Axioms of Probability

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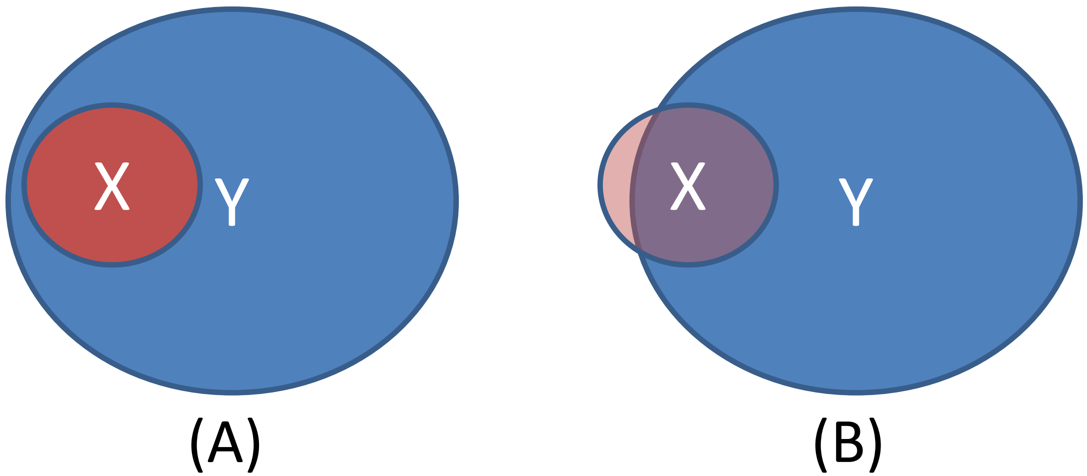
Deterministic vs. Nondeterministic



- Deterministic: in (A), "if Z Є X ➔ Z Є Y"
- Nondeterministic: in (B) the above is not true.
- But in (B), we can say that if Z Є X then it is very likely Z Є Y

The Essentials of Probability

- **Random Experiment:** an experiment whose outcome is not deterministic. It can be repeated under the same condition. (e.g. throwing a dice)
- **Outcome (or observation)**: the result of one single trial of the random experiment. (e.g. 3). *Outcome can be discrete (e.g. dice) or continuous (e.g. height)*
- **Sample space (or outcome space):** all possible outcomes of one single trial of a random experiment. (e.g. 1-6)
- **Event**: any set of possible outcomes of a random experiment (e.g. odd numbers)
  - When a random experiment is performed and the outcome is in an event A, then we say that event A has occurred.
- The **probability** is defined on an EVENT:
  - What is the probability that the outcome belongs to an event
  - (e.g. what is the probability the outcome is odd?)

Example (discrete outcome)

- Random Experiment: toss a coin twice
  - Sample space

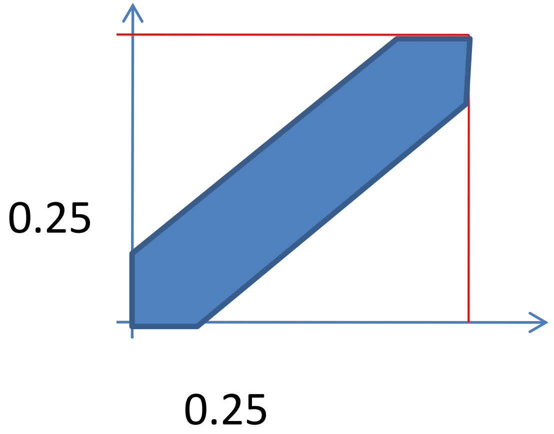
	First Toss	Second Toss
Outcome 1	H	H
Outcome 2	H	T
Outcome 3	T	H
Outcome 4	T	T

  - Events

    {at least one Head}, {two tosses are equal}, {at least one Tail}, {two or more Tails}, {two or more Heads}…etc

Example (continues outcome)

- Random Experiment: randomly pick two real numbers in [0,1]
  - Event E: they are at most 0.25 apart, P(E)=?



P(E)=1-0.75*0.75

Set Theory and Events

- Set (Event) Definitions
  - a set is a collection of objects
  - B is a set that contains 4 elements (1,2,3,4) can be represented as B= {1,2,3,4}
  - $x \in B$ ➔ element x is a member of set B

Set (Event) Definitions

- ***sample space*** **(*S*) :** the collection of all possible outcomes of a random experiment
- ***event*** **(*A,B,C*) :** collections of outcomes

Set (Event) Definitions

- Example:

