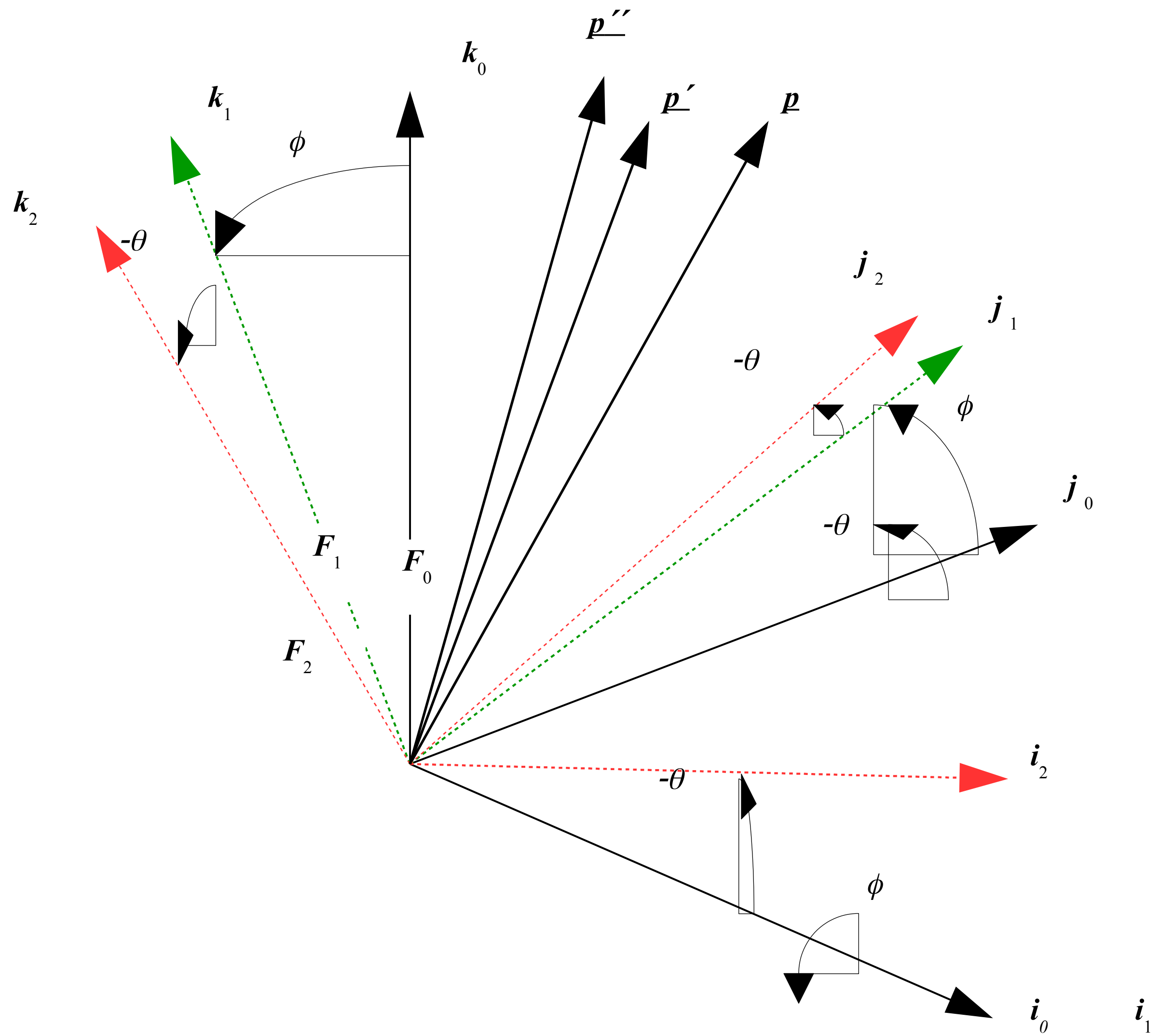
Fixed Frame Rotation - Example Problem

