# Conditional Probability and Independent Events

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# Conditional Probability

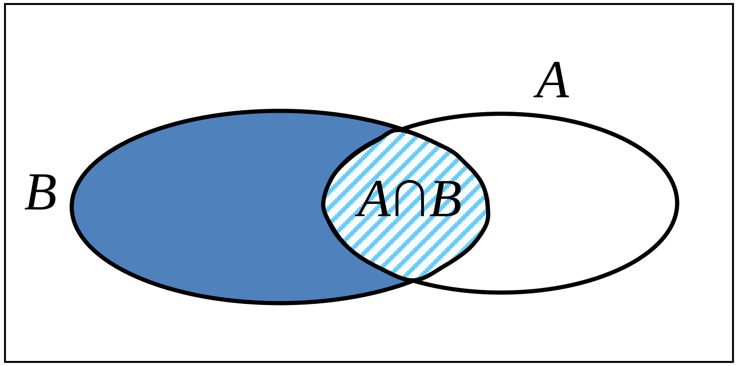
- What's the probability a person X gets 90 in the probability class? [X＝吳侑穎]
- What if we know that X never comes to the class.

[劉冠鴻]

# Conditional Probability

- ***Definition*** **:** Given two events *A*, *B* in a sample space *S* with probability assignment *P*, such that *P*(*B*) > 0, we define the conditional probability of *A* given *B* by the following expression:

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$



# Conditional Probability

a. If $A$ and $B$ are mutually exclusive ($A \cap B = \phi$), $P(A|B) = P(\phi)/\ P(B) = 0$

b. If $A \subset B$, then $(A \cap B) = A$, $\Rightarrow P(B|A) =$ 1

c. If $A \subset B$, then $(A \cap B) = A$, $\Rightarrow P(A|B)$ $(? \cancel{<\ =} >)$ $P(A)$

劉安淩

# Properties of Conditional Probability

- The conditional probability satisfies the following axioms for a probability function.

  with $P(B) > 0$

  (a) $P(A|B) \geq 0$

  (b) $P(B|B) = 1$

  (c) If $A_1, A_2, \ldots, A_k$ are mutually exclusive events, then $P(A_1 \cup A_2 \cup \ldots \cup A_k|B) = P(A_1|B) + P(A_2|B) + \ldots + P(A_k|B)$.

# Conditional Probability

- rolling two dices

  $B$ : sum of two dice is 5

  $B$ = {1 and 4, 2 and 3, 3 and 2, 4 and 1}

  $P(B) = 4/36$

  $A$ = {at least one roll is 2}

  $(A \cap B)$ = {2 and 3, 3 and 2}

  $P(A \cap B) = 2/36$

  $P(A|B) = P(A \cap B)/P(B) = (2/36)/(4/36)$

吳侑穎

# Example

- A dice is rolled. Let A be the event that 3 is rolled, and let B be the event that 5 is rolled.
- Then, in a sequence of rolls the probability that 3 is rolled before 5 is rolled is

$$P(3)/(P(3)+P(5))$$

- Why?

$$P(3)+P(O,3)+P(O,O,3)+P(O,O,O,3)+=P(3)(1+(1-P(3)-P(5))+(1-P(3)-P(5))^2+\ldots.)$$

# Conditional Probability

- $P(A \cap B) = P(A|B)P(B)$

  $P(A \cap B) = P(B|A)P(A)$ $\longrightarrow$ Multiplication Rule

  $P(A|B)P(B) = P(B|A)P(A) = P(A \cap B)$
- If $P(A|B) = P(A)$

  then the occurrence of $B$ has no influence on the occurrence of $A$.

  $A$ and $B$ are ***independent***

# Chain rule (or multiplication rule)

- $P(A \ldots Z) = P(A) * P(B \mid A) * P(C|A, B) * \ldots\ldots * P(Z \mid A \ldots Y)$

Note: people usually use $P(X,Y)$ to denote $P(X \cap Y)$

# Independence

- If $A$ and $B$ are ***independent***

  $P(A \cap B) = P(A)P(B)$

  $P(A|B) = P(A)$ and $P(B|A) = P(B)$

# Independence

- Example: throwing two dices

$B$ : both of them are 6

$C$ : both of them are 5

$P(B \cap C) = 0$
$\neq$
$P(B)P(C) =$ (1/36)(1/36)

$\Rightarrow$ $B$ and $C$ are ***dependent*** ($B$ occurred, $C$ could not have occurred)

[呂紹齊]

# Independence

- Example: throwing two coins

  $A$ = {exactly one Head}

  $B$ = {First toss is Head}

$\Rightarrow$ $A$ and $B$ are dependent or independent? ***Ans: independent***

P(A)= 0.5 , P(B)= 0.5 , $P(A \cap B)$ = 0.25

周俊廷

# Independence

- *If A* and *B* are independent, their complements are also independent.
- *Proof:*