  S={x|90<x<110} A={x|90<x<98} B={x|110>x$\geq$98}

  C={x|90<x<95} D={x|95<x<100} E={x|x=95}

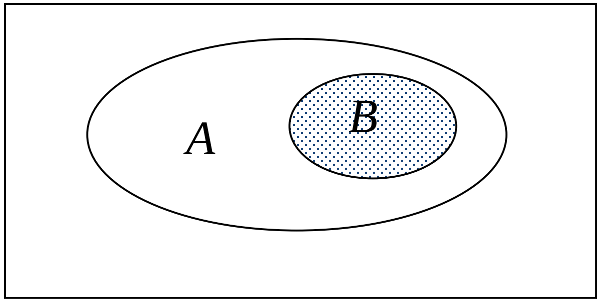
  - $\in$:“belongs”, $\notin$=“does not belong”

    $x_1=97 \Rightarrow x_1 \in A$, $x_2=104 \Rightarrow x_2 \notin A$

  - *S* contains **all** outcomes
  - ϕ contains **no** outcome, ***null*** *event,* ***impossible*** *event*

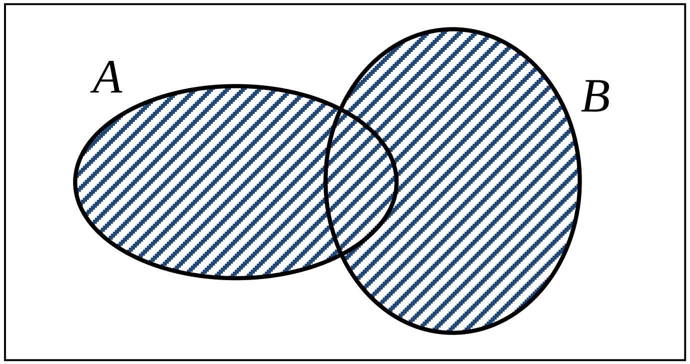
Events Algebra

- $\subseteq$: an event is contained in another event
  - $C \subseteq A$: C is a ***subset*** of A
  - $B \subset A$: B is a proper subset of A, A has at least one element that is not in B

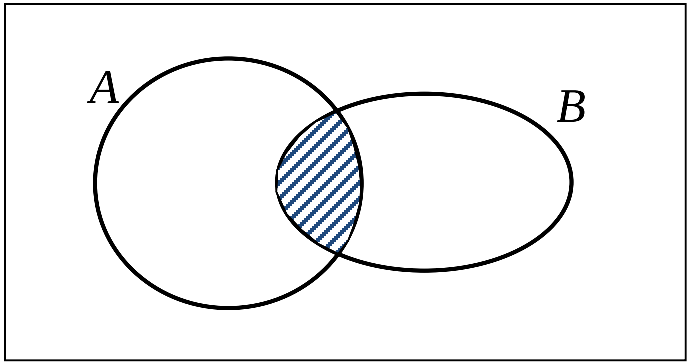


Events Algebra

- ***Union*** : the event that contains all the outcomes that are in either *A* or *B*
- ***Intersection*** : the event that contains all the outcomes that are in both *A* and *B*



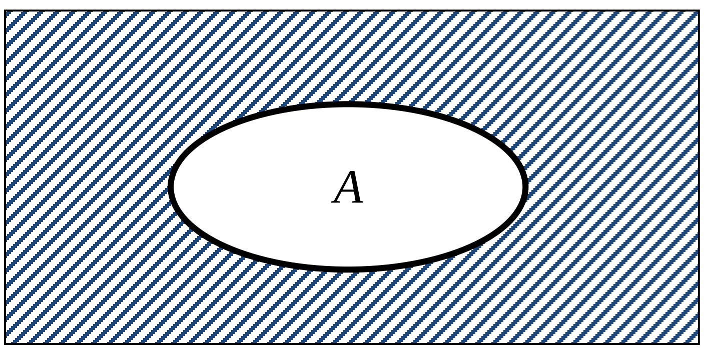
$A \bigcup B$



$A \bigcap B$

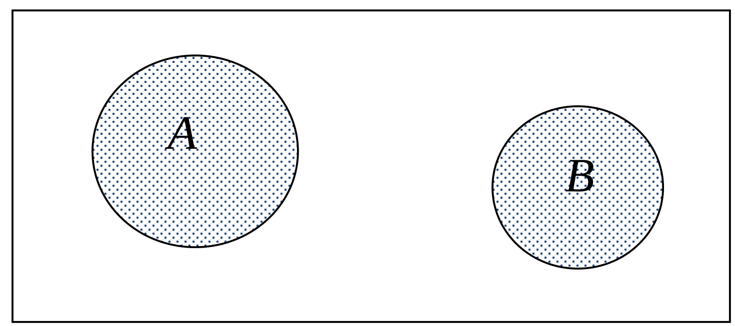
Events Algebra

- ***Complement*** **:** the event containing all outcomes that are not in a given *A*