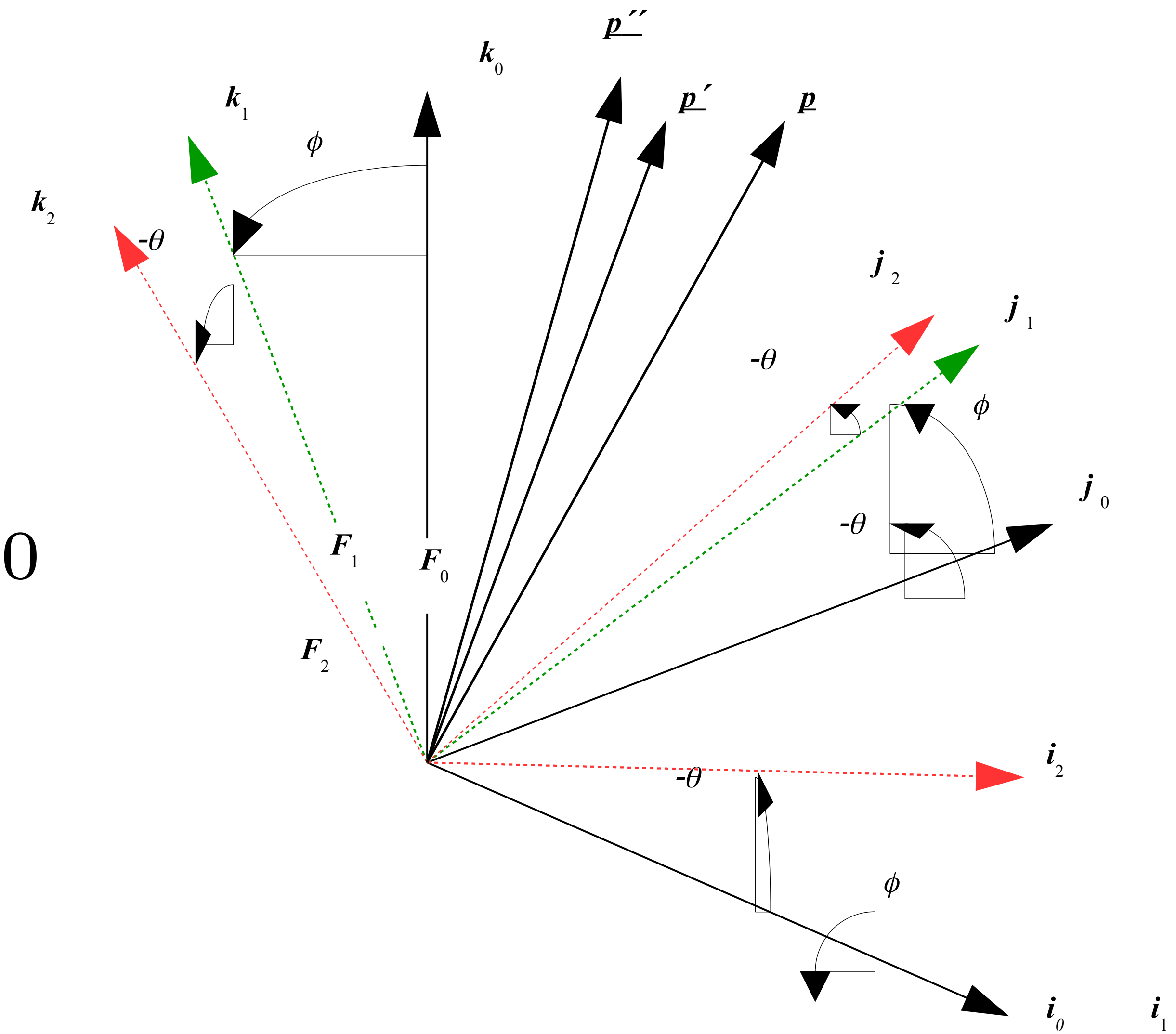
both rotations are done with respect to the F_0 frame.