$P(A' \cap B')=P((A \cup B)')=1-P(A \cup B)$

$=1-(P(A)+P(B)-P(A \cap B))=P(A')-P(B)+P(A)P(B)$

$=P(A')-P(B)(1-P(A))=P(A')(1-P(B))=P(A')P(B')$

- *How about A' and B? A and B'?*

周敦翔

# Examples

- A store sells three different brands of DVD players: A1, A2, and A3. Of the sold players, 50% are A1, 30% are A2, and 20% are A3. Out of the sold products, 25% of brand A1 broke within one year, and there is 20% and 10% chance for A2 and A3 to break. Let B denotes the broken event.
- Q1: What is the probability that a randomly selected purchaser has bought a A1 DVD player that needs repair within a year? $P(A1 \cap B)$
- Q2: What is the probability that a randomly selected purchaser has a DVD player that will need repair within a year? $P(B)$
- Q3: If the customer returns a broken DVD player within one year, what is the probability it is brand A1? $P(A1|B)$

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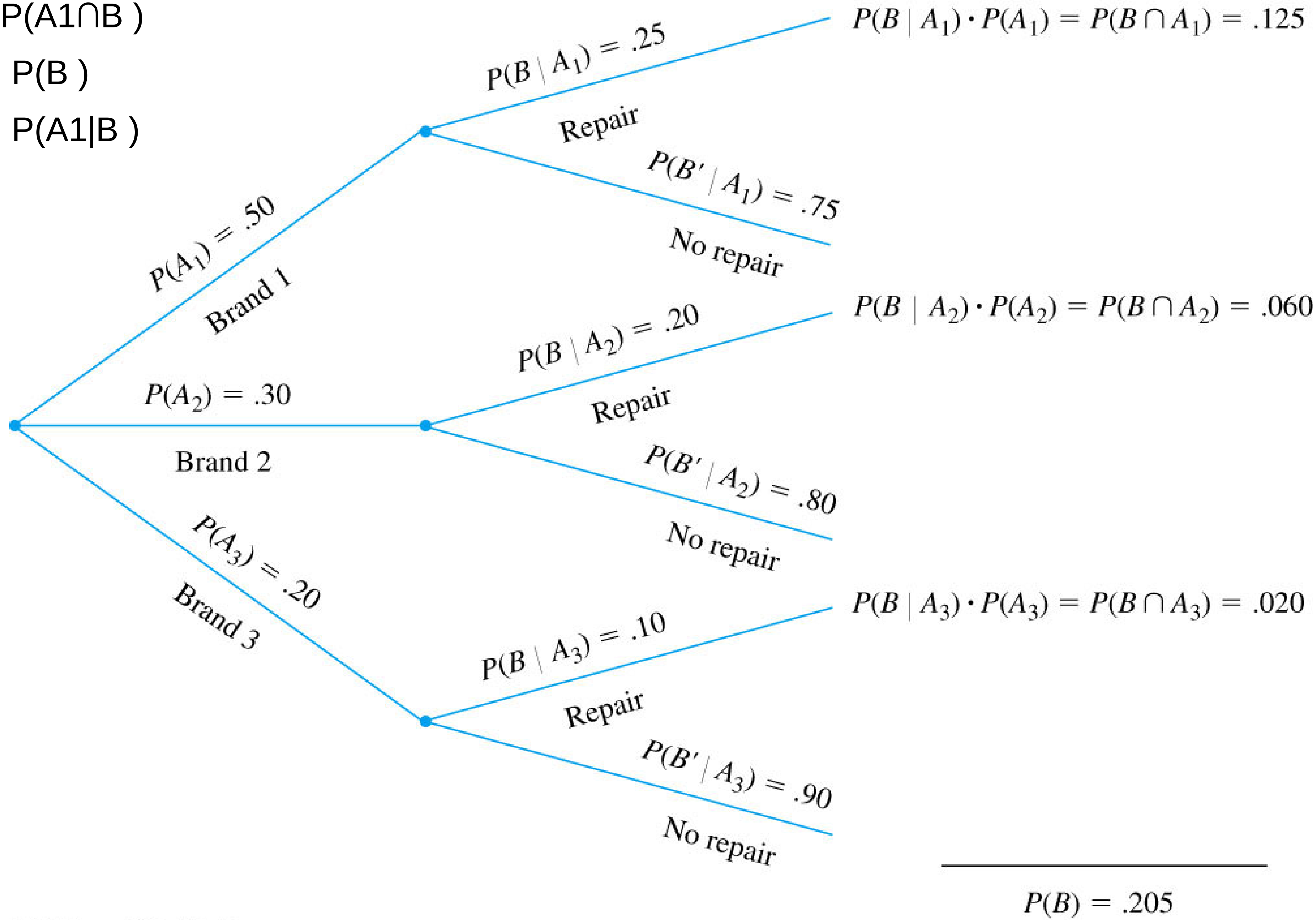
$P(A1 \cap B)$