$\overline{A}$ or A'

- ***Mutually Exclusive Events*** **:** events with no elements in common,(***mutually exclusive*** or ***disjoint***)



$$A \cap B = \phi$$

$$\Rightarrow A \cap \overline{A} = \phi, A \cup \overline{A} = S$$

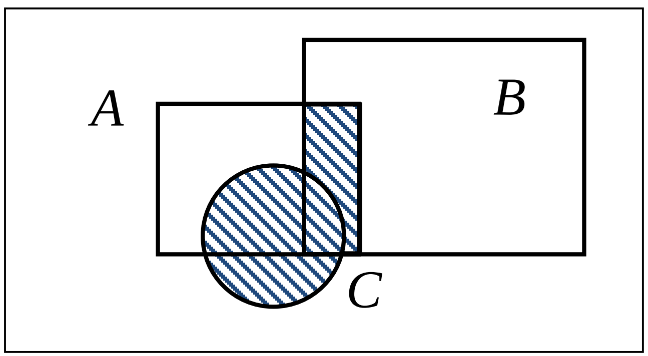
Events Algebra

- ***Properties*** **:**
  - ***Commutative property*** **:** the operations commute
    $A \cup B = B \cup A$, $A \cap B = B \cap A$
  - ***Associative property*** **:** the order of the operations is not important
    $(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$
    $(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$

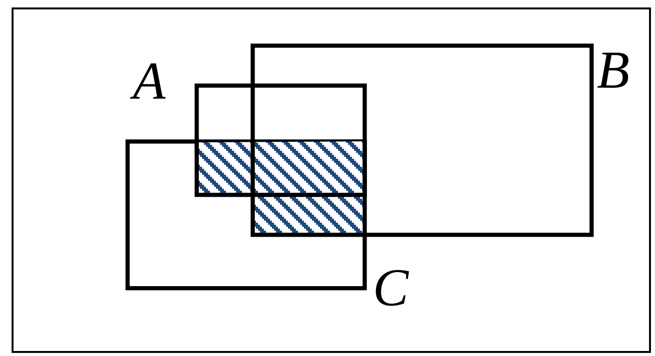
$$A_1 \bigcup A_2 \bigcup A_3 \bigcup \ldots \bigcup A_n = \bigcup_{i=1}^{n} A_i \qquad A_1 \bigcap A_2 \bigcap A_3 \bigcap \ldots \bigcap A_n = \bigcap_{i=1}^{n} A_i$$

Events Algebra

- ***Distributive rule*** **:**
  - union of intersecting
  - intersecting of union



$$C \cup (A \cap B) = (C \cup A) \cap (C \cup B)$$



$$C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$$

Events Algebra

- ***De Morgan's Law*** **:** the complement of an union or an intersection
  - union and intersection interchange
  - event and its complement interchange

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B} \qquad \overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

$$\overline{\bigcup A_i} = \bigcap \overline{A_i} \qquad \overline{\bigcap A_i} = \bigcup \overline{A_i}$$

Axioms of Probability

- Given a sample space S, for any event A, we define the probability of A by $P(A)$ as a real number assigned to the event A

  I. $P(A) \geq 0$

  II. $P(S) = 1$

  III. *if* A_1, A_2, A_3....are events, and $A_i \cap A_j = \phi$ *for all* $i \neq j$, *then* $P(A_1 \cup A_2 \ldots \cup A_k) = P(A_1) + P(A_2) + \ldots P(A_k)$

Properties

$$1: P\left(\overline{A}\right) = 1 - P(A)$$

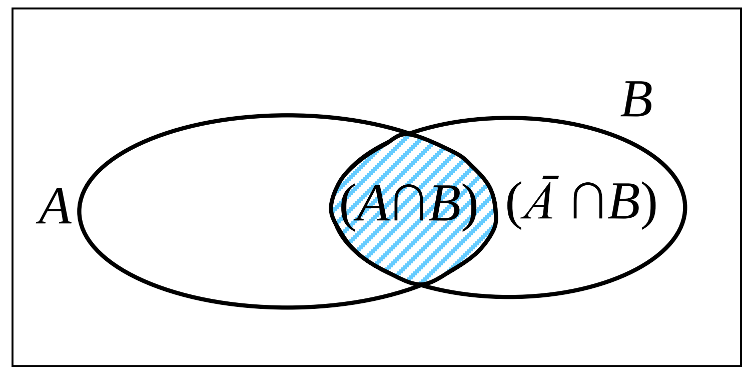
$$2: P(A) \leq 1$$

$$3: \text{if } B \subset A \Rightarrow P(B) < P(A)$$

$$4: P(\phi) = 0$$

Properties

$$5: P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



$$6: P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Game: Non-Transitive Dices