Fixed Frame Rotation - Example Problem

Consider the first rotation which moves ***p*** to ***p'***. As in the previous example:

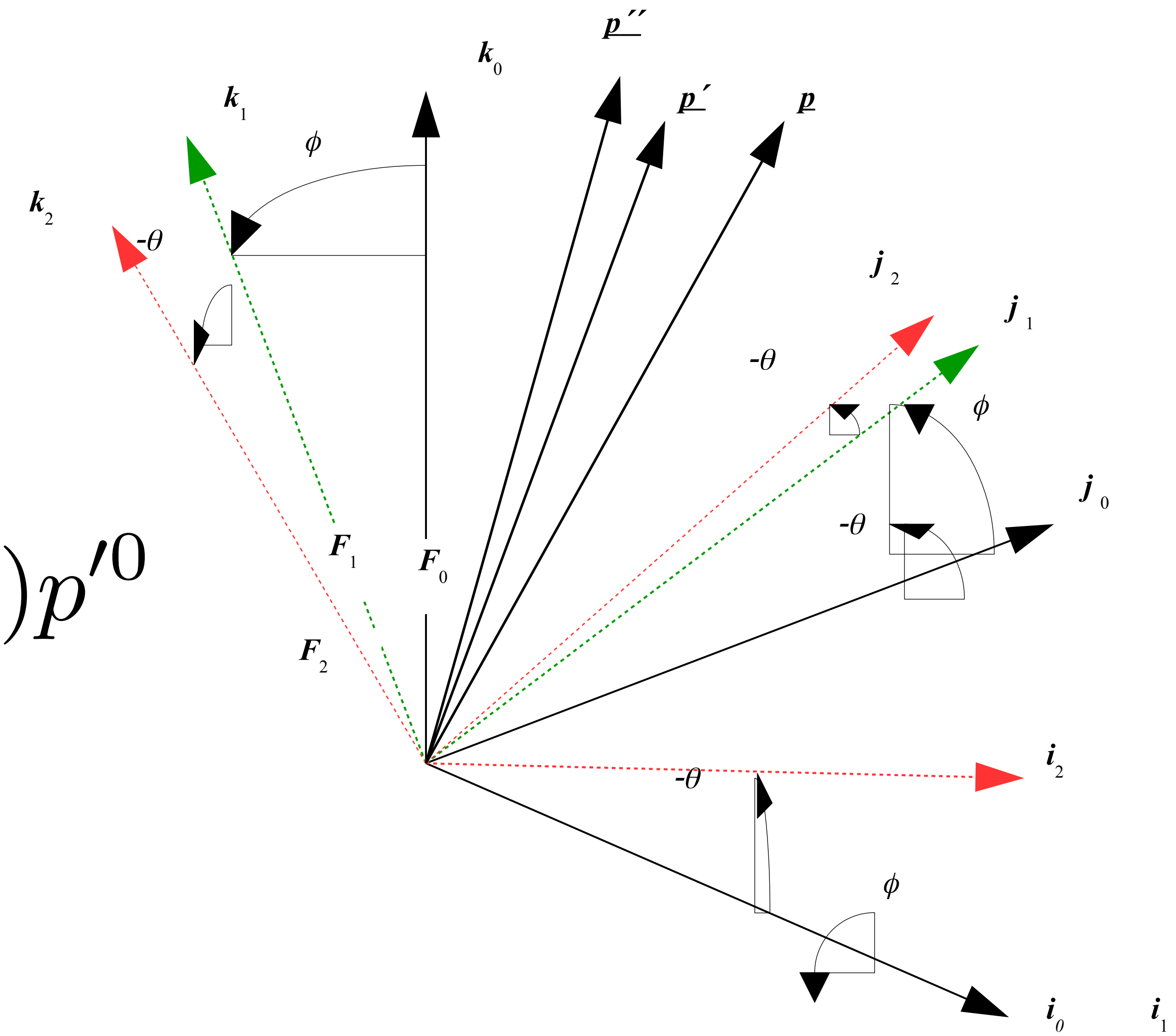
$$p'^0 = R_x(\phi)p^0$$



Fixed Frame Rotation - Example Problem

Now consider the second rotation, which moves ***p***´ to ***p***´´.