$P(B)$

$P(A1|B)$



# Mutual Independence

- ***Independence of n Events***
  - $n$ events $\{A_i, i=1,2,\ldots,n\}$
  - If $n$ events are mutually independent, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2)\cdots P(A_k)$$

  - $\forall$m>=2, for every combination of $C_m^n$ elements, they are mutually independent

# Example

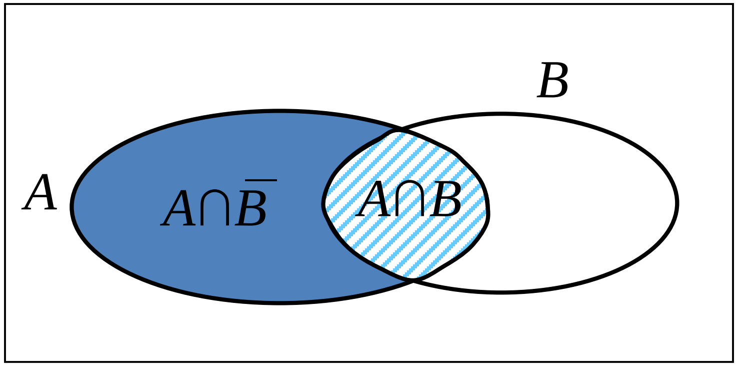
- RE(random experiment): Throwing a 4-sided dice
  - Event A: the outcome is 1 or 2, P(A)= ½
  - Event B: the outcome is 1 or 3, P(B)= ½
  - Event C: the outcome is 1 or 4, P(C)= ½

尚沂瑾

- Q1: Are A, B, and C pairwise independent?
  - P(A, B)=¼=P(A)*P(B) , P(A,C)= ¼ , P(B,C)= ¼
  - Ans: yes
- Q2: Are A, B, and C mutually independent?
  - No, because P(A,B,C)= ¼ ≠ P(A)*P(B)*P(C)

# Law of Total Probability

- ***Total Probability***
  - $A$ is divided into $(A \cap B)$ and $(A \cap \overline{B})$

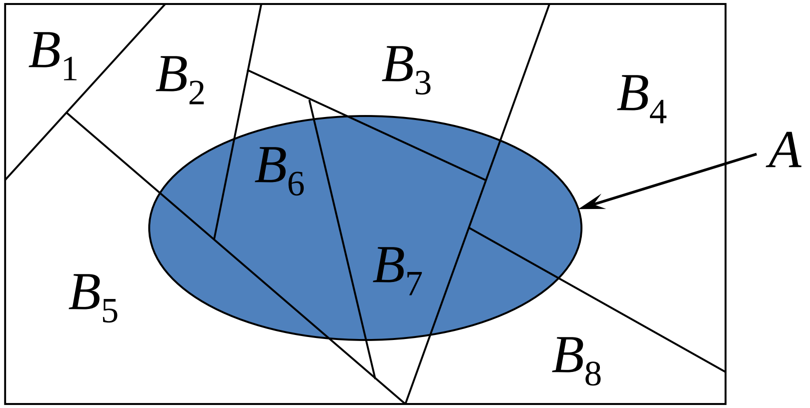



$$A = (A \cap B) \cup (A \cap \overline{B})$$

$$P(A) = P(A \cap B) + P(A \cap \overline{B}) = P(A|B)P(B) + P(A|\overline{B})P(\overline{B})$$

# Law of Total Probability (general)

- ***Total Probability***
  - $n$ mutually exclusive events $B_i$ for $i$=1,2,…,$n$,
  - together form the sample space $S$




$$B_i \cap B_j = \phi \text{ for } i \neq j, \quad S = \bigcup_{i=1}^{n} B_i, \quad A = \bigcup_{i=1}^{n} (A \cap B_i)$$

$$P(A) = P\left\{\bigcup_{i=1}^{n} (A \cap B_i)\right\} = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

# Example

- Dial-up connection with three types of channels

  $A$ = {a bit is received in error}

  $B_i$ = {connection is made via channel of Type $i$}

  $P(B_1) = 0.2,\ P(A|B_1) = 0.01$

  $P(B_2) = 0.3,\ P(A|B_2) = 0.005$

  $P(B_3) = 0.5,\ P(A|B_3) = 0.001$

  $P(A) = 0.01{\times}0.2{+}0.005{\times}0.3{+}0.001{\times}0.5{=}0.004$