$$p''^0 = R_y(-\theta)p'^0$$



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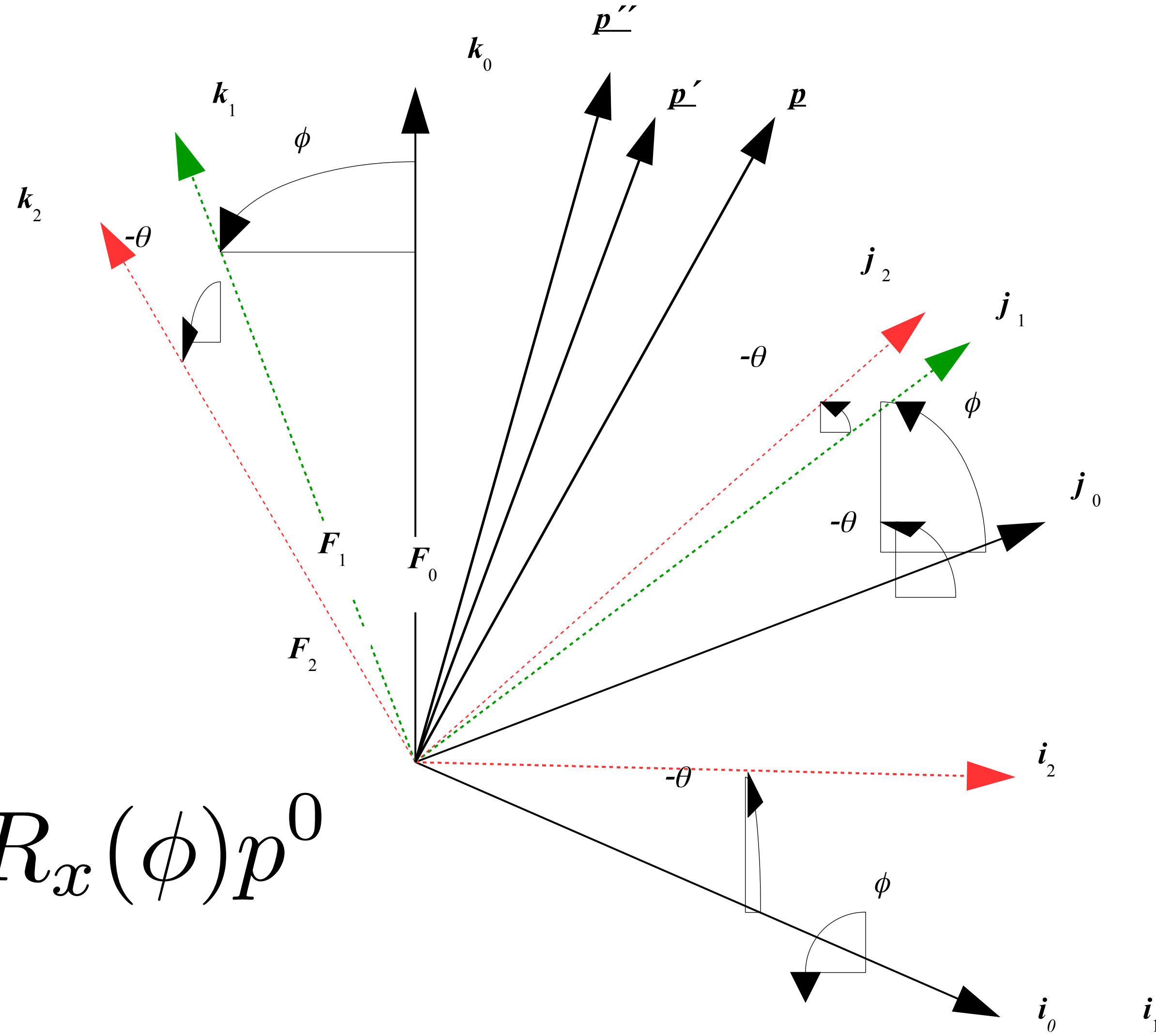
Fixed Frame Rotation - Example Problem

Substituting the value of $\underline{p}'$:



$$p''^{0} = R_y(-\theta)R_x(\phi)p^0$$

Fixed Frame Rotation -Generalized Form

$$R_{ik} = R_{k-1,k} \cdots R_{i+1,i+2} R_{i,i+1}$$

Observe

Fixed-frame (2) is in the reverse order of the current-frame (1) case:

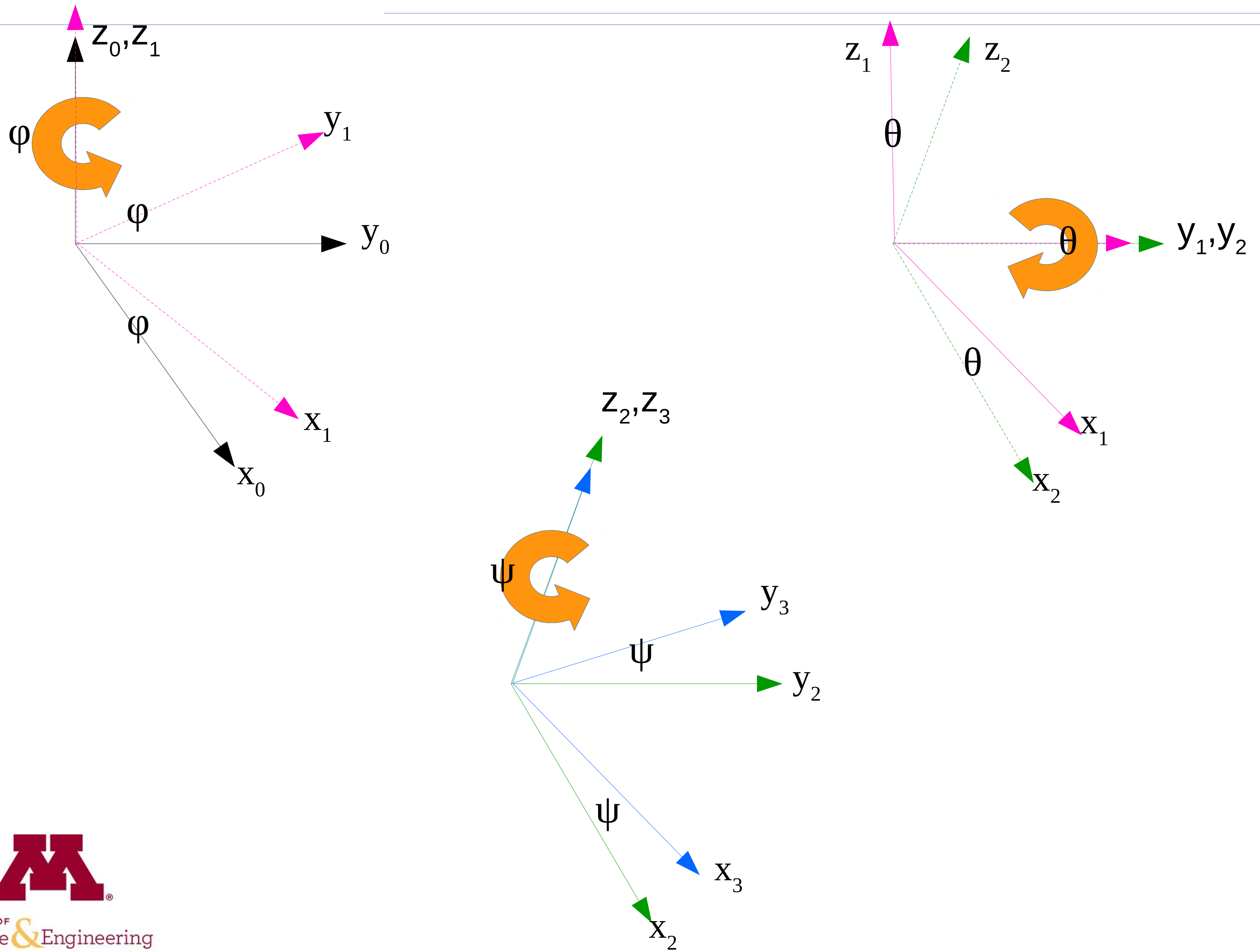
$$R_{ik} = R_{i,i+1} R_{i+1,i+2} \cdots R_{k-1,k} \quad (1)$$

$$R_{ik} = R_{k-1,k} \cdots R_{i+1,i+2} R_{i,i+1} \quad (2)$$

Euler Angles

- Rotations in the Current Frame (Z-Y-Z)
  - Rotate φ about the current (original) z_0 axis to form a new frame F_1
  - Rotate θ about the current y_1 axis to form a new frame F_2
  - Rotate ψ about the current z_2 axis to form a new frame F_3

(Z-Y-Z) Euler Angles



(Z-Y-Z) Euler Angles

$$R_{0,3} = R_z(\phi) R_y(\theta) R_z(\psi),$$

$$R_{0,3} = \begin{bmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix} \begin{bmatrix} cos\psi & -sin\psi & 0 \\ sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{0,3} = \begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & -c\phi c\theta s\psi - s\phi c\psi & c\phi s\theta \\ s\phi c\theta c\psi + c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi & s\phi s\theta \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix}$$

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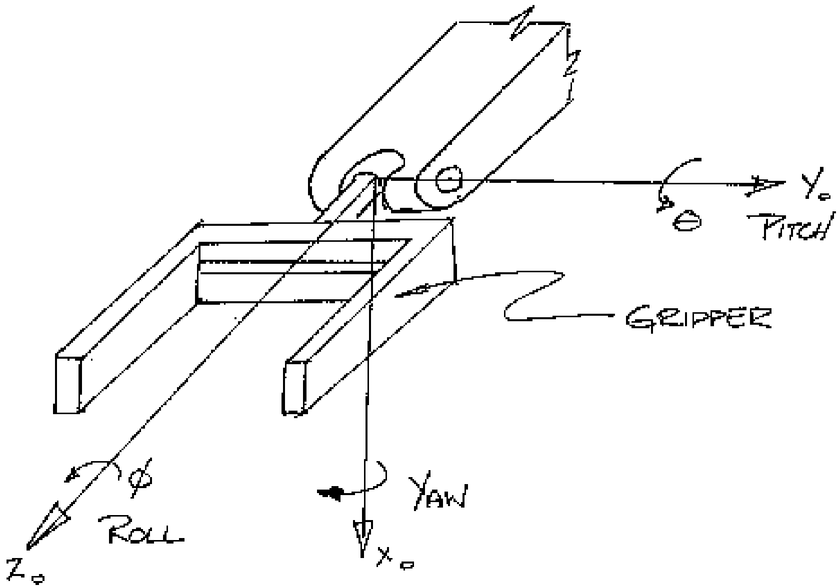
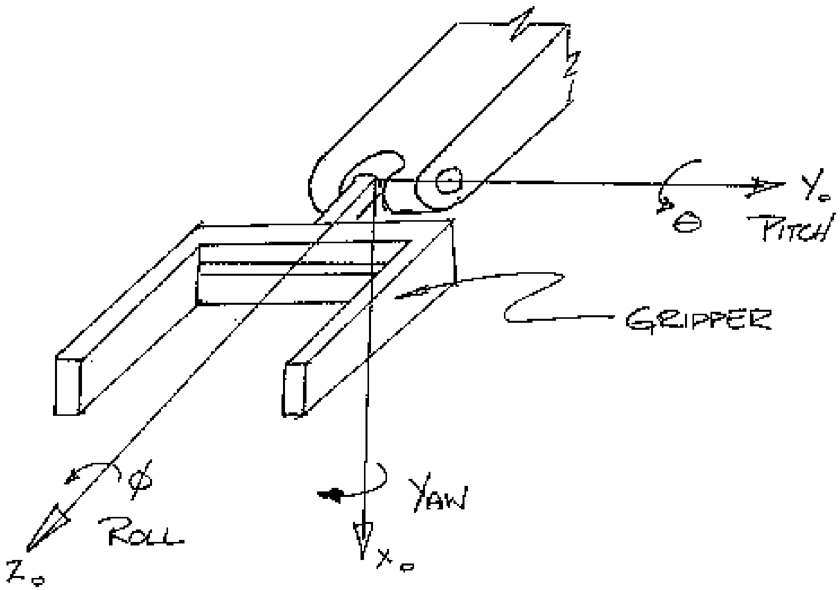
Roll-Pitch-Yaw Representations



- Frame does not move
  - But the gripper does
- Order of fixed-frame rotations:
  - Yaw about x_0
  - Pitch about y_0
  - Roll about z_0

Roll-Pitch-Yaw Representations



$$
\begin{aligned}
R &= R_z(\phi)R_y(\theta)R_x(\psi) \\
&= \begin{bmatrix} c\phi c\theta & c\phi s\theta s\psi - s\phi c\psi & c\phi s\theta c\psi + s\phi s\psi \\ s\phi c\theta & s\phi s\theta s\psi + c\phi c\psi & s\phi s\theta c\psi - c\phi s\psi \\ -s\theta & c\theta s\psi & c\theta c\psi \end{bmatrix}
\end{aligned}
$$

Summary

- Representation of rotations
  - Current frame
  - Fixed frame
  - Euler Angles (Z-Y-Z)
  - Pitch-Roll-Yaw

The Subaru that sucks (dirt)

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